Recitation 1. Solution

Focus: recognizing vector spaces, rules of matrix multiplication.

We will provide a formal definition of a vector space for the sake of honesty, but for the purpose of solving the first problem of this worksheet, you will need to check just three properties: when you add two elements, you get an element of the same set (being <u>closed under addition</u>); when you multiply an element by a scalar, the result is also in the same set (being <u>closed under multiplication by scalars</u>); and <u>zero</u> should belong to this set. These properties are boxed. Once again: we do not expect you to be able to recite all the axioms, although we definitely would be impressed if you are :)

Definition. A *real vector space* V is a set endowed with operations of $\boxed{\text{adding two vectors}}$ and $\boxed{\text{multiplying a vector by a real number}}$ such that the following holds for any three vectors u, v, w and any real numbers a, b:

- $(u + v) + w = u + (v + w)$;
- $u + v = v + u$;
- $\boxed{\text{there exists a } \textit{zero vector } 0 \in V \text{ such that for any vector } v\text{, we have } v + 0 = v}$;
- $a(bv) = (ab)v$;
- $1v = v$;
- $(a + b)v = av + bv$;
- $a(u + v) = au + av$.

Elements of a vector space are called *vectors*.

Remark. The most basic rule that you should remember: $\boxed{\textbf{row column}}$. It shows the order in which you write or compute, e.g.:

- The first index denotes the row, the second number the column.
- You multiply a row by a column to get a number.
- An $n \times m$ matrix has n rows and m columns.

Notation. We will denote by A^T the transpose of a matrix A.

1. Is this a vector space? Why / why not? Which natural operations you considered when checking axioms?

   a) The line $y = x$.
   b) The line $y = x + 1$.
   c) The union of the x and y axes.
   d) The unit circle $\{(x, y) \mid x^2 + y^2 = 1\}$.
   e) The set of 5×5 matrices with the element in position $(3, 3)$ being 0.
   f) Functions of the form $f(x) = ax^2 + bx + c$.
   g) Functions $f(x)$ with $f(7) = 0$.
   h) Functions $f(x)$ with $f(0) = 7$.
   i) *Tricky question.* Newtonian universe.

Solution:

a) Yes.

b) No, because the set is not closed under addition. For example, the points $\binom{-1}{0}$ and $\binom{0}{1}$ belong to the line, but their sum $\binom{-1}{1}$ does not. Neither is the set closed under multiplication or has zero.

c) No, because the set is not closed under addition. Example of points is the same as above. However, it is closed under scaling and contains zero.

d) No, for the same reasons as in part (b).

e) Yes. Adding two matrices or multiplying such a matrix by a number does not affect the property that the middle element is zero.

f) Yes, the set of quadratic polynomials is a vector space.

g) Yes.

h) No, because the set is not closed under addition: if you add two functions f and g with $f(0) = g(0) = 7$, then their sum evaluates to 14 at 0.

i) No, we don't have a zero, because there is no natural reference point.

2. *Rules of matrix multiplication. (Section 2.4 of Strang.)* Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix},\ C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},\ E = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Which of these matrix operations are allowed, and what are the results?

a) AB
b) AB^T
c) B^TA
d) $(A+B)C$
e) $(A+B)C^T$
f) $C(A+B)$
g) DB
h) BD
i) AE
j) EA
k) CAE

Solution: In order to multiply two matrices, number of columns in the first should be equal to number of rows in the second.

a) AB not allowed: we cannot multiply a 2×3 matrix by a 2×3 matrix.

b) $AB^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -1 & -2 \end{pmatrix}.$

c) $B^TA = \begin{pmatrix} -1 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -4 & -1 & -2 \\ 9 & -3 & 6 \\ 7 & 0 & 4 \end{pmatrix}.$

d) $(A+B)C$ not allowed.

e) $(A+B)C^T$ not allowed.

f) $C(A+B) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -2 \\ -4 & -4 & -8 \end{pmatrix}$.

g) DB not allowed.

h) BD not allowed.

i) $AE = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$.

j) EA not allowed.

k) $CAE = C(AE) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} -14 \\ -18 \end{pmatrix}$.

3. When you multiply an $n \times m$ matrix by an $m \times l$ matrix, what are the dimensions of the resulting matrix?

Solution: $(n \times m)(m \times l) \to (n \times l)$.

4. When can a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be written as $X^T X$ for some other matrix $X = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$? Assume that $b \neq 0$. What are p, q, r in terms of a, b, c, d when possible?

Solution: In order to describe the condition, we compare matrix elements in the desired equality $A = X^T X$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & qr \\ qr & r^2 \end{pmatrix}.$$

One can note that matrix elements of $X^T X$ are dot products of the columns of X.

One can immediately observe that A should satisfy $a \geq 0$ and $d \geq 0$, because square of a real number is always nonnegative, and $b = qr = c$, that is A is necessarily symmetric. In addition, if either a or d is zero, then $b = c = 0$ as well, because vanishing of a implies that $q = 0$, so $b = c = qr = 0$, and similarly for d. There will be one more condition which we will find later.

Now we turn to expressing p, q and r in terms of a, b, c and d:

- $r = \sqrt{d}$;
- if $d \neq 0$, then $q = \frac{b}{r} = \frac{b}{\sqrt{d}}$ and $p = \sqrt{a - q^2} = \sqrt{a - \frac{b^2}{d}}$, hence we have an additional constraint that $a - \frac{b^2}{d} \geq 0$;
- if $d = 0$, then $r = 0$ and p and q are any real numbers such that the vector $\begin{pmatrix} p \\ q \end{pmatrix}$ lies on the circle of radius a.

5. Consider the following matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

How is each row of BA, CA, DA related to the rows of A? How is each column of AB, AC, AD related to the columns of A?

Solution: I will only write the solution for rows, because what happens to columns is exactly the same after you change the order of multiplication.

- The first row of BA is twice the first row of A, and the second is minus the second row of A.
- The first row of CA is the second row of A, while the second row is zero.
- The first row of DA is the second row of A and the second row of DA is minus the second row of A.

So you can see that multiplying a matrix A by another matrix on the left performs row operations. Similarly, right multiplication performs column operations. You will see more in the next problem.

6. *In this problem, we will practice block multiplication. (Page 75 of Strang.)* Consider the following column vector c and a 3×3 matrix A with columns a_1, a_2, a_3:

$$c = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}, \; A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}.$$

Write the result of matrix multiplication rA as a linear combination of the column vectors a_1, a_2, a_3. What if we write a matrix R as three rows $R = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix}$ and multiply R by A?

Solution: First compute Ac, and note that a_1, a_2, a_3 are all 3-vectors:

$$Ac = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \lambda a_1 + \mu a_2 + \nu a_3.$$

This is a particular case of block multiplication.

Now calculate RA – this turns out to be the usual rule of matrix multiplication:

$$RA = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix} \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} r_1a_1 & r_1a_2 & r_1a_3 \\ r_2a_1 & r_2a_2 & r_2a_3 \\ r_3a_1 & r_3a_2 & r_3a_3 \end{pmatrix}.$$