18.06 R08 - Recitation 3

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1 Lecture review

1.1 Independence and bases

1. A set of vectors $v_1, v_2, ..., v_n$ is linearly independent if
$$\boxed{a_1v_1 + a_2v_2 + ... + a_nv_n = 0 \iff a_1 = a_2 = ... = a_n = 0}$$
2. A set of vector $v_1, v_2, ..., v_n$ *spans* a vector space if every element of the space can be written as a linear combination of these n vectors.
3. A set of vector $v_1, v_2, ..., v_n$ is a *basis* if the vectors are linearly independent and span the space.
4. The *dimension* of a vector space is the number of elements in a basis for the vector space (this number is well-defined).
5. The *standard basis* for $\mathbb{R}^n$ is the set of vectors $e_1, e_2, ..., e_n$. The vectors e_i have a 1 in their i-th component, and every other component is zero.

1.2 Column spaces and linear systems

1. We say that an $m \times n$ matrix A has full column rank if $r = n$, and full row rank if $r = m$. The rank is always less than the minimum of m and n.
2. There are four possibilities for a linear system $Ax = b$:
   - $Ax = b$ has a unique solution for all b. This can only happen if $r = m = n$ (full column and row rank)
   - $Ax = b$ has either a unique solution or no solution, depending on b. This can only happen if $r = n$, but $m > n$ (full column rank, not full row rank).
   - $Ax = b$ has infinitely many solutions for all b. This can only happen if $r = m$, but $n > m$ (full row rank, not full column rank).
   - $Ax = b$ has either no solution or infinitely many solutions, depending on b. This can only happen if $r < \min(m, n)$ (neither full column nor row rank).

1.3 Projections

1. The projection p of a vector b onto a vector a is given by the formula:
$$\boxed{p = Pb, \quad P = \frac{aa^T}{a^Ta}}$$
2. The projection p of a vector b onto the column space of a matrix A with full column rank is given by the formula:
$$\boxed{p = Pb, \quad P = A(A^TA)^{-1}A^T}$$
3. If $A = QR$ with R invertible, then the projection of p onto the column space of A is given by the simpler formula:
$$\boxed{p = Pb, \quad P = QQ^T}$$

2 Problems

Problem 1.
Can a set of linearly independent vectors contain the zero vector?

Problem 2.
Find a basis for the following vector spaces and state the dimension of the vector space[1]:

1. The set of all polynomials with degree ≤ 3.
2. The set of all vectors whose components are equal.
3. The set of all vectors in $\mathbb{R}^3$ whose components average to zero.
4. The set of all 3×3 antisymmetric matrices.

[1]To attempt these kinds of questions you should do the following: firstly write down the most general vector in your vector space. Then deconstruct this vector into a set of vectors that you can be certain spans the vector space. Then finally test whether they are linearly independent. If they *are* linearly independent then you're all set. If they are not, then try to use your intuition to figure out how to make them independent.

Problem 3.
Consider the following four full SVDs:

$$A_1 = \begin{pmatrix} -0.1965 & -0.3551 & -0.7175 & 0.5661 \\ -0.2649 & -0.7272 & 0.5976 & 0.2094 \\ -0.4527 & -0.3208 & -0.3224 & -0.7669 \\ -0.8284 & 0.4921 & 0.1553 & 0.2179 \end{pmatrix} \begin{pmatrix} 11.0304 & 0 & 0 \\ 0 & 3.2142 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.0318 & -0.8159 & -0.5774 \\ -0.7225 & -0.3804 & 0.5774 \\ -0.6907 & 0.4355 & -0.5774 \end{pmatrix}^T$$

$$A_2 = \begin{pmatrix} -0.1408 & 0.8944 & 0.4245 \\ 0.2816 & 0.4472 & -0.8489 \\ -0.9492 & 0 & -0.3148 \end{pmatrix} \begin{pmatrix} 4.0600 & 0 & 0 & 0 \\ 0 & 1.7321 & 0 & 0 \\ 0 & 0 & 1.2315 & 0 \end{pmatrix} \begin{pmatrix} -0.2685 & 0.5164 & 0.0890 & -0.8083 \\ -0.1644 & 0.2582 & -0.9450 & 0.1155 \\ -0.8054 & 0.2582 & 0.2671 & 0.4619 \\ 0.5022 & 0.7746 & 0.1666 & 0.3464 \end{pmatrix}^T$$

$$A_3 = \begin{pmatrix} -0.7503 & -0.5300 & 0.3951 \\ -0.4961 & 0.8464 & 0.1935 \\ -0.4370 & -0.0509 & -0.8980 \end{pmatrix} \begin{pmatrix} 6.4901 & 0 & 0 \\ 0 & 4.6650 & 0 \\ 0 & 0 & 1.0569 \end{pmatrix} \begin{pmatrix} -0.4796 & 0.4089 & -0.7764 \\ -0.6043 & 0.4876 & 0.6301 \\ -0.6363 & -0.7714 & -0.0132 \end{pmatrix}^T$$

$$A_4 = \begin{pmatrix} -0.5647 & -0.1174 & 0.6460 & -0.5000 \\ 0.0779 & 0.6453 & 0.5723 & 0.5000 \\ -0.1627 & 0.7548 & -0.3921 & -0.5000 \\ -0.8053 & -0.0078 & -0.3184 & 0.5000 \end{pmatrix} \begin{pmatrix} 3.9255 & 0 & 0 \\ 0 & 2.1292 & 0 \\ 0 & 0 & 0.2393 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.0216 & 0.6576 & 0.7531 \\ -0.9446 & 0.2332 & -0.2308 \\ 0.3274 & 0.7164 & -0.6161 \end{pmatrix}^T$$

Decide which of the above matrices corresponds to each of the following situations. In each case state a b for which $Ax = b$ has a solution.

1. $Ax = b$ has 0 or 1 solutions, depending on b
2. $Ax = b$ has infinitely many solutions, regardless of b
3. $Ax = b$ has 0 or infinitely many solutions, depending on b
4. $Ax = b$ has a unique solution, regardless of b.

Problem 4.

1. Find the projection p of the vector b onto the column space of A, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

   Verify that $e = b - p$ is orthogonal to the columns of A.

2. ****If P is a projection matrix, then show that $(I-P)^T = (I-P)$ and $(I-P)^2 = I-P$ (so $I-P$ is also a projection matrix). If P projects onto the column space of a matrix A, then $I-P$ projects onto which subspace? If $P = QQ^T$, where Q is orthogonal, show that $B = I-2P$ is orthogonal.