Summary for Week 7 Recitation

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This week we will go through a very crucial concept in linear algebra. Determinant!

Given $A\vec{x} = \vec{b}$ where A is a square matrix. We know that this equation has a unique solution for arbitrary $\vec{b}$ if and only if A is invertible, that is there exists another matrix B such that $AB = BA = I$. To compute the existence of inverse, we could use the Gauss-Jordan method. In theory, we want a numerical criterion to "determine" whether A is invertible.

There are several ways to define determinant, such as sum of $n!$-terms, cofactor expansion, axiomatic approaches using multilinear property. In my opinion, the easiest way for computation (by hand) is doing row/column operations to reduce the complexity of a matrix. Using cofactor expansion together with induction works quite efficiently sometimes. The axiomatic approach is good for proving statements abstractly. Of course, you should memorize the formula for 2 by 2 matrices and 3 by 3 matrices. For a random matrix of large size, Julia is preferred.

Determinant of a 2 by 2 matrix has a wonderful geometric interpretation. It is equal to the *signed* area of the parallelogram with column vectors as sides. The area becomes zero if two column vectors lie on the same line. Otherwise, the area will be non-zero. In the latter case, it means the two column vectors span the whole plane. In other words, $A\vec{x} = \vec{b}$ is solvable for any $\vec{b}$. There are higher dimensional analogs, though it is quite difficult to visualize a 6-dimension parallelogram.

From this geometric point of view, it is easy to see rotation matrix has determinant of one or minus one. Can you prove the higher dimensional analogue? That is, the determinant of an orthogonal matrix has to be one or minus one.

Here is a side remark, you may think the geometry is quite easy at first glance. Well. Without using the formula, can you give a proof of the triangle with vertices (m_1, n_1), (m_2, n_2), (m_3, n_3) is a half-integer for arbitrary six integers m_1, n_1, m_2, n_2, m_3, n_3?