Rect 5 soul

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1. Linearly independent and span.

   - The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are **linearly independent** if $c_1 v_1 + \dots c_n v_n = 0 \Rightarrow c_1 = \dots = c_n = 0$
   - The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ **span** the vector space S if $\forall s \in S$ is a linear combination of $v_1 \dots v_n$.
   - The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are a **basis for** S if $v_1 \dots v_n$ linearly independent and span S
   - The **dimension of a space** S is n.

2. Solve $Ax = b$. Let A be a rank r $m \times n$ matrix

   - $Ax = b$ is solvable if
     - b is in $col(A)$,
     - $\text{rank}(A) = \text{rank}[A|b]$
     - $U U^T b = b$
   - Suppose $Ax = b$ is solvable. Then
     – if $r < n$, $Ax = b$ has infinitely solutions.
     – if $r = n$, $Ax = b$ has a unique solutions.

Problems

1. (a) Find a basis for symmetric 2×2 matrices. What is the dimension of symmetric 2×2 matrices? Also, do this for skew-symmetric 2×2 matrices.

Symmetric dim=3 basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

skew-symmetric. dim = 1 basis $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b) Let $\vec{v_1}, \vec{v_2}, ..., \vec{v_m}$ be a basis for symmetric 2×2 matrices. Let $\vec{w_1}, \vec{w_2}, ..., \vec{w_n}$ be a basis for skew-symmetric 2×2 matrices. Are $\vec{v_1}, \vec{v_2}, ..., \vec{v_m}, \vec{w_1}, \vec{w_2}, ..., \vec{w_n}$ linearly independent? What subspace of 2×2 matrices do they span?

Linear independent.

$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} a & c+d \\ c-d & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} a = 0 \\ b = 0 \\ c+d = 0 \\ c-d = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ d = 0 \end{matrix}$

They span all the 2x2 matrices

2. A set F consists of functions $f(x) = ae^x + be^{-x}$, where $a, b \in \mathbb{R}$.

   (a) Is F a vector space? Provide a basis of F.

   Yes $f_1(x) = a_1 e^x + b_1 e^{-x}$

   $f_2(x) = a_2 e^x + b_2 e^{-x}$

   $kf_1 + lf_2 = (ka_1 + la_2)e^x + (kb_1 + lb_2)e^{-x}$

   basis: e^x, e^{-x}

   (b) Define a map $V : F \to \mathbb{R}^2$ that sends $f(x) = ae^x + be^{-x}$ to $V(f) = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. What is $V(f')$ for $f \in F$? Find a matrix A satisfying $AV(f) = V(f')$ for all $f \in F$.

   $f(x) = ae^x + be^{-x}$

   $f'(x) = ae^x - be^{-x}$

   $V(f') = (a, -b)$

   $A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$

   $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   (c) Show that $A^2 = I_2$ and thus $\frac{d^2}{dx^2} f(x) = f(x)$.

   $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   $\frac{d^2}{dx^2} f(x) = \frac{d}{dx}(ae^x - be^{-x})$

   $= ae^x + b\cancel{e^{-x}} = f(x)$

3. (a) Find a basis for the space of all the degree ≤ 3 polynomials. $V = \{f(x) = a + bx + cx^2 + dx^3 | a, b, c, d \in \mathbb{R}\}$.

   1. x. x^2 x^3

   (b) Find a basis for the subspace W of V consisting all $f \in V$ such that $f(1) = 0$.

   $(x-1)$, $(x-1)^2$, $(x-1)^3$

   or $x-1$, x^2-1, x^3-1

   or $1-x^3$, $x-x^3$, x^2-x^3

   $f(x) = a + bx + cx^2 + dx^3$

   $f(1) = a + b + c + d = 0$

   So $f(x) = -b - c - d + bx + cx^2 + dx^3$

   $= b(x-1) + c(x^2-1) + d(x^3-1)$

   So basis is $x-1$, x^2-1, x^3-1.

   (c) Find a basis for the subspace U of V consisting all $f \in V$ such that $f(-1) = f(1)$.

   $f(x) = a + bx + cx^2 + dx^3$

   $f(1) = a + b + c + d$

   $f(-1) = a - b + c - d$

   $f(1) = f(-1)$

   $\Rightarrow$ $b + d = 0$

   So $f(x) = a + bx + cx^2 - bx^3$

   basis 1, x^2, $x - x^3$,