1 Lecture Review

1.1 Similar Matrices

1. Two $n \times n$ matrices A and B are *similar* if there exists an invertible matrix X so that
$$A = XBX^{-1}.$$
2. A is similar to itself.
3. If A and B are similar and B and C are similar, then A and C are similar.

1.2 Symmetric Matrices

1. Every real symmetric matrix S can be diagonalized
$$S = Q\Lambda Q^{-1}$$
where Q is orthogonal.
2. A real symmetric S has n real eigenvalues and n orthonormal eigenvectors.

1.3 Systems of Differential Equations

1. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $u(t) = e^{\lambda t}\boldsymbol{x}$ solves $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$.
2. If $A = X\Lambda X^{-1}$ is an eigendecomposition, then
$$e^{At} = I + At + \cdots + (At)^n/n! + \cdots = Xe^{\Lambda t}X^{-1}.$$

2 Problems

1. True or false, explain:
   (a) If A and B are similar, then $A - I$ and $B - I$ are similar.
   (b) There is a matrix $A \neq I$ which is similar to the identity.
   (c) If $A = X\Lambda_A X^{-1}$ and $B = X\Lambda_B X^{-1}$ where Λ_A, Λ_B are diagonal, then $AB = BA$.
   (d) If $A^3 = 0$, then the eigenvalues of A must be 0.
   (e) A matrix with real eigenvalues and n linearly independent eigenvectors is symmetric.
   (f) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.
   (g) The inverse of an invertible symmetric matrix is symmetric.
   (h) The eigenvector matrix Q of a symmetric matrix is symmetric.

2. Suppose A is symmetric. Explain how the diagonalization $Q\Lambda Q^{-1}$ of A can be used to produce a full form SVD $U\Sigma V^T$ of A.

3. Suppose A is $n \times n$ symmetric and B is $m \times n$. Show that BAB^T is symmetric.

4. If A is upper triangular with distinct diagonal entries, is it diagonalizable?

5. Suppose A is an $n \times n$ upper triangular matrix. Show that if A is diagonalizable with n orthonormal eigenvectors, then A is a diagonal matrix.

6. If A is $m \times n$ is $A^T A$ symmetric? If so what are the eigenvalues of $A^T A$ in terms of the (full form) SVD of A? What are the eigenvectors? How about AA^T?

7. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Find e^{At} using the fact that $A^2 = A^3 = \cdots = 0$. How might you compute e^{At} for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$?

8. Explain why $e^{\operatorname{tr} A} = \det e^A$ (Note that $e^{\operatorname{tr} A}$ is a number and e^A is a matrix). You may assume A is diagonalizable.

9. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.
   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors. What can be said about the eigenvectors of different eigenvalues of A and how this connects to $A = A^T$?
   (c) Find two linearly independent solutions to $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$.