1 Lecture Review

1.1 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. A *nonzero* vector $v \in V$ is called an *eigenvector* for the matrix A if for some real or complex scalar λ we have $Av = \lambda v$.

2. The value λ is then called the *eigenvalue* corresponding to this eigenvector v.

3. Since for the eigenvector v we have $(A - \lambda)v = 0$, the matrix $A - \lambda I$ is not invertible, and so an eigenvalue is necessarily a root of the polynomial $\chi_A(\lambda) = \det(A - \lambda I)$.

4. A matrix is *diagonalizable* if $A = X\Lambda X^{-1}$ for some invertible matrix X and some diagonal matrix Λ. In this case, the diagonal entries of Λ are the eigenvalues of A. If λ_i is the ith diagonal entry of Λ, then the ith column vector of X is an eigenvector with eigenvalue λ_i. This representation of A as $X\Lambda X^{-1}$ is called *eigendecomposition*.

5. A matrix is diagonalizable if and only if there exists a linearly independent set of n eigenvectors of A.

6. If A has n distinct eigenvalues (all the roots of $\chi_A(\lambda)$ are different), then A is diagonalizable; note the reverse direction is not true in general.

2 Problems

1. Suppose we have $B = XAX^{-1}$.
   (a) Prove that $\chi_B(\lambda) = \chi_A(\lambda)$.
   (b) How are eigenvalues of B related to those of A?
   (c) How are eigenvectors of B related to those of A?
   (d) Suppose that one of the eigenvalues of A is zero. Does it mean that A is singular? Does it mean that B is singular?

Solution. (a) We have

$$\chi_B(\lambda) = \det(B-\lambda I) = |XAX^{-1}-\lambda I| = |X(A-\lambda I)X^{-1}| = |X||A-\lambda I||X|^{-1} = |A-\lambda I| = \chi_A(\lambda).$$

(b) The eigenvalues of B are the roots of $\chi_B(\lambda)$ and the eigenvalues of A are the roots of $\chi_A(\lambda)$. By the previous part, the eigenvalues of A and B are the same.

(c) If v is an eigenvector of A for an eigenvalue λ, then $w = Xv$ is an eigenvector of B for the eigenvalue λ because

$$Bw = XAX^{-1}Xv = XAv = X\lambda v = \lambda Xv = \lambda w.$$

(d) Suppose 0 is an eigenvalue of A. So if v is an eigenvector of A, then

$$Av = 0v = 0$$

so that null(A) contains a nonzero element. Since the null space is nontrivial, this means A is singular. Since the eigenvalues of A and B are the same, this means B is singular.

□

2. Give an example of a diagonalizable matrix with a pair of equal eigenvalues.

   Solution. The 2×2 identity matrix I has $\chi_I(\lambda) = (\lambda - 1)^2$. So it has eigenvalues $1, 1$. It is diagonalizable because III^{-1}. □

3. Prove that if n is odd and A is $n \times n$, then A has at least one real eigenvalue.

   Solution. We have $\chi_A(\lambda) = \det(A - \lambda I)$ is a degree n polynomial. Any odd degree polynomial must have one real eigenvalue. $\square$

4. *Closed formula for Fibonacci numbers.* Let F_i denote the ith element in the Fibonacci sequence, defined by setting $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all natural values of i (including zero).

   (a) Find a matrix A such that $A\begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix}$.

   Solution. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. □

   (b) Find the eigenvalues of A. Let φ denote the largest eigenvalue.

   Solution. First compute the characteristic polynomial: $\chi_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$.
   Now compute the discriminant $D = 1 + 4 = 5$.
   Then the eigenvalues are $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618$.
   Note that they are related as follows: $\varphi + \overline{\varphi} = 1$, $\varphi\overline{\varphi} = -1$ and $\varphi - \overline{\varphi} = \sqrt{5}$ □

   (c) Find the eigenvectors of A.

   Solution. Since there are two distinct eigenvalues, each of the matrices $A - \varphi I$ and $A - \overline{\varphi} I$ has exactly one-dimensional kernel (nullspace).
   First find eigenvector v_1 for eigenvalue φ. It should satisfy $\begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix} v_1 = 0$. Since we know that the matrix is of rank one, we can look for a vector from the nullspace of the second row, and we see that $v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$.
   Similarly, the vector $v_2 = \begin{pmatrix} \overline{\varphi} \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\overline{\varphi}$.
   For the eigendecomposition, we know that we can write $X = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, then:

$$\begin{aligned} A &= \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\varphi - \overline{\varphi}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} = \\ &= \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix}. \end{aligned}$$

   □

   (d) Compute A^{50} up to nine decimal points. You can only use simple calculators (e.g. Google engine), no matrix calculators are needed.

   Solution.

$$\begin{aligned} A^{50} &= \left(X\Lambda X^{-1}\right)^{50} = X\Lambda^{50}X^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}^{50}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} \approx \\ &\approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & 0 \\ 0 & 4 \cdot 10^{-11} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} \approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} = \\ &= \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & -\varphi^{50} \cdot \overline{\varphi} \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} & -\varphi^{50}\varphi\overline{\varphi} \\ \varphi^{50} & -\varphi^{49}\varphi\overline{\varphi} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} & \varphi^{50} \\ \varphi^{50} & \varphi^{49} \end{pmatrix}. \end{aligned}$$

   □

(e) Using the result of part (c), explain why $\frac{F_{50}}{F_{49}}$ is very close to φ.

Solution. We will compute the approximation of the vector $\begin{pmatrix} F_{51} \\ F_{50} \end{pmatrix}$:

$$\begin{pmatrix} F_{51} \\ F_{50} \end{pmatrix} = A^{50} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} \approx \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{51} & \varphi^{50} \\ \varphi^{50} & \varphi^{49} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{51} \\ \varphi^{50} \end{pmatrix}.$$

Therefore, $\frac{F_{51}}{F_{50}} \approx \frac{\varphi^{51}}{\varphi^{50}} = \varphi$. $\square$