18.06 - Linear Algebra Cheatsheet

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*Remark: This is **not** intended as a definitive list of everything you are meant to memorize for 18.06. Some of these ideas will be familiar, some of them less so. This is a reference document for you to look up definitions if you come across something that seems unfamiliar/confusing.*

1 Vectors

1. When we talk about *vectors* in 18.06, we are usually referring to column vectors.
2. A two-dimensional vector $\mathbf{v}$ is defined by its two components, v_1 and v_2. We write the vector as

$$\boxed{\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}$$

3. The set of all two-dimensional vectors is referred to as $\mathbb{R}^2$
4. In general, a vector $\mathbf{v}$ can have n components, and would then be an n-dimensional vector (a $n \times 1$ array):

$$\boxed{\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}}$$

   The set of all n-dimensional vectors is referred to as $\mathbb{R}^n$.

5. We can always multiply vectors by scalars. We can also add two vectors, provided they have the same dimensions.
6. The zero, n-dimensional vector $\mathbf{0}$ is a vector where every component is 0.
7. The *length* (or *magnitude*) of a vector $\mathbf{v}$ is written $\|\mathbf{v}\|$. It is given by the following formula:

$$\boxed{\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + ... + v_n^2 = \sum_{i=1}^{n} v_i^2}$$

8. A *unit vector* $\mathbf{n}$ is a vector with length $\|\mathbf{n}\| = 1$.

9. Suppose you have two n-dimensional vectors, $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

   Then the *dot product* (or *inner product*) of these two vectors, $\mathbf{u}\cdot\mathbf{v}$, is given by the following formula:

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n}$$

10. The *angle* between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by the following formula:

$$\boxed{\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}}$$

11. We say that a nonzero vector $\mathbf{u}$ is *parallel* to a nonzero vector $\mathbf{v}$ if $\boxed{\mathbf{u} = a\mathbf{v}}$ for some scalar $a \neq 0$. We sometimes say that $\mathbf{u}$ and $\mathbf{v}$ are in the same direction.

12. We say that a vector $\mathbf{u}$ is *perpendicular*, or *orthogonal*, to a vector $\mathbf{v}$ if $\boxed{\mathbf{u} \cdot \mathbf{v} = 0}$.

13. The *transpose* of a vector is an n-dimensional row vector (a $1 \times n$ array):

$$\boxed{\mathbf{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}}$$

14. We can multiply a n-dimensional row vector by a n-dimensional column vector. The order of multiplication matters:

    - $\mathbf{u}^T\mathbf{v}$ has dimensions 1×1, i.e. it is a scalar. In fact $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$:

$$\mathbf{u}^T\mathbf{v} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$$

      We see then that really $\mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, i.e. the *dot product* of $\mathbf{u}$ and $\mathbf{v}$.

    - $\mathbf{u}\mathbf{v}^T$ has dimensions $n \times n$. It is called an *outer product*.

2 Matrices

2.1 General properties

Remark. The most basic rule that you should remember: $\boxed{\textbf{row-column}}$. It shows the order in which you write or compute, e.g.:

- The first index denotes the row, the second number the column.
- You multiply a row by a column to get a number.
- An $m \times n$ matrix has m rows and n columns.

1. A *matrix* is an $m \times n$ array of numbers. An $m \times n$ matrix has m *rows* and n *columns*. A matrix is *square* if $m = n$. Examples:
   - $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ is a 2×2 square matrix.
   - $B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$ is a 2×3 matrix.
   - $C = \begin{pmatrix} 1 & 4 \\ 0 & -3 \\ 1 & 1 \end{pmatrix}$ is a 3×2 matrix.
2. Suppose A is a $m \times n$ matrix and B is a $p \times q$ matrix. We can only multiply these matrices if the dimensions make sense. We can multiply AB only if $n = p$; we can multiply BA only if $m = q$.
3. Suppose A and B are two $n \times n$ square matrices. In general $\boxed{AB \neq BA}$. Matrix multiplication does not *commute*.
4. A *diagonal* matrix is a matrix which only has entries along its diagonal.
5. The *identity* matrix I is a square, diagonal matrix. The entries along the diagonal are all equal to 1. The identity matrix is the only matrix for which $IA = A$ for all square matrices A.
6. We can compute the product of three matrices ABC either as $(AB)C$ (multiply AB and then multiply on the right by C), or as $A(BC)$ (multiply BC and then multiply on the left by A). Matrix multiplication is *associative*.
7. Matrix multiplication is *distributive*. This means that $(A + B)C = AC + BC$.

2.2 Linear systems

1. Often we will be interested in solving equations of the form $\boxed{Ax = b}$, where A is an $m \times n$ matrix, $\mathbf{x}$ is a n-dimensional vector, and $\mathbf{b}$ is a m-dimensional vector. This is usually called a *linear system*.
2. If A is a square matrix, then it might have an *inverse* A^{-1}, so that $AA^{-1} = A^{-1}A = I$. The unique solution of the linear system $A\mathbf{x} = \mathbf{b}$ in this case is $\mathbf{x} = A^{-1}\mathbf{b}$.
3. If A and B are square matrices with inverses A^{-1} and B^{-1}, then $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$.

4. The inverse of a square matrix does not always exist. We have already seen that for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have the following formula

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

   This exists if and only if $\boxed{ad - bc \neq 0}$.

5. Rectangular matrices for which $m \neq n$ will not have inverses.

6. A general linear system $Ax = b$, where A is a rectangular $m \times n$ matrix, *may* have a unique solution. It may also have infinitely many solutions, or no solutions at all.

2.3 Transposes

1. The *transpose* of a matrix A is denoted by A^T. The transpose is the matrix formed by taking the columns of A and making them the rows of A^T.

2. If A has components a_{ij}, then A^T has components a_{ji}.

3. If A is $m \times n$, then A^T is $n \times m$.

4. $A^T A$ and AA^T can always be computed. They are both square matrices, even if A is rectangular. If A is $n \times m$, then $A^T A$ is $m \times m$ and AA^T is $n \times n$.

5. $\boxed{(AB)^T = B^T A^T}$.

6. If A is a square matrix and A^{-1} exists, then $\boxed{(A^T)^{-1} = (A^{-1})^T}$.

2.4 Orthogonal matrices

1. A $n \times n$ square matrix Q is *orthogonal* if $\boxed{Q^T Q = I}$. A rectangular matrix for which $Q^T Q = I$ we will usually refer to as a *tall skinny orthogonal* matrix.

2. An equivalent definition: suppose a matrix Q has n columns given by the vectors $\mathbf{q}_1, ..., \mathbf{q}_n$. Then Q is orthogonal if the column vectors are orthonormal. This means that $q_i^T q_j = 0$ for $i \neq j$, and $\|q_i\| = 1$.

3. If Q is square and orthogonal, then we also have that $QQ^T = I$, and so $Q^{-1} = Q^T$.

4. If Q obey $Q^T Q = I$, but is not square, then $QQ^T \neq I$ generally!

3 Planes, hyperplanes and surfaces

3.1 Planes

1. The general equation of a plane in $\mathbb{R}^3$ is

$$\boxed{ax + by + cd = f,}$$

where $a, b, c, d \in \mathbb{R}$ are scalars. The plane contains the origin if and only if $f = 0$.

2. The normal to this plane is given by the vector $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

3. The equation of the plane may equivalently be written as $\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = f$, where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

3.2 Hyperplanes

1. A hyperplane in $\mathbb{R}^n$ is the set of points with position vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ obeying the equation

$$\boxed{w_1 x_1 + w_2 x_2 + ... w_n x_n = f}$$

for scalars $w_1, w_2, ..., w_n \in \mathbb{R}$.

2. The normal to this plane is given by the vector $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

3. The equation of the hyperplane may equivalently be written as $\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = f$.

3.3 Surfaces

- A surface in three dimensions is described by an equation $f(x, y, z) = Const.$
- The *normal* to a surface is $\boxed{\mathbf{n} = \nabla f}$.