# 1 Lecture Review

## 1.1 Independence, Basis and Dimension

### 1.1.1 Definitions

1. A set of vectors $\{v_1, \dots, v_n\}$ is linearly independent if the only constants $c_1, \dots, c_n$ which solve
$$c_1 v_1 + \cdots + c_n v_n = 0$$
are $c_1 = \cdots = c_n = 0$.

2. A set of vectors $\{v_1, \dots, v_n\}$ is linearly dependent if it is not linearly independent.

3. The span of a set of vectors $\{v_1, \dots, v_n\}$, denoted $\text{span}(v_1, \dots, v_n)$, is the set of linear combinations of $v_1, \dots, v_n$.

4. If $V$ is a vector space, we say that $\{v_1, \dots, v_n\}$ is a basis for $V$ if it is linearly independent and spans $V$.

5. If $V$ is a vector space such that $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are both bases for $V$, then $m = n$. We call the number $n$ (or $m$) the dimension of $V$, denoted $\dim(V)$.

6. If $A$ is a square matrix, we say that $A$ is nonsingular if it is invertible, and otherwise say it is singular.

### 1.1.2 Properties and Examples

Let $A$ be an $m \times n$ matrix.

1. The column space of $A$ is equal to the span of the columns of $A$; if $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are the columns of $A$ this can be expressed as $\text{col}(A) = \text{span}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$.

2. The dimension of the column space is equal to the rank of $A$: $\dim(\text{col}(A)) = \text{rank}(A)$.

3. Any set of $n$ vectors in $\mathbb{R}^m$ is linearly dependent if $n > m$.

4. There is one and only one way to write $v$ as a linear combination of basis vectors.

5. The following statements are equivalent:
   (a) The columns of $A$ are linearly independent.
   (b) The columns of $A$ form a basis for $\text{Col}(A)$.
   (c) The only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = 0$.
   (d) $\text{Null}(A) = \{\boldsymbol{0}\}$.
   (e) The rank of $A$ is $n$ (the number of columns of $A$).
   (f) For any $\boldsymbol{b} \in \mathbb{R}^m$, there is exactly 0 or 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$. There is 1 exactly when $\boldsymbol{b} = UU^T\boldsymbol{b}$.

6. The following statements are equivalent:
   (a) $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ is a basis for $\mathbb{R}^n$.
   (b) The matrix $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$ is invertible.
   (c) For any $\boldsymbol{b} \in \mathbb{R}^n$, the matrix $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution in $\mathbb{R}^n$.
   (d) The full form SVD and compact form SVD of $A$ are the same.

## 2 Problems

1. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be vectors.

   (a) Check that the span of the $\boldsymbol{v}$'s form a vector space.

   (b) If $n = 3$, show that that span is either $\mathbb{R}^3$, a plane, a line, or a point. When is it a point?

*Solution.* (a) We want to show that $\text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ form a vector space. To show this, take two linear combinations: $\sum_{i=1}^n a_i \boldsymbol{v}_i$ and $\sum_{i=1}^n b_i \boldsymbol{v}_i$. Then for any constants $c, d$ we have

$$c \sum_{i=1}^n a_i \boldsymbol{v}_i + d \sum_{i=1}^n b_i \boldsymbol{v}_i = \sum_{i=1}^n (ca_i + db_i) \boldsymbol{v}_i$$

which is still a linear combination of the $\boldsymbol{v}$'s. Thus the span of the $\boldsymbol{v}$'s is a vector space.

(b) If $n = 3$, then we consider $\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$. If the three vectors are linearly independent then this spans $\mathbb{R}^3$. Otherwise, suppose $\boldsymbol{v}_3$ is in the span of the other two, then

$$\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2).$$

The latter is a plane if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent. Otherwise, suppose $\boldsymbol{v}_2$ is in the span of $\boldsymbol{v}_1$, then

$$\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \text{span}(\boldsymbol{v}_1).$$

The latter is a line if $\boldsymbol{v}_1$ is nonzero. Otherwise, it is the point $\{\mathbf{0}\}$. □

2. Describe the subspace of $\mathbb{R}^3$ (is it a line or a plane or $\mathbb{R}^3$) spanned by the following vectors, then identify a basis:

(a) The vectors $(1, 1, -1)$ and $(-1, -1, 1)$.

(b) The vectors $(0, 1, 1)$, $(1, 1, 0)$ and $(0, 0, 0)$.

(c) All vectors in $\mathbb{R}^3$ with integer components.

(d) All vectors with positive components.

*Solution.* (a) The two vectors are multiples of one another, so

$$\text{span}(\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}) = \text{span}(\begin{pmatrix}1\\1\\-1\end{pmatrix})$$

which is a line with basis $\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\}$.

(b) We have

$$\text{span}(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}) = \text{span}(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix})$$

where the latter two vectors are not multiples of one another. This is a plane and $\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\}$ is a basis for this plane.

(c) Let $V$ be the span of all vectors in $\mathbb{R}^3$ with integer components. Note that

$$\text{span}(\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}) = \mathbb{R}^3$$

is contained $V$ (the span of a subset of vectors is contained in the span of a larger set of vectors). Therefore $V = \mathbb{R}^3$ which has a basis $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$.

(d) Let $V$ be the span of all vectors in $\mathbb{R}^3$ with positive components. We show that $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\} \subset V$. This would imply $V = \mathbb{R}^3$ (since $V$ would then be a vector space with $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ contained in it).

Note that

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}$$

where the right hand side a linear combination of elements of $V$. This shows that $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\} \subset V$. $\square$

3. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

What is the span of the $\boldsymbol{v}$'s?

*Solution.* We have $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent because the $3\times 3$ identity matrix has these three vectors as columns, and the identity matrix has trivial null space. However, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent because

$$2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + 4\boldsymbol{v}_3 - \boldsymbol{v}_4 = \boldsymbol{0}.$$

The span is $\mathbb{R}^3$. $\square$

4. Suppose $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors and $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$, $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$, $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$.
   (a) Show that the $\boldsymbol{v}$'s are dependent.
   (b) Which of the following matrices are nonsingular: $A = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}$.
   (c) Explain why we can always find a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b} \in \mathbb{R}^3$.
   (d) Explain (using only linear independence) why Null($B$) contains more than a point. Find a nonzero vector in this null space.

*Solution.* (a) We have

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \boldsymbol{0}$$

which shows the $\boldsymbol{v}$'s are dependent.

(b) $A$ is nonsingular because the $\boldsymbol{w}$'s are independent whereas $B$ is singular because the $\boldsymbol{v}$'s are dependent.

(c) We have a unique solution because $A$ is nonsingular (i.e. invertible).

(d) Null($B$) contains more than a point because the columns of $B$ are linearly dependent. Note that $\left(\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right)$ is in the null space (compare with the coefficients from (a)). $\square$

5. Consider the plane $P$ with equation $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

   (a) Find a basis for the plane $P$.

   (b) Find a basis for the intersection of $P$ with the $xy$-plane.

   (c) Find a basis for all vectors perpendicular to plane $P$.

*Solution.* (a) Note that $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ are both in the plane and are not multiples of one another, so they are linearly independent. Then $\{\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\}$ is a basis for this plane.

(b) Since $z = 0$ in the $xy$-plane, the intersection of $P$ with the $xy$-plane consists of points $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ where

$$x - 2y + 3 \cdot 0 = 0 \qquad \Longrightarrow \qquad x = 2y.$$

Thus the intersection of $P$ with the $xy$-plane is the line in the direction $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. A basis is given by $\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\}$.

(c) In $\mathbb{R}^3$, the set of vectors perpendicular to a plane is given by the line in the direction of the normal vector. The normal to $P$ is $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. So a basis for all vectors perpendicular to $P$ is given by $\{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\}$. $\square$

6. Find a basis and the dimension for the following subspaces of $3 \times 3$ matrices:
   (a) All diagonal matrices
   (b) All symmetric matrices ($A^T = A$).
   (c) All antisymmetric matrices ($A^T = -A$).

*Proof.* (a) The set of all $3 \times 3$ diagonal matrices has basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(one can check these are linearly independent by hand and that they span all $3 \times 3$ diagonal matrices). The dimension is therefore 3.

(b) The set of all $3 \times 3$ symmetric matrices has basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(one can check these are linearly independent by hand and that they span all $3 \times 3$ symmetric matrices). The dimension is therefore 6.

(c) The set of all $3 \times 3$ antisymmetric matrices has basis

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(one can check these are linearly independent by hand and that they span all $3 \times 3$ antisymmetric matrices). The dimension is therefore 3. $\square$

7. Find a basis for the space of $2 \times 3$ matrices whose null space contains $(2, 1, 1)$.

*Solution.* Suppose $A$ is a matrix whose null space contains $(2, 1, 1)$. Then all the rows of $A$ are perpendicular to $(2, 1, 1)$. Note that $(1, -2, 0)$ and $(1, 0, -2)$ form a basis for all vectors perpendicular to $(2, 1, 1)$. We check that

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

is a basis for the space of $2 \times 3$ matrices whose null space contains $(2, 1, 1)$.

(Linear Independence Check) If

$$c_1 \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then the left hand side becomes a matrix whose rows are a linear combination of $(1, -2, 0)$ and $(1, 0, -2)$. Since $(1, -2, 0)$ and $(1, 0, -2)$ are linearly independent, the right hand side can only be the zero matrix if $c_1 = c_2 = c_3 = c_4 = 0$.

(Span Check) As mentioned, the set of all $2 \times 3$ matrices whose null space contains $(2, 1, 1)$ is exactly the set of matrices whose rows are perpendicular to $(2, 1, 1)$. Let $A$ be such a matrix. Then the first row of $A$ is some linear combination

$$c_1(1, -2, 0) + c_2(1, 0, -2)$$

and the second row of $A$ is some linear combination

$$c_3(1, -2, 0) + c_4(1, 0, -2).$$

Thus

$$A = c_1 \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

$\square$