

Topics: Vector spaces, solving  $Ax = b$ , matrix transpose and inverse, orthogonal matrices, block matrices.

## 1. RECOGNIZING VECTOR SPACES

A (real) vector space is a set  $V$  equipped with two operations:

- An operation  $+$  (vector addition) which takes  $v_1, v_2 \in V$  and outputs  $v_1 + v_2 \in V$ .
- An operation  $\cdot$  (scalar multiplication) which takes  $c \in \mathbb{R}$  and  $v \in V$  and outputs  $c \cdot v \in V$ .

(NB: Part of these requirements is that the sum must actually lie in  $V$ , in which case we say that “ $V$  is closed under addition,” and similarly  $V$  must be “closed under scalar multiplication.”) These operations must also satisfy some axioms.<sup>1</sup> Here are two of them:

- Existence of a ‘zero’: there must exist  $v_0 \in V$  such that  $v_0 + v = v$  for all  $v \in V$ .
- Existence of additive inverses: for any  $v \in V$ , there must exist  $v' \in V$  such that  $v + v' = v_0$ .

The ‘zero’ vector  $v_0$  is usually just denoted  $0$ , and the additive inverse of  $v$  is usually denoted  $-v$ .

**Problem 1.** In the vector space consisting of all functions on the real line, what is the zero vector?

**Problem 2.** Consider the set  $V$  consisting of pairs  $(a, b)$  of real numbers satisfying  $a + 2b = 0$ . How should one define the operations of ‘vector addition’ and ‘(real) scalar multiplication’ on  $V$ ? Does this make  $V$  into a vector space? What about when  $V$  is the set of pairs  $(a, b)$  of real numbers satisfying  $a \geq 0$ ?

## 2. SOLVING $Ax = b$

The following four problems are equivalent:

- Basic math: find  $x, y \in \mathbb{R}$  such that

$$\begin{aligned} x + 2y &= 5 \\ 4x + y &= 6 \end{aligned}$$

- Matrix form: find  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$  such that

$$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

- Column view: find  $x, y \in \mathbb{R}$  such that

$$x \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

- Row view: find  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$  such that

- Its dot product with  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is equal to 5.
- Its dot product with  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  is equal to 6.

One way to measure the complexity of an algorithm is to count the number of arithmetic operations (plus, subtract, multiply, divide) that go into it.

**Problem 3.** Consider a 2-variable system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

where  $x_1, x_2$  are the unknowns. Solve it using back substitution, i.e. solve for  $x_2$ , then substitute to find  $x_1$ . How many arithmetic operations do you need? Can you guess how many operations are needed for an  $n$ -variable system of the same “upper triangular” shape?

<sup>1</sup>For a full list, see [https://en.wikipedia.org/wiki/Vector\\_space#Definition](https://en.wikipedia.org/wiki/Vector_space#Definition).

## 3. MATRIX TRANSPOSE AND INVERSE

Given an  $m \times n$  matrix  $A$ , the transpose  $A^\top$  is an  $n \times m$  matrix whose rows are the columns of  $A$ . We have  $(AB)^\top = B^\top A^\top$ .

The matrix  $A$  is *orthogonal* if  $A^\top A = I_{n \times n}$ . Nonobvious fact: this implies that  $m \geq n$ , i.e.  $A$  is square or ‘tall and skinny’. If the  $n$  columns of  $A$  are thought of as individual column vectors  $v_1, \dots, v_n$ , then  $A$  is orthogonal if and only if

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

In this case,  $v_1, \dots, v_n$  are called an *orthonormal* collection of vectors.

**Problem 4.** Find an orthogonal matrix  $Q$  whose rows do not form an orthonormal collection of vectors.

**Problem 5.** Fill in the question marks to make an orthogonal matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & ? \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & ? \\ 0 & -\frac{1}{\sqrt{3}} & ? \end{pmatrix}$$

How many ways are there?

Given an  $n \times n$  matrix  $A$ , an  $n \times n$  matrix  $B$  satisfying  $AB = I_{n \times n}$  is called an *inverse* of  $A$ . (Nonobvious fact: this condition implies  $BA = I_{n \times n}$ , and vice versa.) Any square matrix either has a unique inverse or no inverse at all. The inverse, if it exists, is denoted  $A^{-1}$ , and we say that  $A$  is *invertible*.

Nonobvious fact: a square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .

**Problem 6.** Given *invertible*  $n \times n$  matrices  $A$  and  $B$ , show that  $(AB)^{-1}$  exists and equals  $B^{-1}A^{-1}$ .

**Problem 7.** Show that a square orthogonal matrix  $A$  is invertible, and satisfies  $AA^\top = I_{n \times n}$  and  $A^{-1} = A^\top$ .

**Problem 8.** Show that the rows of a *square* orthogonal matrix form an orthonormal collection of vectors.

## 4. BLOCK MATRICES

A *block decomposition* of an  $m \times n$  matrix  $A$  is a way of writing  $A$  as a ‘matrix of matrices’:

$$A = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}$$

where each  $B_{ij}$  is a matrix. The  $B_{ij}$  don’t need to all have the same size, but they need to be sized so as to fit together into an  $m \times n$  grid.

Fact. Block decompositions are compatible with matrix multiplication, for example:

$$\text{If } A = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \text{ and } C = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

$$\text{then } AC = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} B_{11}D_{11} + B_{12}D_{21} & B_{11}D_{12} + B_{12}D_{22} \\ B_{21}D_{11} + B_{22}D_{21} & B_{21}D_{12} + B_{22}D_{22} \end{pmatrix},$$

provided that the blocks are the right sizes for the matrix multiplications to be possible.

**Problem 9.** Does there exist some block decomposition of some matrix whose set of blocks is as follows:  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, (3 \ 4), (5 \ 6)$ .

**Problem 10.** If  $A = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  is a block decomposition, what is the corresponding block decomposition of  $A^\top$ ?