Recitation 3/2

Sungwoo Jeong Tuesday 10AM, 11AM

March 12, 2020

Four fundamental subspaces of $A \in \mathbb{R}^{m \times n}$

- $\mathrm{col}(A) = \{\text{All linear combinations of columns of } A\} = \{Ax \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$
- $\mathrm{row}(A) = \{\text{All linear combinations of rows of } A\} = \{A^T x \mid x \in \mathbb{R}^m\} = \mathrm{col}(A^T) \subset \mathbb{R}^n$
- $\mathrm{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n$
- $\mathrm{null}(A^T) = \{x \in \mathbb{R}^m \mid A^T x = 0\} \subset \mathbb{R}^m$

SVD and fundamental subspaces

Let $A = U\Sigma V^T \in \mathbb{R}^{m \times n}$ be a full SVD of A and $A = U_1 \Sigma_r V_1^T$ be a Rank-r SVD of A.

Size of matrices ?

- U : __ by __ Matrix
- V : __ by __ Matrix
- U_1 : __ by __ Matrix
- V_1 : __ by __ Matrix

Four fundamental subspaces of A in terms of SVD

- $\mathrm{col}(A) = \mathrm{col}(\_\_\_\_)$
- $\mathrm{row}(A) = \mathrm{col}(\_\_\_\_)$
- $\mathrm{null}(A) = \mathrm{col}(\_\_\_\_)$
- $\mathrm{null}(A^T) = \mathrm{col}(\_\_\_\_)$

Problems

1. Describe four fundamental subspaces of

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$

2. The full SVD of A is given as

$$A = U\Sigma V^T = \begin{pmatrix} 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.732 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4082 & -0.7071 & 0 & 0.5773 \\ 0 & 0 & 1 & 0 \\ 0.4082 & 0.7071 & 0 & 0.5773 \\ 0.8164 & 0 & 0 & -0.5773 \end{pmatrix}$$

(a) What is Rank-r SVD of A?

(b) Desribe four fundamental subspaces of A.

(c) A row space of an square n-by-n orthogonal matrix is $\mathbb{R}^n$ (no need to prove this), which is all the real vectors. That means, any vector $y \in \mathbb{R}^4$ can be obtained by $y = V^T x$. Describe column space of ΣV^T.

(d) What happens if a vector in (c) is multiplied by U on the left? Our A is actually $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Confirm that $\text{col}(A) = \text{col}(U_1)$ and explain why the column space of U_2 does not play a role in column space of A.

(e) Compute $U_1 U_1^T$ without any hard computation. Does $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ satisfy $U_1 U_1^T b = b$? Is $Ax = b$ solvable?

Answers

1.(a) col(A) is all the vectors $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

row(A) is all the vectors $\begin{pmatrix} x \\ x \\ y \end{pmatrix}$

null(A) can be computed from $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x+y \\ 0 \end{pmatrix} = 0$. It is consisted with vectors of shape $\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}$

null(A^T) can be computed similarly, $A^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ x \end{pmatrix} = 0$. So it is vectors of form $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

(b) col(A) is all the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Any linear combination of A has first and third component the same.

row(A) is $\mathbb{R}^2$.

null(A) is obtained from $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 3y \\ x+y \end{pmatrix} = 0$. x, y are zero so nullspace only has a zero vector.

null(A^T) is obtained from $A^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ x+3y+z \end{pmatrix} = 0$. We first get $x = -z$ and we also deduce $y = 0$ since $x + z = 0$. So null(A^T) is all the vectors $\begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}$.

2.(a)

$$A = U_1 \Sigma_r V_1^T = \begin{pmatrix} 0.7071 & -0.7071 & 0 \\ 0 & 0 & 1 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1.732 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.4082 & -0.7071 & 0 & 0.5773 \\ 0 & 0 & 1 & 0 \\ 0.4082 & 0.7071 & 0 & 0.5773 \end{pmatrix}$$

(b) Using the facts above, col(A) is col(U_1) and it is all the vectors of shape $\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$.

row(A) is column space of V_1 and it is all the vectors $\begin{pmatrix} 0.4082x \\ y \\ z \\ .5773x \end{pmatrix}$

null(A) is column space of V_2 so all the vectors $\begin{pmatrix} 0.8164x \\ 0 \\ 0 \\ -0.5773x \end{pmatrix}$.

null(A^T) is column space of U_2 so all the vectors $\begin{pmatrix} 0 \\ 0 \\ 0 \\ x \end{pmatrix}$.

(c) It can be thought as any vector multiplied by a diagonal matrix Σ on the left. So it is any vectors that has the shape of $\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$.

(d) If a vector v in (c) is multiplied by U on the left, the coefficient for the last column of U in the linear combination Uc is always zero. So that means the last column plays no role, which is actually the column space of U_2. We can see that the column space of A is all the vectors of the shape $\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$ so confirming column space of A equals column space of U_1.

(e) It is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Note that U_1 is 4 by 3 matrix so $U_1 U_1^T$ is 4 by 4 square matrix and since the rows 1, 2, 3 are orthonormal and the last row is zero vector, we obtain such matrix. b is not in this space so we deduce from lecture note 13 page 6 that it cannot be solvable.