MIT 18.06 Exam 1, Fall 2018 Solutions
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Problem 1 (30 points):

You have the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

(a) Find matrices P, U, L for a $PA = LU$ factorization of A. Hint: look at A carefully first: if you find the right permutation P (a matrix to *re-order the rows*) it will be simple.

(b) Compute $x = A^{-1}b$ where $b = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 1 \\ -4 \end{pmatrix}$.

Solution:

(a) If we look at this matrix, we see that we can put it in upper triangular form just by reordering the rows. The necessary reordering of the rows can be described by the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The product PA is then in upper triangular form, so that

$$U = \begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) We want to compute $x = A^{-1}b$, which is equivalent to finding $x$such that $Ax = b$. Multiplying this equation by P gives $PAx = Pb = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -4 \\ -2 \end{pmatrix}$, which then gives us the upper-triangular system

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -4 \\ -2 \end{pmatrix}.$$

This gives us a triangular system of equations we can then solve via back substitution:

$$\begin{aligned} x_5 = -2 &\Longrightarrow x_5 = -2 \\ 2x_4 + x_5 = -4 &\Longrightarrow x_4 = -1 \\ x_3 - x_5 = 2 &\Longrightarrow x_3 = 0 \\ 2x_2 + x_4 = 1 &\Longrightarrow x_2 = 1 \\ x_1 + 2x_2 - x_3 + x_5 = 2 &\Longrightarrow x_1 = 2 \end{aligned}$$

The solution is then

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}.$$

Problem 2 (30 points):

A is a 3×5 matrix. One of your Harvard friends performed row operations on A to convert it to rref form, but did something weird—instead of getting the usual $R = \begin{pmatrix} I & F \end{pmatrix}$, they reduced it to a matrix in the form $\begin{pmatrix} F & I \end{pmatrix}$ instead. In particular, their row operations gave:

$$A \rightsquigarrow \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 0 & 1 & 0 \\ 6 & 7 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a basis for $N(A)$.

(b) Give a matrix M so that if you multiply A by M (on the **left or right?**) then the **same** row operations as the ones used by your Harvard friend will give a matrix in the usual rref form:

$$\text{either } MA \text{ or } AM \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \end{pmatrix}.$$

Solution:

(a) Row operations always preserve the null space $N(A)$, i.e. any solution to $Ax = 0$ will be preserved by row operations. Let $H = \begin{pmatrix} F & I \end{pmatrix}$ be the weird row-reduced matrix obtained by our Harvard friend. We can still seek special solutions to $Hx = 0$ using the usual method. Columns 3, 4 and 5 are the pivot columns, while columns 1 and 2 are the free columns. We therefore look for two special solutions:

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ 1 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We can then see that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ -7 \end{pmatrix}$, i.e. the negative entries of each column of F. This gives us a basis for the null space of A:

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ -4 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ -5 \\ -7 \end{pmatrix}.$$

(b) We want to first reorder the columns of H so that it is in the usual rref form. Recall that column operations are equivalent to multiplying on the right by an appropriate matrix. A matrix that will put the columns of H in the correct order is the following permutation matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrix $R = HM$ will then be in the usual rref form. Remember that our Harvard friend performed row operations to put our matrix A into the weird form H, and row operations won't change the column order. In particular, recall that row operations are equivalent to multiplying by an appropriate matrix on the left, so there exists a matrix E so that $EA = H$. The product $R = HM = EAM = E(AM)$ is then in the usual rref form. So peforming the same row operations as our Harvard friend on the matrix AMwill give us a matrix in the usual rref form.

Problem 3 (10 points):

In class, when we derived the LU factorization, we initially found L by multiplying a sequence of elementary elimination matrices, *one to eliminate below each pivot*. (We later found a more clever way to get L just by writing down the multipliers from the elimination steps, no arithmetic required.)

If A is a non-singular $m \times m$ matrix and we compute L in the **"naive" way**, by directly multiplying the elementary elimination matrices (by the usual rows$\times$ columns method, no tricks), how would the cost to compute L (the number of scalar-arithmetic operations) scale with m? (That is, roughly proportional to m, m^2, m^3, m^4, m^5, 2^m, or...?)

Solution:

Suppose A is a non singular, $m \times m$ matrix, and we have performed row operations to put A in upper triangular form. To do this, we need to eliminate every entry below each pivot. Eliminating beneath each pivot can be described by an elementary elimination matrix, so in general we will need $m-1$ elimination matrices to put A in upper triangular form:

$$E_{m-1}...E_1 A = U.$$

We can then find L by calculating the inverse of the product of elimination matrices, i.e. $L = (E_{m-1}...E_1)^{-1} = E_1^{-1}...E_{m-1}^{-1}$. Finding the inverse of each of the elimination matrices is trivial (we just multiply all off diagonal elements by -1). So finding L requires us to multiply $(m-1)$ of these $m \times m$ inverse elimination matrices together. Multiplying two $m \times m$ matrices together requires $\sim m^3$ scalar arithmetic operations (assuming we do not use any tricks to make this matrix multiplication more efficient—in particular, we don't exploit the fact that the E_k matrices have a very special form and are mostly zero). So multiplying $(m-1)$ such matrices will require $\sim m^4$ scalar arithmetic operations. So finding L in this naive way requires a number of scalar-arithmetic operations that scales proportional to m^4.

Problem 4 (30 points):

Here are some miscellaneous questions that require little calculation:

(a) Is V a vector space or not? (For multiplication by real scalars and the usual $\pm$ operations.) If **false**, give a rule of vector spaces that is violated:

(i) A is a 3×6 matrix. $V =$ *all* solutions x to $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(ii) A is a 3×6 matrix. $V =$ *all* 6×2 matrices X where $AX = 0$ (the 3×2 zero matrix).

(iii) $V =$ *all* 3×3 singular matrices A.

(iv) $V =$ *all* 3×3 matries whose diagonal entries average to zero.

(v) $V =$ *all* differentiable functions $f(x)$ with $f'(0) = 2f(0)$. (f' is the derivative.)

(vi) $V =$ *all* functions $f(x)$ with $f(x+y) = f(x)f(y)$.

(b) Give a matrix A whose null space is spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(c) Give a nonzero matrix A whose column space is in $\mathbb{R}^3$ but does *not* include $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Solution:

(a) Is V a vector space or not?

(i) This is **not** a vector space, since it does not contain the zero vector

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(ii) This **is** a vector space (it is closely related to, but is not the *same* as, the null space of A). It definitely contains the zero vector $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, and if we take any two matrices X_1, X_2 in our set V, then the linear combination $aX_1 + bX_2$ will still be in our set V, since $A(aX_1 + bX_2) = aAX_1 + bAX_2 = 0$.

(iii) This is **not** a vector space. The set *does* contain the zero matrix (the zero matrix is singular), and it *is* closed under multiplication by scalars. However, consider two matrices $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which are both singular (they are in rref and neither has three pivots). Their sum is $A_1 + A_2 = I$, and the identity matrix is not singular. Therefore the set is not closed under matrix addition.

(iv) This **is** a vector space. Consider two 3×3 matrices A and B with diagonal entries (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively, where $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$. Any linear combination $\lambda A + \mu B$ will have diagonal entries $(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2, \lambda a_3 + \mu b_3)$. The average of these diagonal entries is then $\frac{1}{3}[(\lambda a_1 + \mu b_1) + (\lambda a_2 + \mu b_2) + (\lambda a_3 + \mu b_3)] = 0$.

(v) This **is** a vector space. Consider two functions $f(x)$ and $g(x)$ obeying this rule. Then any linear combination $\lambda f(x) + \mu g(x)$ will also obey this rule, since $\lambda f'(0) + \mu g'(0) = 2[\lambda f(0) + \mu g(0)]$.

(vi) This is **not** a vector space. It contains the zero function, but is not closed under scalar multiplication (or addition): if $f(x+y) = f(x)f(y)$, and $g(x) = 2f(x)$, then $g(x+y) = 2f(x+y) \neq g(x)g(y) = 4f(x)f(y) = 4f(x+y)$. For example, the function $f(x) = e^x$ is in this set, but the function $g(x) = 2e^x$ is not.

(You might be interested to learn that this is a famous property of exponential functions. In fact, the only real-valued, anywhere-continous functions that satisfy this rule are the zero function $f(x) = 0$ and functions of the form $f(x) = e^{kx}$ for any $k \in \mathbb{R}$.)

(b) We want to find a matrix whose null space is spanned by $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Such a matrix must have three columns, so that this vector can be an element of the null space. It *must have rank two*, so that there are no other vectors in a basis for its null space: the null space *must be one-dimensional* to be spanned by v. Examples of possible matrices are:

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

You might be tempted to write a matrix with one row, like $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$, which indeed has v in its nullspace, but such a matrix has *other* vectors in its null space also—this matrix is rank 1, so it has a two-dimensional nullspace that is not spanned by v. A matrix of all zeros would be even worse—it would have rank 0, with a 3d nullspace that contains v but also contains every other 3-component vector.

(c) We want to find a matrix whose column space is a subspace of $\mathbb{R}^3$, but does not include $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Such a matrix must have three rows, but can have any number of columns, provided that $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is not in the span of the columns, which means that the matrix necessarily has rank less than or equal to two. Examples of possible matrices are:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that, to know that a vector is not in the column space, you must be sure that *any linear combination* of the columns of the matrix cannot give you that vector. So, for example, the matrix

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}$$

does *not* work. Even though $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ does not appear explicitly as one of its columns, this vector is in the column space because you can get it by the linear combination $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. If you pick a 3×2 matrix at random, it is pretty unlikely to have $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ in its column space. On the other hand, if you pick a 3×3 matrix at random, it is probably rank 3 and hence will contain *every* 3-component vector. Another common point of confusion here is that whether a vector is in $C(A)$ is not directly related to whether it is in $N(A)$, so this problem is quite different from the previous part.