18.06 - Recitation 10

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1 Lecture Review

- A symmetric matrix A is positive definite if, and only if, any of the following are true:
  1. All of the eigenvalues of A are strictly positive
  2. All upper left determinants are strictly positive
  3. $x^T Ax > 0$ for all nonzero vectors x
  4. $A = B^T B$, where B has independent columns, but B is not necessarily square.
- A *Markov matrix* has non-negative entries and the elements of each column sum to 1. It always has one eigenvalue equal to 1, and every other eigenvalue has absolute value less than or equal to 1.
- A *positive Markov matrix* has strictly positive entries and the elements of each column sum to 1. It always has one eigenvalue equal to 1, and every other eigenvalue has absolute value strictly less than 1.
- The eigenvector of a Markov matrix corresponding to $\lambda = 1$ is called the *steady state* vector.

2 Problems

Problem 1.

Consider the matrix $A = \begin{pmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with parameter $x \in \mathbb{R}$:

1. Specify all numbers x, if any, for which A is positive definite. (Explain briefly.)
2. Specify all numbers x, if any, for which e^A is positive definite. (Explain briefly.)
3. Find an x, if any, for which $4I - A$ is positive definite. (Explain briefly.)

Problem 2.
True or false? Justify your answer either way.

1. If A and B are invertible, then so is $(A+B)/2$.
2. If A and B are Markov, then so is $(A+B)/2$.
3. If A and B are positive definite, then so is $(A+B)/2$.
4. If A and B are diagonalizable, then so is $(A+B)/2$.
5. If A and B are rank 1, then so is $(A+B)/2$.

Problem 3.
We are told that A is a symmetric Markov matrix. It has an eigenvalue y, where $-1 < y < 1$.

1. Find the matrix A in terms of y.
2. Find the eigenvectors of A.
3. What is $\lim_{n\to\infty} A^n$ in its simplest form?

Problem 4.

1. If A is symmetric then which of these four matrices are necessarily positive definite

$$A^3, \ \ (A^2+I)^{-1}, \ \ A+I, \ \ e^A.$$

2. Suppose C is positive definite and that A has independent columns. Show that $x^T A^T C A x > 0$ for all $x \neq 0$. Hence $S = A^T C A$ is positive definite.