Vector Space of Functions.

Problem 1. *A set F consists of functions $f(x) = ae^x + be^{-x}$ where $a, b \in \mathbb{R}$. Given a function $f(x) = ae^x + be^{-x}$, we define the vector $V_f = \binom{a}{b} \in \mathbb{R}^2$.*

(1) *Is F a vector space? Provide two different basis.*
(2) *Does $\{\sinh x, \cosh x\}$ span F?*
(3) *What is $V_{f'}$ of $f \in F$?*
(4) *Find a matrix A satisfying $AV_f = V_{f'}$ for all $f \in F$.*
(5) *Show that $A^2 = I$, and check $\frac{d^2}{dx^2} f(x) = f(x)$.*

Problem 2. *A set F consists of functions $f(x) = axy + bx + c$ where $a, b, c \in \mathbb{R}$. Given a function $g(x) = axy + bx + cy + d$, we define the vector $V_g = (a\, b\, c\, d)^T \in \mathbb{R}^4$.*

(1) *What is the dimension of F?*
(2) *Does $\{V_f : f \in F\}$ span $\mathbb{R}^4$?*
(3) *What are $V_{\partial_y f}$ and $V_{\partial_x f}$ of $f \in F$? Are they belong to F?*
(4) *Find a matrix A satisfying $AV_f = V_{\partial_y f}$ for all $f \in F$.*
(5) *What is the null space of the operator ∂_y in F?*

Dimension and Rank.

Problem 3. *A matrix is $m \times n$ and has rank r. What are the relations of m, n, r?*

(1) *If $Ax = b$ has no solution for some b.*
(2) *If $Ax = b$ has exactly one solution for some b.*
(3) *If $Ax = b$ has infinitely many solutions for some b.*
(4) *If $Ax = b$ has at least one solution for every b.*
(5) *If $Ax = b$ has exactly one solution for every b.*

Proof. (1) $m > r$. Since $\mathrm{col}(A) \neq \mathbb{R}^m$ and $\mathrm{col}(A) \subset \mathbb{R}^m$, we have

$$r = \dim(\mathrm{col}(A)) < \dim(\mathbb{R}^m) = m.$$

(2) $n = r$. If $\mathrm{null}(A) \neq 0$, then $Ax = b$ can not have only one solution. Hence,

$$n - r = \dim(\mathrm{null}(A)) = 0.$$

(3) $n = r$. If $\mathrm{null}(A) = 0$, then $Ax = b$ can have at most one solution. Hence,

$$n - r = \dim(\mathrm{null}(A)) > 0.$$

(4) $m = r$. Given any $b \in \mathbb{R}^m$, there exists at least one solution $x \in \mathbb{R}^m$ solving $Ax = b$. Namely $b \in \mathrm{col}(A)$, and thus $\mathbb{R}^m \subset \mathrm{col}(A)$. Since $\mathrm{col}(A) \subset \mathbb{R}^m$, we conclude $\mathrm{col}(A) = \mathbb{R}^m$.

$$r = \dim(\mathrm{col}(A)) = \dim(\mathbb{R}^m) = m.$$

(5) $m = n = r$. The result in (2) implies $n = r$, and the result in (4) yields $m = r$.

$\square$

Problem 4. *What is the rank of the matrix*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}?$$

$\mathcal{M}$ is the set of 2×2 matrices, and this is a 4-dimensional vector space. Given any $B \in \mathcal{M}$, the multiplication AB belongs to $\mathcal{M}$ again. Provide a basis of $\mathcal{M}$, and find a matrix represent the operation $B \to AB$. What is the rank of the operation?