18.06 - Recitation 6

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1 Review problems for midterm 2

Problem 1.
The matrix A has a nullspace $N(A)$ spanned by

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and a left nullspace $N(A^T)$ spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

(a) What is the **shape** of the matrix A and what is its **rank**?

(b) If we consider the vector

$$b = \begin{pmatrix} -1 \\ \alpha \\ 0 \\ \beta \end{pmatrix},$$

for **what value(s)** of α and β (if any) is $Ax = b$ solvable? Will the solution (if any) be **unique**?

(c) Give the orthogonal **projections** of

$$y = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

onto **two** of the four fundamental subspaces of A.

Problem 2.
You have a matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Give the **ranks** of A, A^T, and A^TA, and also give **bases** for $C(A)$, $N(A)$, and $N(A^TA)$. (Look carefully at the columns of A, since very little calculation is needed!)

(b) Suppose we are looking for a least squares solution $\hat{x}$ that minimizes $\|b - Ax\|$ for $b = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$. At this minimum, $p = A\hat{x}$ will be the projection of b onto .... ? **Find** p.

Problem 3.

(a) Show that the trace of $A^T A$ must always be ≥ 0 by deriving a simple formula for trace$(A^T A)$ in terms of the matrix entries a_{ij} (i-th row, j-th column) of A. This is called the *Frobenius norm*

$$\|A\|_F = \sqrt{\text{trace}(A^T A)}$$

of the matrix.

(b) Using the compact SVD $A = U\Sigma V^T$, derive a simple relationship between the Frobenius norm $\|A\|_F$ and the singular values $\sigma_1, \ldots, \sigma_r$ of A.

Problem 4.

1. If Q is an orthogonal matrix ($Q^T = Q^{-1}$), explain why it follows from the rules for determinants that $\det Q$ must be .... or ....?
2. If P is a 3×3 projection matrix onto a 2d subspace, then its determinant must be ....?
3. An anti-symmetric matrix is a $n \times n$ matrix A with $A^T = -A$. What is $\det A$ when n is odd?