18.06

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What is the goal of 18.06?

Roughly: To understand matrices

A matrix is much more than just a table of values

A matrix encodes a function

input: a vector

output: a vector

Real goal: To understand these functions

Understanding a function means more than just being able to evaluate it

What was (one of the) goals of 18.01?

Consider a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where each coefficient is a real number

Can specify a polynomial by writing down the list of coefficients a_n,…,a_0

A polynomial is much more than just a list of numbers, it specifies a function that takes as input a real number and produces as output a real number

What was (one of the) goals of 18.01?

Understanding a polynomial means understanding what properties the corresponding function possesses

For which x is $f(x) = 0$?

At which value x does $f(x)$ attain its maximum?

What sort of shape does the curve $y = f(x)$ have?

The Value of Abstraction

Concrete example:

Vectors: elements of $\mathbb{R}^n$ (a list of n real numbers)

Matrices: an $m \times n$ table of real numbers

Function: A matrix A specifies a function $f(\vec{v}) = A\vec{v}$

Abstract version:

Given two vector spaces V, W consider linear maps from V to W

Far more general

Perhaps surprisingly, easier to think about

Examples of Vector Spaces

Standard:

$\mathbb{R}^n$

$\mathbb{C}^n$

Slightly more exotic:

Space of functions $f: \mathbb{R} \to \mathbb{R}$

More exotic, but domain specific:

(Quantum mechanics) Space of observables

Abstraction allows us to think about all of these objects in the same way, as a vector space

General vector spaces behave much like $\mathbb{R}^n$ in many ways

What is a Vector Space?

A vector space is a mathematical object that satisfies a certain set of rules

Informally, these rules enforce the property that a vector space is “similar” to $\mathbb{R}^n$

Slightly-less-informally, a real vector space V is

a set (the elements of which we call vectors)

a vector addition operation: for u, v in V we define $u + v$

a scalar multiplication operation: for v in V and a in $\mathbb{R}$ we define av

Vector space rules: These operation “behave like” ordinary vector addition and scalar multiplication in $\mathbb{R}^n$

What is a Linear Map?

For vector spaces V, W a linear map is a function $f: V \to W$ such that

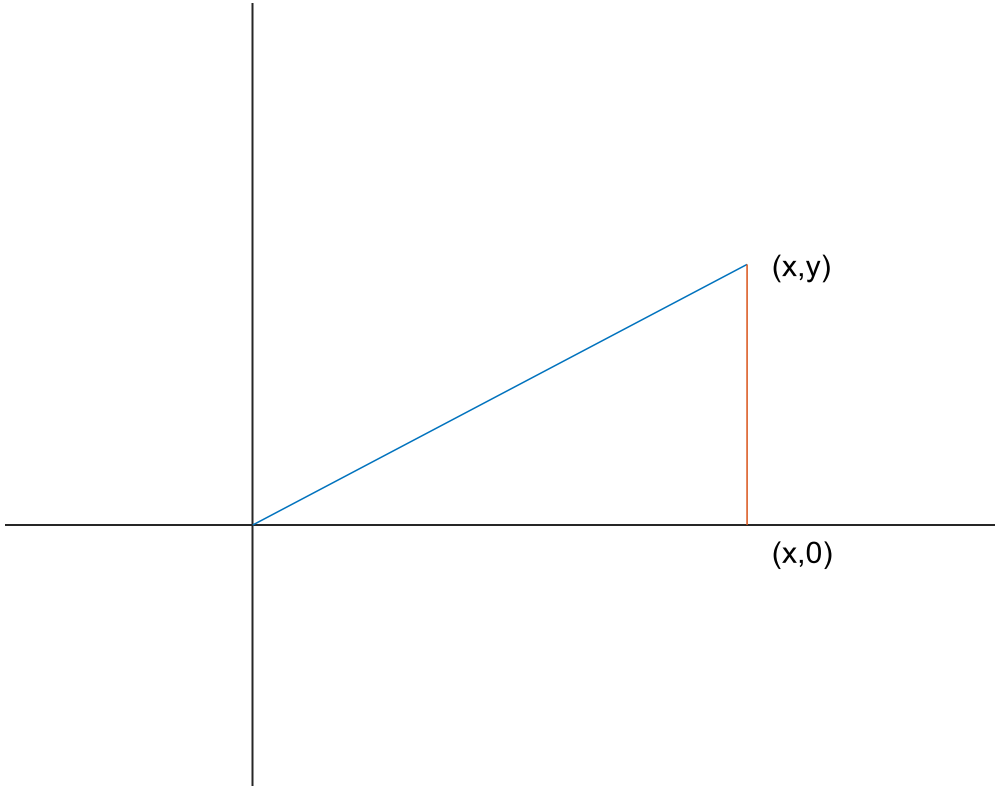
for any u, v in V and a, b in $\mathbb{R}$

$$f(au + bv) = af(u) + bf(v)$$

Example: Projection in $\mathbb{R}^2$

Define $X: \mathbb{R}^2 \to \mathbb{R}^2$ such that, for $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, $X\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

Projection onto x-axis



Example: Projection in $\mathbb{R}^2$

X is linear

$$
\begin{aligned}
X\left(a\begin{bmatrix}x_1\\y_1\end{bmatrix}+b\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) &= X\left(\begin{bmatrix}ax_1+bx_2\\ay_1+by_2\end{bmatrix}\right)\\
&= \begin{bmatrix}ax_1+bx_2\\0\end{bmatrix}\\
&= a\begin{bmatrix}x_1\\0\end{bmatrix}+b\begin{bmatrix}x_2\\0\end{bmatrix}\\
&= aX\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+bX\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)
\end{aligned}
$$

Example: Projection in $\mathbb{R}^2$

X can be encoded by a matrix, which we also denote X

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Could analogously define $Y: \mathbb{R}^2 \to \mathbb{R}^2$ such that, for $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$,

$$Y\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Example: Orthogonal Scaling in $\mathbb{R}^2$

For σ_1, σ_2 in $\mathbb{R}$, define $S_{\sigma_1,\sigma_2}: \mathbb{R}^2 \to \mathbb{R}^2$ by $S_{\sigma_1,\sigma_2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \sigma_1 x \\ \sigma_2 y \end{bmatrix}$

Scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by σ_1 in the x direction and by σ_2 in the y direction

S_{σ_1,σ_2} is linear

Can verify directly

Or note that S_{σ_1,σ_2} can be encoded by the matrix $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$

Or note $S_{\sigma_1,\sigma_2} = \sigma_1 X + \sigma_2 Y$

Example: Rotation in $\mathbb{R}^2$

For θ in $\mathbb{R}$, define $Q_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $Q_\theta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$

Rotation counterclockwise about the origin by angle θ

Can be encoded by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Composition of Linear Maps

For vector spaces, U, V, W with linear maps $f: U \to V$ and $g: V \to W$

We define the composition $g \circ f: U \to W$ by $g \circ f(u) = g(f(u))$ for u in U

First apply f then apply g

Often write gf for $g \circ f$

Example: Projection and Rotation in $\mathbb{R}^2$

Do we have $Q_{\frac{\pi}{2}}X = XQ_{\frac{\pi}{2}}$?

No, $Q_{\frac{\pi}{2}}X\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$=$\begin{bmatrix}0\\1\end{bmatrix}$
but $XQ_{\frac{\pi}{2}}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$=$\begin{bmatrix}0\\0\end{bmatrix}$

Order matters

Matrix multiplication is not commutative