Rect 3 soul

Friday, March 13, 2020 6:02 PM

1. **LU factorization**: A square matrix $A = LU$ where L is a lower-triangular matrix and U is a upper-triangular matrix.

(a) Suppose a 2×2 matrix A has a LU factorization $A = LU$ and $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. How are the column vectors A_1, A_2 of A related to column vectors L_1, L_2 of L?

$$(A_1 A_2) = (L_1 L_2)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = L_1$$
$$A_2 = 2L_1 + L_2$$

2. **QR decomposition**: A $m \times n$ matrix $A = QR$ where Q is a orthogonal matrix and R is a upper-triangular matrix.

(a) Write down a QR decomposition for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hint: what is the dot product of the first column with the second column of A.

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

(b) Let $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Can you solve $Ax = b$?

No since $\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ means

$x_1 - x_2 = 1 \Rightarrow x_1 = 2$
$x_1 + x_2 = 1 \Rightarrow x_1 = 0 \Rightarrow$ No solution,
$x_2 = 1$

(c) Find a vector $\hat{x}$ that minimizes $||Ax - b||$.

$$A^T A x = A^T b$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

3. **SVD:** A $m \times n$ matrix A has the rank-r SVD as $A = U\Sigma V^T$. What are U, V, Σ and what are their matrix size?

U $m \times r$

Σ $r \times r$

V $r \times n$

Suppose that a 3×3 matrix A has the SVD as

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(a) What is the rank of A?

2

(b) Write the column space of A as all the linear combinations of two column vectors of U, Σ or V.

The two columns of U.

(c) Let $B = U\Sigma$. How are the columns of A related to columns of B?

$$A = B V^T$$

$$(A_1 A_2 A_3) = (B_1 B_2)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$A_1 = B_2$, $A_2 = 0$ $A_3 = B_1$

(d) What is the norm of the third column of A?

$$|A_3| = |B_1| = 5 \cdot |A_1| = 5.$$