18.06

Zack Remscrim

Review for Midterm 3

Main Topics:

- Eigenvalues/eigenvectors
- Diagonalization
- Applications of Determinants
  - Volume
  - Cofactors

Eigenvalues and Eigenvectors: Definition

Square matrix A

If $Av = \lambda v$, $v \neq 0$ a vector and λ a scalar

Then v is eigenvector of A with eigenvalue λ

v is a very special vector

Applying A to v only changes the magnitude of v, not direction

Applying A to v produces λv

λv is scaled version of v

Eigenvalues and Eigenvectors: Existence

When will $v \neq 0$ a vector and λ a scalar satisfy $Av = \lambda v$?

$Av = \lambda v$ iff $0 = \lambda v - Av = (\lambda I - A)v$

For any λ, equation $(\lambda I - A)v = 0$ always has solution $v = 0$

When does it have other solutions?

Exactly when $(\lambda I - A)$ is singular

Exactly when $\det(\lambda I - A) = 0$

When does equation (in λ) $\det(\lambda I - A) = 0$ have a solution?

Always!

Why?

Characteristic Polynomial

Why does equation $\det(\lambda I - A) = 0$ always have a solution?

Ex: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\lambda I - A = \begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix}$

$\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$

$\det(\lambda I - A)$ is a polynomial in λ, with real coefficients

True for all (square) matrices, not just this example

Every such polynomial has a (possibly complex) root

Call this the characteristic polynomial of A

Notation: $p_A(\lambda) = \det(\lambda I - A)$

Characteristic Polynomial: Properties

A $n \times n$ matrix, $p_A(\lambda) = \det(\lambda I - A)$ its characteristic polynomial

$p_A(\lambda)$ is a polynomial of degree n, with real coefficients

Compute determinant using cofactors

$p_A(\lambda)$ has exactly n (possibly complex) roots, counting multiplicity

Example: $\lambda^3 - 3\lambda + 2 = (\lambda - 1)^2(\lambda + 2)$

root at $\lambda = -2$, (double) root at $\lambda = 1$

Example: $\lambda^2 - 2\lambda + 2 = (\lambda - (1 + i))(\lambda - (1 - i))$

root at $\lambda = 1 + i$ and $\lambda = 1 - i$

The roots of $p_A(\lambda)$ are the eigenvalues of A

Eigenvalues and Eigenvectors

Square matrix A,

- Vector $v \neq 0$ is an eigenvector with eigenvalue λ if $Av = \lambda v$
- Eigenvalues are the roots of $p_A(\lambda) = \det(\lambda I - A)$
- What about eigenvectors?
- If λ is an eigenvalue,
  - $v \neq 0$ is eigenvector with with eigenvalue λ if $(\lambda I - A)v = 0$
  - All $v \in \text{null}(\lambda I - A)$, except $v = 0$
- Eigenvectors not unique,
  - Each eigenvalue λ has infinitely many corresponding eigenvectors
  - Example: If v eigenvector with eigenvalue λ, so is $2v, \frac{3}{2}v, -7v$, ...
  - Generally just write down a collection of lin. indep. eigenvectors

Eigenvalues and Eigenvectors: Examples

$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, Projection onto x-axis

What are its eigenvalues/eigenvectors?

Eigenvalue 1 with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Vectors pointing along x-axis are not changed by P

Eigenvalue 0 with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Vectors pointing along y-axis are sent to 0 by P

P is 2×2, so these are only eigenvalues

Eigenvalues and Eigenvectors: Examples

$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, diagonal matrix

What are its eigenvalues/eigenvectors?

Eigenvalue d_1 with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Vectors pointing along x-axis are scaled by d_1

Eigenvalue d_2 with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Vectors pointing along y-axis are scaled by d_2

D is 2×2, so these are only eigenvalues

Eigenvalues and Eigenvectors: Examples

$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, a particular lower triangular matrix

What are its eigenvalues/eigenvectors?

Only eigenvalue is 1

$$p_L(\lambda) = \det(\lambda I - L) = (\lambda - 1)^2$$

Only eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (and scalar multiples of course)

Eigenvectors are $v \in \text{null}(1I - A), v \neq 0$

$I - A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, which has rank 1

L does not have two lin. indep. eigenvectors

Eigenvalues and Eigenvectors: Examples

$\frac{d}{dx}$ the derivative operator (on vector space of all differentiable func.)

What are its eigenvalues/eigenvectors?

For which functions $f(x)$ is there a scalar c where $\frac{d}{dx}f(x) = cf(x)$?

Such $f(x)$ is eigenvector with eigenvalue c

Each $c \in \mathbb{C}$ is eigenvalue

with eigenvector (eigenfunction) $f(x) = e^{cx}$

Why are Eigenvectors Useful?

Example: A a 2×2 matrix,

That has two lin. indep. eigenvectors v_1, v_2

With corresponding eigenvalues λ_1, λ_2

v_1, v_2 form a basis of $\mathbb{R}^2$ (eigenbasis)

Can write any $x \in \mathbb{R}^2$ as $c_1 v_1 + c_2 v_2$, for some $c_1, c_2 \in \mathbb{R}$

$$\begin{aligned} Ax &= A(c_1 v_1 + c_2 v_2) \\ &= c_1 A v_1 + c_2 A v_2 \\ &= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \end{aligned}$$

Each component scales by corresponding eigenvector

Why are Eigenvectors Useful?

Example: A an $n \times n$ matrix,

That has n lin. indep. eigenvectors $v_1, \dots, v_n$

With corresponding eigenvalues $\lambda_1, \dots, \lambda_n$

$v_1, \dots, v_n$ form a basis of $\mathbb{R}^n$ (eigenbasis)

Can write any $x \in \mathbb{R}^n$ as $c_1 v_1 + \cdots + c_n v_n$, for some $c_i \in \mathbb{R}$

$$\begin{aligned} Ax &= A(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 A v_1 + \cdots + c_n A v_n \\ &= c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n \end{aligned}$$

Each component scales by corresponding eigenvector

Much easier to compute in an eigenbasis, A behaves like diagonal matrix

Diagonalization

Consider A an $n \times n$ matrix,

That has n lin. indep. eigenvectors $v_1, \ldots, v_n$

With corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$

Let X be $n \times n$ matrix with eigenvectors $v_1, \ldots, v_n$ in the columns

X invertible because it is a square matrix with lin. Indep. cols

Let Λ be $n \times n$ diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ along diagonal

Then $AX = X\Lambda$,

So $A = X\Lambda X^{-1}$

and $\Lambda = X^{-1}AX$

Diagonalization

If we can write $A = X\Lambda X^{-1}$, where Λ is diagonal,

we say we have diagonalized A

because A is "similar" to a diagonal matrix

We say matrices B and C are similar

if there is an invertible matrix T, such that $B = TCT^{-1}$

Similar matrices represent the same function, in different bases

Sometimes one basis is easier to work in than another

Diagonal matrices are especially easy to work with

Eigenvalues and Eigenvectors: Properties

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\lambda I - A = \begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix}$

$$
\begin{aligned}
\det(\lambda I - A) &= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} \\
&= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\
&= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) \\
&= (\lambda - \lambda_1)(\lambda - \lambda_2)
\end{aligned}
$$

Eigenvalues λ_1, λ_2 satisfy

$$\lambda_1 + \lambda_2 = \mathrm{tr}(A)$$

$$\lambda_1 \lambda_2 = \det(A)$$

Eigenvalues and Eigenvectors: Properties

Consider $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicity)

$$\lambda_1 + \cdots + \lambda_n = \mathrm{tr}(A)$$

$$\lambda_1 \cdots \lambda_n = \det(A)$$

If v is an eigenvector of A with eigenvalue λ

For any scalar α, αv is an eigenvector of A with eigenvalue λ

$$A(\alpha v) = \alpha(Av) = \alpha(\lambda v) = \lambda \alpha v$$

v is an eigenvector of A^2 with eigenvalue λ^2

$$A^2 v = AAv = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$$

Eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$, for $k \in \mathbb{N} = \{0,1,2,\dots\}$

Eigenvalues and Eigenvectors: Properties

Consider $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicity)

A is singular if and only if A has eigenvalue 0

A has eigenvalue 0 if and only if $0 = \det(0I - A) = \det(-A)$

If A invertible,

If v is an eigenvector of A with eigenvalue λ

v is an eigenvector of A^{-1} with eigenvalue λ^{-1}

If $Av = \lambda v$, then $A^{-1}Av = A^{-1}\lambda v$

Then $v = \lambda(A^{-1}v)$

So $\lambda^{-1}v = A^{-1}v$

Eigenvalues of A^{-1} are $\lambda_1^{-1}, \dots, \lambda_n^{-1}$

Eigenvalues and Eigenvectors: Properties

For any $n \times n$ matrix A, with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$

Corresponding eigenvectors are $v_1, \ldots, v_n$ are lin. Indep.

So A is diagonalizable

If A does not have distinct eigenvalues

May or may not be diagonalizable

Symmetric matrices always diagonalizable

Determinant: Geometric Meaning in $\mathbb{R}^2$

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, 2×2 real matrix

and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$

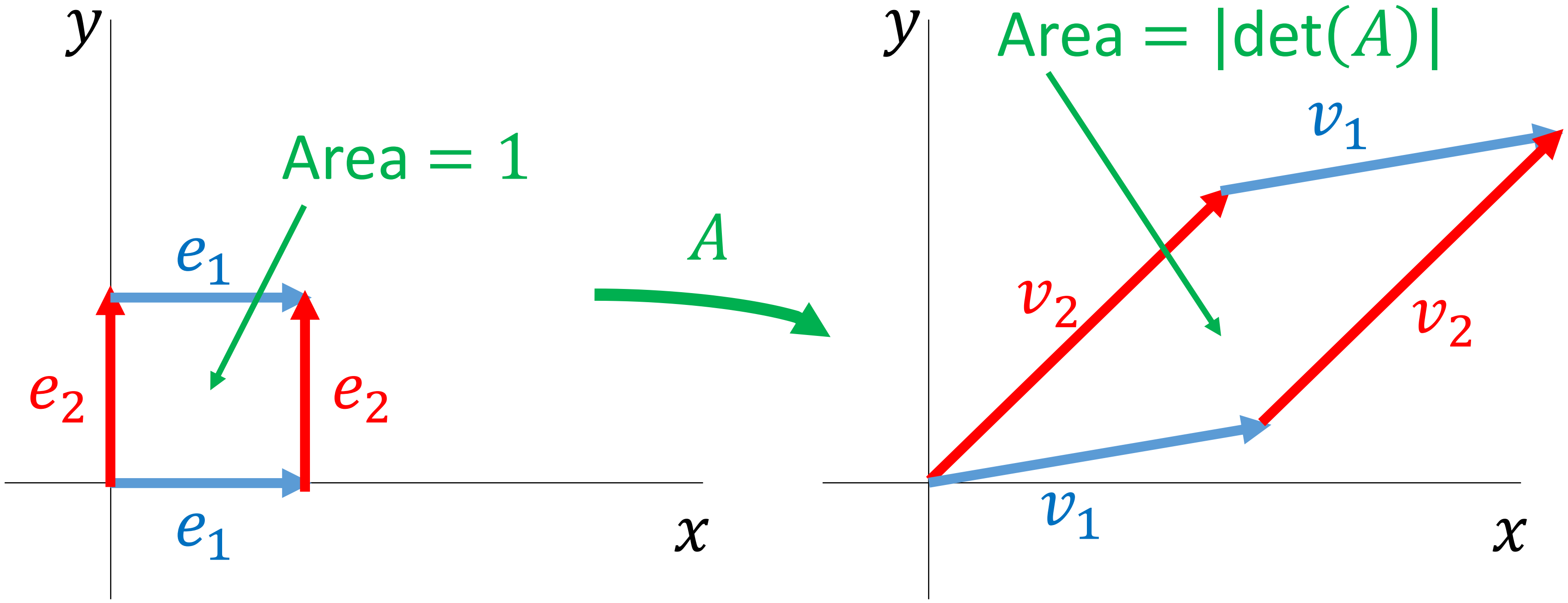
A transforms e_1, e_2 to

$$v_1 = Ae_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$v_2 = Ae_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

1st and 2nd col of A

A transforms square of area 1 to parallelogram of area $|\det(A)|$



Determinant: Geometric Meaning in $\mathbb{R}^n$

Consider A, $n \times n$ real matrix
and $e_1, e_2, \dots, e_n \in \mathbb{R}^n$, where e_i: i^{th} entry 1, all other entries 0

A transforms $e_1, e_2, \dots, e_n$ to

$v_1, v_2, \dots, v_n$, where $v_i = Ae_i$

v_i is i^{th} col of A

$e_1, e_2, \dots, e_n$ specify hypercube in $\mathbb{R}^n$ (n-dimensional version of cube)
$v_1, v_2, \dots, v_n$ specify parallelotope in $\mathbb{R}^n$ (n-dimensional version of parallelogram)

A transforms hypercube $e_1, e_2, \dots, e_n$ of volume 1
to parallelotope $v_1, v_2, \dots, v_n$ of volume $|\det(A)|$

Minors and Cofactors

For an $n \times n$ matrix A

Define $n \times n$ matrix M, matrix of *minors*, by

M_{ij} is det of A with row i and col j deleted

Define $n \times n$ matrix C, matrix of *cofactors*, by

C_{ij} is $(-1)^{i+j} M_{ij}$

Cofactors: Properties

For any $n \times n$ matrix A and any row i

$$|A| = A_{i,1}C_{i,1} + A_{i,2}C_{i,2} + \cdots + A_{i,n}C_{i,n}$$

For any invertible $n \times n$ matrix A

$$A^{-1} = \frac{C^{\top}}{|A|}$$