18.06 R08 - Recitation 2 - SOLUTIONS

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1 Practice problems for midterm 1

Problem 1. (LU factorization)

(a) Compute the LU factorization of an arbitrary 2×2 matrix, i.e. find x, u, v and w for which

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}.$$

Are there any conditions on a, b, c and/or d for this factorization to exist?

(b) Find an LU factorization for the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 5 \end{pmatrix}$$

Solution

(a) We did this on pset 1. We found that

$$\begin{aligned} u &= a \\ v &= b \\ x &= c/a \\ w &= d - bc/a \end{aligned}$$

The only condition for this factorization to exist is that $a \neq 0$.

(b) Although we have only found the LU factorization for 2×2 matrices, we can spot that this 4×4 matrix is a *block* matrix, and the two blocks are both 2×2 matrices. We can therefore write

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 5 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} L_1U_1 & 0 \\ 0 & L_2U_2 \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} = LU$$

where, e.g., L_1U_1 is the LU factorization of A_1. We then find that

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Problem 2. (Vector spaces)
Are the following sets examples of vector spaces? If yes, then show that an arbitrary linear combination of two elements in the set is also in the set. If not, then explain why.

(a) The set of all solutions, x, to the equation $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; where $x \in \mathbb{R}^5$ and A is a 3×5 matrix.

(b) The set of all 3×2 matrices X for which $AX = 0$, where A is a fixed 5×3 matrix.

(c) The set of all 2×2 matrices A for which A^{-1} does *not* exist.

(d) The set of all differentiable functions $f(x)$ for which $f'(0) = 2f(0)$.

(e) The set of all functions $f(x)$ for which $f(x + y) = f(x)f(y)$.[1]

Solution

(a) This is not a vector space. It does not contain the zero vector, because $x = 0$ does not satisfy the equation.

(b) This is a vector space. Take two matrices X_1 and X_2 for which $AX_1 = AX_2 = 0$, and two scalars $a, b \in \mathbb{R}$. Then $A(aX_1 + bX_2) = aAX_1 + bAX_2 = 0$.

(c) This is not a vector space. It does contain the zero matrix. However, $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are not invertible matrices (check $ad - bc = 0$), but $X_1 + X_2 = I$, which is invertible!

(d) This is a vector space. Consider two functions $f_1(x)$ and $f_2(x)$ for which $f_1'(0) = 2f_1(0)$ and $f_2'(0) = 2f_2(0)$, and two scalars $a, b \in \mathbb{R}$. A linear cominbation of f_1 and f_2 is a new function $g(x) = af_1(x) + bf_2(x)$. Then $g'(0) = af_1'(0) + bf_2'(0) = 2af_1(0) + 2bf_2(0) = 2g(0)$.

(e) This is not a vector space. If $f_1(x)$ is a function for which $f_1(x + y) = f(x)f(y)$, then $af_1(x + y) = af_1(x)f_1(y) \neq (af_1(x))(af_2(x))$ for all $a \neq 1$.

Problem 3. (QR factorization)
Consider the following 3×2 matrix A:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(a) Write $A = QR$, where Q is an orthogonal matrix ($Q^TQ = I$) and R is a square, upper triangular matrix. What is R^{-1}? Is Q invertible?

(b) Consider the linear system

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Can we express the right hand side of this linear system as a linear combination of the columns of A?

(c) Check that $QQ^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) Find explicitly the vector $\hat{x}$ that minimizes $\|Ax - b\|^2$.

[1] Aside for those interested: what continuous functions obey this rule?

Solution

(a) The columns of A are already orthogonal. However, they are not normalized. We can then write $A = QR$, where

$$Q = \begin{pmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \quad R = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

See lecture slide 55 for more details. R is a diagonal 2×2, so we can then compute as $R^{-1} = \begin{pmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$. Q is a rectangular matrix, so it cannot be invertible.

(b) This equation does not have any solutions, because there is no linear combination of the columns of A which will give the RHS

(c)

$$QQ^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \\ 2/3 \end{pmatrix}$$

(d) Recall that the least squares solution $\hat{x}$ obeys $\hat{x} = R^{-1}Q^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We have R^{-1} and Q so we can calculate explicitly that $\hat{x} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$.

Problem 4. (SVD)
Consider the 3×3 matrix A with the following full SVD:

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T$$

(a) Describe each of the four fundamental subspaces of A (column space, row space, nullspace, and left nullspace)

(b) Find the best rank-1 approximation to A. Write your answer as a 3×3 matrix.

(c) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ have a solution?

(d) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ have a solution?

Solution

(a)
- The column space of A, $C(A)$, is equal to the column space of U_1. This is all linear combinations of $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$
- The row space of A, $C(A^T)$, is equal to the column space of V_1. This is all linear combinations of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- The nullspace of A, $N(A)$, is equal to the column space of V_2. This is all linear combinations of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
- The left nullspace of A, $N(A^T)$, is equal to the column space of U_2. This is all linear combinations of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(b) The best rank-1 approximation of A is

$$\sigma_1 u_1 v_1^T = 10 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = 10 \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) We can compute $U_1 U_1^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$; and so the system does not have a solution (see lecture slide 68)

(d) We can compute $U_1 U_1^T \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$; and so the system does have a solution.

Problem 5. (Vector algebra)
Consider two vectors u and v. Find a new vector w which is in the direction of v, and for which $w - u$ is orthogonal to v. Your answer should be given in terms of the vectors u and v.[2]

Solution
If w is in the direction of v, then $w = \alpha v$ where $\alpha \in \mathbb{R}$. If $w - u$ is orthogonal to v, then

$$0 = v^T(w-u) = v^T(\alpha v - u) = \alpha v^T v - v^T u$$

$$\implies \boxed{\alpha = \frac{v^T u}{v^T v}}$$

So that $\boxed{w = \frac{v^T u}{v^T v} v}$.

Problem 6. (Least squares)
Suppose we are conducting an experiment and obtain the following m data points $\{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$. We think that our data is close to lying on a curve of the form $f(a) = x_1 + x_2 a + x_3(a-1)^2$. How would we obtain the best fit coefficients $\hat{x}_1, \hat{x}_2$ and $\hat{x}_3$? What quantity does the best fit vector $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$ minimize?

Solution
If a perfect fit existed, then we would want $\hat{x}$ to satisfy the equation:

$$\begin{pmatrix} 1 & a_1 & (a_1-1)^2 \\ 1 & a_2 & (a_2-1)^2 \\ \vdots & \vdots & \vdots \\ 1 & a_m & (a_m-1)^2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\iff A\hat{x} = b$$

However, this system of equations will generally not have a solution for $\hat{x}$. Instead, we seek $\hat{x}$ which minimizes the least squares error $\|Ax - b\|^2$. If we can find a QR factorization $A = QR$, then this least squares solution will obey

$$\boxed{\hat{x} = R^{-1}Q^T b}$$

[2]Your answer w is known as the *projection* of u onto v. More on this in class tomorrow