1 Lecture Review

1.1 Vector Spaces

1. A set V is a vector space if (i) $0 \in V$, (ii) $v, w \in V$ implies $v + w \in V$, (iii) $c \in \mathbb{R}$, $v \in V$ implies $cv \in V$.

1.2 Matrix Transpose

1. The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T defined by $(A^T)_{ij} = A_{ji}$.
2. If A is $m \times n$ and B is $n \times p$, then $(AB)^T = B^T A^T$.

1.3 Matrix Inverse

1. A matrix A is invertible if there exists a matrix A^{-1} so that $AA^{-1} = I = A^{-1}A$.
2. An invertible matrix must be square.
3. If A, B have the same size and are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.
4. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then
$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
given $ad - bc \neq 0$.

1.4 Orthogonal

1. A matrix A is orthogonal if $AA^T = A^T A = I$. That is, $A^T = A^{-1}$.

1.5 Block Matrices

1. A block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a way to write a matrix in terms of smaller matrices A, B, C, D where the sizes are compatible.
2. Block matrices multiply like 2×2 matrices (but you must remember the order)
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

1.6 Matrix Factorizations

1. A matrix factorization of A is a way to express A as a product of matrices.
2. LU Factorization, QR Factorization and SVD (covered later, see Problems 4, 5 and 6 on PSet 1 for 2×2 examples).

2 Problems

1. Determine if the following set is a vector space. Why or why not?
   (a) The line $y = x$.
   (b) The line $y = x + 1$.
   (c) The union of the x and y axes.
   (d) The unit circle (x, y) where $x^2 + y^2 = 1$.
   (e) 5×5 matrices with $(3, 3)$ entry equal to 0.
   (f) Set of functions $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.
   (g) Set of functions $f(x)$ with $f(7) = 0$.
   (h) Set of functions $f(x)$ with $f(0) = 7$.

2. (a) Suppose $\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}$ is the $m \times n$ matrix with columns vectors are $b_1, \ldots, b_n$. Show that if A is $p \times m$, then
$$A \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{pmatrix}.$$
   (b) Suppose $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ is an $m \times n$ matrix with row vectors $a_1, \ldots, a_n$. Show that if B is $n \times p$, then
$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ \vdots \\ a_m B \end{pmatrix}.$$

3. If S_1 stands for the operation of putting on your socks and S_2 the operation of putting on your shoes (so $S_2 \circ S_1$ stands for putting on your socks and then your shoes), what is the inverse of $S_2 \circ S_1$?

4. Show that the rotation matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

5. Suppose $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix where A, B, C, D are matrices with compatible sizes. Write the transpose $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as a block matrix in terms of A^T, B^T, C^T, D^T.

6. When can $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be written as $X^T X$ for $X = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$? Assume $b \neq 0$. What are p, q, r in terms of a, b, c, d when possible?

3 Answers

1. (a) Yes, (b) No, (c) No, (d) No, (e) Yes, (f) Yes, (g) Yes, (h) No
2. -
3. -
4. -
5. $\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$
6. Conditions: $ad - b^2 \geq 0$, $d > 0$, $b = c$. Then $p = \sqrt{\frac{ad-b^2}{d}}$, $q = \frac{b}{\sqrt{d}}$, $r = \sqrt{d}$.