Topics: column space (etc.), QR decomposition, SVD decomposition

1. Vector spaces associated to a matrix A

Given an $m \times n$ matrix A, we have the following four vector spaces:

- Column space (col(A)): all vectors in $\mathbb{R}^m$ of form Ax for some $x \in \mathbb{R}^n$.
- Nullspace (null(A)): all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.
- Row space (row(A)): column space of $A^\top$.
- Left nullspace (leftnull(A)): nullspace of $A^\top$.

We define rank(A) to be the dimension of col(A). Facts which can be proved by SVD:

- $\operatorname{rank}(A) = \operatorname{rank}(A^\top) \leq \min(m, n)$
- $\operatorname{rank}(A) + (\text{dimension of } \operatorname{null}(A)) = n$
- Let A be square, so $n = m$. Then (A is invertible) $\Leftrightarrow$ ($\operatorname{rank}(A) = n$) $\Leftrightarrow$ ($\det(A) \neq 0$).

(Idea: SVD allows one to reduce to the case where A is an $m \times n$ diagonal matrix.)

Problem 1. Let A be a matrix. Explain why the following are true:

(1) (For any $b \in \mathbb{R}^m$, there exists $x \in \mathbb{R}^n$ such that $Ax = b$) if and only if $\operatorname{col}(A) = \mathbb{R}^m$.

(2) (The only solution to $Ax = 0$ is $x = 0$) if and only if $\operatorname{null}(A) = \{0\}$.

(3) If A is square (so $m = n$) and invertible, then $\operatorname{col}(A) = \operatorname{row}(A) = \mathbb{R}^n$ and $\operatorname{null}(A) = \operatorname{leftnull}(A) = \{0\}$.

(4) Let B be an $n \times n$ invertible matrix. Then $\operatorname{col}(A) = \operatorname{col}(AB)$ and $\operatorname{rank}(A) = \operatorname{rank}(AB)$.

(5) Let B be an $m \times m$ invertible matrix. Then $\operatorname{row}(A) = \operatorname{row}(BA)$ and $\operatorname{rank}(A) = \operatorname{rank}(BA)$.

(6) If A is orthogonal (so $m \geq n$), then $\operatorname{null}(A) = \{0\}$ and $\operatorname{rank}(A) = n$.

Problem 2. Let Q be a square orthogonal matrix. Show that $\det(Q) = \pm 1$. (Hint: for any square matrix A, we have $\det(A) = \det(A^\top)$, and if A is invertible then $\det(A^{-1}) = \det(A)^{-1}$.)

2. QR decomposition

Given an $m \times n$ matrix A with $m \geq n$, a QR-decomposition of A is an equation

$$A = QR$$

where Q is an $m \times n$ orthogonal matrix and R is an *invertible* $n \times n$ upper-triangular matrix. The existence of such a decomposition implies the following:

- $\operatorname{rank}(A) = m$, and $\operatorname{null}(A) = \{0\}$.
- The columns of Q are an orthonormal basis in the space col(A).
- Let $b \in \mathbb{R}^m$ be a vector. Then there is a unique vector $x \in \mathbb{R}^n$ which minimizes $\|Ax - b\|$, and that x is given by $x = R^{-1}Q^\top b$. The minimum possible value of $\|Ax - b\|$ is given by $\|QQ^\top b - b\|$.

Problem 3. Let v be the third column of A. What is $\|v\|$ in terms of the entries of R?

Problem 4. Assume that A is a square matrix, and let $A = QR$ be a QR decomposition. Show that $\det(A) = \pm \det(R)$.

Problem 5. Assume $m > n$. Let $A = \begin{pmatrix} R \\ \hline Z \end{pmatrix}$ be a block matrix, where R is an $n \times n$ invertible upper-triangular matrix and Z is the all-zero $(m - n) \times n$ matrix. Write down a QR decomposition for A.

Problem 6. Does the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ have a QR decomposition?

3. Singular value decomposition (SVD)

Given an $m \times n$ matrix A, the (large-format) SVD of A is an equation

$$A = U\Sigma V^\top$$

where U is an $m \times m$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix with entries in (weakly) decreasing order, and V is an $n \times n$ orthogonal matrix. The entries of Σ are the *singular values* of A, the columns of U are the *left-singular vectors*, and the columns of V are the *right-singular vectors*. Every matrix admits at least one SVD.

Since U and V are invertible, we apply Problem 1, parts (4) and (5), to find

$$\mathrm{rank}(A) = \mathrm{rank}(\Sigma) = (\text{number of nonzero entries of } \Sigma).$$

Let $r = \mathrm{rank}(A)$. We define some smaller matrices in order to get rid of the zeros in Σ.

- U' is obtained by truncating U to size $m \times r$ (starting from top-left entry).
- Σ' is obtained by truncating Σ to size $r \times r$.
- V' is obtained by truncating V' to size $n \times r$.

Problem 7. Explain why the following are true:

(1) U' and V' are orthogonal.

(2) Σ' is invertible.

(3) $A = U'\Sigma'(V')^\top$.

The expression $A = U'\Sigma'(V')^\top$ is the small-format SVD of A.

Problem 8. Assume that A is a square matrix, and let $A = U\Sigma V^\top$ be an SVD. Show that $\det(A) = \pm\det(\Sigma)$.

Problem 9. Given $A = U\Sigma V^\top$ as above, find the SVDs of the matrices $A^\top, A^\top A$, and $AA^\top$. Using your answers, determine the ranks of these matrices. Assuming that A is square and invertible, find the SVDs of A^{-1} and $(A^{-1})^\top$.

Problem 10. Let $A = U'\Sigma'(V')^\top$ be a small-format SVD of A. Explain why the following are true:

(1) $\mathrm{col}((V')^\top) = \mathbb{R}^r$. (Hint: find an SVD of $(V')^\top$)

(2) $\mathrm{col}(\Sigma'(V')^\top) = \mathbb{R}^r$.

(3) $\mathrm{col}(A) = \mathrm{col}(U')$.

(4) $\mathrm{row}(A) = \mathrm{col}(V')$.

Problem 11. Find an SVD of the matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$.

(Hint: permutation matrices are orthogonal.)

4. Solutions

1. (1) The condition in parentheses exactly says that every vector in $\mathbb{R}^m$ is of the form Ax for some $x \in \mathbb{R}^n$.

   (2) null(A) is defined to be the space of solutions to $Ax = 0$, so the condition in parentheses exactly says that $\mathrm{null}(A) = \{0\}$.

   (3) If A is invertible, then any vector b is of the form Ax because we can take $x = A^{-1}b$. Therefore $\mathrm{col}(A) = \mathbb{R}^n$. Since $A^\top$ is invertible as well, we also conclude $\mathrm{row}(A) = \mathbb{R}^n$. If $Ax = 0$, then applying A^{-1} yields $x = A^{-1}0 = 0$, so the criterion of the previous part implies that $\mathrm{null}(A) = \{0\}$. Applying this reasoning to $A^\top$ in place of A yields $\mathrm{leftnull}(A) = \{0\}$.

   (4) If $b \in \mathrm{col}(A)$, then $b = Ax$ for some x. Then $b = AB(B^{-1}x)$, so $b \in \mathrm{col}(AB)$. The other direction is similar.

   (5) Take the transpose of the previous part.

   (6) An SVD of A is
   $$A = A\,\mathrm{Id}_{n\times n}\,\mathrm{Id}_{n\times n}$$
   The 'key fact' from the SVD section tells us that $\mathrm{rank}(A) = \mathrm{rank}(\mathrm{Id}_{n\times n}) = n$. (One could also use a QR decomposition instead.)

   To show that $\mathrm{null}(A) = \{0\}$, suppose that $Ax = 0$ for some $x \in \mathbb{R}^n$. Write $x = (x_1, \ldots, x_n)$, and let $v_1, \ldots, v_n$ be the columns of A, which form an orthonormal collection since A is orthogonal. By our hypothesis that $Ax = 0$, we know that
   $$x_1v_1 + \cdots + x_nv_n = 0$$
   as vectors in $\mathbb{R}^n$. Take the dot product with v_i to obtain
   $$\begin{aligned} 0 &= 0 \cdot v_i \\ &= (x_1v_1 + \cdots + x_nv_n) \cdot v_i \\ &= x_1v_1 \cdot v_i + \cdots + x_nv_n \cdot v_i \\ &= x_i \end{aligned}$$
   where we use that $v_j \cdot v_i = 0$ if $j \neq i$ and $v_i \cdot v_i = 1$, by definition of orthonormality. Hence, $x_i = 0$ for all i, so $x = 0$, and this shows that $\mathrm{null}(A) = \{0\}$.

2. Since $Q^{-1} = Q^\top$, we have
   $$\begin{aligned} \det(Q) &= \det(Q^\top) \\ &= \det(Q^{-1}) \\ &= \det(Q)^{-1} \end{aligned}$$
   using the hint. Thus $\det(Q)^2 = 1$, so $\det(Q) = \pm 1$.

3. $\|v\| = \sqrt{R_{13}^2 + R_{23}^2 + R_{33}^2}$

4. We have
   $$\begin{aligned} \det(A) &= \det(Q)\det(R) \\ &= \pm\det(R) \end{aligned}$$
   using the result of Problem 2.

5. A QR decomposition is given by
   $$A = \left(\frac{\mathrm{Id}_{n\times n}}{Z}\right) R,$$
   where Z is as defined in the problem.

6. No, because $\operatorname{rank}(A) = 1$, while any 2×2 matrix with a QR decomposition (in the sense defined in class) has rank 2.

7. (1) Start with the identity $U^\top U = \mathrm{Id}_{n\times n}$. Break U into an $m \times r$ block (namely U') and an $m \times (m-r)$ block, and look at the result of block multiplication. Similarly for V.

   (2) Σ' is a diagonal matrix with nonzero diagonal entries. Its inverse is given by the diagonal matrix whose entries are the reciprocals of the diagonal entries of Σ'.

   (3) Take $A = U\Sigma V^\top$ and look at the result of block multiplication as in the first part.

8. This follows from $\det(U) = \pm 1$ and $\det(V) = \pm 1$.

9. We have
$$\begin{aligned} A^\top &= V\Sigma^\top U^\top \\ A^\top A &= V(\Sigma^\top \Sigma)V^\top \\ AA^\top &= U(\Sigma\Sigma^\top)U^\top. \end{aligned}$$
   These matrices all have the same rank as A.

   If A is square and invertible, so is Σ, and we have
$$\begin{aligned} A^{-1} &= V\Sigma^{-1}U^\top \\ (A^{-1})^\top &= U\Sigma^{-1}V^\top. \end{aligned}$$

10. (1) An SVD of $(V')^\top$ is
$$(V')^\top = \mathrm{Id}_{r\times r}\,\mathrm{Id}_{r\times r}(V')^\top,$$
    so $\operatorname{rank}((V')^\top) = r$. Hence $\operatorname{col}((V')^\top) = \mathbb{R}^r$.

    (2) Since Σ' is invertible, we have $\operatorname{col}(\Sigma' B) = \operatorname{col}(B)$ for any matrix B. (Compare Problem 1(4).)

    (3) See Lecture 7 slides.

    (4) Apply the previous part to $A^\top$ in place of A.

11. We have
$$A = \mathrm{Id}_{4\times 4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$