18.06 - Review for final exam

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(Fall 2018 final, Q2): The real $m \times n$ matrix A has a QR factorization $A = QR$ of the form

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -2 & 2 & 0 & 0 & 0 \\ & 2 & -3 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 3 & 1 & -1 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix},$$

where $q_1, ..., q_6$ are six orthonormal vectors in $\mathbb{R}^m$.

(a) Give as much true information as possible about m, n, and the rank of A.

(b) If a_5 is the 5th column of A, write it in the basis $q_1, ..., q_6$, i.e. write it as $a_5 = c_1 q_1 + c_2 q_2 + ... + c_6 q_6$, by giving the numerical values of the coefficients $c_1, ..., c_6$.

(c) What is $\|a_5\|$?

(d) This pattern of zero entries in R means that columns .... of A must be .... to columns .... of A.

(e) If A is a square matrix, what is $|\det A|$ (the absolute value of the determinant)?

(a) **(Spring 2018 exam 2, Q3):** The compact singular value decomposition of a rank r, $m \times n$ matrix A is $U\Sigma V^T$ where Σ is square r by r with positive diagonal entries, U is $m \times r$ and V is $n \times r$. Write down projection matrices for the four fundamental subspaces of A, in terms of one of U, Σ, or V in each expression. Be sure to clearly identify which fundamental subspace of A goes with which projection matrix.

(b) **(Fall 2013 final, Q1):** Project b onto the column space of A. Do **not** compute a projection matrix for either:

- $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.
- $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$.

(Spring 2017 exam 1, Q2): Circle which of the following statements might possibly be true. Give an example of a possible matrix A for each possibly true statement.

(a) $Ax = b$ has a unique solution for a 5×3 matrix A.

(b) $Ax = b$ has a unique solution for a 3×5 matrix A.

(c) $Ax = b$ is not solvable for any b.

(d) $Ax = b$ is not solvable for any $b \neq 0$.

(Spring 2018 exam 3, Q2, adapted): Prove that $\sum_i \sigma_i^2 = \sum_{i,j} a_{ij}^2$ (Hint: Consider $\mathrm{Tr}(A^T A)$). In all cases find a two by two matrix which has the given eigenvalues and the given singular values or explain why it is impossible.

1. $\lambda = 0, 1$, $\sigma = 1, 1$.
2. $\lambda = 0, 1$, $\sigma = 0, \sqrt{2}$.
3. $\lambda = 0, 0$, $\sigma = 0, 2018$.
4. $\lambda = 4, 4$, $\sigma = 3, 5$.

(Fall 2014 exam 2, Q2): Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

(a) Calculate the determinant $\det(A)$.

(b) Explain why A is an invertible matrix. Find the entry $(2, 3)$ of the inverse matrix A^{-1}.

(c) Notice that all sums of entries in rows of A are the same. Explain why this implies that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue λ_1?

(d) Find two other eigenvalues λ_2 and λ_3 of A.

(e) Find the projection matrix P for the projection onto the column space of A.

(Spring 2019 exam 2, Q2):

1. Compute the gradient of $f(x) = x^T x + \text{sum}(x)$ without the use of indices.
2. Consider the nonlinear matrix function $f(A) = A^T A$. It is possible to write df as a linear transformation of dA. What is that linear transformation?