18.06 - Recitation 9

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Problem 1.

The matrix B has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $x_3 = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

1. Find B using the diagonalization formula $B = X\Lambda X^{-1}$. You can leave your answer as a product of the three matrices, as long as you write down each matrix explicitly (Hint: look at the eigenvectors. Finding X^{-1} should require minimal computation).
2. Let $C = (I - B)(I + B)^{-1}$. What are the eigenvalues of C? (Hint: B and C have the same eigenvectors. Proving this will help you find the eigenvalues).

Problem 2.

The matrix A has diagonalization $A = X\Lambda X^{-1}$ with

$$X = \begin{pmatrix} 1 & 1 & -1 & 0 \\ & 1 & 2 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & -2 & \\ & & & -1 \end{pmatrix}.$$

Give a basis for the nullspace $N(M)$ of the matrix $M = A^4 - 2A^2 - 8I$.

Problem 3.
Let A, B, C and D be 2×2 matrices

1. Use the cofactor expansion to prove that the following block determinant expression holds:

$$\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A||D|$$

2. Verify that if A^{-1} exists, then

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

3. Prove that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

   provided that $AC = CA$.

Problem 4.
Recall that the matrix exponential of A is defined via the infinite series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

1. Explain why e^A is always an invertible matrix (hint: use eigenvalues).
2. There is a result that says that whenever $AB = BA$, it holds that $e^{A+B} = e^A e^B$. Use this result to find the inverse of e^A.
3. Suppose A is a real, antisymmetric matrix so that $A^T = -A$. Show that $U = e^A$ is an orthogonal matrix.
4. If $x(t)$ satisfies
$$\frac{dx}{dt} = Ax,$$
then explain why $\|x(t)\| = \|x(0)\|$ for all t.

Problem 5.
A 3×3 matrix B is known to have eigenvalues $0, 1, 2$. This is enough information to determine 3 of the following. Which are true and what are their values:

1. The rank of B.
2. The determinant of $B^T B$.
3. The eigenvalues of $B^T B$.
4. The eigenvalues of $(B^2 + I)^{-1}$.