1 Lecture Review

1.1 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. A *nonzero* vector $v \in V$ is called an *eigenvector* for the matrix A if for some real or complex scalar λ we have $Av = \lambda v$.

2. The value λ is then called the *eigenvalue* corresponding to this eigenvector v.

3. Since for the eigenvector v we have $(A - \lambda)v = 0$, the matrix $A - \lambda I$ is not invertible, and so an eigenvalue is necessarily a root of the polynomial $\chi_A(\lambda) = \det(A - \lambda I)$.

4. A matrix is *diagonalizable* if $A = X\Lambda X^{-1}$ for some invertible matrix X and some diagonal matrix Λ. In this case, the diagonal entries of Λ are the eigenvalues of A. If λ_i is the ith diagonal entry of Λ, then the ith column vector of X is an eigenvector with eigenvalue λ_i. This representation of A as $X\Lambda X^{-1}$ is called *eigendecomposition*.

5. A matrix is diagonalizable if and only if there exists a linearly independent set of n eigenvectors of A.

6. If A has n distinct eigenvalues (all the roots of $\chi_A(\lambda)$ are different), then A is diagonalizable; note the reverse direction is not true in general.

2 Problems

1. Suppose we have $B = XAX^{-1}$.
   (a) Prove that $\chi_B(\lambda) = \chi_A(\lambda)$.
   (b) How are eigenvalues of B related to those of A?
   (c) How are eigenvectors of B related to those of A?
   (d) Suppose that one of the eigenvalues of A is zero. Does it mean that A is singular? Does it mean that B is singular?
2. Give an example of a diagonalizable matrix with a pair of equal eigenvalues.
3. Prove that if n is odd and A is $n \times n$, then A has at least one real eigenvalue.
4. *Closed formula for Fibonacci numbers.* Let F_i denote the ith element in the Fibonacci sequence, defined by setting $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all natural values of i (including zero).
   (a) Find a matrix A such that $A\begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix}$.
   (b) Find the eigenvalues of A. Let φ denote the largest eigenvalue.
   (c) Find the eigenvectors of A.
   (d) Compute A^{50} up to nine decimal points. You can only use simple calculators (e.g. Google engine), no matrix calculators are needed.
   (e) Using the result of part (c), explain why $\frac{F_{50}}{F_{49}}$ is very close to φ.