1 Lecture Review

1.1 Orthogonality of Subspaces

1. If V, W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if for every $v \in V$ and $w \in W$ we have $v^T w = 0$.

2. Given a vector subspace V of $\mathbb{R}^n$, we denote by $V^\perp$ the set of all vectors $w \in \mathbb{R}^n$ which are orthogonal to all vectors in V; that is $v^T w = 0$ for every $v \in V$. We call $V^\perp$ the orthogonal complement of V.

3. $(V^\perp)^\perp = V$.

4. Given an $m \times n$ matrix A,
$$\mathrm{Col}(A)^\perp = \mathrm{LeftNull}(A), \quad \mathrm{Row}(A)^\perp = \mathrm{Null}(A).$$

1.2 Linear Transformations

1. Let V and W be vector spaces. A function T from V to W is linear if for every $v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{R}$ we have
$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2).$$

2 Problems

1. Let V be a vector subspace of $\mathbb{R}^n$. Check that $V^\perp$ is a vector space.

2. Find bases and dimensions for the four fundamental subspaces associated with the following matrices

$$\text{(a)} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

3. Let A be an $m \times n$ matrix with full form singular value decomposition $U\Sigma V^T$. If A has rank r, find a basis for the following subspaces in terms of columns of U or V: (a) Col(A), (b) LeftNull(A), (c) Row(A), (d) Null(A).

4. Suppose $\boldsymbol{v} = (a, b, c) \in \mathbb{R}^3$ is a nonzero vector. Viewing $\boldsymbol{v}^T$ as a 1×3 matrix, find a basis for: (a) Col($\boldsymbol{v}^T$), (b) LeftNull($\boldsymbol{v}^T$), (c) Row($\boldsymbol{v}^T$). (d) Using the fact that Row$(\boldsymbol{v}^T)^\perp =$ Null($\boldsymbol{v}^T$), explain how this shows that the orthogonal complement of a plane is spanned by its normal vector.

5. Let A be an $n \times n$ orthogonal matrix with column vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Show that the orthogonal complement of span$(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ is span$(\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_n)$.

6. Suppose W is a vector subspace of $\mathbb{R}^n$. For any $\boldsymbol{v} \in \mathbb{R}^n$ check that $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ for some $\boldsymbol{w}_1 \in W$ and $\boldsymbol{w}_2 \in W^\perp$. Using the Pythagorean theorem, show that the vector $\boldsymbol{w} \in W$ which minimizes $\|\boldsymbol{v} - \boldsymbol{w}\|$ is given by $\boldsymbol{w}_1$.

7. Let V and W be vector spaces. Explain why a linear transformation T from V to W must send the zero vector in V to the zero vector in W.

8. Check whether the following maps are linear transformations:

$$\begin{array}{ll} \text{(a) } T(x,y) = (x-y, x+y), & \text{(b) } T(x,y,z) = (x+1, y+1, z+1), \\ \text{(c) } T(x,y) = (xy, x), & \text{(d) } T(x,y,z) = (x+y+z, y+z, z). \end{array}$$

   For the instances where T is a linear transformation, can you find a matrix A such that

$$T(x,y) = A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad T(x,y,z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$