Determine whether or not these objects exist. If so, write down an example. If not, explain why not.

(1) A 2×2 matrix P satisfying $P^2 = P$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathrm{col}(P)$, and $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathrm{null}(P)$.

(2) An invertible matrix V such that

$$V \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} V^{-1} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}.$$

(3) Two vectors $v, w \in \mathbb{R}^2$ such that $v \cdot w = 1$ and $vv^\top + ww^\top = \mathrm{Id}_{2\times 2}$.

(4) A square orthogonal matrix Q such that $Q^3 = \mathrm{Id}$, but neither Q nor Q^2 equal the identity.

(5) An upper triangular[1] matrix U such that $U^3 = \mathrm{Id}$, but but neither U nor U^2 equal the identity.

(6) A square orthogonal matrix Q such that $\det(Q) < 0$.

(7) An orthogonal matrix such that $\det(QQ^\top) < 0$.

(8) A symmetric matrix A such that $\det(A) < 0$.

(9) A matrix A such that $\det(A^\top A) < 0$.

(10) A real number a such that the matrix

$$A = \begin{pmatrix} 3 & a \\ a & 1 \end{pmatrix}$$

transforms a shape with area 1 into a shape with area 4.

(11) A 2×2 matrix which transforms the parallelogram with vertices $(1, 1), (2, -1), (-2, 1), (-1, -1)$ into a square of area 4.

(12) A 2×2 matrix A such that $\det(A) = 1$ and A transforms the square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1)$ into a shape which lies inside the unit disk $D := \{v \in \mathbb{R}^2 \text{ such that } \|v\| \leq 1\}$.

[1] This means that the entries of U lying *strictly* below the main diagonal are zero.

Solutions

DNE stands for 'does not exist.'

(1) $P = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.

(2) DNE. We have

$$\det\left(V\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}V^{-1}\right) = \det(V)\cdot(-2)\cdot\det(V)^{-1} = -2,$$

which contradicts the RHS.

(3) DNE. The second condition would imply

$$vv^\top v + ww^\top v = v.$$

The first condition implies $w^\top v = 1$, so

$$vv^\top v + ww^\top v = \|v\|^2\, v + w.$$

Therefore v and w are linearly dependent. By a homework problem, this implies that

$$\operatorname{rank}(vv^\top + ww^\top) < 2.$$

This contradicts $\operatorname{rank}(\mathrm{Id}_{2\times 2}) = 2$.

(4) Let Q be a rotation matrix with angle $2\pi/3$.

(5) DNE. First, U cannot be 1×1 because $U^3 = 1$ would imply that $U = 1$, contradiction.

Next, let U be an $n\times n$ upper-triangular matrix. We prove, by induction on n, that $U^3 = \mathrm{Id}_{n\times n}$ implies $U = \mathrm{Id}_{n\times n}$. Fix $n \geq 2$ and assume the result holds for $n-1$. Decompose U into blocks of these sizes:

$$\left(\begin{array}{c|c} (n-1)\times(n-1) & (n-1)\times 1 \\ \hline 1\times(n-1) & 1\times 1 \end{array}\right).$$

Since U is upper-triangular, the bottom-left block is zero. This implies that the top-left block of U^3 is the cube of the top-left block of U. By the inductive hypothesis, the top-left block must equal $\mathrm{Id}_{(n-1)\times(n-1)}$. Similarly, the bottom-right block of U^3 is the cube of the bottom-right block of U, so it equals 1. Thus, U looks like

$$U = \left(\begin{array}{c|c} \mathrm{Id}_{(n-1)\times(n-1)} & b \\ \hline 0_{1\times(n-1)} & 1 \end{array}\right)$$

where b is a $(n-1)\times 1$ matrix. Direct multiplication shows that

$$U^3 = \left(\begin{array}{c|c} \mathrm{Id}_{(n-1)\times(n-1)} & 3b \\ \hline 0_{1\times(n-1)} & 1 \end{array}\right),$$

so $U^3 = \mathrm{Id}_{n\times n}$ implies that $b = 0$, hence $U = \mathrm{Id}_{n\times n}$. This completes the induction.

(6) Take $Q = (-1)$.

(7) DNE. If Q is square, then $\det(QQ^\top) = \det(Q)^2 \geq 0$. If Q is $n\times m$ with $m < n$, then $\operatorname{null}(Q^\top) > 0$, so $QQ^\top$ is not invertible, so its determinant equals zero. These are the only two possibilities for the size of Q.

(8) Take $A = (-1)$.

(9) DNE. If $\operatorname{null}(A) > 0$, then $A^\top A$ is not invertible, so its determinant equals zero. If $\operatorname{null}(A) = 0$, then we can write $A = QR$ where Q is orthogonal and R is invertible. Then

$$\det(A^\top A) = \det(R^\top R) = \det(R)^2 \geq 0.$$

(10) Take $a = \sqrt{7}$, so $\det(A) = -4$. This matrix will transform a shape of are 1 into a shape of area 4. The negative sign on the determinant indicates that A reverses orientation, i.e. it transforms a clockwise loop into a counterclockwise loop.

(11) Take $A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$.

(12) DNE. The image of the given square under A is a parallelogram which lies inside the unit disk D. The largest possible area of such a parallelogram is 2, which is attained by a square inscribed in the unit circle. Thus $\det(A) \leq \frac{1}{2}$.