Rect 6 soul

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Solving $Ax = b$

1. When A is invertible, i.e., if $A_{n\times n}$ has rank n, then $Ax = b$ always has a unique solution.

2. When A is singular.

   (a) x is a least squares solution if $\|Ax-b\|$ is as small as possible.

   (b) x is a least norm if furthermore $\|x\|$ is as small as possible.

   (c) Given $A = U\Sigma V^T = U_1\Sigma_r V_1^T$, the solution $x = V_1\Sigma_r^{-1}U_1^T b$ is always a least-squares, least-norm solution.

Orthogonal complement

1. If V and W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if for every $v \in V$, and $w \in W$, we have $v \perp w$.

2. $V^\perp$ is the set of all $w \in \mathbb{R}^n$ such that w is perpendicular to each vector $v \in V$.

3. $V = (V^\perp)^\perp$

   - $\text{Col}(A)^\perp = \text{Leftnull}(A)$
   - $\text{Row}(A)^\perp = \text{Null}(A)$
   - $\text{Null}(A)^\perp = \text{Row}(A)$
   - $\text{Leftnull}(A)^\perp = \text{Col}(A)$

4. Suppose A is a $m \times n$ matrix with rank r, then

   - dim Row$(A) = r$
   - dim Null$(A) = n-r$
   - dim Col$(A) = r$
   - dim Null$(A^T) = m-r$

Problems

1. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (a) What is the rank of A?

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   (b) What is the dimension of the solution space of $Ax = 0$? What is the dimension of the solution space of $A^T x = 0$?

   dim Null$(A) = 3-2=1$

   dim Null$(A^T) = 4-2=2$

   (c) Does $A^T y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ has a solution? If not, find the least norm. (Hint: First, write down a SVD for A^T.)

   No

$$\begin{array}{cccc} & U_1 & \Sigma & V_1^T \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = & \begin{pmatrix} \frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} \end{pmatrix} & \begin{pmatrix} \sqrt2 & 0 \\ 0 & \sqrt2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

So least norm solution

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac12 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & \frac12 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac12 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac12 \\ 0 \\ 0 \end{pmatrix}$$

2. Let V be subspace of $\mathbb{R}^4$ $V = span(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix})$.

   (a) Find the orthogonal complement of V.

$$\text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right)$$

   (b) Write down all the solutions of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + k \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + l \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

3. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Suppose we have $AB = 0$. Show that rank(A) + rank$(B) \le n$.

   Since $AB = 0 \Rightarrow$ rank $B \le$ null (A)

   null$(A) = n -$ rank(A)

   So rank $(B) \le n -$ rank(A)

   $\Rightarrow$ rank (A) + rank $(B) \le n$.