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Four fundamental subspaces of $A \in \mathbb{R}^{m\times n}$

- $\text{col}(A) = \{\text{All linear combinations of columns of } A\} = \{Ax \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$
- $\text{row}(A) = \{\text{All linear combinations of rows of } A\} = \{A^T x \mid x \in \mathbb{R}^m\} = \text{col}(A^T) \subset \mathbb{R}^n$
- $\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n$
- $\text{null}(A^T) = \{x \in \mathbb{R}^m \mid A^T x = 0\} \subset \mathbb{R}^m$

Linearly independent, span, basis, dimension

- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are **linearly independent** if $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n} = \vec{0}$ implies all $c_i = 0$.
- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ **span** the vector space S if S is all linear combinations of the $\vec{v_i}$.
- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are a **basis for** S if they are linearly independent and they span S.
- The **dimension of a space** S is the number of vectors in every basis for S.

Dimension of the four fundamental subspaces

- Recall the full SVD $A_{m\times n} = U\Sigma V^T$ and that
  $\text{col}(A) = \text{col}(U_1)$
  $\text{row}(A) = \text{col}(V_1)$
  $\text{null}(A) = \text{col}(V_2)$
  $\text{null}(A^T) = \text{col}(U_2)$.
- The column vectors of U_1 are orthogonal to each other, so $\dim \text{col}(U_1) = r$.
- $\dim \text{col}(A) = \dim \text{row}(A) = \text{rank}(A)$.
  $\dim \text{null}(A) = n - \text{rank}(A)$
  $\dim \text{null}(A^T) = m - \text{rank}(A)$

Problems

1. (a) Let A be a 3 by 5 matrix, can the dimension of nullspace of A be 1? Explain.

(b) Suppose we know the dimension of nullspace of A is 3. What are the dimensions of other three fundamental subspaces of A? Verifying your answer by constructing an example of A and computing all of its fundamental spaces. Feel free to use Julia.

2. Find a basis for symmetric 3×3 matrices. What is the dimension of symmetric 3×3 matrices? Also,

do this for skew-symmetric 3×3 matrices.

3. Let V be the vector space of degree ≤ 3 polynomials, that is $V = \{f(x) = a + bx + cx^2 + dx^3\}$. Let W be the collection of polynomials in V such that $f(1) = f(-1)$.
(a) Check that W is a subspace of V.

(b) Find a basis for W. What is the dimension of W?

(c) Compare your answer in (b) with your friends now, are they the same? Express the vectors in your friend's answer as linear combinations of yours.

(d) Complete the basis you find in (b) to a basis for V. That is looking for some vectors in V so that their union with your answer in part (b) becomes a basis for V.