Let V be a vector space. For any nonempty subset $S \subseteq V$, we make the following definitions:

(1) If every element of V can be expressed as a linear combination of elements from S in <u>at most</u> one way, then S is *linearly independent.*[1]

(2) If every element of V can be expressed as a linear combination of elements from S in <u>at least</u> one way, then S *spans* V. We also say that S is a *spanning subset* of V.

(3) If both are true, then S is a *basis* of V.

(4) V is *finite-dimensional* if it has a finite spanning subset.

In this case, any two bases of V have the same (finite) size. This number is called the *dimension* of V, denoted $\dim(V)$. Any vector subspace of a finite-dimensional vector space is finite-dimensional.

(5) The *span* of S, denoted $\mathrm{span}(S)$, is the subset of all linear combinations of elements of S. It is a vector subspace of V.

Key facts about linear independence:

(6) If $S \subseteq T$ and T is linearly independent, then S is linearly independent.

(7) Any linearly independent subset of V is contained in a basis of V.

Therefore, if S is linearly independent, then $|S| \leq \dim(V)$.

(8) The following statements about a nonempty subset $S \subseteq V$ are equivalent:

(a) S is linearly independent.

(b) Any nontrivial linear combination of elements in S is nonzero.

(c) $\dim(\mathrm{span}(S)) = |S|$.

(d) The span of any proper subset of S is smaller than $\mathrm{span}(S)$.

(e) S is a basis for the vector space $\mathrm{span}(S)$.

(9) If S is linearly independent, and $v \in V$ is a vector, then $S \cup \{v\}$ is linearly independent if and only if $v \notin \mathrm{span}(S)$. In this case, $\dim(\mathrm{span}(S \cup \{v\})) = \dim(\mathrm{span}(S)) + 1$.

Key facts about spanning subsets:

(10) If $S \subseteq T$ and S spans, then T spans.

(11) Any spanning subset of V contains a basis of V.

Therefore, if S spans, then $|S| \geq \dim(V)$.

Key facts about bases:

(12) The following statements about a nonempty subset $S \subseteq V$ are equivalent:

(a) S is a basis.

(b) S is a linearly independent subset of size $\dim(V)$.

(By (7), this is the largest possible size for a linearly independent subset.)

(c) S is a spanning subset of size $\dim(V)$.

(By (11), this is the smallest possible size for a spanning subset.)

[1] If S is not linearly independent, we say that it is *linearly dependent.* It would be more grammatically correct to say that the *elements* of S are linearly dependent or are linearly independent.

(13) Let $\{v_1, \ldots, v_n\}$ be a basis. Then every element of V can be expressed uniquely as a linear combination of $v_1, \ldots, v_n$, so there is a bijection $V \simeq \mathbb{R}^n$ defined as follows:

$$(\lambda_1, \ldots, \lambda_n) \text{ in } \mathbb{R}^n \qquad \text{matches up with} \qquad \lambda_1 v_1 + \cdots + \lambda_n v_n \text{ in } V,$$

for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

These concepts are related to the column space and null space of matrices. Let A be an $n \times m$ matrix, and let $v_1, \ldots, v_m \in \mathbb{R}^n$ be the columns of A.

(14) $\operatorname{span}(\{v_1, \ldots, v_m\}) = \operatorname{col}(A)$.

Proof. By definition, $\operatorname{col}(A) \subseteq \mathbb{R}^n$ is the set of all linear combinations of the columns of A, which are the $v_1, \ldots, v_m$. □

(15) The following are equivalent:

(a) $v_1, \ldots, v_m$ are linearly independent.

(b) $\operatorname{rank}(A) = m$.

(c) $\operatorname{null}(A) = \{0\}$.

Proof. We have

$$\begin{aligned}\operatorname{rank}(A) &= \dim(\operatorname{col}(A)) \\ &= \dim(\operatorname{span}(\{v_1, \ldots, v_m\}))\end{aligned}$$

by (14). Applying the statement (a) $\Leftrightarrow$ (c) in (8), we conclude that $v_1, \ldots, v_m$ are linearly independent if and only if this number equals m. □

(16) The following are equivalent:

(a) $v_1, \ldots, v_m$ are a spanning subset of $\mathbb{R}^n$.

(b) $\operatorname{rank}(A) = n$.

(c) $\operatorname{col}(A) = \mathbb{R}^n$.

Proof. $\operatorname{rank}(A) = n$ if and only if $\operatorname{col}(A) = \mathbb{R}^n$. Thus the statement follows from (14). □

(17) $v_1, \ldots, v_m$ are a basis if and only if A is invertible.

Proof. By (3), (14), and (15), we see that $v_1, \ldots, v_m$ are a basis if and only if A is square and $\operatorname{rank}(A) = n$. But this is true if and only if A is invertible, as can be seen from the SVD. □

Orthonormality implies linear independence:

(18) If $v_1, \ldots, v_m \in \mathbb{R}^n$ are an orthonormal collection of vectors, then they are linearly independent.

Proof. The $v_1, \ldots, v_m$ are the columns of an $n \times m$ orthogonal matrix Q. We've seen that $\operatorname{rank}(Q) = m$, so the claim follows from (15). □