Recitation 4/7

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Projection Matrix

Let $A \in \mathbb{R}^{m \times n}$, matrix with linearly independent columns. Then, $P = A(A^TA)^{-1}A^T$ is a projection matrix onto the column space of A.

In other words, for $b \in \text{col}(A)$, $Pb = b$, and for $b \notin \text{col}(A)$, Pb is a vector in $\text{col}(A)$ with the property of $||Pb - b||$ being minimum.

Easy to obtain P with QR decomposition - $P = QQ^T$ when $A = QR$.

Determinant

Determinants are scalar real number, only defined on **square matrices**

1. Cofactor expansion formula - Most Linear Algebra Textbooks

2. Product of Pivots(Strang)

3. Product of Singular values is equal to absolute value of determinant

An important properties of determinant
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$

Problems

1.Consider $\mathbb{R}^3$ space.
(a) We have xy-plane and a vector $v = (3, 2, 1)$. Draw a picture and figure out the projected vector of v onto xy-plane without computation.

(b) xy-plane is a span of two vectors. What are those vectors?

(c) Let A be a matrix with two columns obtained in problem (b). What is the QR decomposition of A?

(d) Use formula $P = QQ^T$, compute P and Pv. Does it agree with your result in (a)?

(e) A column space of matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix}$ is also xy-plane. Compute $P = B(B^TB)^{-1}B^T$ and compare it with previous results.

2. (a)Compute the determinant of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ using cofactor expansion formula.

(b) Write down the cofactor expansion of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$.(Don't compute)

3. True or false. Find a counterexample or explain why

(a) $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & \\ & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a rank-r SVD of A. Since the singular values are 3, 2, 0, $\det A = 0$.

(b) A determinant of square orthogonal matrix is 1.

(c) A determinant of projection matrix is 1.

(d) A determinant of diagonal matrix is product of diagonal entries.

(e) A determinant of square matrix with nonzero nullspace is always zero.

(f) A matrix $A \in \mathbb{R}^{n \times n}$ has $(n^2 - n)$ zero entries and n nonzero entries. The determinant is zero unless it is a diagonal matrix.

(g) n-by-n matrix with more than $n^2 - n$ zero entries always has determinant zero.

(h) For non-square matrix A, determinant of $A^T A$ equals determinant of AA^T.

(i) Matrix A has only ones and zeros in it. Its determinant is always one or zero.

(j) A determinant of doubly stochastic matrix(rows and columns have sum one) is always one.

ANSWERS

1.(a) should be (3, 2, 0) by projecting right down to xy plane.
(b)$(1,0,0)^T, (0,1,0)^T$.
(c) The QR decomposition of $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $A = AI_2$ where I_2 is 2 by 2 identity matrix.
(d)$QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Pv = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ and agrees with result in (a).
(e) Computing $B(B^TB)^{-1}B^T$ we get the same P.
2.(a) $1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1 * (-3) - 2 * (-6) + 3 * (-3) = -3 + 12 - 9 = 0$
(b)$1\begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} - 2\begin{vmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{vmatrix} + 3\begin{vmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{vmatrix} - 4\begin{vmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{vmatrix}$
3.(a) False. there is no determinant for nonsquare matrix.
(b) False. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(c) False. it can be zero too(problem(1d)
(d) True, since the pivots are exactly the diagonal entries
(e) True. Nonzero nullspace means linearly dependent columnspace. Linearly dependent columnspace means zero singular value.
(f) False. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(g) True. It always contains row of zeros. Perform cofactor expansion with that row.
(h) False. Think $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
(i) False. same counterexample as (f)
(j) False. same counterexample as (f) or matrix with all entries $\frac{1}{n}$