18.06 R08 - Recitation 1

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1 Important information

Welcome to recitations for 18.06!

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- Office hours: Tuesdays, 3:30-5:30pm, 2-361

We are *not* using stellar for this class (apart from recitation sign-up):

- All course materials, lecture summaries, psets, pset solutions, *etc.* will be posted on the course website (https://mitmath.github.io/1806/).
- HW should be submitted on gradescope (https://www.gradescope.com/courses/40389).

2 Lecture summary

1. In essence, a **vector space** is a set V of vectors for which any linear combination of vectors in V is also in V. Examples:
    - vectors in $\mathbb{R}^n$.
    - matrices in $\mathbb{R}^{m\times n}$
    - real valued functions $f(x)$
2. A **vector subspace**, X, is a set of vectors that satisfies the following requirement: *If v and w are vectors in X, and c is any scalar, then*
    - *$v+w$ is in X*
    - *cv is in X*
3. A **linear system** is an equation of the form $Ax=b$, where A is an $m\times n$ matrix, x is a vector in $\mathbb{R}^n$, and b is a vector in $\mathbb{R}^m$. Much of the purpose of linear algebra is understanding when this equation has solutions, and what these solutions look like.
4. If $m=n$, then it is possible that A will have an **inverse** A^{-1}, so that $A^{-1}A = A^{-1}A = I$. **This does not always exist**. If it does exist, then $x = A^{-1}b$. But this is usually still not a good way to find a solution....
5. If Q is a square matrix with *orthonormal* columns, then $QQ^T = Q^TQ = I$ and we say that Q is an **orthogonal** matrix. In particular, $Q^{-1} = Q^T$.

3 Problems

Problem 1. (Vector spaces)
$\mathbb{R}^3$ is the vector space of all vectors of the form $b = (b_1, b_2, b_3)$. Which of the following are subspaces of $\mathbb{R}^3$?

(a) The set of all vectors (b_1, b_2, b_3) where $b_1 = b_2$.

(b) The set of all vectors with $b_1 = 1$.

(c) The set of all vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$.

Problem 2. (Linear systems and block matrices)

(a) Consider the simultaneous equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned}$$

where a, b, c, d, e and f are real numbers and x, y are scalar unknowns. Write these equations as a single matrix equation $Az = g$. What are A, z and g? What are their dimensions?

(b) If you have two coupled systems of equations

$$\begin{aligned} Bx + Cy &= c \\ Dx + Ey &= d, \end{aligned}$$

where B, C, D and E are 3×3 matrices and x, y, c and d are 3-component vectors, can you write this as a single system of equations,

$$Az = b,$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix}$ is the 6-component vector of x on top of y? What are A and b? What are their dimensions?

Problem 3. (LU factorization)

(a) Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Write the system as three simultaneous equations for the unknowns x_1, x_2 and x_3. Solve these equations explicitly in order to find a solution to the linear system.

(b) Consider the two 3×3 matrices L and U, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute $B = LU$.

(c) Consider the linear system $Bx = b$, where $B = LU$ is what you computed in the previous part, b is the same vector as in part (a), and x is an unknown vector. Find the solution to this linear system using the following steps:

- Explicitly solve the linear system $Ly = b$ to find y.
- Using the y you found in the previous step, explicitly solve $Ux = y$ for x.
- Then this x solves the linear system $Bx = b$. Check this! Why does this work?[1]

[1] Hint: Note that $Bx = b$ is the same as $LUx = L(Ux) = b$.

Problem 4. (More matrix factorization)

Under what conditions on a, b, c and d can a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be written as $A = X^T X$, where $X = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$? What are p, q and r in terms of a, b, c and d?