18.06 - Recitation 10 - Solutions

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Problem 1.

Consider the matrix $A = \begin{pmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with parameter $x \in \mathbb{R}$:

1. Specify all numbers x, if any, for which A is positive definite. (Explain briefly.)
2. Specify all numbers x, if any, for which e^A is positive definite. (Explain briefly.)
3. Find an x, if any, for which $4I - A$ is positive definite. (Explain briefly.)

Solution

1. Note that the second and third columns of A are identical. This means that the rank of A is less than 3 and so $\det A = 0$. This means that A will always have at least one zero eigenvalue and so it cannot be positive definite for any x
2. Recall that if A has an eigenvalue λ, then e^A has an eigenvalue e^λ. Since A is symmetric, $\lambda \in \mathbb{R}$, and so $e^\lambda > 0$ for every eigenvalue. Hence e^A is positive definite.
3. The upper left determinants are $4 - x, 11 - 3x$ and $24 - 8x$. So provided $3 > x$, then $4I - A$ will be positive definite.

Problem 2.

True or false? Justify your answer either way.

1. If A and B are invertible, then so is $(A + B)/2$.
2. If A and B are Markov, then so is $(A + B)/2$.
3. If A and B are positive definite, then so is $(A + B)/2$.
4. If A and B are diagonalizable, then so is $(A + B)/2$.
5. If A and B are rank 1, then so is $(A + B)/2$.

Solution

1. This is false. For example $A = I$ and $B = -I$ are both invertible, but $(A + B)/2 = 0$ which is not invertible.
2. This is true. If and A and B are Markov, then all of their entries are nonnegative. So certainly all the entries of $(A + B)/2$ are nonnegative. Furthermore, The columns of $A + B$ are the sums of the columns of A and the columns of B. If each of their columns sum to 1, then the columns of $(A + B)/2$ will also sum to 1. So $(A + B)/2$ is also Markov.

3. This is true. If A and B are positive definite, then $x^T Ax > 0$ and $x^T Bx > 0$, for all $x \neq 0$. So

$$x^T \left(\frac{A+B}{2} \right) x = \frac{1}{2}(x^T Ax + x^T Bx) > 0.$$

   So $(A+B)/2$ is positive definite.

4. This is false. $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ are both diagonalizable, but

$$(A+B)/2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   is not diagonalizable.

5. This is false. $A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}$ are both rank 1, but

$$(A+B)/2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

   which is rank 2.

Problem 3.
We are told that A is a symmetric Markov matrix. It has an eigenvalue y, where $-1 < y < 1$.

1. Find the matrix A in terms of y.
2. Find the eigenvectors of A.
3. What is $\lim_{n\to\infty} A^n$ in its simplest form?

Solution

1. Since A is symmetric, we know that

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

   Since the columns of A must sum to 1, we know that $a + b = b + d = 1$, from which we can deduce $a = d$. We can then find the characteristic equation of this matrix:

$$\det(A - \lambda I) = (a-\lambda)^2 - b^2 = 0$$

   and so $(a - \lambda) = \pm b$. We know that $a + b = 1$, and so $a - b = y$. Therefore

$$A = \begin{pmatrix} (1+y)/2 & (1-y)/2 \\ (1-y)/2 & (1+y)/2 \end{pmatrix}.$$

2. The eigenvector $\lambda_1 = 1$ is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The eigenvector for $\lambda_2 = y$ is $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3. A is diagonalizable, i.e. we can write $A = X\Lambda X^{-1}$, where

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}, \quad X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then $A^n = X\Lambda^n X^{-1}$. Now $\Lambda^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so

$$\begin{aligned} A^n &\to \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Problem 4.

1. If A is symmetric then which of these four matrices are necessarily positive definite

$$A^3, \;\; (A^2+I)^{-1}, \;\; A+I, \;\; e^A.$$

2. Suppose C is positive definite and that A has independent columns. Show that $x^T A^T C A x > 0$ for all $x \neq 0$. Hence $S = A^T C A$ is positive definite.

Solution

1. $(A^2+I)^{-1}$ and e^A are always positive definite. This is because if A has eigenvalues $\lambda_i \in \mathbb{R}$, then $(A^2+I)^{-1}$ has eigenvalues $\frac{1}{\lambda_i^2+1} > 0$ and e^A has eigenvalues $e^{\lambda_i} > 0$. A^3 has eigenvalues λ_i^3 which need not be positive, and $A+I$ has eigenvalues $\lambda_i + 1$ which also need not be positive.

2. Let $y = Ax$. Then since C is positive definite, we know that $y^T C y = x^T A^T C A x > 0$ whenever $y = Ax \neq 0$. However, $Ax = 0$ only has $x = 0$ as a solution since A has full column rank (independent columns). Hence $y^T C y = x^T A^T C A x > 0$ for all $x \neq 0$ and so $S = A^T C A$ is positive definite.