1 Lecture Review

1.1 Fundamental Vector Spaces

Let A be an $m \times n$ matrix.

1. The column space of A, denoted col(A), is the set $\{Ax : x \in \mathbb{R}^n\}$. The column space is the set of linear combinations of the column vectors of A.
2. The null space of A, denoted null(A) is the set $\{x \in \mathbb{R}^n : Ax = 0\}$.
3. The row space of A, denoted row(A), is the set $\{A^T x : x \in \mathbb{R}^m\}$. The row space is the set of linear combinations of the row vectors of A.

1.2 Singular Value Decomposition (SVD) in Rank r Format (Compact Form)

1. Let A be an $m \times n$ matrix. The SVD of A in rank r format (or compact form) is a factorization of A as
$$A = U\Sigma V^T$$
where $0 \le r \le m, n$ so that
   - U is $m \times r$ with $U^T U = I$,
   - $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$ where $\sigma_1 \ge \cdots \ge \sigma_r > 0$,
   - V is $n \times r$ with $V^T V = I$.
2. We have col$(A) =$ col(U).
3. $Ax = b$ is solvable if and only if $UU^T b = b$.
4. If $\mathbf{u}_1, \dots, \mathbf{u}_r$ and $\mathbf{v}_1, \dots, \mathbf{v}_r$ are the respective column vectors of U and V, then the rank k (for $k \le r$) approximation of A is the $m \times n$ matrix
$$A_k = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} \text{diag}(\sigma_1, \dots, \sigma_k) \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix}^T.$$

1.2.1 Singular Value Decomposition (SVD) in Full Form

1. Let A be an $m \times n$ matrix. The SVD of A in full form is a factorization of A as
$$A = \mathbf{U\Sigma V}$$
where
   - $\mathbf{U}$ is $m \times m$ with $\mathbf{U}^T\mathbf{U} = I$,
   - $\mathbf{\Sigma}$ is $m \times n$ diagonal with $\sigma_1 \ge \cdots \ge \sigma_r > 0$ along the diagonal,
   - $\mathbf{V}$ is $n \times n$ with $\mathbf{V}^T\mathbf{V} = I$.
2. The matrices $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ from the full form SVD are related to the matrices U, Σ, V from the compact form in the following way
   - If $\mathbf{U} = \left(\begin{array}{c|c} U_1 & U_2 \end{array} \right)$ as a block matrix with $m \times r$ matrix U_1 then $U_1 = U$.
   - $\mathbf{\Sigma} = \left(\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array} \right)$.
   - If $\mathbf{V} = \left(\begin{array}{c|c} V_1 & V_2 \end{array} \right)$ as a block matrix with $n \times r$ matrix V_1 then $V_1 = V$.
3. We have null$(A) =$ col(V_2) if V_2 is not an empty block, or equivalently if $r < n$.

2 Computation

2.1 2 Column QR Decomposition

1. Consider a $m \times 2$ matrix A with column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If $\mathbf{a}_1$ is not a multiple of $\mathbf{a}_2$, then the QR decomposition can be computed by the following steps:

   (a) Compute $\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}$.

   (b) Compute $\mathbf{b} = \mathbf{a}_2 - \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1$.

   (c) Compute $\mathbf{q}_2 = \frac{\mathbf{b}}{\|\mathbf{b}\|}$.

   (d) Then

$$Q = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{pmatrix}, \quad R = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 \end{pmatrix}$$

2.2 Rank 1 SVD in Compact Form

1. To check a matrix is rank 1, check that the column vectors are all multiples of one another.

2. Suppose A is an $m \times n$ matrix with rank 1. You can write $A = \mathbf{x}\mathbf{y}^T$ where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ as follows:

   (a) Choose a nonzero column of A, set it equal to $\mathbf{x}$.

   (b) Find $\mathbf{y} = (y_1, \ldots, y_n)$ so that $y_i\mathbf{x}$ is the ith column of A.

3. Given $A = \mathbf{x}\mathbf{y}^T$ nonzero, we can obtain the SVD for A in compact form:

$$A = U\Sigma V^T$$

   where

$$U = \frac{\mathbf{x}}{\|\mathbf{x}\|}, \quad \Sigma = \begin{pmatrix} \|\mathbf{x}\|\|\mathbf{y}\| \end{pmatrix}, \quad V = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

   Above, U is $m \times 1$, Σ is a 1×1 matrix, and V is $n \times 1$.

3 Problems

1. Compute the column spaces of the following matrices

(a) $\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Solution. (a) An element of the column space is of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x = y = 2c_1 + c_2, \quad z = c_1.$$

Thus when $x = y$, we can solve the system by taking $c_1 = z$ and $c_2 = x - 2z$. This implies that the column space is given by the plane $x = y$ in $\mathbb{R}^3$.

(b) An element of the column space is of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x = 3c_1 + c_3, \quad y = 2c_2 + c_3, \quad z = 0.$$

If we take $c_1 = x/3, c_2 = y/2, c_3 = 0$, this solves the equation above, granted $z = 0$. Thus we can solve the system as long as $z = 0$. This implies that the column space is given by the plane $z = 0$ in $\mathbb{R}^3$.

(c) An element of the column space is of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x = c_1 + 2c_2 + c_3, \quad y = c_3, \quad z = c_1.$$

Plugging in the second and third equations into the first gives

$$c_2 = \frac{x - y - z}{2}.$$

Thus we can solve the system for any x, y, z. This implies that the column space is given by $\mathbb{R}^3$. $\square$

2. Compute the null spaces of the following matrices

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$.

Solution. (a) An element of the null space solves

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x + 2y = 0, \quad x + y = 0.$$

Subtracting the second equation from the first gives

$$y = 0$$

and plugging this back into the second equation gives

$$x = 0.$$

Thus the only solution to this system is $x = 0, y = 0$. This implies that the null space is given by $\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$.

(b) An element of the null space solves

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x + y = 0, \quad -x - y = 0.$$

Both equations gives us

$$x = -y.$$

Thus the solutions to this system are given by vectors of the form $\begin{pmatrix} t \\ -t \end{pmatrix}$. This implies that the null space is given by the set of vectors $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $t \in \mathbb{R}$.

(c) An element of the null space solves

$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the set of equations

$$x = 0, \quad 2x = 0.$$

Thus the solutions to this system are given by vectors of the form $\begin{pmatrix} 0 \\ t \end{pmatrix}$. This implies that the null space is given by the set of vectors $t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$. □

3. Find the singular value for the following rank 1 matrices

(a) $\begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{pmatrix}$

Solution. (a) The SVD is given by

$$\begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & -3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} \cdot 10 \cdot \begin{pmatrix} 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix}.$$

Thus the singular value is given by 10.

(b) The SVD is given by

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{21} \\ 1/\sqrt{21} \\ -4/\sqrt{21} \end{pmatrix} \cdot \sqrt{21} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Thus the singular value is given by $\sqrt{21}$.

(c) The SVD is given by

$$\begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \cdot \sqrt{70} \cdot \begin{pmatrix} -2/\sqrt{14} & 1/\sqrt{14} & 3/\sqrt{14} \end{pmatrix}$$

Thus the singular value is given by $\sqrt{70}$. □

4. Find the singular values for the following the matrices

(a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (b) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$

Solution. (a) Notice that
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
is an orthogonal matrix. Then
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$$
is an SVD for A where
$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Thus the singular values of A are given by $1, 1$.

(b) Notice that
$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$$
satisfies $A^T A = I$. Then
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^T$$
is an SVD for A where
$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Thus the singular values of A are given by $1, 1$. $\square$

5. Suppose you are given an $m \times n$ matrix A and its SVD (U, Σ, V). Find the SVD of the following matrices in terms of U, Σ, V and describe the column spaces in terms of the column or row space of A

   (a) A^T,

   (b) A^{-1} assuming $m = n$ and A is invertible,

Solution. (a) We have

$$A^T = (U\Sigma V^T)^T = V\Sigma U^T$$

where we used the fact that $\Sigma^T = \Sigma$ because Σ is diagonal. Thus

$$\mathrm{SVD}(A^T) = (V, \Sigma, U).$$

The column space of A^T is the row space of A.

(b) We have

$$A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T$$

where we used the fact that $U^{-1} = U^T$ and $V^{-1} = V^T$. Notice that Σ^{-1} is diagonal (the entries are the reciprocal of those of Σ). Thus

$$\mathrm{SVD}(A^{-1}) = (V, \Sigma^{-1}, U).$$

Notice that if B is some $n \times n$ invertible matrix, then $\mathrm{col}(B) = \mathbb{R}^n$. This is because for any $y \in \mathbb{R}^n$ we can write $y = Bx$ by taking $x = B^{-1}y$. Since A and A^{-1} are invertible, this implies

$$\mathrm{col}(A) = \mathrm{col}(A^T) = \mathbb{R}^n.$$

□

6. Suppose A is 3×2 with SVD in full form

$$A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Check that this is an SVD in full form for A. How would you find the SVD in compact form for A from the full form?

Solution. This amounts to checking that the first and third matrices are orthogonal and square, as well as noticing that the middle matrix has the dimensions of A and is diagonal.

To find the compact form for A, we turn the middle matrix into a 2×2 square matrix. To do this we must remove the bottom row of the middle matrix. To make the dimensions match, we must remove the third column from the first matrix. The resulting SVD in compact form is

$$A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

□

7. Find the QR decomposition of the following matrices

(a) $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & 3 \\ 2 & 4 \\ 0 & 4 \end{pmatrix}$

Solution. (a) We have

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{q}_1 &= \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ \mathbf{b} &= \mathbf{a}_2 - \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \\ \mathbf{q}_2 &= \frac{\mathbf{b}}{\|\mathbf{b}\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

so that

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad R = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

(b) We have

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{q}_1 &= \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{b} &= \mathbf{a}_2 - \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \frac{8}{4}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \\ \mathbf{q}_2 &= \frac{\mathbf{b}}{\|\mathbf{b}\|} = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} \end{aligned}$$

so that

$$Q = \begin{pmatrix} 0 & 3/5 \\ 1 & 0 \\ 0 & 4/5 \end{pmatrix}, \quad R = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}.$$

□