

1. **LU factorization:** A square matrix $A = LU$ where L is a _____ matrix and U is a _____ matrix.

(a) Suppose a 2×2 matrix A has a LU factorization $A = LU$ and $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. How are the column vectors A_1, A_2 of A related to column vectors L_1, L_2 of L ?

2. **QR decomposition:** A $m \times n$ matrix $A = QR$ where Q is a _____ matrix and R is a _____ matrix.

(a) Write down a QR decomposition for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hint: what is the dot product of the first column with the second column of A .

(b) Let $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Can you solve $Ax = b$?

(c) Find a vector \hat{x} that minimizes $\|Ax - b\|$.

3. **SVD:** A $m \times n$ matrix A has the rank- r SVD as $A = U\Sigma V^T$. What are U, V, Σ and what are their matrix size?

Suppose that a 3×3 matrix A has the SVD as

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- (a) What is the rank of A ?
- (b) Write the column space of A as all the linear combinations of two column vectors of U, Σ or V .
- (c) Let $B = U\Sigma$. How are the columns of A related to columns of B ?
- (d) What is the norm of the third column of A ?