18.06 - Recitation 5

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1 Lecture review

1.1 Linear transformations

- A *linear transformation* T takes vectors $v \in V$ to vectors $T(v) \in W$, where V and W are vector spaces. Linearity requires

$$\boxed{T(cv_1 + dv_2) = cT(v_1) + dT(v_2)}$$

  for any vectors $v_1, v_2 \in V$ and scalars $c, d \in \mathbb{R}$. Note that $T(0) = 0$ necessarily.

- A linear transformation is uniquely defined by its action on a basis, i.e. if $\{v_1, ..., v_n\}$ is a basis for a vector space, then any vector can be written as $v = c_1v_1 + ... + c_nv_n$. Therefore

$$T(v) = c_1T(v_1) + ... + c_nT(v_n),$$

  and so knowing $T(v_1), ..., T(v_n)$ allows us to determine $T(v)$ for any v in the vector space.

- A linear transformation $T(v)$ can be described by a matrix, i.e. $T(v) = Av$. Column j in the matrix A comes from applying T to the basis vector v_j.

2 Problems

Problem 1.
Determine which of the following describe a linear transformation. For those that do, find a matrix that describes the transformation with respect to the standard bases for the underlying vector spaces:

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where
$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 0 \end{pmatrix}$$

2. $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ where
$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

3. $T_3 : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 2}$ where
$$T_3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & 2d \\ 2b - d & -3c \\ 2b - c & -3a \end{pmatrix}$$

4. Let P_4 be the vector space of polynomials of degree less than or equal to 4, and let $T_4 : P_4 \to P_4$, where
$$T_4(f)(x) = f(x) - x - 1$$

5. Let $T_5 : P_3 \to P_5$ where
$$T_5(f)(x) = (x^2 - 2)f(x)$$

.

Problem 2.

1. Show that $f(A) = x^T Ay$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are constant vectors, is a linear transformation from the vector space of $m \times n$ matrices to the real numbers.

2. If $f(A)$ is a scalar function of an $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$, then it is useful to define the gradient *with respect to the matrix* as another $m \times n$ matrix:

$$\nabla_A f = \begin{pmatrix} \frac{\partial f}{\partial a_{11}} & \frac{\partial f}{\partial a_{12}} & \cdots \\ \frac{\partial f}{\partial a_{21}} & \frac{\partial f}{\partial a_{22}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

   Given this definition, give a matrix expression (not in terms of individual components) for $\nabla_A f$ with $f(A) = x^T Ay$ as before.

Problem 3.

Consider the vector space of polynomials of degree less than or equal 2. Let us define a dot product on this vector space[1]:

$$f(x) \cdot g(x) = \int_0^\infty f(x)g(x)e^{-x}\,\mathrm{d}x$$

1. Show that the set of polynomials $\{1, x-1, x^2-4x+2\}$ form an orthogonal basis for the vector space of polynomials of degree less than or equal 2.
2. Normalize these basis polynomials so that $\|f(x)\|^2 = f(x) \cdot f(x) = 1$.
3. Consider the function $f(x) = \begin{cases} x & x < 1 \\ 0 & x \geq 1 \end{cases}$. Find the slope α of the straight line αx that is the *best fit* to $f(x)$ in the sense of minimizing

   $$\|f - \alpha x\|^2 = \int_0^\infty [f(x) - \alpha x]^2 e^{-x} dx$$

   In particular, find α by performing the orthogonal projection (with this dot product) of $f(x)$ onto ....?

[1] You may find it useful to recall that $\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$