Determine whether or not these objects exist. If so, write down an example. If not, explain why not.

(1) A 2×4 matrix A such that $\operatorname{null}(A) = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for all } x \in \mathbb{R} \right\}$.

(2) An invertible matrix of the following form:

$$\begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & 0 & 0 & 0 \\ ? & 0 & 0 & 0 \end{pmatrix}$$

(3) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. A matrix B such that $\operatorname{col}(B) = \operatorname{row}(A)$ and $\operatorname{null}(B) = \operatorname{null}(A^\top)$.

(4) A matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that the system

$$\begin{aligned} a_{11}x + a_{12}y &= 1 \\ a_{21}x + a_{22}y &= 2 \end{aligned}$$

has no solution, the system

$$\begin{aligned} a_{11}x + a_{12}y &= 1 \\ a_{21}x + a_{22}y &= 1 \end{aligned}$$

has exactly one solution, and the system

$$\begin{aligned} a_{11}x + a_{12}y &= 1 \\ a_{21}x + a_{22}y &= 0 \end{aligned}$$

has infinitely many solutions.

(5) Two subspaces $V_1, V_2 \subseteq \mathbb{R}^3$ such that $\dim(V_1) = \dim(V_2) = 2$ and $V_1 \cap V_2 = \{0\}$.

(6) Let $A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. A basis $\{v_1, v_2, v_3\} \subset \mathbb{R}^3$ such that $v_1 \in \operatorname{null}(A)$ and $v_2, v_3 \in \operatorname{row}(A)$.

(7) Let $A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. A basis $\{v_1, v_2, v_3\} \subset \mathbb{R}^3$ such that $v_1, v_2 \in \operatorname{null}(A)$ and $v_3 \in \operatorname{row}(A)$.

(8) An $n \times n$ matrix P such that $P^2 = P$, $\operatorname{rank}(P) = n$, and $P \neq \mathrm{Id}_{n\times n}$.

(9) An $n \times n$ matrix A such that all singular values of A are equal to 1, and $A \neq \mathrm{Id}_{n\times n}$.

(10) Two subspaces $V_1, V_2 \subseteq \mathbb{R}^3$ such that $V_1 \cap V_2 = \{0\}$ and $V_1^\perp \cap V_2^\perp = \{0\}$.

(11) A 3×5 matrix A such that $\dim(\operatorname{null}(A)) + \dim(\operatorname{null}(A^\top)) = 5$.

(12) Matrices A and B such that $\operatorname{pinv}(A) = \operatorname{pinv}(B)$ and $A \neq B$.

(13) Matrices A and B such that $A^\top A = B^\top B$ and $A \neq B$.

(14) A square matrix A such that $A^\top A + AA^\top$ is noninvertible.

(15) An $m \times n$ matrix A and a nonzero vector $v \in \operatorname{row}(A)$ such that $Av \in \operatorname{null}(A^\top)$.

Solutions

DNE stands for 'does not exist.'

(1) DNE. Combine the relation $\text{rank}(A) + \dim(\text{null}(A)) = 4$ with $\text{rank}(A) \leq 2$ and $\dim(\text{null}(A)) = 1$ to get a contradiction.

(2) DNE. The last three columns of A provide three vectors in a 2-dimensional space, so they can't be linearly independent.

(3) Take $B = A^\top$.

(4) DNE. If $\text{null}(A) > 0$, then every system of the form $A\mathbf{x} = \mathbf{b}$ has either zero or infinitely many solutions. If $\text{null}(A) = 0$, then every such system has either zero or one solution.

(5) DNE. Suppose such V_1, V_2 did exist. Take a basis $\{v_1, v_2\}$ for V_1 and a basis $\{w_1, w_2\}$ for V_2. Since $\{v_1, v_2, w_1, w_2\}$ is a list of four vectors in $\mathbb{R}^3$, they must satisfy some nontrivial linear relation. If this linear relation involves both $\{v_1, v_2\}$ and $\{w_1, w_2\}$, then we can rearrange it to look as follows:

$$c_1v_1 + c_2v_2 = d_1w_1 + d_2w_2$$

where c_1 and c_2 are not both zero, and d_1 and d_2 are not both zero. This contradicts $V_1 \cap V_2 = \{0\}$ because the LHS lies in $V_1 \setminus \{0\}$ while the RHS lies in $V_2 \setminus \{0\}$.

The only remaining possibilities are that the linear relation only involves $\{v_1, v_2\}$ or only involves $\{w_1, w_2\}$. Neither is possible, because these are bases.

(6) DNE. Since $\{v_1, v_2, v_3\}$ is a basis, $\{v_2, v_3\}$ must be linearly independent. But $\text{row}(A)$ is one-dimensional, so it doesn't contain a pair of linearly independent vectors.

(7) One may take

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(8) DNE. Since $\text{rank}(P) = n$, we know that $\text{col}(P) = \mathbb{R}^n$. Therefore any vector in $\mathbb{R}^n$ can be written as Px for some $x \in \mathbb{R}^n$.

Next, we claim that $Py = y$ for all $y \in \mathbb{R}^n$. To see this, write $y = Px$ for some x, after which the equation becomes $P^2x = Px$ which is part of the hypothesis.

This implies that $P = \text{Id}_{n\times n}$.

(9) Take A to be any square orthogonal matrix which is not the identity.

(10) Take

$$V_1 = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \qquad \text{and} \qquad V_2 = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Then $V_1^\perp = V_2$ and $V_1 \cap V_2 = 0$, from which it also follows that $V_1^\perp \cap V_2^\perp = \{0\}$.

(11) DNE. If r is the rank of A, we have $\dim(\text{null}(A)) = 5 - r$ and $\dim(\text{null}(A^\top)) = 3 - r$, so the desired equation implies $8 - 2r = 5$. This is impossible since the RHS can't even.

(12) DNE. The construction of the pseudoinverse mentioned in class (based on the SVD) implies that $\text{pinv}(\text{pinv}(A)) = A$ for any matrix A. Therefore, if $\text{pinv}(A) = \text{pinv}(B)$, applying $\text{pinv}(-)$ to both sides implies that $A = B$.

(13) Take A to be any nonidentity orthogonal matrix, and take B to be the identity.

(14) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(15) DNE. Since $Av \in \text{null}(A^\top)$ by hypothesis and $Av \in \text{col}(A)$ by definition, we conclude that $Av \in \text{null}(A^\top) \cap \text{col}(A) = \{0\}$, so $Av = 0$. Therefore $v \in \text{null}(A)$ by definition. Thus, $v \in \text{row}(A) \cap \text{null}(A) = \{0\}$, which contradicts that v is nonzero.