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Four fundamental subspaces of $A \in \mathbb{R}^{m\times n}$

- $\mathrm{col}(A) = \{\text{All linear combinations of columns of } A\} = \{Ax \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$
- $\mathrm{row}(A) = \{\text{All linear combinations of rows of } A\} = \{A^T x \mid x \in \mathbb{R}^m\} = \mathrm{col}(A^T) \subset \mathbb{R}^n$
- $\mathrm{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n$
- $\mathrm{null}(A^T) = \{x \in \mathbb{R}^m \mid A^T x = 0\} \subset \mathbb{R}^m$

SVD and fundamental subspaces

Let $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be a full SVD of A and $A = U_1\Sigma_r V_1^T$ be a Rank-r SVD of A.

Size of matrices

- U : m by m Matrix
- V : n by n Matrix
- U_1 : m by r Matrix
- V_1 : n by r Matrix

Four fundamental subspaces of A in terms of SVD

- $\mathrm{col}(A) = \mathrm{col}(U_1)$
- $\mathrm{row}(A) = \mathrm{col}(V_1)$
- $\mathrm{null}(A) = \mathrm{col}(U_2)$
- $\mathrm{null}(A^T) = \mathrm{col}(V_2)$

invertible matrix

- If A is an n by n invertible matrix, then $\mathrm{col}(A) = \mathrm{col}(A^{-1}) = \mathbb{R}^n, \mathrm{null}(A) = \mathrm{null}(A^T) = 0$

Problems

1. Describe four fundamental subspaces of

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$

2. The full SVD of A is given as

$$A = U\Sigma V^T = \begin{pmatrix} 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.732 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4082 & -0.7071 & 0 & 0.5773 \\ 0 & 0 & 1 & 0 \\ 0.4082 & 0.7071 & 0 & 0.5773 \\ 0.8164 & 0 & 0 & -0.5773 \end{pmatrix}$$

(a) What is Rank-r SVD of A?

(b) Desribe four fundamental subspaces of A.

(c) A row space of an square n-by-n orthogonal matrix is $\mathbb{R}^n$ (no need to prove this), which is all the real vectors. That means, any vector $y \in \mathbb{R}^4$ can be obtained by $y = V^T x$. Describe column space of ΣV^T.

(d) What happens if a vector in (c) is multiplied by U on the left? Our A is actually $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Confirm that $\text{col}(A) = \text{col}(U_1)$ and explain why the column space of U_2 does not play a role in column space of A.

(e) Compute $U_1 U_1^T$ without any hard computation. Does $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ satisfy $U_1 U_1^T b = b$? Is $Ax = b$ solvable?