18.06 - Recitation 4

Sam Turton

March 12, 2019

1 Lecture review

1.1 Linear systems

There are four possibilities for a linear system $Ax = b$:

1. $Ax = b$ has a unique solution for all b. This can only happen if $r = m = n$ (full column and row rank)

2. $Ax = b$ has either a unique solution or no solution, depending on b. This can only happen if $r = n$, but $m > n$ (full column rank, not full row rank).

3. $Ax = b$ has infinitely many solutions for all b. This can only happen if $r = m$, but $n > m$ (full row rank, not full column rank).

4. $Ax = b$ has either no solution or infinitely many solutions, depending on b. This can only happen if $r < \min(m, n)$ (neither full column nor row rank).

1.2 Four fundamental subspaces

1. Let $A = U\Sigma V^T$ be the full SVD for the $m \times n$ matrix A.

2. The **column space** $C(A)$ is the set of vectors in $\mathbb{R}^m$ which are spanned by the columns of A. Equivalently, it is all vectors $u \in \mathbb{R}^m$ that can be written $u = Ax$ for any $x \in \mathbb{R}^n$. The first r columns of U are a basis for $C(A)$.

3. The **row space** $\text{row}(A) = C(A^T)$ is the set of vectors in $\mathbb{R}^n$ which are spanned by the rows of A. Equivalently, it is all vectors $v \in \mathbb{R}^n$ that can be written $v = A^T x$ for any $x \in \mathbb{R}^m$. The first r columns of V are a basis for $C(A)$.

4. The **null space** $N(A)$ is the set of vectors in $\mathbb{R}^n$ which satisfy $Ax = 0$. The last $n - r$ columns of V are a basis for $N(A)$.

5. The **left null space** $N(A^T)$ is the set of vectors in $\mathbb{R}^m$ which satisfy $A^T x = 0$. The last $m - r$ columns of U are a basis for $N(A^T)$.

6. Vectors in $C(A)$ are orthogonal to vectors in $N(A^T)$. Vectors in $C(A^T)$ are orthogonal to vectors in $N(A)$.

2 Problems

Problem 1.
Let A be an $m \times m$ invertible matrix. Describe in words as much as you can about the null space and left nullspace (e.g. dimension, possibly a basis, etc.) of the following:

(a) The matrix A

(b) The matrix $B = \begin{pmatrix} A \\ A \end{pmatrix}$

(c) The matrix $C = \begin{pmatrix} A & 2A \end{pmatrix}$

(d) The matrix $D = \begin{pmatrix} I & A \end{pmatrix}$

Problem 2.

(a) If $AB = 0$, then the columns of B are in which fundamental subspace of A? The rows of A are in which fundamental subspace of B? With $AB = 0$, why can't A and B be 3×3 matrices of rank 2?

(b) If $Ax = b$ has a solution and $A^T y = 0$, then which of the following is true: $y^T x = 0$ *or* $y^T b = 0$?

(c) If $A^T Ax = 0$, then why must $Ax = 0$? Why does this result mean that $N(A^T A) = N(A)$?

Problem 3.
Write down the complete solution to the following linear systems:

1. $A_1 x = b_1$, where:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and the full of SVD of } A_1 \text{ is}$$

$$A_1 = \begin{pmatrix} 0 & 0.4082 & -0.7071 & 0.5774 \\ 0 & 0.8165 & 0 & -0.5774 \\ 0 & 0.4082 & 0.7071 & 0.5774 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2.0000 & 0 & 0 \\ 0 & 1.7321 & 0 \\ 0 & 0 & 1.0000 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \\ 1.0000 & 0 & 0 \end{pmatrix}^T$$

2. $A_2 x = b_2$, where

$$A_2 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and the full of SVD of } A_2 \text{ is}$$

$$A_2 = \begin{pmatrix} 0.8411 & -0.3507 & -0.4117 \\ 0.5332 & 0.4105 & 0.7397 \\ 0.0903 & 0.8417 & -0.5323 \end{pmatrix} \begin{pmatrix} 2.8110 & 0 & 0 & 0 \\ 0 & 1.5773 & 0 & 0 \\ 0 & 0 & 0.7813 & 0 \end{pmatrix} \begin{pmatrix} 0.7881 & -0.1844 & -0.1072 & 0.5774 \\ 0.5211 & 0.5716 & -0.2616 & -0.5774 \\ 0.1897 & 0.2603 & 0.9467 & -0.0000 \\ 0.2671 & -0.7560 & 0.1543 & -0.5774 \end{pmatrix}^T$$

3. $A_3 x = b_3$, where

$$A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and the full of SVD of } A_3 \text{ is}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1.7321 & 0 & 0 \\ 0 & 1.4142 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.5774 & 0.7071 & 0.4082 \\ 0.5774 & -0.0000 & -0.8165 \\ 0.57747 & -0.7071 & 0.4082 \end{pmatrix}^T$$

Problem 4.
Construct matrices with each of the following properties, or explain why it is impossible:

1. Column space contains $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and row space contains $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.
2. Column space has basis $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, nullspace has basis $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.
3. Dimension of nullspace = 1 + dimension of left nullspace.
4. Nullspace contains $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, column space contains $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
5. Row space = column space, nullspace $\neq$ left nullspace.

Problem 5. (Challenge problem)
Write down the QR factorization of an arbitrary 3×3 upper triangular matrix:

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

What conditions are there on possibly a, b, c, d, e and/or f for the QR to exist? How does this generalize to an arbitrary $n \times n$ upper triangular matrix?[1]

[1] Some hints to get started: Are the columns of A linearly independent? What is the column space of A? Can you identify an orthonormal basis for $C(A)$?