18.06

Zack Remscrim

Four Fundamental Spaces

For a matrix A

- Column space: $\text{col}(A)$
- Row space: $\text{row}(A)$
- Nullspace: $\text{null}(A)$
- Left Nullspace: $\text{null}(A^\top)$

What are these spaces?

Why do we study them?

How can we compute them?

Nullspace

A $m \times n$ matrix

Nullspace: $\text{null}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$

Familiar example:

A is a function

input: $x \in \mathbb{R}^n$

output: $y \in \mathbb{R}^m$

$y = Ax$

null(A): $\{x \in \mathbb{R}^n | A(x) = 0\}$

f is a function

input: $x \in \mathbb{R}$

output: $y \in \mathbb{R}$

$y = f(x)$

zeros(f): $\{x \in \mathbb{R} | f(x) = 0\}$

Nullspace

A $m \times n$ matrix: $\text{null}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$

Consider equation $Ax = 0$

This equation always has a solution

Set of all solutions is $\text{null}(A)$

Consider equation $Ax = b$

This equation may or may not have a solution

If has solution, let x_p denote some solution ($Ax_p = b$)

Set of all solutions is $x_p + \text{null}(A)$

$$\{x_p + x_n | x_n \in \text{null}(A)\}$$

Column Space

A $m \times n$ matrix

Column space: $\text{col}(A) = \{$ all linear combinations of columns $\}$

A is a function

input: $x \in \mathbb{R}^n$

output: $y \in \mathbb{R}^m$

$y = Ax$

$\text{col}(A)$: $\{y \in \mathbb{R}^m | Ax = y, x \in \mathbb{R}^n\}$

Familiar example:

f is a function

input: $x \in \mathbb{R}$

output: $y \in \mathbb{R}$

$y = f(x)$

range(f): $\{y \in \mathbb{R} | f(x) = y, x \in \mathbb{R}\}$

Column Space

A $m \times n$ matrix: $\text{col}(A) = \{y \in \mathbb{R}^m | Ax = y, x \in \mathbb{R}^n\}$

Consider equation $Ax = b$

This equation may or may not have a solution

Has at least one solution exactly when $b \in \text{col}(A)$

Row Space

A $m \times n$ matrix

Row space: $\text{row}(A)$ $=$ { all linear combinations of rows }

$= \text{col}(A^\top)$

$A^\top$ is a function

input: $y \in \mathbb{R}^m$

output: $x \in \mathbb{R}^n$

$x = A^\top y$

$\text{row}(A)$: $\{x \in \mathbb{R}^n | A^\top y = x, y \in \mathbb{R}^m\}$

Careful: $A^\top$ is not (generally) the inverse of A

Row Space

A $m \times n$ matrix, $\text{row}(A) = \text{col}(A^\top)$: $\{x \in \mathbb{R}^n | A^\top y = x, y \in \mathbb{R}^m\}$

Consider equation $Ax = b$

Clear what $\text{null}(A)$, $\text{col}(A)$ say about solutions

What does $\text{row}(A)$ tell us?

Idea: $\text{row}(A)$ has important relationship to both $\text{null}(A)$ and $\text{col}(A)$

Recall SVD

Singular Value Decomposition

A $m \times n$ matrix, of rank r

Write $A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$

Σ_r $r \times r$ diagonal matrix, with all positive singular values on diagonal

U_1 first r columns of U

V_1 first r columns of V

U_1, U_2, V_1, V_2 are all orthogonal matrices

Fundamental Theorem of Linear Algebra

A $m \times n$ matrix, of rank r, where $A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$

$\text{col}(A) = \text{col}(U_1)$
$\text{row}(A) = \text{col}(V_1)$
$\text{null}(A) = \text{col}(V_2)$
$\text{null}(A^\top) = \text{col}(U_2)$

Tells us "everything" about A

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

When does $Ax = b$ have at least one solution?

Exactly when $b \in \mathrm{col}(A)$

Exactly when $b \in \mathrm{col}(U_1)$

Exactly when $U_1 {U_1}^\top b = b$

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

If $Ax = b$ has at least one solution, when does it have only one solution?

Exactly when $\mathrm{null}(A) = \{0\}$ (only contains the zero vector)

Exactly when $\mathrm{col}(V_2) = \{0\}$

Exactly when $r = n$

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

If $Ax = b$ has at least one solution, what are all of the solutions?

$x_p = V_1 \Sigma_r^{-1} U_1^\top b$ is a solution ($A = U_1 \Sigma_r V_1^\top$)

$x_p + \text{null}(A)$ is the set of all solutions

$x_p + \text{col}(V_2)$ is the set of all solutions

Orthogonal Spaces

Vector space V

We say vectors $u, w \in V$ are *orthogonal* when $u \cdot w = 0$

Given subspaces R, S of V, say R, S are *orthogonal* when

For every $r \in R$ and $s \in S$

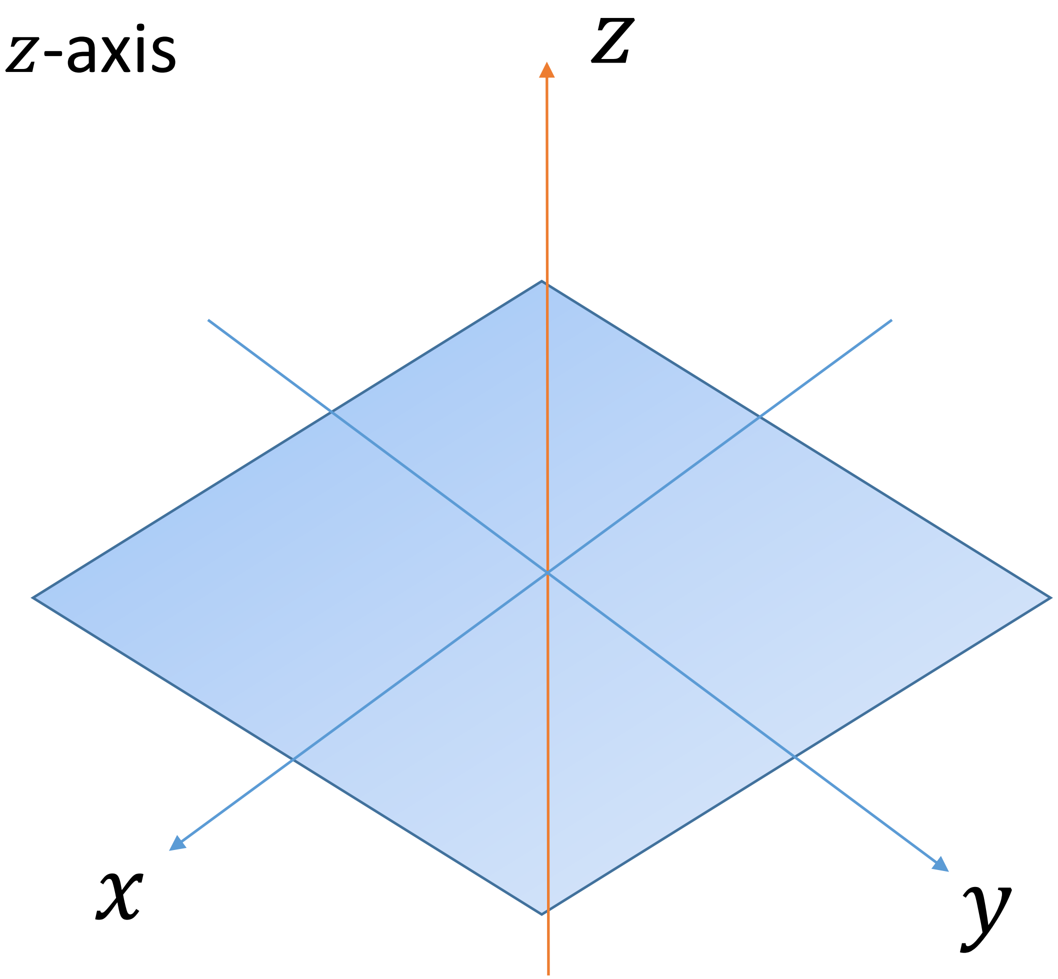
$$r \cdot s = 0$$

Orthogonal Spaces

Example: In $\mathbb{R}^3$, xy-plane is orthogonal to z-axis

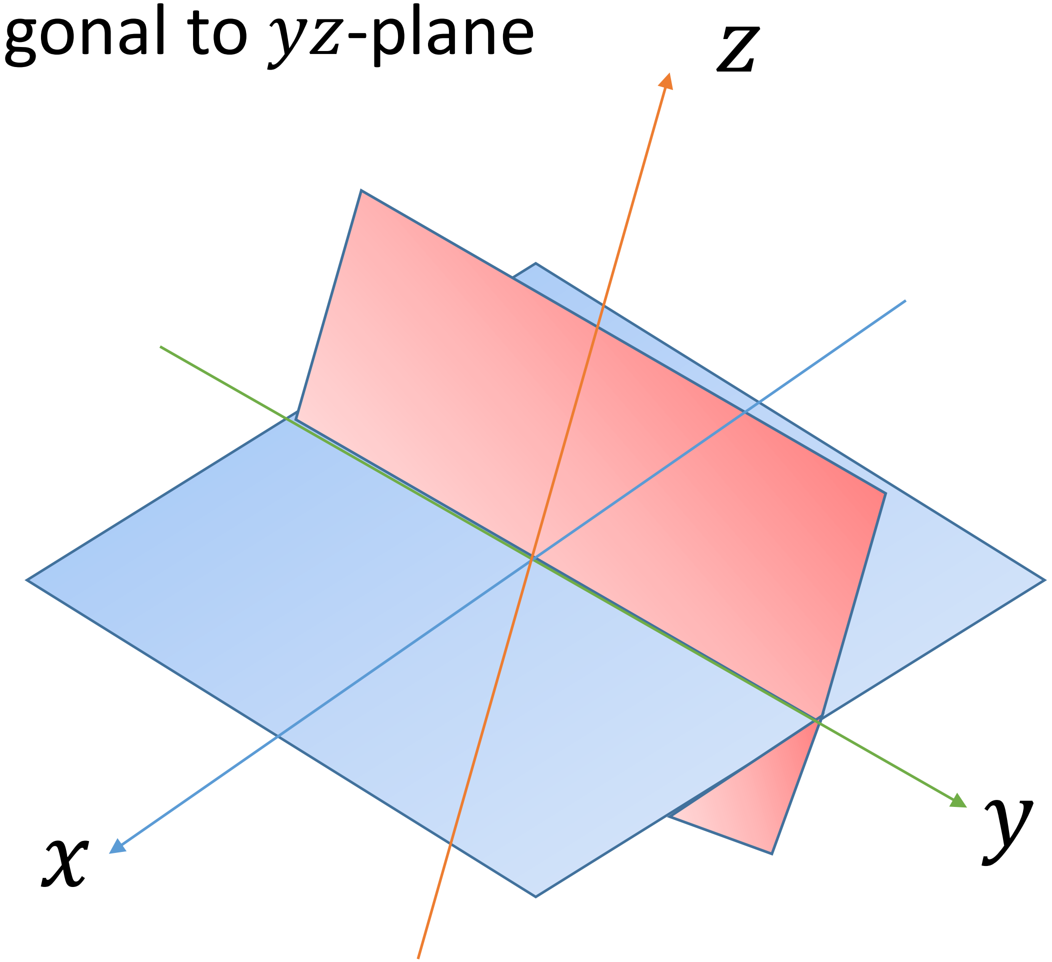
$r = (a, b, 0) \in xy$-plane
$s = (0,0, c) \in z$-axis

$r \cdot s = 0$



Orthogonal Spaces

Non-Example: In $\mathbb{R}^3$, xy-plane is *not* orthogonal to yz-plane



$r = (0,1,0) \in xy$-plane

$s = (0,1,0) \in yz$-plane

$r \cdot s = 1$

Orthogonal Spaces

Vector space V, subspaces R, S

Always have $0 \in R$ and $0 \in S$

If, for a $v \neq 0$, $v \in R$ and $v \in S$

R and S not orthogonal

$$v \cdot v = \|v\|^2 > 0$$

Row Space

A $m \times n$ matrix

$\text{row}(A) = \{x \in \mathbb{R}^n | A^\top y = x, y \in \mathbb{R}^m\}$
$\text{null}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$

$\text{row}(A)$ and $\text{null}(A)$ subspaces of $\mathbb{R}^n$
$\text{row}(A)$ and $\text{null}(A)$ orthogonal
  $\text{row}(A) = \text{col}(V_1)$
  $\text{null}(A) = \text{col}(V_2)$
  V orthogonal matrix

Row Space

A $m \times n$ matrix, row(A) and null(A) orthogonal

$0 \in$ row(A) and $0 \in \text{null}(A)$

But, no $v \neq 0$ in both row(A) and null(A)

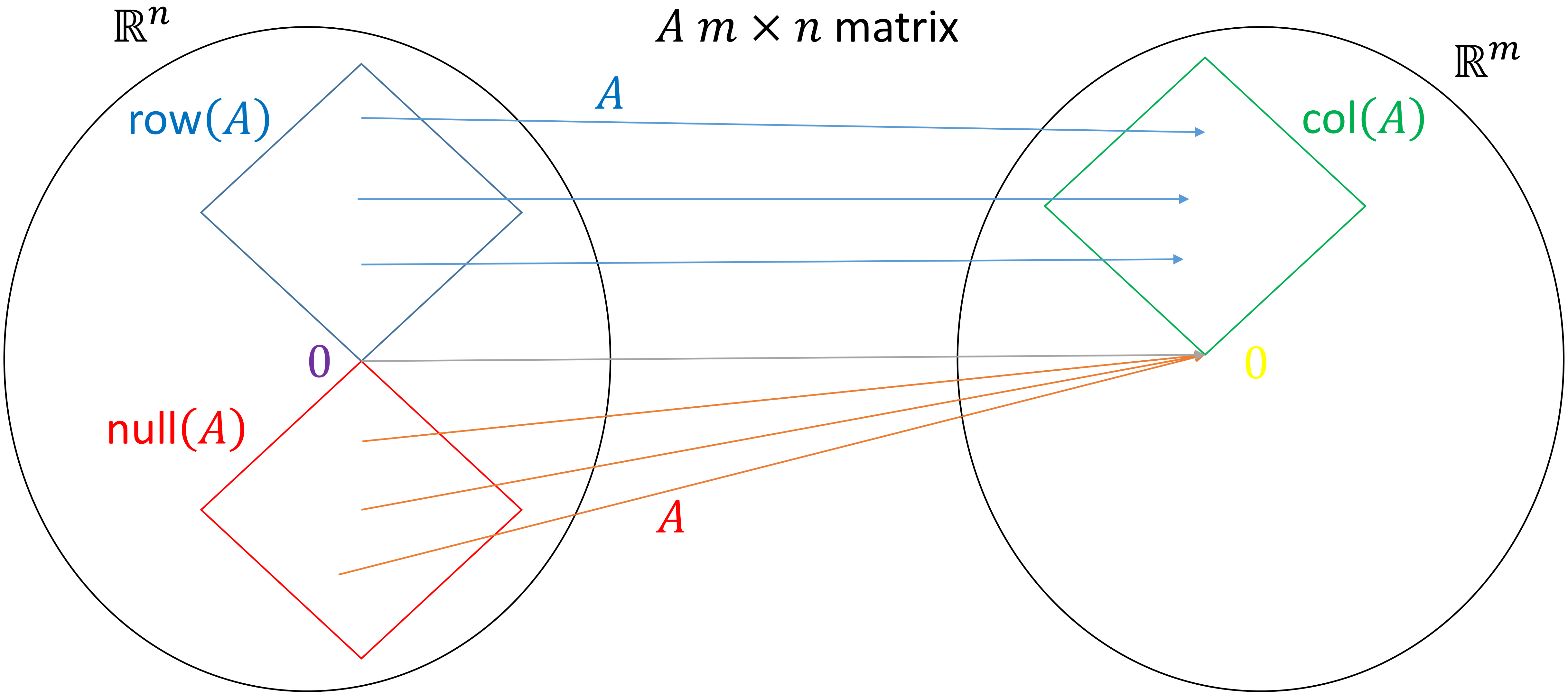
For $x_1, x_2 \in$ row(A)

If $x_1 \neq x_2$ then $Ax_1 \neq Ax_2$

$Ax_1 = Ax_2 \Rightarrow Ax_1 - Ax_2 = 0 \Rightarrow A(x_1 - x_2) = 0 \Rightarrow (x_1 - x_2) \in$ null(A)

$x_1, x_2 \in$ row$(A) \Rightarrow (x_1 - x_2) \in \text{row}(A)$

Three Fundamental Spaces



Left Nullspace

A $m \times n$ matrix

$\text{col}(A) = \{y \in \mathbb{R}^m | Ax = y, x \in \mathbb{R}^n\}$

$\text{null}(A^\top) = \{y \in \mathbb{R}^m | A^\top y = 0\}$

$\text{col}(A)$ and $\text{null}(A^\top)$ subspaces of $\mathbb{R}^m$

$\text{col}(A)$ and $\text{null}(A^\top)$ orthogonal

$\text{col}(A) = \text{col}(U_1)$

$\text{null}(A^\top) = \text{col}(U_2)$

U orthogonal matrix

Four Fundamental Spaces

A $m \times n$ matrix