18.06

Zack Remscrim

Determinant

For a square matrix A, denote determinant of A by

$\det(A)$

or $|A|$

Function:

Input: square matrix

Output: a single scalar

Encodes important information about A

Determinant: Geometric Meaning in $\mathbb{R}^2$

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, 2×2 real matrix

and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$

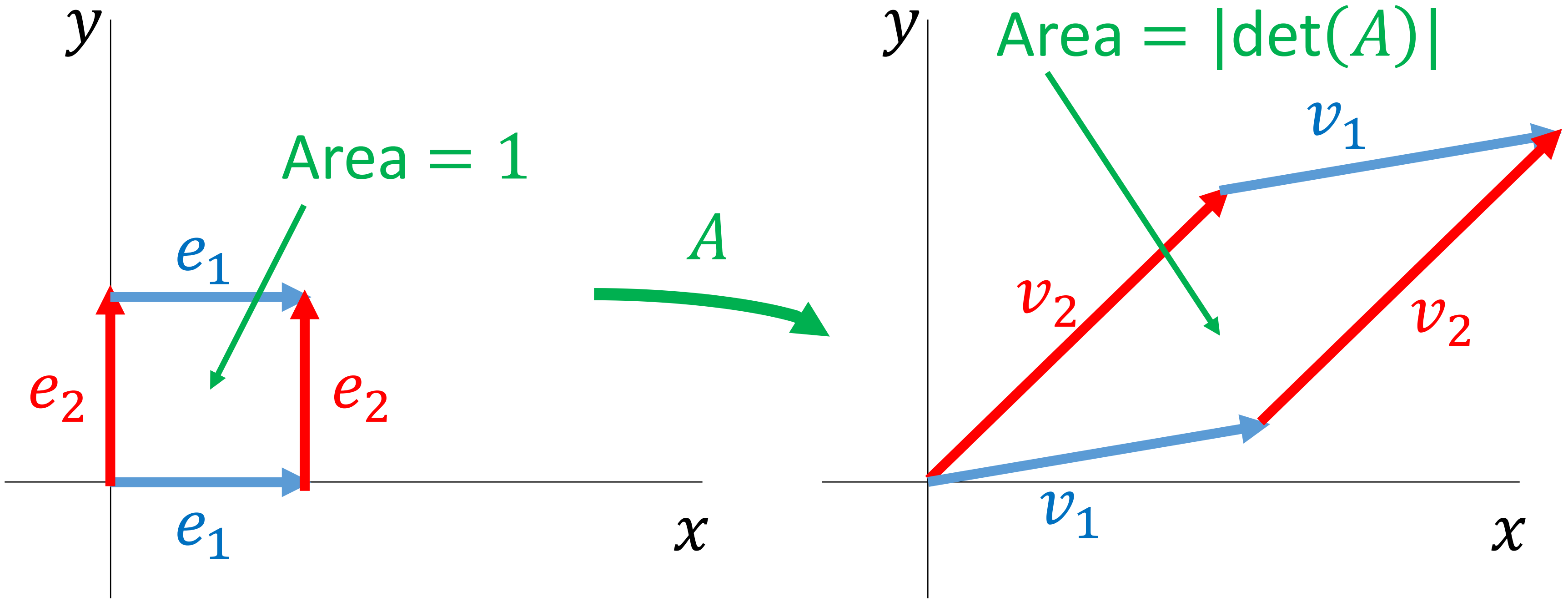
A transforms e_1, e_2 to

$$v_1 = Ae_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$v_2 = Ae_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

1st and 2nd col of A

A transforms square of area 1 to parallelogram of area $|\det(A)|$



Determinant: Geometric Meaning in $\mathbb{R}^n$

Consider A, $n \times n$ real matrix

and $e_1, e_2, \dots, e_n \in \mathbb{R}^n$, where e_i: i^{th} entry 1, all other entries 0

A transforms $e_1, e_2, \dots, e_n$ to

$v_1, v_2, \dots, v_n$, where $v_i = Ae_i$

v_i is i^{th} col of A

$e_1, e_2, \dots, e_n$ specify hypercube in $\mathbb{R}^n$ (n-dimensional version of cube)

$v_1, v_2, \dots, v_n$ specify parallelotope in $\mathbb{R}^n$ (n-dimensional version of parallelogram)

A transforms hypercube $e_1, e_2, \dots, e_n$ of volume 1

to parallelotope $v_1, v_2, \dots, v_n$ of volume $|\det(A)|$

Determinant: Geometric Meaning in $\mathbb{R}^n$

If $\det(A)$ is signed volume of parallelotope, det must satisfy certain properties

1) $\det(I) = 1$

   $v_1, v_2, \dots, v_n$ is $e_1, e_2, \dots, e_n$

2) If any column of A is all zeros, $\det(A) = 0$

   if i^{th} col of A is all zeros, $v_i = 0$, so volume is zero

3) Determinant is linear in each column separately

   $\text{Vol}(v_1, \dots, v_i, cv_{i+1}, v_{i+2}, \dots, v_n) = c\ \text{Vol}(v_1, \dots, v_i, v_{i+1}, v_{i+2}, \dots, v_n)$

   $\text{Vol}(v_1, \dots, v_i, v_{i+1} + v'_{i+1}, v_{i+2}, \dots, v_n)$

   $= \text{Vol}(v_1, \dots, v_i, v_{i+1}, v_{i+2}, \dots, v_n) + \text{Vol}(v_1, \dots, v_i, v'_{i+1}, v_{i+2}, \dots, v_n)$

Determinant: Axioms

In fact, det is the only function that satisfies previous three properties

Will define det as unique functions that satisfies three equivalent axioms

- Consider rows instead of columns
- Replace statement about all zero row,
  - with equivalent statement about swapping rows

Determinant: Axioms

Property 1: Identity matrix

If I is an identity matrix,

Then $\det(I) = 1$

Examples:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Determinant: Axioms

Property 2: Row swapping

For any square matrix A

For any A' obtained from A by swapping any two rows

$$\det(A') = -\det(A)$$

Example:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longrightarrow A' = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

$$\det(A') = -\det(A)$$

Determinant: Axioms

Property 3: Linearity in each row (does not mean $|A+B| = |A| + |B|$)

For any square matrix A and any row i

Fix all of A except row i

Determinant is a linear function of row i

Example:

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, multiply all entries in row 2 by $\alpha \in \mathbb{R}$

$$\begin{vmatrix} a & b & c \\ \alpha d & \alpha e & \alpha f \\ g & h & i \end{vmatrix} = \alpha \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Determinant: Axioms

Property 3: Linearity in each row (does not mean $|A+B| = |A| + |B|$)

For any square matrix A and any row i

Fix all of A except row i

Determinant is a linear function of row i

Example:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, A' = \begin{pmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{pmatrix}$$ identical outside row 2

$$\begin{vmatrix} a & b & c \\ d+d' & e+e' & f+f' \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix}$$

Determinant from the Axioms

These three axioms completely determine the determinant function

Key idea: For any matrix A, to figure out $\det(A)$

Change A little-by-little until it looks like I

Keep track of how determinant changes along the way

Shows determinant is determined by axioms

Gives algorithm for determinant

Why define determinant like this?

Intellectually interesting

Very useful

Product Rule

Claim: $|AB| = |A||B|$

Pf: Easy case, if $|B| = 0$

Then B not invertible

So AB not invertible

$|AB| = 0$

Product Rule

Claim: $|AB| = |A||B|$

Pf: interesting case, if $|B| \neq 0$

Define function $f(A) = \frac{|AB|}{|B|}$, holding B fixed

Want to show $f(A) = |A|$

Suffices to show f satisfies the three axioms

(determinant is unique)

Product Rule

Define function $f(A) = \frac{|AB|}{|B|}$

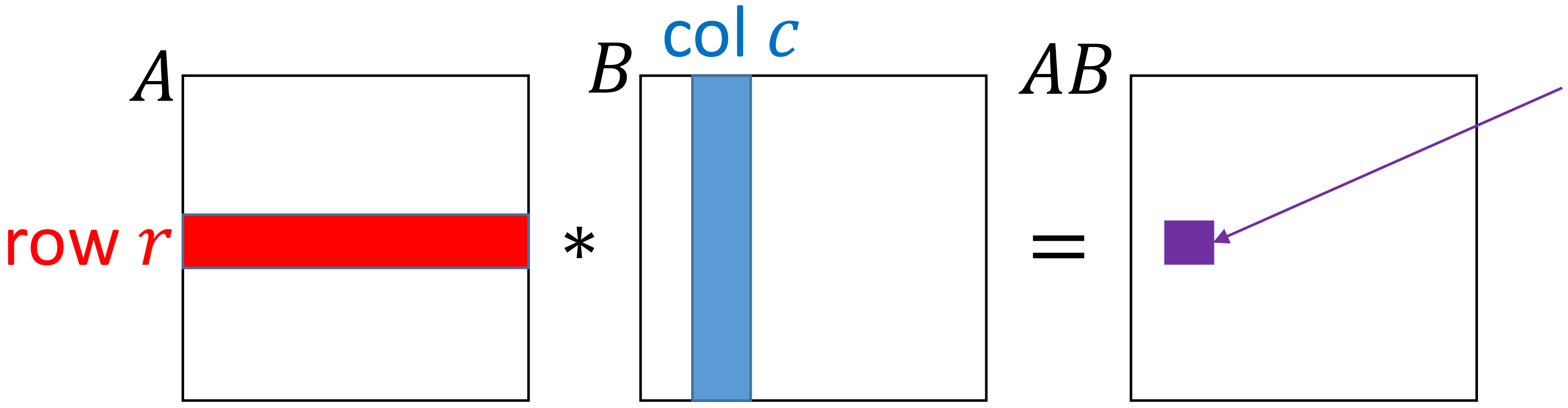
Property 1: $f(I) = \frac{|IB|}{|B|} = \frac{|B|}{|B|} = 1$

Product Rule

Define function $f(A) = \frac{|AB|}{|B|}$

Property 2: Swap rows i and j of A to produce A'

Then $A'B$ obtained from AB by swapping rows i and j



Entry in row r and col c of AB is dot product of row r of A and col c of B

$$f(A') = \frac{|A'B|}{|B|} = -\frac{|AB|}{|B|} = -f(A)$$

Product Rule

Define function $f(A) = \frac{|AB|}{|B|}$

Property 3:

Multiply row i of A by α to get A'

multiplies row i of AB by α

$$f(A') = \frac{|A'B|}{|B|} = \alpha \frac{|AB|}{|B|} = \alpha f(A)$$

If have two matrices C, C' identical outside row i

$A = \mathrm{C} +$ row i of C'

$$f(A) = f(C) + f(C')$$

Determinant Properties

$|A| = 0$ if and only if A singular (not invertible)

Not very useful for computationally testing if A invertible

Important theoretical property

For a lower triangular matrix $L = \begin{pmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}$

det is product of diagonal entries: $|L| = l_{11} l_{22} \cdots l_{nn} = \prod_{i=1}^{n} l_{ii}$

Permutation Matrices

A *permutation matrix* is

a square matrix,

all entries are either 0 or 1

a single 1 in each row and each column

Example: $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Permutation Matrices

Permutation matrix encodes a permutation

A *permutation* on the set of integers $\{1,2,\ldots,n\}$ is a function

$$\sigma: \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$$

where, if $i \neq j$, $\sigma(i) \neq \sigma(j)$

Can think of a permutation as specifying an ordering of n objects

Take n blocks labelled $1,2,\ldots,n$ and put them in a line in some order

Ex: $\boxed{3}$ $\boxed{1}$ $\boxed{4}$ $\boxed{2}$ $\quad \sigma(1) = 2, \sigma(2) = 4, \sigma(3) = 1, \sigma(4) = 3$

σ encodes position of each object in the order

object i is in position $\sigma(i)$

Permutation Matrices

An $n \times n$ permutation matrix P encodes the permutation σ on $\{1,2,\ldots,n\}$ where if in col i, the (unique) row in which P has a 1 is row j then $\sigma(i) = j$

Example: $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ encodes σ on $\{1,2,3,4\}$

where $\sigma(1) = 2, \sigma(2) = 4, \sigma(3) = 1, \sigma(4) = 3$

Equivalently: $Pe_i = e_{\sigma(i)}$

Permutation Matrices: Properties

If P is a permutation matrix, then P is invertible and $P^{-1} = P^\top$

$P^\top$ "undoes" P

If P is a permutation matrix, $|P| = \pm 1$

Swap rows to get to I

Permutation Matrices: Properties

For any invertible matrix A, can write $A = PLU$

- L lower triangular
- U unit upper triangular (diagonal entries all 1)
- P permutation matrix

Saw algorithm for computing det

- Will generally convert A to this form
- Might get stuck when A singular

$$|A| = |PLU| = |P||L||U| = (\pm 1)\left(\prod_{i=1}^{n} l_{ii}\right)(1) = \pm\left(\prod_{i=1}^{n} l_{ii}\right)$$

Minors and Cofactors

For an $n \times n$ matrix A

Define $n \times n$ matrix M, matrix of *minors*, by

M_{ij} is det of A with row i and col j deleted

Define $n \times n$ matrix C, matrix of *cofactors*, by

C_{ij} is $(-1)^{i+j} M_{ij}$

Cofactors: Properties

For any $n \times n$ matrix A and any row i

$$|A| = A_{i,1}C_{i,1} + A_{i,2}C_{i,2} + \cdots + A_{i,n}C_{i,n}$$

For any invertible $n \times n$ matrix A

$$A^{-1} = \frac{C^{\top}}{|A|}$$