1 Lecture Review

1.1 Vector Spaces

1. A set V is a vector space if (i) $0 \in V$, (ii) $v, w \in V$ implies $v + w \in V$, (iii) $c \in \mathbb{R}$, $v \in V$ implies $cv \in V$.

1.2 Matrix Transpose

1. The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T defined by $(A^T)_{ij} = A_{ji}$.
2. If A is $m \times n$ and B is $n \times p$, then $(AB)^T = B^T A^T$.

1.3 Matrix Inverse

1. A matrix A is invertible if there exists a matrix A^{-1} so that $AA^{-1} = I = A^{-1}A$.
2. An invertible matrix must be square.
3. If A, B have the same size and are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.
4. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then
$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
given $ad - bc \neq 0$.

1.4 Orthogonal

1. A matrix A is orthogonal if $AA^T = A^T A = I$. That is, $A^T = A^{-1}$.

1.5 Block Matrices

1. A block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a way to write a matrix in terms of smaller matrices A, B, C, D where the sizes are compatible.
2. Block matrices multiply like 2×2 matrices (but you must remember the order)
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

1.6 Matrix Factorizations

1. A matrix factorization of A is a way to express A as a product of matrices.
2. LU Factorization, QR Factorization and SVD (covered later, see Problems 4, 5 and 6 on PSet 1 for 2×2 examples).

2 Problems

1. Determine if the following set is a vector space. Why or why not?

   (a) The line $y = x$.

   (b) The line $y = x + 1$.

   (c) The union of the x and y axes.

   (d) The unit circle (x, y) where $x^2 + y^2 = 1$.

   (e) 5×5 matrices with $(3, 3)$ entry equal to 0.

   (f) Set of functions $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

   (g) Set of functions $f(x)$ with $f(7) = 0$.

   (h) Set of functions $f(x)$ with $f(0) = 7$.

   Solution.

   (a) $\boxed{\text{Yes.}}$ Any point on the line $y = x$ is of the form (a, a) for some $a \in \mathbb{R}$. Then (i) $(0, 0)$, (ii) $(a, a) + (b, b) = (a + b, a + b)$, (iii) $c(a, a) = (ca, ca)$ are all contained on the line.

   (b) $\boxed{\text{No.}}$ Note that $(0, 0)$ is not contained in the line $y = x + 1$.

   (c) $\boxed{\text{No.}}$ If we take $(a, 0) + (0, b) = (a, b)$, this is not in the union of the x and y axes.

   (d) $\boxed{\text{No.}}$ Observe that $c(1, 0) = (c, 0)$ is on the circle centered at the origin with radius c.

   (e) $\boxed{\text{Yes.}}$ The 5×5 zero matrix satisfies this property. Summing two matrices with $(3, 3)$ entry equal to 0 gives a matrix with $(3, 3)$ entry equal to 0. Likewise when multiplying a matrix with $(3, 3)$ entry equal to 0 by a constant.

   (f) $\boxed{\text{Yes.}}$ Note that 0 is in this set when $a = b = c = 0$. Summing two quadratic polynomials gives a quadratic polynomial. Multiplying a quadratic polynomial gives a quadratic polynomial.

   (g) $\boxed{\text{Yes.}}$ Note that the zero function is in this set (the function defined by $f(x) = 0$). If $f(7) = 0$ and $g(7) = 0$, then $(f + g)(7) = f(7) + g(7) = 0$. If $f(7) = 0$ and $c \in \mathbb{R}$, then $(cf)(7) = cf(7) = 0$.

   (h) $\boxed{\text{No.}}$ Observe that the zero function is not in this set.

   $\square$

2. (a) Suppose $\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}$ is the $m \times n$ matrix with columns vectors are $b_1, \ldots, b_n$. Show that if A is $p \times m$, then

   $$A \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{pmatrix}.$$

   (b) Suppose $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ is an $m \times n$ matrix with row vectors $a_1, \ldots, a_n$. Show that if B is $n \times p$, then

   $$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ \vdots \\ a_m B \end{pmatrix}.$$

   Solution. This follows from the definition of matrix multiplication. $\square$

3. If S_1 stands for the operation of putting on your socks and S_2 the operation of putting on your shoes (so $S_2 \circ S_1$ stands for putting on your socks and then your shoes), what is the inverse of $S_2 \circ S_1$?

   Solution. The inverse of $S_2 \circ S_1$ is $S_1^{-1} \circ S_2^{-1}$, that is taking off your shoes then taking off your socks. $\square$

4. Show that the rotation matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

   Solution. We have
   $$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I.$$
   $\square$

5. Suppose $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix where A, B, C, D are matrices with compatible sizes. Write the transpose $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as a block matrix in terms of A^T, B^T, C^T, D^T.

   Proof. We have
   $$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T = \boxed{\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}}$$
   $\square$

6. When can $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be written as $X^T X$ for $X = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$? Assume $b \neq 0$. What are p, q, r in terms of a, b, c, d when possible?

   Solution. Note that the (i, j) entry of $X^T X$ is given by the dot product between the ith and jth columns of X. Then
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A = X^T X = \begin{pmatrix} p^2 + q^2 & qr \\ qr & r^2 \end{pmatrix}.$$
   From this equality, we see that we require $b = c$. We also require $d = r^2 \geq 0$ or $r = \sqrt{d}$, so $q = \frac{b}{\sqrt{d}}$ and $d > 0$. Moreover,
   $$a = p^2 + q^2 = p^2 + b^2/d \implies p = \sqrt{\frac{ad - b^2}{d}}$$
   from which we get the requirement $ad - b^2 \geq 0$.

   In summary the conditions we require are
   $$\boxed{ad - b^2 \geq 0, \quad d > 0, \quad b = c}$$
   in which case
   $$\boxed{p = \sqrt{\frac{ad - b^2}{d}}, \quad q = \frac{b}{\sqrt{d}}, \quad r = \sqrt{d}}$$
   $\square$