Recitation 3. Solution

Focus: exam comments, linear independence, projections.

Definition. Recall that if A is a matrix with independent columns, then the *projection matrix* P on the column space of A can be written as $P = A(A^TA)^{-1}A^T$.

1. *Recognizing vector spaces.* Is the following set a vector space?

   a) The set of all vectors in $\mathbb{R}^3$ except those of the form $\begin{pmatrix} x & 0 & 0 \end{pmatrix}^T$ with $x > 0$.

   b) The set of 2×3 matrices whose six elements sum to 6.

   c) The set of rank one 3×3 matrices together with the zero matrix.

2. *Vector spaces and bases.* Let V be the space of homogeneous quadratic polynomials in two variables, i.e. polynomials of the form $f(x, y) = ax^2 + bxy + cy^2$.

   a) Are elements x^2, xy, y^2 linearly independent?

   b) What about x^2, $x^2 + xy + y^2$, $xy + y^2$?

   c) And x^2, $x^2 + xy$, $x^2 + xy + y^2$?

   d) What is the dimension of this vector space V?

Solution:

a) Yes. Check the definition: assume that we have a linear combination $f(x, y) = ax^2 + bxy + cy^2 = 0$. Then for every pair of real numbers (x_0, y_0), we have that $f(x_0, y_0)$. In particular, $f(1, 0) = a = 0$, hence the linear combination can be simplified to $f(x, y) = bxy + cy^2$. Similarly, $f(0, 1) = c = 0$ and $f(x, y) = bxy$. And finally, $f(1, 1) = b = 0$, hence all coefficients in the linear combination are zero: $a = b = c = 0$. So by definition, monomials x^2, xy, y^2 are linearly independent.

b) No, because $x^2 - (x^2 + xy + y^2) + (xy + y^2) = 0$.

c) Yes.

d) $\dim V = 3$, because monomials x^2, xy, y^2 generate V and by part (a) are linearly independent, hence form a basis.

3. *Subspaces.* Let V be the space from the previous problem. Consider the subset of V that consists of functions $f \in V$ such that $f(1, 1) = 0$. Denote this subset by W – it is a vector subspace of V.

   a) Prove that W is a vector space.

   b) Find a basis of this vector space.

   c) What is the dimension of W?

Solution:

a)

b) $x^2 - xy$, $y^2 - xy$.

c) $\dim W = 2$.

4. *Projection onto a subspace.* Consider the following matrix A written as a full SVD $A = U\Sigma V^T$:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

a) What is the rank of A?

b) Mark U_1, U_2, V_1 and V_2.

c) Circle columns of U that span $\operatorname{col} A$.

d) Compute the projection matrix on $\operatorname{col} A$.

Solution:

a) $A = 2$.

b) $U_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}$; $U_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$; $V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$; $V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

c) $\operatorname{col} A = \operatorname{col} U_1$, so we should circle $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}$.

d) $\operatorname{col} A = \operatorname{col} U_1$, so we can compute the projection matrix P on $\operatorname{col} U_1$. But U_1 is a matrix with linearly independent columns, so we can use one of the formulas for computing projections. In this case, we can use the formula for the QR decomposition, because in the QR decomposition of $U_1 = QR$, we have $U_1 = Q$ already, because U_1 is tall skinny orthogonal from SVD. Therefore, we can use the formula $P = QQ^T = U_1 U_1^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

5. *Bonus. Might be useful for PSet 4. (Or not.)* If A is decomposed into a product $A = BC$ with C being square invertible, then $\operatorname{col} A = \operatorname{col} B$.

6. *Bonus.* We declare x^2, xy and y^2 to be the standard orthonormal basis in V, that is we write them as follows:

$$x^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; xy = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; y^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compute the projection matrix on W.

Solution: In the given basis, coordinate vectors of the basis of W found in 3(b) is $x^2 - xy = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $y^2 - xy = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Then $W = \operatorname{col} \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}$. Denote this matrix by A, and the projection matrix by P. Now we can either use the formula $P = A(A^T A)^{-1} A^T$ or compute QR decomposition of A and use $P = QQ^T$.