18.06 - Recitation 7

Sam Turton

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1 Lecture review

Let A be a square $n \times n$ matrix.

1. The *determinant* of A is the unique function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ for which
   - $\det I = 1$ for the identity matrix of any dimension
   - $\det A$ changes sign when any two rows of A are interchanged.
   - $\det A$ is a linear transformation of each row of A.
2. The determinant of a 2×2 matrix has a simple formula:
$$\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Longrightarrow \quad \det A = ad - bc}$$
3. Two very useful rules for determinants:
$$\boxed{\det A^T = \det A, \quad \det AB = \det A \det B}$$
4. The *cofactor* C_{ij} is defined by
$$\boxed{C_{ij} = (-1)^{i+j} \det M_{ij}}$$
   where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and jth column from A.
5. **Compute determinant by cofactors:** For any $1 \le i \le n$,
$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.}$$
6. If A is invertible, then
$$\boxed{A^{-1} = \frac{1}{\det A} C^T.}$$
   In terms of entries, $(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$.
7. **Cramer's Rule:** If A is invertible and $Ax = b$, then
$$\boxed{x_i = \frac{\det B_i}{\det A},}$$
   where B_i is the matrix A with the ith column replace by b.

2 Problems

Problem 1.

1. Let A be $n \times n$. Explain using the full form SVD why $\det A \neq 0$ if and only if the rank of A is n.
2. If I is the $n \times n$ identity matrix and a is a scalar, what is $\det(aI)$?
3. Using the cofactor formula, explain why the determinant of an upper triangular matrix is the product of the elements along the diagonal.
4. Find $\det A$ and A^{-1} explicitly, where
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Problem 2.
If A is $n \times n$ and invertible and C is its cofactor matrix, show that

1. $AC^T = \det(A)I$,
2. $\det C = (\det A)^{n-1}$.

Problem 3.
Let Q be a square, orthogonal matrix. Find the cofactor matrix of Q up to sign. Explain how the sign is affected by the sign of $\det Q$.

Problem 4.
Compute the determinant of the tridiagonal matrix

$$A = \begin{pmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & -1 & 1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 1 \end{pmatrix}$$

by using the cofactor expansion.

Problem 5.

Suppose A is an invertible $n \times n$ square matrix and B is a known $n \times m$ matrix.

1. If you want to solve
$$AX = B$$
where X is an $n \times m$ unknown matrix using Cramer's rule, in general how many determinants do you need to compute?

2. How many determinants do you need to compute A^{-1} by cofactors?

3. Compare (6a) and (6b). In particular, how is (6b) a special case of (6a)?