Summary for Week 8 Recitation

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This week we will talk about trace and eigenvalues, which are introduced in lectures on April 10 and 13 respectively.

Given a matrix A, it is sometimes interesting to study the behaviors or limit of A^n, the n-th power of A when n becomes larger and larger. One classic example is random walk problem, such as computing the probability of a drunk person returning home safe if he walks between his home and the cliff randomly? Of course, people shouldn't get drunk and walk outside during these hard time!

For general A, even for 2 by 2 matrices, you will find that it takes so long to compute A^n directly. However, the situation improves when A has more zero entries. It is good when A is an upper triangular matrix. It is even better when A is a diagonal matrix.

So the question is how to transform a general matrix to a upper triangular matrix or diagonal matrix. First, we need to clarify the meaning of transformation. We introduce the concept of conjugacy.

Definition 0.1. We call A and B are conjugate if there exists M such that $A = M^{-1}BM$.

By induction, we know that $A^n = M^{-1}B^nM$. So if A is conjugate to a diagonal matrix, then A^n can be easily computed. In this case, we say A is diagonalizable.

It remains to find what M is. Solving $BM = MA$ where B is diagonal matrix, is equivalent to finding eigenvectors of A. The constraint M being invertible means eigenvectors span the whole space. There are matrices that are non-diagonalizable. The classical example is [1 1; 0 1]. However, it is pretty strange that [1 1; 0 1.0001], which is close to [1 1; 0 1], is diagonalizable. Could you find a M that transforms [1 1; 0 1.0001] to a diagonal matrix in this case?

A remark is that after you know how to compute eigenvalues/eigenvectors of a matrix, you should be able to find SVD of A by hand in theory! Computing Σ in SVD of A is the same as computing eigenvalues of A^TA up to square root. Could you see why?

I find it difficult to give motivation of trace without involving more advanced mathematics like functional analysis or representation theory. Perhaps I will return to this at some point. Anyway, the formal definition of trace is the sum of all diagonal entries. The most important property of trace is the following statement.

Problem 0.2. *Prove that if A and B are conjugate, then they have same trace.*

Therefore trace is an invariant of a matrix in some sense.

Finally, I would upload another document which was prepared for my practice teaching last year. Try to solve the exercises there.