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Topics: Vector spaces, solving $Ax = b$, matrix transpose and inverse, orthogonal matrices, block matrices.

1. Recognizing vector spaces

A (real) vector space is a set V equipped with two operations:

- An operation $+$ (vector addition) which takes $v_1, v_2 \in V$ and outputs $v_1 + v_2 \in V$.
- An operation $\cdot$ (scalar multiplication) which takes $c \in \mathbb{R}$ and $v \in V$ and outputs $c \cdot v \in V$.

(NB: Part of these requirements is that the sum must actually lie in V, in which case we say that "V is closed under addition," and similarly V must be "closed under scalar multiplication.") These operations must also satisfy some axioms.[1] Here are two of them:

- Existence of a 'zero': there must exist $v_0 \in V$ such that $v_0 + v = v$ for all $v \in V$.
- Existence of additive inverses: for any $v \in V$, there must exist $v' \in V$ such that $v + v' = v_0$.

The 'zero' vector v_0 is usually just denoted 0, and the additive inverse of v is usually denoted $-v$.

Problem 1. In the vector space consisting of all functions on the real line, what is the zero vector? Yes.

Problem 2. Consider the set V consisting of pairs (a, b) of real numbers satisfying $a + 2b = 0$. How should one define the operations of 'vector addition' and '(real) scalar multiplication' on V? Does this make V into a vector space? What about when V is the set of pairs (a, b) of real numbers satisfying $a \geq 0$? No since $-1\begin{pmatrix}2\\-1\end{pmatrix}$ is not in the set.

$\begin{pmatrix}a_1\\b_1\end{pmatrix} + \begin{pmatrix}a_2\\b_2\end{pmatrix} = \begin{pmatrix}a_1+a_2\\b_1+b_2\end{pmatrix}$ $\quad k\begin{pmatrix}a_1\\b_1\end{pmatrix} = \begin{pmatrix}ka_1\\kb_1\end{pmatrix}$ Yes. it is a vector space

2. Solving $Ax = b$

The following four problems are equivalent:

- Basic math: find $x, y \in \mathbb{R}$ such that
$$x + 2y = 5$$
$$4x + y = 6$$
- Matrix form: find $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ such that
$$\begin{pmatrix}1 & 2\\4 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$$
- Column view: find $x, y \in \mathbb{R}$ such that
$$x\begin{pmatrix}1\\4\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$$
- Row view: find $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ such that
  - Its dot product with $\begin{pmatrix}1\\2\end{pmatrix}$ is equal to 5.
  - Its dot product with $\begin{pmatrix}4\\1\end{pmatrix}$ is equal to 6.

One way to measure the complexity of an algorithm is to count the number of arithmetic operations (plus, subtract, multiply, divide) that go into it.

Problem 3. Consider a 2-variable system of equations
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_2 = b_2$$
where x_1, x_2 are the unknowns. Solve it using back substitution, i.e. solve for x_2, then substitute to find x_1. How many arithmetic operations do you need? Can you guess how many operations are needed for an n-variable system of the same "upper triangular" shape? 3

$(1+2+\dots+n) = \frac{n(n+1)}{2}$.

[1] For a full list, see https://en.wikipedia.org/wiki/Vector_space#Definition

3. Matrix transpose and inverse

Given an $m \times n$ matrix A, the transpose $A^\top$ is an $n \times m$ matrix whose rows are the columns of A. We have $(AB)^\top = B^\top A^\top$.

The matrix A is *orthogonal* if $A^\top A = I_{n\times n}$. Nonobvious fact: this implies that $m \geq n$, i.e. A is square or 'tall and skinny'. If the n columns of A are thought of as individual column vectors $v_1, \dots, v_n$, then A is orthogonal if and only if
$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.}\end{cases}$$
In this case, $v_1, \dots, v_n$ are called an *orthonormal* collection of vectors.

Problem 4. Find an orthogonal matrix Q whose rows do not form an orthonormal collection of vectors. $\begin{pmatrix}1\\1\end{pmatrix}$

Problem 5. Fill in the question marks to make an orthogonal matrix:
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & ? \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & ? \\ 0 & -\frac{1}{\sqrt{3}} & ? \end{pmatrix} \begin{pmatrix} a \\ -a \\ 2a \end{pmatrix}$$

$a^2 + a^2 + (2a)^2 = 1$

$a = \pm\frac{1}{\sqrt{6}}$ choose $\begin{pmatrix}\frac{1}{\sqrt6}\\-\frac{1}{\sqrt6}\\\frac{2}{\sqrt6}\end{pmatrix}$

How many ways are there? 2.

Given an $n \times n$ matrix A, an $n \times n$ matrix B satisfying $AB = I_{n\times n}$ is called an *inverse* of A. (Nonobvious fact: this condition implies $BA = I_{n\times n}$, and vice versa.) Any square matrix either has a unique inverse or no inverse at all. The inverse, if it exists, is denoted A^{-1}, and we say that A is *invertible*.

Nonobvious fact: a square matrix A is invertible if and only if $\det(A) \neq 0$.

Problem 6. Given *invertible* $n \times n$ matrices A and B, show that $(AB)^{-1}$ exists and equals $B^{-1}A^{-1}$.

Problem 7. Show that a square orthogonal matrix A is invertible, and satisfies $AA^\top = I_{n\times n}$ and $A^{-1} = A^\top$.

Problem 8. Show that the rows of a *square* orthogonal matrix form an orthonormal collection of vectors.

4. Block matrices

A *block decomposition* of an $m \times n$ matrix A is a way of writing A as a 'matrix of matrices':
$$A = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}$$
where each B_{ij} is a matrix. The B_{ij} don't need to all have the same size, but they need to be sized so as to fit together into an $m \times n$ grid.

Fact. Bock decompositions are compatible with matrix multiplication, for example:

If $A = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ and $C = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$,

then $AC = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} B_{11}D_{11} + B_{12}D_{21} & B_{11}D_{12} + B_{12}D_{22} \\ B_{21}D_{11} + B_{22}D_{21} & B_{21}D_{12} + B_{22}D_{22} \end{pmatrix}$,

provided that the blocks are the right sizes for the matrix multiplications to be possible.

Problem 9. Does there exist some block decomposition of some matrix whose set of blocks is as follows: $\begin{pmatrix}1\\2\end{pmatrix}$, $(3 \;\; 4)$, $(5 \;\; 6)$. No

Problem 10. If $A = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ is a block decomposition, what is the corresponding block decomposition of $A^\top$?

$A^\top = \begin{pmatrix} B_{11}^\top & B_{21}^\top \\ B_{12}^\top & B_{22}^\top \end{pmatrix}$