Recitation 1. February 12

Focus: recognizing vector spaces, rules of matrix multiplication.

Definition. A *real vector space* V is a set endowed with operations of adding two vectors and multiplying a vector by a real number such that the following holds for any three vectors u, v, w and any real numbers a, b:

- $(u+v)+w = u+(v+w)$;
- $u+v = v+u$;
- there exists a *zero vector* $0 \in V$ such that $v+0 = v$;
- $a(bv) = (ab)v$;
- $1v = v$;
- $(a+b)v = av+bv$;
- $a(u+v) = au+av$.

Elements of a vector space are called *vectors*.

Remark. The most basic rule that you should remember: $\boxed{\textbf{row column}}$. It shows the order in which you write or compute, e.g.:

- The first index denotes the row, the second number the column.
- You multiply a row by a column to get a number.
- An $n \times m$ matrix has n rows and m columns.

Notation. We will denote by A^T the transpose of a matrix A.

1. Is this a vector space? Why / why not? Which natural operations you considered when checking axioms?

   a) The line $y = x$.
   b) The line $y = x+1$.
   c) The union of the x and y axes.
   d) The unit circle $\{(x,y) \mid x^2+y^2 = 1\}$.
   e) The set of 5×5 matrices with the element in position $(3,3)$ being 0.
   f) Functions of the form $f(x) = ax^2+bx+c$.
   g) Functions $f(x)$ with $f(7) = 0$.
   h) Functions $f(x)$ with $f(0) = 7$.

2. *Rules of matrix multiplication. (Section 2.4 of Strang.)* Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix},\ C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},\ E = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

   Which of these matrix operations are allowed, and what are the results?

   a) AB
   b) AB^T
   c) B^TA
   d) $(A+B)C$
   e) $(A+B)C^T$
   f) $C(A+B)$
   g) DB

h) BD

i) AE

j) EA

k) CAE

3. When you multiply an $n \times m$ matrix by an $m \times l$ matrix, what are the dimensions of the resulting matrix?

Solution:

4. When can a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be written as $X^T X$ for some other matrix $X = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$? Assume that $b \neq 0$. What are p, q, r in terms of a, b, c, d when possible?

Solution:

5. Consider the following matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

How is each row of BA, CA, DA related to the rows of A? How is each column of AB, AC, AD related to the columns of A?

Solution:

6. *In this problem, we will practice block multiplication. (Page 75 of Strang.)* Consider the following column vector c and a 3×3 matrix A with columns a_1, a_2, a_3:

$$c = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}, A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}.$$

Write the result of matrix multiplication rA as a linear combination of the column vectors a_1, a_2, a_3. What if we write a matrix R as three rows $R = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix}$ and multiply R by A?

Solution: