Summary for Week 6 Recitation

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We will talk about least square solution and least norm (cf. Lecture 17). Since not every linear equation $A\vec{x} = \vec{b}$ can be solved, we are looking for the closest solution instead. That is finding some $\vec{x}$ such that the norm of $\vec{b} - A\vec{x}$ is the smallest. $\vec{x}$ is not unique whenever the null space of A is non-zero. It is because adding an element in null space to $\vec{x}$ will not change the value of $A\vec{x}$. In other words, those $\vec{x}$ satisfying norm of $\vec{b} - A\vec{x}$ minimal is an affine space.

What is an affine space? An affine space is very close to a vector subspace, except it does not necessarily contain zero vector. Let us first recall definition of subspace.

A vector subspace W of a vector space V has following properties.

1. W contains the zero vector of V.
2. W is closed under addition.
3. W is closed under scalar multiplication.

An affine space is a translation of some subspace W. i.e. We can find $\vec{v} \in V$ such that the affine space is the same as $\{\vec{v} + \vec{w} : \text{for } \vec{w} \in W\}$. We usually denote this affine space as $\vec{v} + W$.

Example 0.1. $\{(x, y) : x + y = 0\}$ is a subspace of $\mathbb{R}^2$. Hence $\{(x, y) : x + y = 1\}$ is an affine space. What is a translation vector $\vec{v}$?

Going back to our original problem, there are many $\vec{x}$ minimizing norm of $\vec{b} - A\vec{x}$, which one should we choose? There are no obvious preferences. Nevertheless, we could choose the $\vec{x}$ with smallest norm, i.e. closest to the origin.

Exercise 0.2. Which point of $\{(x, y) : x + y = 1\}$ is closest to the origin?

Let's work out an real example, $A = [1, 1; 1, 1]$ and $\vec{b} = [2; 0]$. One can check that $A\vec{x} = \vec{b}$ has no solutions. Null space of A is $\{(x, y) : x + y = 0\}$. The point in the column space of A closest to b is $[1; 1]$. The set $\{\vec{x} : A\vec{x} = [1; 1]\} = \{(x, y) : x + y = 1\}$. So the answer in Exercise 0.2 is what we want, the x minimize norm of $\vec{b} - A\vec{x}$ with smallest norm.

Now use Julia to check you get the correct answer!

To be continued... SVD should appear in somewhere.