Solving $Ax = b$

1. When A is invertible, i.e., if $A_{n\times n}$ has rank n, then $Ax = b$ always has ____________ solution.

2. When A is singular.

   (a) x is a least squares solution if ____________ is as small as possible.

   (b) x is a least norm if furthermore ____________ is as small as possible.

   (c) Given $A = U\Sigma V^T = U_1\Sigma_r V_1^T$, the solution $x =$ ____________ is always a least-squares, least-norm solution.

Orthogonal complement

1. If V and W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if for every $v \in V$, and $w \in W$, we have ______.

2. $V^\perp$ is the set of all $w \in \mathbb{R}^n$ such that w is perpendicular to each vector $v \in V$.

3. $V = (V^\perp)^\perp$

   - $\text{Col}(A)^\perp = \text{Leftnull}(A)$
   - $\text{Row}(A)^\perp = \text{Null}(A)$
   - $\text{Null}(A)^\perp = \text{Row}(A)$
   - $\text{Leftnull}(A)^\perp = \text{Col}(A)$

4. Suppose A is a $m \times n$ matrix with rank r, then

   - $\dim \text{Row}(A) =$
   - $\dim \text{Null}(A) =$
   - $\dim \text{Col}(A) =$
   - $\dim \text{Null}(A^T) =$

Problems

1. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (a) What is the rank of A?

   (b) What is the dimension of the solution space of $Ax = 0$? What is the dimension of the solution space of $A^T x = 0$?

   (c) Does $A^T y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ has a solution? If not, find the least norm. (Hint: First, write down a SVD for A^T.)

2. Let V be subspace of $\mathbb{R}^4$ $V = span(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix})$.

   (a) Find the orthogonal complement of V.

   (b) Write down all the solutions of $\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$.

3. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Suppose we have $AB = 0$. Show that $\text{rank}(A) + \text{rank}(B) \leq n$.