Recitation 3. March 5

Focus: exam comments, linear independence, projections.

Definition. Recall that if A is a matrix with independent columns, then the *projection matrix* P on the column space of A can be written as $P = A(A^T A)^{-1} A^T$.

1. *Recognizing vector spaces.* Is the following set a vector space?

   a) The set of all vectors in $\mathbb{R}^3$ except those of the form $\begin{pmatrix} x & 0 & 0 \end{pmatrix}^T$ with $x > 0$.
   b) The set of 2×3 matrices whose six elements sum to 6.
   c) The set of rank one 3×3 matrices together with the zero matrix.

   Solution:

2. *Vector spaces and bases.* Let V be the space of homogeneous quadratic polynomials in two variables, i.e. polynomials of the form $f(x, y) = ax^2 + bxy + cy^2$.

   a) Are elements x^2, xy, y^2 linearly independent?
   b) What about x^2, $x^2 + xy + y^2$, $xy + y^2$?
   c) And x^2, $x^2 + xy$, $x^2 + xy + y^2$?
   d) What is the dimension of this vector space V?

   Solution:

3. *Subspaces.* Let V be the space from the previous problem. Consider the subset of V that consists of functions $f \in V$ such that $f(1, 1) = 0$. Denote this subset by W – it is a vector subspace of V.

   a) Prove that W is a vector space.
   b) Find a basis of this vector space.
   c) What is the dimension of W?

   Solution:

4. *Projection onto a subspace.* Consider the following matrix A written as a full SVD $A = U\Sigma V^T$:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

a) What is the rank of A?

b) Mark U_1, U_2, V_1 and V_2.

c) Circle columns of U that span $\operatorname{col} A$.

d) Compute the projection matrix on $\operatorname{col} A$.

Solution:

5. *Bonus. Might be useful for PSet 4. (Or not.)* If A is decomposed into a product $A = BC$ with C being square invertible, then $\operatorname{col} A = \operatorname{col} B$.

6. *Bonus.* We declare x^2, xy and y^2 to be the standard orthonormal basis in V, that is we write them as follows:

$$x^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; xy = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; y^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compute the projection matrix on W.