Recitation 4. Solution

Focus: bases, four fundamental subspaces, fitting everything together.

Notation. Let V and W denote two real vector spaces.

Definition (reminder). Vectors $v_1, \ldots, v_k$ are said to be *linearly independent* if the only way to write a zero linear combination $c_1v_1+\cdots+c_kv_k = 0$ is to let all the scalars be zero: $c_1 = \cdots = c_k = 0$.

Definition (reminder). The *span*, or *linear span*, of some set of vectors $S \subset V$ is the set of all possible finite linear combinations of vectors from S, or mathematically:

$$\operatorname{Span} S = \{c_1v_1 + \cdots + c_lv_l \mid l \in \mathbb{Z};\ v_1, \ldots, v_l \in V;\ c_1, \ldots, c_l \in \mathbb{R}\}.$$

The set S can be finite or infinite, and it can be linearly independent or linearly dependent. If $\operatorname{Span} S = V$, then we say that S *generates*, or *spans*, the vector space V.

Definition (reminder). A set of vectors $v_1, \ldots, v_n$ is called a *basis* of V if it these vectors are linearly independent and span V. In this case, we say that V is n-dimensional. All bases in the same vector space have equal number of elements.

Definition. A *linear operator*, or a *linear transformation*, between vector spaces V amd W is a set function $A : V \to W$ that is linear, which means that $A(v + v') = Av + Av'$ for vectors v and v' in V, and $A(\lambda v) = \lambda Av$ for a vector $v \in V$ and a scalar $\lambda \in \mathbb{R}$.

Definition. The *image* of a linear operator $A : V \to W$ is a subset of W that consists of all vectors of the form Av for $v \in V$, or mathematically: $\operatorname{Im} A = \{Av \mid v \in V\}$.

Definition. The *kernel* of a linear operator $A : V \to W$ is a subset of V that consists of all vectors that are sent to zero, or mathematically: $\operatorname{Ker} A = \{v \in V \mid Av = 0\}$.

Definition. The *rank* of a linear operator $A : V \to W$ is the dimension of its image $\dim \operatorname{Im} A$.

1. Prove that $\operatorname{Im} A$ and $\operatorname{Ker} A$ are vector subspaces of W and V, respectively.

> **Solution:** Need to check that both are closed under addition, multiplication by a scalar and contain the zero vector.

2. How can an $m \times n$ matrix be viewed as a linear transformation? What are the dimensions of the two vector spaces?

> **Solution:** Denote this $m \times n$ matrix by A. Then we can define a map $\mathbb{R}^n \to \mathbb{R}^m$ which we also denote by A as follows: whenever we have a vector $v \in \mathbb{R}^n$, we send it to Av as defined by matrix multiplication. So we can use the words "matrix" and "linear transformation" (almost) as synonyms.

3. Let us consider a matrix A as a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $e_1, \ldots, e_n$ be the *standard basis vectors* of $\mathbb{R}^n$, that is: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. Describe $Ae_1, \ldots, Ae_n$ in terms of A. Conclusion: we can define a linear operator $A : V \to W$ by its action on a basis of V.

> **Solution:** Ae_i is the m-vector that is equal to the ith column of A.

4. Let V be the space of polynomials in two variables of the form $f(x, y) = a + bx + cy + dx^2$, and let W be the space of degree one polynomials in two variables.

a) Find (the simplest) bases of V and W. What are the dimensions of these spaces?

b) Consider a linear operator $A = \frac{d}{dx}$ from V to W. Write A as a matrix in the bases that we found in part (a).

c) What are the nullspace and column space of A? What are the kernel and image of $\frac{d}{dx}$? What is the conclusion?

d) What is the rank of A?

e) *Bonus.* Let us add twice the second column of A to the first, and denote the new matrix (linear transformation) by A'. How did the transformation change?

f) *Bonus.* Write A' as a composition of A and some other linear transformation M. What are the vector spaces that M operates between?

`Hint: recall column operations and how they are related to matrix multiplication on the right.`

g) *Added during recitation.* Compute the projection matrix on the image of $\frac{d}{dx}$ in W.

Solution:

a) $V = \text{Span}\left(e_1 = 1, e_2 = x, e_3 = y, e_4 = x^2\right)$; $W = \text{Span}\,(1, x, y)$. $\dim V = 4$; $\dim W = 3$.

b) First use problem 3 to compute columns of A:

- The first column of A is $Ae_1 = \frac{d}{dx}1 = 0 = 0 \cdot 1 + 0x + 0y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;
- The second column of A is $Ae_2 = \frac{d}{dx}x = 1 = 1 \cdot 1 + 0x + 0y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;
- The third column of A is $Ae_3 = \frac{d}{dx}y = 0 = 0 \cdot 1 + 0x + 0y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;
- The fourth column of A is $Ae_4 = \frac{d}{dx}x^2 = 2x = 0 \cdot 1 + 2 \cdot x + 0y = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Since we now know all the columns of A, we can write the matrix: $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

c) $\text{Ker}\,\frac{d}{dx} = \text{nul}\,A = \text{Span}\,(e_1, e_3)$; $\text{Im}\,\frac{d}{dx} = \text{col}\,A = \text{Span}\,(1, x)$. Conclusion: kernel is a coordinate-independent (read: fancy) word for the familiar nullspace, and image is a coordinate-independent incarnation of column space.

d) $A = \dim \text{col}\,A = \dim \text{Im}\,A = 2$.

e)

f)

g) $\text{Im}\,\frac{d}{dx} = \text{Span}\,(1, x) = \text{col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the matrix that appeared is tall skinny orthogonal. Denote the matrix by Q. Then, using that Q is tall skinny orthogonal, the projection matrix is equal to $QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

5. Fix a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ of rank r. Describe the relations between the four fundamental subspaces in terms of kernel and image. *Tricky question:* Would you be able to do that if we said that $A : V \to W$ with the same rank and dimensions of the spaces?

Solution:

- $\operatorname{col} A = \operatorname{Im} A$;
- $\operatorname{nul} A = \operatorname{Ker} A$;
- $\operatorname{row} A^T = \operatorname{Im} A^T$;
- $\operatorname{nul} A^T = \operatorname{Im} A^T$.

We cannot speak of row space and left nullspace of a general linear operator as of subspaces in V and W, respectively, because we cannot define a transpose of a linear transformation. We only know how to transpose matrices, not linear operators.

6. Fix a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Understand that if $b \in \mathbb{R}^m$ is in the image of A, then the system $Ax = b$ has a solution, say x_0. In this case, show in addition that the space of all solutions is $x_0 + \operatorname{Ker} A$. Now conclude that in the case of nonzero kernel (nullspace), the system $Ax = b$ has either infinitely many solutions or no solutions at all, depending on whether $b \in \operatorname{Im} A$ or not.