Recitation 3/9

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Linear Independence

Vectors $v_1, \ldots, v_n$ are linearly independent if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

implies all $c_i = 0$. Otherwise, they are linearly dependent.

In terms of columns of matrix A, columns of A are linearly independent if $Ax = 0$ implies $x = 0$

- QR Test : R is ____________ when the columns of A are linearly independent
- SVD Test : Σ has ________________ when the columns of A are linearly independent

Basis

We say vectors $v_1, \ldots, v_n$ **span** the vector space if every vector in the vector space can be expressed as a linear combination of $v_1, \ldots, v_n$.

$v_1, \ldots, v_n$ is called the **basis** of the vector space V if (both)
1. $v_1, \ldots, v_n$ are linearly independent
2. $v_1, \ldots, v_n$ span V.

Dimension of a vector space V is the number of vectors in the basis, n.

Problems

1. Let Q be m-by-n orthogonal matrix.
(a) Explain why orthonormal vectors are linearly independent. Then explain why it cannot be $m < n$.

(b) Find the rank-r SVD of Q. What is the rank of Q? What are the singular values of Q?

(c) What is the dimension of row(Q) and col(Q)? What about the dimension of the nullspace of Q?

2. Let A be a 3 by 5 matrix. Suppose we know the dimension of its nullspace is 3. What are the dimensions of three other fundamental subspaces? (Hint : Construct a full SVD of A)

3. (a) Find a basis for all 3 by 3 symmetric matrices. What is the dimension?

(b) Find a basis for all 3 by 3 skew-symmetric matrices.($A = -A^T$) What is the dimension?

(c) Combine both basis from (a) and (b). Are they linearly independent? Describe the span of the sum of basis.

ANSWERS

1.(a) For orthonormal vectors $v_1, \ldots, v_n$, $\sum c_i v_i = 0$ means $\sum c_i < v_1, v_i >= 0$ and by orthonormal condition we have $c_1 \times 1 = 0$. So c_1 is 0 and other c_i are also zero by similar argument. So they are linearly independent. Now, $Q^T Q = I$ implies the columns of Q are orthonormal. There can't be more than m columns which are linearly independent so it implies $n \leq m$.
(b) Q has a rank-r SVD, $Q = QII^T$, where I is n-by-n identity. So the rank is n, and all the singular values of Q are 1.
(c) In full-SVD term, $Q = QII^T$ implies $U_1 = Q$, $V_1 = I$ and V_2 doesn't exist. U_2 is a m by $m - n$ orthogonal matrix. The dimensions are dim(row(Q)) = dim(col(V_1)) = n, dim(col(Q)) = n, dim(null(Q)) = 0, dim(null(Q^T)) = dim(col(U_1)) = $m - n$

2. From full SVD, since the dimension of its nullspace is 3, we know that V_2 is 5 by 3 matrix. It implies that V_1 is 5 by 2 matrix, and the rank is 2. So we also deduce that U_1, U_2 are 3 by 2 and 3 by 1 matrix. From this we finally know that the dimension of col(A), row(A), null(A), null(A^T) are 2, 2, 3, 1.
3. (a) The basis is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the dimension is 6
(b) The basis is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the dimension is 3
(c) They are linearly independent. The span of the whole thing is all 3 by 3 matrices. Think about two basis elements

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we can construct any

$$\begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from linear combination of those two matrices. So extending this argument we can create any matrices.