The problems are here: `https://nbviewer.jupyter.org/github/mitmath/1806/blob/master/psets%20spring%202020/quiz1%20study%20questions.ipynb`.

1) a. Yes: given $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ in the space, and constants $c_1, c_2 \in \mathbb{R}$, the linear combination

$$c_1(x_1, y_1, 0) + c_2(x_2, y_2, 0) = (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, 0)$$

is also in the space.

b. Yes: given two such functions y_1, y_2, and constants $c_1, c_2 \in \mathbb{R}$, the linear combination $c_1y_1 + c_2y_2$ is also in the space, by linearity of differentiation:

$$\begin{aligned}(c_1y_1 + c_2y_2)'' &= c_1\, y_1'' + c_2\, y_2'' \\ &= c_1y_1 + c_2y_2.\end{aligned}$$

c. Yes: given two such vectors $x_1, x_2 \in \mathbb{R}^n$, and constants $c_1, c_2 \in \mathbb{R}$, the linear combination $c_1x_1 + c_2x_2$ is also in the space, by distributivity of matrix-vector multiplication, and because scalar multiplication commutes with matrix multiplication:

$$\begin{aligned}(A + B)(c_1x_1 + c_2x_2) &= c_1(A + B)x_1 + c_2(A + B)x_2 \\ &= c_1\, 0 + c_2\, 0 \\ &= 0.\end{aligned}$$

d. No: take two polynomials $P_1(x)$ and $P_2(x)$ in the space whose values at $x = 0$ are 2020. Then $P_1 + P_2$ is a polynomial whose value at $x = 0$ is 4040, so it is not in the space.

e. Yes: given two such polynomials

$$\begin{aligned}&a_1x^2 + b_1x + c_1 \\ &a_2x^2 + b_2x + c_2\end{aligned}$$

in the space, and two constants $\lambda_1, \lambda_2 \in \mathbb{R}$, the linear combination polynomial

$$(\lambda_1a_1 + \lambda_2a_2)x^2 + (\lambda_1b_1 + \lambda_2b_2)x + (\lambda_1c_1 + \lambda_2c_2)$$

is also in the space, as can be seen by multiplying the equations

$$\begin{aligned}a_1\, 1806^2 + b_1\, 1806 + c_1 &= 0 \\ a_2\, 1806^2 + b_2\, 1806 + c_2 &= 0\end{aligned}$$

by λ_1 and λ_2, respectively, and adding them together.

f. Yes: given two such matrices A_1 and A_2, and two constants $c_1, c_2 \in \mathbb{R}$, the linear combination $c_1A_1 + c_2A_2$ is also in the space. Indeed, its entries below the diagonal are linear combinations of the below-diagonal entries of A_1 and A_2, which are zero, and the sum of its entries is

$$c_1(\text{sum of entries of } A_1) + c_2(\text{sum of entries of} A_2)$$

which is zero since each summand is zero.

g. No: the zero matrix is not orthogonal.

2) a. 1, because each column of an orthogonal matrix has length 1.

b. 1.7856, because we have

$$(\text{first column of } A) = (\text{first column of } Q)\, R_{11}.$$

c. 0.824919 because $Q^\top = Q^{-1}$ for any square orthogonal matrix.

d. The answer is

$$\frac{1}{R_{44}} = \frac{1}{-0.537463}.$$

This follows from the identity

$$RR^{-1} = \mathrm{Id}_{4\times 4}$$

by looking at the entry in position $(4,4)$. Indeed, since R is upper triangular, so $R_{41} = R_{42} = R_{43} = 0$, the $(4,4)$-entry on the left hand side is $R_{44}(R^{-1})_{44}$, and this must equal 1.

e. We have

$$\begin{aligned} A^\top A &= (QR)^\top QR \\ &= R^\top Q^\top QR \\ &= R^\top R \end{aligned}$$

where we have used that $Q^\top Q = \mathrm{Id}_{4\times 4}$ since Q is orthogonal.

3) a. No.

b. Yes.

c. No.

d. Yes.

e. Yes.

f. Yes.

4) a. $Ax = b$ has a solution if and only if $b_1 = 0$.

b. The (unique) solution would be given by

$$x = \texttt{b[:,2:n+1]./v}$$

This array-indexing notation means

$$x = \begin{pmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n+1} \end{pmatrix} \texttt{./} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} b_2/v_1 \\ b_3/v_2 \\ \vdots \\ b_{n+1}/v_n \end{pmatrix}$$

Also, $x = [b_2, \ldots, b_{n+1}]\texttt{./}v$ is an acceptable answer.

5) a. Computing Wx uses $n(2n-1)$ operations (n^2 multiplications and $n(n-1)$ additions), which is roughly $2n^2$, so

$$(\text{time when } n \text{ is } 20000) \approx 4 \times (\text{time when } n \text{ is } 10000).$$

b. Solving $Rx = b$ for x takes n^2 operations, so we have the same conclusion as in the previous part.

6) a. $E = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$

b. $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$

7) We have

$$(n \times n \text{ all-ones matrix}) = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} (n) \begin{pmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{pmatrix}$$

where the matrices on the RHS have sizes $n \times 1$, 1×1, and $1 \times n$, respectively.

8) a. The answer is (c), a hyperplane.

b. The vector v.

c. $c = \frac{b}{v \cdot v}$ gives a solution.

d. The length of the vector cv is $\frac{b}{\|v\|}$.

e. The previous part says that the perpendicular line segment from the origin to the solution set of $v^\top x = b$ is given by the vector

$$\frac{b}{v \cdot v} v.$$

The distance from 0 to this solution set is the length of this vector, which was computed in the previous part.

9) The QR decomposition is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \mathrm{Id}_{4\times 4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

since this matrix is already upper-triangular. Here $\mathrm{Id}_{4\times 4}$ plays the role of the orthogonal matrix Q.

An SVD is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \mathrm{Id}_{4\times 4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \mathrm{Id}_{4\times 4}$$

since this matrix is already diagonal. Here the matrices $\mathrm{Id}_{4\times 4}$ on the left and the right play the role of U and V in the SVD.

10) This problem asks for geometric interpretations, so we think of A as giving a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, defined by sending each vector $x \in \mathbb{R}^2$ to the vector $Ax \in \mathbb{R}^2$. As the input x ranges over the unit circle in $\mathbb{R}^2$, the output Ax traces out an ellipse in $\mathbb{R}^2$.

a. The two entries of s give the lengths of the major and minor semi-axes[1] of the ellipse.

b. The columns of U give the unit vectors along the major and minor axes of the ellipse.

c. Considering the analogous transformation $x \mapsto A^\top x$, similar conclusions apply. For the (new) ellipse traced out by $A^\top x$ as x ranges on the unit circle, the entries of s give the lengths of the major and minor semi-axes, and the columns of V give unit vectors along the major and minor axes of the ellipse.

11) The QR decomposition does not exist, since in class R was required to be invertible.

Further explanation:

Recall from the Lecture 7 slides that in the QR decomposition we require R to be an *invertible* upper-triangular matrix. Since Q is orthonormal, its columns are linearly independent, and right-multiplying by the invertible matrix R does not change that property, so any matrix which admits a QR decomposition (in the sense defined in class) must have linearly independent columns. If two columns of A are collinear, then A cannot have linearly independent columns.

Another acceptable answer would be saying that if we write $A = QR$ then R might look like $\begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}$. Such a matrix would not be invertible, and both columns of A would be multiples of the *first* column of Q in this case.

[1] I.e., the major and minor radii.

12) No. For example, Gaussian elimination breaks down for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

since the first entry of the first row is zero (so the first row cannot be used to make the first entry of the second row equal to zero). However, A has an inverse; in fact, $A^{-1} = A$.