18.06 R08 - Recitation 2

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1 Practice problems for midterm 1

Problem 1. (LU factorization)

(a) Compute the LU factorization of an arbitrary 2×2 matrix, i.e. find x, u, v and w for which

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}.$$

Are there any conditions on a, b, c and/or d for this factorization to exist?

(b) Find an LU factorization for the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 5 \end{pmatrix}$$

Problem 2. (Vector spaces)
Are the following sets examples of vector spaces? If yes, then show that an arbitrary linear combination of two elements in the set is also in the set. If not, then explain why.

(a) The set of all solutions, x, to the equation $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; where $x \in \mathbb{R}^5$ and A is a 3×5 matrix.

(b) The set of all 3×2 matrices X for which $AX = 0$, where A is a fixed 5×3 matrix.

(c) The set of all 2×2 matrices A for which A^{-1} does *not* exist.

(d) The set of all differentiable functions $f(x)$ for which $f'(0) = 2f(0)$.

(e) The set of all functions $f(x)$ for which $f(x + y) = f(x)f(y)$.[1]

[1] Aside for those interested: which continuous functions obey this rule?

Problem 3. (QR factorization)
Consider the following 3×2 matrix A:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(a) Write $A = QR$, where Q is an orthogonal matrix ($Q^T Q = I$) and R is a square, upper triangular matrix. What is R^{-1}? Is Q invertible?

(b) Consider the linear system

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Can we express the right hand side of this linear system as a linear combination of the columns of A?

(c) Check that $QQ^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) Find explicitly the vector $\hat{x}$ that minimizes $\|Ax - b\|^2$.

Problem 4. (SVD)
Consider the 3×3 matrix A with the following full SVD:

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T$$

(a) Describe each of the four fundamental subspaces of A (column space, row space, nullspace, and left nullspace)

(b) Find the best rank-1 approximation to A. Write your answer as a 3×3 matrix.

(c) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ have a solution?

(d) Does $Ax = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ have a solution?

Problem 5. (Vector algebra)
Consider two vectors u and v. Find a new vector w which is in the direction of v, and for which $w - u$ is orthogonal to v. Your answer should be given in terms of the vectors u and v.[2]

Problem 6. (Least squares)
Suppose we are conducting an experiment and obtain the following m data points $\{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$. We think that our data is close to lying on a curve of the form $f(a) = x_1 + x_2 a + x_3(a-1)^2$. How would we obtain the best fit coefficients $\hat{x}_1, \hat{x}_2$ and $\hat{x}_3$? What quantity does the best fit vector $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$ minimize?

[2] Your answer w is known as the *projection* of u onto v. More on this in class tomorrow