# 1 Lecture Review

## 1.1 Fundamental Vector Spaces

Let $A$ be an $m \times n$ matrix.

1. The column space of $A$, denoted col($A$), is the set $\{Ax : x \in \mathbb{R}^n\}$. The column space is the set of linear combinations of the column vectors of $A$.
2. The null space of $A$, denoted null($A$) is the set $\{x \in \mathbb{R}^n : Ax = 0\}$.
3. The row space of $A$, denoted row($A$), is the set $\{A^T x : x \in \mathbb{R}^m\}$. The row space is the set of linear combinations of the row vectors of $A$.

## 1.2 Singular Value Decomposition (SVD) in Rank $r$ Format (Compact Form)

1. Let $A$ be an $m \times n$ matrix. The SVD of $A$ in rank $r$ format (or compact form) is a factorization of $A$ as
$$A = U\Sigma V^T$$
where $0 \le r \le m, n$ so that
   - $U$ is $m \times r$ with $U^T U = I$,
   - $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ where $\sigma_1 \ge \cdots \ge \sigma_r > 0$,
   - $V$ is $n \times r$ with $V^T V = I$.
2. We have col($A$) = col($U$).
3. $Ax = b$ is solvable if and only if $UU^T b = b$.
4. If $\mathbf{u}_1, \dots, \mathbf{u}_r$ and $\mathbf{v}_1, \dots, \mathbf{v}_r$ are the respective column vectors of $U$ and $V$, then the rank $k$ (for $k \le r$) approximation of $A$ is the $m \times n$ matrix
$$A_k = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_k \end{pmatrix} \operatorname{diag}(\sigma_1, \dots, \sigma_k) \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix}^T.$$

### 1.2.1 Singular Value Decomposition (SVD) in Full Form

1. Let $A$ be an $m \times n$ matrix. The SVD of $A$ in full form is a factorization of $A$ as
$$A = \mathbf{U\Sigma V}$$
where
   - $\mathbf{U}$ is $m \times m$ with $\mathbf{U}^T\mathbf{U} = I$,
   - $\mathbf{\Sigma}$ is $m \times n$ diagonal with $\sigma_1 \ge \cdots \ge \sigma_r > 0$ along the diagonal,
   - $\mathbf{V}$ is $n \times n$ with $\mathbf{V}^T\mathbf{V} = I$.
2. The matrices $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ from the full form SVD are related to the matrices $U, \Sigma, V$ from the compact form in the following way
   - If $\mathbf{U} = \left( \begin{array}{c|c} U_1 & U_2 \end{array} \right)$ as a block matrix with $m \times r$ matrix $U_1$ then $U_1 = U$.
   - $\mathbf{\Sigma} = \left( \begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array} \right)$.
   - If $\mathbf{V} = \left( \begin{array}{c|c} V_1 & V_2 \end{array} \right)$ as a block matrix with $n \times r$ matrix $V_1$ then $V_1 = V$.
3. We have null($A$) = col($V_2$) if $V_2$ is not an empty block, or equivalently if $r < n$.

# 2 Computation

## 2.1 2 Column QR Decomposition

1. Consider a $m \times 2$ matrix $A$ with column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If $\mathbf{a}_1$ is not a multiple of $\mathbf{a}_2$, then the QR decomposition can be computed by the following steps:

   (a) Compute $\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}$.

   (b) Compute $\mathbf{b} = \mathbf{a}_2 - \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1$.

   (c) Compute $\mathbf{q}_2 = \frac{\mathbf{b}}{\|\mathbf{b}\|}$.

   (d) Then

$$Q = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{pmatrix}, \quad R = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 \end{pmatrix}$$

## 2.2 Rank 1 SVD in Compact Form

1. To check a matrix is rank 1, check that the column vectors are all multiples of one another.

2. Suppose $A$ is an $m \times n$ matrix with rank 1. You can write $A = \mathbf{x}\mathbf{y}^T$ where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ as follows:

   (a) Choose a nonzero column of $A$, set it equal to $\mathbf{x}$.

   (b) Find $\mathbf{y} = (y_1, \ldots, y_n)$ so that $y_i\mathbf{x}$ is the $i$th column of $A$.

3. Given $A = \mathbf{x}\mathbf{y}^T$ nonzero, we can obtain the SVD for $A$ in compact form:

$$A = U\Sigma V^T$$

   where

$$U = \frac{\mathbf{x}}{\|\mathbf{x}\|}, \quad \Sigma = \begin{pmatrix} \|\mathbf{x}\|\|\mathbf{y}\| \end{pmatrix}, \quad V = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

   Above, $U$ is $m \times 1$, $\Sigma$ is a $1 \times 1$ matrix, and $V$ is $n \times 1$.

# 3 Problems

1. Compute the column spaces of the following matrices

(a) $\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

2. Compute the null spaces of the following matrices

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$.

3. Find the singular values for the following rank 1 matrices

(a) $\begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{pmatrix}$

4. Find the singular values of the matrices

(a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (b) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$

5. Suppose you are given an $m \times n$ matrix $A$ and its SVD $(U, \Sigma, V)$. Find the SVD of the following matrices in terms of $U, \Sigma, V$ and describe the column spaces in terms of the column or row space of $A$

   (a) $A^T$,

   (b) $A^{-1}$ assuming $m = n$ and $A$ is invertible,

6. Suppose $A$ is $3 \times 2$ with SVD in full form

$$A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Check that this is an SVD in full form for $A$. How would you find the SVD in compact form for $A$ from the full form?

7. Find the QR decomposition of the following matrices

(a) $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & 3 \\ 2 & 4 \\ 0 & 4 \end{pmatrix}$

# 4 Answers

1. (a) the plane $x - y = 0$, (b) $z = 0$ plane, (c) $\mathbb{R}^3$
2. (a) $\{0\}$, (b) $\{t\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) : t \in \mathbb{R}\}$, (c) $\{t\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) : t \in \mathbb{R}\}$
3. (a) 10, (b) $\sqrt{21}$, (c) $\sqrt{70}$
4. (a) $1, 1$, (b) $1, 1$
5. See solutions
6. -
7. See solutions