18.06

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Review for Midterm

Main Topics:

- Projection
- Span/Linear Independence/Basis
- Four Fundamental Subspaces/Fundamental Theorem of Linear Algebra
- Complete solution to $Ax = b$
- Orthogonal Subspaces
- Fundamental Subspace “picture”
- Abstract Linear Transformations
- Matrix Calculus
- Determinants

Projection

Main idea:

Given vector space V, vector $b \in V$, and subspace $W \subseteq V$

Want to find $\tilde{b} \in W$ that is the best approximation to b

Two Important Cases:

Projection onto line through origin in $\mathbb{R}^2$

Geometric intuition

Projection onto $\mathrm{col}(U)$, U orthogonal matrix

Main use of projection seen in 18.06

Projection Onto Line Through Origin in $\mathbb{R}^2$

Given:

vector $b \in \mathbb{R}^2$

line l through origin

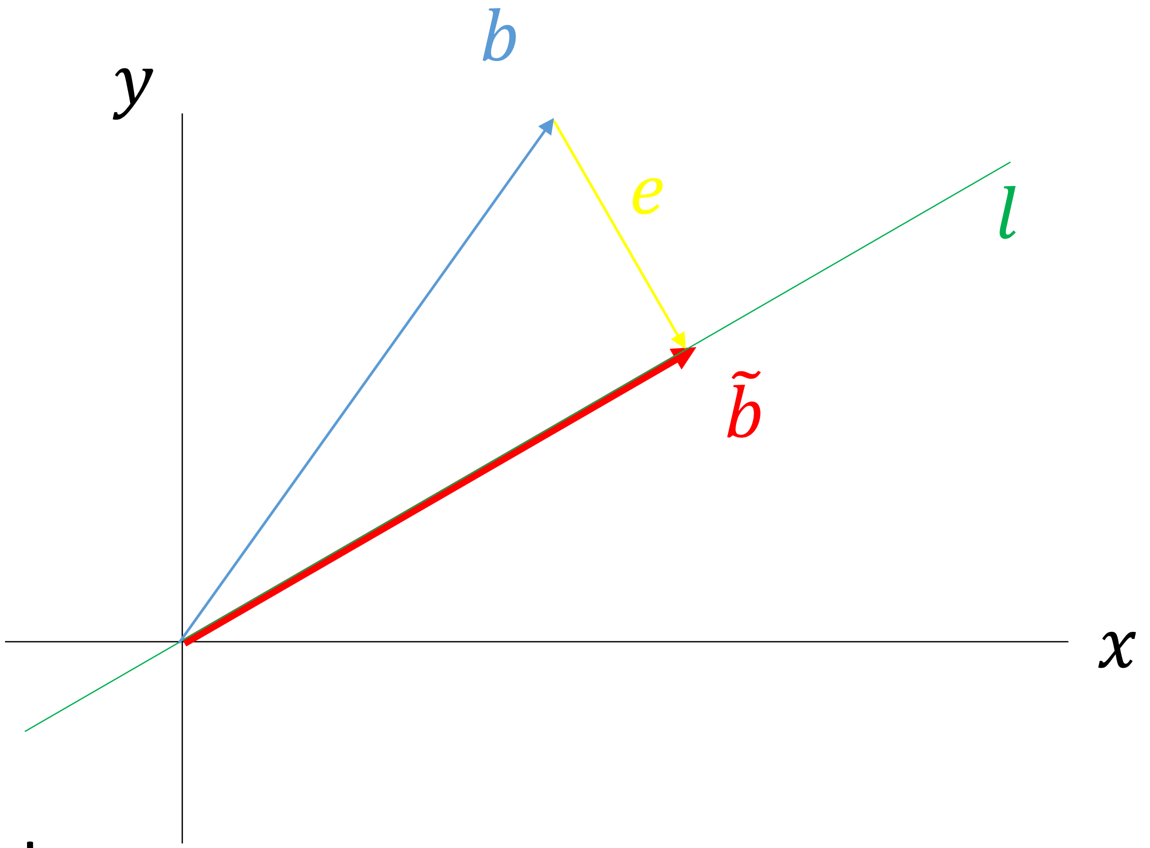
Want:

vector $\tilde{b}$ points along l

$\tilde{b}$ as close to b as possible

$e = \tilde{b} - b$ as small as possible

e orthogonal to $\tilde{b}$



Projection onto col(U): U orthogonal matrix

Given:

vector $b \in \mathbb{R}^m$

orthogonal $m \times n$ matrix U

Want:

vector $\tilde{b} \in$ col(U)

$\tilde{b}$ as close to b as possible

How:

$\tilde{b} = UU^\top b$

$UU^\top$ is "projection matrix" transforms b to $\tilde{b}$

Span/Linear Independence/Basis

Given vector space V over $\mathbb{R}$ (e.g., $\mathbb{R}^3$, col(A))
And vectors $v_1, v_2, \dots, v_k \in V$

Span of $v_1, v_2, \dots, v_k$ is set of all $x \in V$ such that can write x as lin. combo. of $v_1, v_2, \dots, v_k$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k, \text{ for } \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$$

Say $v_1, v_2, \dots, v_k$ *span* W, where W subspace of V if, W contained in span of $v_1, v_2, \dots, v_k$

Say $v_1, v_2, \dots, v_k$ are *linearly independent* if

No non-trivial lin. combo. of $v_1, v_2, \dots, v_k$ is zero (zero vector)

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ only when $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

Say $v_1, v_2, \dots, v_k$ are a *basis* of V if they span V and are linearly independent

Four Fundamental Spaces

A $m \times n$ matrix

$\operatorname{col}(A)$

$\operatorname{row}(A)$

$\operatorname{null}(A)$

$\operatorname{null}(A^{\mathsf{T}})$

Many questions about A can be answered by understanding these spaces

Fundamental Theorem of Linear Algebra

A $m \times n$ matrix, of rank r, where $A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$

col$(A) =$ col(U_1)
row$(A) =$ col(V_1)
null$(A) =$ col(V_2)
null$(A^\top) =$ col(U_2)

Tells us “everything” about A

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

When does $Ax = b$ have at least one solution?

Exactly when $b \in \text{col}(A)$

Exactly when $b \in \text{col}(U_1)$

Exactly when $U_1 {U_1}^\top b = b$

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

If $Ax = b$ has at least one solution, when does it have only one solution?

Exactly when $\text{null}(A) = \{0\}$ (only contains the zero vector)

Exactly when $\text{col}(V_2) = \{0\}$

Exactly when $r = n$

General Solution to $Ax = b$

A $m \times n$ matrix, of rank r

If $Ax = b$ has at least one solution, what are all of the solutions?

$x_p = V_1 \Sigma_r^{-1} U_1^\top b$ is a solution ($A = U_1 \Sigma_r V_1^\top$)

$x_p + \text{null}(A)$ is the set of all solutions

$x_p + \text{col}(V_2)$ is the set of all solutions

Minimization: Subspaces

Consider $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x) = ||Ax - b||^2$

Goal: Minimize $f(x)$

Idea: For any x, if $y = Ax$, then $y \in \mathrm{col}(A)$

and if $y \in \mathrm{col}(A)$, $y = Ax$ for some x

Minimum occurs when $y \in \mathrm{col}(A)$ as close to b as possible

By definition, when $y = \tilde{b}$

So any solution to $Ax = \tilde{b}$ works

Orthogonal Spaces

Vector space V

We say vectors $u, w \in V$ are *orthogonal* when $u \cdot w = 0$

Given subspaces R, S of V, say R, S are *orthogonal* when

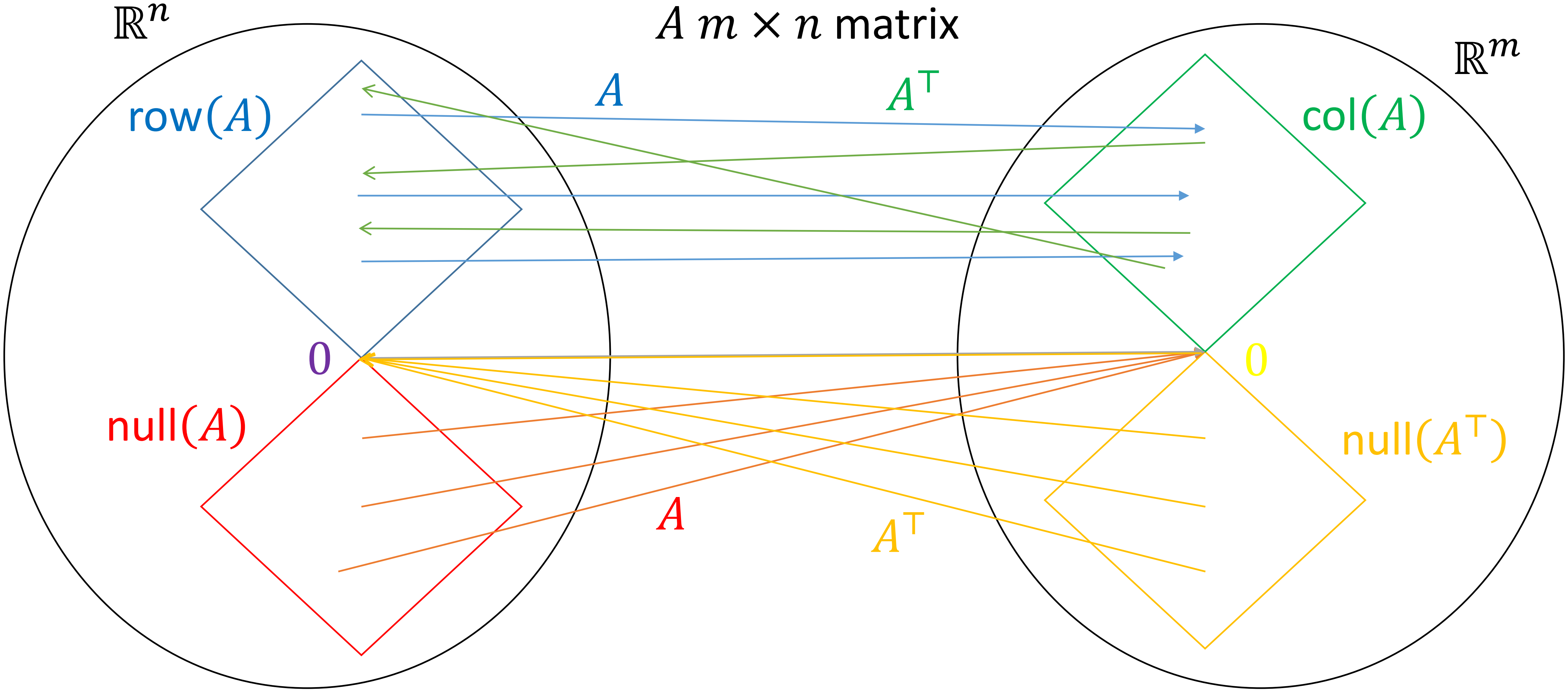
For every $r \in R$ and $s \in S$

$$r \cdot s = 0$$

For R subspace of V, $R^{\perp}$ is space of all $v \in V$ where

$r \cdot v = 0$ For every $r \in R$

Four Fundamental Spaces



Linear Transformations

For V, W vector spaces (over $\mathbb{R}$)

Say a function T from V to W is linear if

For all $x_1, x_2 \in V$

and all $c_1, c_2 \in \mathbb{R}$

$$T(c_1 x_1 + c_2 x_2) = c_1 T(x_1) + c_2 T(x_2)$$

Vector and Matrix Calculus

$f\colon \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta \mathrm{x} \qquad f'\colon \mathbb{R} \longrightarrow \mathbb{R}$$

$$df(x) = f'(x)dx$$

$f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(v+\Delta v) \approx f(v) + \left(\nabla f(v)\right)^{\top}\Delta v \qquad \nabla f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$df(v) = \left(\nabla f(v)\right)^{\top}dx$$

$f\colon \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$

$$f(A+\Delta A) \approx f(A) + \mathrm{tr}((Df(A))^{\top}\Delta \mathrm{A})$$

D differential operator, transforms function f to new function Df

$$Df\colon \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$$

$$df(A) = \mathrm{tr}((Df(A))^{\top}dA)$$

Matrix Calculus

Example: $f(A) = (Ax - b)^\top (Ax - b)$

$x \in \mathbb{R}^n, b \in \mathbb{R}^m$ fixed vectors

$A \in \mathbb{R}^{m \times n}$ varying matrix

$f \colon \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}$

Want expression for $Df(A)$

Matrix Calculus

Example: $f(A) = (Ax - b)^\top (Ax - b)$, Want expression for $Df(A)$

By definition: $df(A) = \mathrm{tr}((Df(A))^\top dA)$

By "product rule":
$$\begin{aligned} df(A) &= ((dA)x)^\top (Ax - b) + (Ax - b)^\top ((dA)x) \\ &= 2(Ax - b)^\top \left((dA)x\right) \\ &= 2(Ax - b)^\top \, (dA)(x) \\ &= \mathrm{tr}(2(Ax - b)^\top (dA)(x)) \\ &= \mathrm{tr}(2(x)(Ax - b)^\top (dA)) \\ &= \mathrm{tr}((2(Ax - b)(x)^\top)^\top (dA)) \end{aligned}$$

$Df(A) = 2(Ax - b)x^\top$

Determinant

det : $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$, denote $\det(A)$ or $|A|$

Defining axioms:

- $\det(I) = 1$
- Exchanging any two rows of A negates $\det(A)$
- det linear function of each row
  - For any row i
    - Fix all entries in A *outside* row i
    - det linear function of row i

Determinant: Important Properties

For any $n \times n$ matrices A, B

- $|A| = 0$ if and only if A is singular
    - When $|A| \neq 0$, $|A^{-1}| = \frac{1}{|A|}$
- $|AB| = |A||B|$
- $|A^{\top}| = |A|$
- If A has a row of all zeros, then $|A| = 0$
- If A has two identical rows, then $|A| = 0$
- Adding a multiple of one row of A to another row does not change det

Minimization: Matrix Calculus

Minimize $f(x) = \|Ax - b\|^2 = (Ax - b)^\top (Ax - b)$

Idea: Compute $\nabla f(x)$, equate to 0, and solve

$$\begin{aligned} df &= (Adx)^\top (Ax - b) + (Ax - b)^\top (Adx) \\ &= (Adx)^\top (Ax - b) + (Adx)^\top (Ax - b) \\ &= 2(Adx)^\top (Ax - b) \\ &= 2(dx)^\top A^\top (Ax - b) \\ &= \left(2A^\top (Ax - b)\right)^\top dx \end{aligned}$$

$$df = \left(\nabla f(x)\right)^\top dx$$

$$\nabla f(x) = 2A^\top (Ax - b)$$

The Equation $A^\top(Ax - b) = 0$

Does this equation always have a solution?

Yes!

Boring case: if $b \in \text{col}(A)$

Any solution to $Ax = b$ works

Interesting case: if $b \notin \text{col}(A)$

$\tilde{b} \in \text{col}(A)$ projection of b onto $\text{col}(A)$

$\tilde{b} - b$ orthogonal to $\text{col}(A)$

$\tilde{b} - b \in \text{null}(A^\top)$

$A^\top\left(\tilde{b} - b\right) = 0$

Any solution to $Ax = \tilde{b}$ works

Minimization: Matrix Calculus

Minimize $f(x) = \|Ax - b\|^2 = (Ax - b)^\top (Ax - b)$

Idea: Compute $\nabla f(x)$, equate to 0, and solve

$\nabla f(x) = 2A^\top (Ax - b)$

Solve $A^\top (Ax - b) = 0$

$\tilde{b}$ projection of b onto col(A)

Any solution to $Ax = \tilde{b}$ works

Can find $\tilde{b}$ and solution to $Ax = \tilde{b}$ with SVD