Summary of 18.06 in first four weeks

Yau Wing Li

March 23, 2020

Since many examples and algorithms illustrated in my recitation can be found in the book/internet, there is no point for re-uploading. Therefore I decide to write a brief summary of what I have said.

Why linear algebra? Well, one reason is it is easier than non-linear algebra. Try to solve $x^2 + 9.81x - 10 = 0$ and $3x + 1 = 0$. Which one is easier? In the language of Physicist, linearity means no acceleration.

The most fundamental objects in linear algebra are matrices and vectors. Given a matrix A and vector b, we would to study $Ax = b$ $(*)$, the most important equation in linear algebra.

What can we ask about this equation? There are so many questions could be asked. Let me try to write down some of them.

1. Is $(*)$ solvable?
2. How many solutions if solvable?
3. How to find solutions explicitly?
4. What can we say when there are no solutions?

Each question above can be answered by software Julia whenever A and b are given explicitly.

These questions motivate many concepts in our class. Here is an explanation. For the illustration, we fix a matrix A.

For question (1), $(*)$ is solvable exactly when b is inside col(A), the **column space** of A. One can further ask how large is this space. Well, one can use dimension or "number of parameters" or "degree of freedom" to describe the size of vector space. In this case, it is called **rank** of the matrix A.

For question (2), which vector b is always solvable no matter what A is? The zero vector! Let's start studying the relatively easier equation $Ax = 0$ first. We give a specific name for the solution space, the **null space** of A. Return to the general equation $(*)$, suppose y is a solution of $Ax = b$, that is $Ay = b$. If z is in null space of A, what is the value of $A(y + z)$? Conversely, if y' is a solution of $Ax = b$, what is the value of $A(y - y')$? After computing these two values, you should be able to conclude that solving $(*)$ amounts to find a particular solution y and the null space of A.

For question (3), Gaussian elimination is the main tool. Doing elementary row operations several times, equivalently **left** multiplying many elementary row matrices, reduces A to **reduced row echelon form**. In some cases, A has a **LU factorization**, then $(*)$ is equivalent to $x = U^{-1}L^{-1}b$. When the sizes of L and U are small, it is very easy to compute their inverses. The underlying philosophy is the number of zeroes that A contains is proportional to the complexity of $(*)$.

For question (4), suppose we know Ax can never be b, the most natural thing we should do is to find an approximate solution. In other words, find an x to minimize the **norm** of $Ax - b$. Let z be such particular choice, Pythagoras theorem implies Az has to be the **orthogonal projection** of b on col(A). It turns out to be the same as solving $A^T b = A^T Az$ $(**)$. Remember the homework question about best-fit line? If we know the **QR factorization** of A, then $A^T A = R^T R$ and $(**)$ simplifies enormously. In the class, I have explained how to find QR factorization via **Gram-Schmidt process**.