Week 3 Review Session

Focus: rules of matrix multiplication, orthogonal matrices, rotation matrices.

1. When you multiply an $n \times m$ matrix by an $m \times l$ matrix, what are the dimensions of the resulting matrix?

Solution:

2. *Zero scalar vs zero vector vs zero matrix.* Let A be an $n \times m$ matrix, B be an $m \times l$ matrix, v be a column m-vector and r be a row m-vector, for example:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{pmatrix},\ v = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix},\ r = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}.$$

In this case, products AB, Av and rv are all zero, so we can write $AB = 0$, $Av = 0$, $rv = 0$. But would it make sense to write $AB = Av = rv = 0$? Why / why not? Do the results of those operations belong to the same vector space?

Solution:

3. Answer the following questions. Provide explanations.

   a) Is the identity matrix always square? (By the way it can be stored with one parameter.)
   b) Do rectangular matrices have inverses?
   c) Do all square matrices have inverses?
   d) What is the condition for a 2×2 matrix to have inverse?

Solution:

4. *Parallel planes.*

   a) Consider the set of points $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ satisfying condition $2x + 3y + 4z = 0$. Describe this geometric object. Find a normal vector to it.

   b) What if we consider the equation $2x + 3y + 4z = 1$? Why is the normal the same?

Solution:

5. *Row and column operations as matrix multiplication. (Inspired by problem 2.4.8 from Strang.)* Consider the following matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

   How is each row of BA, CA, DA related to the rows of A? How is each column of AB, AC, AD related to the columns of A?

Solution:

6. *(Problem 2.4.5 from Strang.)* Compute A^2 and A^3. Make a prediction for A^5 amd A^n:

   a) $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$;

   b) $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$.

Solution:

7. *Binomial formula for matrices. Matrices do not commute. (Problem 2.4.6 from Strang.)* Show that $(A+B)^2$ is different from $A^2+2AB+B^2$ when

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}.$$

Write down the correct rule: $(A+B)^2 = A^2 + \cdots + B^2$. Can you generalize the rule to $(A+B)^n$?

Solution:

8. *(Problem 2.4.7 from Strang.)* Is the following true or false? Give counterexamples when false. Matrices A, B and C are such that all the operations are well-defined.

   a) If columns 1 and 3 of B are the same, then so are columns 1 and 3 of AB.
   b) If rows 1 and 3 of B are the same, then so are rows 1 and 3 of AB.
   c) If rows 1 and 3 of B are the same, then so are rows 1 and 3 of ABC.
   d) $(AB)^2 = A^2B^2$.

Solution:

9. *Orthogonal matrices.* Find A^TA if the columns of A are unit vectors, all mutually perpendicular (in this situation, we say that the vectors are *orthonormal*). What if we ask that the rows of A are orthonormal?

Solution:

10. *Defining a matrix by its image. Rotation matrices.* Work out these questions for 2×2 matrices.

a) If we want a matrix A to send vector $\binom{1}{0}$ to twice itself and vector $\binom{0}{1}$ to $\binom{a}{b}$, then what are the matrix entries of A?

b) If we want a matrix B to send vector $\binom{1}{0}$ $\binom{a}{b}$ and vector $\binom{1}{1}$ to twice itself, then what are the matrix entries of B?

c) What are the coordinates of vectors $\binom{1}{0}$ and $\binom{0}{1}$ after we rotate them by the angle θ?

d) How do you write a matrix that rotates every vector in the plane by the angle θ?

e) Is the matrix in the previous part orthogonal?

11. *Angles between vectors.* Consider an n-cube C – the set of points in $\mathbb{R}^n$ all of whose coordinates vary from 0 to 1.

   a) In a two-dimensional cube, find the angle between the diagonal and an edge.
   b) In a three-dimensional cube, find the angle between the long diagonal and an edge.
   c) In a three-dimensional cube, find the angle between the long diagonal and a face.
   d) In an n-dimensional cube, find the angle between the long diagonal and an edge.