# 1 Lecture Review

## 1.1 Cofactors and Cramer's Rule

Let $A$ be a square $n \times n$ matrix.

1. The *cofactor* $C_{ij}$ is defined by
$$C_{ij} = (-1)^{i+j} \det M_{ij}$$
where $M_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by removing the $i$th row and $j$th column from $A$.

2. (Compute Determinant by Cofactors) For any $1 \leq i \leq n$,
$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

3. If $A$ is invertible, then
$$A^{-1} = \frac{1}{\det A} C^T.$$
In terms of entries, $(A^{-1})ij = C_{ji}/\det A$.

4. (Cramer's Rule) If $A$ is invertible and $A\boldsymbol{x} = \boldsymbol{b}$, then
$$x_j = \det(A \text{ with column } j \text{ changed to } \boldsymbol{b})/\det A.$$

# 2 Problems

1. Let $A$ be $n \times n$. Explain using the full form SVD why $\det A \neq 0$ if and only if the rank of $A$ is $n$.

   *Solution.* Suppose $A = U\Sigma V^T$ is the full form SVD. Since $U$ and $V$ are orthogonal,

   $$\det U = \pm 1, \quad \det V = \pm 1.$$

   Then

   $$\det A = \det(U)\det(\Sigma)\det(V^T) = \pm\det(\Sigma).$$

   Since $\Sigma$ is the diagonal matrix with $\sigma_1, \ldots, \sigma_r$ along the first $r$ diagonal entries and 0 for the rest, we have

   $$\det A = \sigma_1 \cdots \sigma_n \neq 0$$

   if the rank $r = n$, otherwise if $r < n$ then $\det A = 0$. □

2. If $I$ is the $n \times n$ identity and $a$ is a scalar, what is $\det(aI)$?

   *Solution.* We have
   $$\det(aI) = \begin{vmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{vmatrix} = a^n.$$
   $\square$

3. Compute the determinant

$$\begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & -1 & 1 & \cdot & \\ & & \cdot & \cdot & 1 \\ & & & -1 & 1 \end{vmatrix}$$

of the $n \times n$ matrix by cofactor expansion.

*Solution.* Denote the matrix by $T_n$. By cofactor expansion

$$|T_n| = |T_{n-1}| - \left|\begin{array}{c|c} -1 & * \\ \hline 0 & T_{n-2} \end{array}\right| = |T_{n-1}| + |T_{n-2}|$$

where we expanded again to get the second equality. We can compute

$$|T_1| = 1, \quad |T_2| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

Then $|T_n| = F_{n+1}$ where $\{F_n\}$ is the Fibonacci sequence since

$$|T_1| = F_2, |T_2| = F_3$$

and both satisfy the same recurrence, that is $F_{n+1} = F_n + F_{n-1}$ and $|T_n| = |T_{n-1}| + |T_{n-2}|$. $\square$

4. If $A$ is $n \times n$ and invertible and $C$ is its cofactor matrix, show that

   (a) $AC^T = \det(A)I$,

   (b) $\det C = (\det A)^{n-1}$.

   *Solution.* (a) Since $A^{-1} = C^T / \det A$, we have

   $$I = AA^{-1} = \frac{1}{\det A} AC^T.$$

   Then multiply both sides by $\det A$.

   (b) Set $a = \det(A)$. Then

   $$aI = AC^T.$$

   so that

   $$a^n = \det(aI) = \det(A)\det(C^T) = a \det C.$$

   This shows $\det C = a^{n-1} = (\det A)^{n-1}$.

$\square$

5. Let $Q$ be a (square) orthogonal matrix. Find the cofactor matrix of $Q$ up to sign. Explain how the sign is affected by the sign of $\det Q$.

   *Solution.* Since $Q^T Q = I$, we have $Q^T = Q^{-1} = C^T / \det(Q)$. Thus
   $$C = \det(Q) \cdot Q = \pm Q.$$
   In particular, $C = Q$ if $\det Q = 1$ and $C = -Q$ if $\det Q = -1$. $\square$

6. Suppose $A$ is an invertible $n \times n$ square matrix and $B$ is a known $n \times m$ matrix.

   (a) If you want to solve
   $$AX = B$$
   where $X$ is an $n \times m$ unknown matrix using Cramer's rule, in general how many determinants do you need to compute?

   (b) How many determinants do you need to compute $A^{-1}$ by cofactors?

   (c) Compare (6a) and (6b). In particular, how is (6b) a special case of (6a)?

*Solution.* (a) Let $\boldsymbol{x}_i$ denote the $i$th column of our unknown matrix $X$ and $\boldsymbol{b}_i$ denote the $i$th column of our known matrix $B$. The equation $AX = B$ can be reexpressed as $m$ equations
$$A\boldsymbol{x}_i = \boldsymbol{b}_i, \quad 1 \le i \le m.$$
Applying Cramer's rule for each $i$, we need to compute the determinant of $A$ with the $j$th column replaced by $\boldsymbol{b}_i$ for each $j = 1, \ldots, n$; this yields $n$ determinant computations for each $i$. Since there are $m$ different values for $i$, we require $mn$ determinant computations across $1 \le i \le m$, $1 \le j \le n$. Finally, we need to divide by $\det A$ in each application of Cramer's rule so we must compute the determinant of $A$. So we have $mn + 1$ determinants to compute.

(b) To compute $A^{-1}$, we need to compute $C_{ij}$ for $1 \le i, j \le n$. Each $C_{ij}$ is $(-1)^{i+j}$ times the determinant of $A$ with the $i$th row and $j$th column removed. This requires $n^2$ determinants to be computed. Finally, we need to divide by $\det A$ so we must compute the determinant of $A$. So we have $n^2 + 1$ determinants to compute.

(c) If we set $m = n$ and $B = I$, then (6a) is exactly the problem of solving for the inverse matrix $X = A^{-1}$. In this case, both (6a) and (6b) take the same number of determinants. In fact both methods can be seen as computing the same determinants (with a slight modification).

$\square$

7. Let $A$ be an $n \times n$ matrix with row vectors $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_n^T$. The determinant of a matrix is a linear transformation of each row (and column). Consider the function $f(\boldsymbol{x})$ which replaces the first row $\boldsymbol{a}_1^T$ of $A$ with $\boldsymbol{x}^T$, that is

$$f(\boldsymbol{x}) = \begin{vmatrix} - & \boldsymbol{x}^T & - \\ - & \boldsymbol{a}_2^T & - \\ & \vdots & \\ - & \boldsymbol{a}_n^T & - \end{vmatrix}.$$

Since $f$ is a linear transformation from $\mathbb{R}^n \to \mathbb{R}$, we may write

$$f(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

for some $\boldsymbol{w} \in \mathbb{R}^n$. Find $\boldsymbol{w}$ in terms of cofactors of $A$.

*Solution.* Let $A(\boldsymbol{x})$ be the matrix $A$ with the first row replaced by $\boldsymbol{x}^T$. Let $C_{ij}$ denote the cofactors of $A$. By cofactor expansion

$$f(\boldsymbol{x}) = \det A(\boldsymbol{x}) = x_1 C_{11} + x_2 C_{12} + \cdots + x_n C_{1n}.$$

Therefore the vector $\boldsymbol{w}$ we seek is $w_i = C_{1i}$. $\square$