Recitation 6. April 2

Focus: matrix calculus, determinants.

Three axioms of determinants:

(i) Normalization: $\det I = 1$.

(ii) Sign reversal (antisymmetry): when we exchange any two rows, the determinant flips sign.

(iii) The determinant is a linear function of any row, if all other rows are fixed.

Computing Determinant: Suppose A is a square matrix.

(i) Check if the $(1,1)$ entry is nonzero. If not, flip two rows so that this is the case. If this is impossible, then the first column is 0 and $\det(A) = 0$.

(ii) Using row operations, add the first row to the other rows until all the entries below the $(1,1)$ entry in the first column are 0.

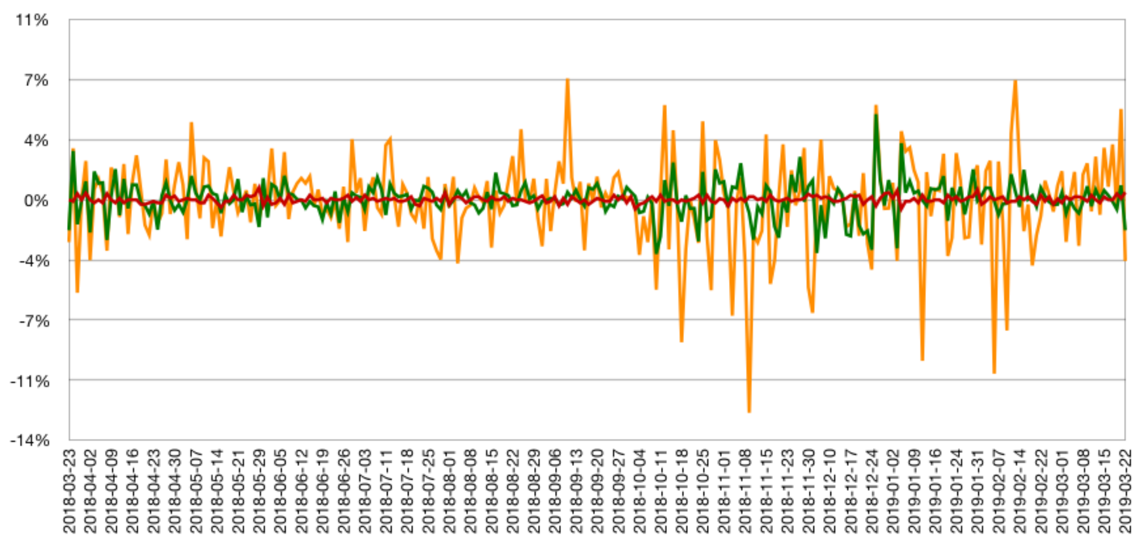
(iii) Repeat from the beginning ignoring the first column and row. If this iterates until the matrix is upper triangular, then

$$\det(A) = (-1)^{\text{number of row flips}} \times \text{product of diagonal entries of the upper triangular matrix.}$$

Note. As a consequence, we have that the same axioms are satisfied by columns as well. Don't forget that determinants are defined only for square matrices.

1. *Portfolio optimization.* In this problem, we will consider three financial instruments: Dow Jones Industrial Average index (first coordinate), Activision Blizzard equities (second coordinate) and Fidelity US Bond Index fund (third coordinate) – and will try to figure out what would be the optimal investment in those. Note that this cannot be used to devise your investment strategy, because the expectation of future returns is very hard to estimate. For the purpose of this problem, we will assume that expected returns are proportional to standard deviation of these instruments.

   a) We won't define what standard deviation means, but we should intuitively understand it as a measure of how far values tend to spread from the mean. Study the following plots of daily percentage gains/losses of the three instruments. (So value 2 on day D means that the price of the instrument rose by 2% compared to day $D-1$.)



   Knowing that standard deviation of gains of Dow Jones is $\sigma_1 = 1$, of Blizzard stock – $\sigma_2 = 2.6$, of US bonds – $\sigma_3 = 0.2$, make a guess which plot corresponds to which instrument.

   b) We write how much we invest in each instrument as a vector $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, so here we invest k\$$w_1$ in Dow Jones, k\$$w_2$ in Activision Blizzard and k\$$w_3$ in US bonds. We call this vector w the *investment portfolio*.

c) Covariance is a measure of the joint variability of two random variables. The (approximate) covariance matrix C for the given instruments, and its inverse are the following:

$$C = \begin{pmatrix} 1 & 1.2 & 0 \\ 1.2 & 6.8 & -0.1 \\ 0 & -0.1 & 0.04 \end{pmatrix}, \; C^{-1} = \begin{pmatrix} 1.3 & -0.23 & -0.6 \\ -0.23 & 0.2 & 0.5 \\ -0.6 & 0.5 & 26.2 \end{pmatrix}.$$

The importance of the covariance matrix is that we can compute variance of a linear combination of stocks (portfolio) w easily via the formula $w^T C w$. Note that variance is the square of standard deviation.
Compute the variance of the investment of k\$1 in Blizzard and k\$1 in US bonds. Explain why the diagonal entries of C are exactly the squares of standard deviations: $C_{ii} = \sigma_i^2$.

d) Modern theory of portfolio optimization says that we want to maximize "Sharpe ratio" – the ratio of expected return to the standard deviation of the portfolio. Assume that the expected return is given by vector $\mu = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$.
Verify that the Sharpe ratio of the portfolio w is given by the following formula:

$$\mathrm{SR} = \frac{\mu^T w}{\sqrt{w^T C w}}.$$

e) If $f(w)$ is a scalar function of w, i.e. its values are 1×1 matrices – real numbers, compute $\mathrm{d}\left(f(w)^{-1}\right)$ using the product formula for $0 = \mathrm{d}1 = \mathrm{d}\left(f(w)f(w)^{-1}\right)$. Differentials are taken with respect to w.

f) Compute the differential $\mathrm{d}(\mathrm{SR})$ with respect to w.

g) Write the condition for finding a local extremum of SR.

h) Conclude that w should be parallel to $C^{-1}\mu$, and argue that scaling w does not affect Sharpe ratio, therefore we can take $w = C^{-1}\mu$.

i) Use the above formula and the given values of μ and C^{-1} to obtain the weights of the optimal portfolio.

2. Using row operations, show that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

3. *(Parts of problems 5.1.14-15 from Strang.)* Use row operations to simplify and compute:

a) $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{vmatrix} =$

b) $\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} =$

4. Prove that $\det\left(A^{-1}\right) = (\det A)^{-1}$.