Recitation 4/14

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Volumes, Matrix Calculus

A region in $\mathbb{R}^n$ is transformed into another region in $\mathbb{R}^n$ under the left multiplication of $A \in \mathbb{R}^{n \times n}$.(i.e., all the vectors inside the region is multiplied on the left by A)

$\det(A)$ is the scaling factor of volumes between two regions.

2 by 2 SVD explanation

Matrix Calculus - Remember $d(AB) = (dA)B + A(dB)$

Eigenvalues

For a square matrix $A \in \mathbb{R}^{n \times n}$, we call λ an **eigenvalue** of A if

$$\text{There exists a vector } x \in \mathbb{R}^n \text{ such that } Ax = \lambda x$$

Moreover, we call x an **eigenvector** of A.

In other words, if a vector is multiplied by A and it remains the scalar multiple of itself, we call such vector an eigenvector, and the scalar factor becomes eigenvalue.

Problems

1. (a)Assume A has QR decomposition $A = QR$. Express dA in terms of Q, R, dQ, dR.

(b) Assume A has SVD $A = U\Sigma V$. Express dA in terms of SVD matrices.

(c) Let A be an orthogonal matrix. Prove that $A^T dA$ is skew-symmetric. Note that $(dA)^T = d(A^T)$.

(d) Let $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and fix A (so $dA = 0$). What is the relationship between dx and dy?

(e) From (d), Assume x_1, x_2, x_3 are unrelated($\frac{dx_i}{dx_j} = 0$ for $i \neq j$). Find a derivative $\frac{dy_i}{dx_j}$ in terms of A_{ij} by using simple division.

2. A tilted square with vertices $(1,0),(0,1),(-1,0),(0,-1)$ is transformed by left multiplication of $A = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$. Given that the image is still a quadrilateral(and the vertices are the images of original four vertices), Compute the volume(area) of the image. Compare it with the determinant of A.

3. (a) What is the volume of square with vertices $(1,0,0),(0,1,0),(0,0,0),(1,1,0)$ in $\mathbb{R}^3$?

(b) Let $A \in \mathbb{R}^{n\times n}$ be a singular matrix, so that the column space has dimension less than n. Explain why determinant is zero in terms of volumes.

4. Compute eigenvalues and corresponding eigenvectors of these matrices. (Multiply $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and let it $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$, then solve it for λ)

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$.

True or False. Explain or give counterexample.

(a) If x is an eigenvector of A then $2x$ is also an eigenvector of A.

(b) If λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of $-A$.

(c) If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.

(d) Assume A is not a symmetric matrix. Then A and A^T cannot have same eigenvalues.

(e) A doubly stochastic matrix always have eigenvalue one.

(f) Eigenvalue of real matrix is always a real number.

(g) At least one eigenvalue of complex matrix is a complex number.

(h) Let x, y be two vectors which are not colinear. They can be both eigenvectors of a same eigenvalue λ of A.

ANSWERS

1. (a) $dA = QdR + (dQ)R$

(b)$dA = (dU)\Sigma V^T + U(d\Sigma)V^T + U\Sigma(dV)^T$

(c) Start from $A^TA = I$, differentiate both sides, $(dA)^TA + A^T(dA) = 0$ and note that $(A^TdA)^T = (dA)^TA$, the equation is just $A^TdA + (A^TdA)^T = 0$, so that A^TdA is skew-symmetric.

(d)$y = Ax$ so $dy = (dA)x + Adx$ and since $dA = 0$ we have $dy = Adx$.

(e) Since $dy_1 = A_{11}dx_1 + A_{12}dx_2 + A_{13}dx_3$ and dividing each side by dx_1 we have $\frac{dy_1}{dx_1} = A_11$ since $\frac{dx_j}{dx_i}$ are all zeros. Similarly, $\frac{dy_i}{dx_j} = A_{ij}$.

2. The four vertices are, $(3,1), (7,4), (-3,-1), (-7,-4)$ and it is a parallelogram. Computing the area by subtracting areas from large square, we have area$= 10$. The determinant is 5 and the original area was 2. So it agrees.

3. (a) It is 2D plane, so the volume is zero.

(b) If you start with a volume in $\mathbb{R}^n$ (for example, unit box $[0,1]^n$), we end up with a region with less than n dimension.(the column space can never be dimension n). So the volume becomes zero on the result. It means the determinant has to be zero.

4. (a) $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_2 \end{pmatrix}$. if $x_2 \neq 0$ then we have $\lambda = 2$ and $x_1 = x_2$. If $x_2 = 0$ then $\lambda = 1$ and x_1 can be any number. So the eigenvalues are 2, 1 and corresponding eigenvectors are $\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}$.

(b). $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ -3x_1 - x_2 \end{pmatrix}$. If $x_1 \neq 0$ then we have $\lambda = 2$ and $x_1 = -x_2$, and if $x_1 = 0$ then $\lambda = -1$. We have eigenvalues $2, -1$ and corresponding eigenvectors are $\begin{pmatrix} x \\ -x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}$.

5. (a) True. $A(2x) = 2Ax = 2\lambda x = \lambda(2x)$.

(b) True. $(-A)x = -\lambda x = (-\lambda)x$.

(c) True. $A^2x = AAx = A\lambda x = \lambda Ax = \lambda^2 x$.

(d) False. Note that eigenvalues of triangular matrices always live on diagonal. You can take any triangular matrix as counterexample.

(e) True. Use eigenvector of all ones.

(f) False. Counterexample $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$.

(g) False. Counterexample $\begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}$ has eigenvalue ± 1. (h) True. Example is I_2. All the eigenvalues are one and any vector can be its eigenvector.