18.06 - Recitation 8 Solutions

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Problem 1.

The 2×2 matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of $B = 2A + 3I$. (Before you jump into solving quadratic equations, think about what happens if you multiply B by x_1 or x_2.)

Solution

Firstly, note that every vector is an eigenvector of the identity matrix with eigenvalue 1. This follows from the fact that $Ix = x$ for all x.

Now consider Bx_1:

$$\begin{aligned} Bx_1 &= (2A + 3I)x_1 && (1) \\ &= 2Ax_1 + 3Ix_1 && (2) \\ &= 2\lambda_1 x_1 + 3x_1 && (3) \\ &= (2\lambda_1 + 3)x_1 && (4) \end{aligned}$$

So x_1 is an eigenvector of B with eigenvalue $2\lambda_1 + 3 = 13$. The same argument shows that x_2 is also an eigenvector of B with eigenvalue $2\lambda_2 + 3 = 1$.

Problem 2.

1. If the eigenvectors of A are the columns of I then A is a .... matrix.
2. If the eigenvector matrix X is invertible and upper triangular, then why must A also be upper triangular? (Note: the inverse of an upper-triangular matrix is upper triangular.)

Solution

1. If the eigenvectors of A are the columns of I then A is a diagonal matrix. This follows from the diagonalization formula $A = X\Lambda X^{-1} = I\Lambda I^{-1} = \Lambda$, so A is a diagonal matrix whose entries are necessarily the eigenvalues.
2. If the eigenvector matrix X is upper triangular, then so too is its inverse X^{-1}. We can then use the diagonalization formula to write $A = X\Lambda X^{-1}$. However, the product of two upper triangular matrices will remain upper triangular, and since Λ is diagonal (and thus upper triangular), the product $X\Lambda X^{-1}$ will be upper triangular. Therefore A must be upper triangular.

Problem 3.

Suppose we form a sequence of numbers g_0, g_1, g_2, g_3 by the rule

$$g_{k+2} = (1-w)g_{k+1} + wg_k$$

for some scalar w. We concentrate on the case where $0 < w < 1$, so that g_{k+2} could be thought of as a *weighted average* of the previous two values in the sequence. For example, for $w = 0.5$ (equal weights) and with $g_0 = 0$ and $g_1 = 1$, this produces the sequence

$$g_0, g_1, g_2, g_3, \ldots = 0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \frac{85}{128}, \frac{171}{256}, \frac{341}{512}, \frac{683}{1024}, \frac{1365}{2048}, \frac{2731}{4096}, \frac{5461}{8192}, \frac{10923}{16384}, \frac{21845}{32768}, \ldots$$

1. If we define $x_k = \begin{pmatrix} g_{k+1} \\ g_k \end{pmatrix}$, then write the rule for the sequence in matrix form: $x_{k+1} = Ax_k$. In particular, what is A?
2. Find the eigenvalues of A (your answers could be a function of w) by computing the characteristic equation. Check that A has corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} w \\ -1 \end{pmatrix}$.
3. What happens to the eigenvalues and eigenvectors as w gets closer and closer to -1? Is there a still a basis of eigenvectors and a diagonalization of A for $w = -1$?
4. Show that $x_n = A^n x_0$. Find the limit as $n \to \infty$ of A^n (for $0 < w < 1$) from the diagonalization of A.
5. For $w = 0.5$, if $g_0 = 0$ and $g_1 = 1$, i.e. $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then show that the sequence g_k approaches $2/3$.

Solution

1. If $x_k = \begin{pmatrix} g_{k+1} \\ g_k \end{pmatrix}$, then we can use the recurrence relation

   $$g_{k+2} = (1-w)g_{k+1} + wg_k$$

   to write

   $$x_{k+1} = \begin{pmatrix} g_{k+2} \\ g_{k+1} \end{pmatrix} = \begin{pmatrix} (1-w) & w \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_{k+1} \\ g_k \end{pmatrix} = Ax_k.$$

2. We can find the eignvalues of our matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

   $$-\lambda(1-w-\lambda) - w = 0 \implies \lambda^2 + (w-1)\lambda - w = 0. \tag{5}$$

   Solving this quadratic yields:

   $$\lambda = \frac{1 - w \pm \sqrt{(w-1)^2 + 4w}}{2} \tag{6}$$
   $$= \frac{1 - w \pm \sqrt{(w+1)^2}}{2} \tag{7}$$
   $$= 1, -w. \tag{8}$$

   To find the eigenvector corresponding to $\lambda_1 = 1$, we solve $(A - I)u_1 = 0$

   $$\begin{pmatrix} -w & w \\ 1 & -1 \end{pmatrix} u_1 = 0 \implies u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9}$$

   To find the eigenvector corresponding to $\lambda_2 = -w$, we solve $(A + wI)u_2 = 0$

   $$\begin{pmatrix} 1 & w \\ 1 & w \end{pmatrix} u_2 = 0 \implies u_2 = \begin{pmatrix} w \\ -1 \end{pmatrix} \tag{10}$$

3. For $w = -1$, then the eigenvalues will coincide, and u_2 will become $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, which is parallel to u_1. For the particular value of w, the matrix A only has one eigenvalue with one linearly independent eigenvector. This means that there is no basis of eigenvectors and that A will not be diagonalizable.

4. If $x_n = Ax_{n-1}$, then $x_n = A^n x_0$. We can write x_0 as a linear combination of the eigenvectors: $x_0 = \alpha_1 u_1 + \alpha_2 u_2$.

   Then $x_n = A^n x_0 = \alpha_1 u_1 + \alpha_2 (-w)^n u_2$. Since $0 < w < 1$, $w^n \to 0$ as $n \to \infty$, and so $x_n \to \alpha_1 u_1$, i.e. g_n tends to a nonzero constant as $n \to \infty$. However, if $\alpha_1 = 0$, then $g_n \to 0$. From the diagonalization formula, we have $A = X\Lambda X^{-1}$. This means that

$$A^n = (X\Lambda X^{-1})^n = (X\Lambda X^{-1})...(X\Lambda X^{-1}) = X\Lambda^n X^{-1}.$$

   We can use the formula for the inverse of a 2×2 matrix to obtain X^{-1}:

$$X = \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \implies X^{-1} = \frac{1}{w+1} \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \tag{11}$$

   So:

$$\begin{aligned} A^n &= \frac{1}{w+1} \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-w)^n \end{pmatrix} \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} && (12) \\ &= \frac{1}{w+1} \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & w \\ (-w)^n & -(-w)^n \end{pmatrix} && (13) \\ &= \frac{1}{w+1} \begin{pmatrix} 1 + w(-w)^n & w - w(-w)^n \\ 1 - (-w)^n & w + (-w)^n \end{pmatrix} && (14) \end{aligned}$$

   But $w^n \to 0$ as $n \to \infty$, and so

$$A^n \to \frac{1}{w+1} \begin{pmatrix} 1 & w \\ 1 & w \end{pmatrix} \tag{15}$$

5. To find the limit of g_n as $n \to \infty$ with $g_0 = 0$ and $g_1 = 1$, we find the limit of $x_n = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $n \to \infty$:

$$\begin{aligned} x_n = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\to \frac{1}{w+1} \begin{pmatrix} 1 & w \\ 1 & w \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (16) \\ &= \frac{1}{w+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (17) \end{aligned}$$

   Substituting $w = 0.5$, we find that $x_n \to \frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and so $g_n \to 2/3$ as $n \to \infty$.