1 Lecture Review

1.1 Lengths and Dot Products

1. Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. The dot product of v and w is

$$v \cdot w = \sum_{i=1}^{n} v_i w_i = v^T w = w^T v.$$

2. The length of a vector $v = (v_1, \ldots, v_n)$ is

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{v \cdot v} = \sqrt{v^T v}.$$

1.2 QR Decomposition

1. If $m \geq n$, a real $m \times n$ square matrix A may be factored into the form $A = QR$ where Q is $m \times n$ satisfying $Q^T Q = I$ and R is $n \times n$ upper triangular.

2. Given $b \in \mathbb{R}^n$, it is possible that $Ax = b$ has no solution. However, $x = R^{-1} Q^T b$ is the "closest" to a solution in the sense that it minimizes $\|Ax - b\|$.

2 Problems

1. True or False. If false, give an example.

   (a) If Q is square and orthogonal then Q^T is square and orthogonal.

   (b) If Q is $m \times n$ with $Q^TQ = I$, then $QQ^T = I$.

   (c) If $Q^TQ = I = QQ^T$, then Q is square.

   (d) If $Ax_1 = y_1$ and $Ax_2 = y_2$, then $A\begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}$ where $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is the matrix with column vectors x_1, x_2 and likewise for $\begin{pmatrix} y_1 & y_2 \end{pmatrix}$.

   Solution. (a) True. This is because $(Q^T)^T = Q$ so that $Q^T(Q^T)^T = I = (Q^T)^TQ^T$.

   (b) False. Consider $Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $Q^TQ = 1$ (the 1×1 identity matrix), but $QQ^T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

   (c) True. This is because the inverse of Q is then Q^T and inverses exist only for square matrices.

   (d) True. This follows from the definition of matrix-vector product. □

2. Let Q be an orthogonal matrix with column vectors $q_1, \ldots, q_n$. Show that $\|q_i\| = 1$ and $q_i \cdot q_j = 0$ if $i \neq j$. Then check that this the case for the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   Solution. We have that Q^TQ is the matrix whose (i,j) entry is $q_i^Tq_j$. Since Q^TQ is the identity matrix, this means that $q_i^Tq_j = 0$ whenever $i \neq j$ and $q_i^Tq_i = 1$.

   For the rotation matrix, this follows from computing and using $\cos^2\theta + \sin^2\theta = 1$. □

3. Let

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 4 \\ 0 & 4 \end{pmatrix}.$$

   (a) Suppose we want the QR decomposition for A and we are given that

$$Q = \begin{pmatrix} 0 & 3/5 \\ 1 & 0 \\ 0 & 4/5 \end{pmatrix}.$$

   What condition should we check that Q satisfies?

   (b) Solve for R so that $A = QR$.

   (c) Show that there is no solution to

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

   (d) Find the best fit x which solves the equation above; i.e. the solution which minimizes $\|Ax - \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)\|$.

Solution. (a) We must check that $Q^TQ = I$.

(b) We want to find $R = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ so that

$$\begin{pmatrix} 0 & 3 \\ 2 & 4 \\ 0 & 4 \end{pmatrix} = A = QR = \begin{pmatrix} 0 & 3/5 \\ 1 & 0 \\ 0 & 4/5 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & c\frac{3}{5} \\ a & b \\ 0 & c\frac{4}{5} \end{pmatrix}.$$

Then $a = 2, b = 4, c = 5$ so that

$$\boxed{R = \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}}.$$

(c) Let $x = (x_1, x_2)$. By definition of matrix-vector product, any solution satisfies

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2\frac{3}{5} \\ x_1 \\ x_2\frac{4}{5} \end{pmatrix}.$$

However, it is impossible to have

$$x_2\frac{3}{5} = 1 = x_2\frac{4}{5}$$

so a solution cannot exist.

(d) Given our QR decomposition, the best fit x is given by

$$R^{-1}Q^T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -2/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -2/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 0 \\ 7/5 \end{pmatrix} = \boxed{\begin{pmatrix} -14/25 \\ 7/25 \end{pmatrix}}.$$

□

4. Let A be an $m \times n$ with $m < n$ and $b \in \mathbb{R}^n$. Use QR decomposition to find $x \in \mathbb{R}^m$ which best fits the equation

$$x^TA = b^T;$$

i.e. find x which minimizes $(x^TA - b^T)(x^TA - b^T)^T$. *Hint: Which matrix should be QR factored?*

Solution. Since A is $m \times n$ with $m < n$, we cannot do QR decomposition on A. However, we can for A^T. Let $A^T = QR$ be this decomposition. We may rewrite our equation as

$$A^Tx = b.$$

We want to minimize

$$(x^TA - b^T)(x^TA - b^T)^T = (A^Tx - b)^T(A^Tx - b) = \|A^Tx - b\|^2$$

which is the same as minimizing $\|A^Tx - b\|$. This is given by

$$x = \boxed{R^{-1}Q^Tb}.$$

where Q, R come from the QR decomposition of A^T (not A). □