1. **Four fundamental subspaces of $A \in \mathbb{R}^{m\times n}$**
   - $\text{col}(A) =$ ________________________________
   - $\text{row}(A) =$ ________________________________
   - $\text{null}(A) =$ ________________________________
   - $\text{null}(A^T) =$ ________________________________

2. **SVD and fundamental subspaces** Let $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be a full SVD of $A$ and $A = U_1\Sigma_r V_1^T$ be a Rank-r SVD of $A$. What are the size of matrices $U, V, \Sigma, U_1$ and $V_1$?

3. **Four fundamental subspaces of $A$ in terms of SVD** Given a full SVD

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix},$$

   - $\text{col}(A) = \text{col}(\_\_\_\_)$
   - $\text{row}(A) = \text{col}(\_\_\_\_)$
   - $\text{null}(A) = \text{col}(\_\_\_\_)$
   - $\text{null}(A^T) = \text{col}(\_\_\_\_)$

4. What is a condition for $Ax = b$ is solvable?

## Problems

1. Describe the null subspace and column subspace of

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$

2. The full SVD of $A$ is given as

$$A = U\Sigma V^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

(a) What is Rank-r SVD of $A$?

(b) Describe fundamental subspaces of $null(A)$, $col(A)$ and $row(A)$.

(c) A row space of an square $n$-by-$n$ orthogonal matrix is $\mathbb{R}^n$(no need to prove this), which is all the real vectors. That means, any vector $y \in \mathbb{R}^4$ can be obtained by $y = V^T x$. Describe column space of $\Sigma V^T$.

(d) What happens if a vector in (c) is multiplied by $U$ on the left? Our $A$ is actually $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Confirm that col($A$) = col($U_1$) and explain why the column space of $U_2$ does not play a role in column space of $A$.

(e) Compute $U_1 U_1^T$ without any hard computation. Does $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ satisfy $U_1 U_1^T b = b$? Is $Ax = b$ solvable?