18.06

Zack Remscrim

Transpose and Conjugate Transpose

Matrix A,

$A^\top$ denotes transpose (flip across main diagonal)

$\bar{A}$ denotes complex conjugate (elementwise) $a + bi \to a - bi$

A^H denotes conjugate transpose $A^H = \bar{A}^T$

Real vector v,

$\|v\|^2 = v^\top v$

Complex vector v,

$\|v\|^2 = v^H v$

Symmetric Matrices

Say matrix A is symmetric if

$A^\top = A$

Only square matrices can possibly be symmetric

All eigenvalues of a symmetric matrix are real

Eigenvalue λ with eigenvector v,

$Av = \lambda v$

Can choose $\|v\| = 1$

$$\lambda = \lambda v^H v = v^H \lambda v = v^H A v = (Av)^H v = (\lambda v)^H v = v^H \bar{\lambda} v = \bar{\lambda}$$

Symmetric Matrices

Eigenvectors corresponding to distinct eigenvalues are orthogonal

Eigenvalues $\lambda_1 \neq \lambda_2$

With $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

Then $v_2^\top A v_1 = v_2^\top \lambda_1 v_1 = \lambda_1 (v_1 \cdot v_2)$

and $v_1^\top A v_2 = v_1^\top \lambda_2 v_2 = \lambda_2 (v_1 \cdot v_2)$

but $(v_1^\top A v_2)^\top = v_2^\top A^\top v_1 = v_2^\top A v_1$

so $\lambda_1 (v_1 \cdot v_2) = \lambda_2 (v_1 \cdot v_2)$

which implies $(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$

therefore $v_1 \cdot v_2 = 0$

Symmetric Matrices

In fact, if A is $n \times n$ symmetric

with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct)

Have corresponding eigenvectors $v_1, \ldots, v_n$ all orthogonal

Can even choose $v_1, \ldots, v_n$ orthonormal

Main idea the same, but details

Symmetric matrices always diagonalizable

even if eigenvalues not distinct

Q, matrix with eigenvectors in columns, orthogonal

$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\top}$

Positive Definite/Semi-definite Matrices

Symmetric matrix A,

A positive definite if all eigenvalues positive

A positive semi-definite if all eigenvalues non-negative

Many equivalent definitions

A positive definite iff $x^\top A x > 0$ for every $x \neq 0$

$\Leftarrow$

eigenvalue λ with eigenvector v

$0 < v^\top A v = v^\top \lambda v = \lambda \|v\|^2$

$\|v\|^2 > 0$, therefore, $\lambda > 0$

Positive Definite/Semi-definite Matrices

A positive definite iff $x^\top Ax > 0$ for every $x \neq 0$

$\Rightarrow$

Eigenvalues $\lambda_1, \dots, \lambda_n > 0$

Corresponding orthonormal eigenvectors $v_1, \dots, v_n$

For any $x \neq 0$, write $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ (eigenbasis)

At least one $\alpha_i \neq 0$

$$(\alpha_1 v_1 + \cdots + \alpha_n v_n)^\top A(\alpha_1 v_1 + \cdots + \alpha_n v_n)$$
$$= (\alpha_1 v_1 + \cdots + \alpha_n v_n)^\top (\alpha_1 \lambda_1 v_1 + \cdots + \alpha_n \lambda_n v_n)$$
$$= (\alpha_1^2 \lambda_1 v_1^\top v_1 + \cdots + \alpha_n^2 \lambda_n v_n^\top v_n)$$
$$= (\alpha_1^2 \lambda_1 + \cdots + \alpha_n^2 \lambda_n) > 0$$

Positive Definite/Semi-definite Matrices

A positive definite iff

- All eigenvalues are positive
- $x^\top A x > 0$ for every $x \neq 0$
- $A = B^\top B$, for B with independent columns
- All principal minors positive
  - determinant of upper left $k \times k$ submatrix, for any k

Positive Definite/Semi-definite Matrices

Properties:

If A, B positive definite, so is $A + B$

$$x^\top (A + B)x = x^\top Ax + x^\top Bx$$

If A positive definite, so is cA, for any scalar $c > 0$

$$x^\top (cA)x = cx^\top Ax > 0$$

Markov Matrices

Say matrix A positive if all entries $A_{ij} > 0$

Write $A > 0$

Say matrix A non-negative if all entries $A_{ij} \geq 0$

Write $A \geq 0$

Say matrix A Markov matrix if

$A > 0$ and each column has sum 1

Say matrix A generalized Markov matrix if

$A \geq 0$ and each column has sum 1

Markov Matrices

Encodes Markov process, example:

- 100 objects, initially divided into two even piles
- Call piles left and right
- Repeatedly redistribute the objects
  - Sequence of steps, at every step
    - move 10% of objects in left pile to right pile
    - and move 5% of objects in right pile to left pile
- What happens after doing the above many times?

Markov Matrices

Example continued:

Encode initial distribution in vector $x = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$

Encode transitions in Markov matrix $A = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix}$

Applying A to x redistributes objects according to previous rule

Applying A^2 does 2 redistribution steps

Applying A^k does k redistribution steps

What does $A^k x$ look like?

Markov Matrices: Steady State

Markov matrix A, dimensions $n \times n$

Eigenvalues $\lambda_1, \ldots, \lambda_n$

Has eigenvalue $\lambda_1 = 1$

$|\lambda_i| < 1$ for all other eigenvalues

Assume A has lin. indep. eigenvectors $v_1, \ldots, v_n$ (it may not)

Can write any $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ (eigenbasis)

$A^k x = A^k(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 \lambda_1^k v_1 + \cdots + \alpha_n \lambda_n^k v_n$

Approaches $\alpha_1 \lambda_1^k v_1 = \alpha_1 v_1$, steady state

Markov Matrices

Example finished:

Encode initial distribution in vector $x = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$

Encode transitions in Markov matrix $A = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix}$

What does $A^k x$ look like?

A has eigenvalues 1 and $.85$

1 has corresponding (steady-state) eigenvector $\begin{pmatrix} .33 \\ .67 \end{pmatrix}$

$A^k x$ approaches $\begin{pmatrix} .33 \\ .67 \end{pmatrix}$