1 Lecture Review

1.1 Cofactors and Cramer's Rule

Let A be a square $n \times n$ matrix.

1. The *cofactor* C_{ij} is defined by
$$C_{ij} = (-1)^{i+j} \det M_{ij}$$
where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and jth column from A.

2. (Compute Determinant by Cofactors) For any $1 \leq i \leq n$,
$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

3. If A is invertible, then
$$A^{-1} = \frac{1}{\det A} C^T.$$
In terms of entries, $(A^{-1})ij = C_{ji}/\det A$.

4. (Cramer's Rule) If A is invertible and $A\boldsymbol{x} = \boldsymbol{b}$, then
$$x_j = \det(A \text{ with column } j \text{ changed to } \boldsymbol{b})/\det A.$$

2 Problems

1. Let A be $n \times n$. Explain using the full form SVD why $\det A \neq 0$ if and only if the rank of A is n.

2. If I is the $n \times n$ identity and a is a scalar, what is $\det(aI)$?

3.

4. Compute the determinant

$$\begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & -1 & 1 & \cdot & \\ & & \cdot & \cdot & 1 \\ & & & -1 & 1 \end{vmatrix}$$

   of the $n \times n$ matrix by cofactor expansion.

5. If A is $n \times n$ and invertible and C is its cofactor matrix, show that

   (a) $AC^T = \det(A)I$,

   (b) $\det C = (\det A)^{n-1}$.

6. Let Q be a (square) orthogonal matrix. Find the cofactor matrix of Q up to sign. Explain how the sign is affected by the sign of $\det Q$.

7. Suppose A is an invertible $n \times n$ square matrix and B is a known $n \times m$ matrix.

   (a) If you want to solve

$$AX = B$$

   where X is an $n \times m$ unknown matrix using Cramer's rule, in general how many determinants do you need to compute?

   (b) How many determinants do you need to compute A^{-1} by cofactors?

   (c) Compare (7a) and (7b). In particular, how is (7b) a special case of (7a)?

8. Let A be an $n \times n$ matrix with row vectors $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_n^T$. The determinant of a matrix is a linear transformation of each row (and column). Consider the function $f(\boldsymbol{x})$ which replaces the first row $\boldsymbol{a}_1^T$ of A with $\boldsymbol{x}^T$, that is

$$f(\boldsymbol{x}) = \begin{vmatrix} — & \boldsymbol{x}^T & — \\ — & \boldsymbol{a}_2^T & — \\ & \vdots & \\ — & \boldsymbol{a}_n^T & — \end{vmatrix}.$$

   Since f is a linear transformation from $\mathbb{R}^n \to \mathbb{R}$, we may write

$$f(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

   for some $\boldsymbol{w} \in \mathbb{R}^n$. Find $\boldsymbol{w}$ in terms of cofactors of A.