1 Lecture Review

1.1 Similar Matrices

1. Two $n \times n$ matrices A and B are *similar* if there exists an invertible matrix X so that
$$A = XBX^{-1}.$$
2. A is similar to itself.
3. If A and B are similar and B and C are similar, then A and C are similar.

1.2 Symmetric Matrices

1. Every real symmetric matrix S can be diagonalized
$$S = Q\Lambda Q^{-1}$$
where Q is orthogonal.
2. A real symmetric S has n real eigenvalues and n orthonormal eigenvectors.

1.3 Systems of Differential Equations

1. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $u(t) = e^{\lambda t}\boldsymbol{x}$ solves $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$.
2. If $A = X\Lambda X^{-1}$ is an eigendecomposition, then
$$e^{At} = I + At + \cdots + (At)^n/n! + \cdots = Xe^{\Lambda t}X^{-1}.$$

2 Problems

1. True or false, explain:

   (a) If A and B are similar, then $A - I$ and $B - I$ are similar.

   (b) There is a matrix $A \neq I$ which is similar to the identity.

   (c) If $A = X\Lambda_A X^{-1}$ and $B = X\Lambda_B X^{-1}$ where Λ_A, Λ_B are diagonal, then $AB = BA$.

   (d) If $A^3 = 0$, then the eigenvalues of A must be 0.

   (e) A matrix with real eigenvalues and n linearly independent eigenvectors is symmetric.

   (f) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.

   (g) The inverse of an invertible symmetric matrix is symmetric.

   (h) The eigenvector matrix Q of a symmetric matrix is symmetric.

   Solution.

   (a) True. We have $A = XBX^{-1}$. Then

   $$A - I = XBX^{-1} - I = XBX^{-1} - XX^{-1} = X(B - I)X^{-1}.$$

   (b) False. If A is similar to the identity, then

   $$A = XIX^{-1} = XX^{-1} = I.$$

   (c) True. Recall that diagonal matrices commute, so $\Lambda_A\Lambda_B = \Lambda_B\Lambda_A$. We have

   $$AB = X\Lambda_A X^{-1}X\Lambda_B X^{-1} = X\Lambda_A\Lambda_B X^{-1} = X\Lambda_B\Lambda_A X^{-1} = X\Lambda_B X^{-1}X\Lambda_A X^{-1} = BA.$$

   (d) True. If $\lambda \neq 0$ is an eigenvalue of A, then we have an eigenvector $\boldsymbol{x}$ of A with eigenvalue λ. This implies

   $$A^3\boldsymbol{x} = \lambda^3\boldsymbol{x} \neq 0.$$

   (e) False. For example,

   $$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

   has eigenvalues 1 and 0 and is therefore diagonalizable. Thus A has 2 real eigenvalues and 2 linearly independent eigenvectors, but is not symmetric.

   (f) True. If S is a matrix with real eigenvalues and n orthonormal eigenvectors, then we can diagonalize $S = X\Lambda X^{-1}$ where the columns of X are orthonormal. This means X is orthogonal so that $X^{-1} = X^T$. Then

   $$S^T = (X\Lambda X^T)^T = X\Lambda^T X^T = X\Lambda X^T = S.$$

   (g) True. If $S = Q\Lambda Q^{-1}$ is symmetric and invertible, then $S^{-1} = Q\Lambda^{-1}Q^{-1}$ is also symmetric since it is diagonalizable with orthogonal eigenvector matrix (note Λ^{-} is diagonal).

   (h) False. An orthogonal matrix Q need not be symmetric, for example

   $$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   However, $Q\Lambda Q^{-1}$ is symmetric for any Q orthogonal and Λ diagonal.

   $\square$

2. Suppose A is symmetric. Explain how the diagonalization $Q\Lambda Q^{-1}$ of A can be used to produce a full form SVD $U\Sigma V^T$ of A.

Solution. If A is symmetric with diagonalization $Q\Lambda Q^{-1} = Q\Lambda Q^T$ where Q is orthogonal, then we can get an SVD $U\Sigma V^T$ by taking (1) $Q = U$, (2) the diagonal entries of Σ are the absolute value of the diagonal entries of Λ:

$$\Sigma = \begin{pmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\lambda_n| \end{pmatrix}, \quad \text{where} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

(3) V is obtained by taking Q and changing the sign of the ith column for each $1 \leq i \leq n$ if $\lambda_i < 0$ and otherwise keeping it the same. (Check the third point with specific examples if you are unsure that this works). $\square$

3. Suppose A is $n \times n$ symmetric and B is $m \times n$. Show that BAB^T is symmetric.

Solution. We have
$$(BAB^T)^T = (B^T)^T A^T B^T = BA^T B^T = BAB^T$$
where the last equality follows from A being symmetric. Thus BAB^T is symmetric. □

4. If A is upper triangular with distinct diagonal entries, is it diagonalizable?

   Solution. Yes. The diagonal entries of an upper triangular matrix are the eigenvalues of the matrix. Thus if the diagonal entries of A are distinct, then A has n distinct eigenvalues. This implies A is diagonalizable. $\square$

5. Suppose A is an $n \times n$ upper triangular matrix. Show that if A is diagonalizable with n orthonormal eigenvectors, then A is a diagonal matrix.

   Solution. If A is diagonalizable with n orthonormal eigenvectors, then we may write $A = Q\Lambda Q^{-1}$ where Q is orthogonal. Then A is symmetric. But a matrix that is both symmetric and upper triangular matrix must be diagonal. Therefore A is diagonal. $\square$

6. If A is $m \times n$ is A^TA symmetric? If so what are the eigenvalues of A^TA in terms of the (full form) SVD of A? What are the eigenvectors? How about AA^T?

   Solution. Let $U\Sigma V^T$ be the full form SVD of A. Then

   $$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$$

   where $\Sigma^T\Sigma$ is a diagonal with squared singular values along the diagonal and possibly 0 (depending on the rank of A and the size of V) – these are the eigenvalues of A^TA. The eigenvectors are given by the columns of V.

   For AA^T, we have similarly

   $$AA^T = U\Sigma\Sigma^T U^T.$$

   $\square$

7. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Find e^{At} using the fact that $A^2 = A^3 = \cdots = 0$. How might you compute e^{Bt} for $B = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

   Solution. We have the formula

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}A^3 + \cdots .$$

   Since $A^2 = A^3 = \cdots = 0$, we have

$$e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

   For B, we can notice that

$$B^3 = 0.$$

   Then

$$e^{Bt} = I + Bt + \frac{(Bt)^2}{2} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

8. Explain why $e^{\operatorname{tr} A} = \det e^A$ (Note that $e^{\operatorname{tr} A}$ is a number and e^A is a matrix). You may assume A is diagonalizable.

Solution. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n, \quad \det A = \lambda_1 \cdots \lambda_n.$$

Since A is diagonalizable, we can write $A = X\Lambda X^{-1}$ and $e^A = Xe^{\Lambda}X^{-1}$. Then

$$e^{\operatorname{tr} A} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\lambda_1} \cdots e^{\lambda_n} = \det e^A.$$

$\square$

9. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) Find the eigenvectors. What can be said about the eigenvectors of different eigenvalues of A and how this connects to $A = A^T$.

(c) Find two linearly independent solutions to $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$.

Solution.

(a) We have

$$\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3.$$

Thus the eigenvalues are $-1, 3$.

(b) To find the eigenvector corresponding to 3, we solve

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}.$$

A solution satisfies

$$x + 2y = 3x, \quad 2x + y = 3y.$$

Setting $x = 1$, we get $y = 1$. Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue 3.

Similarly, we can find that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue -1.

Note that these eigenvalues are orthogonal to one another. This is because A is symmetric. Indeed, if we normalize the eigenvectors then our eigenvector matrix is

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

which is orthogonal, and

$$A = Q\Lambda Q^{-1}, \quad \text{where} \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) We have solutions

$$\boldsymbol{u}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which are linearly independent.

$\square$