18.06 Recitation March 31

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Solving $Ax = b$

- If $A_{n\times n}$ has rank n, then $Ax = b$ always has a unique solution.
- x is a least squares solution if _______ is as small as possible.
- x is a least norm if furthermore _______ is as small as possible.
- Given $A = U\Sigma V^T = U_1\Sigma_r V_1^T$, the solution $x =$ _______ is always a least-squares, least-norm solution.
- The matrix _______ is called the pseudoinverse of A and is also known as the Moore-Penrose inverse.

Basis, dimension

- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are **linearly independent** if $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n} = \vec{0}$ implies all $c_i = 0$.
- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ **span** the vector space S if S is all linear combinations of the $\vec{v_i}$.
- The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are a **basis for** S if they are linearly independent and they span S.
- Every basis has the same number of elements, this number is called the **dimension of a space** S.

Orthogonal complement

- If V and W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if for every $v \in V$, and $w \in W$, we have _______.
- $V^\perp$ is the set of all $w \in \mathbb{R}^n$ such that w is perpendicular to each vector $v \in V$.
- $V = (V^\perp)^\perp$
- $\text{Col}(A)^\perp = \text{Leftnull}(A)$
- $\text{Row}(A)^\perp = \text{Null}(A)$
- $\text{Null}(A)^\perp = \text{Row}(A)$
- $\text{Leftnull}(A)^\perp = \text{Col}(A)$

Problems

1. Let $A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, a is a real number (could be 0).

   (a) What is the colume space of A? What is the rank of A?

   (b) Does $Ax = b$ always has a solution? (not just least norm) Why?

(c) Solve the equation $Ax = b$ where $b = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(a) What is the rank of A?

(b) Write a basis for the solutions of $Ax = 0$. What is the dimension of this space?

(c) Does $A^T y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ has a solution? If not, find the least norm.

(d) Solve $A^T y = 0$. Can you find the result without calculating $A^T y$?

3. (a) Show that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

(b) Show that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

(c) Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Suppose we have $AB = 0$. Show that $\text{rank}(A) + \text{rank}(B) \leq n$.