18.06 - Recitation 9 - SOLUTIONS

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Problem 1.

The matrix B has eigenvalues $\lambda_1 = 2, \lambda_2 = 0$, and $\lambda_3 = 1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $x_3 = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

1. Find B using the diagonalization formula $B = X\Lambda X^{-1}$. You can leave your answer as a product of the three matrices, as long as you write down each matrix explicitly (Hint: look at the eigenvectors. Finding X^{-1} should require minimal computation).
2. Let $C = (I - B)(I + B)^{-1}$. What are the eigenvalues of C? (Hint: B and C have the same eigenvectors. Proving this will help you find the eigenvalues).

Solution

1. $B = X\Lambda X^{-1}$, where

$$\boxed{\Lambda = \begin{pmatrix} 2 & & \\ & 0 & \\ & & 1 \end{pmatrix}}$$

To find X and X^{-1} we can normalize the eigenvectors so that they form an orthonormal set:

$$q_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad q_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$$

. Then we can let

$$X = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{pmatrix},$$

so that X is an orthogonal matrix for which $X^{-1} = X^T$. Then $B = X\Lambda X^T$.

2. Let $Bx = \lambda x$. We can then show that:

$$(I + B)x = x + \lambda x = (1 + \lambda)x \tag{1}$$
$$(I - B)x = x - \lambda x = (1 - \lambda)x \tag{2}$$

so that x is an eigenvector of $I \pm B$. By inverting the first of these equations, we also have that $(I+B)^{-1}x = \frac{1}{1+\lambda}x$. Putting this all together we can then show that

$$Cx = (I - B)(I + B)^{-1}x = \frac{1 - \lambda}{1 + \lambda}x$$

So then the eigenvalues of C are

$$\frac{1-\lambda_1}{1+\lambda_1} = -\frac{1}{3}, \quad \frac{1-\lambda_2}{1+\lambda_2} = 1, \quad \frac{1-\lambda_3}{1+\lambda_3} = 0$$

Problem 2.
The matrix A has diagonalization $A = X\Lambda X^{-1}$ with

$$X = \begin{pmatrix} 1 & 1 & -1 & 0 \\ & 1 & 2 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & -2 & \\ & & & -1 \end{pmatrix}.$$

Give a basis for the nullspace $N(M)$ of the matrix $M = A^4 - 2A^2 - 8I$.

Solution
The eigenvalues of A are $\lambda = 1, 2, -2, -1$. The eigenvalues of M are then $\lambda^4 - 2\lambda^2 - 8$, with the same corresponding eigenvectors. M therefore has two zero eigenvalues (which come from the ± 2 eigenvalues of A), and so the corresponding eigenvectors are a basis for $N(M)$, i.e.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Problem 3.
Let A, B, C and D be 2×2 matrices

1. Use the cofactor expansion to prove that the following block determinant expression holds:

$$\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A||D|$$

2. Verify that if A^{-1} exists, then

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

3. Prove that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

provided that $AC = CA$.

Solution

1. Let[1]

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad D = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

[1]You don't really need all the components to show this, but I figured it would be a little bit more transparent to write everything out.

Then

$$\begin{aligned}
\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} &= \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \alpha & \beta & u & v \\ \gamma & \delta & w & x \end{vmatrix} \\
&= a \begin{vmatrix} d & 0 & 0 \\ \beta & u & v \\ \delta & w & x \end{vmatrix} - b \begin{vmatrix} c & 0 & 0 \\ \alpha & u & v \\ \gamma & w & x \end{vmatrix} \\
&= ad \begin{vmatrix} u & v \\ w & x \end{vmatrix} - bc \begin{vmatrix} u & v \\ w & x \end{vmatrix} \\
&= (ad - bc)|D| \\
&= |A||D|
\end{aligned}$$

Note that a similar process allows us to show that

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D|$$

which we will need in the last part.

2. Multiplying out the left hand side gives the same as the right hand side (Be careful to do matrix multiplication in the correct order!)

3. Taking the determinant of both sides of the equation in part (2) shows that

$$\begin{aligned}
\begin{vmatrix} I & 0 \\ -CA^{-1} & I \end{vmatrix} \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} \\
\implies \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= |A||D - CA^{-1}B| \\
\implies \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= |A\left(D - CA^{-1}B\right)| \\
&= |AD - ACA^{-1}B| \\
&= |AD - CB|
\end{aligned}$$

where the last line holds provided that $AC = CA$.

Problem 4.
Recall that the matrix exponential of A is defined via the infinite series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

1. Explain why e^A is always an invertible matrix (hint: use eigenvalues).

2. There is a result that says that whenever $AB = BA$, it holds that $e^{A+B} = e^A e^B$. Use this result to find the inverse of e^A.

3. Suppose A is a real, antisymmetric matrix so that $A^T = -A$. Show that $U = e^A$ is an orthogonal matrix.

4. If $x(t)$ satisfies

$$\frac{dx}{dt} = Ax,$$

then explain why $\|x(t)\| = \|x(0)\|$ for all t.

Solution

1. Suppose that $Ax = \lambda x$. Then we can show that
$$e^A x = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) x = \sum_{n=0}^{\infty} \frac{A^n x}{n!} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) x = e^{\lambda} x$$
So the eigenvalues of e^A are given by e^{λ} for each eigenvalue λ of A. Hence e^A can never have a zero eigenvalue (since $|e^{\lambda}| > 0 \;\; \forall \lambda \in \mathbb{C}$). Therefore e^A is always invertible for any matrix A.

2. Notice that
$$e^{-A} e^{A} = e^{-A+A} = e^0 = I$$
and so
$$\left(e^A\right)^{-1} = e^{-A}$$

3. Notice that
$$U^T = \left(e^A\right)^T = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right)^T = \left(\sum_{n=0}^{\infty} \frac{(A^T)^n}{n!}\right) = \left(\sum_{n=0}^{\infty} \frac{(-A)^n}{n!}\right) = e^{-A} = U^{-1}$$
and so $U^T U = I$. Therefore U is an orthogonal matrix.

4. Recall that the solution of a matrix ODE $\frac{dx}{dt} = Ax$ is
$$x(t) = e^{At} x(0).$$
Therefore
$$\|x(t)\|^2 = \left(e^{At}x(0)\right)^T e^{At}x(0) = x^T(0)\left(e^{At}\right)^T e^{At}x(0) = x^T(0)e^{-At}e^{At}x(0) = x^T(0)x(0) = \|x(0)\|^2$$

Problem 5.
A 3×3 matrix B is known to have eigenvalues $0, 1, 2$. This is enough information to determine 3 of the following. Which are true and what are their values:

1. The rank of B.
2. The determinant of $B^T B$.
3. The eigenvalues of $B^T B$.
4. The eigenvalues of $(B^2 + I)^{-1}$.

Solution

1. Since B has 3 distinct eigenvalues, exactly one of which is 0, we know that the dimension of the nullspace is 1. Since B is 3×3, we can deduce that $r = 3 - 1 = 2$.

2. The determinant of $B^T B$ is
$$|B^T B| = |B^T||B| = |B|^2$$
But we know that $|B| = 0$, since the determinant of a matrix is the product of its eigenvalues. Therefore $|B^T B| = 0$.

3. The eigenvalues of $B^T B$ cannot be determined from this information

4. Suppose $Bx = \lambda x$ so that λ is an eigenvalue of B with eigenvector x. Then

$$(B^2 + I)x = B^2 x + x = \lambda^2 x + x = (\lambda^2 + 1)x$$

and so x is also an eigenvector of $B^2 + I$, but with eigenvalue $\lambda^2 + 1$. By inverting this equation, we then also have that

$$(B^2 + I)^{-1} x = \frac{1}{\lambda^2 + 1}.$$

Hence if B has eigenvalues $0, 1, 2$, then $(B^2 + I)^{-1}$ must have eigenvalues $1, 1/2, 1/5$.