1 Lecture Review

1.1 Orthogonality of Subspaces

1. If V, W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if for every $v \in V$ and $w \in W$ we have $v^T w = 0$.

2. Given a vector subspace V of $\mathbb{R}^n$, we denote by $V^\perp$ the set of all vectors $w \in \mathbb{R}^n$ which are orthogonal to all vectors in V; that is $v^T w = 0$ for every $v \in V$. We call $V^\perp$ the orthogonal complement of V.

3. $(V^\perp)^\perp = V$.

4. Given an $m \times n$ matrix A,
$$\mathrm{Col}(A)^\perp = \mathrm{LeftNull}(A), \quad \mathrm{Row}(A)^\perp = \mathrm{Null}(A).$$

1.2 Linear Transformations

1. Let V and W be vector spaces. A function T from V to W is linear if for every $v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{R}$ we have
$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2).$$

2 Problems

1. Let V be a vector subspace of $\mathbb{R}^n$. Check that $V^\perp$ is a vector space.

   Solution. To check $V^\perp$ is a vector space, we check that if w_1, w_2 are in $V^\perp$ then $c_1 w_1 + c_2 w_2$ is in $V^\perp$ for any constants c_1, c_2.

   Take $w_1, w_2 \in V^\perp$. This means that

   $$w_1^T v = 0, \quad w_2^T v = 0$$

   for any $v \in V$. Then for any $v \in V$, we have

   $$(c_1 w_1 + c_2 w_2)^T v = c_1 w_1^T v + c_2 w_2^T v = 0$$

   so that $c_1 w_1 + c_2 w_2 \in V^\perp$. This shows that $V^\perp$ is a vector space. $\square$

2. Find bases and dimensions for the four fundamental subspaces associated with the following matrices

$$\text{(a)}\ \begin{pmatrix}1&2&4\\2&4&8\end{pmatrix},\quad \text{(b)}\ \begin{pmatrix}1&2&4\\2&5&8\end{pmatrix},\quad \text{(c)}\ \begin{pmatrix}0&3&3&3\\0&0&0&0\\0&1&0&1\end{pmatrix},\quad \text{(d)}\ \begin{pmatrix}1\\4\\5\end{pmatrix}.$$

Solution. (a) Set $A = \begin{pmatrix}1&2&4\\2&4&8\end{pmatrix}$. Since the columns are multiples of one another and the rows are multiples of one another, we have

$$\mathrm{Col}(A) = \mathrm{span}\{\begin{pmatrix}1\\2\end{pmatrix}\},\quad \mathrm{Row}(A) = \mathrm{span}\{\begin{pmatrix}1\\2\\4\end{pmatrix}\}.$$

Since $\mathrm{LeftNull}(A) = \mathrm{Col}(A)^T$, we have that $\mathrm{LeftNull}(A)$ consist of vectors (x, y) in $\mathbb{R}^2$ such that

$$0 = \begin{pmatrix}x\\y\end{pmatrix}^T \begin{pmatrix}1\\2\end{pmatrix} = x + 2y$$

which is spanned by $(-2, 1)$, thus

$$\mathrm{LeftNull}(A) = \mathrm{span}\{\begin{pmatrix}-2\\1\end{pmatrix}\}.$$

Since $\mathrm{Null}(A) = \mathrm{Row}(A)^T$, we have that $\mathrm{Null}(A)$ consists of vectors (x, y, z) in $\mathbb{R}^3$ such that

$$0 = \begin{pmatrix}x\\y\\z\end{pmatrix}^T \begin{pmatrix}1\\2\\4\end{pmatrix} = x + 2y + 4z.$$

Setting $y = 0$, we get a solution $(-4, 0, 1)$. Setting $z = 0$, we get a solution $(-2, 1, 0)$. Since $(-4, 0, 1)$ and $(-2, 1, 0)$ are linearly independent (they are not multiples of each other) and since a plane has dimension two, we have that they form a basis for the plane $x + 2y + 4z = 0$. Thus

$$\mathrm{Null}(A) = \mathrm{span}\{\begin{pmatrix}-4\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}\}.$$

Altogether, we have

	Col(A)	Row(A)	LeftNull(A)	Null(A)
basis	$\{\begin{pmatrix}1\\2\end{pmatrix}\}$	$\{\begin{pmatrix}1\\2\\4\end{pmatrix}\}$	$\{\begin{pmatrix}-2\\1\end{pmatrix}\}$	$\{\begin{pmatrix}-4\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}\}$
dimension	1	1	1	2

(b) Set $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$. Note that the third column is a multiple of the first, but the first and second columns are linearly indepedent. Similarly, the first and second rows are linearly independent. Thus

$$\mathrm{Col}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\}, \quad \mathrm{Row}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}\}.$$

Since $\mathrm{Col}(A)$ is spanned by two linearly independent vectors in $\mathbb{R}^2$, we must have $\mathrm{Col}(A) = \mathbb{R}^2$. Then

$$\mathrm{LeftNull}(A) = \mathrm{Col}(A)^T = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

Since $\mathrm{Null}(A) = \mathrm{Row}(A)^T$, we have that $\mathrm{Null}(A)$ consists of vectors (x, y, z) in $\mathbb{R}^3$ such that

$$0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = x + 2y + 4z, \quad \text{and} \quad 0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = 2x + 5y + 8z.$$

We can find such a vector by taking the cross product of $(1, 2, 4)$ and $(2, 5, 8)$, or alternatively solve the system

$$\begin{array}{r} x + 2y + 4z = 0 \\ 2x + 5y + 8z = 0 \end{array}.$$

Multiplying the first equation by 2 and subtracting it from the second gives $y = 0$. Plugging this into the first equation gives

$$x + 4z = 0.$$

Set $z = t$, then $x = -4t$, so a general solution to the system is given by $t(-4, 0, 1)$. Thus

$$\mathrm{Null}(A) = \mathrm{span}\{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}\}.$$

Altogether, we have

	$\mathrm{Col}(A)$	$\mathrm{Row}(A)$	$\mathrm{LeftNull}(A)$	$\mathrm{Null}(A)$
basis	$\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\}$	$\{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}\}$	$\{ \quad \}$	$\{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}\}$
dimension	2	2	0	1

(c) Set $A = \begin{pmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Note the first column is the zero column and the second and fourth columns are the same. Also, the second row is the zero row, thus

$$\mathrm{Col}(A) = \mathrm{span}\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\}, \quad \mathrm{Row}(A) = \mathrm{span}\{\begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}.$$

Since $\mathrm{Col}(A)$ is a plane in $\mathbb{R}^3$, we have $\mathrm{LeftNull}(A) = \mathrm{Col}(A)^\perp$ is spanned by the normal vector to this plane. This can be obtained by taking the cross product of $(3, 0, 1)$ and $(3, 0, 0)$. Alternatively, find $(x, y, z) \in \mathbb{R}^3$ solving

$$0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 3x + z, \quad \text{and} \quad 0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3x.$$

The second equation gives $x = 0$. Substituting this into the first equation gives $z = 0$. Thus $\mathrm{LeftNull}(A)$ is the line $(0, t, 0)$, that is

$$\mathrm{LeftNull}(A) = \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}.$$

Since $\mathrm{Null}(A) = \mathrm{Row}(A)^T$, we have that $\mathrm{Null}(A)$ consists of vectors (x, y, z, w) in $\mathbb{R}^4$ such that

$$0 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^T \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 3y + 3z + 3w, \quad \text{and} \quad 0 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = y + w.$$

Multiplying the second equation by 3 and subtracting it from the first yields $z = 0$. The second equation also yields $y = -w$. We have $(1, 0, 0, 0)$ and $(0, 1, 0, -1)$ satisfy these constraints. Observe that $\mathrm{Null}(A)$ has dimension 2 since the dimension of null space is $\dim \mathbb{R}^4 - \dim \mathrm{Row}(A) = 4 - 2 = 2$, thus these two vectors form a basis. So

$$\mathrm{Null}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\}.$$

Altogether, we have

	$\mathrm{Col}(A)$	$\mathrm{Row}(A)$	$\mathrm{LeftNull}(A)$	$\mathrm{Null}(A)$
basis	$\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\}$	$\{\begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}$	$\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$	$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\}$
dimension	2	2	1	2

(d) Set $A = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$. We have

$$\mathrm{Col}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}\}, \quad \mathrm{Row}(A) = \mathrm{span}\{1\}.$$

Note that $\mathrm{Row}(A) = \mathbb{R}^1 = \mathbb{R}$.

Since $\mathrm{LeftNull}(A) = \mathrm{Col}(A)^{\perp}$, we have that (x, y, z) is in $\mathrm{LeftNull}(A)$ exactly when

$$x + 4y + 5z = 0.$$

Since this is a plane, it suffices to find 2 linearly independent vectors satisfying the equation above to obtain a basis. Taking $y = 0$, we get $x + 5z = 0$ so that $(-5, 0, 1)$ is a solution. Taking $z = 0$, we get $x + 4y = 0$ so that $(-4, 1, 0)$. Thus

$$\mathrm{LeftNull}(A) = \mathrm{span}\{\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}\}.$$

Since $\mathrm{Row}(A) = \mathrm{span}\{(1)\} = \mathbb{R}^1$, we have

$$\mathrm{Null}(A) = \mathrm{Row}(A)^{\perp} = \{0\}.$$

Altogether, we have

	$\mathrm{Col}(A)$	$\mathrm{Row}(A)$	$\mathrm{LeftNull}(A)$	$\mathrm{Null}(A)$
basis	$\{\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}\}$	$\{1\}$	$\{\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}\}$	$\{\ \}$
dimension	1	1	2	0

$\square$

3. Let A be an $m \times n$ matrix with full form singular value decomposition $U\Sigma V^T$. If A has rank r, find a basis for the following subspaces in terms of columns of U or V: (a) Col(A), (b) LeftNull(A), (c) Row(A), (d) Null(A).

   Solution. Recall that we may write $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$ in block matrix notation so that U_1 is $m \times r$ and V_1 is $n \times r$ and these are the left and right matrices which appear in the compact form SVD of A. Furthermore, we have

   $$\mathrm{Col}(A) = \mathrm{Col}(U_1), \quad \mathrm{LeftNull}(A) = \mathrm{Col}(U_2), \quad \mathrm{Row}(A) = \mathrm{Col}(V_1), \quad \mathrm{Null}(A) = \mathrm{Col}(V_2).$$

   Note that the columns of U_1 are linearly independent and likewise for U_2, V_1, V_2. So we have:

   (a) the first r columns of U form a basis for Col(A),

   (b) the last $m - r$ columns of U form a basis for LeftNull(A),

   (c) the first r columns of V form a basis for Row(A),

   (d) the last $n - r$ columns of V form a basis for Null(A). □

4. Suppose $\boldsymbol{v} = (a, b, c) \in \mathbb{R}^3$ is a nonzero vector. Viewing $\boldsymbol{v}^T$ as a 1×3 matrix, find a basis for: (a) $\mathrm{Col}(\boldsymbol{v}^T)$, (b) $\mathrm{LeftNull}(\boldsymbol{v}^T)$, (c) $\mathrm{Row}(\boldsymbol{v}^T)$. (d) Using the fact that $\mathrm{Row}(\boldsymbol{v}^T)^\perp = \mathrm{Null}(\boldsymbol{v}^T)$, explain how this shows that the orthogonal complement of a plane is spanned by its normal vector.

Solution. (a) Since $\mathrm{Col}(\boldsymbol{v}^T) = \mathbb{R}^1$ a basis is given by $\{1\}$.

(b) The orthogonal complement of $\mathbb{R}^1$ is $\{0\}$, thus the empty basis $\{\ \}$ is the basis for $\mathrm{LeftNull}(A)$.

(c) A basis for $\mathrm{Row}(A)$ is given by $\{\boldsymbol{v}\}$.

(d) The null space is given by all vectors orthogonal to $\boldsymbol{v}$. This is the plane

$$ax + by + cz = 0.$$

Since $(V^\perp)^\perp = V$, we have that the orthogonal complement of the plane $ax + by + cz = 0$ is given by $\mathrm{Row}(A)$ which is the line spanned by $\boldsymbol{v}$, the normal vector to this plane. □

5. Let A be an $n \times n$ orthogonal matrix with column vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Show that the orthogonal complement of $\operatorname{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ is $\operatorname{span}(\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_n)$.

Solution. Write $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ in block matrix notation where A_1 is the $n \times r$ matrix formed by the first r columns of A and A_2 is the $n \times (n-r)$ matrix formed by the last $n-r$ columns of A. Let Σ be the $n \times r$ matrix $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ in block notation where I is the $r \times r$ identity and 0 denotes the $(n-r) \times r$ zero matrix. Then

$$A_1 = U \Sigma V^T$$

if we set $U = A$ and V to be the $r \times r$ identity. Note that this gives a full form SVD for A_1. The compact form SVD is given by

$$A_1 = U_1 \Sigma_1 V_1^T.$$

where $U_1 = A_1$, $\Sigma_1 = I_r$ and $V_1 = I_r$. In particular, we have $U = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ in block matrix notation (here A_1 is our usual U_1 and A_2 is our usual U_2) so that

$$\operatorname{Col}(A_1)^\perp = \operatorname{LeftNull}(A_1) = \operatorname{Col}(A_2).$$

This is exactly the statement that the orthogonal complement of $\operatorname{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$ is $\operatorname{span}(\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_n)$. $\square$

6. Suppose W is a vector subspace of $\mathbb{R}^n$. For any $\boldsymbol{v} \in \mathbb{R}^n$ check that $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ for some $\boldsymbol{w}_1 \in W$ and $\boldsymbol{w}_2 \in W^\perp$. Using the Pythagorean theorem, show that the vector $\boldsymbol{w} \in W$ which minimizes $\|\boldsymbol{v} - \boldsymbol{w}\|$ is given by $\boldsymbol{w}_1$.

   Solution. Recall that we can project $\boldsymbol{v}$ onto the subspace W which gives a vector $\boldsymbol{w}_1 \in W$ so that $\boldsymbol{v} - \boldsymbol{w}_1$ is orthogonal to W. Setting $\boldsymbol{w}_2 = \boldsymbol{v} - \boldsymbol{w}_1$ we have

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

   where $\boldsymbol{w}_1 \in W$ and $\boldsymbol{w}_2 \in W^\perp$.

   We want to minimize $\|\boldsymbol{v} - \boldsymbol{w}\|$ over all $\boldsymbol{w} \in W$. This is the same as minimizing $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ over all $\boldsymbol{w}$. Writing

$$\boldsymbol{v} - \boldsymbol{w} = (\boldsymbol{w}_1 - \boldsymbol{w}) + \boldsymbol{w}_2$$

   we have that $\boldsymbol{w}_1 - \boldsymbol{w} \in W$, so by the Pythagorean theorem (using that $\boldsymbol{w}_1 - \boldsymbol{w}$ and $\boldsymbol{w}_2$ are perpendicular)

$$\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{w}_1 - \boldsymbol{w}\|^2 + \|\boldsymbol{w}_2\|^2.$$

   This is minimized when $\|\boldsymbol{w}_1 - \boldsymbol{w}\|^2 = 0$ which is when $\boldsymbol{w} = \boldsymbol{w}_1$. $\square$

7. Let V and W be vector spaces. Explain why a linear transformation T from V to W must send the zero vector in V to the zero vector in W.

Solution. If $\mathbf{0}$ is the zero vector in V and $0 \in \mathbb{R}$ is the zero scalar (difference is in bolding), then

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0T(\mathbf{0}) = 0$$

where we used linearity in the second equality. $\square$

8. Check whether the following maps are linear transformations:

$$\begin{array}{ll} \text{(a) } T(x,y) = (x-y, x+y), & \text{(b) } T(x,y,z) = (x+1, y+1, z+1), \\ \text{(c) } T(x,y) = (xy, x), & \text{(d) } T(x,y,z) = (x+y+z, y+z, z). \end{array}$$

For the instances where T is a linear transformation, can you find a matrix A such that

$$T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad T(x,y,z) = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution. Note that if $T(\boldsymbol{v}) = A\boldsymbol{v}$ for some $m \times n$ matrix A, then T is a linear transformation. Indeed if $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$, then

$$T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = A(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = c_1 A\boldsymbol{v}_1 + c_2 A\boldsymbol{v}_2$$

by linearity of matrix-vector product.

(a) This is a linear transformation because we can write

$$T(x,y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) This is not a linear transformation since $T(0,0,0) = (1,1,1)$ and linear transformations must send the zero vector to the zero vector.

(c) This is not a linear transformation. Indeed, we have

$$T(2,2) = (4,2), \quad T(1,1) = (1,1) \quad \implies \quad T(2,2) \neq 2T(1,1).$$

(d) This is a linear transformation because we can write

$$T(x,y,z) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$\square$