# Rect 4 solu

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1. **Four fundamental subspaces of $A \in \mathbb{R}^{m\times n}$**
   - $\text{col}(A) =$ all linear combinations of columns of $A = \{Ax \mid x \in \mathbb{R}^n\}$
   - $\text{row}(A) =$ — — — — — rows of $A = \{A^T x \mid x \in \mathbb{R}^m\}$
   - $\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
   - $\text{null}(A^T) = \{x \in \mathbb{R}^m \mid A^T x = 0\}$

2. **SVD and fundamental subspaces** Let $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be a full SVD of $A$ and $A = U_1\Sigma_r V_1^T$ be a Rank-r SVD of $A$. What are the size of matrices $U, V, \Sigma, U_1$ and $V_1$?

   $U$ $m\times m$ $U_1$ $m\times r$

   $\Sigma$ $m\times n$

   $V$ $n\times n$ $V_1$ $n\times r$

3. **Four fundamental subspaces of $A$ in terms of SVD** Given a full SVD

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix},$$

   - $\text{col}(A) = \text{col}(U_1)$
   - $\text{row}(A) = \text{col}(V_1)$
   - $\text{null}(A) = \text{col}(V_2)$
   - $\text{null}(A^T) = \text{col}(U_2)$

4. What is a condition for $Ax = b$ is solvable?

   b is in the column space of A

   $UU^Tb = b$

## Problems

1. Describe the null subspace and column subspace of

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Null space $= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \ x_1 = x_2 = 0 \right\}$

$\text{col}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \middle| \ x_1, x_2 \in \mathbb{R} \right\}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$ $\text{Null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\text{Col}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix} \middle| \ x_1, x_2 \in \mathbb{R} \right\}$





2. The full SVD of $A$ is given as

$$A = U\Sigma V^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

(a) What is Rank-r SVD of $A$?

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(b) Describe fundamental subspaces of $null(A)$, $col(A)$ and $row(A)$.

$\text{Null}(A) = \left\{ k \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \middle| \ k \in \mathbb{R} \right\}$ $\text{Col}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \middle| \ x_1, x_2, x_3 \in \mathbb{R} \right\}$

$\text{row}(A) = \left\{ (x_1\ x_2\ x_3\ x_4) \middle| \ x_1 + x_2 - x_4 = 0 \right\}$

(c) A row space of an square $n$-by-$n$ orthogonal matrix is $\mathbb{R}^n$(no need to prove this), which is all the real vectors. That means, any vector $y \in \mathbb{R}^4$ can be obtained by $y = V^T x$. Describe column space of $\Sigma V^T$.

$\text{col}(\Sigma V^T) = \{\Sigma V^T x \mid x \in \mathbb{R}^4\} = \{\Sigma y \mid y \in \mathbb{R}^4\} = \text{col}(\Sigma)$

(d) What happens if a vector in (c) is multiplied by $U$ on the left? Our $A$ is actually $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Confirm that $\text{col}(A) = \text{col}(U_1)$ and explain why the column space of $U_2$ does not play a role in column space of $A$.

$\text{Col}(A) = \{Ax \mid x \in \mathbb{R}^4\} = \{U\Sigma y \mid y \in \mathbb{R}^4\} = \left\{ U_1 \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} y \middle| \ y \in \mathbb{R}^4 \right\}$

$= \{U_1 z \mid z \in \mathbb{R}^4\} = \text{col}(U_1)$

(e) Compute $U_1U_1^T$ without any hard computation. Does $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ satisfy $U_1U_1^Tb = b$? Is $Ax = b$ solvable?

$U_1U_1^T + U_2U_2^T = UU^T = I_4$ $U_2U_2^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

So $U_1U_1^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $U_1U_1^T \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \Rightarrow$ Not soluable.