1 Lecture Review

1.1 Lengths and Dot Products

1. Let $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$. The dot product of v and w is

$$v \cdot w = \sum_{i=1}^{n} v_i w_i = v^T w = w^T v.$$

2. The length of a vector $v = (v_1, \dots, v_n)$ is

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{v \cdot v} = \sqrt{v^T v}.$$

1.2 QR Decomposition

1. If $m \geq n$, a real $m \times n$ square matrix A may be factored into the form $A = QR$ where Q is $m \times n$ satisfying $Q^T Q = I$ and R is $n \times n$ upper triangular.

2. Given $b \in \mathbb{R}^n$, it is possible that $Ax = b$ has no solution. However, $x = R^{-1} Q^T b$ is the "closest" to a solution in the sense that it minimizes $\|Ax - b\|$.

2 Problems

1. True or False. If false, give an example.
   (a) If Q is square and orthogonal then Q^T is square and orthogonal.
   (b) If Q is $m \times n$ with $Q^TQ = I$, then $QQ^T = I$.
   (c) If $Q^TQ = I = QQ^T$, then Q is square.
   (d) If $Ax_1 = y_1$ and $Ax_2 = y_2$, then $A\begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}$ where $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is the matrix with column vectors x_1, x_2 and likewise for $\begin{pmatrix} y_1 & y_2 \end{pmatrix}$.

2. Let Q be an orthogonal matrix with column vectors $q_1, \ldots, q_n$. Show that $\|q_i\| = 1$ and $q_i \cdot q_j = 0$ if $i \neq j$. Then check that this the case for the rotation matrix
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3. Let
$$A = \begin{pmatrix} 0 & 3 \\ 2 & 4 \\ 0 & 4 \end{pmatrix}.$$
   (a) Suppose we want the QR decomposition for A and we are given that
$$Q = \begin{pmatrix} 0 & 3/5 \\ 1 & 0 \\ 0 & 4/5 \end{pmatrix}.$$
   What condition should we check that Q satisfies?
   (b) Solve for R so that $A = QR$.
   (c) Show that there is no solution to
$$Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
   (d) Find the best fit x which solves the equation above; i.e. the solution which minimizes $\|Ax - \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)\|$.

4. Let A be an $m \times n$ with $m < n$ and $b \in \mathbb{R}^n$. Use QR decomposition to find $x \in \mathbb{R}^m$ which best fits the equation
$$x^T A = b^T;$$
i.e. find x which minimizes $(x^TA - b^T)(x^TA - b^T)^T$. *Hint: Which matrix should be QR factored?*

3 Answers

1. (a) True, (b) False $Q = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, (c) True, (d) True
2. -
3. (a) $Q^TQ = I$, (b) $\left(\begin{smallmatrix} 2 & 4 \\ 0 & 5 \end{smallmatrix}\right)$, (c) -, (d) $\begin{pmatrix} -14/25 \\ 7/25 \end{pmatrix}$
4. -