Recitation 4. March 12

Focus: bases, four fundamental subspaces, fitting everything together.

Notation. Let V and W denote two real vector spaces.

Definition (reminder). Vectors $v_1, \dots, v_k$ are said to be *linearly independent* if the only way to write a zero linear combination $c_1v_1+\dots+c_kv_k = 0$ is to let all the scalars be zero: $c_1 = \dots = c_k = 0$.

Definition (reminder). The *span*, or *linear span*, of some set of vectors $S \subset V$ is the set of all possible finite linear combinations of vectors from S, or mathematically:

$$\operatorname{Span} S = \{c_1v_1 + \dots + c_lv_l \mid l \in \mathbb{Z};\ v_1, \dots, v_l \in V;\ c_1, \dots, c_l \in \mathbb{R}\}.$$

The set S can be finite or infinite, and it can be linearly independent or linearly dependent. If $\operatorname{Span} S = V$, then we say that S *generates*, or *spans*, the vector space V.

Definition (reminder). A set of vectors $v_1, \dots, v_n$ is called a *basis* of V if it these vectors are linearly independent and span V. In this case, we say that V is n-dimensional. All bases in the same vector space have equal number of elements.

Definition. A *linear operator*, or a *linear transformation*, between vector spaces V amd W is a set function $A : V \to W$ that is linear, which means that $A(v + v') = Av + Av'$ for vectors v and v' in V, and $A(\lambda v) = \lambda Av$ for a vector $v \in V$ and a scalar $\lambda \in \mathbb{R}$.

Definition. The *image* of a linear operator $A : V \to W$ is a subset of W that consists of all vectors of the form Av for $v \in V$, or mathematically: $\operatorname{Im} A = \{Av \mid v \in V\}$.

Definition. The *kernel* of a linear operator $A : V \to W$ is a subset of V that consists of all vectors that are sent to zero, or mathematically: $\operatorname{Ker} A = \{v \in V \mid Av = 0\}$.

Definition. The *rank* of a linear operator $A : V \to W$ is the dimension of its image $\dim \operatorname{Im} A$.

1. Prove that $\operatorname{Im} A$ and $\operatorname{Ker} A$ are vector subspaces of W and V, respectively.

   Solution:

2. How can an $m \times n$ matrix be viewed as a linear transformation? What are the dimensions of the two vector spaces?

   Solution:

3. Let us consider a matrix A as a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $e_1, \dots, e_n$ be the *standard basis vectors* of $\mathbb{R}^n$, that is: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\dots$, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. Describe $Ae_1, \dots, Ae_n$ in terms of A. Conclusion: we can define a linear operator $A : V \to W$ by its action on a basis of V.

   Solution:

4. Let V be the space of polynomials in two variables of the form $f(x, y) = a + bx + cy + dx^2$, and let W be the space of degree one polynomials in two variables.

   a) Find (the simplest) bases of V and W. What are the dimensions of these spaces?

   b) Consider a linear operator $A = \frac{d}{dx}$ from V to W. Write A as a matrix in the bases that we found in part (a).

   c) What are the nullspace and column space of A? What are the kernel and image of $\frac{d}{dx}$? What is the conclusion?

   d) What is the rank of A?

   e) *Bonus.* Let us add twice the second column of A to the first, and denote the new matrix (linear transformation) by A'. How did the transformation change?

   f) *Bonus.* Write A' as a composition of A and some other linear transformation M. What are the vector spaces that M operates between?

   Hint: recall column operations and how they are related to matrix multiplication on the right.

Solution:

5. Fix a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ of rank r. Describe the relations between the four fundamental subspaces in terms of kernel and image. *Tricky question:* Would you be able to do that if we said that $A : V \to W$ with the same rank and dimensions of the spaces?

Solution:

6. Fix a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Understand that if $b \in \mathbb{R}^m$ is in the image of A, then the system $Ax = b$ has a solution, say x_0. In this case, show in addition that the space of all solutions is $x_0 + \operatorname{Ker} A$. Now conclude that in the case of nonzero kernel (nullspace), the system $Ax = b$ has either infinitely many solutions or no solutions at all, depending on whether $b \in \operatorname{Im} A$ or not.