Determine whether or not these objects exist. If so, write down an example. If not, explain why not.

(1) A 3×2 matrix whose columns are linearly independent.

(2) A 2×3 matrix whose columns are linearly independent.

(3) A noninvertible 4×4 matrix whose columns span $\mathbb{R}^4$.

(4) A basis of the vector space $\mathrm{null}(\begin{pmatrix} 1 & 1 & 2 \end{pmatrix})$.

(5) An orthogonal matrix whose rows are linearly dependent.

(6) A nonidentity matrix which equals its own inverse.

(7) A matrix A such that $\mathrm{null}(A) = \mathrm{col}(A)$.

(8) A basis $\{v_1, v_2, v_3\} \in \mathbb{R}^3$ such that $\|v_i - v_j\| = 1$ for all $i \neq j$.

(9) An orthogonal matrix Q such that $\mathrm{null}(QQ^\top) \cap \mathrm{col}(QQ^\top)$ is larger than $\{0\}$.

(10) A 3×2 matrix A and a 2×3 matrix B such that $AB = \mathrm{Id}_{3\times 3}$.

(11) Two matrices A, B such that $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathrm{null}(B)$ is larger than $\{0\}$.

(12) Two matrices A, B such that $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathrm{null}(B)$ is larger than $\{0\}$.

(13) Two linearly independent vectors in $\mathrm{null}(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix})$ which are both perpendicular to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(14) A rank-one matrix whose columns are linearly independent.

(15) An nonzero upper-triangular matrix whose columns are linearly dependent.

(16) Two matrices A, B such that $AB = \mathrm{Id}_{4\times 4}$, the matrix A is not invertible, and the columns of B are linearly independent.

(17) A diagonal matrix Σ such that $P\Sigma P$ is not diagonal, where $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(18) A 3×4 matrix A and a 3-vector b such that $Ax = b$ has a unique solution.

(19) A spanning set $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ with $v_i + v_j + v_k = 0$ whenever i, j, k are all different.

(20) Nonzero 2×2 projection matrices P, Q, R satisfying $P + Q + R = \mathrm{Id}_{2\times 2}$.

(21) Nonzero 2×2 projection matrices P, Q, R satisfying $P + Q + R = \frac{3}{2}\,\mathrm{Id}_{2\times 2}$.

Solutions

DNE stands for 'does not exist.'

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(2) DNE. If the columns were independent, the rank would be 3, but the rank of a 2×3 matrix is ≤ 2.

(3) DNE. If the columns span $\mathbb{R}^4$, then the rank is 4, so the matrix is invertible (e.g. because Σ in the full SVD is invertible).

(4) $\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$.

(5) Any $n \times m$ orthogonal matrix with $n > m$ works.

(6) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(7) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(8) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

(9) DNE. If $P^2 = P$, then $\text{null}(P) \cap \text{col}(P) = \{0\}$. Indeed, any x which lies in the intersection must satisfy $Px = 0$ and $x = Py$ for some y. But applying P to the second equation gives $0 = Px = Py = x$, so $x = 0$. Now apply this to $P = QQ^\top$.

(10) DNE. Since B is 2×3, its null space has dimension $\geq 3 - 2 = 1$. Since $\text{null}(AB) \supseteq \text{null}(B)$, the former has dimension ≥ 1. But $\text{null}(\text{Id}_{3\times 3}) = \{0\}$, contradiction.

(11) DNE. Since $\text{null}(AB) \supseteq \text{null}(B)$, we conclude that $\text{null}(B) = \{0\}$ since the nullspace of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $\{0\}$.

(12) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(13) DNE. The two vectors must lie in the nullspace of $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Since this nullspace is 1-dimensional, it cannot contain two linearly independent vectors.

(14) Any nonzero matrix of size 1×1 works.

(15) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

(16) Take B to be any 5×4 orthogonal matrix, and take $A = B^\top$.

(17) DNE. For the multiplication to make sense, Σ must be a 4×3 diagonal matrix, and one can check that $P\Sigma P = \Sigma$ by directly doing the multiplication.

(18) DNE. Since $\text{null}(A)$ is bigger than $\{0\}$ (it has dimension $\geq 4 - 3 = 1$), a solution to $Ax = b$ is never unique; if x is a solution, then $x + v$ is another solution, for any $v \in \text{null}(A)$.

(19) DNE. By using $v_1 = -v_2 - v_3$, we conclude that

$$\mathrm{span}(v_1, v_2, v_3, v_4, v_5) = \mathrm{span}(v_2, v_3, v_4, v_5).$$

By using $v_2 = -v_3 - v_4$, we conclude that

$$\mathrm{span}(v_2, v_3, v_4, v_5) = \mathrm{span}(v_3, v_4, v_5).$$

But the latter has dimension ≤ 3, so it cannot equal $\mathbb{R}^4$.

(20) DNE. Since P, Q, R are nonzero, each must have rank 1 or 2. We break into two cases.

Case 1. Suppose at least one of P, Q, R has rank 2, e.g. P has rank 2. Then P is invertible, so $P^2 = P$ implies that $P = \mathrm{Id}_{2\times 2}$, and hence $Q + R = 0$. But then setting $Q = -R$ in $Q^2 = Q$ yields $R^2 = -R$, which contradicts $R^2 = R$ since R is nonzero. So this case can't happen.

Case 2. Suppose each of P, Q, R has rank 1. Then $Q + R = \mathrm{Id}_{2\times 2} - P$. Define $\tilde{P} := \mathrm{Id}_{2\times 2} - P$, and observe that $\tilde{P}$ is a projection matrix of rank 1. Let $v \in \mathrm{null}(\tilde{P})$ be a nonzero vector. The relation $Q + R = \tilde{P}$ implies that $Qv = -Rv$, and now we break into two subcases.

Subcase 2a. Assume that $Qv = 0$, so $Rv = 0$. This implies that $\mathrm{null}(Q) = \mathrm{null}(R) = \mathrm{null}(\tilde{P})$. For any $x \in \mathbb{R}^2$ not in $\mathrm{null}(\tilde{P})$, we have

$$\begin{aligned} Qx - x &\in \mathrm{null}(Q) = \mathrm{null}(\tilde{P}) \\ Rx - x &\in \mathrm{null}(R) = \mathrm{null}(\tilde{P}) \\ -\tilde{P}x + x &\in \mathrm{null}(\tilde{P}). \end{aligned}$$

Adding these and using $Q + R = \tilde{P}$ shows that $-x \in \mathrm{null}(\tilde{P})$, contradicting our choice of x. So this subcase can't happen.

Subcase 2b. Assume that $Qv \neq 0$, so $Rv \neq 0$. This implies that $\mathrm{col}(Q) = \mathrm{col}(R) = \mathrm{col}(\tilde{P})$. For any nonzero $x \in \mathrm{col}(\tilde{P})$, we have $Qx = Rx = \tilde{P}x = x$, but then the relation $Q + R = \tilde{P}$ implies that $2x = x$, contradicting our choice of x. So this subcase can't happen.

A simpler solution can be given by looking at the trace of these matrices – the sum of the diagonal entries. The key idea is that the trace of a projection matrix is equal to its rank.

(21) Take three 1-dimensional subspaces of $\mathbb{R}^2$ at $60°$-angles to one another, and let P, Q, R be orthogonal projections onto those subspaces.