♡ Week 2 Extra ♡

Focus: LU decomposition (aka Gaussian elimination), orthogonal matrices.

LU factorization of a matrix A is a way of writing A as a product of two matrices $\boxed{A = LU}$, where L is a lower-diagonal matrix with units on the diagonal and U is an upper-diagonal matrix.

Definition. A square matrix A is called *orthogonal* if $A^T A = I$. (The unit matrix is denoted by I.)

1. Compute the result of multiplying a row $r = \begin{pmatrix} r_1 & \cdots & r_n \end{pmatrix}$ by a matrix M with rows $M_1, \ldots, M_n$.

   Solution:

2. If R is a $k \times n$ matrix and M is a matrix with n rows $M_1, \ldots, M_n$, what are the rows of RM in terms of $M_1, \ldots, M_n$ and the matrix coefficients of R?

   Solution:

3. *Bonus. LU factorization = Gaussian elimination.* Solve the system of linear equations using LU factorization:

$$\begin{cases} x + 2y + 3z = 1, \\ \quad\;\; y + \;\; z = 2, \\ 3x + \;\; y - \;\; z = 3. \end{cases}$$

   Solution:

4. *Not all matrices can be written in LU form.* Show directly why these matrix equations are both impossible (empty spaces mean zeroes):

   a) $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$;

   b) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{pmatrix} \begin{pmatrix} d & e & g \\ & f & h \\ & & i \end{pmatrix}$.

Solution:

5. *Orthogonal matrices.* Find $A^T A$ if the columns of A are unit vectors, all mutually perpendicular (in this situation, we say that the vectors are *orthonormal*). What if we ask that the rows of A are orthonormal?

Solution:

6. *Bonus.* Say that a square matrix A is factored as a product $A = B^{-1}C$. Perform the same row operation on both B and C, for example add the first row to the second: $B \mapsto B'$, $C \mapsto C'$. Show that this operation does not change the result, that is $A = B^{-1}C = (B')^{-1}C'$.

Hint: think about the first two problems.

Solution:

7. *Binomial formula for matrices.* Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$ when

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}.$$

Write down the correct rule: $(A+B)^2 = A^2 + \cdots + B^2$. Can you generalize the rule to $(A+B)^n$?

Solution: