Recitation 3/31

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Σ as coordinate transformation

- Lecture slide 23 page 3, Summary on page 6

- A be m-by-n matrix, $u \in \text{col}(A) = \text{col}(U_1) \subset \mathbb{R}^m$, $v \in \text{row}(A) = \text{col}(V_1) \subset \mathbb{R}^n$

Since u is a vector in column space and v is a vector in row space, we have coordinates b, c(coefficients of linear combination), so that

$$u = U_1 b, \quad v = V_1 c, \quad b, c \in \mathbb{R}^r$$

If $u = Av$, we can say $b = \Sigma_r c$

Orthogonal Subspaces

If V and W are vector subspaces of $\mathbb{R}^n$, we say that V and W are orthogonal if

$$\forall v \in V, \forall w \in W \text{ we have } < v, w >= 0 \text{ (or } v \perp w \text{ or } v^T w = 0)$$

In other words, every vectors from V and W are perpendicular to each other. Denote $V \perp W$

Orthogonal Complement

Given vector subspace $V \subset \mathbb{R}^n$, the **Orthogonal complement** of V is denoted by $V^\perp$, and defined as the set of all $w \in \mathbb{R}^n$ such that w is perpendicular to all vectors in V.

It can be thought as the largest subspace orthogonal to V.

Problems

1.(a) Find any 3 orthogonal subspaces of $V \in \mathbb{R}^5$, where $V = \text{span}(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix})$

(b) What is the orthogonal complement of V?

2. Full SVD of $A \in \mathbb{R}^{4\times 5}$ is given as,

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 0 & -1\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) What is the rank? What are U_1, V_1?

(b) $u = \begin{pmatrix} 7/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 5/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$ satisfies $u = Av$. Find $b, c \in \mathbb{R}^3$ such that $u = U_1 b$ and $v = V_1 c$.

(c) Find simple relationship between b and c.

(d) Let x be $\begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Compute $y = Ax$ without hard computation.

3. Why are col(A) and null(A^T) orthogonal complements to each other? What about row(A) and null(A^T)? Explain in terms of SVD.

ANSWERS

1.(a) span$\left(\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right)$, span$\left(\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\right)$, span$\left(\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right)$, $\cdots$

(b) Any vectors that are perpendicular to all vectors of $\begin{pmatrix}x\\0\\0\\y\\0\end{pmatrix}$ has nonzero elements on 2, 3, 5th entries.

So the orthogonal complement of V is,

$$\text{span}\left(\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right)$$

2.(a) Rank is number of nonzers on the diagonal, 3. U_1 and V_1 are,

$$U_1 = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, V_1 = \begin{pmatrix}0 & 1 & 0 & 0 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0\\ 1/\sqrt{2} & 0 & -1\sqrt{2} & 0 & 0\end{pmatrix}^T$$

(b) You can either utilize the formula, $b = U_1^T u$ or set up a equation

$$x\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix} + y\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = u$$

where $b = \begin{pmatrix}x\\y\\z\end{pmatrix}$. You can solve it with Gaussian elimination or etc... Also same for v. The answer is,

$$b = \begin{pmatrix}3\\4\\3\end{pmatrix}, c = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

(c) We can see that $b = \Sigma_r c$

3. It is because the matrix U_1 and U_2 have orthogonal column spaces. In other words, all the vectors in basis of U_1 and all the vectors in basis of U_2 are perpendicular to each other, because they together form a orthogonal matrix. So col$(U_1) \perp$ col(U_2) which means col$(A) \perp$ null(A^T). Also if there is any vector v outside null(A^T) that is perpendicular to col(A), Q of QR factorization of $\begin{pmatrix}U_1 & U_2 & v\end{pmatrix}$ becomes a n by $(n+1)$ orthogonal matrix, which cannot exist.