1 Lecture Review

1.1 Positive Definite Matrices

If a symmetric matrix S has one of these properties, it has them all:

1. All eigenvalues are > 0 (S is positive definite).
2. $S = LU$ where L is unit lower triangular and U is upper triangular with positive diagonal entries.
3. $S = LDL^T$ where L is unit lower triangular and D is a diagonal with positive diagonal entries.
4. All n upper left determinants are positive.
5. $x^T Sx > 0$ unless $x = 0$.
6. $S = A^T A$ for A with independent columns.
7. The compact SVD = the full SVD = an eigenfactorization.

2 Problems

1. Show that if S is positive semidefinite, then $x^T Sx \geq 0$ for any vector x. Then show the other direction: if $x^T Sx \geq 0$ for any vector x, then S is positive semidefinite.

2. Show that if $S = A^T A$ for some matrix A, then $x^T Sx \geq 0$ for any vector x.

3. Let P be the $n \times n$ matrix
$$\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}.$$
Is this matrix: (i) symmetric, (ii) positive semi-definite, (iii) positive-definite, (iv) Markov, (v) positive Markov, (vi) a projection matrix, (vii) rank 1? Explain, then determine the eigenvalues of P.

4. Let P be the matrix from the previous problem. Let $M = I - P$ and answer (i)-(vii) from the previous problem for this matrix. What is the rank of M and what are its eigenvalues?

5. True or False:
   (a) Every positive definite matrix is invertible.
   (b) The only positive definite projection matrix is $P = I$.
   (c) Every projection matrix is positive semidefinite.
   (d) A diagonal matrix with positive diagonal entries is positive definite.
   (e) A symmetric matrix with positive determinant is positive definite.

6. Without multiplying
$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$
find
   (a) the determinant of S
   (b) the eigenvalues of S
   (c) the eigenvectors of S
   (d) a reason why S is positive definite.

7. Suppose $A = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$. Explain why $ad - b^2 > 0$ and $a + d > 0$ imply A is positive definite. How should we change these conditions to ensure that the eigenvalues have opposite sign.

8. Find an example of a 3×3 matrix with positive determinant and positive trace which is not positive definite.

9. True or False (assume A is $n \times n$):
   (a) If A is a matrix whose columns sum to 0, then $A + I$ is a Markov matrix.
   (b) If A is a diagonal matrix and is a Markov matrix, then $A = I$.
   (c) If A is Markov then $I - A$ is positive semidefinite.
   (d) If A is positive Markov then $I - A$ has rank $n - 1$.