18.06 R08 - Recitation 3 - SOLUTIONS

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1 Problems

Problem 1.
Can a set of linearly independent vectors contain the zero vector?

Solution:
A sequence of vectors $v_1, ..., v_n$ is linearly independent when

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0, \quad a_1 = a_2 = ... = a_n = 0$$

Suppose one of these vectors is the zero vector, say $v_1 = 0$. Then a_1 can be any real number, while the other coefficients are zero. But then we have found a non trivial linear combination that gives the zero vector. So a set of vectors containing the zero vector can never be linearly independent.

Problem 2.
Find a basis for the following vector spaces and state the dimension of the vector space[1]:

1. The set of all polynomials with degree ≤ 3.
2. The set of all vectors in $\mathbb{R}^3$ whose components are equal.
3. The set of all vectors in $\mathbb{R}^3$ whose components average to zero.
4. The set of all 3×3 antisymmetric matrices.

Solution:

1. The most general polynomial of degree ≤ 3 takes the form $ax^3 + bx^2 + cx + d$. We can therefore write any such polynomial as a linear combination of the functions in the set $\{x^3, x^2, x, 1\}$. This set therefore spans the vector space. Furthermore, an arbitrary linear combination of these spanning vectors yields the zero function, i.e. $ax^3 + bx^2 + cx + d = 0$, only if $a = b = c = d = 0$. Hence this set is also linearly independent. The set

   $$\boxed{\{x^3, x^2, x, 1\}}$$

   is therefore a basis for this vector space, and the vector space has dimension $\boxed{4}$.

2. The most general vector in $\mathbb{R}^3$ whose components are equal is

   $$\begin{pmatrix} a \\ a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

[1] To attempt these kinds of questions you should do the following: firstly write down the most general vector in your vector space. Then deconstruct this vector into a set of vectors that you can be certain spans the vector space. Then finally test whether they are linearly independent. If they *are* linearly independent then you're all set. If they are not, then try to use your intuition to figure out how to make them independent.

. Therefore the set

$$\boxed{\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}}$$

spans this vector space. Furthermore, any set containing only one nonzero vector is a linearly independent set. So this is a basis for this vector space, which has dimension $\boxed{1}$.

3. The most general vector in $\mathbb{R}^3$ whose components all average to zero takes the form

$$\begin{pmatrix}a\\b\\-(a+b)\end{pmatrix}=a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

The set

$$\boxed{\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}}$$

therefore spans the vector space. It is also a linear independent set, since the only linear combination that gives the zero vector is when $a=b=0$. This set is then a basis for the vector space, which has dimension $\boxed{2}$.

4. The most general 3×3 anti-symmetric matrix takes the form

$$\begin{pmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{pmatrix}=a\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}+b\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}+c\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

The set

$$\boxed{\left\{\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\right\}}$$

therefore spans this vector space. This set is also linearly independent, since the only way a linear combination of these matrices will give the zero matrix is if $a=b=c=0$. This set is then a basis for the vector space, which has dimension $\boxed{3}$.

Problem 3.
Consider the following four full SVDs:

$$A_1 = \begin{pmatrix} -0.1965 & -0.3551 & -0.7175 & 0.5661 \\ -0.2649 & -0.7272 & 0.5976 & 0.2094 \\ -0.4527 & -0.3208 & -0.3224 & -0.7669 \\ -0.8284 & 0.4921 & 0.1553 & 0.2179 \end{pmatrix} \begin{pmatrix} 11.0304 & 0 & 0 \\ 0 & 3.2142 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.0318 & -0.8159 & -0.5774 \\ -0.7225 & -0.3804 & 0.5774 \\ -0.6907 & 0.4355 & -0.5774 \end{pmatrix}^T$$

$$A_2 = \begin{pmatrix} -0.1408 & 0.8944 & 0.4245 \\ 0.2816 & 0.4472 & -0.8489 \\ -0.9492 & 0 & -0.3148 \end{pmatrix} \begin{pmatrix} 4.0600 & 0 & 0 & 0 \\ 0 & 1.7321 & 0 & 0 \\ 0 & 0 & 1.2315 & 0 \end{pmatrix} \begin{pmatrix} -0.2685 & 0.5164 & 0.0890 & -0.8083 \\ -0.1644 & 0.2582 & -0.9450 & 0.1155 \\ -0.8054 & 0.2582 & 0.2671 & 0.4619 \\ 0.5022 & 0.7746 & 0.1666 & 0.3464 \end{pmatrix}^T$$

$$A_3 = \begin{pmatrix} -0.7503 & -0.5300 & 0.3951 \\ -0.4961 & 0.8464 & 0.1935 \\ -0.4370 & -0.0509 & -0.8980 \end{pmatrix} \begin{pmatrix} 6.4901 & 0 & 0 \\ 0 & 4.6650 & 0 \\ 0 & 0 & 1.0569 \end{pmatrix} \begin{pmatrix} -0.4796 & 0.4089 & -0.7764 \\ -0.6043 & 0.4876 & 0.6301 \\ -0.6363 & -0.7714 & -0.0132 \end{pmatrix}^T$$

$$A_4 = \begin{pmatrix} -0.5647 & -0.1174 & 0.6460 & -0.5000 \\ 0.0779 & 0.6453 & 0.5723 & 0.5000 \\ -0.1627 & 0.7548 & -0.3921 & -0.5000 \\ -0.8053 & -0.0078 & -0.3184 & 0.5000 \end{pmatrix} \begin{pmatrix} 3.9255 & 0 & 0 \\ 0 & 2.1292 & 0 \\ 0 & 0 & 0.2393 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.0216 & 0.6576 & 0.7531 \\ -0.9446 & 0.2332 & -0.2308 \\ 0.3274 & 0.7164 & -0.6161 \end{pmatrix}^T$$

Decide which of the above matrices corresponds to each of the following situations. In each case state a b for which $Ax = b$ has a solution.

1. $Ax = b$ has 0 or 1 solutions, depending on b
2. $Ax = b$ has infinitely many solutions, regardless of b
3. $Ax = b$ has 0 or infinitely many solutions, depending on b
4. $Ax = b$ has a unique solution, regardless of b.

Solution

1. The matrix A_4 is a 4×3 matrix with rank $r = 3$. The system $A_4x = b$ can then have 0 or 1 solution depending on whether or not $b \in C(A)$. This will happen when b can be expressed as a linear combination of the first 3 columns of U. This matrix only has the zero vector in its nullspace, since A has full column rank, and so when solutions exist they are necessarily unique.
2. The matrix A_2 is a 3×4 matrix with rank $r = 3$. The system $A_2x = b$ will always have a solution, since A_2 has full row rank, and so every b is in $C(A)$. However, A_2 does not have full column rank, and so has a non trivial nullspace. Therefore this system will have infinitely many solutions, regardless of b
3. The matrix A_1 is a 4×3 matrix with rank $r = 2$. The system $A_1x = b$ does not necessarily have a solution, but if it does have a solution it will not be unique since A has a nontrivial subspace. This system therefore has 0 or infinitely many solutions.
4. The matrix A_3 is a 3×3 matrix with rank $r = 3$. The system $A_3x = b$ always has a unique solution, since A_3 is then an invertible matrix.

The first column of U for each of these examples will always be in the column space, and so this will always be a possible b for which $Ax = b$ is solvable.

Problem 4.

1. Find the projection p of the vector b onto the column space of A, where
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$
Verify that $e = b - p$ is orthogonal to the columns of A.

2. ****If P is a projection matrix, then show that $(I-P)^T = (I-P)$ and $(I-P)^2 = I - P$ (so $I - P$ is also a projection matrix). If P projects onto the column space of a matrix A, then $I - P$ projects onto which subspace? If $P = QQ^T$, where Q is orthogonal, show that $B = I - 2P$ is orthogonal.

Solution

1. One way to approach this is to recall that the projection of a vector b onto the column space of A (provided A has independent columns), will be given by
$$p = A(A^TA)^{-1}A^Tb.$$
In practice, however, we very rarely want to *explicitly* calculate an inverse. Therefore, it is usually better to solve the system
$$A^TA\hat{x} = A^Tb,$$
and then the projection is
$$p = A\hat{x}.$$
Formally these two processes are the same, but numerically it is usually much easier to solve $A^TA\hat{x} = A^Tb$ than it is to explicitly find $(A^TA)^{-1}$. In this problem
$$A^TA = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}, \quad A^Tb = \begin{pmatrix} 7 \\ 10 \end{pmatrix}.$$
We can then solve the 2×2 system $A^TA\hat{x} = A^Tb$ using whatever method we like, to find that
$$\boxed{\hat{x} = \begin{pmatrix} 1/2 \\ 3 \end{pmatrix}}$$
Finally we can compute
$$\boxed{p = A\hat{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}}$$

2. The first part is on your problem set. If P projects onto the column space of a matrix A, then $I - P$ projects onto the left nullspace. If $P = QQ^T$, where Q is orthogonal, then
$$\begin{aligned} B^TB &= (I-2P)^T(I-2P) \\ &= I - 2P^T - 2P + 4P^TP \\ &= I - 2QQ^T - 2QQ^T + 4QQ^T \\ &= I, \end{aligned}$$
and so B is orthogonal.