1 Lecture Review

1.1 Independence, Basis and Dimension

1.1.1 Definitions

1. A set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent if the only constants $c_1, \ldots, c_n$ which solve
$$c_1 v_1 + \cdots + c_n v_n = 0$$
are $c_1 = \cdots = c_n = 0$.
2. A set of vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if it is not linearly independent.
3. The span of a set of vectors $\{v_1, \ldots, v_n\}$, denoted $\mathrm{span}(v_1, \ldots, v_n)$, is the set of linear combinations of $v_1, \ldots, v_n$.
4. If V is a vector space, we say that $\{v_1, \ldots, v_n\}$ is a basis for V if it is linearly independent and spans V.
5. If V is a vector space such that $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are both bases for V, then $m = n$. We call the number n (or m) the dimension of V, denoted $\dim(V)$.
6. If A is a square matrix, we say that A is nonsingular if it is invertible, and otherwise say it is singular.

1.1.2 Properties and Examples

Let A be an $m \times n$ matrix.

1. The column space of A is equal to the span of the columns of A; if $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are the columns of A this can be expressed as $\mathrm{col}(A) = \mathrm{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$.
2. The dimension of the column space is equal to the rank of A: $\dim(\mathrm{col}(A)) = \mathrm{rank}(A)$.
3. Any set of n vectors in $\mathbb{R}^m$ is linearly dependent if $n > m$.
4. There is one and only one way to write v as a linear combination of basis vectors.
5. The following statements are equivalent:
   (a) The columns of A are linearly independent.
   (b) The columns of A form a basis for $\mathrm{Col}(A)$.
   (c) The only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = 0$.
   (d) $\mathrm{Null}(A) = \{\boldsymbol{0}\}$.
   (e) The rank of A is n (the number of columns of A).
   (f) For any $\boldsymbol{b} \in \mathbb{R}^m$, there is exactly 0 or 1 solution to $A\boldsymbol{x} = \boldsymbol{b}$. There is 1 exactly when $\boldsymbol{b} = UU^T\boldsymbol{b}$.
6. The following statements are equivalent:
   (a) $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is a basis for $\mathbb{R}^n$.
   (b) The matrix $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$ is invertible.
   (c) For any $\boldsymbol{b} \in \mathbb{R}^n$, the matrix $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution in $\mathbb{R}^n$.
   (d) The full form SVD and compact form SVD of A are the same.

2 Problems

1. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be vectors.
   (a) Check that the span of the $\boldsymbol{v}$'s form a vector space.
   (b) If $n = 3$, show that that span is either $\mathbb{R}^3$, a plane, a line, or a point. When is it a point?

2. Describe the subspace of $\mathbb{R}^3$ (is it a line or a plane or $\mathbb{R}^3$) spanned by the following vectors, then identify a basis:
   (a) The vectors $(1, 1, -1)$ and $(-1, -1, 1)$.
   (b) The vectors $(0, 1, 1)$, $(1, 1, 0)$ and $(0, 0, 0)$.
   (c) All vectors in $\mathbb{R}^3$ with integer components.
   (d) All vectors with positive components.

3. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

   What is the span of the $\boldsymbol{v}$'s?

4. Suppose $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors and $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$, $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$, $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$.
   (a) Show that the $\boldsymbol{v}$'s are dependent.
   (b) Which of the following matrices are nonsingular: $A = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix}, B = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}$.
   (c) Explain why we can always find a unique solution to $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b} \in \mathbb{R}^3$.
   (d) Explain (using only linear independence) why $\text{Null}(B)$ contains more than a point. Find a nonzero vector in this null space.

5. Consider the plane P with equation $x - 2y + 3z = 0$ in $\mathbb{R}^3$.
   (a) Find a basis for the plane P.
   (b) Find a basis for the intersection of P with the xy-plane.
   (c) Find a basis for all vectors perpendicular to plane P.

6. Find a basis and the dimension for the following subspaces of 3×3 matrices:
   (a) All diagonal matrices
   (b) All symmetric matrices ($A^T = A$).
   (c) All antisymmetric matrices ($A^T = -A$).

7. Find a basis for the space of 2×3 matrices whose nullspace contains $(2, 1, 1)$.