Recitation 8. April 22

Focus: eigenvectors, eigenvalues and eigendecomposition.

Notation. For the rest of this worksheet, let A be an $n \times n$ matrix operating on an n-dimensional vector space V, so $A : V \to V$.

Definition. A nonzero vector $v \in V$ is called an *eigenvector* for the matrix A if for some real or complex scalar λ we have $Av = \lambda v$.

Definition. The value λ is then called the *eigenvalue* corresponding to this eigenvector v.

Remark. Since for the eigenvector v we have $(A - \lambda)v = 0$, the matrix $A - \lambda I$ is not invertible, and so an eigenvalue is necessarily a root of the polynomial $\chi_A(\lambda) = \det(A - \lambda I)$.

Definition. If all roots of $\chi_A(\lambda)$ are different, then A is *diagonalizable*, which means that we can write $A = X\Lambda X^{-1}$ for some diagonal matrix Λ and invertible matrix X. This representation of A as $X\Lambda X^{-1}$ is called *eigendecomposition*.

1. Suppose we have $B = XAX^{-1}$.
   a) Prove that $\chi_B(\lambda) = \chi_A(\lambda)$.
   b) How are eigenvalues of B related to those of A?
   c) How are eigenvectors of B related to those of A?
   d) Suppose that one of the eigenvalues of A is zero. Does it mean that A is singular? Does it mean that B is singular?

Solution:

2. Give an example of a diagonalizable matrix with a pair of equal eigenvalues.

Solution:

3. Prove that if V is odd-dimensional, then $A : V \to V$ has at least one real eigenvalue.

Solution:

4. *Closed formula for Fibonacci numbers.* Let F_i denote the ith element in the Fibonacii sequence, defined by setting $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all natural values of i (including zero).

a) Find a matrix A such that $A \begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix}$.

b) Find the eigenvalues of A. Let φ denote the largest eigenvalue.

c) Find the eigenvectors of A.

d) Compute A^{50} up to nine decimal points. You can only use simple calculators (e.g. Google engine), no matrix calculators are needed.

e) Using the result of part (c), explain why $\frac{F_{51}}{F_{50}}$ is very close to φ.