1 Lecture Review

1.1 Positive Definite Matrices

If a symmetric matrix S has one of these properties, it has them all:

1. All eigenvalues are > 0 (S is positive definite).
2. $S = LU$ where L is unit lower triangular and U is upper triangular with positive diagonal entries.
3. $S = LDL^T$ where L is unit lower triangular and D is a diagonal with positive diagonal entries.
4. All n upper left determinants are positive.
5. $x^T Sx > 0$ unless $x = 0$.
6. $S = A^T A$ for A with independent columns.
7. The compact SVD = the full SVD = an eigenfactorization.

2 Problems

1. Show that if S is positive semidefinite, then $x^T Sx \geq 0$ for any vector x. Then show the other direction: if $x^T Sx \geq 0$ for any vector x, then S is positive semidefinite.

   Solution. Assume first that S is positive semidefinite. Since S is symmetric, we have a diagonalization $S = Q\Lambda Q^{-1}$ with an orthogonal Q, so in fact $S = Q\Lambda Q^T$. Since S is positive semidefinite, the diagonal entries λ_i of Λ are nonnegative. Let us denote by q_i the ith column of Q – the ith eigenvector of S, so $Sq_i = \lambda_i q_i$.

   Now take any vector x. Since S is diagonalizable, its eigenvectors span the whole vector space, so $x = \sum_i c_i q_i$. Then compute:

$$x^T Sx = x^T S \sum_i c_i q_i = \sum_i c_i x^T \lambda_i q_i = \sum_i c_i \lambda_i \left(\sum_j c_j q_j^T q_i \right).$$

   Since Q is orthogonal, the system of vectors $q_1, \ldots, q_n$ is orthonormal, so we can continue the sequence of equalities:

$$x^T Sx = \sum_i c_i \lambda_i \left(c_i q_i^T q_i \right) = \sum_i c_i^2 \lambda_i.$$

   Since we know that $\lambda_i \geq 0$ and c_i^2 is always nonnegative, we can conclude that $x^T Sx \geq 0$ for every vector x.

   Now prove the reverse direction. Assume that for any vector x, we have $x^T Sx \geq 0$. Now, if x is an eigenvector with eigenvalue λ, then $0 \leq x^T Sx = x^T \lambda x = \lambda \, ||x||^2$. Therefore, the eigenvalue λ must be nonnegative. □

2. Show that if $S = A^T A$ for some matrix A, then $x^T S x \geq 0$ for any vector x.

   Solution. $x^T S x = x^T A^T A x = (Ax)^T (Ax) = ||Ax||^2 \geq 0$. □

3. Let P be the $n \times n$ matrix

$$\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}.$$

Is this matrix: (i) symmetric, (ii) positive semi-definite, (iii) positive-definite, (iv) Markov, (v) positive Markov, (vi) a projection matrix, (vii) rank 1? Explain, then determine the eigenvalues of P.

Solution. (i) Yes.

(ii) Observe that $v = (1, \ldots, 1)^T$ is an eigenvector with eigenvalue 1. Moreover, P is of rank one, so the nullspace is of dimension $n-1$, so 0 is an eigenvalue with multiplicity $n-1$. So the eigenvalues are 0 and 1, both nonnegative, so P is positive semidefinite.

(iii) No, because 0 is an eigenvalue. Unless $n = 1$, then $P = (1)$ is positive definite.

(iv) Yes.

(v) Yes.

(vi) Yes, because $P^2 = P$ and $P^T = P$.

(vii) Yes, because all the columns are equal.

□

4. Let P be the matrix from the previous problem. Let $M = I - P$ and answer (i)-(vii) from the previous problem for this matrix. What is the rank of M and what are its eigenvalues?

Solution. (i) Yes, because it is a difference between two symmetric matrices and taking transpose is a linear operation.

(ii) If x is an eigenvector of P with eigenvalue λ, then it is also an eigenvector of M with a different eigenvalue $1-\lambda$, because $Mx = (I-P)x = x - Px = x - \lambda x = (1-\lambda)x$. So M has one eigenvalue 0 and the eigenvalue of 0 has multiplicity $n-1$. So it is also positive semidefinite.

(iii) No, because 0 is an eigenvalue.

(iv) No, because it has negative entries $-\frac{1}{n}$ (if $n > 1$) and moreover the sum of entries in each column is equal to 0, not to 1.

(v) No, because it is not even Markov.

(vi) Yes, because $(I-P)^2 = I - P - P + P^2 = I - P$, using $P^2 = P$, and $(I-P)^T = I - P$.

(vii) No, unless $n = 2$. The rank of the matrix M is $\operatorname{rk} M = \dim \operatorname{Col} M = n - \dim \operatorname{Nul} M = n - 1$.

□

5. True or False:

   (a) Every positive definite matrix is invertible.

   (b) The only positive definite projection matrix is $P = I$.

   (c) Every projection matrix is positive semidefinite.

   (d) A diagonal matrix with positive diagonal entries is positive definite.

   (e) A symmetric matrix with positive determinant is positive definite.

Solution. (a) True. Because if S is positive definite, then $S = Q\Lambda Q^{-1}$ – a product of three invertible matrices, and here Λ is invertible, because it is diagonal with nonzero diagonal entries.

(b) True. If P is a projection matrix, then from $P^2 = P$ we conclude that its eigenvalues are necessarily 0 or 1. If P is in addition positive definite, then they should all be 1. So P is similar to the identity matrix I, hence must be I itself.

(c) True, because the possible eigenvalues are 0 and 1.

(d) True, because it is symmetric and the diagonal entries are its eigenvalues.

(e) False, because we can take the 2×2 matrix $-I$ which is negative definite and whose determinant is 1.

$\square$

6. Without multiplying

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

find

(a) the determinant of S

(b) the eigenvalues of S

(c) the eigenvectors of S

(d) a reason why S is positive definite.

Solution. Note that we can introduce the notation $S = Q\Lambda Q^T$. Here Q is an orthogonal matrix, so we also have $S = Q\Lambda Q^{-1}$. Therefore, we are given a diagonalization of S.

(a) $\det S = \det \Lambda = 2 \cdot 5 = 10$, and here we used the fact that $\det S$ is equal to the determinant of its diagonalized form.

(b) Since $S = Q\Lambda Q^{-1}$ is a diagonalization, the eigenvalues are 2 and 5 – the diagonal entries of Λ.

(c) The eigenvectors of S are the two columns of Q.

(d) The matrix S is symmetric, because $S^T = (QQ^T)^T = Q\Lambda Q^T = S$, and has positive eigenvalues 2 and 5, and therefore is positive definite.

□

7. Suppose $A = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$. Explain why $ad - b^2 > 0$ and $a + d > 0$ imply A is positive definite. How should we change these conditions to ensure that the eigenvalues have opposite sign.

   Solution. Let λ and μ be the two eigenvalues of the symmetric matrix A. Then $\lambda\mu = \det A = ad - b^2 > 0$ and hence λ and μ must have the same sign. If in addition $\lambda + \mu = \operatorname{tr} A = a + d > 0$, then these values must be positive. So A is necessarily positive definite.

   If we want opposite signs of the eigenvalues, we must require $\det A = ad - b^2 < 0$. □

8. Find an example of a 3×3 matrix with positive determinant and positive trace which is not positive definite.

   Solution. Empty spaces denote zeroes:

$$\begin{pmatrix} 3 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

□

9. True or False (assume A is $n \times n$):

   (a) If A is a matrix whose columns sum to 0, then $A + I$ is a Markov matrix.

   (b) If A is a diagonal matrix and is a Markov matrix, then $A = I$.

   (c) If A is Markov then $I - A$ is positive semidefinite.

   (d) If A is positive Markov then $I - A$ has rank $n - 1$.

Solution. (a) False, for example take $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

(b) True, because then the only nonzero entry in each column is one on the diagonal, hence it must be 1 to ensure that the sum is equal to 1.

(c) True. If A is Markov, then any of its eigenvalues is at most one. So if $Ax = \lambda x$, then $(I - A)x = (1 - \lambda)x$, and $1 - \lambda > 0$.

(d) True. If A is positive Markov, then there is only one eigenvalue equal to 1. If λ is an eigenvalue of A, then $1 - \lambda$ is an eigenvalue of $I - A$, and all eigenvalues of $I - A$ can be obtained this way. So the only way to get a zero eigenvalue for $I - A$ is to take $\lambda = 1$ in A, but there is exactly one eigenvalue of A equal to 1. So the nullspace of $I - A$ is one-dimensional, therefore its rank is $n - 1$.

□