18.06 - Recitation 8

Sam Turton

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1 Lecture review

1. If we have that
$$Ax = \lambda x, \quad x \neq 0,$$
then λ is an **eigenvalue** of A, and x is the corresponding **eigenvector**.

2. The eigenvalues are solutions of the n-th degree polynomial equation
$$\det(A - \lambda I) = 0.$$
This is known as the *characteristic equation.*

3. Warning: The eigenvalues and eigenvectors need not be real, even if all the entries of A are real !

4. A is invertible if, and only if, 0 is **not** an eigenvalue.

5. Powers of A have the same eigenvectors as A; the corresponding eigenvalues are raised to the same power, i.e.
$$Ax = \lambda x \quad \Longrightarrow \quad A^n x = \lambda^n x$$

6. The eigenvalues of a triangular matrix are on the diagonal.

7. The determinant of A is equal to the product of all of its eigenvalues.

8. The trace of A is equal to the sum of its eigenvalues.

9. **Diagonalisation:** Suppose A has n-independent eigenvectors $x_1, x_2, ..., x_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let X be a matrix whose columns are the eigenvectors in this order, and Λ be a diagonal matrix with the corresponding eigenvalues along its diagonal. Then we can write
$$\boxed{A = X\Lambda X^{-1}}$$

10. Diagonalisation is always possible if A has n distinct eigenvalues; it may or may not be possible if it has repeated eigenvalues.

2 Problems

Problem 1.

The 2×2 matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of $B = 2A + 3I$. (Before you jump into solving quadratic equations, think about what happens if you multiply B by x_1 or x_2.)

Problem 2.

1. If the eigenvectors of A are the columns of I then A is a .... matrix.
2. If the eigenvector matrix X is invertible and upper triangular, then why must A also be upper triangular? (Note: the inverse of an upper-triangular matrix is upper triangular.)

Problem 3.

Suppose we form a sequence of numbers g_0, g_1, g_2, g_3 by the rule

$$g_{k+2} = (1-w)g_{k+1} + wg_k$$

for some scalar w. We concentrate on the case where $0 < w < 1$, so that g_{k+2} could be thought of as a *weighted average* of the previous two values in the sequence. For example, for $w = 0.5$ (equal weights) and with $g_0 = 0$ and $g_1 = 1$, this produces the sequence

$$g_0, g_1, g_2, g_3, \ldots = 0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \frac{85}{128}, \frac{171}{256}, \frac{341}{512}, \frac{683}{1024}, \frac{1365}{2048}, \frac{2731}{4096}, \frac{5461}{8192}, \frac{10923}{16384}, \frac{21845}{32768}, \ldots$$

1. If we define $x_k = \begin{pmatrix} g_{k+1} \\ g_k \end{pmatrix}$, then write the rule for the sequence in matrix form: $x_{k+1} = Ax_k$. In particular, what is A?
2. Find the eigenvalues of A (your answers could be a function of w) by computing the characteristic equation. Check that A has corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} w \\ -1 \end{pmatrix}$.
3. What happens to the eigenvalues and eigenvectors as w gets closer and closer to -1? Is there a still a basis of eigenvectors and a diagonalization of A for $w = -1$?
4. Show that $x_n = A^n x_0$. Find the limit as $n \to \infty$ of A^n (for $0 < w < 1$) from the diagonalization of A.
5. For $w = 0.5$, if $g_0 = 0$ and $g_1 = 1$, i.e. $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then show that the sequence g_k approaches $2/3$.