

Elements of Matrix Algebra

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Foreword

These lecture notes are supposed to summarize the main results concerning matrix algebra as they are used in econometrics and economics. For a deeper discussion of the material, the interested reader should consult the references listed at the end.

1 Definitions

A matrix is a rectangular array of numbers. Here we consider only real numbers. If the matrix has n rows and m columns, we say that the matrix is of dimension $(n \times m)$. We denote matrices by capital bold letters:

$$\mathbf{A} = (\mathbf{A})_{ij} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

The numbers a_{ij} are called the elements of the matrix.

An $(n \times 1)$ matrix is a column vector with n elements. Similarly, a $(1 \times m)$ matrix is a row vector with m elements. We denote vectors by bold letters.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \mathbf{b} = (b_1 \quad b_2 \quad \dots \quad b_m).$$

A (1×1) matrix is a scalar which is denoted by an italic letter.

The *null matrix* (\mathbf{O}) is a matrix whose elements are all equal to zero, i.e. $a_{ij} = 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

A *square matrix* is a matrix with the same number of columns and rows, i.e. $n = m$.

A *symmetric matrix* is a square matrix such that $a_{ij} = a_{ji}$ for all $i = 1, \dots, n$ and $j = 1, \dots, n$.

A *diagonal matrix* is a square matrix such that the off-diagonal elements are all equal to zero, i.e. $a_{ij} = 0$ for $i \neq j$.

The *identity matrix* is a diagonal matrix with all diagonal elements equal to one. The identity matrix is denoted by \mathbf{I} or \mathbf{I}_n .

A square matrix is said to be *upper triangular* whenever $a_{ij} = 0$ for $i > j$ and *lower triangular* whenever $a_{ij} = 0$ for $i < j$.

Two vectors \mathbf{a} and \mathbf{b} are said to be *linearly dependent* if there exist scalars α and β both not equal to zero such that $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$. Otherwise they are said to be *linearly independent*.

2 Matrix Operations

2.1 Equality

Two matrices or two vectors are equal if they have the same dimension and if their respective elements are all equal:

$$\mathbf{A} = \mathbf{B} \quad \Longleftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i \text{ and } j$$

2.2 Transpose

Definition 1. The matrix \mathbf{B} is called the transpose of matrix \mathbf{A} if and only if

$$b_{ij} = a_{ji} \quad \text{for all } i \text{ and } j.$$

The matrix \mathbf{B} is denoted by \mathbf{A}' or \mathbf{A}^T .

Taking the transpose of a matrix is equivalent to interchanging rows and columns. If \mathbf{A} has dimension $(n \times m)$ then \mathbf{A}' has dimension $(m \times n)$. The transpose of a column vector is a row vector and vice versa. Note:

$$\bullet \quad (\mathbf{A}')' = \mathbf{A} \text{ for any matrix } \mathbf{A} \quad (2.1)$$

$$\bullet \quad \mathbf{A}' = \mathbf{A} \text{ for a symmetric matrix } \mathbf{A} \quad (2.2)$$

2.3 Addition and Subtraction

The addition and subtraction of matrices is only defined for matrices with the same dimension.

Definition 2. The sum of two matrices \mathbf{A} and \mathbf{B} of the same dimensions is given by the sum of their elements, i.e.

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \Longleftrightarrow \quad c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i \text{ and } j$$

The sum of a matrix \mathbf{A} and a scalar b is a matrix $\mathbf{C} = \mathbf{A} + b$ with $c_{ij} = a_{ij} + b$. Note that $\mathbf{A} + b = b + \mathbf{A}$.

We have the following calculation rules if matrix dimensions agree:

$$\bullet \quad \mathbf{A} + \mathbf{O} = \mathbf{A} \quad (2.3)$$

$$\bullet \quad \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \quad (2.4)$$

$$\bullet \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (2.5)$$

$$\bullet \quad (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad (2.6)$$

$$\bullet \quad (\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}' \quad (2.7)$$

2.4 Product

Definition 3. The *inner product* (dot product, scalar product) of two vectors \mathbf{a} and \mathbf{b} of the same dimension $(n \times 1)$ is a scalar (real number) defined as:

$$\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a} = a_1b_1 + a_2b_2 + \dots + a_nb_n = \sum_{i=1}^n a_ib_i.$$

The product of a scalar c and a matrix \mathbf{A} is a matrix $\mathbf{B} = c\mathbf{A}$ with $b_{ij} = ca_{ij}$. Note that $c\mathbf{A} = \mathbf{A}c$ when c is a scalar.

Definition 4. The product of two matrices \mathbf{A} and \mathbf{B} with dimensions $(n \times k)$ and $(k \times m)$, respectively, is given by the matrix \mathbf{C} with dimension

$(n \times m)$ such that

$$\mathbf{C} = \mathbf{A}\mathbf{B} \iff c_{ij} = \sum_{s=1}^k a_{is}b_{sj} \quad \text{for all } i \text{ and } j$$

Remark 1. The matrix product is only defined if the number of columns of the first matrix is equal to the number of rows of the second matrix. Thus, although $\mathbf{A}\mathbf{B}$ may be defined, $\mathbf{B}\mathbf{A}$ is only defined if $n = m$. Thus for square matrices both $\mathbf{A}\mathbf{B}$ and $\mathbf{B}\mathbf{A}$ are defined.

Remark 2. The product of two matrices is in general **not** commutative, i.e. $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$.

Remark 3. The product $\mathbf{A}\mathbf{B}$ may also be defined as

$$c_{ij} = (\mathbf{C})_{ij} = \mathbf{a}'_{i\bullet} \mathbf{b}_{\bullet j}$$

where $\mathbf{a}'_{i\bullet}$ denotes the i -th row of \mathbf{A} and $\mathbf{b}_{\bullet j}$ the j -th column of \mathbf{B} .

We have the following calculation rules if matrix dimensions agree:

$$\bullet \quad \mathbf{A}\mathbf{I} = \mathbf{A}, \quad \mathbf{I}\mathbf{A} = \mathbf{A} \quad (2.8)$$

$$\bullet \quad \mathbf{A}\mathbf{O} = \mathbf{O}, \quad \mathbf{O}\mathbf{A} = \mathbf{O} \quad (2.9)$$

$$\bullet \quad (\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C}) = \mathbf{A}\mathbf{B}\mathbf{C} \quad (2.10)$$

$$\bullet \quad \mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C} \quad (2.11)$$

$$\bullet \quad (\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{B}\mathbf{A} + \mathbf{C}\mathbf{A} \quad (2.12)$$

$$\bullet \quad c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \quad (2.13)$$

$$\bullet \quad (\mathbf{A}\mathbf{B})' = \mathbf{B}'\mathbf{A}' \quad (\text{order!}) \quad (2.14)$$

$$\bullet \quad (\mathbf{A}\mathbf{B}\mathbf{C})' = \mathbf{C}'\mathbf{B}'\mathbf{A}' \quad (\text{order!}) \quad (2.15)$$

3 Rank of a Matrix

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is *linearly independent* if $\sum_{i=1}^n c_i \mathbf{x}_i = \mathbf{0}$ implies $c_i = 0$ for all $i = 1, \dots, n$.

The *column rank* of a matrix is the maximal number of linearly independent columns. The *row rank* of a matrix is the maximal number of linearly independent rows. A matrix is said to have full column (row) rank if the column rank (row rank) equals the number of columns (rows).

The column rank of an $n \times k$ matrix \mathbf{A} is equal to its row rank. We can therefore just speak of the rank of a matrix denoted by $\text{rank}(\mathbf{A})$.

For an $(n \times k)$ matrix \mathbf{A} , a $(k \times m)$ matrix \mathbf{B} and an $(n \times n)$ square matrix \mathbf{C} , we have

$$\bullet \quad \text{rank}(\mathbf{A}) \leq \min(n, k) \quad (3.1)$$

$$\bullet \quad \text{rank}(\mathbf{A}') = \text{rank}(\mathbf{A}) \quad (3.2)$$

$$\bullet \quad \text{rank}(\mathbf{A}'\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}') = \text{rank}(\mathbf{A}) \quad (3.3)$$

$$\bullet \quad \text{rank}(\mathbf{A}\mathbf{B}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (3.4)$$

$$\bullet \quad \text{rank}(\mathbf{A}\mathbf{B}) = \text{rank}(\mathbf{B}) \text{ if } \mathbf{A} \text{ has full column rank} \quad (3.5)$$

$$\bullet \quad \text{rank}(\mathbf{A}\mathbf{B}) = \text{rank}(\mathbf{A}) \text{ if } \mathbf{B} \text{ has full row rank} \quad (3.6)$$

$$\bullet \quad \text{rank}(\mathbf{A}'\mathbf{C}\mathbf{A}) = \text{rank}(\mathbf{C}\mathbf{A}) \text{ if } \mathbf{C} \text{ is nonnegative definite} \quad (3.7)$$

$$\bullet \quad \text{rank}(\mathbf{A}'\mathbf{C}\mathbf{A}) = \text{rank}(\mathbf{A}) \text{ if } \mathbf{C} \text{ is positive definite} \quad (3.8)$$

4 Special Functions of Square Matrices

In this section only square ($n \times n$) matrices are considered.

4.1 Trace of a Matrix

Definition 5. The *trace* of a matrix \mathbf{A} , denoted by $\text{tr}(\mathbf{A})$, is the sum of its diagonal elements:

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$$

The following calculation rules hold if matrix dimensions agree:

$$\bullet \quad \text{tr}(c\mathbf{A}) = c \text{tr}(\mathbf{A}) \quad (4.1)$$

$$\bullet \quad \text{tr}(\mathbf{A}') = \text{tr}(\mathbf{A}) \quad (4.2)$$

$$\bullet \quad \text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B}) \quad (4.3)$$

$$\bullet \quad \text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad (4.4)$$

$$\bullet \quad \text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{CAB}) \quad (4.5)$$

4.2 Determinant

The *determinant* of a ($n \times n$) matrix \mathbf{A} with $n > 1$ can be computed according to the following formula:

$$|\mathbf{A}| = \sum_{i=1}^n a_{ij}(-1)^{i+j}|\mathbf{A}_{ij}| \quad \text{for some arbitrary } j$$

The determinant, computed as above, is said to be developed according to the j -th column. The term $(-1)^{i+j}|\mathbf{A}_{ij}|$ is called the cofactor of the element a_{ij} . Thereby \mathbf{A}_{ij} is a matrix of dimension $((n-1) \times (n-1))$ which is obtained by deleting the i -th row and the j -th column.

$$\mathbf{A}_{ij} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

For $n = 1$, i.e. if \mathbf{A} is a scalar, the determinant $|\mathbf{A}|$ is defined as the absolute value. For $n = 2$, the determinant is given by:

$$|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

If at least two columns (rows) are linearly dependent, the determinant is equal to zero and the inverse of \mathbf{A} does not exist. The matrix is called *singular* in this case. If the matrix is *nonsingular* then all columns (rows) are linearly independent. If a column or a row has just zeros as its elements, the determinant is equal to zero. If two columns (rows) are interchanged, the determinant changes its sign.

Calculation rules for the determinant are:

$$\bullet \quad |\mathbf{A}| = |\mathbf{A}'| \quad (4.6)$$

$$\bullet \quad |\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| \quad (4.7)$$

$$\bullet \quad |c\mathbf{A}| = c^n |\mathbf{A}| \quad (4.8)$$

4.3 Inverse of a Matrix

If \mathbf{A} is a square matrix, there may exist a matrix \mathbf{B} with property $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. If such a matrix exists, it is called the *inverse* of \mathbf{A} and is denoted by \mathbf{A}^{-1} , hence $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. The inverse of a matrix can be computed as follows

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} (-1)^{1+1}|\mathbf{A}_{11}| & (-1)^{2+1}|\mathbf{A}_{21}| & \cdots & (-1)^{n+1}|\mathbf{A}_{n1}| \\ (-1)^{1+2}|\mathbf{A}_{12}| & (-1)^{2+2}|\mathbf{A}_{22}| & \cdots & (-1)^{n+2}|\mathbf{A}_{n2}| \\ \vdots & \ddots & \ddots & \vdots \\ (-1)^{1+n}|\mathbf{A}_{1n}| & (-1)^{2+n}|\mathbf{A}_{2n}| & \cdots & (-1)^{n+n}|\mathbf{A}_{nn}| \end{pmatrix}$$

where \mathbf{A}_{ij} is the matrix of dimension $(n-1) \times (n-1)$ obtained from \mathbf{A} by deleting the i -th row and the j -th column.

$$\mathbf{A}_{ij} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

The term $(-1)^{i+j}|\mathbf{A}_{ij}|$ is called the cofactor of a_{ij} .

For $n = 2$, the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

We have the following calculation rules if both \mathbf{A}^{-1} and \mathbf{B}^{-1} exist and matrix dimensions agree:

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (4.9)

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (order!) (4.10)

- $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$ (4.11)

- $|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1} = \frac{1}{|\mathbf{A}|}$ (4.12)

4.4 Nonsingular Square Matrices

The following statements about a square $(n \times n)$ matrix \mathbf{A} are equivalent:

- \mathbf{A} is nonsingular (4.13)

- $|\mathbf{A}| \neq 0$ (4.14)

- \mathbf{A}^{-1} exists (4.15)

- $\text{rank}(\mathbf{A}) = n$ (full rank) (4.16)

- $\lambda_i \neq 0$ for all $i = 1, \dots, n$ (4.17)

5 Systems of Equations

Consider the following system of m equations in n unknowns x_1, \dots, x_n :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

If we collect the unknowns into a vector $\mathbf{x} = (x_1, \dots, x_n)'$, the coefficients b_1, \dots, b_m into a vector \mathbf{b} , and the coefficients (a_{ij}) into a matrix \mathbf{A} , we can rewrite the equation system compactly in matrix form as follows:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}$$

$\mathbf{Ax} = \mathbf{b}$

This equation system has a unique solution if $m = n$, i.e. if \mathbf{A} is a square matrix, and \mathbf{A} is nonsingular, i.e. \mathbf{A}^{-1} exists. The solution is then given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Remark 4. To achieve numerical accuracy it is preferable not to compute the inverse explicitly. There are efficient numerical algorithms which can solve the equation system without computing the inverse.

6 Eigenvalue, -vector and Decomposition

6.1 Eigenvalue and Eigenvector

A scalar λ is said to be an *eigenvalue* of the square matrix \mathbf{A} if there exists a vector $\mathbf{x} \neq 0$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

The vector \mathbf{x} is called an *eigenvector* corresponding to λ . If \mathbf{x} is an eigenvector then $\alpha\mathbf{x}$, $\alpha \neq 0$, is also an eigenvector. Eigenvectors are therefore not unique. It is sometimes useful to normalize the length of the eigenvectors to one, i.e. to choose the eigenvector such that $\mathbf{x}'\mathbf{x} = 1$.

6.2 Characteristic Equation

In order to find the eigenvalues and eigenvectors of a square matrix, one has to solve the equation system

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} \quad \Longleftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0.$$

This equation system has a nontrivial solution, $\mathbf{x} \neq 0$, if and only if the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular, or equivalently if and only if the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is equal to zero. This leads to an equation in the unknown parameter λ :

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

This equation is called the *characteristic equation* of the matrix \mathbf{A} and corresponds to a polynomial equation of order n . The n solutions of this equation (roots) are the eigenvalues of the matrix. The solutions may be complex numbers. Some solutions may appear several times. Eigenvectors corresponding to some eigenvalue λ can be obtained from the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$.

We have the following relations for an $(n \times n)$ matrix \mathbf{A} :

$$\bullet \quad \text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i \quad (6.1)$$

$$\bullet \quad |\mathbf{A}| = \prod_{i=1}^n \lambda_i \quad (6.2)$$

6.3 Decomposition of Symmetric Matrices

If \mathbf{A} is a symmetric $(n \times n)$ matrix, all n eigenvalues $\lambda_1, \dots, \lambda_n$ are real and there exist n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ with the properties $\mathbf{x}_i'\mathbf{x}_j = 0$ for $i \neq j$ and $\mathbf{x}_i'\mathbf{x}_i = 1$, i.e. the eigenvectors are orthogonal to each other and of length one. The eigenvector \mathbf{x}_i corresponds to the eigenvalue λ_i .

A symmetric $(n \times n)$ matrix \mathbf{A} can be diagonalized as

$$\mathbf{H}'\mathbf{A}\mathbf{H} = \mathbf{\Lambda}, \quad (6.3)$$

where the diagonal matrix $\mathbf{\Lambda}$ collects the eigenvalues of \mathbf{A}

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

and the $(n \times n)$ matrix $\mathbf{H} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ collecting the corresponding eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ is orthogonal

$$\mathbf{H}'\mathbf{H} = \mathbf{I},$$

hence $\mathbf{H}^{-1} = \mathbf{H}'$ and $\mathbf{H}\mathbf{H}' = \mathbf{I}$. We can therefore decompose \mathbf{A} into the sum of n matrices:

$$\mathbf{A} = \mathbf{H}\mathbf{\Lambda}\mathbf{H}' = \sum_{i=1}^n \lambda_i \mathbf{x}_i \mathbf{x}_i'$$

where the matrices $\mathbf{h}_i \mathbf{h}_i'$ have all rank one. This decomposition is called the *spectral decomposition* or *eigendecomposition* of \mathbf{A} .

The inverse of a nonsingular symmetric matrix \mathbf{A} can be calculated as

$$\mathbf{A}^{-1} = \mathbf{H}\mathbf{\Lambda}^{-1}\mathbf{H}' = \sum_{i=1}^n \frac{1}{\lambda_i} \mathbf{x}_i \mathbf{x}_i'.$$

Remark 5. Beside symmetric matrices, many other matrices, but not all matrices, are also diagonalizable.

7 Quadratic Forms

For a vector $\mathbf{x} \in \mathbb{R}^n$ and a symmetric matrix \mathbf{A} of dimension $(n \times n)$ the scalar function

$$f(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{j=1}^n \sum_{i=1}^n x_i x_j a_{ij}$$

is called a *quadratic form*.

The quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ and therefore the matrix \mathbf{A} is called *positive (negative) definite*, if and only if

$$\mathbf{x}'\mathbf{A}\mathbf{x} > 0 (< 0) \quad \text{for all } \mathbf{x} \neq 0.$$

The property that \mathbf{A} is positive definite implies that

$$\bullet \quad \lambda_i > 0 \text{ for all } i = 1, \dots, n \quad (7.1)$$

$$\bullet \quad |\mathbf{A}| > 0 \quad (7.2)$$

$$\bullet \quad \mathbf{A}^{-1} \text{ exists and is positive definite} \quad (7.3)$$

$$\bullet \quad \text{tr}(\mathbf{A}) > 0 \quad (7.4)$$

The first property is an alternative definition for a positive definite matrix.

The quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ and therefore the matrix \mathbf{A} is called *non-negative definite* or *positive semi-definite*, if and only if

$$\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0 \quad \text{for all } \mathbf{x}.$$

For nonnegative definite matrices we have:

$$\bullet \quad \lambda_i \geq 0 \text{ for all } i = 1, \dots, n \quad (7.5)$$

$$\bullet \quad |\mathbf{A}| \geq 0 \quad (7.6)$$

$$\bullet \quad \text{tr}(\mathbf{A}) \geq 0 \quad (7.7)$$

The first property is an alternative definition for nonnegative definiteness.

For an $(n \times m)$ matrix \mathbf{B} ,

$$\bullet \quad \mathbf{B}'\mathbf{B} \text{ is nonnegative definite} \quad (7.8)$$

$$\bullet \quad \mathbf{B}'\mathbf{B} \text{ is positive definite if } \mathbf{B} \text{ has full column rank} \quad (7.9)$$

$$\bullet \quad \mathbf{B}\mathbf{B}' \text{ is nonnegative definite} \quad (7.10)$$

If the $(n \times m)$ matrix \mathbf{B} has rank m (full column rank) and the $(n \times n)$ matrix \mathbf{A} is positive definite then

$$\bullet \quad \mathbf{B}'\mathbf{A}\mathbf{B} \text{ is positive definite} \quad (7.11)$$

The inverse of a positive definite $(n \times n)$ matrix \mathbf{A} can be decomposed into

$$\mathbf{A}^{-1} = \mathbf{C}'\mathbf{C} \quad \text{where} \quad \mathbf{C}\mathbf{A}\mathbf{C}' = \mathbf{I}.$$

where \mathbf{C} is a $(n \times n)$ matrix.

8 Partitioned Matrices

Consider a square matrix \mathbf{P} of dimensions $((p+q) \times (r+s))$ which is partitioned into the $(p \times r)$ matrix \mathbf{P}_{11} , the $(p \times s)$ matrix \mathbf{P}_{12} , the $(q \times r)$ matrix \mathbf{P}_{21} and the $(q \times s)$ matrix \mathbf{P}_{22} :

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

Assuming that dimensions in the involved multiplications agree, two partitioned matrices are multiplied as

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{11}\mathbf{Q}_{11} + \mathbf{P}_{12}\mathbf{Q}_{21} & \mathbf{P}_{11}\mathbf{Q}_{12} + \mathbf{P}_{12}\mathbf{Q}_{22} \\ \mathbf{P}_{21}\mathbf{Q}_{11} + \mathbf{P}_{22}\mathbf{Q}_{21} & \mathbf{P}_{21}\mathbf{Q}_{12} + \mathbf{P}_{22}\mathbf{Q}_{22} \end{pmatrix}$$

Assuming that \mathbf{P}_{11}^{-1} exists, the determinant of a partitioned matrix is

$$\begin{vmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{vmatrix} = |\mathbf{P}_{11}| \cdot |\mathbf{P}_{22} - \mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}| \quad (8.1)$$

and the inverse is

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{P}_{11}^{-1} + \mathbf{P}_{11}^{-1}\mathbf{P}_{12}\mathbf{F}^{-1}\mathbf{P}_{21}\mathbf{P}_{11}^{-1} & -\mathbf{P}_{11}^{-1}\mathbf{P}_{12}\mathbf{F}^{-1} \\ -\mathbf{F}^{-1}\mathbf{P}_{21}\mathbf{P}_{11}^{-1} & \mathbf{F}^{-1} \end{pmatrix} \quad (8.2)$$

where $\mathbf{F} = \mathbf{P}_{22} - \mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}$ is assumed nonsingular.

The determinant of a block diagonal matrix is

$$\begin{vmatrix} \mathbf{P}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}_{22} \end{vmatrix} = |\mathbf{P}_{11}| \cdot |\mathbf{P}_{22}|$$

and its inverse is, assuming that \mathbf{P}_{11}^{-1} and \mathbf{P}_{22}^{-1} exist,

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{P}_{11}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}_{22}^{-1} \end{pmatrix}.$$

9 Derivatives with Matrix Algebra

A linear function f from the n -dimensional vector space of real numbers, \mathbb{R}^n , to the real numbers, \mathbb{R} , $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is determined by the coefficient vector $\mathbf{a} = (a_1, \dots, a_n)'$:

$$y = f(\mathbf{x}) = \mathbf{a}'\mathbf{x} = \sum_{i=1}^n a_i x_i = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

where \mathbf{x} is a column vector of dimension n and y a scalar.

The derivative of $y = f(\mathbf{x})$ with respect to the column vector \mathbf{x} is defined as follows:

$$\frac{\partial y}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}'\mathbf{a}}{\partial \mathbf{x}} = \begin{pmatrix} \partial y / \partial x_1 \\ \partial y / \partial x_2 \\ \vdots \\ \partial y / \partial x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \mathbf{a}$$

and with respect to the row vector \mathbf{x}' as follows:

$$\frac{\partial y}{\partial \mathbf{x}'} = \frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}'} = \frac{\partial \mathbf{x}'\mathbf{a}}{\partial \mathbf{x}'} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \dots & \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} = \mathbf{a}'$$

The simultaneous equation system $\mathbf{y} = \mathbf{A}\mathbf{x}$ can be viewed as m linear functions $y_i = \mathbf{a}_i'\mathbf{x}$ where \mathbf{a}_i' denotes the i -th row of the $(m \times n)$ dimensional matrix \mathbf{A} . Thus the derivative of y_i with respect to \mathbf{x} is given by

$$\frac{\partial y_i}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}_i'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}_i$$

Consequently the derivative of $\mathbf{y} = \mathbf{A}\mathbf{x}$ with respect to row vector \mathbf{x}' can be defined as

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}'} = \frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}'} = \begin{pmatrix} \partial y_1 / \partial \mathbf{x}' \\ \partial y_2 / \partial \mathbf{x}' \\ \vdots \\ \partial y_m / \partial \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_m' \end{pmatrix} = \mathbf{A}.$$

The derivative of $\mathbf{y} = \mathbf{Ax}$ with respect to column vector \mathbf{x} is therefore

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial \mathbf{Ax}}{\partial \mathbf{x}} = \mathbf{A}'.$$

For a square matrix \mathbf{A} of dimension $(n \times n)$ and the quadratic form $\mathbf{x}'\mathbf{Ax} = \sum_{j=1}^n \sum_{i=1}^n x_i x_j a_{ij}$ the derivative with respect to the column vector \mathbf{x} is defined as

$$\frac{\partial \mathbf{x}'\mathbf{Ax}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}')\mathbf{x}.$$

If \mathbf{A} is a symmetric matrix this reduces to

$$\frac{\partial \mathbf{x}'\mathbf{Ax}}{\partial \mathbf{x}} = 2\mathbf{Ax}.$$

The derivative of the quadratic form $\mathbf{x}'\mathbf{Ax}$ with respect to the matrix elements a_{ij} is given by

$$\frac{\partial \mathbf{x}'\mathbf{Ax}}{\partial a_{ij}} = x_i x_j.$$

Therefore the derivative with respect to the matrix \mathbf{A} is given by

$$\frac{\partial \mathbf{x}'\mathbf{Ax}}{\partial \mathbf{A}} = \mathbf{xx}'.$$

10 Kronecker Product

The Kronecker Product of a $m \times n$ Matrix \mathbf{A} with a $p \times q$ Matrix \mathbf{B} is a $mp \times nq$ Matrix $\mathbf{A} \otimes \mathbf{B}$ defined as follows:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix}$$

The following calculation rules hold if matrix dimensions agree:

$$\bullet (\mathbf{A} \otimes \mathbf{B}) + (\mathbf{C} \otimes \mathbf{B}) = (\mathbf{A} + \mathbf{C}) \otimes \mathbf{B} \quad (10.1)$$

$$\bullet (\mathbf{A} \otimes \mathbf{B}) + (\mathbf{A} \otimes \mathbf{C}) = \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) \quad (10.2)$$

$$\bullet (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \quad (10.3)$$

$$\bullet (\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \quad (10.4)$$

$$\bullet \text{tr}(\mathbf{A} \otimes \mathbf{B}) = \text{tr}(\mathbf{A})\text{tr}(\mathbf{B}) \quad (10.5)$$

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Formula Sources and Proofs

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