18.06 - Recitation 5 SOLUTIONS

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Problem 1.
Determine which of the following describe a linear transformation. For those that do, find a matrix that describes the transformation with respect to the standard bases for the underlying vector spaces:

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where
$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 0 \end{pmatrix}$$
2. $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ where
$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$
3. $T_3 : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 2}$ where
$$T_3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 2d \\ 2b-d & -3c \\ 2b-c & -3a \end{pmatrix}$$
4. Let P_4 be the vector space of polynomials of degree less than or equal to 4, and let $T_4 : P_4 \to P_4$, where
$$T_4(f)(x) = f(x) - x - 1$$
5. Let $T_5 : P_3 \to P_5$ where
$$T_5(f)(x) = (x^2 - 2)f(x)$$
.

Solution

1. This is a linear transformation. We can verify this explicitly:
$$T_1 \left(c_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = c_1 \begin{pmatrix} 2x_1 + y_1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2x_2 + y_2 \\ 0 \end{pmatrix}.$$
We can represent this transformation using a matrix by using the standard basis for $\mathbb{R}^2$, $\{e_1, e_2\}$, where
$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
Then $T_1(e_1) = 2e_1$ and $T_1(e_2) = e_1$, and so this linear transformation can be represented by the matrix
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$
2. This is *not* a linear transformation. Notice that
$$T_2 \left(c \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} c(x+y) \\ c^2 xy \end{pmatrix} \neq c \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

3. This is a linear transformation. We can verify explicitly that

$$T_3\left(\lambda_1\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix} + \lambda_2\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix}\right) = \lambda_1\begin{pmatrix} a_1+b_1 & 2d_1 \\ 2b_1-d_1 & -3c_1 \\ 2b_1-c_1 & -3a_1\end{pmatrix} + \lambda_2\begin{pmatrix} a_2+b_2 & 2d_2 \\ 2b_2-d_2 & -3c_2 \\ 2b_2-c_2 & -3a_2\end{pmatrix}$$

To write a matrix to describe this linear transformation, we need to provide a basis for both the input and output vector spaces. The standard basis for $\mathbb{R}^{2\times 2}$ is the set $\{e_1, e_2, e_3, e_4\}$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}.$$

The standard basis for $\mathbb{R}^{3\times 2}$ is the set $\{f_1, f_2, f_3, f_4, f_5, f_6\}$, where

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0\end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0\end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0\end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0\end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0\end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1\end{pmatrix}.$$

We can represent the linear transformation T_3 as a matrix with respect to these bases by considering the action of T_3 on each of the input basis elements. For example, $T_3(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -3\end{pmatrix} = f_1 - 3f_6$, and so the first column of A is given by the column vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3\end{pmatrix}$$

Repeating this for the other three basis elements yields the 6×4 matrix for this linear transformation

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ -3 & 0 & 0 & 0\end{pmatrix}$$

4. This is *not* a linear transformation. This transformation maps the zero polynomial $0(x)$ onto $-x-1 \neq 0$.

5. This is a linear transformation, since

$$T_5(c_1 f_1 + c_2 f_2)(x) = (x^2-2)(c_1 f_1(x) + c_2 f_2(x)) = c_1(x^2-2)f_1(x) + c_2(x^2-2)f_2(x)$$

The standard basis for P_3 is given by the polynomials $\{x^3, x^2, x, 1\}$ and the standard basis for P_5 is given by the polynomials $\{x^5, x^{\cdot}x^3, x^2, x, 1\}$. We can construct a matrix that describes this linear transformation with respect to these bases by considering the action of the linear transformation on each of the input basis elements. For example:

$$T_5(x^3) = (x^2-2)x^3 = x^5 - 2x^3,$$

and so the first column in a matrix representation of this transformation is given by

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0\end{pmatrix}.$$

We can repeat this for each of the input basis elements to derive the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

Problem 2.

1. Show that $f(A) = x^T Ay$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are constant vectors, is a linear transformation from the vector space of $m \times n$ matrices to the real numbers.
2. If $f(A)$ is a scalar function of an $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$, then it is useful to define the gradient *with respect to the matrix* as another $m \times n$ matrix:

$$\nabla_A f = \begin{pmatrix} \frac{\partial f}{\partial a_{11}} & \frac{\partial f}{\partial a_{12}} & \cdots \\ \frac{\partial f}{\partial a_{21}} & \frac{\partial f}{\partial a_{22}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Given this definition, give a matrix expression (not in terms of individual components) for $\nabla_A f$ with $f(A) = x^T Ay$ as before.

Solution

1. To show this is a linear transformation, we consider $f(cA + dB)$:

$$f(cA + dB) = x^T(cA + dB)y = cx^T Ay + dx^T By = cf(A) + df(B).$$

2. We have $f(A) = x^T Ay = \sum_{p=1}^m \sum_{q=1}^n x_p a_{pq} y_q$. It then follows that $\frac{\partial f}{\partial a_{ij}} = x_i y_j$. We can then write

$$\nabla_A f = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots \\ x_2 y_1 & x_2 y_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

We can then identify this matrix as

$$\nabla_A f = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} = xy^T$$

Problem 3.

Consider the vector space of polynomials of degree less than or equal 2. Let us define a dot product on this vector space[1]:

$$f(x) \cdot g(x) = \int_0^\infty f(x)g(x)e^{-x}\ \mathrm{d}x$$

1. Show that the set of polynomials $\{1, x-1, x^2-4x+2\}$ form an orthogonal basis for the vector space of polynomials of degree less than or equal 2.
2. Normalize these basis polynomials so that $\|f(x)\|^2 = f(x) \cdot f(x) = 1$ for each element in the basis.
3. Consider the function $f(x) = \begin{cases} x & x < 1 \\ 0 & x \geq 1 \end{cases}$. Find the slope α of the straight line αx that is the *best fit* to $f(x)$ in the sense of minimizing
$$\|f - \alpha x\|^2 = \int_0^\infty [f(x) - \alpha x]^2\, e^{-x} dx$$
In particular, find α by performing the orthogonal projection (with this dot product) of $f(x)$ onto ....?

Solution

1. This set of polynomials is clearly linearly independent and it spans the vector space of polynomials. It is therefore definitely a basis. To show that it is orthogonal, we just have to check that the dot product between any two of the basis elements is 0:
$$\begin{aligned}
1 \cdot (x-1) &= \int_0^\infty (x-1)e^{-x}\ \mathrm{d}x \\
&= \int_0^\infty xe^{-x}\ \mathrm{d}x - \int_0^\infty e^{-x}\ \mathrm{d}x \\
&= 1! - 0! \\
&= 0 \\
1 \cdot (x^2-4x+2) &= \int_0^\infty x^2e^{-x}\ \mathrm{d}x - 4\int_0^\infty xe^{-x}\ \mathrm{d}x + 2\int_0^\infty e^{-x}\ \mathrm{d}x \\
&= 2! - 4\cdot 1! + 2\cdot 0! \\
&= 0 \\
(x-1)\cdot(x^2-4x+2) &= \int_0^\infty (x^3-5x^2+6x-2)e^{-x}\ \mathrm{d}x \\
&= 3! - 5\cdot 2! + 6\cdot 1! - 2\cdot 0! \\
&= 0.
\end{aligned}$$
2. We can find the norm of each of these basis elements:
$$\begin{aligned}
\|1\|^2 &= \int_0^\infty e^{-x}\ \mathrm{d}x = 1 \\
\|(x-1)\|^2 &= \int_0^\infty (x-1)^2e^{-x}\ \mathrm{d}x = 1 \\
\|(x^2-4x+2)\|^2 &= \int_0^\infty (x^2-4x+2)^2e^{-x}\ \mathrm{d}x = 4.
\end{aligned}$$
So an orthonormal basis for this vector space with respect to this dot product is given by
$$\left\{1, x-1, \frac{x^2-4x+2}{2}\right\}$$

[1] You may find it useful to recall that $\int_0^\infty x^n e^{-x}\ \mathrm{d}x = n!$

3. We find α by calculating the orthogonal projection of $f(x)$ onto the function x, so that

$$\alpha = \frac{x \cdot f(x)}{x \cdot x} \tag{1}$$

Now $x \cdot x = \int_0^\infty x^2 e^{-x} dx = 2$, and

$$\begin{aligned} x \cdot f(x) &= \int_0^\infty x f(x) e^{-x} dx \qquad (2) \\ &= \int_0^1 x^2 e^{-x} dx \qquad (3) \\ &= 2 - 5e^{-1} \qquad (4) \end{aligned}$$

So that

$$\alpha = 1 - 5/2e^{-1}. \tag{5}$$