Recitation 8. Solution

Focus: eigenvectors, eigenvalues and eigendecomposition.

Notation. For the rest of this worksheet, let A be an $n \times n$ matrix operating on an n-dimensional vector space V, so $A : V \to V$.

Definition. A nonzero vector $v \in V$ is called an *eigenvector* for the matrix A if for some real or complex scalar λ we have $Av = \lambda v$.

Definition. The value λ is then called the *eigenvalue* corresponding to this eigenvector v.

Remark. Since for the eigenvector v we have $(A - \lambda)v = 0$, the matrix $A - \lambda I$ is not invertible, and so an eigenvalue is necessarily a root of the polynomial $\chi_A(\lambda) = \det(A - \lambda I)$.

Definition. If all roots of $\chi_A(\lambda)$ are different, then A is *diagonalizable*, which means that we can write $A = X\Lambda X^{-1}$ for some diagonal matrix Λ and invertible matrix X. This representation of A as $X\Lambda X^{-1}$ is called *eigendecomposition*.

1. Suppose we have $B = XAX^{-1}$.

   a) Prove that $\chi_B(\lambda) = \chi_A(\lambda)$.

   b) How are eigenvalues of B related to those of A?

   c) How are eigenvectors of B related to those of A?

   d) Suppose that one of the eigenvalues of A is zero. Does it mean that A is singular? Does it mean that B is singular?

Solution:

a) $\chi_B(\lambda) = \det(B-\lambda I) = \det(XAX^{-1}-\lambda XX^{-1}) = \det\left(X(A - \lambda I)X^{-1}\right) = \det X \cdot \det(A-\lambda I) \cdot \det X^{-1} = \chi_A(\lambda)$.

b) Since eigenvalues correspond to roots of the characteristic polynomial, and those are equal for A and B, as follows from part (a), we can conclude that eigenvalues of B coincide with those of A, counted with multiplicities.

c) If v is and eigevector of A with eigenvalue λ_0, then Xv is an eigenvector of B with eigenvalue λ_0: $B(Xv) = XAX^{-1}Xv = XAv = \lambda_0 Xv$.

d) Yes, because then $\det(A - 0 \cdot I) = \det A = 0$, so A is singular. And since B is a matrix similar to A, it is also singular.

2. Give an example of a diagonalizable matrix with a pair of equal eigenvalues.

Solution: For example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Prove that if V is odd-dimensional, then $A : V \to V$ has at least one real eigenvalue.

Solution: First way. All complex eigenvalues come in pairs, so if the dimension is odd, then one eigenvalue will have to be real.

Second way. Since V is odd-dimensional, the degree of $\chi_A(\lambda)$ is odd. Also note that the leading coefficient is -1. So $\chi_A(\lambda)$, considered as a function of single variable λ, is positive when λ is large negative and negative when λ is sufficiently big positive. Therefore, it must have a zero.

4. *Closed formula for Fibonacci numbers.* Let F_i denote the ith element in the Fibonacii sequence, defined by setting $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all natural values of i (including zero).

a) Find a matrix A such that $A\begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix}$.

Solution. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

b) Find the eigenvalues of A. Let φ denote the largest eigenvalue.

Solution. First compute the characteristic polynomial: $\chi_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$.
Now compute the discriminant $D = 1 + 4 = 5$.
Then the eigenvalues are $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618$.
Note that they are related as follows: $\varphi + \overline{\varphi} = 1$, $\varphi\overline{\varphi} = -1$ and $\varphi - \overline{\varphi} = \sqrt{5}$

c) Find the eigenvectors of A and the eigendecomposition $A = X\Lambda X^{-1}$.

Solution. Since there are two distinct eigenvalues, each of the matrices $A - \varphi I$ and $A - \overline{\varphi} I$ has exactly one-dimensional kernel (nullspace).
First find eigenvector v_1 for eigenvalue φ. It should satisfy $\begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix} v_1 = 0$. Since we know that the matrix is of rank one, we can look for a vector from the nullspace of the second row, and we see that $v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$.
Similarly, the vector $v_2 = \begin{pmatrix} \overline{\varphi} \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\overline{\varphi}$.
For the eigendecomposition, we know that we can write $X = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, then:

$$A = \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\varphi - \overline{\varphi}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} =$$
$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix}.$$

d) Compute A^{50} up to 9 decimal points. You can only use simple calculators (e.g. Google engine), no matrix calculators are needed.

Solution.

$$A^{50} = (X\Lambda X^{-1})^{50} = X\Lambda^{50}X^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}^{50}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} \approx$$
$$\approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & 0 \\ 0 & 4 \cdot 10^{-11} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix}.$$

Now note that all quantities that $4 \cdot 10^{-11}$ gets multiplied with are smaller than 10 in absolute value, so we can approximate this number with 0:

$$A^{50} \approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & 0 \\ 0 & 4 \cdot 10^{-11} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} \approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{pmatrix} =$$
$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi^{50} & -\varphi^{50} \cdot \overline{\varphi} \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} & -\varphi^{50}\varphi\overline{\varphi} \\ \varphi^{50} & -\varphi^{49}\varphi\overline{\varphi} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} & \varphi^{50} \\ \varphi^{50} & \varphi^{49} \end{pmatrix}.$$

e) Using the result of part (c), explain why $\frac{F_{51}}{F_{50}}$ is very close to φ.

Solution. We will compute the approximation of the vector $\begin{pmatrix} F_{51} \\ F_{50} \end{pmatrix}$:

$$\begin{pmatrix} F_{51} \\ F_{50} \end{pmatrix} = A^{50}\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} \approx \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} & \varphi^{50} \\ \varphi^{50} & \varphi^{49} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{51} \\ \varphi^{50} \end{pmatrix}.$$

Therefore, $\frac{F_{51}}{F_{50}} \approx \frac{\varphi^{51}}{\varphi^{50}} = \varphi$.