Derive marginalised probability

Ultimately want to maximise $P(\theta|\{\log L_i\})$. Bayes' rule to invert in terms of known quantities

$$P(\theta \mid \{\log L_i\}) = \frac{P(\{\log L_i\} \mid \theta) \times P(\theta)}{P(\{\log L_i\})} \tag{1}$$

Assume an uninformative constant prior $P(\theta)$, and disregard the total evidence $P(\{\log L_i\})$ which is independent of θ. The remaining quantity is the evidence for θ, which is calculated by marginalising over $\boldsymbol{X}$

$$P(\theta \mid \{\log L_i\}) \propto P(\{\log L_i\} \mid \theta) = \int P(\{\log L_i\} \mid \theta, \boldsymbol{X}) P(\boldsymbol{X})\ d\boldsymbol{X} \tag{2}$$

$$P(\boldsymbol{X}) = \frac{1}{2\pi|\Sigma|} \exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^\intercal \Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right] \tag{3}$$

$$P(\{\log L_i\} \mid \boldsymbol{X}, \theta) = \prod_i \delta(\log L_i - f(X_i, \theta)) \tag{4}$$

$$\therefore P(\theta \mid \{\log L_i\}) = \int \frac{1}{2\pi|\Sigma|} \exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^\intercal \Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right] \left(\prod_i \delta(\log L_i - f(X_i, \theta))\right)\ dX_1 \cdots\ dX_n \tag{5}$$

$$f(X_i, \theta) = \log L_{\max} - \frac{X_i^{2/d}}{2\sigma^2} \tag{6}$$

Evaluate integral

Consider the identity

$$\delta(g(x)) = \sum_{\text{roots } j} \frac{\delta(x - x_j)}{|g'(x_j)|} \tag{7}$$

For the above case we expect only one root for each X_i, namely at $\log L_i = f(X_i^*, \theta)$ (assuming that f is one-to-one, which it is for the form we are using). (4) reduces to

$$\prod_i \frac{\delta(X_i - X_i^*)}{|f'(X_i^*, \theta)|} \tag{8}$$

Putting this together, the integral evaluates to

$$P(\theta \mid \log \boldsymbol{L}) = \frac{1}{2\pi|\Sigma|} \left(\prod_i \frac{1}{|f'(X_i^*, \theta)|}\right) \exp[-\frac{1}{2}(\boldsymbol{X^*}-\boldsymbol{\mu})^\intercal \Sigma^{-1}(\boldsymbol{X^*}-\boldsymbol{\mu})] \tag{9}$$

where the $\boldsymbol{X^*}$ are given by the inverse of f. Discard normalising term:

$$\log P(\theta \mid \log \boldsymbol{L}) = -\left(\sum_i \log|f'(X_i^*, \theta)|\right) - \frac{1}{2}(\boldsymbol{X^*}-\boldsymbol{\mu})^\intercal \Sigma^{-1}(\boldsymbol{X^*}-\boldsymbol{\mu}) \tag{10}$$

This is the expression I am maximising, for various forms of f e.g. $f(X_i, \theta) = aX_i$, $f = -X_i^{2/d}$

Specific form of f for simplified case

Invert f to find X_i^* and $|f'(X_i^*)|$

$$f(X_i, \theta) = f(X_i, d) = -X_i^{2/d} \tag{11}$$

$$X_i^* = (-\log L_i)^{d/2} \tag{12}$$

$$f'(X_i, d) = -\frac{2}{d} X_i^{2/d-1} \tag{13}$$

$$\therefore \log P(\theta \mid \log \boldsymbol{L}) = -\left(\sum_i \log \left|\frac{2}{d} X_i^{*\,2/d-1}\right|\right) - \frac{1}{2}(\boldsymbol{X^*} - \boldsymbol{\mu})^\intercal \Sigma^{-1} (\boldsymbol{X^*} - \boldsymbol{\mu}) \tag{14}$$