Midterm Review

1. Brownian Motions
   Property 1:
   $$dW \sim \mathcal{N}(0, dt)$$
   (a) Multiplication rules (3.10.1 [Shreve])
   $$dW(t)dW(t) = dt, dW(t)dt = 0, dtdt = 0$$
   (b) Derive the formula for $d(W^2)$, $d(W^4)$.

Property 2: all the increment dWs are independent from each other.

(a) Suppose $X(T) = \int_0^T f(t)dt + \int_0^T g(t)dW_t$. Find $Var(X)$

Property 3: Brownian Motions are a Martingale.
Suppose a random walk on 1D-axis with $X(0) = 0$ and

$$dX = X(t+1) - X(t) = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$$

2. Ito's Formula

   (a) 1D case

   $$df(t,x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$$

   Compute the stochastic differential dZ when

   i. $Z(t) = exp(\alpha t)$

   ii. $Z(t) = exp(\alpha X(t))$ with

   $$dX(t) = \mu dt + \sigma dB(t)$$

   iii. $Z(t) = 1/X(t)$ with

   $$dX(t) = aX(t)dt + \sigma X(t)dW(t)$$

   (b) 2D case

   $$df(t,x,y) = f_t dt + (f_x dx + f_y dy) + \frac{1}{2}(f_{xx} dx dx + 2 f_{xy} dx dy + f_{yy} dy dy)$$

   Derive the Ito's product rule $d(XY) = XdY + YdX + dXdY$

3. Geometric Brownian Motions

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \leq t \leq T$$

Set

$$D(t) = exp\left(-\int_0^t R(s)ds\right)$$

(a) Derive a formula for $S(t)$

(b) Derive a formula for $d(D(t)S(t))$ by Ito's product rule.

(c) Derive a formula for $d(D(t)S(t))$ by Ito's formula (Exercise 5.1 [Shreve]).
Hint: Consider $f(x) = S(0)e^x$ and set

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)$$

(d) Show that S is log-normally distributed. i.e., show that $log(S)$ is normally distributed.

4. Black-Scholes-Merton Equation
Let $c(t, x)$ denote the value of an option at time t with current price $S(t) = x$. A profolio $X(t)$ with hedging strategy $\Delta(t)$ should satisfy

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, x))$$

Use Ito's formula to compute both sides to get

$$\begin{cases} \Delta(t) = c_x \\ rc = c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} \end{cases}$$

For European call options, we have

$$c(t, 0) = 0$$

$$c(T, x) = max(S(T) - K, 0)$$

We solve the equation with the boundary conditions above to get

$$c(t, x) = xN(d+) - Ke^{-rt}N(d-),$$

where

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}(T-t)\right)\right)$$

The problems about the BSM equations could be extremely difficult. For a reference, you may take a look at the problem as an excerpt of Exercise 4.9 [Shreve]

(a) Show that for $x > K$, $\lim_{t\to T^-} d_{\pm} = \infty$, but for $0 < x < K$, $\lim_{t\to T^-} d_{\pm} = -\infty$

(b) Show that for $0 \leq t < T$, $\lim_{x\to 0^+} d_{\pm} = -\infty$

(c) Show that for $0 \leq t < T$, $\lim_{x\to\infty} d_{\pm} = \infty$.

(d) Use (c) to verify

$$lim_{x\to\infty}\left(c(t, x) - (x - e^{-r(T-t)K})\right) = 0$$

.

Hint: In this verification, you will need to show that

$$\lim_{x\to\inf} \frac{N(d_+) - 1}{x^{-1}}.$$

Use the L'Hopital's rule and the fact

$$x = Kexp\left(\sigma\sqrt{T-t}d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right)\right)$$