Math of Finance

Recitation - Feb 11
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Office Hours - W 9-11 pm via zoom, or by appointments

Piazza - https://piazza.com/NYU/spring2022/mathua250

Simplest Binary Model — Chptr 1,2. txbk.

$$S(0) = S_0 \begin{cases} S(1,+) = S_{1H} \\ S(1,-) = S_{1T} \end{cases}$$

Binary Tree Model   Sect. 3.2

S_0

S_{1H} S_{2HH} S_{2HT}

S_{1T} S_{2TH} S_{2TT}

$\underbrace{\quad}_{dt}$

$n \to \infty$
$dt \to 0$

Continuous Time-Limit Model   Sect 3.3.2

SDEs.   (lecture)

Brownian Motion / Wiener Process

Definition

(P4) almost surely

Brownian Motion is a continuous stochastic process $(W_t)_{t \leq T}$ s.t.

~~W not differentiable a.s.~~

(P1) $W_0 = 0,$

(P2) $W_t - W_s, W_{t'} - W_{s'}$ independent, $0 \leq s' < t' < s < t \leq T$

(P3) $W_t - W_s \sim \mathcal{N}(0, t - s), 0 \leq s < t \leq T$

"σ^2" s.dv. $= \sqrt{t-s}$

pdf: $\frac{1}{\sqrt{2\pi(t-s)}} \cdot \exp\left(-\frac{x^2}{2(t-s)}\right)$ "σ" $\rightarrow \frac{1}{\sqrt{2\pi}\,\sigma}$

$$\mathrm{Var}(dW_t) = dt$$

$$\mathrm{Var}\left(\int_0 dW_t\right) \overset{(P2)}{=} \int \mathrm{Var}(dW_t)$$

$$= \int dt = T.$$

$$\boxed{\begin{array}{l} X \sim N(\mu_1, \sigma_1^2) \\ Y \sim N(\mu_2, \sigma_2^2) \\ X, Y \text{ indep.} \\ X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2) \end{array}}$$

Brownian Motion / Wiener Process

Random Walk

special case $x_i = \begin{cases} 1 & \text{w/ prob. } 1/2 \\ -1 & \text{w/ prob. } 1/2 \end{cases}$ $\quad \mathbb{E}X = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$

$n \to \infty$ $\quad \mathrm{Var} = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$

Let $x_1, x_2, \ldots, x_n$ be i.i.d. Random variables with mean 0 and var 1.

$x_1 + x_2 + \cdots + x_{nt}$

Define a stochastic process $W_t = \frac{1}{\sqrt{n}} \sum_{i \le nt} x_i$

$\lfloor nt \rfloor = nt$.

What can we say about $W_t - W_s$ when $n \to \infty$ according to the CLT?

$W_t - W_s \sim N(0, t-s)$ (P3)

$W_{t'} - W_{s'}$, $W_t - W_s$ $\quad s' < t' < s < t$. - independent

$x_1 \ x_2 \ x_3 \cdots \underbrace{x_{ns'+1} \cdots x_{nt'}}_{W_{t'} - W_{s'}} \cdots \underbrace{x_{ns+1} \cdots x_{nt}}_{W_t - W_s} \cdots$

W_t. random variable → distribution, mean, var $\qquad W_s$.

$$W_t - W_s = \frac{1}{\sqrt{n}} \sum_{i=ns+1}^{nt} X_i \sim N\left(\sqrt{n}(t-s)\cdot\mu,\ n(t-s)^2 \cdot \frac{\sigma^2}{n(t-s)}\right) \qquad N(0, t-s)$$

$$\times\sqrt{n}\cancel{(t-s)} \qquad \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \qquad \| \ (t-s)\sigma^2$$

$$\boxed{\begin{aligned} X &\sim N(\mu, \sigma^2) \\ aX &\sim N(a\mu, a^2\sigma^2) \end{aligned}}$$

$\|$

$\sqrt{n}(t-s)$

CLT $$\frac{1}{n\cancel{(t-s)}} \sum_{i=ns+1}^{nt} X_i \sim N\left(\mu, \frac{\sigma^2}{n(t-s)}\right)$$

$$N = nt - ns = n(t-s)$$

Brownian Motion / Wiener Process

CLT

, central limit theorem

Let $x_1, \ldots, x_n$ be i.i.d. Random variables with mean μ and variance σ^2

sample mean

$$\frac{1}{N}\sum x_i \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

random variable

Brownian Motion / Wiener Process

Property - Martingale

Martingale - $\mathbb{E}(W_t \,|\, W_1, \ldots W_s) = W_S$

Itô Intergral

Construction and properties

f: non-stochastic

$\mathbb{E}\int_0^T f^2(t)\,dt < \infty$

$$\int_0^T f(t)dW_t = \lim_{N\to 0}\sum_{n=0}^{n-1} f(t_n)(W_{t_n+1} - W_{t_n})$$

Martingale if $\mathbb{E}\int_0^T f^2(t)\,dt < \infty$, $\Rightarrow \mathbb{E}\left(\int_0^T f(t)dW_t \,\middle|\, W_1 \cdots W_s\right) = \int_0^s f(t)\,dW_t$

Itô Isometry $\mathbb{E}\int_0^T f^2(t)\,dt<\infty$, $\mathbb{E}\int_0^T g^2(t)\,dt<\infty$. $\therefore \mathbb{E}\int_0^T f(t)\,dW_t \cdot \int_0^T g(t)\,dW_t = \mathbb{E}\int_0^T f(t)g(t)\,dt$

Now, what can you say about the distributions of the Itô intergral?

random variable

$$(W_{t_{n+1}} - W_{t_n}) \sim \mathcal{N}(0,\ dt) \qquad t_{n+1} - t_n$$

$$f(t_n)(W_{t_{n+1}} - W_{t_n}) \sim \mathcal{N}(0 \cdot f(t_n),\ dt \cdot f^2(t_n))$$

$$\sim N(0,\ f^2(t_n)\cdot dt).$$

$$\int f(t_n)(W_{t_{n+1}} - W_{t_n}) \sim N\Big(0,\ \underbrace{\int f^2(t_n)\cdot dt}_{<\infty}\Big)$$

Stochastic Differential Equations

Itô's Lemma

SDEs.

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

$$\rightarrow \quad X_t = \int dX_t$$

$$= X_0 + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s$$

Itô's Lemma.

$$df = \frac{\partial f}{\partial t}dt + a\frac{\partial f}{\partial x}dt + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}dt + b\frac{\partial f}{\partial x}dw \quad \leftarrow$$

derive from

$$f(W_{t_{n+1}}) - f(W_{t_n}) = f'(W_{t_n})(W_{t_{n+1}} - W_{t_n}) + \frac{1}{2}f''(W_{t_n})(W_{t_{n+1}} - W_{t_n})^2$$

$$t_{n+1} \to t_n$$

Stochastic Differential Equations

Some Examples

$\int_0^t b \cdot dW_s$ Itô's integral

random variable.

$$dX_t = adt + bdW_t \quad a, b \in \mathbb{R}$$

$a \cdot dt + b \cdot dW_t$

$\int_0^t dW_s \sim \mathcal{N}(0, t)$

$X_t = X_0 + \int_0^t a\, ds + \int_0^t b \cdot dW_s$.

$\int_0^t b dW_s \sim \mathcal{N}(0, b^2 t)$

$= X_0 + at + b\sqrt{t}N$, $N = \mathcal{N}(0, 1)$

s.d.v. $b\sqrt{t}$

$EX_t = X_0 + at$

$\operatorname{Var} X_t = \operatorname{Var}(b\sqrt{t}N) = b^2 t$.

solution - random variable.

Stochastic Differential Equations

Some Examples

$X' = aX.$

$$dX_t = aXdt + bdW_t \quad a, b \in \mathbb{R}. \quad \Rightarrow \quad X = e^{at}$$

Apply change of variable $Y = e^{-at}X$, what can we say about dY by Itô Lemma?

Do the same for $dX_t = (aX + c)dt + bdW_t \quad a, b, c \in \mathbb{R}$

$X = e^{at}(\quad \cdots \quad)$

$y = e^{-at} X$

∂t

$dy = -ae^{-at} X \cdot dt + e^{-at} \cdot dX + 0.$

$= -ae^{-at} X \cdot dt + e^{-at} \cdot (aX\, dt + b\, dW_t)$

$= e^{-at} b \cdot dW_t$

$y = y_0 + \int e^{-at} b \cdot dW_t$

$Var(y) = Var\left(\int e^{-at} b \cdot dW_t\right)$

$\mathbb{E}y = y_0$

$= \left(e^{-at} \cdot b\right)^2 \cdot t$

$\downarrow$

dt-term 0

SDEs w/o dt-term martingale.

$X = e^{at} \cdot y$ $\quad y = e^{-at} X$

$Var(X) = (e^{at})^2 \cdot Var\, y$

$= (\cancel{e^{at}})^2 \cdot (\cancel{e^{-at}} b)^2 \cdot t$

$= b^2 t$

Stochastic Differential Equations

Some Examples

In general, $dX_t = (aX + c)dt + (bX + d)W_t \quad a, b, c, d \in \mathbb{R}$.

Apply change of variable $Y = exp(-at + b^2/2 + b\sqrt{T} \cdot N)X$, where $N \sim \mathcal{N}(0,1)$

what can we say about dY by Itô Lemma?