Mathematics of Finance
Exercises 1 Brownian Motions

- **Change in Office Hours - 3.30 - 5pm on Mondays at CIWW 805 (in person only), or by appointment. The office hours on Wednesdays are cancelled.**
- **Recommended Textbook - Stochastic Calculus for Finance, Vol II, by S.E. Shreve; Springer Verlag.**
- **Problems on the handout are drawn from or inspired by the exercises in the book.**

1. (Exercise 4.7 [Shreve] Calculations on Brownian Motions)
   (a) Compute dW^4 and then write W^4 as the sum of an ordinary (Lebesgue) integral.
   (b) Take expectations on both sides to derive the formula $\mathbb{E}W^4(T) = 3T^2$.
   (c) Deduce a formula for $\mathbb{E}W^6$.

2. (Exercise 4.19 [Shreve]) Let $W(t)$ be a Brownian motion and define

$$B(t) = \int_0^t sign(W(s))dW(s),$$

where

$$sign(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

   (a) Show that $(dB(t))^2 = dt$. Hence $B(t)$ is a Brownian motion by Levy's theorem.
   (b) Show the *Itô's product rule* $d(XY) = XdY + YdX + dXdY$ for stochastic process $X(t), Y(t)$.
   (c) Use (b) to compute $d(B(t)W(t))$. Conclude that $B(t)$ and $W(t)$ are uncorrelated normal random variables by showing $\mathbb{E}(B(t)W(t)) = 0$.
   (d) Compute $dW^2(t)$ and conclude that $B(t)$ and $W(t)$ are not independent by showing $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W(t)$. Why does this happen to uncorrelated normal variables?

3. (Geometric Brownian Motions) Assume a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

   (a) Apply the Itô's lemma to solve for S.
   (b) Compute $d(S^p(t))$.

   (Exercise 4.18 [Shreve]) Let X denote the value of an investor's profolio with a hedging strategy of $\Delta(t)$.

   (c) Find dX.

   Denote $\theta = (\alpha - r)/\sigma$ as the *market price of risk*, where r denotes the interest rate. Define the *state price desity process* as $\zeta(t) = exp\left\{-\theta W(t) - \left(r + \theta^2/2\right)t\right\}$.

   (d) Find $d\zeta$. Hint: use two different ways to express $d(e^{rt}\zeta)$
   (e) Show that $\zeta(t)X(t)$ is a martingale. (i.e. $d(\zeta(t)X(t))$ has no dt-terms).

   From (c), the *present value* at $t = 0$ of the random payment $V(T)$ at $t = T$ is $X(0) = \mathbb{E}(\zeta(T)V(T))$. Hence it is valid to call $\zeta(t)$ the *state price density process*.

4. (Exercises 4.9-4.11 [Shreve] Black-Scholes-Merton Equation) For a European call with mature time T and strike price K, the BSM price at time t is

$$c(t,x) = xN(d_+) - Ke^{-r(T-t)}N(d_-),$$

where

$$d_\pm = \frac{1}{\sigma_1\sqrt{r}}\left(log\frac{x}{K} + (r \pm \frac{1}{2}\sigma_1^2)r\right),$$

However, the underlying asset is indeed a geometric Brownian motion with volatility

$$\sigma_2 > \sigma_1 : dS(t) = \alpha S(t)dt + \sigma_2 S(t)dW(t).$$

We set up a profolio with value denoted by $X(t)$.
We remove cash from this portfolio at a rate $(\sigma_2^2 - \sigma_1^2)S^2c_{xx}/2 > 0$. Hence,

$$dX = dc - c_x dS + r(X - c + Sc_x)dt - (\sigma_2^2 - \sigma_1^2)S^2c_{xx}/2$$

   (a) Show that $dX = rXdt$.
   (b) Write out the Itô's formula for $d(e^{-rt}X(t))$. Deduce $dX = 0$.
   This implies the existence of an arbitrage opportunity.

5. (Exercise 4.20 [Shreve] Local Time) The Itô's Lemma in differential form says that

$$df(x,t) = f_t dt + f_x dx + \frac{1}{2}f_{xx}dxdx.$$

Plug in $x = W(t)$ to get

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt. \tag{1}$$

   (a) Let $K > 0$ a constant, and define $f(x) = max(x - K, 0)$. Compute $f'(x), f''(x)$. Be careful about the points when either differential is not defined.
   (b) Show that Equation 1 does not hold for $f(x) = max(x-K,0)$. Hint: Consider taking expected values and integrals on both sides.

To get some idea of what is going on here, we define a sequence of functions $\{f_n\}_{n=1}^\infty$ by

$$f_n(x) = \begin{cases} 0 & x \le K_{n-} \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & K_{n-} \le K \le K_{n+} \\ x - K & x \ge K_{n+} \end{cases}$$

where $K_{n-} = K - 1/(2n), K_{n+} = K + 1/(2n)$.

   (c) Show that

$$\lim_{n\to\infty} f_n(x) = max(x-K, 0),$$

   and that

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & x < K \\ 1/2 & x = K \\ 1 & x > K. \end{cases}$$

The value of $\lim_{n\to\infty} f'(x)$ at a single point will not matter when we integrate. We are constructing a continuous function $f_n(x)$ and $f_n'(x)$ is defined everywhere. Note further that $f_n''(x)$ is defined for $x \in \mathbb{R}\backslash\{K_+, K_-\}$, and $|f_n''(x)|$ is bounded above by n. Hence, the Itô's Lemma applies to the function f_n because the intergrals are well defined.

1. (Exercise 4.7 [Shreve] Calculations on Brownian Motions)

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(a). Itô's formula.

$$df(t,x) = f_t\,dt + f_x\,dx + \frac{1}{2} f_{xx}\,dx\,dx.$$

$$f(t,x) = x^4, \quad f_t = 0, \quad f_x = 4x^3, \quad f_{xx} = 12x^2.$$

$$dw^4 = df(t,w) = \left(0\cdot dt + 4x^3 dx + 6x^2 dx dx\right)\Big|_w$$

$$= 4w^3 dw + 6w^2 \underline{dw dw}$$

$$= 4w^3 dw + 6w^2 dt,$$

$$\mathbb{E}dw^2 = dt.$$

$$dw \sim N(0, dt)$$

$$dt = Var(dw) = \mathbb{E}(dw)^2 - \underbrace{(\mathbb{E}dw)^2}_{0}$$

$$\rightarrow \underline{dw\cdot dw} = \underline{dt}$$

$$w^4 = 4\int w^3 dw + 6\int w^2 dt$$

$$\mathbb{E}w^4 = 6\cdot \mathbb{E}\int w^2 dt$$

$$= 6\cdot \int \mathbb{E}w^2 dt$$

$$= 6\int t\,dt.$$

$$\mathbb{E}w^4(T) = 6\int_0^T t\,dt = 6\cdot\left[\frac{t^2}{2}\right]_0^T = 6\cdot\frac{T^2}{2} = 3T^2$$

$\mathbb{E}w^6$ $$dw^6 = 6w^5 dw + 15w^4 dt$$

$$w^6 = \int \cdots dw + \int 15 w^4 dt.$$

$$\mathbb{E}w^6 = \int \mathbb{E}\,15 w^4 dt$$

$$= \int 15\cdot 3t^2 dt.$$

$$\mathbb{E}w^6(T) = 15T^3$$

2. (Exercise 4.19 [Shreve]) Let $W(t)$ be a Brownian motion and define

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where

$$sign(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

(a) Show that $(dB(t))^2 = dt$. Hence $B(t)$ is a Brownian motion by Levy's theorem.

(b) Show the *Itô's product rule* $d(XY) = XdY + YdX + dXdY$ for stochastic process $X(t), Y(t)$.

(c) Use (b) to compute $d(B(t)W(t))$. Conclude that $B(t)$ and $W(t)$ are uncorrelated normal random variables by showing $\mathbb{E}(B(t)W(t)) = 0$.

(d) Compute $dW^2(t)$ and conclude that $B(t)$ and $W(t)$ are not independent by showing $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t)$. Why does this happen to uncorrelated normal variables?

(a) $dB(t) = sign(W(t)) \cdot dW(t)$ $\qquad dw \cdot dw = dt$.

$(dB(t))^2 = 1 \cdot dW(t) \cdot dW(t) = dt$

(b). $d(XY) = Xdy + ydX + dXdy$.

$df(t,x,y) = \cancel{f_t dt} + f_x dx + f_y dy + \cancel{\frac{1}{2} f_{xx} dxdx} + f_{xy} dxdy + \cancel{\frac{1}{2} f_{yy} dydy}$.

$f(t,x,y) = xy \qquad f_t = 0, \; f_x = y, \; f_y = x,$

$f_{xx} = 0, \; f_{yy} = 0, \; f_{xy} = 1.$

$dxy = df(t,x,y) = ydx + xdy + dxdy$.

(c). $d(BW) = BdW + WdB + dBdW$

$= \left(\int_0^t sign(W(s))dW(s)\right) dW + W \cdot sign\, W dW.$

$+ \; sign(W) dw \cdot dw$.

$BW = \int \left(\int_0^t sign(w) \cdot dw\right) dw + \int w \cdot sign(w) \cdot dw$

$+ \int sign(w) \cdot dt$

$$\mathbb{E}BW = \int \left(\cancel{\mathbb{E}\int_0^t sign(w)\, dw}\right) dw + \int \mathbb{E}\, sign(w)\, dt$$

$$= 0$$

$$sign(w) = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$$\mathbb{E}\, sign(w) = 1\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = 0$$

$\mathbb{E}BW = 0 \quad \rightarrow \quad B$ & W uncorrelated.

A, B independent $\rightarrow$ A, B uncorrelated. Always holds.

A, B uncorrelated $\not\rightarrow$ A, B independent.

Exception

A, B Jointly Normal
uncorrelated $\longrightarrow$ A, B independent.

$$A, B \sim N(\mu, \Sigma)$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\rho = 0$$

$$A, B \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$$

$$\Rightarrow \begin{cases} A \sim N(\mu_1, \sigma_1^2) \\ B \sim N(\mu_2, \sigma_2^2) \end{cases}$$

A, B normal
uncorrelated $\not\rightarrow$ A, B independent.

(d). $\mathbb{E}(BW^2) \neq \underline{\mathbb{E}B} \cdot \mathbb{E}W^2 = 0.$ $\qquad \mathbb{E}\int dw$

$$d(W^2) = 2W\,dw + \underbrace{(dw)^2}_{= dt}.$$

$$d(BW^2) = B\,d(W^2) + W^2 dB + dB \cdot d(W^2)$$

$$= \text{--}\,dw + \quad w \cdot \operatorname{sign}(w)\,dt$$

$$BW^2 = \cancel{\int \text{--}\,dw} + \int \underline{w \cdot \operatorname{sign}(w)}\,dt$$

$$\mathbb{E}BW^2 = \int \mathbb{E} \cdot \underbrace{\cancel{w \cdot \operatorname{sign}(w)}}_{= |w|} dt \quad > 0.$$

$> 0 \quad w \neq 0.$

$$\underbrace{\mathbb{E}BW^2}_{>0} \overset{?}{\neq} \mathbb{E}B \cdot \mathbb{E}W^2 = 0$$

B, W not independent

3. (Geometric Brownian Motions) Assume a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

(a) Apply the Itô's lemma to solve for S. Hint, $d(\log S)$.

(b) Compute $d(S^p(t))$.

(Exercise 4.18 [Shreve]) Let X denote the value of an investor's profolio with a hedging strategy of $\Delta(t)$.

(c) Find dX.

Denote $\theta = (\alpha - r)/\sigma$ as the *market price of risk*, where r denotes the interest rate. Define the *state price desity process* as $\zeta(t) = exp\left\{-\theta W(t) - \left(r + \theta^2/2\right)t\right\}$.

(d) Find $d\zeta$. Hint: use two different ways to express $d(e^{rt}\zeta)$

(e) Show that $\zeta(t)X(t)$ is a martingale. (i.e. $d(\zeta(t)X(t))$ has no dt-terms).

From (c), the *present value* at $t = 0$ of the random payment $V(T)$ at $t = T$ is $X(0) = \mathbb{E}(\zeta(T)V(T))$. Hence it is valid to call $\zeta(t)$ the *state price density process.*

(a) $$d\log S = \frac{1}{S}\underline{dS} + \frac{1}{2}\left(-\frac{1}{S^2}\right)\underline{dSdS}.$$

$$= \sigma dW + \left(\alpha - \frac{1}{2}\sigma^2\right)dt.$$

$$\log S = \int \sigma dW + \left(\alpha - \frac{1}{2}\sigma^2\right)dt$$

$$S(T) = S(0)\cdot \exp\left(\int_0^T \underline{\sigma(t)}\, dW(t) + \int_0^T \alpha(t) - \frac{1}{2}\sigma^2(t)dt\right)$$

if α, σ constants

$$\rightarrow S(T) = S(0)\cdot \exp\left(\sigma W + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right)$$

(b). $$d(S^p(t)) = pS^{p-1}dS + \frac{1}{2}p(p-1)S^{p-2}dSdS$$

$$= pS^p\left(\sigma dW + \left(\alpha + \frac{p-1}{2}\sigma^2\right)dt\right)$$

(c). $$dX = \underline{rXdt} + \underline{\Delta(\alpha - r)Sdt} + \underline{\Delta\sigma S dW}$$

(d). $$\theta = \frac{\alpha - r}{\sigma} \qquad Z(t) = \exp\left\{-\theta W - \left(\cancel{r} + \frac{\theta^2}{2}\right)t\right\}.$$

$$d\left(e^{rt}Z(t)\right) = d\cdot\left(\exp\left(-\theta W - \frac{\theta^2}{2}\right)t\right)$$

$$= (-\theta)\cdot \exp\left(-\theta W - \frac{\theta^2}{2}\right)t\, dW.$$

$$= -\theta e^{rt} Z(t)\, dW.$$

$$d(e^{rt}Z(t)) = re^{rt}\cdot Z\,dt + e^{rt}\cdot dZ.$$

$f(t,x) = e^{rt}\cdot x$

$f_t = re^{rt}\cdot x$

$f_x = e^{rt}$

$f_{xx} = 0.$

$$-\theta \cancel{e^{rt}} Z(t)\,dW = r\cancel{e^{rt}} Z\,dt + \cancel{e^{rt}}\,dZ$$

$$dZ = -\theta Z\,dW - rZ\,dt.$$

(e) $d(ZX) = Z\,dX + X\,dZ + dX\,dZ.$

$$= (\Delta\sigma S - \theta X)Z\,dW.$$

dt

$$\mathbb{E}(ZX \mid \mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_t) = Z(t)X(t)$$

martingale.

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However, the underlying asset is indeed a geometric Brownian motion with volatility

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We set up a profolio with value denoted by $X(t)$.
We remove cash from this portfolio at a rate $(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2 > 0$. Hence,

$$dX = dc - c_x dS + r(X - c + Sc_x)dt - (\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2$$

(a) Show that $dX = rXdt$.

(b) Write out the Itô's formula for $d(e^{-rt}X(t))$. Deduce $dX = 0$.
This implies the existence of an arbitrage opportunity.

(a) $dc = c_t dt + c_x dS + \frac{1}{2}c_{xx}dSdS$ (Itô)

$= \left(rSc_x + \frac{1}{2}\sigma_1^2 S^2 c_{xx} - rc\right)dt + c_x\left(\alpha S dt + \sigma_2 S dW\right) + \frac{1}{2}c_{xx}\sigma_2^2 S^2 dt$

(plug in $\underbrace{c_t = rSc_x + \frac{1}{2}\sigma_1^2 S^2 c_{xx} - rc}_{\text{BSM}}$, $dS = \alpha S dt + \sigma_2 S dW$)

Now, $dX = dc - c_x dS + r(X - c + S c_x)dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}$

$ds = (\alpha S dt + \sigma_2 S dW)$

$= \left(rSc_x + \frac{1}{2}\sigma_1^2 S^2 c_{xx} - rc\right)dt + c_x\left(\alpha S dt + \sigma_2 S dW\right) + \frac{1}{2}c_{xx}\sigma_2^2 S^2 dt$

$- c_x dS + r(X - c + S c_x)dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}$

$= rXdt$

(terms with the same underlined patterns are canceled.

(b) $d(e^{-rt}X) = -re^{-rt}X\,dt + e^{-rt}\underline{dX}$

$= -re^{-rt}X\,dt + e^{-rt}\underline{rX\,dt}$

(plug in $dX = rX\,dt$)

$= 0$

$\Rightarrow$ $e^{-rt}X(t)$ is a constant.

$\Rightarrow$ $e^{-rt}X(t) = e^{-r\cdot 0}X(0) = 0$

$\Rightarrow$ $X(t) = 0$.

$\Rightarrow$ $dX = 0$

$$\underbrace{Ef(W)}_{>0} = E\int df(W) = E\int \frac{1}{2} f''(W)\,dt \quad =0.$$

5. (Exercise 4.20 [Shreve] Local Time) The Itô's Lemma in differential form says that

$$df(x,t) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx. \qquad f_1 \cdots f_n \to f$$

Plug in $x = W(t)$ to get

$$df(W(t)) = f'(W(t))\cancel{dW(t)} + \frac{1}{2} f''(W(t))dt. \tag{1}$$

(a) Let $K > 0$ a constant, and define $f(x) = max(x-K, 0)$. Compute $f'(x), f''(x)$. Be careful about the points when either differential is not defined.

(b) Show that Equation 1 does not hold for $f(x) = max(x-K, 0)$. Hint: Consider taking expected values and integrals on both sides.

To get some idea of what is going on here, we define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ by

$$f_n(x) = \begin{cases} 0 & x \le K_{n-} \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & K_{n-} \le K \le K_{n+} \\ x - K & x \ge K_{n+} \end{cases}$$

where $K_{n-} = K - 1/(2n), K_{n+} = K + 1/(2n)$.

$$f_1, f_2 \cdots \to f$$

(c) Show that

$$\lim_{n\to\infty} f_n(x) = max(x-K, 0),$$

$$f_1' \cdots f_n' \to f'$$

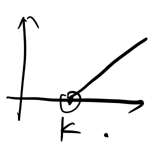
and that

$$f_1'' \cdots f_n'' \to f''$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & x < K \\ 1/2 & x = K \\ 1 & x > K. \end{cases}$$

The value of $\lim_{n\to\infty} f'(x)$ at a single point will not matter when we integrate. We are constructing a continuous function $f_n(x)$ and $f_n'(x)$ is defined everywhere. Note further that $f_n''(x)$ is defined for $x \in \mathbb{R}\backslash\{K_+, K_-\}$, and $|f_n''(x)|$ is bounded above by n. Hence, the Itô's Lemma applies to the function f_n because the intergrals are well defined.

(1). $f(x) = max(x-K, 0)$.



$$f'(x) = \begin{cases} 1 & x > K \\ 0 & x < K \\ \text{not def.} & x = K \end{cases}$$

$$f''(x) = \begin{cases} 0 & x \ne K \\ \text{not defied} & x = K \end{cases}$$

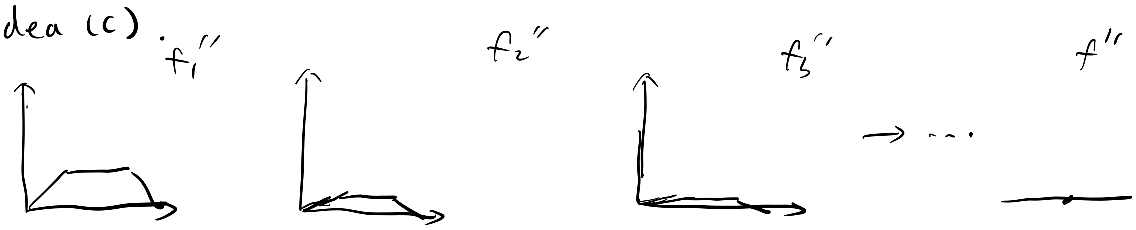
$$\Rightarrow f''(x) = 0$$

$$Ef(W) = \int_{-\infty}^{\infty} f(x) \cdot \underset{\uparrow \text{pdf}}{p(x)}\, dx$$

$$= \int_{-\infty}^{K} f(x)\cdot p(x)\,dx + \int_{K}^{+\infty} f(x)\cdot p(x)\,dx$$

$$= \int_{-\infty}^{K} 0 \cdot p(x)\,dx + \int_{K}^{+\infty} (x-K)\cdot \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}\,dx > 0$$

Idea (c).



$f_1 \cdots f_n$ $f = \max(x-K, 0)$.

$$\mathbb{E}\, g(\cancel{X}) = \int g(x) \cdot f(x)\, dx.$$

$\uparrow$

pdf

$$f_t = \frac{\partial f}{\partial t}$$

$$f_x = \frac{\partial f}{\partial x}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$