Mathematics of Finance Review 1

Review plans - 1.5 sessions on the handout & 1.5 sessions on practical exercises.

1. [Martingales] [Definitions] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider an adapted stochastic process $M(t), 0 \leq t \leq T$. If $\forall 0 \leq s \leq t \leq T$:
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) = M(s)$, then we say this process is a martingale. It has no tendency to rise or fall;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \leq M(s)$, then we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \geq M(s)$, then we say this process is a supermartingale. It has no tendency to rise; it may have a tendency to fall;

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step symmetric random walk as follows.

   $$W_n(t) = \sum_{i=1}^{n} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Show that $W_n(t)$ is a martingale.

   (b) For American Options, we have $V_t = max(\tilde{\mathbb{E}}(V_{t+1}|\mathcal{F}(t)), G(t)) \forall t$. Classify V_t as a type of martingale.

2. [Scaled Symmetric Random Walks]

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step scaled symmetric random walk as follows.

   $$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Deduce that $E(W_n(t) - W_n(s)) = 0$ and $Var(W_n(t) - W_n(s)) = t - s$.

   (b) Show that $W(t) := \lim_{n\to\infty} W_n(t) = \mathcal{N}(0, t)$.

3. [Binary and Log-Normal Markets] Consider an n-step binary market with no interest rate ($R = 1.0$)

   (a) Set $u = 3/2, d = 1/2$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S(t)$ at time t.

   (b) Set $u = 1 + \sigma/\sqrt{n}, d = 1 - \sigma/\sqrt{n}$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S_n(t)$ at time t. Show that

   $$\lim_{n\to\infty} S_n(t) = S(0) \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right),$$

   where $W(t)$ is defined in 2(b).

4. [Brownian Motions - Calculations] [Definition] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$. Then $W(t), t \geq 0$ is a Brownian motion if $\forall i \in \{0, 1, \ldots m\}$, the increments $W(t_{i+1}) - W(t_i)$ are independent and each of these increments is normally distributed with $\mathbb{E} = 0$ and $Var = t_{i+1} - t_i$.

   (a) Set $dW(t) = W(t + dt) - W(t)$. From the defination, find $\mathbb{E}(dW), Var(dW)$.
   Find $\mathbb{P}\{W(0.25) \leq 0.2\}$

   (b) Show that $\mathbb{E}(W(t)W(s)) = t \wedge s$. Deduce the covariance of $W(t)$ and $W(s)$.

   (c) Show that $W(t)$ is a martingale, and so is $Z(t) = exp(\sigma W(t) - 1/2 * \sigma^2 t)$

5. [Ito's integral] Consider the following Ito's integral:

$$I(t) = \int_0^t \Delta(u)dW(u) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1} - W(t_j))$$

(a) Show that, $I(t)$ is a martingale. Remark: An Ito's integral with zero $dt-$term is a martingale.

(b) Show that,

$$f(T, W(T)) = f(0, W(0) + \int_0^T f_t(t, W(t)dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

.

Hint: We may apply the Taylor's formula:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2$$

Note that the reminder contains a sum of terms $(W(t_{j+1} - W_j)^3$ which has limit 0.

Further Hint: $dW(t)dW(t) = dt, \quad dtdW(t) = 0, \quad dtdt = 0.$

Remark: We can rewrite the formula as the differential term:

$$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2} f_{xx} dX(t)dX(t)$$

(c) Deduce that $d(AB) = AdB + BdA + dAdB$ for stochastic process $A(t), B(t)$.

6. [Probability Measures] Consider the geometric brownian motions $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$. We define a disounted process $D(t) = exp(-rt)$.
We define $X(t)$ as the total profolio under the hedging strategy $\Delta(t)$.

(a) Find $d(D(t)S(t)$ and $d(D(t)X(t))$.

(b) Show that, if $d(D(t)S(t)$ is a martingale under some probability measure $\tilde{\mathbb{P}}$, then so is $d(D(t)X(t))$.

7. [Change of Probability Measures] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$. Then, $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then $\tilde{\mathbb{E}}X = \mathbb{E}(XZ)$.

(a) Show that $\mathbb{E}\, Z = 0$ for $Z(\omega) = \exp(-\theta X(\omega) - 1/2 * \theta^2)$

(b) Show that $\mathbb{E}\, Z = 0$ for

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right)$$

8. [Binary Markets] Consider a 2-step binary market with $S_0 = 4, u = 2, d = 0.5, R = 1.1$

(a) Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.

(b) Find the fair prices of a European call with $K = 4, T = 2$.

(c) Derive a hedging strategy $\Delta(t)$.

9. [Log-Normal Markets] Consider a geometric brownian motion with $dS(t) = 0.1 * S(t)dt + 0.3 * S(t)dW(t)$. Set the interest factor $R(t) = \exp(1.05t)$.

(a) Set $S(0) = 1$, consider a European call with $K = 1, T = 2$. Find the fair price of such an option. You may proceed with either Black-Shore, or the risk-neutral probability measure.