

Midterm Review

1. Brownian Motions

Property 1:

$$dW \sim \mathcal{N}(0, dt)$$

(a) Multiplication rules (3.10.1 [Shreve])

$$dW(t)dW(t) = dt, dW(t)dt = 0, dt dt = 0$$

(b) Derive the formula for $d(W^2)$, $d(W^4)$.

Property 2: all the increment dW s are independent from each other.

(a) Suppose $X(T) = \int_0^T f(x)dt + \int_0^T g(x)dW_t$. Find $Var(X)$

2. Ito's Formula

(a) 1D case

$$df(t, x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$$

Compute the stochastic differential dZ when

i. $Z(t) = \exp(\alpha t)$

ii. $Z(t) = \exp(\alpha X(t))$ with

$$dX(t) = \mu dt + \sigma dB(t)$$

iii. $Z(t) = 1/X(t)$ with

$$dX(t) = aX(t)dt + \sigma X(t)dW(t)$$

(b) 2D case

$$df(t, x, y) = f_t dt + (f_x dx + f_y dy) + \frac{1}{2}(f_{xx} dx dx + 2f_{xy} dx dy + f_{yy} dy dy)$$

Derive the Ito's product rule $d(XY) = XdY + YdX + dXdY$

3. Geometric Brownian Motions

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \leq t \leq T$$

Set

$$D(t) = \exp\left(-\int_0^t R(s)ds\right)$$

- (a) Derive a formula for $S(t)$
- (b) Derive a formula for $d(D(t)S(t))$ by Ito's product rule.
- (c) Derive a formula for $d(D(t)S(t))$ by Ito's formula (Exercise 5.1 [Shreve]).
Hint: Consider $f(x) = S(0)e^x$ and set

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds$$

- (d) Show that S is log-normally distributed. i.e., show that $\log(S)$ is normally distributed.

4. Black-Scholes-Merton Equation

Let $c(t, x)$ denote the value of an option at time t with current price $S(t) = x$. A portfolio $X(t)$ with hedging strategy $\Delta(t)$ should satisfy

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, x))$$

Use Ito's formula to compute both sides to get

$$\begin{cases} \Delta(t) = c_x \\ rc = c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} \end{cases}$$

For European call options, we have

$$c(t, 0) = 0$$

$$c(T, x) = \max(S(T) - K, 0)$$

We solve the equation with the boundary conditions above to get

$$c(t, x) = xN(d_+) - Ke^{-rt}N(d_-),$$

where

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}(T-t) \right) \right)$$

The problems about the BSM equations could be extremely difficult. For a reference, you may take a look at the problem as an excerpt of Exercise 4.9 [Shreve]

- (a) Show that for $x > K$, $\lim_{t \rightarrow T^-} d_{\pm} = \infty$, but for $0 < x < K$, $\lim_{t \rightarrow T^-} d_{\pm} = -\infty$
- (b) Show that for $0 \leq t < T$, $\lim_{t \rightarrow 0^+} d_{\pm} = -\infty$
- (c) Show that for $0 \leq t < T$, $\lim_{t \rightarrow \infty} d_{\pm} = \infty$.
- (d) Use (c) to verify

$$\lim_{x \rightarrow \infty} (c(t, x) - (x - e^{-r(T-t)K})) = 0$$

Hint: In this verification, you will need to show that

$$\lim_{x \rightarrow \infty} \frac{N(d_+) - 1}{x^{-1}}.$$

Use the L'Hopital's rule and the fact

$$x = K \exp \left(\sigma \sqrt{T-t} d_+ - (T-t) \left(r + \frac{1}{2} \sigma^2 \right) \right)$$