Risk Neutral Probability Measure in Binary Markets.

$S_0 = 4$; $S_1(H) = 8$ ($\tilde{p}$) ; $S_1(T) = 2$ ($\tilde{q}$)

$V_0 = ?$

$\tilde{p}$ $\underline{V_1(H)} = \max(S_1(H) - K, 0)$

$= \max(8-4, 0)$

$= 4$

$\underline{V_1(T)} = \max(S_1(T) - K, 0)$

$= \max(2-4, 0)$

$= 0.$

European call $K = 4.$

at $T = 1$

$\tilde{p} + \tilde{q} = 1$

$V_0 = \tilde{p} V_1(H) + \tilde{q} V_1(T)$

F.I. sells such a call for V_0, Δ_0

$$X_1 = V_1$$

$$\underbrace{(V_0 - \Delta_0 S_0) R}_{\text{cash}} + \underbrace{\Delta_0 \cdot S_1}_{\text{stock}} = \underbrace{V_1}_{\text{payoff}}$$

$$\begin{cases} (V_0 - \Delta_0 S_0) R + \Delta_0 \cdot S_1(H) = V_1(H) & \times \tilde{p} \\ (V_0 - \Delta_0 S_0) R + \Delta_0 \cdot S_1(T) = V_1(T) & \times \tilde{q} \end{cases}$$

$\tilde{p} + \tilde{q} = 1$

$\Downarrow$

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

$$(V_0 - \Delta_0 S_0) R \underbrace{(\cancel{\tilde{p} + \tilde{q}})}_{1} + \Delta_0 \cdot (\tilde{p} S_1(H) + \tilde{q} S_1(T)) = \tilde{p} V_1(H) + \tilde{q} V_1(T)$$

$$V_0 - \Delta_0 S_0 + \Delta_0 \cdot \underbrace{\frac{1}{R}(\tilde{p} S_1(H) + \tilde{q} S_1(T))}_{\tilde{E}(\frac{1}{R} S_1)} = \underbrace{\frac{1}{R}(\tilde{p} V_1(H) + \tilde{q} V_1(T))}_{\tilde{E}(\frac{1}{R} V_1)}.$$

$$V_0 - \Delta_0 S_0 + \Delta_0 \cdot \tilde{\mathbb{E}}\left(\frac{S_1}{R}\right) = \tilde{\mathbb{E}}\left(\frac{V_1}{R}\right) \quad *$$

holds for $\forall \tilde{p}, \tilde{q}: \tilde{p}+\tilde{q}=1$

some $\tilde{p}, \tilde{q}$: $\quad S_0 = \tilde{\mathbb{E}}\left(\frac{S_1}{R}\right) \Leftrightarrow$ discounted stock price is martingale

$* \Rightarrow \quad V_0 = \tilde{\mathbb{E}}\left(\frac{V_1}{R}\right) \Leftrightarrow$ discounted payoff is martingale

$K=4$

$S_1(H)=8, \quad V_1(H)=4 \qquad r=0.1$

$S_0 = 4 \qquad\qquad\qquad\qquad\quad R = 1+r = 1.1$

$S_1(T)=2. \quad V_1(T)=0.$

$\tilde{p}, \tilde{q}: \to S_0 = \tilde{\mathbb{E}}\left(\frac{S_1}{R}\right) \Leftrightarrow V_0 = \tilde{\mathbb{E}}\left(\frac{V_1}{R}\right)$

$$= \frac{1}{R} \cdot \left(\tilde{p} S_1(H) + \tilde{q} S_1(T)\right). \qquad \tilde{p}+\tilde{q}=1.$$

$$\tilde{p} = \frac{R S_0 - S_1(T)}{S_1(H) - S_1(T)} = \frac{1.1 \times 4 - 2}{8-2} = 0.4$$

$$V_0 = \tilde{\mathbb{E}}\left(\frac{V_1}{R}\right)$$

$$= \frac{1}{R}\left(\tilde{p} V_1(H) + \tilde{q} V_1(T)\right)$$

$$= \frac{1}{1.1}\left(0.4 \times 4 + 0.6 \times 0\right) = \frac{1.6}{1.1} = \ldots$$

$$dS = \alpha S\,dt + \sigma S\,dW.$$

$$d\left(e^{-rt} S(t)\right) = \sigma(t)\cdot e^{-rt} S(t) \underbrace{\left(\theta(t)\,dt + dW(t)\right)}_{d\widetilde{W}(t)}.$$

$$\underline{d\widetilde{W}(t)} = dW(t) + \theta(t)\,dt$$

no dt term $\Leftrightarrow$ martingale

$$\underline{d\left(e^{-rt} X(t)\right)} = \Delta(t)\,\sigma(t)\cdot e^{-rt} S(t)\,\underline{d\widetilde{W}(t)}$$

no dt term

$$X(t) = V(t)$$

$e^{-rt} V(t)$ also martingale (under $\widetilde{P}$).

$$e^{-rt} V(t) = \widetilde{\mathbb{E}}\left(e^{-rT} \underline{V(T)} \,\middle|\, \mathcal{F}_t\right)$$

* holds for all securities.

$$V(t) = \widetilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \,\middle|\, \mathcal{F}_t\right).$$

$$V(t) = \hat{\mathbb{E}}\left(e^{-r(T-t)} V(T) \,\middle|\, \mathcal{F}_t\right)$$

$$V(T) = \max\left(S(T) - K, 0\right).$$ for European call.

$$S(T) = S(0) \exp\left(\sigma \tilde{W}(T) + (r - \tfrac{1}{2}\sigma^2) T\right)$$

$$S(T) = S(t) \exp\left(\sigma\left[\tilde{W}(T) - \tilde{W}(t)\right] + (r - \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$= S(t) \exp\left(-\sigma\sqrt{T-t} \cdot n + (r - \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$n = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T-t}} \sim N(0,1)$$

$$\tilde{W}(T) - \tilde{W}(t) \sim N(0, T-t)$$

$S(T) - K > 0$ iff.

$$S(t) \exp\left(-\sigma\sqrt{T-t} \cdot n + (r - \tfrac{1}{2}\sigma^2)(T-t)\right) > K.$$

iff $\Leftrightarrow$ $$n < \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \ln \frac{K}{S(t)}}{\sigma\sqrt{T-t}}$$

$$\shortparallel$$

$$d_-$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$$

$S(T) - K > 0$ iff $n < d_-$

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)} \cdot \max(S(T) - K, 0) \,\middle|\, \mathcal{F}_t\right)$$

$$= \int_{-\infty}^{d_-} \underbrace{e^{-r(T-t)} \cdot \left(S(t) \exp\left(-\sigma\sqrt{T-t} \cdot n + (r - \tfrac{1}{2}\sigma^2)(T-t)\right) - K\right)}_{X} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}n^2}}_{p(X)}\, dn$$

$$= S(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-}\exp\Big(\underbrace{-\tfrac{1}{2}\sigma^2(T-t)-\sigma\sqrt{T-t}\,n-\tfrac{1}{2}n^2}_{-\frac{1}{2}(\sigma\sqrt{T-t}+n)^2}\Big)\,dn$$

$m = n + \sigma\sqrt{T-t}$

$$-K\cdot e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-}\exp(-\tfrac{1}{2}n^2)\,dn$$

$$= S(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\underbrace{d_-+\sigma\sqrt{T-t}}=d_+}\exp(-\tfrac{1}{2}m^2)\,dm - Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-}\cdots dn$$

$$V(t) = S(t)\,\underline{N(d_+)} - K\cdot e^{-r(T-t)}\,\underline{N(d_-)}$$

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}}\Big(\log\frac{S(t)}{K} + (r\pm\tfrac{1}{2}\sigma^2)(T-t)\Big)$$

$$d_+ - d_- = \sigma^2(T-t)\frac{1}{\sigma\sqrt{T-t}}$$

$$= \sigma\sqrt{T-t}.$$

$$V(t) = \hat{\mathbb{E}}\left(e^{-r(T-t)} V(T) \,\middle|\, \mathcal{F}_t\right)$$

up & out call, with barrier B.

$$\max_{t\in[0,T]} (S(t)) \le B$$

$$S(t) = S(0)\exp\left(\sigma W(t) + (r-\tfrac{1}{2}\sigma^2)t\right)$$

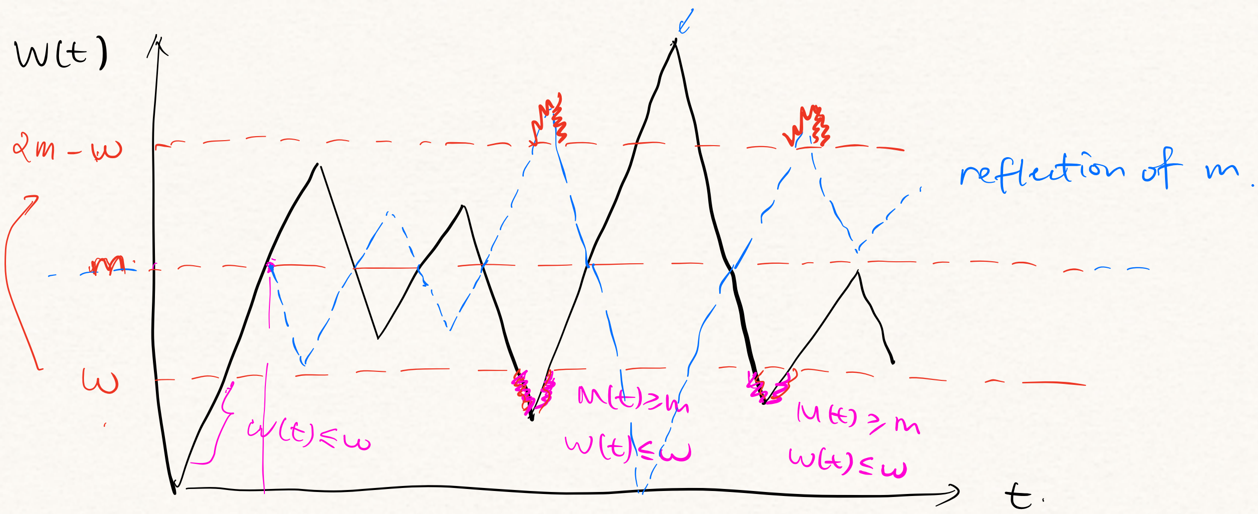
$$= S(0)\exp\left(\sigma \tilde{W}(t)\right).$$

$$\tilde{M}(t) = \max_{u\in[0,t]} \tilde{W}(u)$$

$$\tilde{W}(0) = 0, \quad \tilde{M}(t) \ge 0, \quad \tilde{M}(t) \ge \tilde{W}(t).$$

$$P(M(t) \ge m,\ W(t) \le w) = \int_m^\infty \int_{-\infty}^{w} f_{M(t),W(t)}(m,w)\,dw\,dm$$

$$P(M(t) \ge m,\ W(t) \le w) = P(W(t) \ge 2m-w)$$



$$P(M(t) \geq m, \quad W(t) \leq w) = \int_m^\infty \int_{-\infty}^{w} f_{M(t), W(t)}(m, w)\, dw\, dm$$

$$P(M(t) \geq m, \quad W(t) \leq w) = P(W(t) \geq 2m - w) \qquad \text{(reflection)}$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^{\infty} \exp\left(-\frac{z^2}{2t}\right) dz$$

$$W(t) \sim N(0, t)$$

$$f_{M(t), W(t)}(m, w)\, dw\, dm = \frac{2(2m-w)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2m-w)^2}{2t}\right)$$

$$\sigma \widetilde{W}(t) = \sigma W(t) + (r - \tfrac{1}{2}\sigma^2)\, t.$$

$$\hat{f}_{\widehat{M}(t), \widehat{W}(t)}(m, w)\, dw\, dm = \exp\left(\alpha m - \frac{1}{2}\alpha^2 T\right) f_{M(t), W(t)}(m, w)\, dw\, dm$$

$$\widetilde{P}(\widetilde{M}(t) \leq m, \quad \widetilde{W}(t) \geq w) \qquad = \int_m^\infty \int_{-\infty}^{w} \tilde{f}$$

not reach barriers.

$$S(T) - K > 0$$

$$V(t) = \widetilde{E}(e^{-rT} V(T) \mid \mathcal{F}(t))$$

Knock-out:

$$V(T) = \max\left(S(0)\, e^{\sigma \widetilde{W} T} - K, 0\right) \cdot \mathbb{1}_{\{S(0) e^{\sigma \widehat{M}(T)} \leq B\}}$$

$$= \left(S(0)\, e^{\sigma \widetilde{W}(T)} - K\right) \mathbb{1}_{\{S(0) e^{\sigma \widehat{M}(T)} \leq B, \quad S(0) e^{\sigma \widetilde{W}(T)} \geq K\}}$$

$$P\{\widetilde{W}(T) \geq k, \ \widetilde{M}(T) \leq b\} = \iint \tilde{f}$$

$$= \left(S(0)\, e^{\sigma \widetilde{W}(T)} - K\right) \cdot \mathbb{1}_{\left\{\widetilde{W}(T) \geq \underbrace{\frac{1}{\sigma} \log \frac{K}{S(0)}}_{= k}, \quad \widehat{M}(T) \leq \underbrace{\frac{1}{\sigma} \log \frac{B}{S(0)}}_{= b}\right\}}$$

lookback $V(T) = \max_{t\in[0,T]} \big(\underbrace{S(t)}_{\geq S(T)} \big) - S(T) \geq 0$

$= S(0)\, e^{\sigma \hat{M}(T)} - S(T)$