Exotic Options (Review)

$$V(t) = \tilde{E}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$$

European: $V(T) = \max\left(S(T)-K, 0\right)$ $\quad \left\{ \begin{array}{l} S(T) > K \\ \text{"n"} < d_- \end{array}\right.$

$$V(t) = S(t)\,N(d_+) - Ke^{-r(T-t)}N(d_-)$$

Knockout (up & out)

$$V(T) = \max(S(T)-K, 0) \quad \text{iff} \quad S(0)e^{\sigma \tilde{M}(T)} \le B.$$

$$V(T) = \left(S(0)\exp\left(\sigma\tilde{W}(T)\right) - K\right) \cdot \mathbb{1}_{\{\tilde{W}(T) \ge k,\ \tilde{M}(T) \le b\}}.$$

$$k = \frac{1}{\sigma}\log\frac{K}{S(0)}, \quad b = \frac{1}{\sigma}\log\frac{B}{S(0)}$$

$$\tilde{P}\left(\hat{M}(T) \le m,\ \tilde{W}(T) \le w\right) = \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \cdot \frac{2(2m-w)}{T\sqrt{2\pi T}}\, dm\, dw$$

$$\Rightarrow V(t) = S(t)\left(N(d_{1+}) - N(d_{2+})\right)$$
$$- e^{-r(T-t)}K\left(N(d_{1-}) - N(d_{2-})\right)$$
$$- B\left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}}\left(N(d_{3+}) - N(d_{4+})\right)$$
$$+ e^{-r(T-t)}K\left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left(N(d_{3-}) - N(d_{4-})\right)$$

$$d_{1\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S(t)}{K} + \left(r \pm \tfrac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{2\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S(t)}{B} + \left(r \pm \tfrac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{3\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{B^2}{KS(t)} + \left(r \pm \tfrac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{4\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{B}{S(t)} + \left(r \pm \tfrac{1}{2}\sigma^2\right)(T-t)\right)$$

LookBack:

$$V(T) = \max_{t\in[0,T]} S(t) - S(T)$$

$$= S(0)\exp\left(\sigma \widehat{M}(t)\right) - S(T)$$

$$V(t) = \widetilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$$

$$V(t, y) \qquad y(t) = \max_{u\in[0,t]} S(u)$$

$$= \left(1+\frac{\sigma^2}{2r}\right) S(t) N(d_+) + e^{-r(T-t)} y N(-d_-)$$

$$- \frac{\sigma^2}{2r} e^{-r(T-t)} \left(\frac{y}{S(t)}\right)^{\frac{2r}{\sigma^2}} S(t) N(+d_-) - S(t);$$

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S(t)}{y} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

American Options

Binary market, put $K=5 \Rightarrow G = 5 - S_n$.

$\{ R = 1.25$.

$S_0 = 4$

$G_0 = 1$.

$S_1(H) = 8$

$G_1(H) = -3$

$S_1(T) = 2$

$G_1(T) = 3$

$S_2(HH) = 16$ $\quad G_2(HH) = -11$

$S_2(HT) = 4$ $\quad G_2(HT) = 1$

$S_2(TH) = 4$ $\quad G_2(TH) = 1$

$S_2(TT) = 1$ $\quad G_2(TH) = 4$

Stopping time