Mathematics of Finance
Exercises 1   Brownian Motions

- **Change in Office Hours - 3.30 - 5pm on Mondays at CIWW 805 (in person only), or by appointment. The office hours on Wednesdays are cancelled.**
- **Recommended Textbook - Stochastic Calculus for Finance, Vol II, by S.E. Shreve; Springer Verlag.**
- **Problems on the handout are drawn from or inspired by the exercises in the book.**

1. (Exercise 4.7 [Shreve] Calculations on Brownian Motions)
   (a) Compute dW^4 and then write W^4 as the sum of an ordinary (Lebesgue) integral.
   (b) Take expectations on both sides to derive the formula $\mathbb{E}W^4(T) = 3T^2$.
   (c) Deduce a formula for $\mathbb{E}W^6$.

2. (Exercise 4.19 [Shreve]) Let $W(t)$ be a Brownian motion and define

$$B(t) = \int_0^t sign(W(s))dW(s),$$

where

$$sign(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

   (a) Show that $(dB(t))^2 = dt$. Hence $B(t)$ is a Brownian motion by Levy's theorem.
   (b) Show the *Itô's product rule* $d(XY) = XdY + YdX + dXdY$ for stochastic process $X(t), Y(t)$.
   (c) Use (b) to compute $d(B(t)W(t))$. Conclude that $B(t)$ and $W(t)$ are uncorrelated normal random variables by showing $\mathbb{E}(B(t)W(t)) = 0$.
   (d) Compute $dW^2(t)$ and conclude that $B(t)$ and $W(t)$ are not independent by showing $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t)$. Why does this happen to uncorrelated normal variables?

3. (Geometric Brownian Motions) Assume a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

   (a) Apply the Itô's lemma to solve for S.
   (b) Compute $d(S^p(t))$.

   (Exercise 4.18 [Shreve]) Let X denote the value of an investor's profolio with a hedging strategy of $\Delta(t)$.

   (c) Find dX.

   Denote $\theta = (\alpha - r)/\sigma$ as the *market price of risk*, where r denotes the interest rate. Define the *state price desity process* as $\zeta(t) = exp\left\{-\theta W(t) - \left(r + \theta^2/2\right)t\right\}$.

   (d) Find $d\zeta$. Hint: use two different ways to express $d(e^{rt}\zeta)$
   (e) Show that $\zeta(t)X(t)$ is a martingale. (i.e. $d(\zeta(t)X(t))$ has no dt-terms).

   From (c), the *present value* at $t = 0$ of the random payment $V(T)$ at $t = T$ is $X(0) = \mathbb{E}(\zeta(T)V(T))$. Hence it is valid to call $\zeta(t)$ the *state price density process.*

4. (Exercises 4.9-4.11 [Shreve] Black-Scholes-Merton Equation) For a European call with mature time T and strike price K, the BSM price at time t is

$$c(t,x) = xN(d_+) - Ke^{-r(T-t)}N(d_-),$$

where

$$d_{\pm} = \frac{1}{\sigma_1\sqrt{r}}\left(log\frac{x}{K} + (r \pm \frac{1}{2}\sigma_1^2)r\right),$$

However, the underlying asset is indeed a geometric Brownian motion with volatility

$$\sigma_2 > \sigma_1 : dS(t) = \alpha S(t)dt + \sigma_2 S(t)dW(t).$$

We set up a profolio with value denoted by $X(t)$.
We remove cash from this portfolio at a rate $(\sigma_2^2 - \sigma_1^2)S^2c_{xx}/2 > 0$. Hence,

$$dX = dc - c_x dS + r(X - c + Sc_x)dt - (\sigma_2^2 - \sigma_1^2)S^2c_{xx}/2$$

(a) Show that $dX = rXdt$.

(b) Write out the Itô's formula for $d(e^{-rt}X(t))$. Deduce $dX = 0$.
This implies the existence of an arbitrage opportunity.

5. (Exercise 4.20 [Shreve] Local Time) The Itô's Lemma in differential form says that

$$df(x,t) = f_t dt + f_x dx + \frac{1}{2}f_{xx}dxdx.$$

Plug in $x = W(t)$ to get

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt. \tag{1}$$

(a) Let $K > 0$ a constant, and define $f(x) = max(x-K, 0)$. Compute $f'(x), f''(x)$. Be careful about the points when either differential is not defined.

(b) Show that Equation 1 does not hold for $f(x) = max(x-K, 0)$. Hint: Consider taking expected values and integrals on both sides.

To get some idea of what is going on here, we define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ by

$$f_n(x) = \begin{cases} 0 & x \le K_{n-} \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & K_{n-} \le K \le K_{n+} \\ x - K & x \ge K_{n+} \end{cases}$$

where $K_{n-} = K - 1/(2n), K_{n+} = K + 1/(2n)$.

(c) Show that

$$\lim_{n\to\infty} f_n(x) = max(x-K, 0),$$

and that

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & x < K \\ 1/2 & x = K \\ 1 & x > K. \end{cases}$$

The value of $\lim_{n\to\infty} f'(x)$ at a single point will not matter when we integrate. We are constructing a continuous function $f_n(x)$ and $f_n'(x)$ is defined everywhere. Note further that $f_n''(x)$ is defined for $x \in \mathbb{R}\backslash\{K_+, K_-\}$, and $|f_n''(x)|$ is bounded above by n. Hence, the Itô's Lemma applies to the function f_n because the intergrals are well defined.