Exotic Options (Review)

geometric B.M. $dS(t) = \exp(\sigma \tilde{W}(t))$

$d\tilde{W} = dW + \theta(t)\,dt$ $\tilde{P}$

$\tilde{W}$ is brownian under $\tilde{P}$ $d\tilde{W}$

$d(e^{-rt}S(t)) = \sigma(t)e^{-rt}S(t)\left(\theta(t)dt + dW\right)$

under $\tilde{P}$: $\begin{cases} e^{-rt}S(t) \text{ is martingale} \\ e^{-rt}V(t) \text{ is martingale} \end{cases}$

$\theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$

$d(e^{-rt}V(t)) = \sigma(t)e^{-rt}S(t) \cdots d\tilde{W}$

$$\Rightarrow e^{-rt}V(t) = \tilde{E}\left(e^{-rT}V(T) \mid \mathcal{F}(t)\right)$$

$$V(t) = \tilde{E}\left(e^{-r(T-t)}V(T) \mid \mathcal{F}(t)\right)$$

$S(t)$

$S(0) \sim S(t)$

$$V(t) = \widetilde{E}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$$

European: $V(T) = \max\left(S(T) - K, 0\right)$ $\left\{\begin{array}{l} S(T) > K \\ \text{"}n\text{"} < d_- \end{array}\right.$

$$V(t) = S(t) N(d_+) - Ke^{-r(T-t)} N(d_-)$$

cdf of standard normal.

Knockout (up & out) $\widetilde{M}(t) = \max_{u \in [0,t]} \widetilde{W}(u)$

$$V(T) = \max(S(T) - K, 0) \quad \text{iff} \quad S(0) e^{\sigma \widetilde{M}(T)} \le B.$$

$$V(T) = \left(S(0) \exp\left(\sigma \widetilde{W}(T)\right) - K\right) \cdot \mathbb{1}_{\{\widetilde{W}(T) \ge k,\ \widetilde{M}(T) \le b\}}.$$

$$k = \frac{1}{\sigma} \log \frac{K}{S(0)}, \quad b = \frac{1}{\sigma} \log \frac{B}{S(0)}$$

$$\Rightarrow \widetilde{P}\left(\widetilde{M}(T) \le m,\ \widetilde{W}(T) \le w\right) = \int_{-\infty}^{w} \int_{-\infty}^{m} e^{\alpha m - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \cdot \frac{2(2m-w)}{T\sqrt{2\pi T}} \, dm \, dw$$

$$\begin{aligned}
\Rightarrow V(t) = {} & S(t)\left(N(d_{1+}) - N(d_{2+})\right) \\
& - e^{-r(T-t)} K \left(N(d_{1-}) - N(d_{2-})\right) \\
& - B\left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}} \left(N(d_{3+}) - N(d_{4+})\right) \\
& + e^{-r(T-t)} K \left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}+1} \left(N(d_{3-}) - N(d_{4-})\right)
\end{aligned}$$

exercise 7.1

$$d_{1\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{2\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{B} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{3\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{B^2}{KS(t)} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{4\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{B}{S(t)} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

LookBack:

$$V(T) = \max_{u \in [0,T]} S(u) - S(T) \qquad \geq S(T) - S(T) = 0$$

$$= S(0) \exp(\sigma \hat{M}(t)) - S(T)$$

$\tilde{\mathbb{P}}(\hat{M}(t), \tilde{W}(t))$

$\Downarrow$

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right) \qquad \mathbb{P}(\hat{M}(t)) = \int P(\quad) dw.$$

$$V(t, y) \qquad y(t) = \max_{u \in [0,t]} S(u)$$

$$= \left(1 + \frac{\sigma^2}{2r}\right) S(t) N(d_+) + e^{-r(T-t)} y N(-d_-)$$

$$- \frac{\sigma^2}{2r} e^{-r(T-t)} \left(\frac{y}{S(t)}\right)^{\frac{2r}{\sigma^2}} S(t) N(+d_-) - S(t);$$

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{y} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

American Options

options than can be exercised at ANY time

up to time T.

(European: can only be exercised at T).

- ① $V_0(\text{Am}) \geq V_0(\text{Eu})$ same K, T

- ②. For American Options:

$V_0(\text{Am}) \geq G_0$ → instric value "payoff at $t=0$"

"value for the option at $t=0$"

Binary market, put $K=5 \Rightarrow G = 5 - S$

$\{ R = 1.25$.

$2.

$S_0 = 4$

$G_0 = 1$.

$S_1(H) = 8$

$G_1(H) = \underline{-3}$ ✗

$S_1(T) = 2$

$G_1(T) = 3$

$S_2(HH) = 16$ $\quad G_2(HH) = -11$ ✗

$S_2(HT) = 4$ $\quad G_2(HT) = 1$ ✓

$S_2(TH) = 4$ $\quad G_2(TH) = 1$

$S_2(TT) = 1$ $\quad G_2(TH) = 4$

G:

$2

$t=0$: 1

$t=1$: -3, $\underline{3}$

$t=2$: -11, 1 ←, 1, $\underline{4}$

$t = 1$, if "H" × (-3)

$t = 2$, if "HH" × (-11)

if "HT" ✓ (1)

$t = 1$ if "T" $\begin{cases} \text{exercise: } G_1(T) = 3. \\ \text{not exercise } G_2 = \begin{cases} 1 \\ 4 \end{cases} \end{cases}$

$3 \times 1.25 = 3.75$

$\max(3,2) = 3$

$V_1(T) = ?$

$S_1(T) = 2$

$G_1(T) = 3$

$\widetilde{\mathbb{E}}\left(\frac{1}{R} V_2 \mid \mathcal{F}_1 = T\right) = 2.$

$S_2(TH) = 4$ $\quad \widetilde{p}$

$G_2(TH) = 1$

$S_2(TT) = 1$ $\quad \widetilde{q}$

$G_2(TT) = 4.$

1-period American $\Leftrightarrow$ European option

$$\frac{1}{R} V_1 = \widetilde{\mathbb{E}}\left(\frac{1}{R^2} V_2 \mid \mathcal{F}_1 = T\right) \quad \text{martingale.}$$

$$V_1 = \widetilde{\mathbb{E}}\left(\frac{1}{R} V_2 \mid \mathcal{F}_1 = T\right)$$

$$= \widetilde{p} \cdot \frac{1}{R} V_2(TH) + \widetilde{q} \cdot \frac{1}{R} V_2(TT). \quad *$$

$\widetilde{p}, \widetilde{q}$

$$\frac{1}{R} S_1 = \widetilde{\mathbb{E}}\left(\frac{1}{R^2} S_2 \mid \mathcal{F}_1(T)\right).$$

$$S_1 = \widetilde{p} \cdot \frac{1}{R} S_2(TH) + \widetilde{q} \frac{1}{R} S_2(TT).$$

$$\widetilde{p} + \widetilde{q} = 1.$$

$$\widetilde{p} = \frac{R \cdot S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)}$$

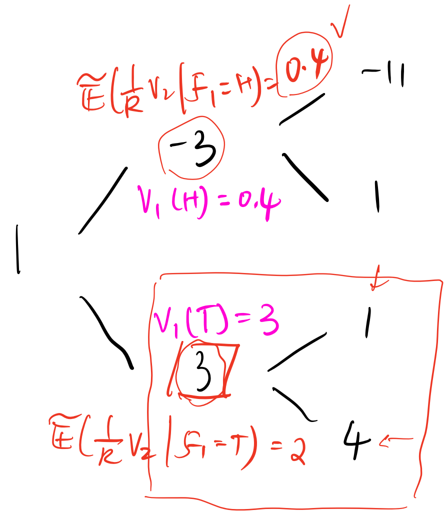
in 1 stage

$$\widetilde{p} = \frac{R \cdot S_0 - S_1(T)}{S_1(H) - S_1(T)}$$

$$= \frac{1.25 \times 2 - 1}{4 - 1} = \frac{1.5}{3} = 0.5, \quad \widetilde{q} = 0.5$$

$$* \quad \widetilde{\mathbb{E}}\left(\frac{1}{R} V_2 \mid \mathcal{F}_1 = T\right) = 0.5 \times \frac{1}{1.25} \times 4 + 0.5 \times \frac{1}{1.25} \times 1 = 2$$

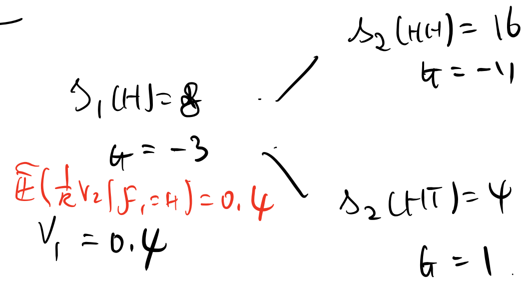
G.



if "T"

$$V_t = \max\left(G_t, \tilde{E}\left(\frac{1}{R}V_{t+1} \middle| F_t\right)\right)$$

if "H"



$V_2(HH) = \max(-11, 0) = 0$

$V_2(HT) = \max(1, 0) = 1$

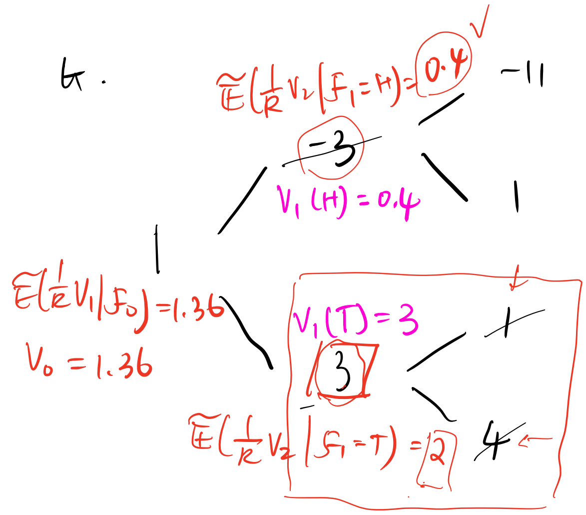
$$\tilde{E}\left(\frac{1}{R}V_2 \middle| F_1 = H\right)$$

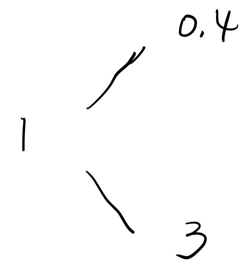
$$= \tilde{p} \cdot \frac{1}{R} V_2(HH) + \tilde{q} \cdot \frac{1}{R} V_2(HT)$$

$$\tilde{p} = \frac{R \cdot S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)} = \frac{R - \frac{S_2(TT)}{S_1(T)}}{\frac{S_2(TH)}{S_1(T)} - \frac{S_2(TT)}{S_1(T)}} = \frac{R - \frac{1}{2}}{2 - \frac{1}{2}} = 0.5$$

$$\tilde{E}\left(\frac{1}{R}V_2 \middle| F_1 = H\right) = 0.5 \times \frac{1}{1.25} \times 0 + 0.5 \times \frac{1}{1.25} \times 1 = 0.4$$

6.





$$\widetilde{\mathbb{E}}\left(\frac{1}{R}V_1 \,\middle|\, \mathcal{F}_0\right) = \widetilde{\mathbb{E}}\left(\frac{1}{R}V_1\right)$$

$$= \tilde{p}\cdot\frac{1}{R}V_1(H) + \tilde{q}\cdot\frac{1}{R}V_1(T)$$

$$= 0.5\times\frac{1}{1.25}\times 0.4 + 0.5\times\frac{1}{1.25}\times 3$$

$$= 1.36$$

European $K=5$, $T=2$.

$V_0(Eu) = 0.96$.

$$V_t = \max\left(G_t,\ \widetilde{\mathbb{E}}\left(\tfrac{1}{R} V_{t+1} \middle| \mathcal{F}_t\right)\right)$$

$$V_t \geq \widetilde{\mathbb{E}}\left(\tfrac{1}{R} V_{t+1} \middle| \mathcal{F}_t\right)$$

$$\frac{1}{R^t} V_t \geq \widetilde{\mathbb{E}}\left(\frac{1}{R^{t+1}} V_{t+1} \middle| \mathcal{F}_t\right) \Rightarrow \frac{1}{R^t} V_t \text{ is supermartingale}$$

(M_t) stochastic process.

sub-martingale $\quad M_t \leq \widetilde{\mathbb{E}}\left(M_{t+1} \middle| \mathcal{F}_t\right)$

super-martingale $\quad M_t \geq \widetilde{\mathbb{E}}\left(M_{t+1} \middle| \mathcal{F}_t\right)$.

f: convex $\quad$ [Jenson's inequality: $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$]

f convex, $\quad M$ martingale $\Rightarrow f(M)$ super-martingale

f concave $\quad M$ martingale $\Rightarrow f(M)$ sub-martingale

Random walk in 1D: (X_n)
(B.M.)

$$X_n - X_{n-1} = \begin{cases} 1 & \text{Prob} = 1/2 \\ -1 & \text{Prob} = 1/2 \end{cases} \qquad *$$

X_n is martingale $\Leftarrow \quad E(X_{n+1} \mid \mathcal{F}_n) = X_n$

$$\Uparrow$$

$$* \Rightarrow E(X_{n+1} - X_n \mid \mathcal{F}_n) = 0$$

$X_n^2 \qquad f(x) = x^2$ convex

$X_n^2 \to$ sub-martingale

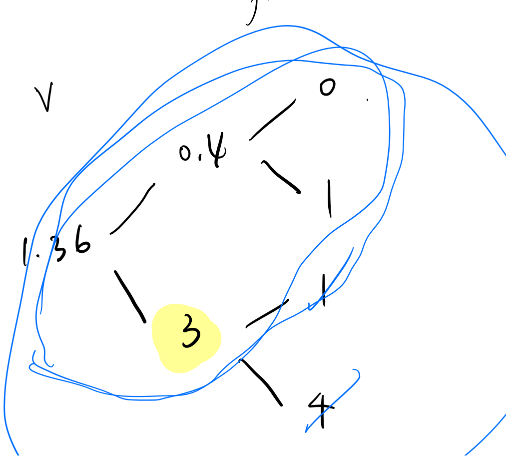
$$X_{n+1} = \begin{cases} X_n + 1 & \text{prob} = 1/2 \\ X_n - 1 & \text{prob} = 1/2 \end{cases}$$

$$E(X_{n+1} \mid \mathcal{F}_n) = \frac{1}{2}(X_n+1)^2 + \frac{1}{2}(X_n-1)^2 = X_n^2 + 1 > X_n^2$$

$\to \quad X_n^2$ sub-martingale

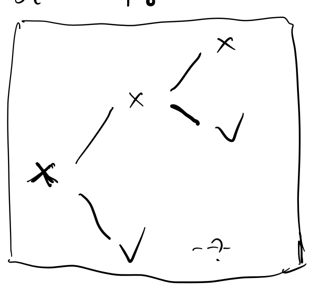
$\text{Var}(W(t)) = t$

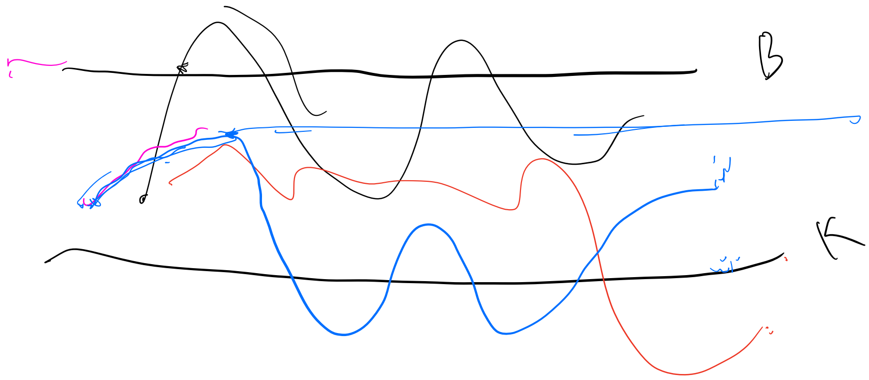
V



(u^+)

strategy (investor)





$$\widetilde{M}(t) = \max_{u \in [0,t]} \widetilde{W}(u)$$

$$\max\left(S(T) - K, 0\right)$$

$$\max_{u \in [0,t]} \left(S(u)\right) < B$$

$$\Updownarrow$$

$$S(0)\exp\left(\sigma \widetilde{M}(T)\right) <$$

$$dS = \alpha S\,dt + \sigma S\,dW$$

$$\Rightarrow \quad dS = \sigma S\,d\widetilde{W}$$

$$S(t) = S(0)\exp\left(\sigma \widetilde{W}(t)\right)$$