HW due Mar 11

Chapter 5 - Risk-Neutral Measure

$dS(t) = \alpha(t) S(t)\, dt + \sigma(t) S(t)\, dt$

Interest rate $R(t)$

discounted value $\dfrac{S(T)}{\exp\left(\int_0^T R(t)\,dt\right)}$

$D(t) = -\exp\left(\int_0^t R(s)\,ds\right)$

$\uparrow$

Discounted

discounted value: $S(t)D(t)$

$I(t) = \int_0^t R(s)\,ds \rightarrow dI = R(t)dt$

$f(x) = e^{-x}$

$\boxed{f_x(x) = -e^{-x} = -f(x)}$, $f_{xx}(x) = e^{-x}$

$f_t(x) = 0$

$dD = df(I)$

$= f_x(I) \cdot dI + \frac{1}{2} f_{xx}(I)\cancel{dI\,dI} + \cancel{f_t\,dt}$

$= -D(t)R(t)\,dt$

$dDS = DdS + SdD + dDdS.$ ex 5.1

$= DS\left((\alpha - R)dt + \sigma(t)\,dW\right) \leftarrow$

$= \sigma DS\left(\frac{\alpha - R}{\sigma}dt + dW\right)$

$\downarrow$

risk $\theta(t)$

$d\tilde{W} = \theta(t)dt + dW.$

$d\tilde{W} \cdot d\tilde{W} = (\theta\, dt + dW)^2$

$= dW^2 = dt$

Probability Measures (Chapter 1)

$W \to P$ $\qquad \tilde{P} = P \cdot Z$

$\tilde{W} \to \tilde{P}$ $\qquad \tilde{P}$ prob. measure $\iff \begin{cases} \mathbb{E}Z = 1 \\ Z > 0 \text{. a.s.} \end{cases}$

$$Z(\omega) = \exp\left(-\theta X - \tfrac{1}{2}\theta^2\right) \qquad X(\omega) \sim N(0,1)$$

$$\mathbb{E}Z = \int_{-\infty}^{\infty} \exp(-\theta x - \tfrac{1}{2}\theta^2) \cdot \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(-\tfrac{1}{2}x^2 - \theta x - \tfrac{1}{2}\theta^2\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(-\tfrac{1}{2}(x-\theta)^2\right) dx$$

$$= 1$$

$$\begin{cases} Z(t) = \exp\left(-\int_0^t \theta(u)\,du - \frac{1}{2}\int_0^t \theta^2(u)\,du\right) \\ \tilde{W} = W + \int_0^t \theta(u)\,du. \end{cases}$$

$\begin{cases} \tilde{P} = P \cdot Z \text{ prob. measure} \\ \tilde{W} \text{ martingale under } \tilde{P} \end{cases}$

Fundamental Theorems of asset pricing
If a market model has a risk-neutral probability measure, then
1the market does not admit arbitrage
2 all derivative security can be hedged.