Math of Finance

Recitation - Feb18
BSM equations & Multiple Brownian Motions
Zixiao Yang zy1032@nyu.edu

Geometric Brownian Motion

Definition

A stochastic process $S(t)$ follows GBM if

$$dS = \alpha S dt + \sigma S dW,$$

Where α (percentage drift) and σ (percentage volatility) are constants.

$$dS = \alpha S\,dt + \sigma S\,dW.$$

$$d(\ln S) = \frac{1}{S}\,dS - \frac{1}{2}\frac{dS\,dS}{S^2}$$

$$df(S,t)$$

$$f(x,t) = \ln x$$

$$\left\{\begin{aligned} f_t &= 0 \\ f_x &= \frac{1}{x} \\ f_{xx} &= -\frac{1}{x^2} \end{aligned}\right.$$

$$\left(\frac{dS}{S}\right)^2 = (\alpha\,dt + \sigma\,dW)^2$$

$$= \alpha^2 \cancel{\underbrace{dt^2}_{\to 0}} + 2\alpha\sigma \cancel{\underbrace{dt}_{\to 0}\,\underbrace{dW}_{\to 0}} + \sigma^2 dW^2.$$

$$\underbrace{dW}_{\to 0} \sim N(0, \underbrace{dt}_{\to 0})$$

$$= \sigma^2 dW^2$$

$$= \sigma^2 dt \quad \leftarrow \quad \mathrm{Var}(dW) = dt$$

$$\mathrm{Var}(dW) = E\,dW^2 - \underbrace{(E\,dW)^2}_{=0}$$

$$d(\ln S) = \frac{dS}{S} - \frac{1}{2}\left(\frac{dS}{S}\right)^2$$

$$= \alpha\,dt + \sigma\,dW - \frac{1}{2}\sigma^2 dt$$

$$= \underbrace{\left(\alpha - \frac{1}{2}\sigma^2\right)}_{\mu} dt + \sigma\,dW.$$

$$= \mu\,dt + \sigma\,dW \Rightarrow \ln S = \mu t + \sigma W + c_0$$

$$S(t) = S(0)\exp(\mu t + \sigma W)$$

In real life, how to determine μ, σ?

$$S(0),\ S(\Delta t),\ S(2\Delta t),\ \cdots,\ S(N\Delta t)$$

$$d\ln S = \ln S(t+\Delta t) - \ln S(t)$$
$$= \mu dt + \sigma \underline{dW}. \sim N(\mu, \sigma^2 dt)$$

If $x_1, x_2, \cdots, x_n$ normally distributed

μ, σ^2: $\quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\hat{\sigma}^2 = \frac{1}{\underbrace{n-1}_{\frac{1}{n}}} \sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

$$E\hat{\sigma}^2 = \sigma^2$$

$$\ln S(\Delta t) - \ln S(0),\ \ln S(2\Delta t) - \ln S(\Delta t),\ \ln S(3\Delta t) - \ln S(2\Delta t),\ \cdots$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \ln S((i+1)\Delta t) - \ln S(i\Delta t)$$
$$= \frac{1}{n} \ln S(N\Delta t) - \ln S(0)$$

$$S(0),\ S(\Delta t),\ S(2\Delta t).\ \cdots\ S(N\Delta t) \leftarrow$$

$N \to \infty \quad \Delta t \to 0$

$\left(\ln S(\Delta t)-\ln S(0)\right)^2, \quad \left(\ln S(2\Delta t)-\ln S(\Delta t)\right)^2 \ \cdots$

$\underline{\left(\ln S(t+\Delta t)-\ln S(t)\right)^2} = \left(\mu dt+\sigma dW\right)^2$

$= \mu^2 \cancel{dt^2} + 2\mu\sigma \cancel{dt\,dW} + \sigma^2 \underline{dW^2}$

$\to 0 \qquad \to 0 \ \to 0$

$= \sigma^2 dt + o(dt^2) \qquad \sim o(dt^2)$

$dW \sim N(0, dt)$

$\downarrow 0 \qquad \downarrow 0$

$\sum \left(\ln S(t+\Delta t)-\ln S(t)\right)^2 = \sigma^2 \cdot T$

$\left(\ln \frac{S(t+\Delta t)}{S(t)}\right)^2$

$\underline{\sigma^2 dW}$

BSM

Equation

pay off: $c(T, x) = \max(x-K, 0)$

Δ $x = S(t)$

Suppose a stock is modeled by the GBM:

$$dS = \alpha S dt + \sigma S dW$$

An investor purchase an ~~European~~ call with strike value K.

For a finance institute, what is the hedging strategy and what is the value $\underbrace{c(t, S(t))}$ for the call?

FI: purchase Δ shares of stock

X: ~~pro~~folio value for FI: $\begin{cases} \text{stock} \\ \text{cash/bond} \end{cases}$

$$dX = d(\text{Stock}) + d(\text{cash})$$
$$= \Delta dS + r(X - \Delta S) \cdot dt$$

$f(x,t) = e^{-rt} x$

$f_t = e^{-rt} x(-r)$

$f_x = e^{-rt}$

$f_{xx} = 0$

discounted value at $t=0$

$$d(e^{-rt} X) = e^{-rt} X(-r)\,dt + e^{-rt} dX$$
$$= e^{-rt} X(-r)\,dt + e^{-rt} \Delta \underbrace{dS}_{\alpha S dt + \sigma S dW}$$
$$+ e^{-rt} r(X - \Delta S)\,dt$$
$$= rX dt + \Delta(\alpha - r) S dt + \Delta \cdot \sigma S dW$$

avg. return risk stock volatility. "standard dev"

Δ: chosen

$$d(e^{-rt} X) - d(e^{-rt} C) = 0 \cdot dt + 0 \cdot dW$$

value c, investor

$$d(e^{-rt}c) = e^{-rt}c(-r)\,dt + e^{-rt}\,dC.$$

$$dC = C_t\,dt + C_x\,\underbrace{dS}_{\alpha S dt + \sigma S dW} + \frac{1}{2}C_{xx}\,\underbrace{dS\,dS}_{= \sigma^2 S^2 dt}$$

$$= \left(C_t + \alpha S C_x + \frac{1}{2}\sigma^2 S^2 C_{xx}\right)dt + \sigma S C_x\,dW$$

$$d(e^{-rt}X) - d(e^{-rt}c) = 0, \quad \forall t \in [0,T)$$

$$\_\_\_\_\,dt + \_\_\_\_\,dW = 0.$$

dW: $\quad \Delta(t) - C_x = 0$

$\rightarrow \Delta(t) = C_x$

dt: $\quad (\alpha - r)S C_x - \left(-rC + C_t + \alpha S C_x + \frac{1}{2}\sigma^2 S^2 C_{xx}\right) = 0.$

$\Delta \Rightarrow C_x$

$$rC = rS\,C_x + C_t + \frac{1}{2}\sigma^2 S^2 C_{xx}$$

— BSM eq.

$$\begin{cases} c(t,0) = 0. & \text{true for all calls} \\ c(T,x) = \cancel{(x-k)^+} = \max(x-k, 0) \end{cases}$$

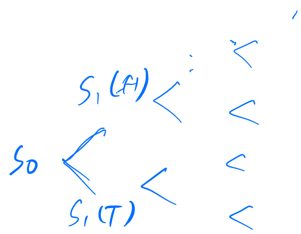
↓ European calls.

$$C = xN(d_+) - Ke^{-rt}N(d_-), \text{ where}$$

$$d_\pm = \frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)t\right)$$

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz .$$

cdf



$\rightarrow$ $d_+ = \frac{S_1(H)}{S_0}$ $\quad d_- = \frac{S_1(T)}{S_0}$

ODE with Boundary Values

Example

$$y''(t) + 4y(t) = 0,\ y(0) = 0,\ y(\pi/4) = 5$$

$c_1 = 0,\ c_2 = 5$

$\rightarrow\ y = 5\sin 2x$

$y'' + 4y = 0 \qquad m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$y(t) = a e^{i2x} + b e^{-i2x}$$

$$= a(\cos 2x + i\sin 2x) + b(\cos 2x - i\sin 2x)$$

$c_1 = a + b$

$c_2 = i(a - b)$

$$= c_1 \cos 2x + c_2 \sin 2x$$

BSM

Solution

Put-Call Parity

Suppose $f(t,x) = x - e^{-r(T-t)}K$ denotes the value of a forward contract. Then what is the value of a put $p(t,x)$ given $c(t,x)$?

ignore r: replace with discounted value.

$f(T,x) = x - K$

	$x > K$	$x < K$
$c(T,x) = \max(x-K, 0)$	$x-K$	0
$p(T,x) = \max(K-x, 0)$	0	$K-x$

$f = c - p$ (T) $\longrightarrow$ $p = c - f \quad \forall t.$

Multiple Brownian Motions

Definition

A n-dimensional Brownian motion is a process $W(t) = (W_1(t), \ldots, W_n(t))$ such that:

1. Each $W_i(t)$ is a 1-d Brownian motion
2. $W_i(t)$ and $W_j(t)$ are independent for $i \neq j$
3. $\mathscr{F}(s) \subset \mathscr{F}(t)$ if $0 \leq s < t$ $\qquad \mathcal{F}$
4. $W(t)$ is $\mathscr{F}(t)$ measurable for each t
5. $W(t + \Delta t) - W(t)$ independent of $\mathscr{F}(t)$ for $t, \Delta t > 0$

Multiple Brownian Motions

Calculations of BMs

What is $dW_i(t)dW_j(t)$?

$dW_i(t)\, dW_i(t) = dt$

$i \neq j \quad dW_i(t)\, dW_j(t) = 0$

$C_{ij} = \sum_{k=0}^{n-1} (dW_i(t_k))(dW_j(t_k))$

$C_{ij}^2 = \sum_{k=0}^{n-1} \underline{(dW_i(t_k))^2} \underline{(dW_j(t_k))^2}$

$+ 2 \sum_{l \le k} (\underbrace{dW_i(t_l)}_{\to 0} \cdot \underbrace{dW_j(t_l)}_{\to 0} \cdot \underbrace{dW_i(t_k)}_{\to 0} \cdot \underbrace{dW_j(t_k)}_{\to 0})$

$\sum dW_i(t_l) \sum dW_j(t_l) \cdot \sum dW_j(t_k) \sum \ldots$

$C_{ij}^2 = \sum dt \cdot dt = \frac{T}{dt} dt^2 = T dt \to 0$

Multiple Brownian Motions

Itô's Lemma in 2d

$$df(t,X) = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX dX$$

$$df(t,X,Y) = f_t dt + f_x dX + f_y dY + \frac{1}{2}(f_{xx} dX dX + 2 f_{xy} dX dY + f_{yy} dY dY)$$

Multiple Brownian Motions

Itô's Lemma in 2d - example

W_1, W_2 independent.

Suppose $dS(t)/S = \alpha dt + \sigma \left(\rho dW_1(t) + \sqrt{1-\rho^2} W_2(t) \right)$

$\rho \in [-1, 1]$.

$dW = \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)$

$\underline{dW^2} = \rho^2 \underbrace{dW_1^2}_{dt} + 2\rho\sqrt{1-\rho^2} \underbrace{dW_1 dW_2}_{0} + (1-\rho^2) \underbrace{dW_2^2}_{dt}$

$= \rho^2 dt + (1-\rho^2) dt = \underline{dt}$

1D Lévy thm

$$dw^2 = c\,dt.$$

$$\left(\frac{dw}{\sqrt{c}}\right)^2 = dt$$

$$f(t,x) = \ln x$$

$$f(t,x) \qquad f_t = 0,\; f_x = \frac{1}{x},\; f_{xx} = -\frac{1}{x^2}$$

$$\Rightarrow \quad df = f_t\,dt + f_x\,dx + \frac{1}{2} f_{xx}\,dx\,dx$$

$$df(t,S) = \cancel{\frac{1}{x}dx}\;\frac{1}{S}dS - \frac{1}{2}\frac{1}{S^2}dS\,dS.$$