SDE (Chapter 6 [Shreve, Vol. II])

[Feynman-Kac] Consider the SDE.

$$dX(t) = \beta(t, X(t))\,dt + \gamma(t, X(t))\,dW(t).$$

Define $g(t,x) = \mathbb{E}\left(h(X(T)) \,\middle|\, t,x\right)$. for some function h.

Then $g(t,x)$ satisfies

$$g_t + \beta \cdot g_x + \frac{1}{2}\gamma^2 g_{xx} = 0$$

with $g(T,x) = h(x)$

[Discounted Feynman-Kac]

Define $f(t,x) = \mathbb{E}\left(e^{-r(T-t)} h(X(T)) \,\middle|\, t,x\right)$ for some h,

Then $f(t,x)$ satisfies

$$f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx} - rf = 0 \qquad (*)$$

with $f(T,x) = h(x)$

[outline of proof]

Consider $d(e^{-rt} f)$ by Itô's formula,

and set $dt \to 0$.

[example]. Let $h(S(T))$ be the payoff of a derivative security, whose underlying asset is GBM:

$$dS = \alpha S\,dt + \sigma S\,dW$$
$$= rS\,dt + \sigma S\,d\tilde{W}$$ for $\tilde{W}$ under a risk-neutral probability measure $\tilde{P}$

Now, $V(t) = \tilde{E}\left(e^{-r(T-t)} h(S(T)) \,\middle|\, \mathcal{F}(t)\right)$ by martingale

Then V satisfies $(*)$:

$$V_t + rxV_x + \frac{1}{2}\sigma^2 x^2 f_{xx} - rf$$

(we plug in $\beta = rx$, $\gamma = \sigma x$ for the result)

This coincides with the Black-Scholes-Merton Equation.