Why does it make sense to calculate V based on the risk-neutral probability measure?

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dt.$$

$$S(t) = S(0)\cdot\exp\left(\int_0^t \sigma(u)\,dW(u) + \int_0^t\left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right).$$

$$\rightarrow\quad D(t) = e^{-\int_0^t R(u)\,du}.$$

$$dD(t) = -R(t)D(t)\,dt.$$

$$
\begin{aligned}
d(D(t)S(t)) &= D(t)\,dS(t) + S(t)\,dD(t) + \underbrace{dD(t)\cdot dS(t)}_{=0} \\
&= D(t)\left(\alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)\right) - S(t)R(t)D(t)\,dt \\
&= D(t)S(t)\left(\alpha(t)\,dt + \sigma(t)\,dW(t) - R(t)\,dt\right) \\
&= D(t)S(t)\left((\alpha(t) - R(t))\,dt + \sigma(t)\,dW(t)\right) \\
&= \sigma(t)D(t)S(t)\cdot\left(\frac{\alpha(t) - R(t)}{\sigma(t)}\,dt + dW(t)\right) \\
&= \sigma(t)D(t)S(t)\,d\tilde{W}
\end{aligned}
$$

$$dX(t) = \Delta(t)\underline{dS(t)} + R(t)\big(X(t) - \Delta(t)S(t)\big)dt$$

$$= \Delta(t)\big(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\big) + R(t)\big(X(t) - \Delta(t)S(t)\big)dt$$

$$= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)\left(\frac{\alpha(t)-R(t)}{\sigma(t)}dt + dW(t)\right)$$

$$= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)d\widetilde{W}(t)$$

$$d(D(t)X(t))$$

$$= D(t)dX(t) + X(t)dD(t) + \underbrace{dD(t)dX(t)}^{=0}.$$

$$= \underline{D(t)\big(R(t)X(t)dt} + \Delta(t)\sigma(t)S(t)d\widetilde{W}(t)\big) - \underline{X(t)R(t)D(t)dt}.$$

$$= \Delta(t)\sigma(t)S(t)d\widetilde{W}(t).$$

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\widetilde{W}(t)$$

$$d(D(t)X(t)) = \Delta(t)\sigma(t)S(t)d\widetilde{W}(t)$$

$$d\widetilde{W}(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}dt + dW(t)$$

if $\widetilde{P}$: DS is a martingale under $\widetilde{P}$; (*)
then so is DX $(=DV)$. ($X = V$ almost surely).

$$Z(t) = \exp\left(-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right), \quad \theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$$

$\widetilde{P} = Z \cdot P$ satisfies (*).

$$\tilde{P}(\alpha) = Z(\alpha)\cdot P(\alpha).$$

$$d\big(D(t)S(t)\big) = \sigma(t)D(t)S(t)\,d\tilde{W}(t)$$

$$d\big(D(t)X(t)\big) = \Delta(t)\,\sigma(t)S(t)\,d\tilde{W}(t)$$

$$\Rightarrow \quad D(t)S(t) = S(0)\exp\left(\int_0^t \sigma(t)\,d\tilde{W}(t)\right) \qquad (*)$$

$$D(t) = \exp\left(-\int_0^t R(u)\,du\right) = e^{-rt}$$

if $R(t)=r$, constant

(**) $e^{-rt}V(t)$ martingale

$$e^{-rt}V(t) = \tilde{\mathbb{E}}\left(e^{-rT}V(T)\,\middle|\,\mathcal{F}(t)\right)$$

on binary market

$$\begin{tikzcd}[column sep=small]
 & S_1(H),\ V_1(H) \quad \tilde{p} \\
S_0 \quad V_0 \arrow[ur] \arrow[dr] & \\
 & S_1(T),\ V_1(T) \quad \tilde{q}
\end{tikzcd}$$

$$V_0 = \tilde{\mathbb{E}}\left(\frac{1}{R}V_1 \,\middle|\, \mathcal{F}_0\right) = \tilde{p}\cdot\frac{1}{R}V_1(H) + \tilde{q}\cdot\frac{1}{R}V_1(T)$$

$$S_0 = \tilde{\mathbb{E}}\left(\frac{1}{R}S_1 \,\middle|\, \mathcal{F}_0\right) = \tilde{p}\cdot\frac{1}{R}S_1(H) + \tilde{q}\cdot\frac{1}{R}S_1(T)$$

$$\begin{cases} V_0 = \tilde{p}\,\frac{1}{R}\,\underline{V_1(H)} + \tilde{q}\,\frac{1}{R}\,\underline{V_1(T)} \\ \underline{S_0} = \tilde{p}\,\frac{1}{R}\,\underline{S_1(H)} + \tilde{q}\,\frac{1}{R}\,\underline{S_1(T)} \\ \tilde{p} + \tilde{q} = 1 \end{cases}$$

" ___ " known

" ~ " unknown

$$e^{-rt} V(t) = \tilde{E}\left(e^{-rT} V(T) \,\middle|\, \mathcal{F}(t)\right)$$

European: $V(T) = \max(S(T) - K, 0)$.

$$V(t) = S(t) N(d_+) - K e^{-r(T-t)} N(d_-)$$

American under Binary:

$R = 1.25$, $G = 5 - S$

$S_0 = 4$

$G_0 = 1$

$\widetilde{E}\left(\frac{1}{R} V_1 \,\middle|\, \mathcal{F}_0\right) = 1.36$

$V_0 = \max(1, 1.36) = 1.36$

$S_1(H) = 8$

$G_1(H) = -3$

$\widetilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = H\right) = 0.4$

$V_1(H) = \max(0.4, -3) = 0.4$

$S_1(T) = 2$

$G_1(T) = 3$

$\widetilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = T\right) = 2,$

$V_1(T) = \max(3, 2) = 3.$

$S_2(HH) = 16$

$G_2(HH) = -11$

$V_2(HH) = \max(G_2(HH), 0) = 0$

$S_2(HT) = 4$

$G_2(HT) = 1$

$V_2(HT) = \max(G_2(HT), 0) = 1$

$S_2(TH) = 4$

$G_2(TH) = 1$

$V_2(TH) = \max(G_2(TH), 0) = 1$

$S_2(TT) = 1$

$G_2(TT) = 4$

$V_2(TT) = \max(G_2(TT), 0) = 4$

If this is a European put with $V(2) = \max(5 - S_2, 0)$, then:

$S_0 = 4$

$\cancel{G_0 = 1}$

$\tilde{E}\left(\frac{1}{R} V_1 \,\middle|\, \mathcal{F}_0\right) = \cancel{1.36}$ 0.96

$V_0 = \cancel{\max(1, 1.36) = 1.36}$

0.96.

$S_1(H) = 8$

$\cancel{G_1(H) = -3}$

$\tilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = H\right) = 0.4$

$V_1(H) = \max(0.4, -3) = 0.4$

$S_1(T) = 2$

$\cancel{G_1(T) = 3}$

$\tilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = T\right) = 2,$

$V_1(T) = \cancel{\max(3, 2) = 3}$.

2.

$S_2(HH) = 16$

$G_2(HH) = -11$

$V_2(HH) = \max(G_2(HH), 0) = 0$

$S_2(HT) = 4$

$G_2(HT) = 1$

$V_2(HT) = \max(G_2(HT), 0) = 1$

$S_2(TH) = 4$

$G_2(TH) = 1$

$V_2(TH) = \max(G_2(TH), 0) = 1$

$S_2(TT) = 1$

$G_2(TT) = 4$

$V_2(TT) = \max(G_2(TT), 0) = 4$

European: $V_0^{Eu} = 0.96$

American: $V_0^{Am} = 1.36$

$V_0^{Am} - V_0^{Eu} = 0.4$

$$V_0^{Am} = V_*(x) = \max_{t \in [0,T] \cup \{\infty\}} \widetilde{E}\left(e^{-rt}(K-S(t))\right)$$

t: stop tie (when the option is exercised).

$$t = \infty, \quad e^{-rt}(K-S(t)) = 0$$

Investor Strategy:

Suppose: $L < K$.

exercise the put when stock price falls to L:

$$t_L = \min(t \geq 0,\ S(t) = L).$$

payoff: $V_L(S(0)) = (K-L)\,\widetilde{E}\left(e^{-rt_L}\right)$, $\quad S(0) \geq L$.

Problem: Evaluate $\widetilde{E}\left(e^{-rt_L}\right)$

$S(t) = L$ iff $\quad m = -\widetilde{W}(t) - \underbrace{\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)}_{\mu.}t = \frac{1}{\sigma}\log\frac{x}{L}.$

$$-\mu + \sqrt{\mu^2 + 2\lambda} = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\left(r + \frac{1}{2}\sigma^2\right) = \frac{2r}{\sigma}.$$

$$\widetilde{E}\left(e^{-rt_L}\right) = \exp\left(-\left(\frac{1}{\sigma}\log\frac{x}{L}\right)\frac{2r}{\sigma}\right) = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$$

$$\underset{\substack{\downarrow \\ x = S(0).}}{V_L(x)} = \begin{cases} K - x & x \leq L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x \geq L \end{cases}$$

Lemma: $t_m = \min\{t \geq 0,\ \widetilde{W}(t) + \mu t = m\}$.

$t_m = \infty$ if $\widetilde{W}(t) + \mu t$ does not achieve m.

$$\widetilde{\mathbb{E}}\left(e^{-\lambda t_m}\right) = e^{-m\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)}, \quad \forall \lambda > 0.$$

Proof (Lemma) $\sigma = -\mu + \sqrt{\mu^2 + 2\lambda}$

$$\lambda = \mu\sigma + \frac{1}{2}\sigma^2.$$

$e^{\sigma \widetilde{W}(t)}$ martingale under $\widetilde{P}$.

$$1 = \widetilde{\mathbb{E}}\left(e^{\sigma \widetilde{W}(t)}\right) = \widetilde{\mathbb{E}}\left(e^{\sigma m - \lambda t_m}\right)$$

$$\widetilde{\mathbb{E}}\left(e^{-\lambda t_m}\right) = e^{-m\sigma} = e^{-m\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)}$$

$$V_L(x) = \begin{cases} K - x & x \leq L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x \geq L \end{cases}$$

$x = S(0)$.

The value for American put (for T infinitely long)

$$V^{Am}(x) = \max_L V_L(x).$$

$$V_L'(x) = \begin{cases} -1 & x \le L \\ -(K-L)\frac{2r}{\sigma^2 x}\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x > L \end{cases}$$

$$V_L'(L+) = -\frac{2r}{\sigma^2}\cdot\frac{2r+\sigma^2}{2r} + \frac{2r}{\sigma^2} = -1.$$

$$V_L''(x) = \begin{cases} 0 & x < L \\ (K-L)\frac{2r(2r+\sigma^2)}{\sigma^4 x^2}\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x > L \end{cases}$$

Coments:

$V_L''(L+) > 0$ $\quad$ V'' not defined at L.

$$rV_L(x) - rxV_L'(x) - \frac{1}{2}\sigma^2x^2V_L''(x) = \begin{cases} rK & x < L \\ 0 & x > L \end{cases}$$

(BSS)

Finite Expiration American Put:

(Infinite) $t_L = \min_t \{ S(t) = L \}$:

$$t_L = \min_t \{ S(t) = L \} \wedge T.$$

Recall: $V(t) \geq (K - S(t))$.

$e^{-rt} V(t)$ supermartingale under $\widetilde{P}$.

$$V(t,x) = \widetilde{E}\left(e^{-r(t_* - t)} (K - S(t_*)) \mid S(t) = x\right).$$

$$e^{-r(t\wedge r)} v(t\wedge r, S(t\wedge r)) \geq \widetilde{E}\left(e^{-r(T\wedge r)} v(T\wedge r, S(T\wedge r)) \mid \mathcal{F}(t)\right).$$

$$\Downarrow e^{-rt} V^{(Am)}(t,x) \geq \widetilde{E}\left(e^{-rt_{stp}} (K - S(t_{stp})) \mid S(t) = x\right). \quad 1$$

$t_* = \min \{u \in (t, T), (u, S(u)) \in \mathcal{S}\}$.

$\mathcal{S}$ (stopping set) $:= \{(t,x);\ v(t,x) = \max(K-x, 0)\}$

$e^{-r(u\wedge t_*)} v(u, S(u\wedge t_*))$, $u \in [t, T]$. is martingale

$$\Downarrow e^{-rt} V^{(Am)}(t,x) \leq \widetilde{E}\left(e^{-rt_*} (K - S(t_*)) \mid S(t) = x\right) \quad 2$$

Note: t_* also satisfies 1

$$\Rightarrow e^{-rt} V^{(Am)}(t,x) = \widetilde{E}\left(e^{-rt_*} (K - S(t_*)) \mid S(t) = x\right)$$

$$V^{(Am)}(t,x) = \widetilde{E}\left(e^{-r(t_* - t)} (K - S(t_*)) \mid S(t) = x\right)$$