Exotic Options

1. [A Simple Binary Market] Suppose the current price of a stock is $S_0 = 4$. It is predicted that after some time, the price of such a stock will be either $S_1(H) = 8$ or $S_1(T) = 2$. The interest rate for bonds will be $r = 0.1$ over this period.
   Suppose an finance institute short sell a European call with strike value $K = 2$.

   (a) Discuss a hedging strategy in terms of Δ_0.

   (b) Show that, if the discounted stock price is a martingale under some probability measure $\tilde{\mathbb{P}}$, then the discounted value of the option is also a martingale under the same measure.
   Hint: for the binary market, if we set $\tilde{\mathbb{P}}(H) = \tilde{p}$, then a process (M) is said to be martingale if $M_0 = \tilde{\mathbb{E}}(M_1) = \tilde{p}M_1(H) + (1-\tilde{p})M_1(T)$.

   (c) Derive the fair value for the option $V(t)$ from the previous conclusion.

2. We continue our discussions in a GBM model: $dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW$. We assume the interest rate for bonds is $R(t)$, and the corresponding discounted process is calculated as $D(t) = \int_0^t R(t)dt$. Suppose a finance institute short sells a European call with payoff $V(T,x) = \max(x-K,0)$.

   (a) Discuss a hedging strategy in terms of the Δ_0.
   Hint: set X as the total portfolio of the institute. Write out the formula for dX.

   (b) Show that, if the discounted stock price $D(t)S(t)$ is a martingale under some probability measure $\tilde{\mathbb{P}}$, then the discounted value of the option $D(t)V(t)$ is also a martingale under the same measure.

   (c) Derive the fair value for the option $V(t)$ from the previous conclusion.

3. [Knockout Options] We repeat the discussions with an up-and-out call with strike value K and an up-and-out barrier $B > K$. The payoff is $V(T) = max(S(T)-K,0)$ if $S(t) < B, \forall t \in [0,T]$

4. [Lookback Options] We repeat the discussions with a lookback call with payoff

$$V(T,x) = \max_{0 \le u \le T} S(u) - S(T)$$

5. [Asian Options] We repeat the discussions with an Asian call with payoff

$$V(T,x) = \max\left(\frac{1}{T}\int_0^T S(t)dt - K, 0\right)$$