Exotic Options (Review)

geometric B.M. $dS(t) = \exp(\sigma \tilde{W}(t))$

$d\tilde{W} = dW + \Theta(t)\,dt$

$\tilde{W}$ is brownian under $\tilde{\mathbb{P}}$

under $\tilde{P}$: $\begin{cases} e^{-rt}S(t) \text{ is martingale} \\ e^{-rt}V(t) \text{ is martingale} \end{cases}$

$\Rightarrow \quad e^{-rt}V(t) = \tilde{\mathbb{E}}\left(e^{-rT}V(T) \,\middle|\, \mathcal{F}(t)\right)$

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)}V(T) \,\middle|\, \mathcal{F}(t)\right)$$

$$V(t) = \tilde{E}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$$

European: $V(T) = \max(S(T)-K, 0)$ $\quad \left\{ \begin{array}{l} S(T) > K \\ \text{"}n\text{"} < d_- \end{array} \right.$

$$V(t) = S(t) N(d_+) - Ke^{-r(T-t)} N(d_-)$$

Knockout (up & out) $\quad \tilde{M}(t) = \max_{u \in [0,t]} \tilde{W}(t)$

$$V(T) = \max(S(T)-K, 0) \quad \text{iff} \quad S(0) e^{\sigma \tilde{M}(T)} \leq B.$$

$$V(T) = \left(S(0) \exp\left(\sigma \tilde{W}(T)\right) - K\right) \cdot \mathbb{1}_{\{\tilde{W}(T) \geq k,\ \tilde{M}(T) \leq b\}}.$$

$$k = \frac{1}{\sigma} \log \frac{K}{S(0)}, \quad b = \frac{1}{\sigma} \log \frac{B}{S(0)}$$

$$\Rightarrow \tilde{\mathbb{P}}\left(\hat{M}(T) \leq m,\ \tilde{W}(T) \leq w\right) = \int_{-\infty}^{w} \int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \cdot \frac{2(2m-w)}{T\sqrt{2\pi T}}\, dm\, dw$$

$$\begin{aligned} \Rightarrow V(t) = {} & S(t)\left(N(d_{1+}) - N(d_{2+})\right) \\ & - e^{-r(T-t)} K \left(N(d_{1-}) - N(d_{2-})\right) \\ & - B\left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}} \left(N(d_{3+}) - N(d_{4+})\right) \\ & + e^{-r(T-t)} K \left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}+1} \left(N(d_{3-}) - N(d_{4-})\right) \end{aligned}$$

exercise 7.1

$$d_{1\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{2\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{B} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{3\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{B^2}{KS(t)} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_{4\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{B}{S(t)} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

LookBack:

$$V(T) = \max_{t\in[0,T]} S(t) - S(T) \quad \geq 0 \quad S(T)-S(T) = 0$$

$$= S(0)\exp(\sigma \tilde{M}(t)) - S(T)$$

$\tilde{\mathbb{P}}(\tilde{M}(t), \tilde{W}(t))$

$\Downarrow$

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right) \qquad \mathbb{P}(\tilde{M}(t)) = \int P(\quad) dw.$$

$V(t, y)$ $\qquad y(t) = \max_{u\in[0,t]} S(u)$

$$= \left(1+\frac{\sigma^2}{2r}\right) S(t) N(d_+) + e^{-r(T-t)} y N(-d_-)$$

$$- \frac{\sigma^2}{2r} e^{-r(T-t)} \left(\frac{y}{S(t)}\right)^{\frac{2r}{\sigma^2}} S(t) N(+d_-) - S(t);$$

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}} \left(\log\frac{S(t)}{y} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right)$$

American Options

options than can be exercised at ANY time

up to time T.

(European: can only be exercised at T).

- ① $V_0(Am) \geq V_0(Eu)$ same K, T

- ②. For American Option:

instric value

$V_0(Am) \geq G_0$ "payoff at t=0"

"value for the option at t=0"

Binary market, put $K=5 \Rightarrow G = 5 - S$

$\{ R = 1.25$.

$S_0 = 4$
$G_0 = 1$.

$S_1(H) = 8$
$G_1(H) = -3$

$S_1(T) = 2$
$G_1(T) = 3$

$S_2(HH) = 16$ $\quad G_2(HH) = -11$

$S_2(HT) = 4$ $\quad G_2(HT) = 1$

$S_2(TH) = 4$ $\quad G_2(TH) = 1$

$S_2(TT) = 1$ $\quad G_2(TH) = 4$

G:

\$2

1

-3 $\qquad$ -11, $\;$ 1 ←.

3 $\qquad$ 1, $\;$ 4

$t=0$ $\qquad$ $t=1$ $\qquad$ $t=2$.

$t=1$, if "H" × (-3)

$t=2$, if "HH" × (-11)

if "HT" ✓ (1)

$t=1$ if "T" $\begin{cases} \text{exercise: } G_1(T) = 3. \\ \text{not exercise } G_2 = \begin{cases} 1 \\ 4 \end{cases} \end{cases}$

$3 \times 1.25 = 3.75$

$V_1(T) = ?$ $\qquad\qquad$ $S_2(TH) = 4$ $\qquad$ $\tilde{p}$

$S_1(T) = 2$ $\qquad\qquad$ $G_2(TH) = 1$

$G_1(T) = 3$

$\uparrow$ $\qquad\qquad$ $S_2(TT) = 1$ $\qquad$ $\tilde{q}$

$\qquad\qquad$ $G_2(TT) = 4$.

1-period American $\Leftrightarrow$ European option

$$\frac{1}{R} V_1 = \tilde{E}\left(\frac{1}{R^2} V_2 \,\middle|\, \mathcal{F}_1 = T\right) \qquad \text{martingale.}$$

$$V_1 = \tilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = T\right)$$

$$= \tilde{p} \cdot \frac{1}{R} V_2(TH) + \tilde{q} \cdot \frac{1}{R} V_2(TT) . \; *$$

$\tilde{p}, \tilde{q}$

$$\frac{1}{R} S_1 = \tilde{E}\left(\frac{1}{R^2} S_2 \,\middle|\, \mathcal{F}_1(T)\right).$$

$$S_1 = \tilde{p} \cdot \frac{1}{R} S_2(TH) + \tilde{q} \frac{1}{R} S_2(TT).$$

$$\tilde{p} + \tilde{q} = 1.$$

$$\tilde{p} = \frac{R \cdot S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)}$$

in 1 stage

$$\tilde{p} = \frac{R \cdot S_0 - S_1(T)}{S_1(H) - S_1(T)}$$

$$= \frac{1.25 \times 2 - 1}{4 - 1} = \frac{1.5}{3} = 0.5, \qquad \tilde{q} = 0.5$$

$$* \; \tilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = T\right) = 0.5 \times \frac{1}{1.25} \times 4 + 0.5 \times \frac{1}{1.25} \times 1 = 2$$

G.

$\tilde{E}\left(\frac{1}{R}V_2 \mid \mathcal{F}_1 = H\right) = 0.4$ ✓ $\quad -11$

$-3 \quad V_1(H) = 0.4 \quad 1$

1

$V_1(T) = 3 \quad 1$

$\boxed{3}$

$\tilde{E}\left(\frac{1}{R}V_2 \mid \mathcal{F}_1 = T\right) = 2 \quad 4$

if "T" ✓

$$V_t = \max\left(G_t, \tilde{E}\left(\frac{1}{R}V_{t+1} \mid \mathcal{F}_t\right)\right)$$

if "H"

$S_2(HH) = 16$, $G = -11$ $\quad V_2(HH) = \max(-11, 0) = 0$

$S_1(H) = 8$, $G = 3$, $V_1 = 0.4$

$S_2(HT) = 4$, $G = 1$ ✓ $\quad V_2(HT) = \max(1, 0) = 1$

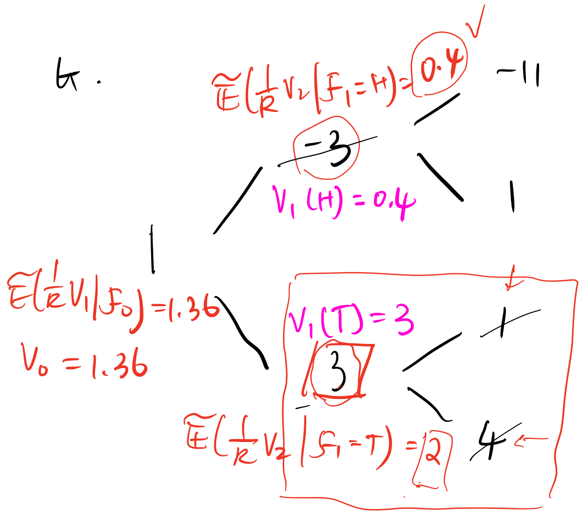
$$V_1(H) = \tilde{E}\left(\frac{1}{R}V_2 \mid \mathcal{F}_1 = H\right)$$

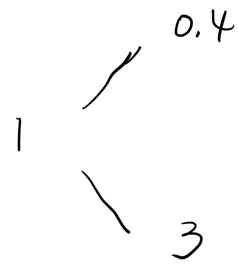
$$= \tilde{p} \cdot \frac{1}{R} V_2(HH) + \tilde{q} \cdot \frac{1}{R} V_2(HT)$$

$$\tilde{p} = \frac{R \cdot S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)} = \frac{R - \frac{S_2(TT)}{S_1(T)}}{\frac{S_2(TH)}{S_1(T)} - \frac{S_2(TT)}{S_1(T)}} = \frac{R - \frac{1}{2}}{2 - \frac{1}{2}} = 0.5$$

$$\tilde{E}\left(\frac{1}{R}V_2 \mid \mathcal{F}_1 = H\right) = 0.5 \times \frac{1}{1.25} \times 0 + 0.5 \times \frac{1}{1.25} \times 1 = 0.4$$

6.





$$\tilde{\mathbb{E}}\left(\frac{1}{R}V_1 \,\middle|\, \mathcal{F}_0\right) = \tilde{\mathbb{E}}\left(\frac{1}{R}V_1\right)$$

$$= \tilde{p}\cdot\frac{1}{R}V_1(H) + \tilde{q}\cdot\frac{1}{R}V_1(T)$$

$$= 0.5\times\frac{1}{1.25}\times 0.4 + 0.5\times\frac{1}{1.25}\times 3$$

$$= 1.36$$

European $K=5$, $T=2$

$V_0(Eu) = 0.96$.

$$V_t = \max\left(G_t ,\ \widetilde{\mathbb{E}}\left(\tfrac{1}{R} V_{t+1} \,\middle|\, \mathcal{F}_t\right)\right)$$

$$V_t \geq \widetilde{\mathbb{E}}\left(\tfrac{1}{R} V_{t+1} \,\middle|\, \mathcal{F}_t\right)$$

$$\frac{1}{R^t} V_t \geq \widetilde{\mathbb{E}}\left(\frac{1}{R^{t+1}} V_{t+1} \,\middle|\, \mathcal{F}_t\right) \Rightarrow \frac{1}{R^t} V_t \text{ is supermartingale}$$

(M_t) stochastic process.

sub-martingale $\quad M_t \leq \widetilde{\mathbb{E}}\left(M_{t+1} \,\middle|\, \mathcal{F}_t\right)$

super-martingale $\quad M_t \geq \widetilde{\mathbb{E}}\left(M_{t+1} \,\middle|\, \mathcal{F}_t\right)$.

f: convex $\quad$ [Jenson's inequality: $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$]

f convex, M martingale $\Rightarrow f(M)$ super-martingale

f concave M martigale $\Rightarrow f(M)$ sub-martigale

Random walk in 1D: (X_n)

(B.M.)

$$X_n - X_{n-1} = \begin{cases} 1 & \text{Prob} = 1/2 \\ -1 & \text{Prob} = 1/2 . \end{cases} \qquad *$$

X_n is martingale $\Leftarrow$ $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n$

$$\Uparrow$$

$$* \Rightarrow \mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0$$

X_n^2 $\quad f(x) = x^2$ convex

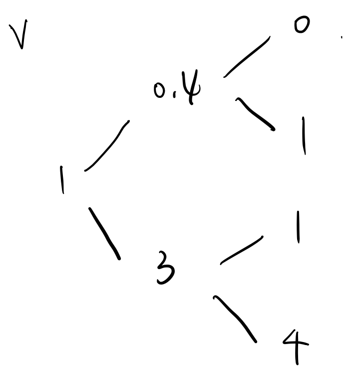
$X_n^2 \rightarrow$ sub-martingale

$$X_{n+1} = \begin{cases} X_n + 1 & \text{prob} = 1/2 \\ X_n - 1 & \text{prob} = 1/2 \end{cases}$$

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \frac{1}{2}(X_n+1)^2 + \frac{1}{2}(X_n-1)^2 = X_n^2 + 1 > X_n^2$$

$\rightarrow$ X_n^2 sub-martingale

$$\mathrm{Var}(W(t)) = t$$



strategy (investor)

(u^t)