Mathematics of Finance Review 1

Review plans - 1.5 sessions on the handout & 1.5 sessions on practical exercises.

1. [Martingales] [Definitions] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider an adapted stochastic process $M(t), 0 \leq t \leq T$. If $\forall 0 \leq s \leq t \leq T$:
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) = M(s)$, then we say this process is a martingale. It has no tendency to rise or fall;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \leq M(s)$, then we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \geq M(s)$, then we say this process is a supermartingale. It has no tendency to rise; it may have a tendency to fall;

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step symmetric random walk as follows.

   $$W_n(t) = \sum_{i=1}^{n} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Show that $W_n(t)$ is a martingale.

   (b) For American Options, we have $V_t = max(\tilde{\mathbb{E}}(V_{t+1}|\mathcal{F}(t)), G(t))\forall t$. Classify V_t as a type of martingale.

2. [Scaled Symmetric Random Walks]

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step scaled symmetric random walk as follows.

   $$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Deduce that $E(W_n(t) - W_n(s)) = 0$ and $Var(W_n(t) - W_n(s)) = t - s$.

   (b) Show that $W(t) := \lim_{n\to\infty} W_n(t) = \mathcal{N}(0, t)$.

3. [Binary and Log-Normal Markets] Consider an n-step binary market with no interest rate ($R = 1.0$)

   (a) Set $u = 3/2, d = 1/2$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S(t)$ at time t.

   (b) Set $u = 1 + \sigma/\sqrt{n}, d = 1 - \sigma/\sqrt{n}$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S_n(t)$ at time t. Show that

   $$\lim_{n\to\infty} S_n(t) = S(0) \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right),$$

   where $W(t)$ is defined in 2(b).

4. [Brownian Motions - Calculations] [Definition] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$. Then $W(t), t \geq 0$ is a Brownian motion if $\forall i \in \{0, 1, \ldots m\}$, the increments $W(t_{i+1}) - W(t_i)$ are independent and each of these increments is normally distributed with $\mathbb{E} = 0$ and $Var = t_{i+1} - t_i$.

   (a) Set $dW(t) = W(t + dt) - W(t)$. From the defination, find $\mathbb{E}(dW), Var(dW)$.
   Find $\mathbb{P}\{W(0.25) \leq 0.2\}$

   (b) Show that $\mathbb{E}(W(t)W(s)) = t \wedge s$. Deduce the covariance of $W(t)$ and $W(s)$.

   (c) Show that $W(t)$ is a martingale, and so is $Z(t) = exp(\sigma W(t) - 1/2 * \sigma^2 t)$

5. [Ito's integral] Consider the following Ito's integral:

$$I(t) = \int_0^t \Delta(u) dW(u) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1} - W(t_j))$$

   (a) Show that, $I(t)$ is a martingale. Remark: An Ito's integral with zero $dt-$term is a martingale.

   (b) Show that,

$$f(T, W(T)) = f(0, W(0) + \int_0^T f_t(t, W(t) dt + \int_0^T f_x(t, W(t)) dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t)) dt$$

   .

   Hint: We may apply the Taylor's formula:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2$$

   Note that the reminder contains a sum of terms $(W(t_{j+1} - W_j)^3$ which has limit 0.

   Further Hint: $dW(t)dW(t) = dt, \quad dtdW(t) = 0, \quad dtdt = 0.$

   Remark: We can rewrite the formula as the differential term:

$$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$$

   (c) Deduce that $d(AB) = AdB + BdA + dAdB$ for stochastic process $A(t), B(t)$.

6. [Probability Measures] Consider the geometric brownian motions $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$. We define a disounted process $D(t) = exp(-rt)$.
We define $X(t)$ as the total profolio under the hedging strategy $\Delta(t)$.

   (a) Find $d(D(t)S(t)$ and $d(D(t)X(t))$.

   (b) Show that, if $d(D(t)S(t)$ is a martingale under some probability measure $\tilde{\mathbb{P}}$, then so is $d(D(t)X(t))$.

7. [Change of Probability Measures] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$. Then, $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then $\tilde{\mathbb{E}}X = \mathbb{E}(XZ)$.

   (a) Show that $\mathbb{E}\, Z = 0$ for $Z(\omega) = \exp(-\theta X(\omega) - 1/2 * \theta^2)$

   (b) Show that $\mathbb{E}\, Z = 0$ for

$$Z(t) = \exp\left(-\int_0^t \Theta(u) dW(u) - \frac{1}{2}\int_0^t \theta^2(u) du\right)$$

8. [Binary Markets] Consider a 2-step binary market with $S_0 = 4, u = 2, d = 0.5, R = 1.1$

   (a) Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.

   (b) Find the fair prices of a European call with $K = 4, T = 2$.

   (c) Derive a hedging strategy $\Delta(t)$.

9. [Log-Normal Markets] Consider a geometric brownian motion with $dS(t) = 0.1 * S(t)dt + 0.3 * S(t)dW(t)$. Set the interest factor $R(t) = \exp(1.05t)$.

   (a) Set $S(0) = 1$, consider a European call with $K = 1, T = 2$. Find the fair price of such an option. You may proceed with either Black-Shore, or the risk-neutral probability measure.

known information

1. [Martingales] [Definitions] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider an adapted stochastic process $M(t), 0 \le t \le T$. If $\forall 0 \le s \le t \le T$:

– $\mathbb{E}(M(t)|\mathcal{F}(s)) = M(s)$, then we say this process is a martingale. It has no tendency to rise or fall;

– $\mathbb{E}(M(t)|\mathcal{F}(s)) \le M(s)$, then we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise;

– $\mathbb{E}(M(t)|\mathcal{F}(s)) \ge M(s)$, then we say this process is a supermartingale. It has no tendency to rise; it may have a tendency to fall;

(a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step symmetric random walk as follows.

$$W_{\cancel{n}}(t) = \sum_{i=1}^{\cancel{n}\ t} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

Show that $W_{\cancel{n}}(t)$ is a martingale.

(b) For American Options, we have $V_t = max(\tilde{\mathbb{E}}(V_{t+1}|\mathcal{F}(t)), G(t))\forall t$. Classify V_t as a type of martingale.

Need to prove

(a) $\mathbb{E}(W(t) | \mathcal{F}(t-1)) \overset{?}{=} W(t-1)$

$= \mathbb{E}(W(t-1) + X_t | \mathcal{F}(t-1))$

$= \mathbb{E}(W(t-1) | \mathcal{F}(t-1)) + \mathbb{E}(X_t | \mathcal{F}(t-1))$

$= \mathbb{E}(X_t)$

$= W(t-1) + 0$

$= W(t-1)$

induction, $\forall s \le t$

$\Rightarrow$ martingale

(b) $V_t = max(\tilde{\mathbb{E}}(V_{t+1} | \mathcal{F}_t), G(t))$

$V_t \ge \tilde{\mathbb{E}}(V_{t+1} | \mathcal{F}_t)$ – submartingale

2. [Scaled Symmetric Random Walks]

(a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2$, $\forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step scaled symmetric random walk as follows.

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\overset{M_{nt}}{nt}} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

$W_n(t)$ not Brownian Motion, because not normally distributed

Deduce that $\mathbb{E}(W_n(t) - W_n(s)) = 0$ and $Var(W_n(t) - W_n(s)) = t - s$.

(b) Show that $W(t) := \lim_{n\to\infty} W_n(t) = \mathcal{N}(0, t)$. $W(t)$ Brownian i.i.d.

(a).

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}}\big(\underbrace{X_{ns+1} + X_{ns+2} + \cdots + X_{nt}}_{\text{each of them}}\big).$$

each of them $\mathbb{E}=0$, $Var = 1$

$(nt-ns)$ numbers, indep.

$\mathbb{E}=0$, $Var=1$

$$\Rightarrow \quad \frac{1}{\sqrt{n}}\sum X_i \quad \Rightarrow \quad \mathbb{E}=0$$

$$Var = \frac{1}{n}\cdot(nt-ns) = t-s$$

(b).

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i$$

$$W(t) = \lim_{n\to\infty} W_n(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \underline{\sum_{i=1}^{nt} X_i}$$

By CLT:
$$\lim_{n\to\infty} \frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}} \sim \mathcal{N}(0,1)$$

$$W(t) := \lim_{n\to\infty} \frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}} \sim \mathcal{N}(0,t).$$

3. [Binary and Log-Normal Markets] Consider an n-step binary market with no interest rate ($R = 1.0$)

(a) Set $u = 3/2, d = 1/2$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$. (for European Calls).
Find the stock price $S(t)$ at time t.

(b) Set $u = 1 + \sigma/\sqrt{n}, d = 1 - \sigma/\sqrt{n}$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
Find the stock price $S_n(t)$ at time t. Show that

$$\lim_{n\to\infty} S_n(t) = S(0)\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right),$$

where $W(t)$ is defined in 2(b).

S_0; $\tilde{p}$: $S_1(H) = uS_0$; $\tilde{q}$: $S_1(T) = dS_0$; $u^2 S_0$, udS_0, udS_0, $d^2 S_0$

solve $\tilde{p}, \tilde{q}$

$$\tilde{p} + \tilde{q} = 1$$

$$S_0 = \tilde{E}\left(\frac{1}{R} S_1 \mid \mathcal{F}_0\right) = \tilde{p}\frac{1}{R}S_1(H) + \tilde{q}\frac{1}{R}S_1(T)$$

$$V_0 = \tilde{E}\left(\frac{1}{R} V_1 \mid \mathcal{F}_0\right) = \tilde{p}\frac{1}{R}V_1(H) + \tilde{q}\frac{1}{R}V_1(T)$$

$r = 0$, $R = 1 + r = 1$

(a) $\tilde{p} = \dfrac{1+r-d}{u-d} = \dfrac{1}{2}$. $\tilde{q} = \dfrac{1}{2}$.

$S(t) = S(0) \cdot u^H \cdot d^T$. H, T: #

$$\begin{cases} H + T = t \\ H - T = M_t = \sum_{i=1}^{t} X_i \end{cases}$$

(X_i defined in Prob 1).

$X_i = \begin{cases} 1 \\ -1 \end{cases}$ $\tilde{p} = 1/2$, $\tilde{q} = 1/2$.

$H = \frac{1}{2}(t + M_t)$

$T = \frac{1}{2}(t - M_t)$

$$S(t) = S(0)\, u^{\frac{1}{2}(t+M_t)}\, d^{\frac{1}{2}(t-M_t)}$$

(b) $\tilde{p} = \frac{1+r-d}{u-d} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$.

$$\sum_{i=1}^{nt} X_i = M_{nt} = H_{nt} - T_{nt}.$$

$$\left\{ nt = H_{nt} + T_{nt} \right.$$

$$S_n(t) = S(0)\, u^{\frac{1}{2}(nt + M_{nt})} d^{\frac{1}{2}(nt - M_{nt})}$$

$$\log S_n(t) = \log S(0) + \frac{1}{2}(nt + M_{nt}) \log\left(1 + \frac{\sigma}{\sqrt{n}}\right) + \frac{1}{2}(nt - M_{nt}) \log\left(1 - \frac{\sigma}{\sqrt{n}}\right)$$

$\frac{\sigma}{\sqrt{n}} \to 0$ for $n \to \infty$

$\log(1+x) = x - \frac{1}{2}x^2 + o(x^3)$, apply to $x = \frac{\sigma}{\sqrt{n}}$, $x = -\frac{\sigma}{\sqrt{n}}$

$$\begin{aligned}
\log S_n(t) &= \log S(0) + \frac{1}{2}(nt + M_{nt})\left(\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n}\right) \\
&\quad + \frac{1}{2}(nt - M_{nt})\left(-\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n}\right) \\
&= \log S(0) + \frac{1}{2}nt\left(-\frac{\sigma^2}{n}\right) + \frac{1}{2}M_{nt}\left(\frac{2\sigma}{\sqrt{n}}\right) \\
&= \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W_n(t) \qquad W_n(t) = \frac{1}{\sqrt{n}} M_{nt}. \\
&\to \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W(t)
\end{aligned}$$

$$S_n(t) \to S(0) \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right).$$

log-normal model.

4. [Brownian Motions - Calculations] [Definition] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$. Then $W(t), t \geq 0$ is a Brownian motion if $\forall i \in \{0, 1, \dots m\}$, the increments $W(t_{i+1}) - W(t_i)$ are independent and each of these increments is normally distributed with $\mathbb{E} = 0$ and $Var = t_{i+1} - t_i$.

(a) Set $dW(t) = W(t+dt) - W(t)$. From the defination, find $\mathbb{E}(dW), Var(dW)$.
Find $\mathbb{P}\{W(0.25) \leq 0.2\}$

$= \min(t, s)$

(b) Show that $\mathbb{E}(W(t)W(s)) = t \wedge s$. Deduce the covariance of $W(t)$ and $W(s)$.

(c) Show that $W(t)$ is a martingale, and so is $Z(t) = exp(\sigma W(t) - 1/2 * \sigma^2 t)$

(a). $dW = W(t+dt) - W(t) \sim N(0, dt)$

$$\mathbb{E}(dW) = 0, \quad Var(dW) = dt$$

$$W(0.25) = W(0.25) - \underbrace{W(0)}_{=0} \sim N(0, 0.25)$$

$$W(0.25) \sim N(0, 0.25)$$

$$\mathbb{P}(W(0.25) \leq 0.2) = \mathbb{P}(W(1) \leq 0.4) = N(0.4)$$

$$N(0.4) = \int_{-\infty}^{0.4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

(b). Suppose $0 \leq s < t$

increments.

$$\mathbb{E}(W(s)W(t)) = \mathbb{E}(W(s)(W(t) - W(s)) + W^2(s))$$

$$= \mathbb{E}(W(s)) \cdot \mathbb{E}(W(t) - W(s)) + \mathbb{E}(W^2(s)),$$

$$= \mathbb{E}(W^2(s)) - (\mathbb{E}\,W(s))^2$$

≈ 0

$$= Var(W(s))$$

$$= s.$$

A, B indep

$$\mathbb{E}(AB) = \mathbb{E}(A) \cdot \mathbb{E}(B).$$

$$\operatorname{cov}(W(s), W(t)) = \mathbb{E}(W(s)W(t)) - \underbrace{\mathbb{E}W(s)}_{=0} \cdot \underbrace{\mathbb{E}W(t)}_{=0}$$

$$= s$$

If $\ell = [W(t_1), W(t_2), \dots W(t_j)]$ $\quad t_1 < t_2 < \dots < t_j$

then covariance matrix of ℓ is found as

$$\begin{array}{c|c}
 & W(t_1) \quad W(t_2) \quad \dots \quad W(t_j) \\
\begin{array}{c} W(t_1) \\ \vdots \\ W(t_j) \end{array} & \begin{pmatrix} t_1 & t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & t_2 & \dots & t_2 \\ \vdots & t_2 & t_3 & \dots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \\ t_1 & t_2 & t_3 & & t_j \end{pmatrix}
\end{array} \qquad (*)$$

If W adapted process with covariance matrix of form $(*)$

then W brownian motion.

(c). $\mathbb{E}\left(W(t) \mid \mathcal{F}(s)\right) = \mathbb{E}\left(W(t) - W(s) + W(s) \mid \mathcal{F}(s)\right)$

$$= \mathbb{E}\Big(\underbrace{W(t) - W(s) \mid \mathcal{F}(s)}_{\text{indep. of the filter.}}\Big) + \mathbb{E}\Big(\underbrace{W(s) \mid \mathcal{F}(s)}_{\text{in the filter}}\Big)$$

$$= \mathbb{E}\left(W(t) - W(s)\right) + W(s)$$

$$= W(s)$$

$\rightarrow$ martingale

(d) $Z(t) = \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$

$$\mathbb{E}\left(Z(t) \mid \mathcal{F}(s)\right)$$

$$= \mathbb{E}\left(\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right) \mid \mathcal{F}(s)\right)$$

constant

$$= \mathbb{E}\left(\exp\left(\sigma(W(t)-W(s))\right) \cdot \exp\left(\underline{W(s) - \frac{1}{2}\sigma^2 t}\right) \mid \mathcal{F}(s)\right)$$

$$= \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 t\right) \cdot \underline{\mathbb{E}\left(\exp\left(\sigma(W(t)-W(s))\right) \mid \mathcal{F}(s)\right)}$$

$$\mathbb{E}\left(\exp\left(\sigma(\underbrace{W(t)-W(s)}_{x \sim N(0,t-s)})\right)\right) = \int_{-\infty}^{\infty} \exp(\sigma x) \cdot \underbrace{\frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)}\right)}_{\text{pdf of } N(0,t-s)} dx$$

$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot \text{pdf}(x)\, dx$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)} + \sigma x - \frac{\sigma^2(t-s)^2}{2(t-s)}\right) \exp\left(\frac{\sigma^2}{2}(t-s)\right) dx.$$

$$= \exp\left(\frac{\sigma^2(t-s)}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x-\sigma(t-s))^2}{2(t-s)}\right) dx$$

$$= \exp\left(\frac{\sigma^2(t-s)}{2}\right)$$

$\int_{-\infty}^{\infty} \text{pdf}(x)\, dx = 1$

pdf: $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma}\right)$

$$\mathbb{E}\left(Z(t) \mid \mathcal{F}(s)\right) = \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 t\right) \cdot \exp\left(\frac{\sigma^2(t-s)}{2}\right)$$

$$= \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 s\right)$$

$$= Z(s)$$

$\rightarrow$ Z is a martingale

5. [Ito's integral] Consider the following Ito's integral:

$$I(t) = \int_0^t \Delta(u)dW(u) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))$$

(a) Show that, $I(t)$ is a martingale. Remark: An Ito's integral with zero $dt-$term is a martingale.

(b) Show that, (Itô's formula). $I(t) = \int \_ \, dt + \int \_ \, dW$

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t)dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

.

Hint: We may apply the Taylor's formula:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2 + O((x_{j+1}-x_j)^3)$$

Note that the reminder contains a sum of terms $(W(t_{j+1} - W_j)^3$ which has limit 0. $W(t_j)$.

Further Hint: $dW(t)dW(t) = dt, \quad dtdW(t) = 0, \quad dtdt = 0.$

$(dW)^2 = dt$

$(dW)^3 = dW \cdot dt = 0$

Remark: We can rewrite the formula as the differential term:

$$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2} f_{xx} dX(t)dX(t)$$

(c) Deduce that $d(AB) = AdB + BdA + dAdB$ for stochastic process $A(t), B(t)$.

(a). $\mathbb{E}(I(t) | \mathcal{F}(s))$

$$= \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \mathbb{E}\left(\sum_{j=l}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) \middle| \mathcal{F}(s)\right)$$

in the filter — known process (not stochastic) — indep. of filter.

$(t_l < t_k)$.

$$= I(s) + \sum_{j=l}^{k-1} \Delta(t_j) \cdot \mathbb{E}(W(t_{j+1}) - W(t_j) | \mathcal{F}(s))$$

$= I(s)$.

$dS = \alpha S dt + \sigma S dW$.

$dX = \_ dt + \_ dW$.

(b). $df = f_t dt + f_x dX + \frac{1}{2}\left(f_{xx} dX dX + 2f_{xt} \cancel{dXdt} + f_{tt} \cancel{dtdt}\right)$

$dtdW = 0, \quad dtdt = 0$

$df(t, X(t))$ — stochastic

Result: $df = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX dX$.

(c) plug in for $df(t, (A,B)) = f_t\, dt + f_A\, dA + f_B\, dB$

$$+ \frac{1}{2}\left(f_{AA}\, dA\, dA + 2 f_{AB}\, dA\, dB + f_{BB}\, dB\, dB\right).$$

$dA\,dt = 0$, $dB\,dt = 0$, $dt\,dt = 0$

$f(t, A, B) = AB$. $f_t = 0$, $f_A = B$, $f_B = A$

$f_{AA} = f_{BB} = 0$, $f_{AB} = 1$

$$\Rightarrow \quad d(AB) = A\,dB + B\,dA + dA\,dB.$$

6. [Probability Measures] Consider the geometric brownian motions $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$. We define a disounted process $D(t) = exp(-rt)$.
We define $X(t)$ as the total profolio under the hedging strategy $\Delta(t)$.

(a) Find $d(D(t)S(t)$ and $d(D(t)X(t))$.

(b) Show that, if $d(D(t)S(t)$ is a martingale under some probability measure $\tilde{\mathbb{P}}$, then so is $d(D(t)X(t))$.

Itô's formula: $df(t,x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$

Itô's product rule: $d(AB) = A dB + B dA + dA dB$.

$dS(t) = \alpha S(t) dt + \sigma S(t) dW$

$$d(e^{-rt} S(t)) = S(t)\, d(e^{-rt}) + e^{-rt} dS(t) + \underbrace{dS(t)}_{\_dt + \_dW}\, \underbrace{de^{-rt}}_{-re^{-rt}dt}$$

$$= -rS(t) e^{-rt} dt + e^{-rt}(\alpha S(t) dt + \sigma S(t) dW)$$

$$= e^{-rt} S(t) \big((\alpha - r) dt + \sigma dW \big)$$

$$= e^{-rt} S(t)\, \sigma \cdot d\tilde{W}$$

$d\tilde{W} = dW + \frac{\alpha - r}{\sigma} dt$.

$$dX = \Delta(t)\, dS(t) + r \cdot (X(t) - \Delta(t) S(t)) dt$$

$$= \Delta(t)(\alpha(t) S(t) dt + \sigma \cdot S(t) dW)$$
$$+ r(X(t) - \Delta(t) S(t)) dt$$

$$= rX(t) dt + \Delta(t) S(t) \big((\alpha - r) dt + \sigma dW \big)$$

$$= rX(t) dt + \Delta(t) \cdot S(t)\, \sigma \cdot d\tilde{W}$$

$(d\tilde{W} = dW + \frac{\alpha - r}{\sigma} dt)$

$$d(e^{-rt}X) = X \cdot d(e^{-rt}) + e^{-rt}dX + \underline{dX\,d(e^{-rt})} = 0$$

$$= -rXe^{-rt}dt + e^{-rt}(rX(t)dt + \Delta(t)S(t)\sigma d\tilde{W})$$

$$= e^{-rt}\Delta(t)S(t)\sigma \cdot d\tilde{W}.$$

$$d(e^{-rt}S(t)) = e^{-rt}S(t)\sigma d\tilde{W}$$

$$d(e^{-rt}X(t)) = \Delta(t)e^{-rt}S(t)\sigma d\tilde{W}$$

Recall: Martingale $\Longleftrightarrow$ no dt-term.

if $d(e^{-rt}S(t))$ martingale under $\tilde{P}$,

then so is $d(e^{-rt}X(t))$

Existence of $\tilde{P}$ (optional)

$$\tilde{P} = ZP,$$

$$Z(t) = \exp\left(-\int_0^t \theta(u)dW - \frac{1}{2}\int_0^t \theta^2(u)du\right), \quad \theta = \frac{\alpha - r}{\sigma}$$

7. [Change of Probability Measures] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$. Then, $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then $\tilde{\mathbb{E}}X = \mathbb{E}(XZ)$.

(a) Show that $\mathbb{E}\,Z = 0$ for $Z(\omega) = \exp(-\theta X(\omega) - 1/2 * \theta^2)$

(b) Show that $\mathbb{E}\,Z = 0$ for

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right)$$

(easy & not important)

8. [Binary Markets] Consider a 2-step binary market with $S_0 = 4, u = 2, d = 0.5, R = 1.1$

(a) Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.

(b) Find the fair prices of a European call with $K = 4, T = 2$.

(c) Derive a hedging strategy $\Delta(t)$.

Applications in the binary markets:

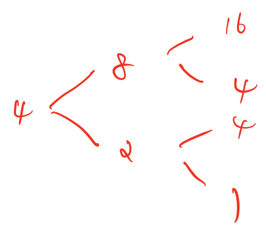
$e^{-rt} S(t)$ martingale $\Rightarrow$ $S(t-1) = \tilde{E}\left(\frac{1}{R} S(t) \,\middle|\, \mathcal{F}_t\right)$

$e^{-rt} V(t)$ martingale $\Rightarrow$ $V(t-1) = \tilde{E}\left(\frac{1}{R} V(t) \,\middle|\, \mathcal{F}_t\right)$

$$S(t-1) = \tilde{E}\left(\frac{1}{R} S(t) \,\middle|\, \mathcal{F}_t\right)$$

Binary tree for $S(t)$

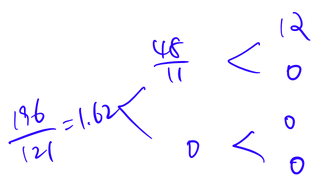
$$S(t-1) = \tilde{p} \cdot \frac{1}{R} S(t,H) + \tilde{q} \frac{1}{R} S(t,T)$$



$$1 = \tilde{p} \cdot \frac{1}{R} \frac{S(t,H)}{S(t-1)} + \tilde{q} \cdot \frac{1}{R} \frac{S(t,T)}{S(t-1)}$$

$$1 = \tilde{p} \cdot \frac{1}{R} \cdot u + \tilde{q} \cdot \frac{1}{R} \cdot d$$

Binary tree for $V(t)$



$$\Rightarrow \quad \tilde{p} = \frac{R-d}{u-d} = 0.4 \quad (\text{we know } \tilde{p} + \tilde{q} = 1)$$

$$\tilde{q} = 0.6$$

$$V_1(H) = \tilde{E}\left(\frac{1}{R} V_2(t) \,\middle|\, \mathcal{F}_1 = H\right)$$

$$= \tilde{p} \cdot \frac{1}{R} \cdot V_2(HH) + \tilde{q} \cdot \frac{1}{R} \cdot V_2(HT)$$

$$= 0.4 \times \frac{1}{1.1} \times 12 + 0.6 \times \frac{1}{1.1} \times 0 = \frac{48}{11}$$

$$\Rightarrow \quad V_0 = \frac{196}{121} = 1.62$$

9. [Log-Normal Markets] Consider a geometric brownian motion with $dS(t) = 0.1 * S(t)dt + 0.3 * S(t)dW(t)$. Set the interest factor $R(t) = \exp(\cancel{1.05t})$. $\exp(0.05t)$

(a) Set $S(0) = 1$, consider a European call with $K = 1, T = 2$. Find the fair price of such an option. You may proceed with either Black-Shore, or the risk-neutral probability measure.

$$V(0) = \tilde{E}\left(e^{-r\cdot 2} \cdot \max(S(2) - 1, 0)\right).$$

$$S(2) = S(0)\exp\left(-\sigma \tilde{W}(2) + (r - \tfrac{1}{2}\sigma^2)\cdot 2\right)$$

$W(2) \sim N(0, 2)$.

$$S(2) \geq 1 \iff S(0)\exp\left(-\sigma\sqrt{2}\cdot n + (r - \tfrac{1}{2}\sigma^2)\cdot 2\right) > 1.$$

$n \sim N(0,1)$

$$\iff n < \frac{(r - \frac{1}{2}\sigma^2)\cdot 2 - \ln\frac{1}{S(0)}}{\sigma\sqrt{2}} = \frac{\sqrt{2}(r - \frac{1}{2}\sigma^2)}{\sigma}.$$

$$\text{set } d_- = \frac{\sqrt{2}(r - \frac{1}{2}\sigma^2)}{\sigma}.$$

$$V(0) = \tilde{E}\left(e^{-r\cdot 2} \cdot \max(S(2) - 1, 0)\right)$$

$$= \int_{-\infty}^{d_-} e^{-r\cdot 2}\,(S(2) - 1)\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}n^2}\,dn.$$

$$= \int_{-\infty}^{d_-} e^{-2r}\left(\exp\left[-\sigma\sqrt{2}\cdot n + 2(r - \tfrac{1}{2}\sigma^2)\right] - 1\right)\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}n^2}\,dn$$

$$= \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} \exp\left(-\sigma\sqrt{2}\cdot n - \sigma^2 - \tfrac{1}{2}n^2\right)$$

$$- e^{-2r}\int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}n^2\right)dn.$$

$$= \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\sigma\sqrt{2}+n)^2\right) dn$$

$$- e^{-2r} \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}n^2\right) dn .$$

set $d_+ = d_- + \sigma\sqrt{2} = \frac{\sqrt{2}(r - \frac{1}{2}\sigma^2)}{\sigma} + \sigma\sqrt{2} = \frac{\sqrt{2}(r + \frac{1}{2}\sigma^2)}{\sigma}$

Then $V(0) = \int_{-\infty}^{d_+} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}m^2) dm - e^{-2r} \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}n^2) dn$

$$= N(d_+) - e^{-2r} \cdot N(d_-)$$

plug in $r = 0.05$ $\sigma = 0.3$ to get:

$$d_+ = 0.68, \quad d_- = 0.26$$

$$N(d_+) = 0.75 \quad N(d_-) = 0.60$$

$$\rightarrow \quad V(0) = 0.75 - e^{-0.2} \times 0.6 = 0.26 .$$

In general, $V(t) = S(t) N(d_+) - e^{-r(T-t)} K N(d_-)$.

(proved in notes/exotic.pdf, or P219 – P220, [Shreve])

Exercises (Not that important)

[Itô's formula]. Compute $d(W^4)$, $d(W^6)$.

Hint: $dt \cdot dt = 0$, $dt \cdot dW = 0$, $dW \cdot dW = dt$

[put-call parity] Try to show that, $f = c - p$,

where f: payoff for forward contract;

c: payoff for European call;

p: payoff for European put.

& f, c, p same K, T.

[Feynman-Kac]. $dX = \beta\, du + \gamma\, dW$; β, γ functions of $(t, X(t))$.

Define $f(t,x) = \mathbb{E}\left(e^{-r(T-t)} h(X(T))\right)\Big|_{x,t}$

Then $f(t,x)$ satisfies

$$f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx} = rf, \quad \text{with } f(T,x) = h(x) \ \forall x$$

Portfolio Theory:

Single Period, 2 risky assets.

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \quad \mathbb{E}R = \begin{pmatrix} 0.20 \\ 0.05 \end{pmatrix} \quad \mathrm{Cov}R = \begin{pmatrix} 100 & -20 \\ -20 & 1 \end{pmatrix}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{weights}, \quad w_1 + w_2 = 1.$$

$\Rightarrow$ Return $\quad R_w = w_1 R_1 + w_2 R_2 = w^T R$

$$\mathbb{E}(R_w) = \mathbb{E}(w_1 R_1 + w_2 R_2)$$
$$= w_1 \cdot \mathbb{E}R_1 + w_2 \cdot \mathbb{E}R_2$$
$$= w^T \cdot (\mathbb{E}R)$$

$$\mathrm{Var}(R_w) = \mathrm{Var}(w_1 R_1 + w_2 R_2)$$
$$= w_1^2 \mathrm{Var}(R_1) + 2 w_1 w_2 \cdot \mathrm{Cov}(R_1, R_2)$$
$$+ w_2^2 \mathrm{Var}(R_2)$$
$$= (w_1 \quad w_2) \begin{pmatrix} \mathrm{Var}(R_1) & \mathrm{Cov}(R_1, R_2) \\ \mathrm{Cov}(R_1, R_2) & \mathrm{Var}(R_2) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
$$= w^T \mathrm{Cov}(R) \cdot w.$$

Suppose we want to minimize risk:

$$\min \operatorname{Var}(R_w) = 100 w_1^2 - 40 w_1 w_2 + w_2^2.$$

subject to $\begin{cases} w_1 + w_2 = 1 \\ 0.2 w_1 + 0.05 w_2 = \alpha_0. \end{cases}$ $\Rightarrow$

Lagrange multiplier:

$$\overset{f}{w^T \Sigma w} \rightarrow \overset{f'}{2 \Sigma w}.$$

$$f = (100 w_1^2 - 40 w_1 w_2 + w_2^2)$$
$$- \lambda_1 (w_1 + w_2) - \lambda_2 (0.2 w_1 + 0.05 w_2 - \alpha_0)$$

$$\begin{cases} \dfrac{\partial f}{\partial w_1} = 0 \quad 200 w_1 - 40 w_2 - \lambda_1 - 0.2 \lambda_2 = 0 \\ \dfrac{\partial f}{\partial w_2} = 0 \quad 40 w_1 + 2 w_2 - \lambda_1 - 0.05 \lambda_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = -0.021 \lambda_1 - 0.0012 \lambda_2 .. \\ w_2 = 0.08 \lambda_1 + 0.001 \lambda_2 \end{cases}$$