

## Midterm Review

### 1. Brownian Motions

Property 1:

$$dW \sim \mathcal{N}(0, dt)$$

(a) Multiplication rules (3.10.1 [Shreve])

$$dW(t)dW(t) = dt, dW(t)dt = 0, dt dt = 0$$

(b) Derive the formula for  $d(W^2)$ ,  $d(W^4)$ .

Property 2: all the increment  $dW$ s are independent from each other.

(a) Suppose  $X(T) = \int_0^T f(t)dt + \int_0^T g(t)dW_t$ . Find  $Var(X)$

Property 3: Brownian Motions are a Martingale.

Suppose a random walk on 1D-axis with  $X(0) = 0$  and

$$dX = X(t+1) - X(t) = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$$

## 2. Ito's Formula

(a) 1D case

$$df(t, x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$$

Compute the stochastic differential  $dZ$  when

i.  $Z(t) = \exp(\alpha t)$

ii.  $Z(t) = \exp(\alpha X(t))$  with

$$dX(t) = \mu dt + \sigma dB(t)$$

iii.  $Z(t) = 1/X(t)$  with

$$dX(t) = aX(t)dt + \sigma X(t)dW(t)$$

(b) 2D case

$$df(t, x, y) = f_t dt + (f_x dx + f_y dy) + \frac{1}{2}(f_{xx} dx dx + 2f_{xy} dx dy + f_{yy} dy dy)$$

Derive the Ito's product rule  $d(XY) = XdY + YdX + dXdY$

### 3. Geometric Brownian Motions

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \leq t \leq T$$

Set

$$D(t) = \exp\left(-\int_0^t R(s)ds\right)$$

- (a) Derive a formula for  $S(t)$
- (b) Derive a formula for  $d(D(t)S(t))$  by Ito's product rule.
- (c) Derive a formula for  $d(D(t)S(t))$  by Ito's formula (Exercise 5.1 [Shreve]).  
Hint: Consider  $f(x) = S(0)e^x$  and set

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds$$

- (d) Show that  $S$  is log-normally distributed. i.e., show that  $\log(S)$  is normally distributed.

#### 4. Black-Scholes-Merton Equation

Let  $c(t, x)$  denote the value of an option at time  $t$  with current price  $S(t) = x$ . A portfolio  $X(t)$  with hedging strategy  $\Delta(t)$  should satisfy

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, x))$$

Use Ito's formula to compute both sides to get

$$\begin{cases} \Delta(t) = c_x \\ rc = c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} \end{cases}$$

For European call options, we have

$$c(t, 0) = 0$$

$$c(T, x) = \max(S(T) - K, 0)$$

We solve the equation with the boundary conditions above to get

$$c(t, x) = xN(d_+) - Ke^{-rt}N(d_-),$$

where

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}} \left( \log \frac{x}{K} + \left( r \pm \frac{\sigma^2}{2}(T-t) \right) \right)$$

The problems about the BSM equations could be extremely difficult. For a reference, you may take a look at the problem as an excerpt of Exercise 4.9 [Shreve]

- (a) Show that for  $x > K$ ,  $\lim_{t \rightarrow T^-} d_{\pm} = \infty$ , but for  $0 < x < K$ ,  $\lim_{t \rightarrow T^-} d_{\pm} = -\infty$
- (b) Show that for  $0 \leq x < T$ ,  $\lim_{t \rightarrow 0^+} d_{\pm} = -\infty$
- (c) Show that for  $0 \leq x < T$ ,  $\lim_{t \rightarrow \infty} d_{\pm} = \infty$ .
- (d) Use (c) to verify

$$\lim_{x \rightarrow \infty} (c(t, x) - (x - e^{-r(T-t)K})) = 0$$

Hint: In this verification, you will need to show that

$$\lim_{x \rightarrow \infty} \frac{N(d_+) - 1}{x^{-1}}.$$

Use the L'Hopital's rule and the fact

$$x = K \exp \left( \sigma \sqrt{T-t} d_+ - (T-t) \left( r + \frac{1}{2} \sigma^2 \right) \right)$$