Exotic Options

geometric B.M. $\quad dS(t) = S(0)\exp(\sigma \widetilde{W}(t))$

$d\widetilde{W} = dW + \theta(t)\,dt$,

$\widetilde{W}$ is B.M. under $\widetilde{P}$ (risk-neutral prob. measure)

under $\widetilde{P}$: $\quad e^{-rt}S(t)$ is a martingale

$\quad\quad e^{-rt}V(t)$ is a martingale

$$\Rightarrow \quad e^{-rt}V(t) = \widetilde{E}\left(e^{-rT}V(T)\,\middle|\,\mathcal{F}(t)\right), \quad 0 \le t \le T$$

$$V(t) = \widetilde{E}\left(e^{-r(T-t)}\underline{V(T)}\,\middle|\,\mathcal{F}(t)\right)$$

$$V(t) = \tilde{E}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$$

European: $V(T) = \max\left(S(T)-K, 0\right)$ $\left\{ \begin{array}{l} S(T) > K \\ \text{"n"} < d_- \end{array} \right.$

$$V(t) = S(t) N(d_+) - Ke^{-r(T-t)} N(d_-)$$

Knockout (up & out)

$$V(T) = \max(S(T)-K, 0) \quad \text{iff} \quad S(0) e^{\sigma \tilde{M}(T)} \le B.$$

$$V(T) = \left(S(0)\exp\left(\sigma\tilde{W}(T)\right) - K\right) \cdot \mathbb{1}_{\{\tilde{W}(T) \ge k,\ \tilde{M}(T) \le b\}}.$$

$$k = \frac{1}{\sigma}\log\frac{K}{S(0)}, \quad b = \frac{1}{\sigma}\log\frac{B}{S(0)}$$

$$\tilde{\mathbb{P}}\left(\hat{M}(T) \le m,\ \tilde{W}(T) \le w\right) = \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \cdot \frac{2(2m-w)}{T\sqrt{2\pi T}}\, dm\, dw$$

$$\Rightarrow V(t) = S(t)\left(N(d_{1+}) - N(d_{2+})\right)$$
$$- e^{-r(T-t)} K \left(N(d_{1-}) - N(d_{2-})\right)$$
$$- B\left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}} \left(N(d_{3+}) - N(d_{4+})\right)$$
$$+ e^{-r(T-t)} K \left(\frac{S(t)}{B}\right)^{-\frac{2r}{\sigma^2}+1} \left(N(d_{3-}) - N(d_{4-})\right)$$

$$d_{1\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S(t)}{K} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$d_{2\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S(t)}{B} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$d_{3\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{B^2}{KS(t)} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$d_{4\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{B}{S(t)} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

LookBack:

$$V(T) = \max_{t \in [0,T]} S(t) - S(T)$$

$$= S(0) \exp(\sigma \widehat{M}(t)) - S(T)$$

$$V(t) = \widetilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \,\middle|\, \mathcal{F}(t)\right)$$

$$V(t, y) \qquad y(t) = \max_{u \in [0,t]} S(u)$$

$$= \left(1 + \frac{\sigma^2}{2r}\right) S(t) N(d_+) + e^{-r(T-t)} y N(-d_-)$$

$$- \frac{\sigma^2}{2r} e^{-r(T-t)} \left(\frac{y}{S(t)}\right)^{\frac{2r}{\sigma^2}} S(t) N(-d_-) - S(t);$$

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{y} + (r \pm \tfrac{1}{2}\sigma^2)(T-t)\right)$$

American Options

options can be exercised at any time, up to T.

European Options
If you choose to exercise the option, then can only do this at T.

for investor extra freedom

- $V_0(A) \geq V_0(E)$ same K, T
- (since it can be exercised immediately)

"value for option" $V_0 \geq G_0$. instrinsic value. "payoff according to the rule"

Binary market, $R = 1.25$. put, $K=5 \Rightarrow G = 5 - S_n$.

$S_0 = 4$, $G_0 = 1$

$S_1(H) = 8$, $G_1(H) = -3$

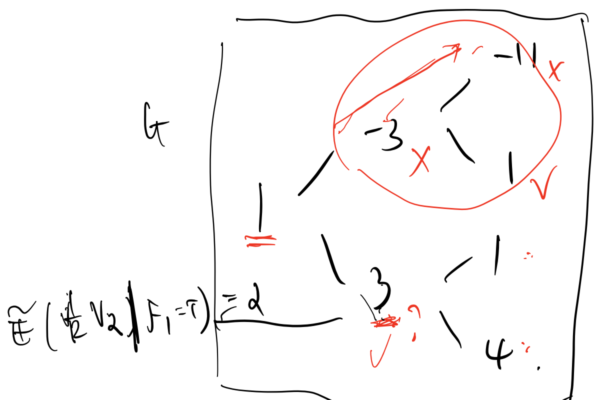
$S_1(T) = 2$, $G_1(T) = 3$

$S_2(HH) = 16$, $G_2(HH) = -11$

$S_2(HT) = 4$, $G_2(HT) = 1$

$S_2(TH) = 4$, $G_2(TH) = 1$

$S_2(TT) = 1$, $G_2(TH) = 4$

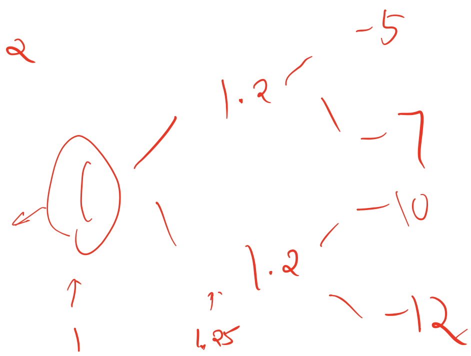


$3 \times 1.25 = 3.75$

if H. ×
HH ×
HT ✓

if T ? exercise payoff = 3
not exercise payoff $\{1, 4\}$

pay \$2



$R = 1.25$

$s_1(T) = 2$ — $s_2(TH) = 4$ | $G_2(TH) = 1$ $\tilde{p}$

$G_1(T) = 3$ — $s_2(TT) = 1$ | $G_2(TH) = 4$ $\tilde{q}$

$V_1(T) = 2$.

2-stage American $\Leftrightarrow$ European.

$$\frac{1}{R} V_1 = \tilde{E}\left(\frac{1}{R^2} V_2 \,\middle|\, \mathcal{F}_1 = T\right) \quad \text{martingale}$$

$$V_1 = \tilde{E}\left(\frac{1}{R} V_2 \,\middle|\, \mathcal{F}_1 = T\right)$$

$$= \tilde{p}\frac{1}{R} V_2(TH) + \tilde{q}\frac{1}{R} V_2(TT)$$

$V_2(HT)$
$V_2(HH)$

$$\tilde{p}, \tilde{q}: \begin{cases} \tilde{p} + \tilde{q} = 1 \\ \frac{1}{R} S_1 = \tilde{E}\left(\frac{1}{R^2} S_2 \,\middle|\, \mathcal{F}_1 = T\right) \end{cases} \rightarrow \tilde{p} = \frac{R \cdot S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)}$$

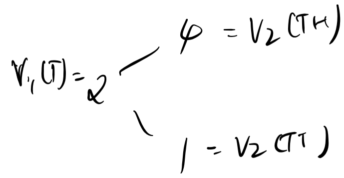
$$\tilde{p} = \frac{1.25 \times 2 - 1}{4 - 1} = \frac{1.5}{3} = 0.5 \qquad \tilde{p} = \frac{R \cdot S_0 - S_1(T)}{S_1(H) - S_1(T)}$$

$$\tilde{q} = 0.5$$

$$V_1 = \tilde{p} \cdot \frac{1}{R} V_2(TH) + \tilde{q} \cdot \frac{1}{R} V_2(TT)$$

$$= 0.5 \, \frac{1}{1.25} \times 4 + 0.5 \times \frac{1}{1.25} \times 1$$

$$= 2.$$



$$V_t = \max\left(G_t , \tilde{E}\left(\frac{1}{R} V_{t+1} \middle| F_t \right) \right).$$

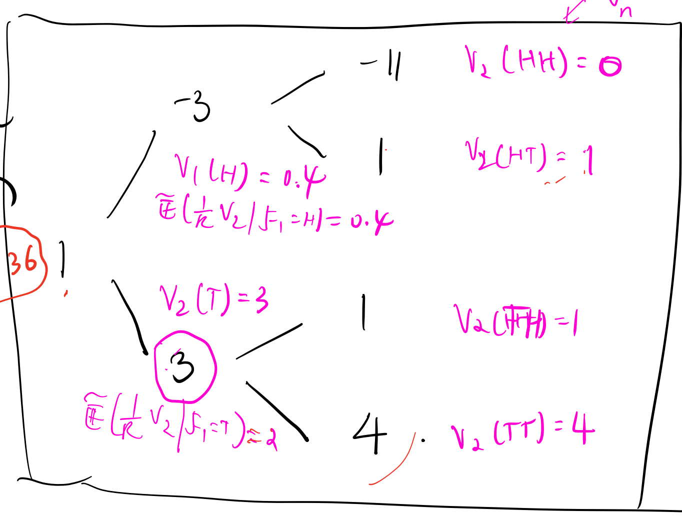
$$V_n = \max(G_n, 0)$$

G:

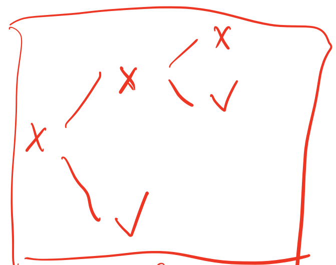
intrinsic value

(payoff by rule)

$V_0 = 1.36$

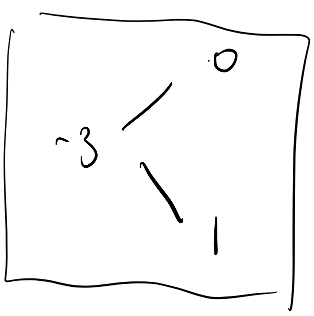


investor strategy



Stopped process

G

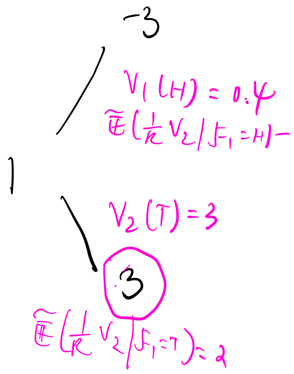


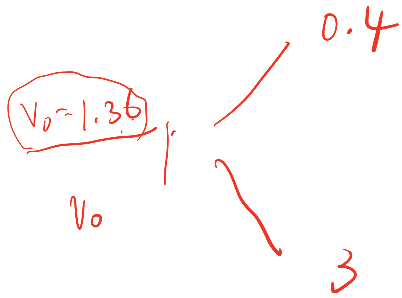
$$V_1(H) = \tilde{E}\left(\frac{1}{R} V_2 \middle| F_1 = H \right)$$

$$= \tilde{p} \cdot \frac{1}{R} V_2(HH) + \tilde{q} \cdot \frac{1}{R} V_2(HT)$$

$$= \frac{1}{2} \cdot \frac{1}{1.25} \cdot 0 + \frac{1}{2} \cdot \frac{1}{1.25} \cdot 1$$

$$= 0.4$$





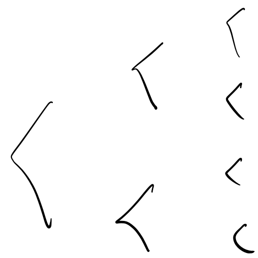
$$\widetilde{E}\left(\frac{1}{R} V_1 \mid \mathcal{F}_0\right)$$

$$= \tilde{p} \frac{1}{R} V_1(H) + \tilde{q} \frac{1}{R} V_1(T)$$

$$= \frac{1}{2} \cdot \frac{1}{1.25} \times 0.4 + \frac{1}{2} \cdot \frac{1}{1.25} \times 3$$

$$= 1.36$$

American securities.



- European in each.
- hedge.
  choose large one
- step by step.

"$V_t = \max\left(\tilde{\mathbb{E}}\left[\frac{1}{R} V_{t+1} \middle| \mathcal{F}_t\right], G_t\right)$"

V_0 no longer a martingale

sub-martingale $\quad V_t \le \tilde{\mathbb{E}}\left(\frac{1}{R} V_{t+1} \middle| \mathcal{F}_t\right)$.

super-martingale $\quad M_t \ge \tilde{\mathbb{E}}\left(M_{t+1} \middle| \mathcal{F}_t\right)$

f convex: $f(\mathbb{E}(X)) \le \mathbb{E}(f(X))$.

- Jensen's inequality. M martingale

f: convex, $f(M)$ sub-martingale

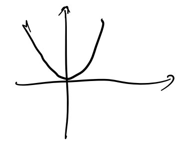
f: concave: $f(M)$ super-martingale

Random walk in 1D: $X_n - X_{n-1} = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$.

X_n is a martingale

$\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$.

X_n^2 martingale?



X_n^2 → sub-martingale

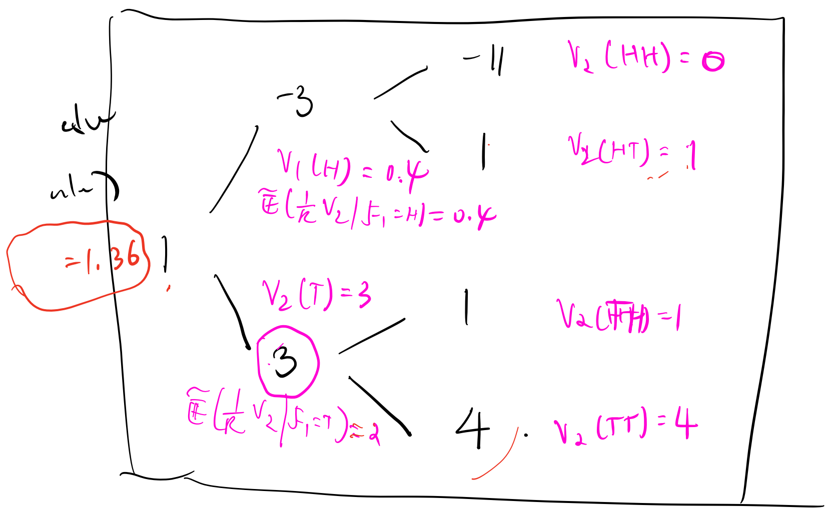
$Var(W(t)) = t.$

$$X_{n+1} = \begin{cases} X_n + 1 & p = 1/2 \\ X_n - 1 & p = 1/2 \end{cases}$$

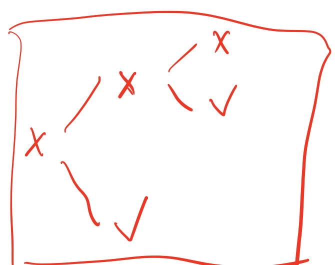
$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) = \frac{1}{2}(X_n+1)^2 + \frac{1}{2}(X_n-1)^2 = X_n^2 + 1 > X_n^2$$

↑ X_n known

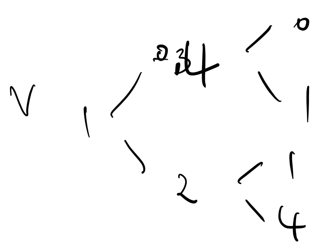
Model:



investor strategy



stopped process



V is a super-martingale

stopped process also super-martingale