Mathematics of Finance Review 1

Review plans - 1.5 sessions on the handout & 1.5 sessions on practical exercises.

1. [Martingales] [Definitions] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider an adapted stochastic process $M(t), 0 \leq t \leq T$. If $\forall 0 \leq s \leq t \leq T$:
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) = M(s)$, then we say this process is a martingale. It has no tendency to rise or fall;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \leq M(s)$, then we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise;
   – $\mathbb{E}(M(t)|\mathcal{F}(s)) \geq M(s)$, then we say this process is a supermartingale. It has no tendency to rise; it may have a tendency to fall;

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2, \ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step symmetric random walk as follows.

   $$W_n(t) = \sum_{i=1}^{n} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Show that $W_n(t)$ is a martingale.

   (b) For American Options, we have $V_t = max(\tilde{\mathbb{E}}(V_{t+1}|\mathcal{F}(t)), G(t)) \forall t$. Classify V_t as a type of martingale.

2. [Scaled Symmetric Random Walks]

   (a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2, \ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step scaled symmetric random walk as follows.

   $$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

   Deduce that $E(W_n(t) - W_n(s)) = 0$ and $Var(W_n(t) - W_n(s)) = t - s$.

   (b) Show that $W(t) := \lim_{n \to \infty} W_n(t) = \mathcal{N}(0, t)$.

3. [Binary and Log-Normal Markets] Consider an n-step binary market with no interest rate ($R = 1.0$)

   (a) Set $u = 3/2, d = 1/2$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S(t)$ at time t.

   (b) Set $u = 1 + \sigma/\sqrt{n}, d = 1 - \sigma/\sqrt{n}$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
   Find the stock price $S_n(t)$ at time t. Show that

   $$\lim_{n \to \infty} S_n(t) = S(0) \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right),$$

   where $W(t)$ is defined in 2(b).

4. [Brownian Motions - Calculations] [Definition] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$. Then $W(t), t \geq 0$ is a Brownian motion if $\forall i \in \{0, 1, \ldots m\}$, the increments $W(t_{i+1}) - W(t_i)$ are independent and each of these increments is normally distributed with $\mathbb{E} = 0$ and $Var = t_{i+1} - t_i$.

   (a) Set $dW(t) = W(t + dt) - W(t)$. From the defination, find $\mathbb{E}(dW), Var(dW)$.
   Find $\mathbb{P}\{W(0.25) \leq 0.2\}$

   (b) Show that $\mathbb{E}(W(t)W(s)) = t \wedge s$. Deduce the covariance of $W(t)$ and $W(s)$.

   (c) Show that $W(t)$ is a martingale, and so is $Z(t) = exp(\sigma W(t) - 1/2 * \sigma^2 t)$

5. [Ito's integral] Consider the following Ito's integral:

$$I(t) = \int_0^t \Delta(u)dW(u) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1} - W(t_j))$$

   (a) Show that, $I(t)$ is a martingale. Remark: An Ito's integral with zero $dt-$term is a martingale.

   (b) Show that,

$$f(T, W(T)) = f(0, W(0) + \int_0^T f_t(t, W(t)dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

   .

   Hint: We may apply the Taylor's formula:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2$$

   Note that the reminder contains a sum of terms $(W(t_{j+1} - W_j)^3$ which has limit 0.

   Further Hint: $dW(t)dW(t) = dt, \quad dtdW(t) = 0, \quad dtdt = 0.$

   Remark: We can rewrite the formula as the differential term:

$$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2}f_{xx}dX(t)dX(t)$$

   (c) Deduce that $d(AB) = AdB + BdA + dAdB$ for stochastic process $A(t), B(t)$.

6. [Probability Measures] Consider the geometric brownian motions $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$. We define a disounted process $D(t) = exp(-rt)$.
We define $X(t)$ as the total profolio under the hedging strategy $\Delta(t)$.

   (a) Find $d(D(t)S(t)$ and $d(D(t)X(t))$.

   (b) Show that, if $d(D(t)S(t)$ is a martingale under some probability measure $\tilde{\mathbb{P}}$, then so is $d(D(t)X(t))$.

7. [Change of Probability Measures] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$. Then, $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then $\tilde{\mathbb{E}}X = \mathbb{E}(XZ)$.

   (a) Show that $\mathbb{E}\, Z = 0$ for $Z(\omega) = \exp(-\theta X(\omega) - 1/2 * \theta^2)$

   (b) Show that $\mathbb{E}\, Z = 0$ for

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right)$$

8. [Binary Markets] Consider a 2-step binary market with $S_0 = 4, u = 2, d = 0.5, R = 1.1$

   (a) Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.

   (b) Find the fair prices of a European call with $K = 4, T = 2$.

   (c) Derive a hedging strategy $\Delta(t)$.

9. [Log-Normal Markets] Consider a geometric brownian motion with $dS(t) = 0.1 * S(t)dt + 0.3 * S(t)dW(t)$. Set the interest factor $R(t) = \exp(1.05t)$.

   (a) Set $S(0) = 1$, consider a European call with $K = 1, T = 2$. Find the fair price of such an option. You may proceed with either Black-Shore, or the risk-neutral probability measure.

1. [Martingales] [Definitions] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider an adapted stochastic process $M(t), 0 \leq t \leq T$. If $\forall 0 \leq s \leq t \leq T$:
– $\mathbb{E}(M(t)|\mathcal{F}(s)) = M(s)$, then we say this process is a martingale. It has no tendency to rise or fall;
– $\mathbb{E}(M(t)|\mathcal{F}(s)) \leq M(s)$, then we say this process is a submartingale. It has no tendency to fall; it may have a tendency to rise;
– $\mathbb{E}(M(t)|\mathcal{F}(s)) \geq M(s)$, then we say this process is a supermartingale. It has no tendency to rise; it may have a tendency to fall;

(a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2$, $\forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step symmetric random walk as follows.

$$W_{\cancel{n}}(t) = \sum_{i=1}^{\cancel{n}\,t} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

Show that $W_{\cancel{n}}(t)$ is a martingale.

(b) For American Options, we have $V_t = max(\tilde{\mathbb{E}}(V_{t+1}|\mathcal{F}(t)), G(t))\forall t$. Classify V_t as a type of martingale.

(a) $$\mathbb{E}\left(W(t) \,\middle|\, \mathcal{F}(t-1)\right) \overset{\text{Need to prove}}{=} W(t-1)$$

$$= \mathbb{E}\left(W(t-1) + X_t \,\middle|\, \mathcal{F}(t-1)\right)$$

$$= \mathbb{E}\left(W(t-1) \,\middle|\, \mathcal{F}(t-1)\right) + \underbrace{\mathbb{E}\left(X_t \,\middle|\, \mathcal{F}(t-1)\right)}_{= \mathbb{E}(X_t)}$$

$$= W(t-1) + 0$$

$$= W(t-1) \qquad \text{martingale}$$

(b) $$V_t = \max\left(\tilde{\mathbb{E}}\left(V_{t+1} \,\middle|\, \mathcal{F}_t\right), G(t)\right)$$

$$V_t \geq \tilde{\mathbb{E}}\left(V_{t+1} \,\middle|\, \mathcal{F}_t\right) \quad \text{– submartingale}$$

2. [Scaled Symmetric Random Walks]

(a) Consider a fair coin with $\mathbb{P}(w_i = H) = \mathbb{P}(w_i = T) = 1/2,\ \forall i \in \mathbb{R}$. According to the first n results of the coin toss, we define an n-step scaled symmetric random walk as follows.

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\overset{M_{nt}}{nt}} X_i, \text{ where } X_i = \begin{cases} 1, & w_i = H, \\ -1, & w_i = T \end{cases}$$

Deduce that $E(W_n(t) - W_n(s)) = 0$ and $Var(W_n(t) - W_n(s)) = t - s$.

(b) Show that $W(t) := \lim_{n\to\infty} W_n(t) = \mathcal{N}(0, t)$.

(a).

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}}\underbrace{\left(X_{ns+1} + X_{ns+2} + \cdots + X_{nt}\right)}_{\text{each of them}}.$$

each of them $E=0$, $Var = 1$

$(nt-ns)$ numbers, indep.

$E=0$, $Var=1$

$$\Rightarrow \frac{1}{\sqrt{n}}\sum X_i \Rightarrow \begin{array}{l} E=0 \\ Var = \frac{1}{n}\cdot(nt-ns) = t-s \end{array}$$

(b).

$$W_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{nt} X_i$$

$$W(t) = \lim_{n\to\infty} W_n(t) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{i=1}^{nt} X_i$$

By CLT: $\displaystyle\lim_{n\to\infty}\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}} \sim N(0,1)$

$$W(t) := \lim_{n\to\infty}\frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}} \sim N(0, t).$$

3. [Binary and Log-Normal Markets] Consider an n-step binary market with no interest rate ($R = 1.0$)

(a) Set $u = 3/2, d = 1/2$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$. (for European calls)
Find the stock price $S(t)$ at time t.

(b) Set $u = 1 + \sigma/\sqrt{n}, d = 1 - \sigma/\sqrt{n}$. Derive the risk-neutral probabilities $\tilde{p}, \tilde{q}$.
Find the stock price $S_n(t)$ at time t. Show that

$$\lim_{n\to\infty} S_n(t) = S(0)\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right),$$

where $W(t)$ is defined in 2(b).

$S_0 \xrightarrow{\tilde{p}} S_1(H) = uS_0 \to u^2 S_0,\ udS_0$

$S_0 \xrightarrow{\tilde{q}} S_1(T) = dS_0 \to udS_0,\ d^2 S_0$

solve $\tilde{p}, \tilde{q}$ $\begin{cases} \tilde{p} + \tilde{q} = 1 \\ S_0 = \tilde{\mathbb{E}}\left(\frac{1}{R} S_1 \mid \mathcal{F}_0\right) = \tilde{p}\frac{1}{R} S_1(H) + \tilde{q}\frac{1}{R} S_1(T) \end{cases}$

$V_0 = \tilde{\mathbb{E}}\left(\frac{1}{R} V_1 \mid \mathcal{F}_0\right) = \tilde{p}\frac{1}{R} V_1(H) + \tilde{q}\frac{1}{R} V_1(T).$

$r = 0, \quad R = 1 + r = 1$

(a) $\tilde{p} = \dfrac{1+r-d}{u-d} = \dfrac{1}{2}. \qquad \tilde{q} = \dfrac{1}{2}.$

$S(t) = S(0) \cdot u^H \cdot d^T. \qquad H, T: \#$

$\begin{cases} H + T = t \\ H - T = M_t = \sum_{i=1}^{t} X_i \end{cases}$ (X_i defined in Prob 1).

$X_i = \begin{cases} 1 & \tilde{p} = 1/2 \\ -1 & \tilde{q} = 1/2. \end{cases}$

$H = \frac{1}{2}(t + M_t)$

$T = \frac{1}{2}(t - M_t)$

$S(t) = S(0)\, u^{\frac{1}{2}(t+M_t)}\, d^{\frac{1}{2}(t-M_t)}$

(b) $\tilde{p} = \frac{1+r-d}{u-d} = \frac{1}{2}, \quad \tilde{q} = \frac{1}{2}.$

$$\sum_{i=1}^{nt} X_i = M_{nt} = H_{nt} - T_{nt}.$$

$$\begin{cases} nt = H_{nt} + T_{nt} \end{cases}$$

$$S_n(t) = S(0)\, u^{\frac{1}{2}(nt + M_{nt})}\, d^{\frac{1}{2}(nt - M_{nt})}$$

$$\log S_n(t) = \log S(0) + \frac{1}{2}(nt + M_{nt}) \log\left(1 + \underbrace{\frac{\sigma}{\sqrt{n}}}_{\to 0}\right)$$

for $n \to \infty$

$$+ \frac{1}{2}(nt - M_{nt}) \log\left(1 - \frac{\sigma}{\sqrt{n}}\right)$$

$\log(1+x) = x - \frac{1}{2}x^2 + o(x^3)$, apply to $x = \frac{\sigma}{\sqrt{n}}$, $x = -\frac{\sigma}{\sqrt{n}}$

$$\log S_n(t) = \log S(0) + \frac{1}{2}(nt + M_{nt})\left(\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n}\right)$$

$$+ \frac{1}{2}(nt - M_{nt})\left(-\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n}\right)$$

$$= \log S(0) + \frac{1}{2}nt\left(-\frac{\sigma^2}{n}\right) + \frac{1}{2}M_{nt}\left(\frac{2\sigma}{\sqrt{n}}\right)$$

$$= \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W_n(t) \qquad W_n(t) = \frac{1}{\sqrt{n}} M_{nt}.$$

$$\to \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W(t)$$

$$S_n(t) \to S(0) \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right).$$

log-normal model.

4. [Brownian Motions - Calculations] [Definition] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$. Then $W(t), t \geq 0$ is a Brownian motion if $\forall i \in \{0, 1, \dots m\}$, the increments $W(t_{i+1}) - W(t_i)$ are independent and each of these increments is normally distributed with $\mathbb{E} = 0$ and $Var = t_{i+1} - t_i$.

(a) Set $dW(t) = W(t + dt) - W(t)$. From the defination, find $\mathbb{E}(dW), Var(dW)$. Find $\mathbb{P}\{W(0.25) \leq 0.2\}$

$= \min(t, s)$

(b) Show that $\mathbb{E}(W(t)W(s)) = t \wedge s$. Deduce the covariance of $W(t)$ and $W(s)$.

(c) Show that $W(t)$ is a martingale, and so is $Z(t) = exp(\sigma W(t) - 1/2 * \sigma^2 t)$

(a). $dW = W(t+dt) - W(t) \sim N(0, dt)$

$$\mathbb{E}(dW) = 0, \quad Var(dW) = dt$$

$$W(0.25) = W(0.25) - \underbrace{W(0)}_{=0} \sim N(0, 0.25).$$

$$W(0.25) \sim N(0, 0.25)$$

$$\mathbb{P}(W(0.25) \leq 0.2) = \mathbb{P}(W(1) \leq 0.4) = N(0.4)$$

$$N(0.4) = \int_{-\infty}^{0.4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

(b). Suppose $0 \leq s < t$

increments.

$$\mathbb{E}(W(s)W(t)) = \mathbb{E}\left(W(s)(\underline{W(t) - W(s)}) + W^2(s)\right)$$

A, B indep

$\mathbb{E}(AB) = \mathbb{E}(A) \cdot \mathbb{E}(B)$.

$$= \mathbb{E}(W(s)) \cdot \mathbb{E}(W(t) - W(s)) + \mathbb{E}(W^2(s)),$$

$$= \mathbb{E}(W^2(s)) - \underbrace{(\mathbb{E}W(s))^2}_{=0}$$

$$= Var(W(s))$$

$$= s.$$

$$\operatorname{cov}(W(s), W(t)) = \mathbb{E}\big(W(s)W(t)\big) - \underbrace{\mathbb{E}W(s)}_{=0} \cdot \underbrace{\mathbb{E}W(t)}_{=0}$$

$$= s$$

If $\ell = [W(t_1), W(t_2), \cdots W(t_j)]$ $\quad t_1 < t_2 < \cdots < t_j$

then covariance matrix of ℓ is found as

$$\begin{array}{c|c}
 & \begin{array}{ccccc} W(t_1) & W(t_2) & \cdots & & W(t_j) \end{array} \\
\begin{array}{c} W(t_1) \\ \vdots \\ \vdots \\ \vdots \\ W(t_j) \end{array} & \begin{pmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ \vdots & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \\ t_1 & t_2 & t_3 & & t_j \end{pmatrix}
\end{array} \qquad (*)$$

If W adapted process with covariance matrix of form $(*)$

then W brownian motion.

(c). $\mathbb{E}(W(t) \mid \mathcal{F}(s)) = \mathbb{E}(W(t) - W(s) + W(s) \mid \mathcal{F}(s))$

$$= \underbrace{\mathbb{E}(W(t) - W(s) \mid \mathcal{F}(s))}_{\text{indep. of the filter.}} + \mathbb{E}(\underbrace{W(s) \mid \mathcal{F}(s)}_{\text{in the filter}})$$

$$= \mathbb{E}(W(t) - W(s)) + W(s)$$

$$= W(s)$$

$\rightarrow$ martingale

(d) $Z(t) = \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$

$$\mathbb{E}\left(Z(t) \mid \mathcal{F}(s)\right)$$

$$= \mathbb{E}\left(\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right) \mid \mathcal{F}(s)\right)$$

$$= \mathbb{E}\left(\exp\left(\sigma(W(t) - W(s))\right) \cdot \exp\Big(\underbrace{W(s) - \frac{1}{2}\sigma^2 t}_{\text{constant}} \Big| \mathcal{F}(s)\right)$$

$$= \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 t\right) \cdot \mathbb{E}\left(\exp\left(\sigma(W(t) - W(s))\right) \mid \mathcal{F}(s)\right)$$

$$\mathbb{E}\left(\exp\left(\sigma(\underbrace{W(t) - W(s)}_{X \sim N(0, t-s)})\right)\right) = \int_{-\infty}^{\infty} \exp(\sigma x) \cdot \underbrace{\frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)}\right)}_{\text{pdf of } N(0,t-s)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)} + \sigma x - \frac{\sigma^2(t-s)^2}{2(t-s)}\right) \exp\left(\frac{\sigma^2}{2}(t-s)\right) dx.$$

$$= \exp\left(\frac{\sigma^2(t-s)}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x-\sigma(t-s))^2}{2(t-s)}\right) dx$$

$$= \exp\left(\frac{\sigma^2(t-s)}{2}\right).$$

$\int_{-\infty}^{\infty} \text{pdf}(x)\,dx = 1$

pdf: $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma}\right)$

$$\mathbb{E}\left(Z(t) \mid \mathcal{F}(s)\right) = \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 t\right) \cdot \exp\left(\frac{\sigma^2(t-s)}{2}\right)$$

$$= \exp\left(\sigma W(s) - \frac{1}{2}\sigma^2 s\right)$$

$$= Z(s)$$

$\rightarrow Z$ is a martingale

5. [Ito's integral] Consider the following Ito's integral:

$$I(t) = \int_0^t \Delta(u)dW(u) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1} - W(t_j))$$

(a) Show that, $I(t)$ is a martingale. Remark: An Ito's integral with zero $dt-$term is a martingale.

(b) Show that, (Itô's formula).

$$f(T, W(T)) = f(0, W(0) + \int_0^T f_t(t, W(t)dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

.

Hint: We may apply the Taylor's formula:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2 + O((x_{j+1}-x_j)^3)$$

Note that the reminder contains a sum of terms $(W(t_{j+1} - W_j)^3$ which has limit 0. $W(j)$.

Further Hint: $dW(t)dW(t) = dt, \quad dtdW(t) = 0, \quad dtdt = 0$.

Remark: We can rewrite the formula as the differential term:

$$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2} f_{xx} dX(t)dX(t)$$

(c) Deduce that $d(AB) = AdB + BdA + dAdB$ for stochastic process $A(t), B(t)$.

(a). $\mathbb{E}(I(t) | \mathcal{F}(s))$

$$= \underbrace{\sum_{j=0}^{l-1} \Delta(t_j)\left(W(t_{j+1}) - W(t_j)\right)}_{\text{in the filter}} + \mathbb{E}\left(\sum_{j=l}^{k-1} \underbrace{\Delta(t_j)}_{\text{known process (not stochastic)}} \underbrace{\left(W(t_{j+1}) - W(t_j)\right)}_{\text{indep. of filter.}} \Big| \mathcal{F}(s)\right)$$

$(t_l < t_k)$.

$$= I(s) + \sum_{j=l}^{k-1} \Delta(t_j) \cdot \mathbb{E}\left(W(t_{j+1}) - W(t_j) \mid \mathcal{F}(s)\right)$$

$$= I(s).$$

$dS = \alpha S dt + \sigma S dW$.

$dX = \_ dt + \_ dW$.

(b). $df = f_t dt + f_x dX + \frac{1}{2}\left(f_{xx} dX dX + 2 f_{xt} \cancel{dX dt} + f_{tt} \cancel{dt dt}\right)$

$dt dW = 0$, $dt dt = 0$

$df(t, X(t))$ stochastic

Result: $df = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX dX$.

(c) plug in for $df(t, (A, B)) = f_t\,dt + f_A\,dA + f_B\,dB$

$$+ \frac{1}{2}\left(f_{AA}\,dA\,dA + f_{AB}\,dA\,dB + f_{BB}\,dB\,dB\right).$$

$dA\,dt = 0$, $dB\,dt = 0$, $dt\,dt = 0$

$f(t, A, B) = AB$. $f_t = 0$, $f_A = B$, $f_B = A$

$f_{AA} = f_{BB} = 0$, $f_{AB} = 1$

$$\Rightarrow \quad d(AB) = A\,dB + B\,dA + dA\,dB.$$