Math of Finance

Recitation - Feb 4
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Office Hours - W 9-11 pm via zoom, or by appointments

Piazza - TBA

Logistics

Meeting time -
F 11 - 12.15, 194M 203, or https://nyu.zoom.us/j/92338445220
F 12.30 - 1.45, 194M 204, or https://nyu.zoom.us/j/91604832921

Grader
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Risk-free securities

Credits

Stochastic Calculus for Finance, Vols. I and II, by S.E. Shreve; Springer Verlag

not textbook.

Risk-free securities

Growth factors

Risk free - fixed and predicable return, must grow at a sufficient rate to avoid depreciation.

$R(t)$

The rate is called growth factor $R(t)$ interest of government-issued bonds.

Mathematically, a risk-free security with initial value V_0 acquire a value of $V(t) = V_0 R(t)$ at time t

Risk-free securities

Estimations of $R(t)$ under easy models

Suppose a yearly interest rate r, $= 0.06$ then after t years,

Simple Interest $R(t) = 1 + rt$, usually for $t < 1$.

$R(\frac{1}{2}) = 1 + \frac{0.06}{2} = 1.03$

Compound interest

1. interest compunded once per year.

$R(1) = 1 + r$.

$R(2) = (1+r)(1+r) = (1+r)^2$

...

$R(t) = (1+r)^t$

interest compounded twice per year.

$$R(t) = \left(1+\frac{r}{2}\right)^{2t}$$

interest compounded m tims per year

$$R(t) = \left(1+\frac{r}{m}\right)^{mt}$$

Continuously compound interest

$$R(t) = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} = e^{rt}$$

Risk-free securities

Discounted Values

$$V_T = V_0 R(T)$$

$$\Delta$$

$$V_0 = \frac{V_T}{R(T)}$$

fair price of a future value

Risk-free securities

Example

none-coupon bonds.

Bonds

coupon bonds

Annuities / Loans

Deposit plans

None-Coupon.

investor: pay $V(0)$

recieve F at T

face value

maturity time

$$V(0) = \frac{F}{R(T)}$$

Coupon bonds.

investor: pay $V(0)$

receive periodic payments.

$C, C, C, C \cdots C, F$

coupon value

face value

time (t) $(2t)$ $(3t)$ $\cdots$ (Nt) (Nt)

Fair price for coupon bonds:

$$V_0 = \frac{C}{R(t)} + \frac{C}{R(2t)} + \cdots + \frac{C}{R(Nt)} + \frac{F}{R(Nt)}$$

$$R(2t) = R^2(t) \qquad R(Nt) = R^N(t)$$

Annuities: – bonds, $F=0$

pay $V(0)$, recieve payments $C, C, C \cdots C$

Loan:

recieve P, pay $C, C, C \cdots, C$ (F)

Deposit Plans / Retirement Plans.

pay $C, C, C \cdots$, $\begin{cases} \text{recieve } P. \\ \text{recieve } C', C', C', \cdots C' \end{cases}$

$V(0)$

Arbitrage

- imbalance of markets

Example

$x > 1.38$

1 pounds = 10 USD.

borrow £1 in Britain

$\to$ £6 ~ 7

1 Euro = 1.15 USD (France)

1 Pound = 1.2 Euro (Britain)

1 Pound = ~~?~~ x USD (Britain)

$1.2 \times 1.15 = 1.38$

$x < 1.38$

1 pound = 0.1 USD.

Def - fair price is a price that does not lead to arbitrages

borrow \$1 in Britain

$\to$ \$13.

$x \to 1.38$

Arbitrage

Example

recieve V_0 at $t=0$

V_T at $t=T$.

Investor borrows V_0 to be returned in one payment at time T

A bond for such a period pays $R(T)$ — growth factor $R(T)$

V_T - fair amount investor must return at time T ?

$$\boxed{V_T = V_0 \cdot R(T)}$$

$V_T < V_0 \cdot R(T)$

investor → borrow money to purchase bonds.

$V_0 \cdot R(T) - V_T > 0 \Rightarrow$ arbitrage for investor

$V_T > V_0 \cdot R(T)$

FI (Finance Institute)

~~sell~~ short-sell bonds;

$-V_0 \cdot R(T) + V_T > 0 \Rightarrow$ arbitrage for FI

Arbitrage

(assumptions)

No arbitrage principle

Markets exhibit no arbitrages.

Assumptions

Infinite divisibility of assets - no minimal units, no need to round off

mobilized capitals

Infinite liquidity — case

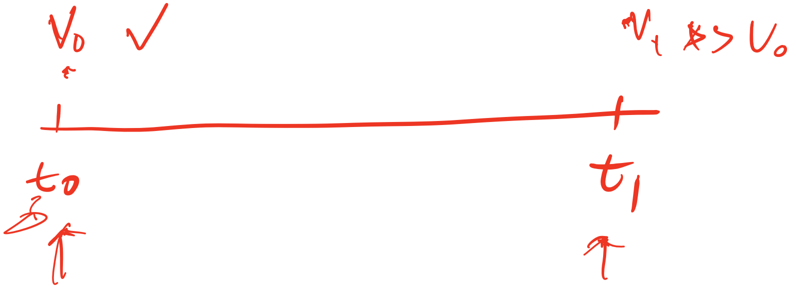
Infinite availability — stock

No interest spread / price spread

$R(T)$ $S(T)$

Simple Markets

Single period derivatives



Two time involved - t_0, t_1

Investor needs some asset at t_1.

The present value $V(0) = V_0$ is known while the future value $V_1 := V(t_1)$ is unknown.

The investor want to buy the asset at time t_1 with no more than K

strike value

if $K \geq V_0$.

simply purchase at $t = 0$, innoboured

Simple Markets

Fair prices and hedging strategies

$S_0 = 2$

$S_1 = 4$

$S_1 = 1$

$k = 2$ (-2)

Current price $S_0 = 2$

Future price: either $S_1 = 1$ or $S_1 = 4$.

An investor wished to purchase the stock at t_1 for no more than $K = 2$.

$R = 1.0$ hedge by purchasing Δ_0 some shares of stock.

- Contract - investor is obliged to spend $K = 2$ to purchase the stock at t_1
- Call option - investor has the right to spend $K = 2$ to purchase the stock at t_1 if it is beneficial. has higher fair price

Contract. (hedge by purchasing Δ_0 shares) $\rightarrow t_1$

scenario	bonds	stock	profit from contract.
$S_1 = 4$	$V_0 - 2\Delta_0$	$4\Delta_0$	-2
$S_1 = 1$	$V_0 - 2\Delta_0$	Δ_0	1

$$\begin{cases} S_1 = 4 & V_0 - 2\Delta_0 + 4\Delta_0 - 2 = 0 \\ S_1 = 1 & V_0 - 2\Delta_0 + \Delta_0 + 1 = 0 \end{cases}$$

$$\rightarrow \begin{cases} V_0 = 0 \\ \Delta_0 = 1 \end{cases}$$

at t_0: purchase 1 share of stock,

Contract · (hedge by purchasing Δ_0 shares)

$R = 1.1$ $\rightarrow t_1$

scenario	bonds	stock	profit from contract.
$S_1 = 4$	$(V_0 - 2\Delta_0) \cdot R$	$4\Delta_0$	-2
$S_1 = 1$	$(V_0 - 2\Delta_0) \cdot R$	Δ_0	1

$S_1 = 4$ $\quad R(V_0 - 2\Delta_0) + 4\Delta_0 - 2 = 0$

$S_1 = 1$ $\quad R(V_0 - 2\Delta_0) + \Delta_0 + 1 = 0$

$$\rightarrow \begin{cases} V_0 = \cancel{0}\ 0.18 \\ \Delta_0 = 1 \end{cases}$$

at t_0: purchase 1 share of stock,

Call Option · (hedge by purchasing Δ_0 shares

$\rightarrow t_1$

scenario	bonds	stock	profit from contract.
$S_1 = 4$	$V_0 - 2\Delta_0$	$4\Delta_0$	-2
$S_1 = 1$	$V_0 - 2\Delta_0$	Δ_0	$\cancel{1}$ 0

$$\begin{cases} S_1 = 4 & V_0 - 2\Delta_0 + 4\Delta_0 - 2 = 0 \\ S_1 = 1 & V_0 - 2\Delta_0 + \Delta_0 \cancel{+1} = 0 \end{cases}$$

$$\rightarrow \begin{cases} V_0 = \underline{0.66} \\ \Delta_0 = 0.66 \end{cases}$$

at t_0: purchase $\underline{0.66}$ shares of stock, Call Option. (hedge by purchasing Δ_0 shares)

$R = 1.1$ → t_1

scenario	bonds	stock	profit from contract
$S_1 = 4$	$(V_0 - 2\Delta_0)R$	$4\Delta_0$	-2
$S_1 = 1$	$(V_0 - 2\Delta_0)R$	Δ_0	~~1~~ 0

$S_1 = 4 \quad R(V_0 - 2\Delta_0) + 4\Delta_0 - 2 = 0$

$S_1 = 1 \quad R(V_0 - 2\Delta_0) + \Delta_0 \cancel{+1} = 0$

$\rightarrow \begin{cases} V_0 = \cancel{0.66}\ 1.93 \\ \Delta_0 = 0.66 \end{cases}$

at t_0: purchase 0.66 shares of stock,

Simple Markets

Contract & Options

Suppose: stock S_1

strike value K ← contract/option.

Investors:

Securities	V_1: profit from contract/option	
Contract to buy	$S_1 - K$	could be neg.
Call option	$\max(S_1 - K, 0)$	
Contract to sell	$K - S_1$	could be neg.
Put option	$\max(K - S_1, 0)$	