Midterm Review

1. Brownian Motions
   Property 1:

$$dW \sim \mathcal{N}(0, dt)$$

   (a) Multiplication rules (3.10.1 [Shreve])

$$dW(t)dW(t) = dt, \underline{dW(t)dt} = 0, \underline{dtdt} = 0$$

   (b) Derive the formula for $d(W^2)$, $d(W^4)$.

(a) $dt = Var(dW) = \mathbb{E}dW^2 - \underbrace{(\mathbb{E}dW)^2}_{0}$

$dt \to 0 \quad \underline{dW} \sim N(0, \underbrace{dt}_{\to 0}) \quad dW \to 0$

(b). $f(t,x) = x^2$

$f_t = 0, \quad f_x = 2x, \quad f_{xx} = 2$

$$\begin{aligned} d(W^2) &= df(t, W) \\ &= \left.\left(2x\,dx + \tfrac{1}{2} \cdot 2\,dx\,dx\right)\right|_{x=W}. \\ &= 2W\,dW + dW\,dW \\ &= 2W\,dW + dt. \end{aligned}$$

$d(W^4) \quad f(x) = x^4, \ f_x = 4x^3, \ f_{xx} = 12x^2, \ f_t = 0.$

$d(W^2), \qquad dW\,dW = dt$

$\qquad\qquad\qquad (dW)^2 = dt$

Property 2: all the increment dWs are independent from each other.

(a) Suppose $X(T) = \int_0^T f(t)dt + \int_0^T g(t)dW_t$. Find $Var(X)$

If $X_1, X_2, X_3 \cdots X_n$ indep.

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} Var(X_i).$$

(a) $$X(T) = \underbrace{\int_0^T f(t)dt}_{\text{constant}} + \int_0^T g(t)\,dW_t$$

$$Var(X) = Var\left(\int_0^T g(t)\,dW_t\right).$$

$$= \int_0^T Var(g(t)\,dW_t).$$

$$= \int_0^T g^2(t)\,dt.$$

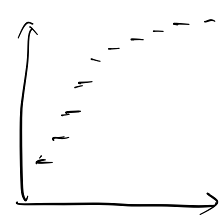
$Var(dW_t) = dt$

$Var(g(t)dW_t) = g^2(t)\,dt.$

$g(t)$

$Var(X) = \sigma^2$

$Var(aX) = a^2\sigma^2$



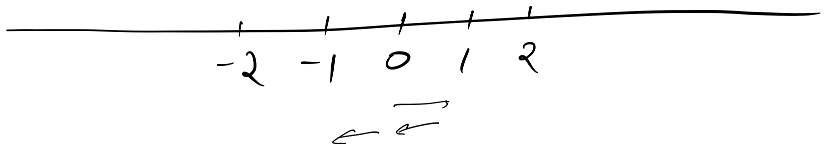
$g(t)$ constant on dt. $dt \to 0$

$VarX = \mathbb{E}X^2 - (\mathbb{E}X)^2$

X^2 $\begin{cases} dt^2 \to 0 \\ dt \cdot dw \to 0 \\ dw^2 \end{cases}$

Property 3: Brownian Motions are a Martingale.
Suppose a random walk on 1D-axis with $X(0) = 0$ and

$$\underset{(t)}{dX} = X(t+1) - X(t) = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$$



$$\mathbb{E}(dX(t)) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0.$$

$$\mathbb{E}(X(t+1) - X(t)) = 0.$$

$$\mathbb{E}(X(t+1)) = \mathbb{E}(X(t)).$$

$$\mathbb{E}(X(t+5)) = \mathbb{E}(X(t))$$

$$\Downarrow$$

$$\mathbb{E}(X(t+1) \mid X(t) = X_t) = X_t.$$

$$\underline{\mathbb{E}(X(t+5) \mid X(t) = X_t) = X_t}$$

$$\mathbb{E}(X(t+5) \mid \underline{X(t) = X_t}, \cancel{X(0) = X_0}, \cancel{X(t-20) = X_{t-20}}) = \underline{X_t}$$

$$dW \sim N(0, dt).$$

$$\mathbb{E}(dW) = 0.$$

$$\Downarrow$$

$$\mathbb{E}(W(t+5) \mid W(t) = W_t, W(0) = W_0) = W_t. \quad \text{martingale.}$$

$$\mathbb{E}(\underline{\int a\,dt} + \int b\,dW; \quad \text{martingale} \iff a = 0$$

no dt-term.

2. Ito's Formula

(a) 1D case

$$df(t, x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$$

Compute the stochastic differential dZ when

i. $Z(t) = exp(\alpha t)$

ii. $Z(t) = exp(\alpha X(t))$ with

$$dX(t) = \mu dt + \sigma dB(t)$$

iii. $Z(t) = 1/X(t)$ with

$$dX(t) = aX(t)dt + \sigma X(t)dW(t)$$

(b) 2D case

$$df(t, x, y) = f_t dt + (f_x dx + f_y dy) + \frac{1}{2}(f_{xx} dx dx + 2 f_{xy} dx dy + f_{yy} dy dy)$$

Derive the Ito's product rule $d(XY) = XdY + YdX + dXdY$

iii. $Z(t) = \frac{1}{X(t)}$, $f(t,x) = \frac{1}{x}$. $f_t = 0$, $f_x = -\frac{1}{x^2}$, $f_{xx} = \frac{2}{x^3}$.

$$dZ(t) = df(t, X(t)) = -\frac{1}{X^2} dX + \frac{1}{2} \cdot \frac{2}{X^3} dX dX.$$

plug in dX for the result. $dW^2 = dt$

$dt^2 = 0$, $dt \cdot dW = 0$.

(b). $d(XY)$, $f(t, x, y) = xy$, $f_t = 0$, $f_x = y$, $f_y = x$

$f_{xx} = 0$, $f_{xy} = 1$, $f_{yy} = 0$

$$dxy = df(t, x, y) = xdy + ydx + dxdy.$$

3. Geometric Brownian Motions

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \le t \le T$$

Set

$$D(t) = exp\left(-\int_0^t R(s)ds\right)$$

R constant,

$D(t) = exp(-Rt)$.

(a) Derive a formula for $S(t)$ $d(\log S)$

(b) Derive a formula for $d(D(t)S(t))$ by Ito's product rule. $dD = R(t)dt$

(c) Derive a formula for $d(D(t)S(t))$ by Ito's formula (Exercise 5.1 [Shreve]).
Hint: Consider $f(x) = S(0)e^x$ and set

$d(DS) = DdS + SdD + dDdS$.

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)$$

(d) Show that S is log-normally distributed. i.e., show that $log(S)$ is normally distributed.

$$d(DS) = \sigma DS\left(\frac{\alpha - R}{\sigma}dt + dW\right).$$

$\theta = \frac{\alpha - R}{\sigma}$ risk.

$d\tilde{W} = \theta dt + dW$.

$P \to \tilde{P}$

$\tilde{P} = P \cdot exp\left(-\int_0^t \theta dW - \frac{1}{2}\int_0^t \theta^2 du\right)$.

$d\tilde{W}$ martingale under $\tilde{P}$.

$dX = rXdt + \Delta(\alpha - r)Sdt + \Delta\sigma S dW.$

avg return　　risk　　vol.

4. Black-Scholes-Merton Equation

Let $c(t,x)$ denote the value of an option at time t with current price $S(t) = x$. A profolio $X(t)$ with hedging strategy $\Delta(t)$ should satisfy

porfolio　　payoff　　$N(①, ②^2)$　　$N(①, ②^2)$

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t,x))$$

$① dt + ② dW = ① dt + ② dW$

Use Ito's formula to compute both sides to get

$\int (\ \ ) dt + \int (\ \ ) dW$　　$E(\ \ ) dW = 0$

delta

$$\begin{cases} \Delta(t) = c_x \\ rc = c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} \end{cases}$$

theta

$\begin{cases} ① = ① \\ ② = ② \end{cases}$

For European call options, we have

(1)
$$c(t,0) = 0$$

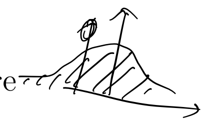
(2)
$$c(T,x) = max(S(T) - K, 0)$$

$\max\left(\int_0^T S(T) - K, 0\right)$ for Asian call

We solve the equation with the boundary conditions above to get

(3)
$$c(t,x) = xN(d+) - Ke^{-rt}N(d-),$$

cdf　　cdf

where 

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}(T-t)\right)\right)$$

$t \approx 0$

The problems about the BSM equations could be extremely difficult. For a reference, you may take a look at the problem as an excerpt of Exercise 4.9 [Shreve]

(a) Show that for $x > K$, $\lim_{t\to T^-} d_{\pm} = \infty$, but for $0 < x < K$, $\lim_{t\to T^-} d_{\pm} = -\infty$

(b) Show that for $0 \le t < T$, $\lim_{x\to 0^+} d_{\pm} = -\infty$ → $\lim_{x\to 0^+} c(t,x) = 0$

(c) Show that for $0 \le t < T$, $\lim_{x\to\infty} d_{\pm} = \infty$.　　$\lim_{t\to T^-} c(t,x) = \max(S(t) - K, 0)$

(d) Use (c) to verify　　(3) holds for $x \to \infty$.

$$lim_{x\to\infty}\left(c(t,x) - (x - e^{-r(T-t)K})\right) = 0$$

.

Hint: In this verification, you will need to show that

$$\lim_{x\to\inf}\frac{N(d_+) - 1}{x^{-1}}.$$

Use the L'Hopital's rule and the fact

$$x = Kexp\left(\sigma\sqrt{T-t}d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right)\right)$$

(a). $d_{\pm} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{X}{K} + r\right) \pm \frac{\sigma}{2}\sqrt{T-t}$　$\to \infty$.　$\to -\infty$

$\to 0$　　$\log\frac{X}{K} > 0$ if $X > K$　　$\to 0$

$t \to T^-$