

Mathematics of Finance

Exercises 1 Brownian Motions

- **Change in Office Hours** - 3.30 - 5pm on Mondays at CIWW 805 (in person only), or by appointment. The office hours on Wednesdays are cancelled.
- **Recommended Textbook** - Stochastic Calculus for Finance, Vol II, by S.E. Shreve; Springer Verlag.
- **Problems on the handout** are drawn from or inspired by the exercises in the book.

1. (Exercise 4.7 [Shreve] Calculations on Brownian Motions)

- (a) Compute dW^4 and then write W^4 as the sum of an ordinary (Lebesgue) integral.
- (b) Take expectations on both sides to derive the formula $\mathbb{E}W^4(T) = 3T^2$.
- (c) Deduce a formula for $\mathbb{E}W^6$.

2. (Exercise 4.19 [Shreve]) Let $W(t)$ be a Brownian motion and define

$$B(t) = \int_0^t \text{sign}(W(s))dW(s),$$

where

$$\text{sign}(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

- (a) Show that $(dB(t))^2 = dt$. Hence $B(t)$ is a Brownian motion by Levy's theorem.
 - (b) Show the *Itô's product rule* $d(XY) = XdY + YdX + dXdY$ for stochastic process $X(t), Y(t)$.
 - (c) Use (b) to compute $d(B(t)W(t))$. Conclude that $B(t)$ and $W(t)$ are uncorrelated normal random variables by showing $\mathbb{E}(B(t)W(t)) = 0$.
 - (d) Compute $dW^2(t)$ and conclude that $B(t)$ and $W(t)$ are not independent by showing $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t)$. Why does this happen to uncorrelated normal variables?
3. (Geometric Brownian Motions) Assume a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

- (a) Apply the Itô's lemma to solve for S .
- (b) Compute $d(S^p(t))$.

(Exercise 4.18 [Shreve]) Let X denote the value of an investor's portfolio with a hedging strategy of $\Delta(t)$.

- (c) Find dX .

Denote $\theta = (\alpha - r)/\sigma$ as the *market price of risk*, where r denotes the interest rate. Define the *state price density process* as $\zeta(t) = \exp\{-\theta W(t) - (r + \theta^2/2)t\}$.

- (d) Find $d\zeta$. Hint: use two different ways to express $d(e^{rt}\zeta)$
- (e) Show that $\zeta(t)X(t)$ is a martingale. (i.e. $d(\zeta(t)X(t))$ has no dt -terms).

From (c), the *present value* at $t = 0$ of the random payment $V(T)$ at $t = T$ is $X(0) = \mathbb{E}(\zeta(T)V(T))$. Hence it is valid to call $\zeta(t)$ the *state price density process*.

4. (Exercises 4.9-4.11 [Shreve] Black-Scholes-Merton Equation) For a European call with mature time T and strike price K , the BSM price at time t is

$$c(t, x) = xN(d_+) - Ke^{-r(T-t)}N(d_-),$$

where

$$d_{\pm} = \frac{1}{\sigma_1 \sqrt{r}} \left(\log \frac{x}{K} + (r \pm \frac{1}{2} \sigma_1^2) r \right),$$

However, the underlying asset is indeed a geometric Brownian motion with volatility

$$\sigma_2 > \sigma_1 : dS(t) = \alpha S(t)dt + \sigma_2 S(t)dW(t).$$

We set up a portfolio with value denoted by $X(t)$.

We remove cash from this portfolio at a rate $(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2 > 0$. Hence,

$$dX = dc - c_x dS + r(X - c + S c_x)dt - (\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2$$

(a) Show that $dX = rXdt$.

(b) Write out the Itô's formula for $d(e^{-rt}X(t))$. Deduce $dX = 0$.

This implies the existence of an arbitrage opportunity.

5. (Exercise 4.20 [Shreve] Local Time) The Itô's Lemma in differential form says that

$$df(x, t) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx.$$

Plug in $x = W(t)$ to get

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2} f''(W(t))dt. \quad (1)$$

(a) Let $K > 0$ a constant, and define $f(x) = \max(x - K, 0)$. Compute $f'(x), f''(x)$. Be careful about the points when either differential is not defined.

(b) Show that Equation 1 does not hold for $f(x) = \max(x - K, 0)$. Hint: Consider taking expected values and integrals on both sides.

To get some idea of what is going on here, we define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ by

$$f_n(x) = \begin{cases} 0 & x \leq K_{n-} \\ \frac{n}{2}(x - K)^2 + \frac{1}{2}(x - K) + \frac{1}{8n} & K_{n-} \leq x \leq K_{n+} \\ x - K & x \geq K_{n+} \end{cases}$$

where $K_{n-} = K - 1/(2n), K_{n+} = K + 1/(2n)$.

(c) Show that

$$\lim_{n \rightarrow \infty} f_n(x) = \max(x - K, 0),$$

and that

$$\lim_{n \rightarrow \infty} f'_n(x) = \begin{cases} 0 & x < K \\ 1/2 & x = K \\ 1 & x > K. \end{cases}$$

The value of $\lim_{n \rightarrow \infty} f'_n(x)$ at a single point will not matter when we integrate. We are constructing a continuous function $f_n(x)$ and $f'_n(x)$ is defined everywhere. Note further that $f''_n(x)$ is defined for $x \in \mathbb{R} \setminus \{K_+, K_-\}$, and $|f''_n(x)|$ is bounded above by n . Hence, the Itô's Lemma applies to the function f_n because the integrals are well defined.

1. (Exercise 4.7 [Shreve] Calculations on Brownian Motions)

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- (c) Deduce a formula for $\mathbb{E}W^6$.

(a). Itô's formula.

$$df(t, x) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx.$$

$$f(t, x) = x^4, \quad f_t = 0, \quad f_x = 4x^3, \quad f_{xx} = 12x^2.$$

$$dW^4 = df(t, w) = (0 \cdot dt + 4x^3 dx + \frac{1}{2} 12x^2 dx dx) \Big|_w.$$

$$= 4w^3 dw + 6w^2 dw dw$$

$$= 4w^3 dw + 6w^2 dt$$

$$W^4 = 4 \int w^3 dw + 6 \int w^2 dt$$

$$\mathbb{E}W^4 = 6 \cdot \mathbb{E} \int_0^T w^2 dt$$

$$= 6 \cdot \int_0^T \mathbb{E}W^2 dt$$

$$= 6 \int_0^T t dt.$$

$$\underbrace{d(w^2) = 2w dw + dt}_{\mathbb{E}dw^2 = dt}$$

$$\mathbb{E}dw^2 = dt.$$

$$dw \sim N(0, dt)$$

$$dt = \text{Var}(dw) = \mathbb{E}\{dw\}^2 - (\mathbb{E}dw)^2$$

$$\rightarrow \underline{dw \cdot dw = dt}$$

$$\mathbb{E}W^4(T) = 6 \int_0^T t dt = 6 \cdot \left[\frac{t^2}{2} \right]_0^T = 6 \cdot \frac{T^2}{2} = 3T^2$$

$$\mathbb{E}W^6 \quad dw^6 = 6w^5 dw + 15w^4 dt$$

$$W^6 = \int_0^T \dots dw + \int_0^T 15w^4 dt.$$

$$\mathbb{E}W^6 = \int_0^T \mathbb{E}15w^4 dt$$

$$= \int_0^T 15 \cdot 3t^2 dt.$$

$$\mathbb{E}W^6(T) = 15T^3$$

2. (Exercise 4.19 [Shreve]) Let $W(t)$ be a Brownian motion and define

$$\Rightarrow B(t) = \int_0^t \text{sign}(W(s)) dW(s),$$

where

$$\text{sign}(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

- (a) Show that $(dB(t))^2 = dt$. Hence $B(t)$ is a Brownian motion by Levy's theorem.
 (b) Show the Itô's product rule $d(XY) = XdY + YdX + dXdY$ for stochastic process $X(t), Y(t)$.
 (c) Use (b) to compute $d(B(t)W(t))$. Conclude that $B(t)$ and $W(t)$ are uncorrelated normal random variables by showing $\mathbb{E}(B(t)W(t)) = 0$.
 (d) Compute $dW^2(t)$ and conclude that $B(t)$ and $W(t)$ are not independent by showing $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t)$. Why does this happen to uncorrelated normal variables?

(a) $dB(t) = \text{sign}(W(t)) \cdot dW(t) \quad dW \cdot dW = dt$

$$(dB(t))^2 = 1 \cdot dW(t) \cdot dW(t) = dt$$

(b). $d(XY) = XdY + YdX + dXdY$

$$df(t, x, y) = \cancel{f_t dt} + \cancel{f_x dx} + \cancel{f_y dy} + \cancel{\frac{1}{2} f_{xx} dx dx} + \cancel{f_{xy} dx dy} + \cancel{\frac{1}{2} f_{yy} dy dy}$$

$$f(t, x, y) = xy \quad f_t = 0, f_x = y, f_y = x, \\ f_{xx} = 0, f_{yy} = 0, f_{xy} = 1.$$

$$dxy = df(t, x, y) = ydx + xdy + dx dy.$$

(c). $d(BW) = B dW + W dB + dB dW$
 $= \left(\int_0^t \text{sign}(W(s)) dW(s) \right) dW + W \cdot \text{sign}(W) dW + \text{sign}(W) dW \cdot dW$

$$BW = \int_0^t \left(\int_0^s \text{sign}(W(u)) dW(u) \right) dW + \int_0^t W \cdot \text{sign}(W) dW + \int_0^t \text{sign}(W) dt$$

$$\mathbb{E}Bw = \int (\underbrace{\mathbb{E} \int_0^t \text{sign}(w) dw}_{=0}) dw + \int \mathbb{E} \text{sign}(w) dt$$

$$= 0$$

$$\text{sign}(w) = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$$\mathbb{E} \text{sign}(w) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$$

$\mathbb{E}Bw = 0 \rightarrow B \text{ \& } w \text{ uncorrelated.}$

$A, B \text{ independent} \rightarrow A, B \text{ uncorrelated.}$ Always holds.

$A, B \text{ uncorrelated} \not\Rightarrow A, B \text{ independent.}$

Exception

A, B Jointly Normal
uncorrelated $\rightarrow A, B \text{ independent.}$

$$A, B \sim N(\mu, \Sigma)$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$A, B \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$$

$$\Rightarrow \begin{cases} A \sim N(\mu_1, \sigma_1^2) \\ B \sim N(\mu_2, \sigma_2^2) \end{cases}$$

$$\rho = 0$$

A, B normal
uncorrelated $\rightarrow A, B \text{ independent.}$

$$(d). \quad \mathbb{E}(BW^2) \neq \underline{\mathbb{E}B} \cdot \mathbb{E}W^2 = 0. \quad \mathbb{E} \int dw$$

$$\underline{d(w^2)} = 2w dw + \underbrace{(dw)^2}_{\frac{1}{dt} dt}.$$

$$d(BW^2) = B \underline{d(w^2)} + w^2 \underline{dB} + dB \cdot d(w^2) \quad \rightarrow$$

$$= - \underline{dw} + w \cdot \text{sign}(w) dt$$

$$BW^2 = \int - \underline{dw} + \int \underline{w \cdot \text{sign}(w) dt}$$

$$\mathbb{E}BW^2 = \int \mathbb{E} \cdot \underline{w \cdot \text{sign}(w) dt} > 0.$$

$$\quad \quad \quad \parallel > 0 \quad w \neq 0.$$

$$\quad \quad \quad |w|$$

$$\underline{\mathbb{E}BW^2} \neq \mathbb{E}B \cdot \mathbb{E}W^2 = 0$$

$$> 0$$

B, W not independent

3. (Geometric Brownian Motions) Assume a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

(a) Apply the Itô's lemma to solve for S . *Hint: $d(\log S)$.*

(b) Compute $d(S^p(t))$.

(Exercise 4.18 [Shreve]) Let X denote the value of an investor's portfolio with a hedging strategy of $\Delta(t)$.

(c) Find dX .

Denote $\theta = (\alpha - r)/\sigma$ as the *market price of risk*, where r denotes the interest rate. Define the *state price density process* as $\zeta(t) = \exp\{-\theta W(t) - (r + \theta^2/2)t\}$.

(d) Find $d\zeta$. Hint: use two different ways to express $d(e^{rt}\zeta)$

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From (c), the *present value* at $t = 0$ of the random payment $V(T)$ at $t = T$ is $X(0) = \mathbb{E}(\zeta(T)V(T))$. Hence it is valid to call $\zeta(t)$ the *state price density process*.

$$\begin{aligned} (a) \quad d \log S &= \frac{1}{S} dS + \frac{1}{2} \left(-\frac{1}{S^2}\right) dS dS \\ &= \sigma dW + \left(\alpha - \frac{1}{2}\sigma^2\right) dt \\ \log S &= \int \sigma dW + \left(\alpha - \frac{1}{2}\sigma^2\right) dt \\ S(T) &= S(0) \cdot \exp\left(\int_0^T \sigma(t) dW(t) + \int_0^T \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt\right) \\ &\text{if } \alpha, \sigma \text{ constants} \\ \rightarrow S(T) &= S(0) \cdot \exp\left(\sigma W + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right) \end{aligned}$$

$$\begin{aligned} (b). \quad d(S^p(t)) &= p S^{p-1} dS + \frac{1}{2} p(p-1) S^{p-2} dS dS \\ &= p S^p \left(\sigma dW + \left(\alpha + \frac{p-1}{2}\sigma^2\right) dt\right) \end{aligned}$$

$$(c). \quad dX = rX dt + \Delta(\alpha - r) S dt + \Delta \sigma S dW$$

$$(d). \quad \theta = \frac{\alpha - r}{\sigma} \quad \zeta(t) = \exp\left\{-\theta W - \left(r + \frac{\theta^2}{2}\right)t\right\}$$

$$\begin{aligned} d(e^{rt} \zeta(t)) &= d\left(\exp\left(-\theta W - \frac{\theta^2}{2}t\right)\right) \\ &= (-\theta) \cdot \exp\left(-\theta W - \frac{\theta^2}{2}t\right) dW \\ &= -\theta e^{rt} \zeta(t) dW \end{aligned}$$

$$d(e^{rt} Z(t)) = re^{rt} \cdot Z dt + e^{rt} \cdot dZ$$

$$f(t, x) = e^{rt} \cdot x$$

$$f_t = re^{rt} \cdot x$$

$$f_x = e^{rt}$$

$$f_{xx} = 0$$

$$-\theta e^{rt} Z(t) dW = re^{rt} Z dt + e^{rt} dZ$$

$$dZ = -\theta Z dW - rZ dt$$

$$(e) \quad d(Zx) = Z dx + x dZ + dx dZ$$

$$= (\Delta \sigma S - \theta X) Z dW \quad \xrightarrow{dt}$$

$$E(Zx | F_0, F_1, \dots, F_t) = Z(t)X(t) \quad \text{martingale.}$$

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We set up a portfolio with value denoted by $X(t)$.

We remove cash from this portfolio at a rate $(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2 > 0$. Hence,

$$dX = dc - c_x dS + r(X - c + S c_x)dt - (\sigma_2^2 - \sigma_1^2)S^2 c_{xx}/2$$

(a) Show that $dX = rXdt$.

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This implies the existence of an arbitrage opportunity.

$$(a) \quad dc = c_t dt + c_x dS + \frac{1}{2} c_{xx} dS dS \quad (It\hat{o})$$

$$= \left(rSc_x + \frac{1}{2} \sigma_1^2 S^2 c_{xx} - rc \right) dt + c_x (\alpha S dt + \sigma_2 S dW) + \frac{1}{2} c_{xx} \sigma_2^2 S^2 dt$$

$$(\text{plug in } c_t = rSc_x + \frac{1}{2} \sigma_1^2 S^2 c_{xx} - rc, dS = \alpha S dt + \sigma_2 S dW) \\ \uparrow \\ \text{BSM.}$$

$$\begin{aligned} \text{Now, } dX &= dc - c_x dS + r(X - c + S c_x)dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx} \\ &= \left(\underbrace{rSc_x + \frac{1}{2} \sigma_1^2 S^2 c_{xx} - rc}_{\text{wavy}} \right) dt + \underbrace{c_x (\alpha S dt + \sigma_2 S dW)}_{\text{wavy}} + \underbrace{\frac{1}{2} c_{xx} \sigma_2^2 S^2 dt}_{\text{wavy}} \\ &\quad - \underbrace{c_x dS}_{\text{wavy}} + \underbrace{r(X - c + S c_x)dt}_{\text{wavy}} - \underbrace{\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}}_{\text{wavy}} \\ &= rXdt \end{aligned}$$

(terms with the same underlined patterns are canceled.)

$$\begin{aligned}
 (b) \quad d(e^{-rt} X) &= -re^{-rt} X dt + e^{-rt} \underline{dX} \\
 &= -re^{-rt} X dt + e^{-rt} \underline{rX dt}
 \end{aligned}$$

(plug in $dX = rX dt$)

$$= 0$$

$\Rightarrow e^{-rt} X(t)$ is a constant.

$$\Rightarrow e^{-rt} X(t) = e^{-r \cdot 0} X(0) = 0$$

$$\Rightarrow X(t) = 0.$$

$$\Rightarrow dX = 0$$

$$Ef(w) = \mathbb{E} \int df(w) = \mathbb{E} \int \frac{1}{2} f''(w) dt = 0.$$

5. (Exercise 4.20 [Shreve] Local Time) The Itô's Lemma in differential form says that

$$df(x, t) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx. \quad f_1 \dots f_n \rightarrow f$$

Plug in $x = W(t)$ to get

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- (b) Show that Equation 1 does not hold for $f(x) = \max(x - K, 0)$. Hint: Consider taking expected values and integrals on both sides.

To get some idea of what is going on here, we define a sequence of functions $\{f_n\}_{n=1}^\infty$ by

$$f_n(x) = \begin{cases} 0 & x \leq K_{n-} \\ \frac{n}{2}(x - K)^2 + \frac{1}{2}(x - K) + \frac{1}{8n} & K_{n-} \leq x \leq K_{n+} \\ x - K & x \geq K_{n+} \end{cases}$$

where $K_{n-} = K - 1/(2n), K_{n+} = K + 1/(2n)$.

(c) Show that

$$\lim_{n \rightarrow \infty} f_n(x) = \max(x - K, 0),$$

and that

$$\lim_{n \rightarrow \infty} f'_n(x) = \begin{cases} 0 & x < K \\ 1/2 & x = K \\ 1 & x > K. \end{cases}$$

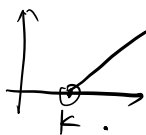
$$f_1, f_2 \dots \rightarrow f$$

$$f'_1 \dots f'_n \rightarrow f'_0$$

$$f''_1 \dots f''_n \rightarrow f''$$

The value of $\lim_{n \rightarrow \infty} f'_n(x)$ at a single point will not matter when we integrate. We are constructing a continuous function $f_n(x)$ and $f'_n(x)$ is defined everywhere. Note further that $f''_n(x)$ is defined for $x \in \mathbb{R} \setminus \{K_+, K_-\}$, and $|f''_n(x)|$ is bounded above by n . Hence, the Itô's Lemma applies to the function f_n because the integrals are well defined.

1). $f(x) = \max(x - K, 0)$.



$$f'(x) = \begin{cases} 1 & x > K \\ 0 & x < K \\ \text{not def.} & x = K \end{cases}$$

$$f''(x) = \begin{cases} 0 & x \neq K \\ \text{not defined} & x = K \end{cases}$$

$$\Rightarrow f''(x) = 0$$

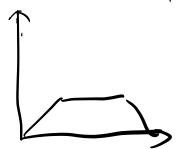
$$Ef(w) = \int_{-\infty}^{\infty} f(x) \cdot p(x) dx$$

↑
pdf

$$= \int_{-\infty}^K f(x) \cdot p(x) dx + \int_K^{+\infty} f(x) \cdot p(x) dx$$

$$= \int_{-\infty}^K 0 \cdot p(x) dx + \int_K^{+\infty} (x - K) \cdot \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx > 0$$

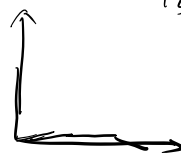
Idea (c). f_1''



f_1, \dots, f_n



f_2''



f_3''

$\rightarrow \dots$

f''



$$f = \max(x - K, 0).$$

$$\mathbb{E}g(x) = \int g(x) \cdot \underset{\substack{\uparrow \\ \text{pdf}}}{f(x)} dx.$$

$$f_t = \frac{\partial f}{\partial t}$$

$$f_x = \frac{\partial f}{\partial x}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$