TP4 UNSTEADY DIFFUSION

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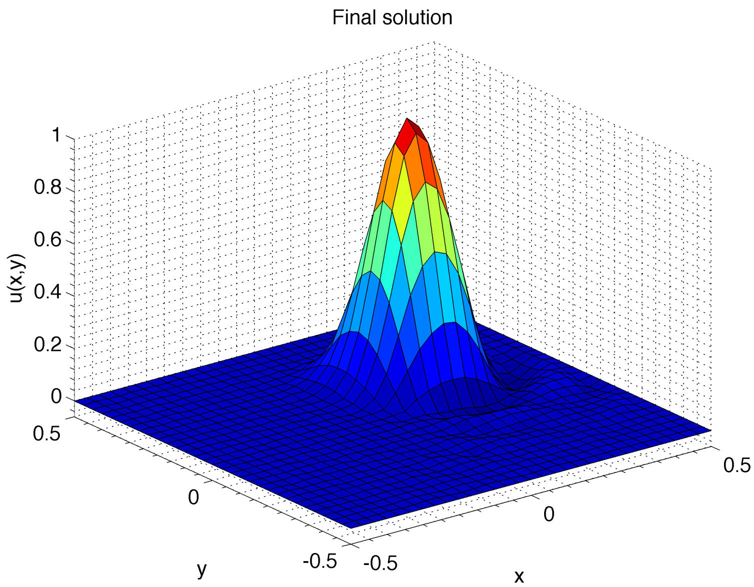

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1 Introduction

In this TP, we will solve the diffusion equation.
Consider the following equation in 1D:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \quad 0 \leq x \leq 1;\ t \geq 0$$

with the initial and boundary conditions as:

$$u(0,t) = u(1,t) = 1 \;\; u(x,0) = \sin(\pi x) + 1$$

2 Solve the problem using a collocated mesh and finite difference

First, we will define the grid for this problem. This system is defined on the spatial interval L, which discrete into N parts.

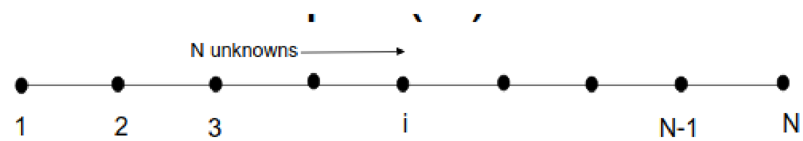

Figure 1:

Next, we will use the Second order finite differences:

$$\frac{\partial^2}{\partial x^2} f_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

to obtain the central difference approximation of the second derivative of u.
This form can be written as a matrix B1 multiply by a vector with source of Western boundary (SbcW) condition and source of Eastern boundary condition (SbcE).
So the second derivative of u can be written as:

$$\frac{\partial^2}{\partial x^2} u = \begin{bmatrix} -2 & 1 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & \cdots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & & 1 & -2 & 1 \\ 0 & \cdots & & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \\ \vdots \\ \\ u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} \frac{\Phi_0}{\Delta x^2} \\ 0 \\ \\ \vdots \\ \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \\ \vdots \\ \\ 0 \\ \frac{\Phi_L}{\Delta x^2} \end{bmatrix}$$

In this time, the matrix B1 is :

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & \cdots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & & 1 & -2 & 1 \\ 0 & \cdots & & & 0 & 1 & -2 \end{bmatrix}$$

The second derivative of u_1 and u_{N-1} are:

$$\frac{\partial^2}{\partial x^2} u_1 = \frac{u_2 - 2u_1 + \Phi_0}{\Delta x^2} = \frac{u_2 - 2u_1}{\Delta x^2} + \frac{\Phi_0}{\Delta x^2} + 0(\Delta x^2)$$

$$\frac{\partial^2}{\partial x^2} u_{N-1} = \frac{\Phi_L - 2u_{N-1} + u_{N-2}}{\Delta x^2} = \frac{2u_{N-1} + u_{N-2}}{\Delta x^2} + \frac{\Phi_L}{\Delta x^2} + 0(\Delta x^2)$$

This method is in order 2. So, the maximum eigenvalue is proportional to $\frac{1}{\Delta x^2}$ and the CFL is proportional to $\frac{\Delta t}{\Delta x^2}$.

Next step, we will use Crank-Nicolson scheme for time integration.

$$\frac{\partial u}{\partial t} = f(u)$$

So, the original equation can be written as:

$$\frac{u^{n-1} - u^n}{\Delta t} = \nu \frac{1}{2}[f^{n+1}(u) + f^n(u)]$$

We replace the $f^{n+1}(u) + f^n(u)$ by $\frac{\partial^2}{\partial x^2} u_{N+1} + \frac{\partial^2}{\partial x^2} u_N$, So:

$$\frac{u^{n-1} - u^n}{\Delta t} = \nu \frac{1}{2}[B1u^{n+1} + SbuW + SbcE + B1u^n + SbuW + SbcE]$$

Finally, we can get:

$$(I - \frac{\Delta t}{2} B1)u^{n+1} = \nu(I + \frac{\Delta t}{2} B1)u^n + \nu(SbcW + SbcE)\Delta t$$

We can solve this equation with the same method for $AX = b$ This equation is numerically stable if and only if the following condition is satisfied:

$$r = \frac{\nu \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

3 Solve the problem using a staggered mesh and finite difference

This time, we will define a new grid. This system is defined on the spatial interval L, which discrete into N parts, we choice the centre of every part.

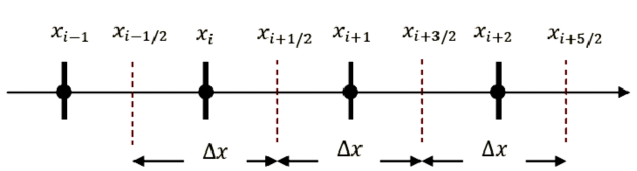

Figure 2:

Here, we can write the the Second order finite differences of u as:

$$\frac{\partial^2}{\partial x^2} u_{i+1/2} = \frac{u_{i+3/2} - 2u_{i+1/2} + u_{i-1/2}}{\Delta x^2}$$

This form can be written as a matrix B2 multiply by a vector with source of Western boundary (SbcW) condition and source of Eastern boundary condition (SbcE).(we set a,b,c as three unknown number)

$$\frac{\partial^2}{\partial x^2} u_{x+1/2} = \left[\begin{array}{cccccccc} -b & -c & 0 & \cdots & & & & 0 \\ \hline 1 & -2 & 1 & \cdots & & & & 0 \\ 0 & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ 0 & \cdots & & & & 1 & -2 & 1 \\ \hline 0 & \cdots & & & & 0 & -c & -b \end{array}\right] \begin{bmatrix} u_{\frac{1}{2}} \\ u_{\frac{3}{2}} \\ \\ \vdots \\ \\ u_{N-\frac{3}{2}} \\ u_{N-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} -a\frac{\Phi_0}{\Delta x^2} \\ 0 \\ \\ \vdots \\ \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \\ \vdots \\ \\ 0 \\ -a\frac{\Phi_L}{\Delta x^2} \end{bmatrix}$$

In this time, :

$$\frac{\partial^2}{\partial x^2} u_{\frac{1}{2}} = \frac{-a\Phi_L - bu_{N-\frac{1}{2}} - cu_{N-\frac{3}{2}}}{\Delta x^2} + 0(\Delta x)$$

$$\frac{\partial^2}{\partial x^2} u_{N-\frac{1}{2}} = \frac{-a\Phi_0 - bu_{\frac{1}{2}} - cu_{\frac{3}{2}}}{\Delta x^2} + 0(\Delta x)$$

This method is in first order. We try to find the three coefficients a, b, c with Taylor's theorem:

$$af_0 = af_{1/2} - a\frac{\Delta x}{2} f'_{1/2} + a\frac{\Delta x^2}{8} f''_{1/2} - a\frac{\Delta x^3}{48} f'''_{1/2}$$

$$cf_{3/2} = cf_{1/2} + c\Delta x f'_{1/2} + c\frac{\Delta x^2}{2} f''_{1/2} + c\frac{\Delta x^3}{6} f'''_{1/2}$$

	$f_{1/2}$	$f'_{1/2}$	$f''_{1/2}$	$f'''_{1/2}$
$\Delta x^2 f"_{1/2}$	0	0	Δx^2	0
af_0	a	$-a\frac{\Delta x}{2}$	$a\frac{\Delta x^2}{8}$	$-a\frac{\Delta x^3}{48}$
$bf_{1/2}$	b	0	0	0
$cf_{3/2}$	c	$c\Delta x$	$c\frac{\Delta x^2}{2}$	$c\frac{\Delta x^3}{6}$

We can solve this system as :

$$\begin{cases} a + b + c = 0 \\ -\frac{a}{2}\Delta x + c\Delta x = 0 \\ \Delta x^2 + \frac{a}{8}\Delta x^2 + \frac{1}{2}c\Delta x^2 = 0 \end{cases}$$

$$\begin{cases} a = \frac{8}{3} \\ b = 4 \\ c = \frac{4}{3} \end{cases}$$

For this method, we should set the central part of the B2, it is a diagonal matrix:

$$\begin{bmatrix} 1 & -2 & 1 & \cdots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & & 1 & -2 & 1 \end{bmatrix}_{n-2,n}$$

Then, we add the first line and the last line of the matrix B2. We will use Crank-Nicolson scheme for time integration too, so the second part is the same as the first.

4 Solve the problem using a staggered mesh and finite volume

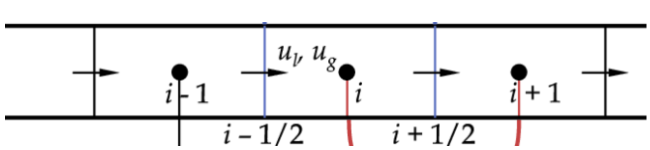

Figure 3:

Compared with the finite difference, it is based on the differential form of the equation, so that when the initial conditions are discontinued, an undesired unstable effect is generated.
Therefore, the key of the finite volume method is to discrete based on the integral form of the equation to approximate the true solution.
For any position at any time, the solution U_i^n on each grid point is the volume average on the x-axis interval $[x_{i-1/2}, x_{i+1/2}]$:

$$U_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n) dx$$

The value of the flow at the boundary of each x-axis interval $I_i = [x_{i-1/2}, x_{i+1/2}]$ is the average value over the time t interval:

$$F_{i-1/2}^{n+1/2} = \frac{1}{\Delta t} \int_{t_{i-n}}^{t_{n+1}} f(u(x_{i-1/2}, t)) dt$$

Then, update the solution U_i^{n+1} at the next time n + 1:

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} \left(F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2} \right)$$