Honest Causal Tree

Presenter: FU Zixuan & SONG Shuo

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Motivation

- Why study heterogeneous treatment effects?
- Research question:
    - Estimate heterogeneity by covariates or features.
    - Conduct inference about the magnitude of the differences in treatment effects across subsets of the population.
- Athey and Imbens (2016): a data-driven approach to partition the data.

Contribution:

- Provide valid CIs w/o restrictions on the number of covariates or the complexity of the DGP.
- Discover subpopulations with lower-than-average or higher-than-average treatment effects.

Estimand

- Object: CATE, $\tau(x) \equiv \mathbb{E}[Y_i(1) - Y_i(0)|X_i = x]$.
  - How **large** is the effect τ for a specific x?
  - How **different** is the effect across x?
- After having the estimates:
  - Improvement on **treatment design** itself.
  - Improvement of **assigment** of treatment.

On a side note

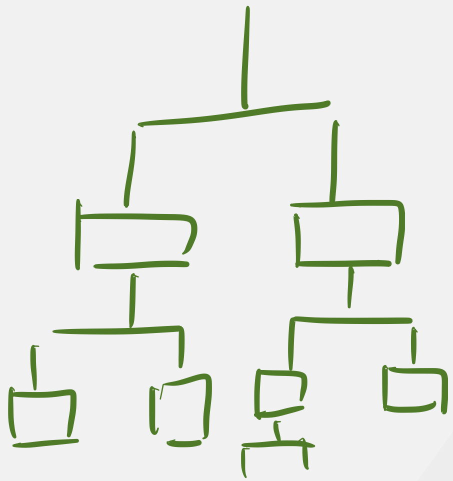
- Causal Trees: focuses on the heterogeneity of treatment effects. How different is subgroup l_1 from subgroup l_2? Which group is benefiting the most?
- Causal Forests: focuses on the precision. personalized treatment effect for each individual x_i. Can be used as input into estimation of lower dimentsioal objects, such as to estimate optimal policy.

Trees: an overview of regression trees

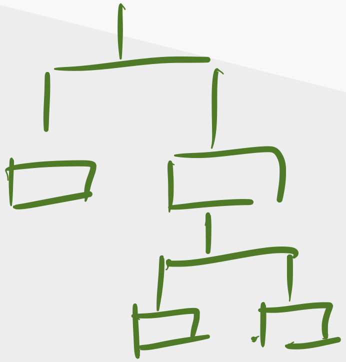
> “ The best time to plant a tree was 20 years ago. The second best time is now. ”

1. **Build a tree** (partition) π square based on data.
2. Given the tree, **construct an estimator** $\hat{\mu}$ by calculating the mean outcome in each leaf in the tree (box in the square).

Grow



Prune



Trees: a walkthrough of the Conventional

Sample splitting

1. Split the data into training $S^{tr,tr}$,and validation set $S^{tr,cv}$ and test set S^{te}.
2. Hold out the test set. This is not going to be used in following steps except the final evaluation.

Training

Testing

Tree growing

1. Grow the tree π_0 on the entire traning set S^{tr} fold by minimizing the MSE until the leaves contain a minimum number of samples.

$$\widehat{\mathrm{MSE}}(S^{tr}, S^{tr}) = \frac{1}{|S^{tr}|} \sum_{i \in S^{tr}} (Y_i - \underbrace{\hat{\mu}_\pi(X_i; S^{tr})}_{\text{Mean of the leaf}})^2$$

where

$$\hat{\mu}_\pi(x; S^{tr}) = \frac{1}{|S_l^{tr}|} \sum_{i \in S_l^{tr}} Y_i, \quad x \in l$$

Tree pruning

1. Get a set of effective α_k to prune the tree π_0.
2. For one α_k. Split the traning set into two parts: $S^{tr,tr}$ and $S^{tr,cv}$. Find the pruned tree π^* that minimizes the following.

$$C_{\alpha_k}(\pi') = \frac{1}{|S^{\boxed{tr,cv}}|} \sum_{i \in S^{\boxed{tr,cv}}} (Y_i - \hat{\mu}_{\pi'}(X_i; S^{\boxed{tr,tr}}))^2 + \alpha_k |\pi'|$$

3. Pick the α_k (along with its corresponding pruned tree π_{α_k}) that gives the smallest cost-complexity measure.

	terror	xerror.
α_1		
α_2		✓
α_3		

Final evaluation

1. Construct the final estimator $\hat{\mu}_{\pi^*}(x; S^{tr})$ based on the optimal tree $\pi^* = \pi_{\alpha_k}$
2. Evalute the estimator $\hat{\mu}_{\pi^*}$ on the hidden test set.

A conundrum

We shift our focus from predicting outcome Y_i to predicting effect τ_i. However, while Y_i is observed in the data, the τ_i is not!

Therefore, we need to adapt the evaluation criteria from $\widehat{\mathrm{MSE}}$ to $\widehat{\mathrm{EMSE}}$.

Adapt the criterion

S_1 evaluation | S_2 estimate

$$\widehat{\mathrm{MSE}}_\pi(\boxed{S_1}, \boxed{S_2}) = \frac{1}{|\boxed{S_1}|}\sum_{i\in S_1}\left(Y_i - \hat{\mu}_\pi(X_i; \boxed{S_2})\right)^2$$

To evaluate

To estimate

Expectation over S_2

$$\mathrm{MSE}_\pi(S_1) = \frac{1}{|S_1|}\sum_{i\in S_1}\mathbb{E}_{S_2}\left(Y_i - \hat{\mu}_\pi(X_i; S_2)\right)^2$$

because we want to evaluate μ but we only have $\hat{\mu}(\cdot\,; S_2)$

In the case of conventional,

$$\mathrm{MSE}_\pi = \mathbb{E}_S\left\{\frac{1}{|S|}\sum_{i\in S}\left(Y_i - \hat{\mu}_\pi(X_i; S)\right)^2\right\} = \mathbb{E}(Y_i^2) - 2\mathbb{E}(Y_i\hat{\mu}_\pi(X_i; S)) + \mathbb{E}(\hat{\mu}_\pi(X_i; S))^2$$

Expectation $\Longrightarrow$ over S_1

because now we don't observe Y_i $i \in S_1$

$$\mathrm{EMSE}_\pi = \mathbb{E}_{S_1,S_2}\left(\frac{1}{|S_1|}\sum_{i\in S_1}(Y_i - \hat{\mu}_\pi(X_i;S_2))^2\right)$$

$$= \mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i;S_2))\} - \mathbb{E}_{S_1}(\mu_\pi(X_i))^2 + \mathbb{E}_{S_1}(Y_i^2)$$

Why different set?

We want to make use the independence between sets.

First

The estimand we want to evaluate

$$\mathbb{E}\left\{(Y_i - \underbrace{\mu_\pi(X_i)} + \underbrace{\mu_\pi(X_i)} - \hat{\mu}_\pi(X_i; S_2))^2\right\}$$

$$= \mathbb{E}\left\{(Y_i - \mu_\pi(X_i))^2\right\} + \mathbb{E}\left\{(\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))^2\right\}$$

$$+ \mathbb{E}_{S_1, S_2}\left\{(Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))\right\}$$

The third term is zero because

$$\mathbb{E}_{S_1}\left\{(Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \underbrace{\mathbb{E}_{S_2}\hat{\mu}_\pi(X_i; S_2)}_{=\mu_\pi(X_i)})\right\} = 0$$

where we make use of the independence between S_1 and S_2!

The second term is $\mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i; S_2))\}$

The first term can be simplified to

$$\mathbb{E}_{S_1}\left\{(Y_i^2) - 2Y_i\mu_\pi(X_i) + \mu_\pi(X_i)^2\right\}$$

$$= \mathbb{E}_{S_1}\left\{Y_i^2 - 2\mathbb{E}(Y_i|X_i)\mu_\pi(X_i) + \mu_\pi(X_i)^2\right\} = \mathbb{E}_{S_1}\left\{Y_i^2 - \mu_\pi(X_i)^2\right\}$$

Putting it together

We estimate EMSE by the estimating the two components.

$$\mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i; S_2))\}, \mathbb{E}_{S_1}[\mu_\pi(X_i)^2]$$

Therefore, we can estimate the EMSE by

$$\widehat{\mathrm{EMSE}}_\pi(S^{tr}, S^{est})$$

Therfore, in the tree growing step, we replace

$$\widehat{\text{MSE}}_\pi(S^{tr}, S^{tr}) \quad \text{by} \quad \widehat{\text{EMSE}}_\pi(S^{est}, S^{tr})$$

Similarly in the tree pruning step, we replace

$$C_{\alpha_k}(\pi') = \widehat{\text{MSE}}_\pi(S^{tr,cv}, S^{tr,tr}) + \alpha_k|\pi'|$$

by

$$C_{\alpha_k}(\pi') = \widehat{\text{EMSE}}_\pi(S^{tr,cv}, S^{est}) + \alpha_k|\pi'|$$

The conundrum?

We adapt the criterion because we don't observe the treatment effect τ_i directly. But how does it solve the problem?

Then $\hat{E}\left(\hat{V}_{S_1}\left(\hat{\mu}^2(X_i; S_2)\right)\right) - \hat{E}\left(\mu^2(X_i; \Pi)\right)$

$$\widehat{\text{EMSE}}_\tau(S^{\text{tr}}, S^{\text{est}}, \Pi) \equiv \frac{1}{N_{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\tau}^2(X_i; S^{\text{tr}}, \Pi)$$

$$- \left(\frac{1}{N_{\text{tr}}} + \frac{1}{N_{\text{est}}}\right) \cdot \sum_{\ell \in \Pi} \left(\frac{S^{2,\text{tr}}_{\text{treat}}(\ell)}{p} + \frac{S^{2,\text{tr}}_{\text{control}}(\ell)}{1-p} \right)$$

$$\widehat{\mathrm{EMSE}}_\mu(S^{\mathrm{tr}}, N^{\mathrm{est}}, \Pi) \equiv \frac{1}{N_{\mathrm{tr}}} \sum_{i \in S^{\mathrm{tr}}} \hat{\mu}^2(X_i; S^{\mathrm{tr}}, \Pi)$$

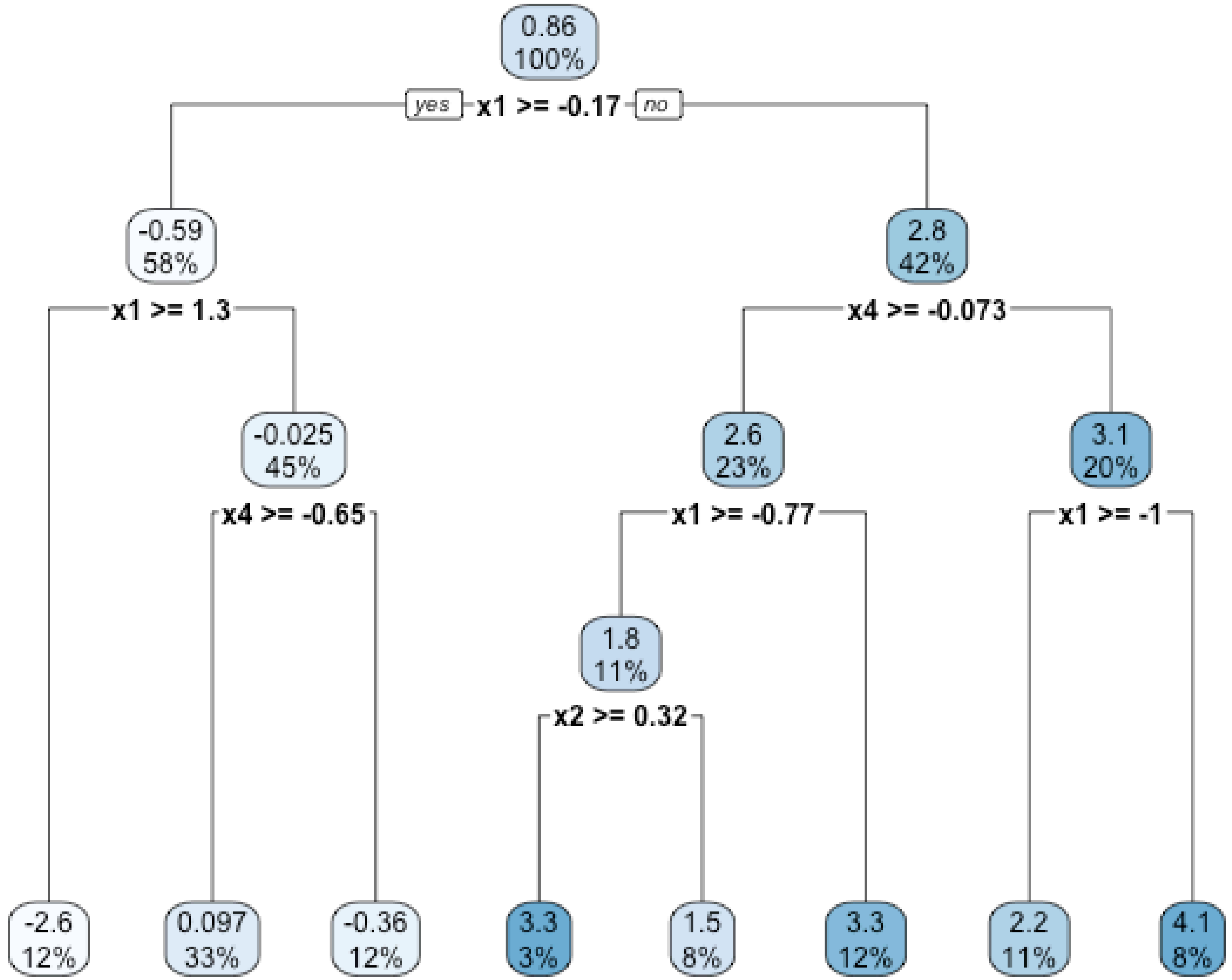
$$- \left(\frac{1}{N_{\mathrm{tr}}} + \frac{1}{N_{\mathrm{est}}} \right) \cdot \sum_{\ell \in \Pi} S^2_{\mathrm{tr}} \left(\ell(x; \Pi) \right)$$

Taking stock

1. Tree splitting **criterion** for treatment effect estimation. $\widehat{EMSE}$
2. Honest tree growing and estimation thus **valid inference** procedure.

Implementation

```
honestTree <- honest.causalTree(y ~ x1 + x2 + x3 + x4,
    data = train_data,
    treatment = train_data$treatment,
    est_data = est_data,
    est_treatment = est_data$treatment,
    split.Rule = "CT", split.Honest = T, HonestSampleSize = nrow(est_data),
    split.Bucket = T, cv.option = "CT",
    cv.Honest = T
)
opcp <- honestTree$cptable[, 1][which.min(honestTree$cptable[, 4])] # pruning parameter
opTree <- prune(honestTree, opcp) # prune the tree
rpart.plot(opTree) # plot the tree
```


0.86
100%
yes
x1 >= -0.17
no
-0.59
58%
x1 >= 1.3
2.8
42%
x4 >= -0.073
-0.025
45%
x4 >= -0.65
2.6
23%
x1 >= -0.77
3.1
20%
x1 >= -1
1.8
11%
x2 >= 0.32
-2.6
12%
0.097
33%
-0.36
12%
3.3
3%
1.5
8%
3.3
12%
2.2
11%
4.1
8%

Other methods

1. Transformed Outcome Trees: create a pseudo outcome $Y_i^* = Y_i \times \frac{W_i - p}{p(1-p)}$ that encodes the treatment effect, and then run a regular regression tree.
2. Fit-Based Trees: grow a tree by looking at improvement in the outcome fit when allowing treatment effects at the leaves.
3. Squared T-Statistic Trees: split whenever the difference in leaf means is statistically significant.

Conclusion

1. An "honest" data-driven approach.
2. Bridging the gap between predictive machine learning and causal inference.
3. Example.

Subsequent work

1. Inference.
2. Causal Forest
3. Generalized Forest: generalized splitting rule (the gradient of the estimating moment condition).

Thanks!