



# Honest Causal Tree

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# Preliminaries

- Object:  $\tau(x)$ .
- Goal:
  - How **large** is the effect  $\tau$  for a specific  $x$ ?
  - How **different** is the effect across  $x$ ?
- Use:
  - Improvement on treatment.
  - Improvement of assignment of treatment.

## On a side note

- Causal Trees: focuses on the heterogeneity of treatment effects. How different is subgroup  $l_1$  from subgroup  $l_2$ ? Which group is benefiting the most?
- Causal Forests: focuses on the precision. personalized treatment effect for each individual  $x_i$ . Can be used as input into estimation of lower dimensional objects, such as to estimate optimal policy.

# Trees: an overview of regression trees

“ The best time to plant a tree was 20 years ago. The second best time is now. ”

1. **Build a tree** (partition)  $\pi$  square based on data.
2. Given the tree, **construct an estimator**  $\hat{\mu}$  by calculating the mean outcome in each leaf in the tree (box in the square).

# Trees: a walkthrough of the Conventional

## Sample splitting

1. Split the data into training  $S^{tr, tr}$ , and validation set  $S^{tr, cv}$  and test set  $S^{te}$ .
2. Hold out the test set. This is not going to be used in following steps except the final evaluation.

# Tree growing

1. Grow the tree  $\pi_0$  on the entire training set  $S^{tr}$  fold by minimizing the MSE until the leaves contain a minimum number of samples.

$$\widehat{\text{MSE}}(S^{tr}, S^{tr}) = \frac{1}{|S^{tr}|} \sum_{i \in S^{tr}} (Y_i - \hat{\mu}_{\pi}(X_i; S^{tr}))^2$$

where

$$\hat{\mu}_{\pi}(x; S^{tr}) = \frac{1}{|S_l^{tr}|} \sum_{i \in S_l^{tr}} Y_i, \quad x \in l$$

# Tree pruning

1. Get a set of effective  $\alpha_k$  to prune the tree  $\pi_0$ .
2. For one  $\alpha_k$ . Split the training set into two parts:  $S^{tr,tr}$  and  $S^{tr,cv}$ . Find the pruned tree  $\pi^*$  that minimizes the following.

$$C_{\alpha_k}(\pi') = \frac{1}{|S^{tr,cv}|} \sum_{i \in S^{tr,cv}} (Y_i - \hat{\mu}_{\pi'}(X_i; S^{tr,tr}))^2 + \alpha_k |\pi'|$$

3. Pick the  $\alpha_k$  (along with its corresponding pruned tree  $\pi_{\alpha_k}$ ) that gives the smallest cost-complexity measure.

# Final evaluation

1. Grow the tree again with the entire training set  $S^{tr}$  and the selected  $\alpha_k$  to get the final tree  $\pi^*$ . Construct the final estimator  $\hat{\mu}_{\pi^*}(x; S^{tr})$ .
2. Evaluate the estimator  $\hat{\mu}_{\pi^*}$  on the hidden test set.



# A conundrum

We shift our focus from predicting outcome  $Y_i$  to predicting effect  $\tau_i$ . However, while  $Y_i$  is observed in the data, the  $\tau_i$  is not!

Therefore, we need to adapt the evaluation criteria from  $\widehat{\text{MSE}}$  to  $\widehat{\text{EMSE}}$ .

# Adapt the criterion

$$\widehat{\text{MSE}}_{\pi}(S_1, S_2) = \frac{1}{|S_1|} \sum_{i \in S_1} (Y_i - \hat{\mu}_{\pi}(X_i; S_2))^2$$

$$\text{MSE}_{\pi}(S_1) = \frac{1}{|S_1|} \sum_{i \in S_1} \mathbb{E}_{S_2} (Y_i - \hat{\mu}_{\pi}(X_i; S_2))^2$$

In the case of conventional,

$$\text{MSE}_{\pi} = \mathbb{E}_S \left\{ \frac{1}{|S|} \sum_{i \in S} (Y_i - \hat{\mu}_{\pi}(X_i; S))^2 \right\} = \mathbb{E}(Y_i^2) - 2\mathbb{E}(Y_i \hat{\mu}_{\pi}(X_i; S)) + \mathbb{E}(\hat{\mu}_{\pi}(X_i; S))^2$$

$$\begin{aligned}\text{EMSE}_\pi &= \mathbb{E}_{S_1, S_2} \left( \frac{1}{|S_1|} \sum_{i \in S_1} (Y_i - \hat{\mu}_\pi(X_i; S_2))^2 \right) \\ &= \mathbb{V}_{S_1, S_2}(\hat{\mu}_\pi(X_i; S_2)) - \mathbb{E}_{S_1}(\mu_\pi(X_i))^2 + \mathbb{E}_{S_1}(Y_i^2)\end{aligned}$$

# Why different set?

We want to make use the independence between sets.

First

$$\begin{aligned} & \mathbb{E} \left\{ (Y_i - \mu_\pi(X_i) + \mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))^2 \right\} \\ &= \mathbb{E} \left\{ (Y_i - \mu_\pi(X_i))^2 \right\} + \mathbb{E} \left\{ (\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))^2 \right\} \\ & \quad + \mathbb{E}_{S_1, S_2} \left\{ (Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2)) \right\} \end{aligned}$$

The third term is zero because

$$\mathbb{E}_{S_1} \{ (Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \mathbb{E}_{S_2} \hat{\mu}_\pi(X_i; S_2)) \} = 0$$

**where we make use of the independence between  $S_1$  and  $S_2$ !**

The second term is  $\mathbb{E}_{S_2} \{ \mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i; S_2)) \}$

The first term can be simplified to

$$\begin{aligned} & \mathbb{E}_{S_1} \{ (Y_i^2) - 2Y_i\mu_\pi(X_i) + \mu_\pi(X_i)^2 \} \\ &= \mathbb{E}_{S_1} \{ Y_i^2 - 2\mathbb{E}(Y_i|X_i)\mu_\pi(X_i) + \mu_\pi(X_i)^2 \} = \mathbb{E}_{S_1} \{ Y_i^2 - \mu_\pi(X_i)^2 \} \end{aligned}$$

# Putting it together

We estimate EMSE by the estimating the two components.

$$\mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_{\pi}(X_i; S_2))\}, \mathbb{E}_{S_1}[\mu_{\pi}(X_i)^2]$$

Therefore, we can estimate the EMSE by

$$\widehat{\text{EMSE}}_{\pi}(S^{tr}, S^{est})$$

Therefore, in the tree growing step, we replace

$$\widehat{\text{MSE}}_{\pi}(S^{tr}, S^{tr}) \quad \text{by} \quad \widehat{\text{EMSE}}_{\pi}(S^{est}, S^{tr})$$

Similarly in the tree pruning step, we replace

$$C_{\alpha_k}(\pi') = \widehat{\text{MSE}}_{\pi}(S^{tr,cv}, S^{tr,tr}) + \alpha_k |\pi'|$$

by

$$C_{\alpha_k}(\pi') = \widehat{\text{EMSE}}_{\pi}(S^{tr,cv}, S^{est}) + \alpha_k |\pi'|$$

# The conundrum?

We adapt the criterion because we don't observe the treatment effect  $\tau_i$  directly. But how does it solve the problem?

Then

$$\widehat{\text{EMSE}}_{\tau}(S^{\text{tr}}, S^{\text{est}}, \Pi) \equiv \frac{1}{N_{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\tau}^2(X_i; S^{\text{tr}}, \Pi) \\ - \left( \frac{1}{N_{\text{tr}}} + \frac{1}{N_{\text{est}}} \right) \cdot \sum_{\ell \in \Pi} \left( \frac{S_{\text{treat}}^{2, \text{tr}}(\ell)}{p} + \frac{S_{\text{control}}^{2, \text{tr}}(\ell)}{1 - p} \right)$$



$$\widehat{\text{EMSE}}_{\mu}(S^{\text{tr}}, N^{\text{est}}, \Pi) \equiv \frac{1}{N_{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\mu}^2(X_i; S^{\text{tr}}, \Pi) \\ - \left( \frac{1}{N_{\text{tr}}} + \frac{1}{N_{\text{est}}} \right) \cdot \sum_{\ell \in \Pi} S_{\text{tr}}^2(\ell(x; \Pi))$$

# Other methods

1. Single trees (S-learner): does not split on treatment...
2. Two trees (T-learner): split on different features therefore can not compare...
3. X-learner
4. Transform the LHS variable from outcome into treatment effect
5. Other splitting criteria:
  1. split on the outcome
  2. split based on some  $T$  statistics.

