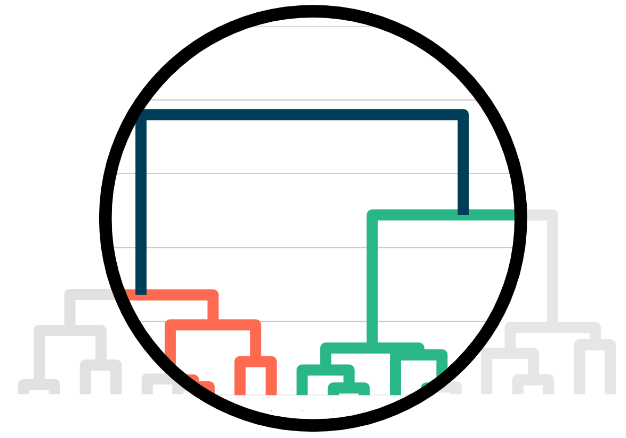

Honest Causal Tree

Presenter: FU Zixuan & SONG Shuo

Date: 2025-07-24

Preliminaries

- Object: $\tau(x)$.
- Goal:
  - How **large** is the effect τ for a specific x?
  - How **different** is the effect across x?
- Use:
  - Improvement on **treatment design** itself.
  - Improvement of **assigment** of treatment.

On a side note

- Causal Trees: focuses on the heterogeneity of treatment effects. How different is subgroup l_1 from subgroup l_2? Which group is benefiting the most?
- Causal Forests: focuses on the precision. personalized treatment effect for each individual x_i. Can be used as input into estimation of lower dimentsioal objects, such as to estimate optimal policy.

Trees: an overview of regression trees

> “ The best time to plant a tree was 20 years ago. The second best time is now. ”

1. **Build a tree** (partition) π square based on data.
2. Given the tree, **construct an estimator** $\hat{\mu}$ by calculating the mean outcome in each leaf in the tree (box in the square).

Trees: a walkthrough of the Conventional

Sample splitting

1. Split the data into training $S^{tr,tr}$,and validation set $S^{tr,cv}$ and test set S^{te}.
2. Hold out the test set. This is not going to be used in following steps except the final evaluation.

Tree growing

1. Grow the tree π_0 on the entire traning set S^{tr} fold by minimizing the MSE until the leaves contain a minimum number of samples.

$$\widehat{\text{MSE}}(S^{tr}, S^{tr}) = \frac{1}{|S^{tr}|} \sum_{i \in S^{tr}} (Y_i - \hat{\mu}_\pi(X_i; S^{tr}))^2$$

where

$$\hat{\mu}_\pi(x; S^{tr}) = \frac{1}{|S_l^{tr}|} \sum_{i \in S_l^{tr}} Y_i, \quad x \in l$$

Tree pruning

1. Get a set of effective α_k to prune the tree π_0.
2. For one α_k. Split the traning set into two parts: $S^{tr,tr}$ and $S^{tr,cv}$. Find the pruned tree π^* that minimizes the following.

$$C_{\alpha_k}(\pi') = \frac{1}{|S^{tr,cv}|} \sum_{i \in S^{tr,cv}} (Y_i - \hat{\mu}_{\pi'}(X_i; S^{tr,tr}))^2 + \alpha_k |\pi'|$$

3. Pick the α_k (along with its corresponding pruned tree π_{α_k}) that gives the smallest cost-complexity measure.

Final evaluation

1. Construct the final estimator $\hat{\mu}_{\pi^*}(x; S^{tr})$ based on the optimal tree $\pi^* = \pi_{\alpha_k}$
2. Evalute the estimator $\hat{\mu}_{\pi^*}$ on the hidden test set.

A conundrum

We shift our focus from predicting outcome Y_i to predicting effect τ_i. However, while Y_i is observed in the data, the τ_i is not!

Therefore, we need to adapt the evaluation criteria from $\widehat{\mathrm{MSE}}$ to $\widehat{\mathrm{EMSE}}$.

Adapt the criterion

$$\widehat{\mathrm{MSE}}_\pi(S_1, S_2) = \frac{1}{|S_1|}\sum_{i\in S_1}(Y_i - \hat{\mu}_\pi(X_i; S_2))^2$$

$$\mathrm{MSE}_\pi(S_1) = \frac{1}{|S_1|}\sum_{i\in S_1}\mathbb{E}_{S_2}(Y_i - \hat{\mu}_\pi(X_i; S_2))^2$$

In the case of conventional,

$$\mathrm{MSE}_\pi = \mathbb{E}_S\left\{\frac{1}{|S|}\sum_{i\in S}(Y_i - \hat{\mu}_\pi(X_i; S))^2\right\} = \mathbb{E}(Y_i^2) - 2\mathbb{E}(Y_i\hat{\mu}_\pi(X_i; S)) + \mathbb{E}(\hat{\mu}_\pi(X_i; S))^2$$

$$\mathrm{EMSE}_\pi = \mathbb{E}_{S_1,S_2}\left(\frac{1}{|S_1|}\sum_{i\in S_1}(Y_i - \hat{\mu}_\pi(X_i;S_2))^2\right)$$

$$= \mathbb{V}_{S_1,S_2}(\hat{\mu}_\pi(X_i;S_2)) - \mathbb{E}_{S_1}(\mu_\pi(X_i))^2 + \mathbb{E}_{S_1}(Y_i^2)$$

Why different set?

We want to make use the independence between sets.
First

$$\mathbb{E}\left\{(Y_i - \mu_\pi(X_i) + \mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))^2\right\}$$

$$= \mathbb{E}\left\{(Y_i - \mu_\pi(X_i))^2\right\} + \mathbb{E}\left\{(\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))^2\right\}$$

$$+\mathbb{E}_{S_1,S_2}\left\{(Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \hat{\mu}_\pi(X_i; S_2))\right\}$$

The third term is zero because

$$\mathbb{E}_{S_1}\left\{(Y_i - \mu_\pi(X_i))(\mu_\pi(X_i) - \mathbb{E}_{S_2}\hat{\mu}_\pi(X_i; S_2))\right\} = 0$$

where we make use of the independence between S_1 and S_2!

The second term is $\mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i; S_2))\}$

The first term can be simplified to

$$\mathbb{E}_{S_1}\left\{(Y_i^2) - 2Y_i\mu_\pi(X_i) + \mu_\pi(X_i)^2\right\}$$

$$= \mathbb{E}_{S_1}\left\{Y_i^2 - 2\mathbb{E}(Y_i|X_i)\mu_\pi(X_i) + \mu_\pi(X_i)^2\right\} = \mathbb{E}_{S_1}\left\{Y_i^2 - \mu_\pi(X_i)^2\right\}$$

Putting it together

We estimate EMSE by the estimating the two components.

$$\mathbb{E}_{S_2}\{\mathbb{V}_{S_1}(\hat{\mu}_\pi(X_i; S_2))\}, \mathbb{E}_{S_1}[\mu_\pi(X_i)^2]$$

Therefore, we can estimate the EMSE by

$$\widehat{\text{EMSE}}_\pi(S^{tr}, S^{est})$$

Therfore, in the tree growing step, we replace

$$\widehat{\mathrm{MSE}}_\pi(S^{tr}, S^{tr}) \quad \text{by} \quad \widehat{\mathrm{EMSE}}_\pi(S^{est}, S^{tr})$$

Similarly in the tree pruning step, we replace

$$C_{\alpha_k}(\pi') = \widehat{\mathrm{MSE}}_\pi(S^{tr,cv}, S^{tr,tr}) + \alpha_k|\pi'|$$

by

$$C_{\alpha_k}(\pi') = \widehat{\mathrm{EMSE}}_\pi(S^{tr,cv}, S^{est}) + \alpha_k|\pi'|$$

The conundrum?

We adapt the criterion because we don't observe the treatment effect τ_i directly. But how does it solve the problem?

Then

$$\widehat{\text{EMSE}}_\tau(S^{\text{tr}}, S^{\text{est}}, \Pi) \equiv \frac{1}{N_{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\tau}^2(X_i; S^{\text{tr}}, \Pi)$$

$$- \left(\frac{1}{N_{\text{tr}}} + \frac{1}{N_{\text{est}}} \right) \cdot \sum_{\ell \in \Pi} \left(\frac{S^{2,\text{tr}}_{\text{treat}}(\ell)}{p} + \frac{S^{2,\text{tr}}_{\text{control}}(\ell)}{1-p} \right)$$

$$\widehat{\mathrm{EMSE}}_\mu(S^{\mathrm{tr}}, N^{\mathrm{est}}, \Pi) \equiv \frac{1}{N_{\mathrm{tr}}} \sum_{i \in S^{\mathrm{tr}}} \hat{\mu}^2(X_i; S^{\mathrm{tr}}, \Pi)$$

$$-\left(\frac{1}{N_{\mathrm{tr}}} + \frac{1}{N_{\mathrm{est}}}\right) \cdot \sum_{\ell \in \Pi} S^2_{\mathrm{tr}}\left(\ell(x; \Pi)\right)$$

Contribution

1. Tree splitting **criterion** for treatment effect estimation.
2. Honest tree growing and estimation thus **valid inference** procedure.

Other methods

1. Single trees (S-learner, e.g. Imai and Ratkovic (25)): does not split on treatment...
2. Two trees (T-learner, e.g. Foster et al. (24)): split on different features therefore can not compare...
3. Transfrom the LHS variable from outcome into treatment effect

$$Y_i^* = Y_i/p_i * W_i - Y_i/(1-p_i) * (1-W_i) \quad E(Y_i^*|X_i = x) = \tau(x)$$

4. Other splitting criteria:
    1. split on the outcome,
    2. split based on some T statistics.

Subsequent work

1. Inference.
2. Causal Forest
3. Generalized Forest: generalized splitting rule (the gradient of the estimating moment condition).

Thanks!