

# **L'Hôpital's (Selection) Rule**

## **An Empirical Bayes Application to French Hospitals**

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## Literature: Measuring efficiency of individual units

- ▶ *Productivity/Efficiency*: Factories, Schools, Hospitals etc.
- ▶ *Ownership*: Public (Teaching, Ordinary) vs. Private (For profit, Non-profit).
- ▶ *Methodology*: Following Croiset and Gary-Bobo (2024), we use the *conditional input demand function*. The less input is needed to produce the same amount of output, the more efficient the hospital is. [▶ Reasons](#)

## Literature: Invidious decision

- ▶ *League table mentality*: Ranking & Selection.(Gu and Koenker, 2023)
- ▶ *Noisy estimates*: e.g. estimated fixed effect  $\hat{\theta}_i$ . (Chetty et al., 2014; Kline et al., 2022)
- ▶ *Compound Decision/ Empirical Bayesian*: Compound decision framework (Robbins, 1956), (Non-parametric) Estimation of the prior distribution of  $\theta_i$ . (Koenker and Mizera, 2014; Gu and Koenker, 2017)

## Questions

- ▶ Out of the top 20% hospitals in France in terms of labor employment efficiency, how many of them are public hospitals/private hospitals?
- ▶ What would be the selection outcome if I want to control the number of mistakes that I make?
- ▶ Does different selection rule produce different results? And to what degree?

## Recap

1. **Estimate the efficiency** with input demand function.

- LHS  $X$ : Labor input (number of full time equivalent nurses).
- RHS  $Y$ : Hospital output (e.g., inpatient/outpatient stays, medical sessions).

$$\log(x_{it,\text{nurses}}) = \log(y_{it,\text{output}})\beta + \theta_i + \varepsilon_{it} \quad \text{where} \quad \varepsilon_{it} \sim \mathcal{N}(0, \sigma_i^2)$$

2. **Estimate the prior distribution  $G$  of  $\theta_i$**  with nonparametric maximum likelihood estimator (NPMLE).

3. **Select the 20% most efficient hospitals** utilizing the estimated  $G$ .

## Hospital Types

The Annual Statistics of Health Establishments (SAE)<sup>1</sup>, 2013-2022 <sup>2</sup>.

Year	Teaching	Normal Public	Private For Profit	Private Non Profit	Total
2013	198	1312	1305	1382	4197
2014	201	1274	1293	1349	4117
2015	211	1275	1297	1349	4132
2016	212	1266	1297	1313	4088
2017	211	1249	1297	1306	4063
2018	214	1247	1296	1288	4045
2019	214	1236	1287	1281	4018
2021	219	1222	1293	1264	3998
2022	220	1220	1296	1259	3995

- Teaching hospitals may be innately very different from others (training, research).

► Appendix

<sup>1</sup>La Statistique annuelle des établissements (SAE)

<sup>2</sup>2020 missing due to Covid-19

## Output

The list of 8 medical output.

Output	Teaching	Normal Public	Private For Profit	Private Non Profit	Total
STAC inpatient	25.17%	43.09%	23.64%	8.1%	100%
STAC outpatient	18.4%	19.46%	52.95%	9.18%	100%
Sessions	14.49%	21.96%	34.4%	29.16%	100%
Outpatient Consultations	36.8%	52.45%	0.23%	10.52%	100%
Emergency	21.4%	60.06%	13.37%	5.17%	100%
Follow-up care and Long-term care	7.6%	19.47%	37.95%	34.98%	100%
Home hospitalization	13%	17.38%	12.4%	57.22%	100%
Psychiatry stays	6.53%	62.26%	12.93%	18.28%	100%

- Hospitals differ not only in efficiency but also in the mix of services they provide.

# Compound Decision Framework

Observe:

$$\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \dots, \hat{\theta}_n)$$

$$\text{where } \hat{\theta}_i | \theta_i \sim P_{\theta_i}$$

Decision:

$$\delta(\hat{\boldsymbol{\theta}}) = (\delta_1(\hat{\boldsymbol{\theta}}), \dots, \delta_n(\hat{\boldsymbol{\theta}}))$$



## Compound Loss and Risk

Loss:

$$L_n(\theta, \delta(\hat{\theta})) = \sum_{i=1}^n L(\theta_i, \delta_i(\hat{\theta})).$$

Risk (Expectation of loss):

$$\begin{aligned} R_n(\theta, \delta(\hat{\theta})) &= \mathbb{E}[L_n(\theta, \delta(\hat{\theta}))] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\theta_i | \hat{\theta}_i} [L(\theta_i, \delta_i(\hat{\theta}))] \quad \text{Separable decision rule } \delta \\ &= \frac{1}{n} \sum_{i=1}^n \int L(\theta_i, \delta_i(\hat{\theta}_i)) dP_{\theta_i}(\hat{\theta}_i) \\ &= \int \int L(\theta_i, \delta(\hat{\theta}_i)) dP_{\theta_i}(\hat{\theta}_i) dG_n(\theta) \end{aligned}$$

where  $G_n(\theta)$  is the empirical distribution (Frequentist View) of  $\theta \sim G$ .

→ Bayesian view: replace  $G_n$  by a distribution  $G$ . → Empirical Bayes: Estimate the  $G$ .

## Estimate $G$

Kiefer and Wolfowitz (1956) established the nonparametric maximum likelihood estimator (NPMLE)

$$\hat{G} = \arg \min_{G \in \mathcal{G}} \left\{ - \sum_{i=1}^n \log g(\hat{\theta}_i) \middle| g(\hat{\theta}_i) = \int \mathbb{P}(\hat{\theta}_i | \theta) dG(\theta) \right\}$$

where  $\mathbb{P}(\hat{\theta}_i | \theta)$  is the probability density function of  $\hat{\theta}_i$  conditional on the true parameter  $\theta \longrightarrow g(\hat{\theta}_i)$  is the marginal pdf of  $\hat{\theta}_i$ .

## Estimate $G$

This is an **infinite-dimensional** convex optimization problem with a strictly convex objective subject to linear constraints.

$$\min_{f=dG} \left\{ - \sum_i \log g(y_i) \middle| g(y_i) = T(f), K(f) = 1, \forall i \right\}$$

where  $T(f) = \int \mathbb{P}(y_i|\theta) f d\theta$  and  $K(f) = \int f d\theta$ .

Consistency is proven by Kiefer and Wolfowitz (1956). Efficient computation method introduced by Koenker and Mizera (2014). Implemented with MOSEK created by Andersen and Andersen (2010).

## The Selection Task

- ▶ Select the bottom 20% (the smaller the  $\theta_i$ , the more efficient) of the true  $\theta_i$ . Since we assume that  $\theta_i \sim G$ , those  $i$  whose  $\theta_i < G^{-1}(0.2)$
- ▶ Control the overall false discovery rate at 20%,

$$\frac{\mathbb{E}_G [1 \{ \theta_i > \theta_\alpha, \delta_i = 1 \}]}{\mathbb{E}_G [\delta_i]} \leq \gamma$$

1. Nominator: Selected but whose true value  $> G^{-1}(0.2)$ .
2. Denominator: Selected.

## Problem Formulation

The **loss** function is

$$L(\delta, \theta) = \sum h_i(1 - \delta_i) + \tau_1 \left( \sum (1 - h_i)\delta_i - \gamma\delta_i \right) + \tau_2 \left( \sum \delta_i - \alpha n \right)$$

where  $h_i = 1 \{ \theta_i < \theta_\alpha = G^{-1}(\alpha) \}$ .  $h_i$  is an indicator of whether the true value belong to the set.  $\delta_i$  is an indicator of whether  $i$  is being selected. Therefore, the **problem** is to find  $\delta$  such that

$$\begin{aligned} \min_{\delta} \quad & \mathbb{E}_G \mathbb{E}_{\theta|\hat{\theta}} [L(\delta, \theta)] \\ = & \mathbb{E}_G \sum \mathbb{E}(h_i)(1 - \delta_i) + \tau_1 \left( \sum (1 - \mathbb{E}(h_i))\delta_i - \gamma\delta_i \right) \\ & + \tau_2 \left( \sum \delta_i - \alpha n \right) \\ = & \mathbb{E}_G \sum v_\alpha(\hat{\theta})(1 - \delta_i) + \tau_1 \left( \sum (1 - v_\alpha(\hat{\theta}))\delta_i - \gamma\delta_i \right) + \tau_2 \left( \sum \delta_i - \alpha n \right) \end{aligned}$$

where  $v_\alpha(\hat{\theta}) = \mathbb{P}(\theta < \theta_\alpha | \hat{\theta})$  is the **posterior tail probability**.

## Derive tail probability $v_\alpha$

Pick hospital  $i$  whose true inefficiency value is  $\theta_i$ , which we don't observe. We only observe a sequence of  $Y_{it}$  where

$$Y_{it} = \theta_i + \varepsilon_{it} \quad \varepsilon_{it} \sim \mathcal{N}(0, \sigma_i^2) \quad (\theta_i, \sigma_i^2) \sim G$$

Neither  $\theta_i$  nor  $\sigma_i^2$  is known. But the sufficient statistics are

$$Y_i = \frac{1}{T_i} \sum_{t=1}^{T_i} Y_{it} \quad \text{where} \quad Y_i | \theta_i, \sigma_i^2 \sim \mathcal{N}(\theta_i, \sigma_i^2 / T_i)$$

$$S_i = \frac{1}{T_i - 1} \sum_{t=1}^{T_i} (Y_{it} - Y_i)^2 \quad \text{where} \quad S_i | \sigma_i^2 \sim \Gamma(r_i = (T_i - 1)/2, 2\sigma_i^2 / (T_i - 1))$$

In our input demand function specification, we have  $Y_{it} = \log(x_{it}) - \beta \log(y_{it})$ . [► Appendix](#)

## TP and Constraints

Given the two sufficient statistics, the posterior tail probability is

$$\begin{aligned}v_{\alpha}(\hat{\theta}_i) &= v_{\alpha}(Y_i, S_i) \\&= P(\theta_i < \theta_{\alpha} | Y_i, S_i) \\&= \frac{\int_{-\infty}^{\theta_{\alpha}} \Gamma(s_i | r_i, \sigma_i^2) f(y_i | \theta_i, \sigma_i^2) dG(\theta_i, \sigma_i^2)}{\int_{-\infty}^{\infty} \Gamma(s_i | r_i, \sigma_i^2) f(y_i | \theta_i, \sigma_i^2) dG(\theta_i, \sigma_i^2)}\end{aligned}$$

We want to find a cutoff  $\lambda$  such that both constraints are satisfied <sup>3</sup>:

- ▶ Capacity:  $\int \int P(v_{\alpha}(Y_i, S_i) > \lambda) dG(\theta_i, \sigma_i^2) \leq \alpha$
- ▶ FDR:  $\int \int \frac{E[1\{v_{\alpha}(Y_i, S_i) > \lambda\}(1 - v_{\alpha}(Y_i, S_i))]}{E[1\{v_{\alpha}(Y_i, S_i) > \lambda\}]} dG(\theta_i, \sigma_i^2) \leq \gamma$

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<sup>3</sup>Relaxed discrete optimization problem, following (Basu et al., 2018)

## Recap

1. We have a  $N \times T$  panel.  $Y_{it}$  is an observation of hospital  $i$ 's inefficiency term  $\theta_i$  at time  $t$ . Say  $Y_{it}|\theta_i, \sigma_i \sim \mathcal{N}(\theta_i, \sigma_i^2)$ .
2. Given a panel of  $Y_{it}$ , perform NPMLE to get an estimate of  $G(\theta, \sigma^2)$ .
3. Given the estimated prior  $G$ , derive the explicit form of posterior tail probability  $v_\alpha(Y_i, S_i)$  and the two constraints.
4. Solve the selection problem and find the optimal  $\delta^*$

$$\min_{\delta} \mathbb{E}_G \sum v_\alpha(\hat{\theta})(1 - \delta_i) + \tau_1 \left( \sum (1 - v_\alpha(\hat{\theta}))\delta_i - \gamma\delta_i \right) + \tau_2 \left( \sum \delta_i - \alpha n \right)$$

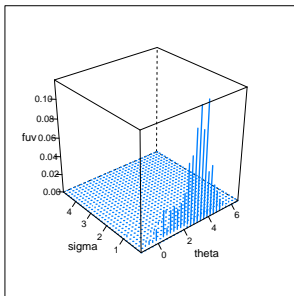
5. The decision rule is defined by the cutoff  $\lambda^*$

$$\delta^*(y_i, s_i) = 1 \{v_\alpha(y_i, s_i) > \lambda^*\}$$

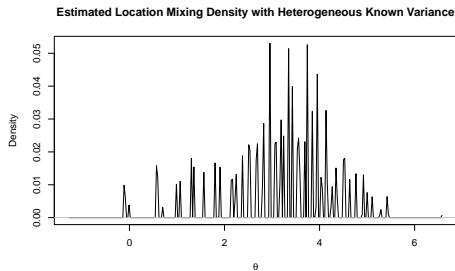


## The estimated $\hat{G}$

Case 1:  $G(\theta, \sigma^2)$  for  $v_\alpha(Y_i, S_i)$

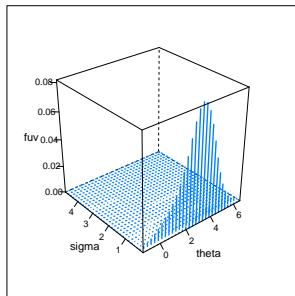


Case 2:  $G(\theta)$  for  $v_\alpha(Y_i)$

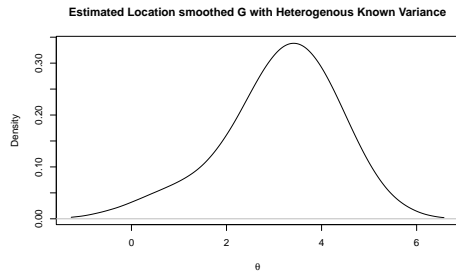


## The estimated $\hat{G}$

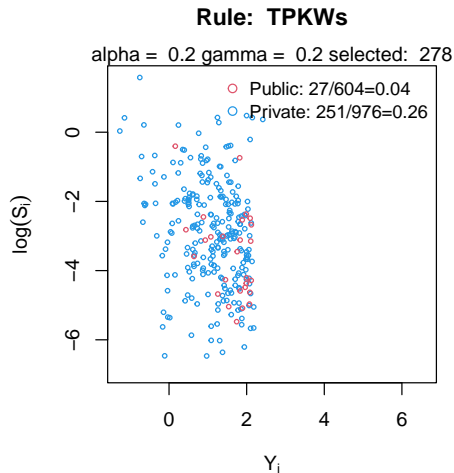
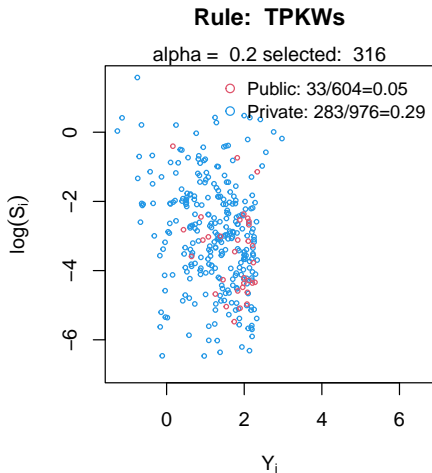
Case 1:  $G(\theta, \sigma^2)$  for  $v_\alpha(Y_i, S_i)$



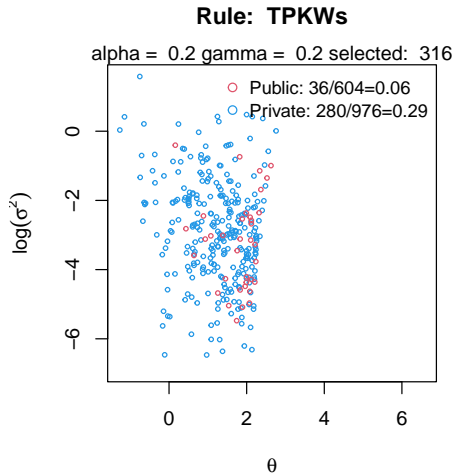
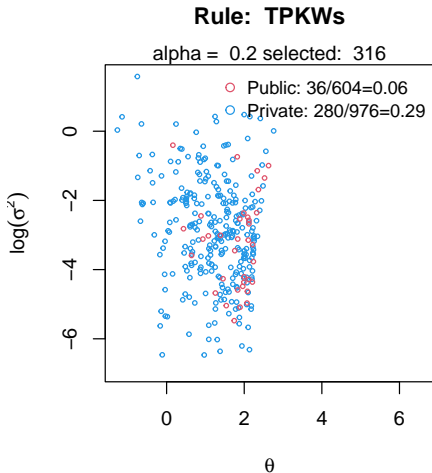
Case 2:  $G(\theta)$  for  $v_\alpha(Y_i)$



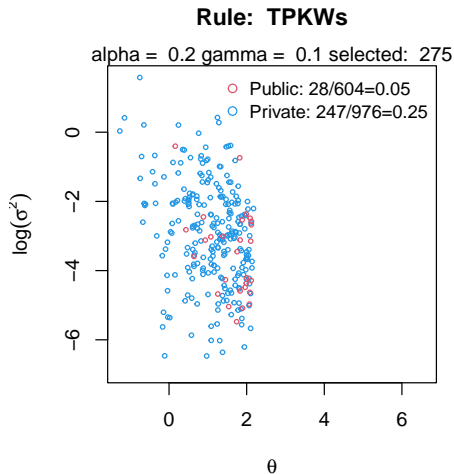
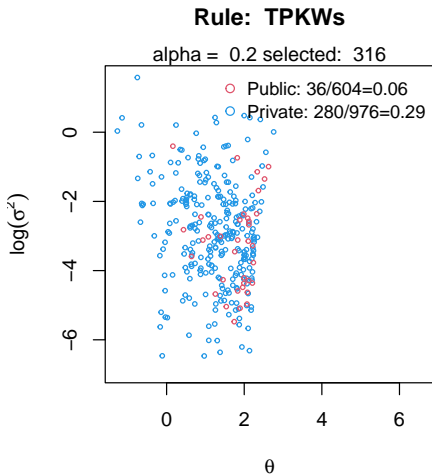
## $G(\theta, \sigma^2)$ : Posterior Tail probability (0.2,0.2)



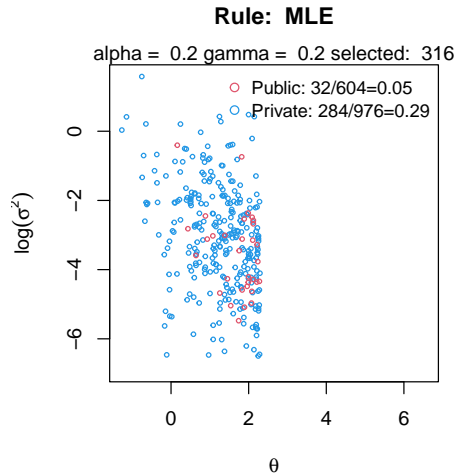
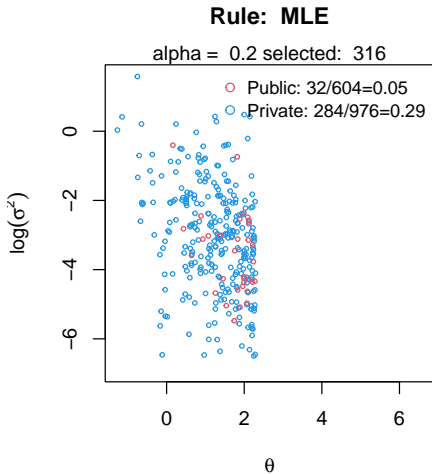
## $G(\theta)$ : Posterior Tail probability (0.2,0.2)



## $G(\theta)$ : Posterior Tail probability (0.2,0.1)



## "Face value"



## Fixed effect estimation

Assume that  $\mathbb{E}[\varepsilon_{it}|\theta_i, x_{i1}, \dots, x_{i,t-1}] = 0$ .

**First Difference GMM:** use lagged level as instruments for current difference

$$\mathbb{E}[x_{i,t-2}(\Delta y_{it} - \beta \Delta x_{it})]$$

System GMM: use lagged difference as instruments for current levels

$$\mathbb{E}[\Delta x_{i,t-1}(y_{it} - \beta x_{it})] \quad \text{if} \quad \mathbb{E}[\Delta x_{i,t-1}(\theta_i + \varepsilon_{i,t})] = 0$$

## Results

Dependent Variable:	Nurses			
Model:	Within Group (1)	First Difference (2)	FD GMM (3)	SYS GMM (4)
<i>Variables</i>				
STAC inpatient	0.10*** (0.00)	0.07*** (0.01)	0.13*** (0.03)	0.48*** (0.02)
STAC outpatient	0.02*** (0.00)	0.01*** (0.00)	0.02 (0.01)	0.05* (0.02)
Medical sessions	0.02*** (0.00)	0.02*** (0.00)	0.02*** (0.00)	0.06*** (0.01)
External consultations	0.00 (0.00)	0.00 (0.00)	0.01 (0.01)	0.07*** (0.01)
Emergency	0.01*** (0.00)	0.01 (0.00)	0.05 (0.04)	-0.04** (0.01)
Long-term & follow-up	0.01*** (0.00)	0.01*** (0.00)	-0.06** (0.02)	0.07*** (0.01)
Home care	0.01*** (0.00)	0.02** (0.01)	-0.00 (0.02)	0.01 (0.02)
Psychiatric care	0.02*** (0.00)	0.01 (0.01)	0.01 (0.02)	0.07** (0.02)
<i>Fit statistics</i>				
n	1690	1690	1690	1690
T	9	9	9	9
Sargan Test: chisq			128.32	360.97
Sargan Test: df			48.00	104.00

*Heteroskedasticity-robust standard-errors in parentheses*  
*Signif. Codes: \*\*\*: 0.01, \*\*: 0.05, \*: 0.1*



## Conclusion

- ▶ Whether to control for **False discovery rate** → Control for FDR shrinks the selection set.
- ▶ Whether to assume known  $\sigma_i$  makes a difference → Assume unknown  $\sigma_i$  makes the FDR constraints bind, thus less selected than assuming  $\sigma_i$  known.
- ▶ Private (FP and NP) hospitals are indeed more "efficient" → Caution.

## Limitation

- ▶ Interpretation of the  $\theta_i$ : The Schmidt and Sickles/Pitt and Lee models treat all time invariant effects as inefficiency. Greene (2005) treats time invariant components as only unobserved heterogeneity.
- ▶ Specification, endogeneity, normality assumption on  $\varepsilon_{it}$ .etc. [▶ Next](#)

# Thanks

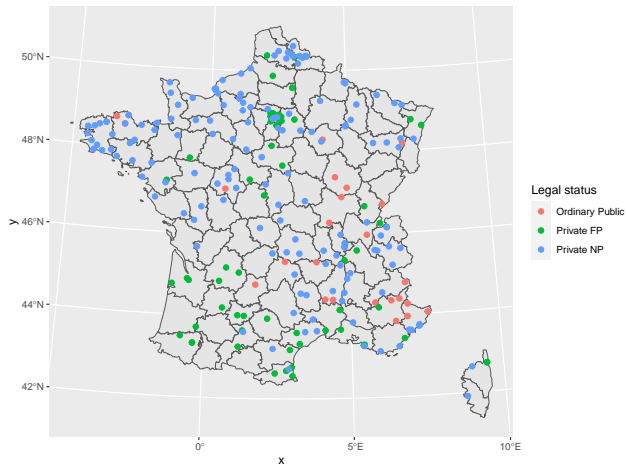


Figure: Just for fun:)

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## Conditional Input Demand Function

In standard microeconomics, the profit maximization problem is

$$\max_{\vec{y}} \sum k_i y_i - \sum p_i x_i \quad \text{subject to} \quad f_i(x_1, x_2, \dots, x_n) = y_i$$

where  $p_i$  is the price of input  $i$  and  $f$  is the cost function.

The cost minimization problem is thus

$$\min_{\vec{x}} \sum p_i x_i \quad \text{subject to} \quad f_i(x_1, x_2, \dots, x_n) = y_i$$

Thus, the factor demand function/correspondence is

$$x_i = x_i(p_1, p_2, \dots, p_n, y_1, y_2, \dots, y_m)$$

## Input demand function vs Production function

- ▶ We can remain agnostic as to the nature of the appropriate formula for the aggregation of outputs and use as many different products as desired.
- ▶ When input prices have low variability. Conditional factor demand can be estimated without information on input prices. Even if we add prices, due a lack of variability, the price parameters will be poorly estimated.
- ▶ From  $x_i = x_i(p_1, p_2, \dots, p_n, y_1, y_2, \dots, y_m)$ , we do not need to observe a complete list of inputs. But we do need to observe all input prices (can be ignored if almost no variability) and all outputs. While in the production function, it is the other way around (need to observe all inputs). Since, in our case, output is more *observable* than input (because capital is not easily observed), this approach is preferred.



# First glance

Dependent Variable:	Nurses	
Model:	Pool (1)	Pool exclude (2)
<i>Variables</i>		
Constant	1.39*** (0.025)	1.36*** (0.024)
STAC inpatient	0.279*** (0.005)	0.293*** (0.005)
STAC outpatient	0.050*** (0.003)	0.034*** (0.003)
Medical sessions	0.061*** (0.002)	0.063*** (0.002)
External consultations	0.057*** (0.002)	0.040*** (0.001)
Emergency	0.016*** (0.001)	0.023*** (0.001)
Long-term & follow-up	0.076*** (0.002)	0.072*** (0.002)
Home care	0.016*** (0.003)	0.028*** (0.003)
Psychiatric care	0.073*** (0.003)	0.075*** (0.004)
<i>Fit statistics</i>		
Observations	13,402	12,279
R <sup>2</sup>	0.821	0.819

*Heteroskedasticity-robust standard-errors in parentheses*  
*Signif. Codes: \*\*\*: 0.01, \*\*: 0.05, \*: 0.1*

Dependent Variable:	Nurses	
Model:	Dummy (1)	Dummy exclude (2)
<i>Variables</i>		
Constant	1.58*** (0.027)	1.50*** (0.028)
STAC inpatient	0.276*** (0.005)	0.290*** (0.005)
STAC outpatient	0.057*** (0.004)	0.048*** (0.004)
Medical sessions	0.063*** (0.002)	0.068*** (0.002)
External consultations	0.027*** (0.002)	0.028*** (0.002)
Emergency	0.021*** (0.001)	0.018*** (0.001)
Long-term & follow-up	0.069*** (0.002)	0.067*** (0.002)
Home care	0.026*** (0.003)	0.025*** (0.003)
Psychiatric care	0.063*** (0.003)	0.071*** (0.004)
Private Forprofit	-0.270*** (0.027)	-0.245*** (0.027)
Private Nonprofit	-0.180*** (0.021)	-0.160*** (0.022)
Teaching	0.707*** (0.021)	
<i>Fit statistics</i>		
Observations	13,402	12,279
R <sup>2</sup>	0.838	0.821

*Heteroskedasticity-robust standard-errors in parentheses*  
*Signif. Codes: \*\*\*: 0.01, \*\*: 0.05, \*: 0.1*

## Second glance

Dependent Variable:	Nurses			
Model:	Teaching (1)	Public (2)	Forprofit (3)	Nonprofit (4)
<i>Variables</i>				
Constant	3.28*** (0.123)	1.40*** (0.099)	1.41*** (0.041)	1.00*** (0.059)
STAC inpatient	0.108*** (0.016)	0.328*** (0.018)	0.261*** (0.007)	0.343*** (0.012)
STAC outpatient	0.131*** (0.013)	0.078*** (0.005)	0.048*** (0.005)	0.046*** (0.010)
Medical sessions	0.058*** (0.008)	0.049*** (0.003)	0.075*** (0.002)	0.093*** (0.006)
External consultations	0.018** (0.009)	0.025*** (0.004)	-0.003 (0.006)	0.002 (0.005)
Emergency	0.049*** (0.004)	-0.008** (0.003)	0.034*** (0.002)	0.024*** (0.004)
Long-term & follow-up	0.058*** (0.005)	0.051*** (0.003)	0.057*** (0.003)	0.117*** (0.007)
Home care	0.020** (0.010)	0.029*** (0.003)	0.049*** (0.007)	-0.012 (0.009)
Psychiatric care	0.029*** (0.005)	0.071*** (0.004)	0.076*** (0.007)	0.049*** (0.017)
<i>Fit statistics</i>				
Observations	1,123	5,260	4,415	2,604
R <sup>2</sup>	0.780	0.863	0.742	0.754

*Heteroskedasticity-robust standard-errors in parentheses*  
*Signif. Codes: \*\*\*: 0.01, \*\*: 0.05, \*: 0.1*

## Panel data Estimator

- ▶ Strict exogeneity: Within Group/First Difference

$$E[\epsilon_{it}|x_{i1}, \dots, x_{iT}, \theta_i] = 0$$

- ▶ Relaxed: First Difference GMM (Arellano and Bond, 1991), System GMM (Arellano and Bover, 1995; Blundell and Bond, 1998).

$$E[\epsilon_{it}|x_{i1}, \dots, x_{it-p}, \theta_i] = 0$$

Issues: Weak instruments (?) and the proliferation of instruments (Roodman, 2007).

$$\mathbb{E}[\Delta x_{i,t-1}(y_{it} - \beta x_{it})] \quad \text{if} \quad \mathbb{E}[\Delta x_{i,t-1}(\theta_i + \varepsilon_{i,t})] = 0$$

## NPMLE Computation Methods

The primal problem:

$$\min_{f=dG} \left\{ - \sum_i \log g(y_i) \middle| g(y_i) = T(f), K(f) = 1, \forall i \right\}$$

where  $T(f) = \int p(y_i|\theta) f d\theta$  and  $K(f) = \int f d\theta$ .

Discretize the support:

$$\min_{f=dG} \left\{ - \sum_i \log g(y_i) \middle| g = Af, 1^T f = 1 \right\}$$

where  $A_{ij} = p(y_i|\theta_j)$  and  $f = (f(\theta_1), f(\theta_2), \dots, f(\theta_m))$ .

The dual problem:

$$\max_{\lambda, \mu} \left\{ \sum_i \log \lambda_1(i) \middle| A^T \lambda_1 < \lambda_2 1, (\lambda_1 > 0) \right\}$$

## Normality assumption on $\varepsilon_{it}$

Estimate the fixed effect  $\theta_i$  by

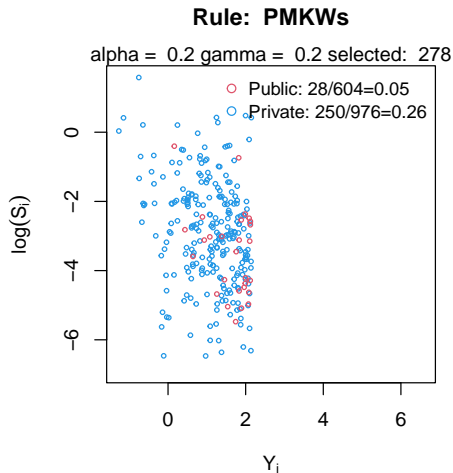
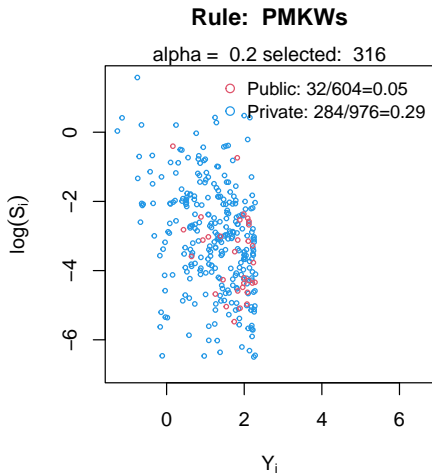
$$\hat{\theta}_i = \frac{1}{T} \sum (\theta_i + \varepsilon_{it} + x_{it}(\beta - \hat{\beta}))$$
$$\xrightarrow{N \rightarrow \infty} \theta_i + \frac{1}{T} \sum_t \varepsilon_{it}$$

When  $T$  is relatively small (or even fixed), can't use central limit theorem to claim that  $\hat{\theta}_i \xrightarrow{d} \mathcal{N}(\theta_i, \frac{\sigma_i^2}{T})$ .  $\longrightarrow$  Assume that  $\varepsilon_{it} \sim \mathcal{N}(0, \sigma_i^2)$  .

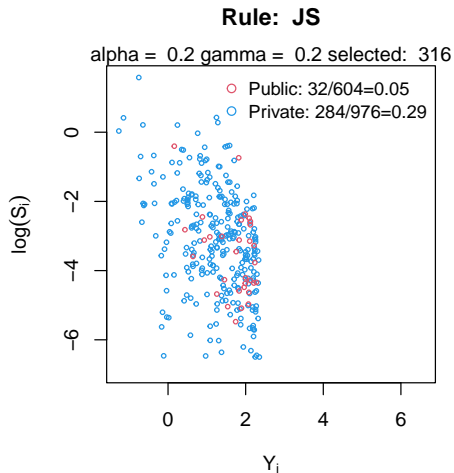
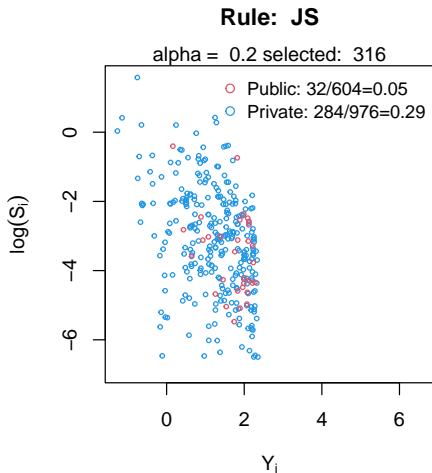
[► Back \(main\)](#)

[► Back \(end\)](#)

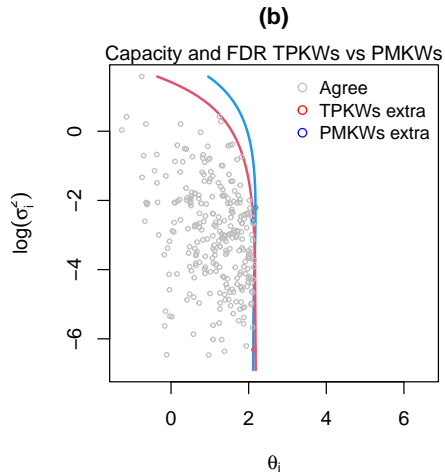
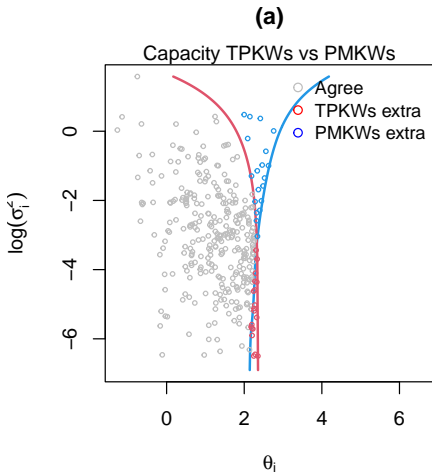
## $G(\theta, \sigma)$ : Posterior Mean



## $G(\theta, \sigma)$ : James-Stein Shrinkage

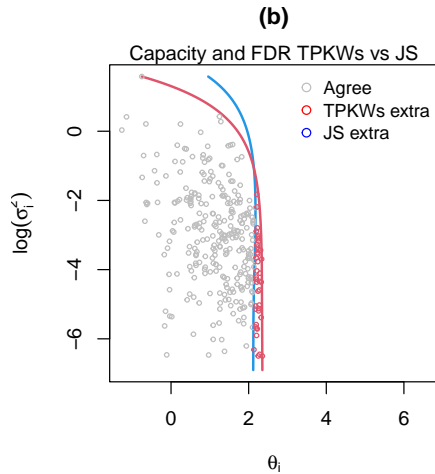
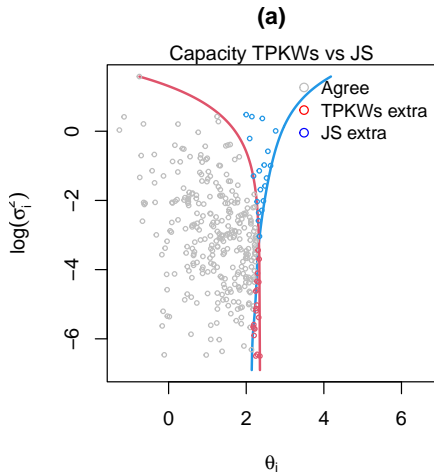


# TP vs PM





## TP vs JS



# TP vs MLE

