Estimate Dynamic Models with CCP

Problem Set 1
Empirical Industrial Organization 2025 Srping

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Utility (Profit) The profit function of the Harold Zurcher is

$$u(s, Y) = (-RC)Y + (-\mu s)(1 - Y) = (-RC + \mu s)Y - \mu s$$

The realized profit is

$$v(s, Y, \nu_0, \nu_1) = u(s, Y) + \nu_Y$$

Variables For **observed variables**, we have

- s: the state variable, the mileage at the end of period t
- Y: the decision variable, the decision to replace the bus at the end of period t

For **unobserved variables**, we have ν_0, ν_1.

State transition The state variable s evolves according to this transition function

$$s_{t+1} = s_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(\rho, \sigma_\rho^2)$$

Parameters The profit function parameters:

- μ: the cost of maintaining the bus
- RC: the replacement cost of the bus

The state transition parameters: ρ and σ_ρ^2. Discount factor $\beta = 0.99$ is given.

Value function The value function is

$$\begin{aligned} V(s,\nu) &= \max_{Y} \{v(s,Y,\nu) + \beta \mathbb{E}[V(s',\nu')|s,Y]\} \\ &= \max_{Y} \left\{ u(s,Y) + \nu_Y + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')] f(s'|s) ds' \right\} \end{aligned}$$

To understand this value function, notice that at the end of each period t Harold Zurcher is essentially comparing the **continuation value** from replacing $Y=1$ and that from not replacing $Y=0$. We define the **continuation value** as $\tilde{v}(s,Y,\nu)$. The value function is the maximum of them.

$$\begin{aligned} \tilde{v}(s,1,\nu) &= \tilde{u}(s,1) + \nu_1 = (-RC + \mu s) + \nu_1 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,1] f(s'|s) ds' \\ \tilde{v}(s,0,\nu) &= \tilde{u}(s,0) + \nu_0 = \nu_0 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,0] f(s'|s) ds' \end{aligned}$$

Therefore the value function can be rewritten as

$$V(s,\nu) = \max_{Y} \{\tilde{v}(s,1,\nu), \tilde{v}(s,0,\nu)\}$$

If we know the $\tilde{u}(s,1)$ and $\tilde{u}(s,0)$, and assume ν are iid from T1EV, we are back to the static logit discrete choice model such that

$$\Pr(Y=1|s) = \frac{\exp(\tilde{u}(s,1))}{\exp(\tilde{u}(s,1)) + \exp(\tilde{u}(s,0))}$$

We follow Bajari et al. (2007) to approximate the continuation value $\tilde{v}$ by

$$\begin{aligned} \tilde{v}(s,1,\nu) = \tilde{u}(s,1) + \nu_1 = x_{i1}^1 \ + x_{i1}^2 \beta_2 + x_{i1}^3 + \nu_1 \\ \tilde{v}(s,0,\nu) = \tilde{u}(s,0) + \nu_0 = x_{i0}^1 \beta_1 + x_{i0}^2 \beta_2 + x_{i0}^3 + \nu_0 \end{aligned}$$

Where

- $x_i^1 = -\frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T Y_{it}$
- $x_i^2 = -\frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T s_{it}(1-Y_{it})$
- $x_i^3 = \frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T (\gamma - \ln(\Pr(Y_{it}|s_{it})))$: **I was tripped by this term.** It is $\ln(\Pr(Y=1|s))$ when $Y_{it}=1$ and $\ln(\Pr(Y=0|s))$ when $Y_{it}=0$.

Therefore, we obtain $RC = \beta_1, \mu = \beta_2$.
The optimal θ^* is the one that maximizes the likelihood function

$$\max_{\theta} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln \Pr(Y_{it}|s_{it},\theta) \tag{1}$$

Question 1 Reduced Form CCP (Probit)

Estimating a reduced form conditional choice probability is the preparation step for simulation. Experimenting with 4 different specifications.

- `probit1`: $\Pr(Y = 1|s) = \Phi(\gamma_0 + \gamma_1 s)$
- `probit2`: $\Pr(Y = 1|s) = \Phi(\gamma_0 + \gamma_1 s + \gamma_2 s^2)$
- `probit3`: $\Pr(Y = 1|s) = \Phi(\gamma_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3)$
- `probitlog`: $\Pr(Y = 1|s) = \Phi(\gamma_0 + \gamma_1 \ln(s))$

Then, we plot the estimated CCPs against a range of value for $s \in [0, 40]$. Results are shown in Figure 1. I chose the `probit1` for later simulation.

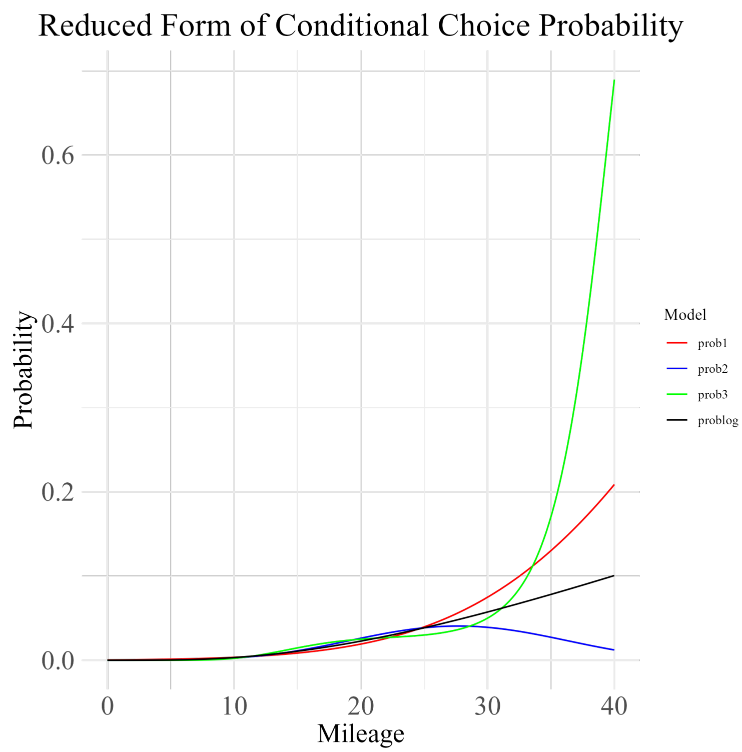

Figure 1: Estimated Reduced Form CCPs from Probit Models

Question 2 State Transition

The transition of states is guided by $\epsilon \in N(\rho, \sigma_\rho^2)$. This is a somewhat unrealistic assumption since the mileage can only increase. However, we proceed as instructed.

- $\rho = 0.19$
- $\sigma_\rho^2 = 0.0129$

Question 3 Simulation

For each observation bus $\times$ period, we perform simulation to get the 6 variables $x_{i1}^1, x_{i1}^2, x_{i1}^3$ and $x_{i0}^1, x_{i0}^2, x_{i0}^3$. Then, we take average over all observations to get Table 1.

	Replace	Non Replace
x_i^1	-1.19	-0.57
x_i^2	-436.09	-702.38
x_i^3	37.81	39.42

Table 1: Average of Simulation Results

Question 4 Estimation

We estimate by Equation 1 using the simulated approximation. Results are shown in Table 2.

	Dependent variable:
	choice
RC	8.812***
	(0.700)
μ	0.005***
	(0.001)
Observations	7,250
Log Likelihood	−319.721
Akaike Inf. Crit.	643.442
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 2: Estimation Results

Question 5 Comparison

In order to plot the CCPs estimated from Question 4 against mileage s, we first simulate for each $s \in [0, 40]$ exactly as in Question 3. Then we use the estimated parameters θ^* to compute the CCPs. The results are shown in Figure 2. Alternatively, one could take the

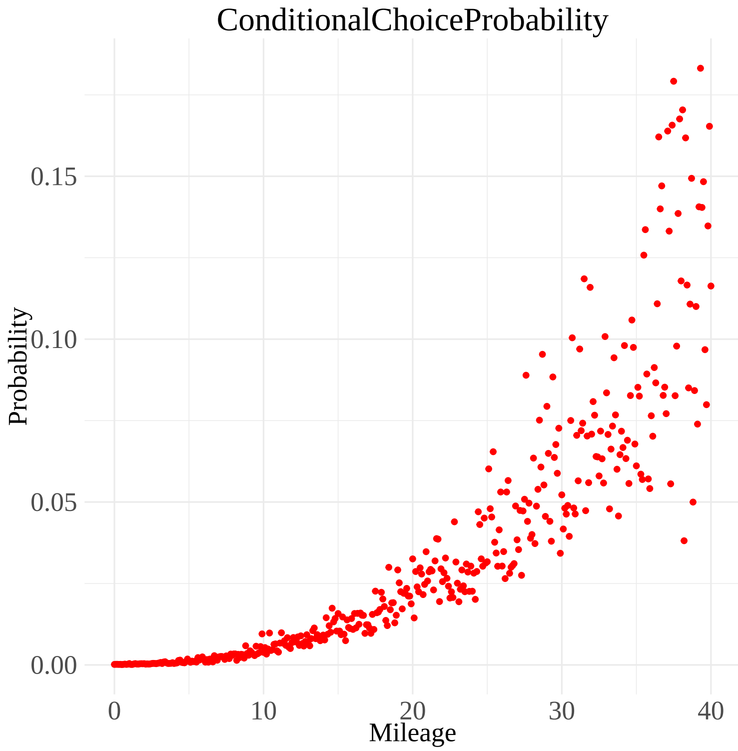

Figure 2: Estimated dynamic CCPs following Bajari et al. (2007).

parameters θ^* and solve the dynamic programming problem analytically as in Rust (1987). Solving for $\bar{V}(s)$ from Equation 2 and then compute CCP from Equation 3.

$$\bar{V}(s) = \gamma + \ln\left(\sum_{Y} \exp\left(u(s,Y) + \beta \int \bar{V}(s')f(s'|s)ds'\right)\right) \tag{2}$$

$$\Pr(Y=1|s) = \frac{\exp\left\{u(s,Y=1) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,Y)\right\}}{\sum_{Y\in\{0,1\}} \exp\left\{u(s,Y) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,Y)\right\}} \tag{3}$$

References

Bajari, P., Benkard, C. L., and Levin, J. (2007). Estimating dynamic models of imperfect competition. *Econometrica*, 75(5):1331–1370.

Rust, J. (1987). Optimal replacement of gmc bus engines: An empirical model of harold zurcher. *Econometrica: Journal of the Econometric Society*, pages 999–1033.