Estimate Dynamic Models with CCP

Problem Set 1
Empirical Industrial Organization 2025 Spring

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Utility (Profit) The profit function of the Harold Zurcher is

$$u(s, Y) = (-RC)Y + (-\mu s)(1 - Y) = (-RC + \mu s)Y - \mu s$$

The realized profit is

$$v(s, Y, \nu_0, \nu_1) = u(s, Y) + \nu_Y$$

Variables For **observed variables**, we have

- s: the state variable, the mileage at the end of period t
- Y: the decision variable, the decision to replace the bus at the end of period t

For **unobserved variables**, we have ν_0, ν_1.

State transition The state variable s evolves according to this transition function

$$s_{t+1} = s_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(\rho, \sigma_\rho^2)$$

Parameters The profit function parameters:

- μ: the cost of maintaining the bus
- RC: the replacement cost of the bus

The state transition parameters: ρ and σ_ρ^2. Discount factor $\beta = 0.99$ is given.

Value function The value function is

$$\begin{aligned} V(s,\nu) &= \max_Y \{v(s,Y,\nu) + \beta \mathbb{E}[V(s',\nu')|s,Y]\} \\ &= \max_Y \left\{ u(s,Y) + \nu_Y + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')] f(s'|s) ds' \right\} \end{aligned}$$

To understand this value function, notice that at the end of each period t Harold Zurcher is essentially comparing the **continuation value** from replacing $Y = 1$ and that from not replacing $Y = 0$. We define the **continuation value** as $\tilde{v}(s, Y, \nu)$.

$$\begin{aligned} \tilde{v}(s,1,\nu) &= \tilde{u}(s,1) + \nu_1 = (-RC + \mu s) + \nu_1 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,1] f(s'|s) ds' \\ \tilde{v}(s,0,\nu) &= \tilde{u}(s,0) + \nu_0 = \nu_0 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,0] f(s'|s) ds' \end{aligned}$$

Therefore the value function, being the maximum of the two, can be rewritten as

$$V(s,\nu) = \max_Y \{\tilde{v}(s,1,\nu), \tilde{v}(s,0,\nu)\}$$

If we know the $\tilde{u}(s,1)$ and $\tilde{u}(s,0)$, and assume ν are iid from T1EV, we are back to the static logit discrete choice model such that

$$\Pr(Y=1|s) = \frac{\exp(\tilde{u}(s,1))}{\exp(\tilde{u}(s,1)) + \exp(\tilde{u}(s,0))}$$

We follow Bajari et al. (2007) to approximate the continuation value $\tilde{v}$ by

$$\begin{aligned} \tilde{v}(s,1,\nu) = \tilde{u}(s,1) + \nu_1 = x_{i1}^1 \quad + x_{i1}^2 \beta_2 + x_{i1}^3 + \nu_1 \\ \tilde{v}(s,0,\nu) = \tilde{u}(s,0) + \nu_0 = x_{i0}^1 \beta_1 + x_{i0}^2 \beta_2 + x_{i0}^3 + \nu_0 \end{aligned}$$

Where

- $x_i^1 = -\frac{1}{K} \sum_{k=1}^K \sum_{t=1}^T Y_{it}$
- $x_i^2 = -\frac{1}{K} \sum_{k=1}^K \sum_{t=1}^T s_{it}(1 - Y_{it})$
- $x_i^3 = \frac{1}{K} \sum_{k=1}^K \sum_{t=1}^T (\gamma - \ln(\Pr(Y_{it}|s_{it})))$: **I was tripped by this term.** It is $\ln(\Pr(Y = 1|s))$ when $Y_{it} = 1$ and $\ln(\Pr(Y = 0|s))$ when $Y_{it} = 0$.

Therefore, we obtain $RC = \beta_1, \mu = \beta_2$.
The optimal θ^* is the one that maximizes the likelihood function

$$\max_\theta \sum_{i=1}^N \sum_{t=1}^T \ln \Pr(Y_{it}|s_{it}, \theta) \tag{1}$$

Question 1 Reduced Form CCP (Probit)

Estimating a reduced form conditional choice probability is the preparation step for simulation. Experimenting with 4 different specifications.

- `probit1`: $\Pr(Y=1|s) = \Phi(\gamma_0 + \gamma_1 s)$
- `probit2`: $\Pr(Y=1|s) = \Phi(\gamma_0 + \gamma_1 s + \gamma_2 s^2)$
- `probit3`: $\Pr(Y=1|s) = \Phi(\gamma_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3)$
- `probitlog`: $\Pr(Y=1|s) = \Phi(\gamma_0 + \gamma_1 \ln(s))$

Then, we plot the estimated CCPs against a range of value for $s \in [0, 40]$. Results are shown in Figure 1. The AIC of the 4 models seem close. Because of simplicity, I chose the `probit1` for later simulation. In Table of appendix, I perform some robustness check under different reduced form CCPs specifications.

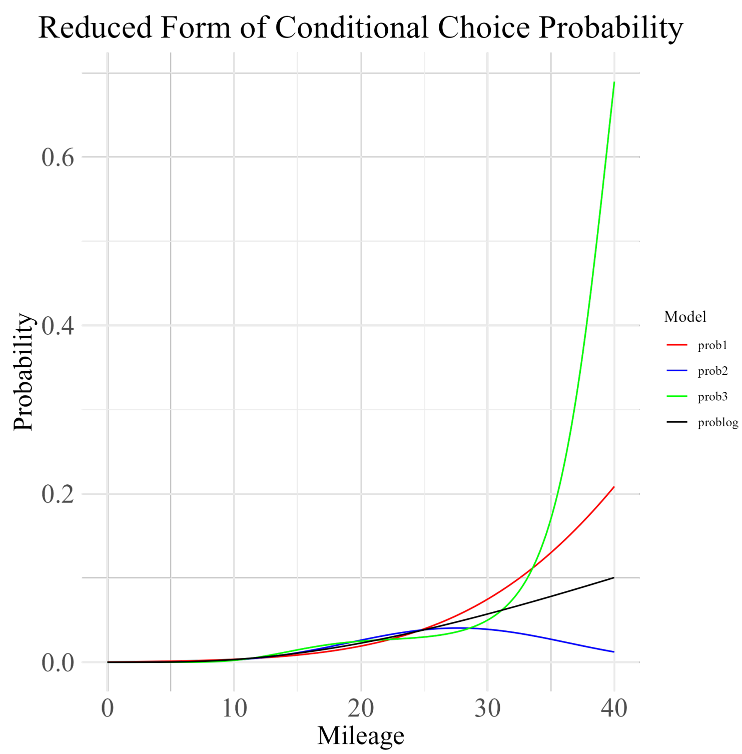

Figure 1: Estimated Reduced Form CCPs from Probit Models

Question 2 State Transition

The transition of states is guided by $\epsilon \in N(\rho, \sigma_\rho^2)$. This is a somewhat unrealistic assumption since the mileage can only increase. However, we proceed as instructed.

- $\rho = 0.19$
- $\sigma_\rho^2 = 0.0129$

Question 3 Simulation

For each observation bus × period, we perform simulation to get the 6 variables $x_{i1}^1, x_{i1}^2, x_{i1}^3$ and $x_{i0}^1, x_{i0}^2, x_{i0}^3$. Then, we take average over all observations to get Table 1.

	Replace	Non Replace
x_i^1	-1.20	-0.57
x_i^2	-435.64	-702.74
x_i^3	37.82	39.41

Table 1: Average of Simulation Results

Question 4 Estimation

We estimate by Equation 1 using the simulated approximation. Results are shown in Table 2.

	Dependent variable:
	choice
RC	8.795***
	(0.733)
μ	0.005***
	(0.001)
Observations	7,250
Log Likelihood	−319.716
Akaike Inf. Crit.	643.432
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 2: Estimation Results

Question 5 Comparison

In order to plot the CCPs estimated from Question 4 against mileage s, we first simulate for each $s \in [0, 40]$ exactly as in Question 3. Then we use the estimated parameters θ^* to compute the CCPs. The results are shown in Figure 2. Alternatively, one could take the

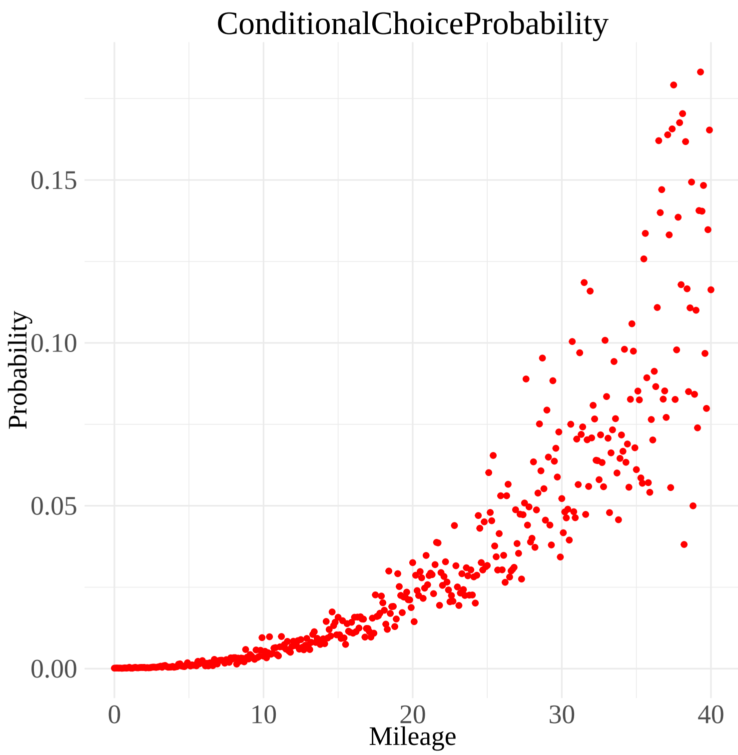

Figure 2: Estimated dynamic CCPs following Bajari et al. (2007).

parameters θ^* and solve the dynamic programming problem analytically as in Rust (1987). Solve for $\bar{V}(s)$ from Equation 2 and then compute CCP from Equation 3.

$$\bar{V}(s) = \gamma + \ln\left(\sum_Y \exp\left(u(s,Y) + \beta \int \bar{V}(s')f(s'|s)ds'\right)\right) \tag{2}$$

$$\Pr(Y=1|s) = \frac{\exp\left\{u(s,Y=1) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,Y)\right\}}{\sum_{Y\in\{0,1\}}\exp\left\{u(s,Y) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,Y)\right\}} \tag{3}$$

In this case, I guess the graph would not be the *scatter plot* like as of now. Because the CCPs should be a relatively monotonic function of s while currently it goes a bit up and down.

Remark: If the question is asking why using the full solution method (maximize likelihood from 3 obtained from solving for 2), is different from the this approximation method, it is because we get different x_i^1, x_i^2, x_i^3 from simulation than the true values and the approximation is subject to a lot of randomness.

References

Bajari, P., Benkard, C. L., and Levin, J. (2007). Estimating dynamic models of imperfect competition. *Econometrica*, 75(5):1331–1370.

Rust, J. (1987). Optimal replacement of gmc bus engines: An empirical model of harold zurcher. *Econometrica: Journal of the Econometric Society*, pages 999–1033.

A Tables

	model	K	T	time	RC	\mu
1	probit1	10.00	100.00	31.77	9.98	0.01
2	probit2	10.00	100.00	37.59	8.85	0.01
3	probit3	10.00	100.00	36.63	8.98	0.01
4	probitlog	10.00	100.00	34.73	10.00	0.01
5	probit1	20.00	100.00	59.63	8.48	0.00
6	probit2	20.00	100.00	1.05	8.63	0.01
7	probit3	20.00	100.00	1.10	9.23	0.01
8	probitlog	20.00	100.00	1.08	9.51	0.01
9	probit1	10.00	200.00	58.68	9.05	0.01
10	probit2	10.00	200.00	1.02	7.73	0.00
11	probit3	10.00	200.00	1.11	9.30	0.01
12	probitlog	10.00	200.00	1.04	10.34	0.01
13	probit1	20.00	200.00	1.49	8.98	0.01
14	probit2	20.00	200.00	1.54	7.73	0.00
15	probit3	20.00	200.00	1.64	9.03	0.01
16	probitlog	20.00	200.00	1.16	6.48	0.00

Table 3: Comparison of Parameter Estimates under Different Reduced Form CCPs and Simulation Rounds

The simulation time seems counterintuitive. Maybe this is due to parallel computing.