Solving Dynamic Oligopoly Game

Problem Set 1
Empirical Industrial Organization 2025 Spring

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1 Framework

In this section, I want to review what we learned in class. Typing them out as a way to organize my thoughts.

1.1 Markov Perfect Nash Equilibrium

Definition Each player's strategy is memoryless such that the each player's strategy s_t^i at time t is a function of only the current state ω_t. MPNE is a set of $\{s_t^i\}_{i,t}$ (strategy profile) such that given s_t^{-i} of the rest of players, s_t^i is best responding (profit maximizing).

Examples For a certain game, there may exist multiple equilibrium (strategy profile). Some do not satisfy the definition of MPNE.

- Asymmetric information
- Collusion

1.2 Dynamic Oligopoly Model

In an active market with multiple incumbent firms and a set of potential entrants, Ericson and Pakes (1995) proposed a framework to model the market conditions. That is

- firms enter or exit the market, leaving N many active firms.
- firms make investment so as to increase their productivity/decrease their cost of production.
- firms produce goods and services, set prices and take profits. The price they can set and the quantity they can sell are jointly determined by the prices and quantitie of all other firms.

The model can be classified as multiagent dynamic problem with infinite horizon. While the name is mouthful, the idea is essentially that each agent is solving its own optimization problem in the face of others, while taking into account the future.

Rules of Game The games repeats for inifite number of periods. Inside each period, the game is played in the following way.

1. Stage 1: Each firm i observes the industry state ω and its own productivity ω^i.[1]
2. Stage 2: (simultaneous decision)
   - Incumbent firms first make decision χ about whether to stay in the market and if so, how much to invest x
   - Potential entrants first make decision χ about whether to enter the market and if so, how much to invest x
3. Stage 3: Only the incumbent firms that have decided to stay will produce while the rest of incumbents and all potential entrants do not produce anything. The industry profit is $\Pi(\omega) = (\Pi_1(\omega), \dots, \Pi_n(\omega))$ and a firm in position j will get $\Pi_j(\omega)$.
4. Stage 4: As if the market is closed for a short period of time, and now let's update the market conditions.
   - We kick out those incumbents who have decided to exit and welcome those potential entrants who have decided to enter. We update the number of firms N'.
   - We let the investment to take effect on the productivity. We update the productivity ω'. This transition of productivity follows the folloing equation

     $$\omega' = \omega + \tau_i - \nu.$$

     If we specify that $\Pr(\tau_i = 1) = \frac{ax}{1+ax}$ and $\Pr(\nu = 1) = \delta$ then

     $$\omega' = \begin{cases} \omega + 1 & \text{with probability } \frac{ax}{1+ax}(1-\delta) \\ \omega - 1 & \text{with probability } \frac{1}{1+ax}\delta \\ \omega & \text{with probability } \frac{ax}{1+ax}\delta + \frac{1}{1+ax}(1-\delta) \end{cases}$$

Now we are ready to introduce the framework similar to single-agent dynamic problem as in Rust (1987).

Utility (Profit) The profit function of one firm is

$$\Pi_j(\omega)(\omega)$$

[1] Let's say there are firm A, B, C with productivity 1, 2, 3. From the perspective of A, this is equivalent to another state which is (1,3,2) because the industry profit is determined sole by the industry state and my profit is determined by how we split the profit, which is solely determined by my position in the industry.

Incumbent Value function The value function is

$$V((\omega_j, \omega_{-j}), \phi) = \max\left\{\phi, \max_{x_i} -x_i + \beta\mathbb{E}_{\omega',j',\phi'}[V(\omega'_j, \omega'_{-j}, \phi')|(\omega_j, \omega_{-j}), x_j]\right\} + \Pi_j(\omega)$$

This value function looks quite different from that of Rust (1987). It incorporates two decisions. The first max refers to the choice to exit or stay. The second max refers to the choice of investment if staying.

Let's look at the second max **first which looks more friendly.**

1. Take expecation over ϕ' (no need of conditioning because of independnce) gives

$$V(\omega'_j, \omega'_{-j}) = \int V(\omega'_j, \omega'_{-j}, \phi')dF(\phi').$$

2. Take expectation over ω'_{-j} conditioned on current industry state, and next industry shock $\omega_j, \omega_{-j}, \nu$ gives

$$\int V(\omega'_j, \omega'_{-j})dF(\omega'_{-j}|\omega_j, \omega_{-j}, \nu).$$

3. Take expectation over ω'_j (to be more specific, over τ_j and ν) conditioned on ω_j, x_j gives

$$\int_\nu \int_{\tau_j} V(\omega_j + \tau_j - \nu, \omega'_{-j})dF(\tau_j|x_j)dF(\nu).$$

Putting together we get

$$\mathbb{E}_{\omega',j',\phi'}[V(\omega'_j, \omega'_{-j}, \phi')|(\omega_j, \omega_{-j}), x_j] = \int_\nu \int_{\tau_j} \int V(\omega_j+\tau_j-\nu, \omega'_{-j})dF(\omega'_{-j}|\omega_j, \omega_{-j}, \nu)dF(\tau_j|x_j)dF(\nu).$$

Since both τ_j and ν are binary, and ω_{-j} discrete, we can simplify the above equation to

$$\int_{\tau_j} \Pr(\nu = 1) \sum_{\omega'_{-j}} f(\omega_{-j}|\omega_j, \omega_{-j}, \nu = 1)V(\omega_j + \tau_j - 1, \omega_{-j}) + \Pr(\nu = 0) \sum_{\omega_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 0)V(\omega_j + \tau_j,$$

$$\begin{aligned}
&= \Pr(\tau_j = 1)\Pr(\nu = 1) \sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)V(\omega_j, \omega'_{-j}) \\
&+ \Pr(\tau_j = 1)\Pr(\nu = 0) \sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 0)V(\omega_j + 1, \omega'_{-j}) \\
&+ \Pr(\tau_j = 0)\Pr(\nu = 1) \sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)V(\omega_j - 1, \omega'_{-j}) \\
&+ \Pr(\tau_j = 0)\Pr(\nu = 0) \sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 0)V(\omega_j, \omega'_{-j})
\end{aligned}$$

Reorganizing the terms, we get

$$\delta\left[p(x_j)\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j,\omega_{-j},\nu=1)V(\omega_j,\omega'_{-j})+(1-p(x_j))\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j,\omega_{-j},\nu=1)V(\omega_j-1,\omega'_{-j})\right]$$
$$+(1-\delta)\left[p(x_j)\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j,\omega_{-j},\nu=0)V(\omega_j+1,\omega'_{-j})+(1-p(x_j))\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j,\omega_{-j},\nu=0)V(\omega_j,\omega'_{-j})\right] \tag{1}$$

which is how we compute things in the code. (finally understand the logic behind the code).

Now let's look at the first max. Basically, the firm will choose to stay if

$$\phi < \max_{x_i} -x_i + \beta\mathbb{E}_{\omega',j',\phi'}[V(\omega'_j,\omega'_{-j},\phi')|(\omega_j,\omega_{-j}),x_j].$$

We denote the probability of staying by $r_j = F_\phi(\max_{x_i} -x_i + \beta\mathbb{E}_{\omega',j',\phi'}[V(\omega'_j,\omega'_{-j},\phi')|(\omega_j,\omega_{-j}),x_j])$.

Potential Entrant Value function The value function is

$$V(\omega,\phi^e) = \max\left\{0, \max_{x_i}\left\{-\phi^e - x_i + \beta\mathbb{E}V(\omega'_i,\omega'_{-i},\phi')|(\omega,x_i)\right\}\right\}$$

Similarly, we denote the probability of entering by $r^e = F_{\phi^e}(-x_i + \beta\mathbb{E}V(\omega'_i,\omega'_{-i},\phi')|(\omega,x_i))$.

1.3 Computation

I defer the discussion of equilibrium existence (which will be discussed anyway when I submit the referee report of Doraszelski and Satterthwaite (2010)) and talk about computation first.

Profit functions Interprete `static_profit` and `ccprofit`. For each $n = 1,\dots,\bar{n}$:

1. Loop over all industry structures:
   (a) Decode the structure into:
   $$(w_1, w_2, \dots, w_n)$$
   (b) For this structure, solve:
      i. Compute marginal costs for each firm:
      $$\theta_i = \gamma\exp(-(w_i-4))$$
      ii. Solve for static Cournot equilibrium:
         A. Compute equilibrium price:
         $$p = \frac{D+\sum_{i=1}^n \theta_i}{n+1}$$

B. Compute quantities for each firm (if feasible):

$$q_i = p - \theta_i$$

C. Compute profits for each firm:

$$\pi_i = (p - \theta_i)q_i - f$$

(c) Save computed profits into the profit table.

Continuation value function Interprete `calcval`. Refer to equation **??**. Consider the terms inside the bracket of δ, this is the case when the aggregate shock is 1, then all firms would only have two possibility, either staying at the same level (probability $p(x_j)$) or going down by 1 (probability $1 - p(x_j)$).

Fix $\nu = 1$, for each $m = 1, \ldots, 2^{n-1}$:

1. Compute the probability of all other firms transition from ω_k to ω'_k given investment x_k, that is $f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)$ in equation **??**.

$$f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1) = \Pi_{k \neq j} \Pr(\omega'_k|\omega_k, x_k, \nu = 1)$$

Look at firm k, we if k's efficiency stays the same, that is $m_k = 1$, then we have

$$\Pr(\omega'_k|\omega_k, x_k, \nu = 1) = \frac{ax_k}{1 + ax_k}$$

Similarly for the case when efficienty goes down $m_k = 0$, putting both cases together we have

$$\Pr(\omega'_k|\omega_k, x_k, \nu = 1) = m_k \frac{ax_k}{1 + ax_k} + (1 - m_k)(1 - \frac{ax_k}{1 + ax_k})$$

2. Now that we have computed each $f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)$, we can compute $\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)V(\omega_j, \omega'_{-j})$ and $\sum_{\omega'_{-j}} f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)V(\omega_j - 1, \omega'_{-j})$.

   (a) $V(\omega_j, \omega'_{-j})$: For each (ω_j, ω'_{-j}) we need to sort the ω vector and encode it into a code of the state plus the index of firm j's position. Then we look up the old value table to get $V(\omega_j, \omega'_{-j})$. Sum the product of $f(\omega'_{-j}|\omega_j, \omega_{-j}, \nu = 1)$ and $V(\omega_j, \omega'_{-j})$.

   (b) $V(\omega_j - 1, \omega'_{-j})$: For each $(\omega_j - 1, \omega'_{-j})$ we do exactly the same thing. Look up the old value table to get $V(\omega_j - 1, \omega'_{-j})$. Sum the product.

3. we get the value inside the bracket of δ.

Fix $\nu = 0$, same as above. we get the term inside the bracket of $1 - \delta$.

Adding them together, we get the continuation value function for a given investment vector x...[2]

[2] how hard...

References

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