Estimate Dynamic Models with CCP

Problem Set 1
Empirical Industrial Organization 2025 Srping

Zixuan

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Utility (Profit) The profit function of the Harold Zurcher is

$$\pi(s, Y) = (-RC)Y + (-\mu s)(1 - Y) = (-RC + \mu s)Y - \mu s$$

The realized profit is

$$\Pi(s, Y, \nu_0, \nu_1) = u(s, Y) + \nu_Y$$

Since only the difference of profit matters for the replacement decision, we can normalize such that

$$\begin{aligned} u(s, Y) &= (-RC + \mu s)Y \\ v(s, Y, \nu) &= \Pi(s, 1, \nu_0, \nu_1) - \Pi(s, 0, \nu_0, \nu_1) \\ &= (-RC + \mu s)Y + \nu \end{aligned}$$

Variables For **observed variables**, we have

- s: the state variable, the mileage at the end of period t
- Y: the decision variable, the decision to replace the bus at the end of period t

For **unobserved variables**, we have ν.

State transition The state variable s evolves according to this transition function

$$s_{t+1} = s_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(\rho, \sigma_\rho^2)$$

Parameters The profit function parameters:

- μ: the cost of maintaining the bus
- RC: the replacement cost of the bus

The state transition parameters: ρ and σ_ρ^2. Discount factor $\beta = 0.99$ is given.

Value function The value function is

$$\begin{aligned} V(s,\nu) &= \max_Y \{v(s,Y,\nu) + \beta\mathbb{E}[V(s',\nu')|s,Y]\} \\ &= \max_Y \left\{u(s,Y) + \nu_Y + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')]f(s'|s)ds'\right\} \end{aligned}$$

To understand this value function, notice that at the end of each period t Harold Zurcher is essentially comparing the **continuation value** from replacing $Y = 1$ and that from not replacing $Y = 0$. We define the **continuation value** as $\tilde{v}(s, Y, \nu)$. The value function is the maximum of them.

$$\begin{aligned} \tilde{v}(s,1,\nu) &= \tilde{u}(s,1) + \nu = (-RC + \mu s) + \nu + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,1]f(s'|s)ds' \\ \tilde{v}(s,0,\nu) &= \tilde{u}(s,0) = \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,0]f(s'|s)ds' \end{aligned}$$

Therefore the value function can be rewritten as

$$V(s,\nu) = \max_Y \{\tilde{v}(s,1,\nu), \tilde{v}(s,0,\nu)\}$$

If we know the $\tilde{u}(s,1)$ and $\tilde{u}(s,0)$, and assume ν are iid from T1EV, we are back to the static logit discrete choice model such that

$$\Pr(Y = 1|s) = \frac{\exp(\tilde{u}(s,1))}{\exp(\tilde{u}(s,1)) + \exp(\tilde{u}(s,0))}$$

We follow Bajari et al. (2007) to approximate the continuation value $\tilde{v}$ by

$$\begin{aligned} \tilde{v}(s,1,\nu) = \tilde{u}(s,1) + \nu &= x_{i1}^1\beta_1 + x_{i1}^2\beta_2 + x_{i1}^3 + \nu \\ \tilde{v}(s,0,\nu) = \tilde{u}(s,0) &= x_{i0}^1\beta_1 + x_{i0}^2\beta_2 + x_{i0}^3 \end{aligned}$$

where

- $x_i^1 = \frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T Y_{it}$
- $x_i^2 = \frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T s_{it}Y_{it}$
- $x_i^3 = \frac{1}{K}\sum_{k=1}^K \sum_{t=1}^T s_{it}$

Therefore, we obtain $RC = -\beta_1, \mu = \beta_2$.
The optimal θ^* is the one that maximizes the likelihood function

$$\max_\theta \sum_{i=1}^N \sum_{t=1}^T \ln \Pr(Y_{it}|s_{it},\theta)$$

References

Bajari, P., Benkard, C. L., and Levin, J. (2007). Estimating dynamic models of imperfect competition. *Econometrica*, 75(5):1331–1370.