Estimate Dynamic Models with CCP

Problem Set 1
Empirical Industrial Organization 2025 Srping

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Utility (Profit) The profit function of the Harold Zurcher is

$$\pi(s, Y) = (-RC)Y + (-\mu s)(1 - Y) = (-RC + \mu s)Y - \mu s$$

The realized profit is

$$\Pi(s, Y, \nu_0, \nu_1) = u(s, Y) + \nu_Y$$

Since only the difference of profit matters for the replacement decision, we can normalize such that

$$\begin{aligned} u(s, Y) &= (-RC + \mu s)Y \\ v(s, Y, \nu) &= \Pi(s, Y, \nu_0, \nu_1) - \Pi(s, 0, \nu_0, \nu_1) \\ &= (-RC + \mu s)Y + \nu \end{aligned}$$

Variables For **observed variables**, we have

- s: the state variable, the mileage at the end of period t
- Y: the decision variable, the decision to replace the bus at the end of period t

For **unobserved variables**, we have ν.

State transition The state variable s evolves according to this transition function

$$s_{t+1} = s_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(\rho, \sigma_\rho^2)$$

Parameters The profit function parameters:

- μ: the cost of maintaining the bus
- RC: the replacement cost of the bus

The state transition parameters: ρ and σ_ρ^2. Discount factor $\beta = 0.99$ is given.

Value function The value function is

$$\begin{aligned} V(s,\nu) &= \max_{Y} \{v(s,Y,\nu) + \beta \mathbb{E}[V(s',\nu')|s,Y]\} \\ &= \max_{Y} \left\{ u(s,Y) + \nu_Y + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')] f(s'|s) ds' \right\} \end{aligned}$$

To understand this value function, notice that at the end of each period t Harold Zurcher is essentially comparing the **continuation value** from replacing $Y = 1$ and that from not replacing $Y = 0$. We define the **continuation value** as $\tilde{v}(s,Y,\nu)$. The value function is the maximum of them.

$$\begin{aligned} \tilde{v}(s,1,\nu) &= \tilde{u}(s,1) + \nu_1 = (-RC + \mu s)Y + \nu_1 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,1] f(s'|s) ds' \\ \tilde{v}(s,0,\nu) &= \tilde{u}(s,0) + \nu_0 = \nu_0 + \beta \int \mathbb{E}_{\nu'}[V(s',\nu')|s,0] f(s'|s) ds' \end{aligned}$$

Therefore the value function can be rewritten as

$$V(s,\nu) = \max_{Y} \{\tilde{v}(s,1,\nu), \tilde{v}(s,0,\nu)\}$$

If we know the $\tilde{u}(s,1)$ and $\tilde{u}(s,0)$, and assume ν_1, ν_0 are iid from T1EV, we are back to the static logit discrete choice model such that

$$\Pr(Y = 1|s) = \frac{\exp(\tilde{u}(s,1))}{\exp(\tilde{u}(s,1)) + \exp(\tilde{u}(s,0))}$$

We follow Bajari et al. (2007) to approximate the continuation value $\tilde{v}$ by

$$\begin{aligned} \tilde{v}(s,1,\nu) &= \tilde{u}(s,1) + \nu_1 = x_{i1}^1 \beta_1 + x_{i1}^2 \beta_2 + x_{i1}^3 \beta_3 + \nu_1 \\ \tilde{v}(s,0,\nu) &= \tilde{u}(s,0) + \nu_0 = x_{i0}^1 \beta_1 + x_{i0}^2 \beta_2 + x_{i0}^3 \beta_3 + \nu_0 \end{aligned}$$

Therefore, we obtain $\beta_1/\beta_3 = RC, \beta_2/\beta_3 = \mu$.
The optimal θ^* is the one that maximizes the likelihood function

$$\max_{\theta} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln \Pr(Y_{it}|s_{it},\theta)$$

References

Bajari, P., Benkard, C. L., and Levin, J. (2007). Estimating dynamic models of imperfect competition. *Econometrica*, 75(5):1331–1370.