No assumption on the DGP of ξ_{jt}

After recovering δ_{jt} from the observed market share s_{jt}. We regress it on the product characteristics.

The **estimation equation** is

$$\delta_{jt} = x_{jt}\beta + p_{jt}\alpha + \xi_{jt}$$

where x_j is the product characteristics, p_j is the price, and ξ_{jt} unobserved product characteristics.

We can use the following moment conditions to estimate the parameters:

$$E(Z'_{jt}\xi_{jt}) = 0$$

where Z_{jt} is the exogenous variable + instruments.

The empirical moments condition is

$$\frac{1}{N}Z'\xi = 0$$

where Z is $N \times J$ and ξ is $N \times 1$.

Impose AR(1) structure on ξ_{jt}

Now, we assume that ξ_{jt} follows an AR(1) process:

$$\xi_{jt} = \rho\xi_{j,t-1} + \omega_{jt}$$

where ω_{jt} is the innovation term (error term) that is uncorrelated with x_{jt} and p_{jt}.

The new **estimation equation** is

$$\delta_{jt} = x_{jt}\beta + p_{jt}\alpha + \rho\xi_{j,t-1} + \omega_{jt}$$

$$\rho\xi_{j,t-1} = \rho x_{j,t-1}\beta + \rho p_{j,t-1}\alpha + \rho\xi_{j,t-1}$$

$$\delta_{j,t} - \rho\delta_{j,t-1} = \beta x_{jt} - \rho\beta x_{j,t-1} + \alpha p_{jt} - \rho\alpha p_{j,t-1} + \omega_{jt}$$

Thus the parameters we want to estimate is β, α, and ρ. This corresponds exactly to the model in blundellbond2000.

We rewrite as

$$\delta_{j,t} = \pi_1 x_{jt} + \pi_2 x_{j,t-1} + \pi_3 p_{jt} + \pi_4 p_{j,t-1} + \pi_5\delta_{j,t-1} + \omega_{jt}$$

where $\omega_{jt} \sim \text{MA}(0)$. The restriction that

$$\pi_4/\pi_3 = \pi_2/\pi_1 = -\pi_5$$

can be tested and imposed in the estimation.

Now we can perform standard GMM without any additional instruments because ω_{jt} is orthogonal to

$$z_{jt} = [x_{jt}, x_{j,t-1}, p_{jt}, p_{j,t-1}, \delta_{j,t-1}]$$