Dynamic Discrete Choice

Problem Set 2
Empirical Industrial Organization

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The value function is given by

$$\begin{aligned} V(i,c,p,\epsilon_t) &= \max_{x\in\{0,1\}} \{u(i,c,p,x) + \epsilon(x) + \beta\mathbb{E}[V(i',c',p',\epsilon_{t+1})|i,c,p,x]\} \\ &= \max_{x\in\{0,1\}} \left\{u(i,c,p,x) + \epsilon(x) + \beta \sum_{i',c',p'} \mathbb{E}_{\epsilon_{t+1}}[V(i',c',p',\epsilon_{t+1})|i,c,p,x] \Pr(i',c',p'|i,c,p,x)\right\} \end{aligned}$$

The utility function $u(i,c,p,x)$ is given by

$$u(i,c,p,x) = -\lambda \mathbb{1}(c>0)\mathbb{1}(i=0) + \alpha c - xp$$

In terms of the variables (data that we have),

- i is the inventory level.
- c is the consumer's purchase decision (firm's sales).
- p is the price.
- x is the firm's purchase decision.
- $\epsilon(x)$ is choice specific random utility shock

In terms of the parameters (to be estimated, but actually given in this problem),

- $\lambda = 3$ is the penalty of stocking out (the consumer wants to buy, but the firm does not have the product).
- $\alpha = 2$ is the marginal utility of selling the product.
- $\beta = 0.99$ is the discount factor.

The variables follow a certain process. Here, we assume that the variables follow discrete Markov process. The variables in the next period:

- Inventory i will be the current level + firm's purchase - sales:

$$i' = \min\{\bar{i} = 4, i + x - c\}$$

- Consumer's purchase decision c (firm's sales):

$$c' = \begin{cases} 0 & \text{with probability } \gamma = 0.5 \\ 1 & \text{with probability } 1 - \gamma = 0.5 \end{cases}$$

- Price p with two discrete states $p_s = 0.5$ and $p_r = 2$:

$$\Pi = \begin{pmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{pmatrix}$$

1 Question 1: Transition Probability

Since we have discrete state variables (i, c, p), the transition probability can be expressed in matrix form. Moreover, the transition of c and p are independent of each other, i and x. We only specify the transition of i here which takes values from 0 to 4.
When $x = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix} \tag{1}$$

When $x = 1$,

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Then the transition probability matrix for state s is given by the Kronecker product of the transition matrices of i, c, and p, which is $P_s(x) = P_i(x) \otimes P_c \otimes P_p$.

2 Question 2: Expected Value Function

2.1 Expected/Intermediate Value Function $\bar{V}(i, c, p)$

We denote $\bar{V}(i, c, p) = \mathbb{E}_\epsilon[V(i, c, p, \epsilon)]$ as the expected value function (I used to call it intermediate value function).

$$\bar{V}(i,c,p) = \sum_{x\in\{0,1\}} P(x|i,c,p)\left\{u(i,c,p,x) + \mathbb{E}[\epsilon(x)|i,c,p,x] + \beta\sum \bar{V}(i',c',p')\Pr(i',c',p'|i,c,p,x)\right\} \tag{3}$$

Note that the terms that are known are

- $u(i,c,p,x)$ is the **utility function** which is explicitly given.
- $\Pr(i',c',p'|i,c,p,x)$ is the **transition probability** of i,c,p given i,c,p,x (see equation 1 and 2).

The unknown terms are

- $P(x|i,c,p)$ is the **choice probability**.
- $E(\epsilon(x)|i,c,p,x)$ is the expectation of $\epsilon(x)$ conditional on i,c,p and x being the optimal choice.
- $\bar{V}(i,c,p)$ is the **expected value function**.

In the binary case with $\epsilon \sim T1EV$, instead of solving $V(s)$ as a function of $P(x|s)$ from the equation 3, we now have a simplified expression for $\bar{V}(s)$.
Let us denote $v(i,c,p,x) = u(i,c,p,x) + \beta\sum\bar{V}(i',c',p')\Pr(i',c',p'|i,c,p,x)$. Then we have

$$\begin{aligned}\bar{V}(s) &= \gamma + \ln(1 - P(x=1|s)) \\ &= \gamma + \ln(\exp(v(i,c,p,0)) + \exp(v(i,c,p,1))) \\ &= \gamma + \ln\left(\exp\left(u(s,0) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,0)\right) + \exp\left(u(s,1) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,1)\right)\right)\end{aligned} \tag{4}$$

Or should I write it as

$$\begin{aligned}\bar{V}(s) &= \gamma + v(s,0) + \ln(1 - P(x=1|s)) \\ &= \gamma + v(s,0) + \ln(1 + \exp(v(i,c,p,1) - v(i,c,p,0))) \\ &= \gamma + u(s,0) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,0) + \ln\left(1 + \exp(u(s,1) - u(s,0) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,1) - \beta\sum_{s'}\right.\end{aligned} \tag{5}$$

2.2 Numerical solution of $\bar{V}(i,c,p)$

We use the equation 4 to solve for $\bar{V}(i,c,p)$ numerically. I want rewrite the equations for all s in matrix form. Since we have a total of $20 = 5\times 2\times 2$ discrete state s, denote

- $\bar{V}$ as a vector of length 20.

	Inventory	Consumer purchase	Price	Expected value function
1	0	0	0.50	172.41
2	0	0	2.00	171.53
3	0	1	0.50	171.41
4	0	1	2.00	170.53
5	1	0	0.50	175.39
6	1	0	2.00	174.29
7	1	1	0.50	177.39
8	1	1	2.00	176.29
9	2	0	0.50	177.07
10	2	0	2.00	176.41
11	2	1	0.50	179.07
12	2	1	2.00	178.41
13	3	0	0.50	178.09
14	3	0	2.00	177.60
15	3	1	0.50	180.09
16	3	1	2.00	179.60
17	4	0	0.50	178.64
18	4	0	2.00	178.29
19	4	1	0.50	180.64
20	4	1	2.00	180.29

- u_0 as a vector of length 20 where the i-th element is $u(s, 0)$.
- M_0 as a matrix of size 20×20 where the i-th row is the vector of $\Pr(s'|s, 0)$

Then we have

$$\bar{V} = \gamma + \ln\left(\exp\left(u_0 + \beta M_0 \bar{V}\right) + \exp\left(u_1 + \beta M_1 \bar{V}\right)\right) \tag{6}$$

The goal is to numerically solve for this equation 6 for $\bar{V}$. The result is shown in table **??**.

3 Question 3: Simulation

1. At period $t = 0$, simulate state s as well as the shock $\epsilon \sim T1EV$.
2. Find the optimal choice x given current s and ϵ by the following

$$x^* = \arg\max_{x \in \{0,1\}} \left\{ u(s, x) + \epsilon + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x) \right\}$$

3. Given x^*, simulate a new state s' from the transition matrix M_{x^*}.
4. Repeat step 2 and 3 for T periods.

The simulation results are shown in the table **??**.

	statistic	value
1	Frequency of purchase	0.55
2	Probability of purchase when sales	0.62
3	Average duration between sales	1.26
4	Average duration between purchases	1.81

	i	c	p	V_ss	hat_V_ss
1	0	0	0.50	172.41	18.34
2	0	0	2.00	171.53	13.05
3	0	1	0.50	171.41	12.86
4	0	1	2.00	170.53	10.02
5	1	0	0.50	175.39	15.99
6	1	0	2.00	174.29	15.08
7	1	1	0.50	177.39	17.38
8	1	1	2.00	176.29	16.10
9	2	0	0.50	177.07	19.46
10	2	0	2.00	176.41	18.52
11	2	1	0.50	179.07	21.29
12	2	1	2.00	178.41	19.94
13	3	0	0.50	178.09	22.10
14	3	0	2.00	177.60	21.38
15	3	1	0.50	180.09	24.22
16	3	1	2.00	179.60	23.03
17	4	0	0.50	178.64	24.19
18	4	0	2.00	178.29	22.87
19	4	1	0.50	180.64	26.13
20	4	1	2.00	180.29	25.18

4 Question 4: Estimate $\bar{V}(i, c, p)$ using CCP method

In this question we focus on the first line of equation 4 to estimate the $\bar{V}(i, c, p)$. We first estimate the choice probability $\hat{P}(x|s)$ and then recover $\hat{\bar{V}}(s)$. That is

$$\hat{\bar{V}}(s) = \gamma + u(s, 0) + \beta \sum_{s'} \hat{\bar{V}}(s') \hat{\Pr}(s'|s, 0) + \ln(1 - \hat{P}(x = 1|s))$$

The estimated transition probability matrix M_0 and M_1, as well as the conditional choice probability are presented in the appendix. Table **??** shows the estimated $\bar{V}(i, c, p)$ using the CCP method against the full solution method in the previous section.

	1	2	3	4	5	6	7
1	0.375 (0.8)	0.125 (0.1)	0.375 (0)	0.125 (0)	0 (0)	0 (0)	0
2	0.475 (0.333)	0.025 (0)	0.475 (0.444)	0.025 (0)	0 (0)	0 (0)	0
3	0.375 (0.4)	0.125 (0)	0.375 (0.3)	0.125 (0)	0 (0)	0 (0)	0
4	0.475 (0.364)	0.025 (0.091)	0.475 (0.182)	0.025 (0)	0 (0)	0 (0)	0
5	0.1875 (0.185)	0.0625 (0.037)	0.1875 (0.185)	0.0625 (0.037)	0.1875 (0.148)	0.0625 (0.111)	0
6	0.2375 (0.25)	0.0125 (0)	0.2375 (0.222)	0.0125 (0)	0.2375 (0.111)	0.0125 (0)	0
7	0.1875 (0.121)	0.0625 (0.091)	0.1875 (0.152)	0.0625 (0.03)	0.1875 (0.121)	0.0625 (0.03)	0
8	0.2375 (0.15)	0.0125 (0)	0.2375 (0.3)	0.0125 (0)	0.2375 (0.125)	0.0125 (0)	0
9	0 (0)	0 (0)	0 (0)	0 (0)	0.1875 (0.179)	0.0625 (0.048)	0
10	0 (0)	0 (0)	0 (0)	0 (0)	0.2375 (0.211)	0.0125 (0.015)	0
11	0 (0)	0 (0)	0 (0)	0 (0)	0.1875 (0.186)	0.0625 (0.069)	0
12	0 (0)	0 (0)	0 (0)	0 (0)	0.2375 (0.246)	0.0125 (0.006)	0
13	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
14	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
15	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
16	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
17	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
18	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
19	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
20	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0

Table 1: Transition Probability Matrix M_0

A Transition Probability Matrix

The number in parentheses is the estimated transition probability.

B Conditional Choice Probability

	1	2	3	4	5	6	7
1	0.1875 (0.164)	0.0625 (0.055)	0.1875 (0.182)	0.0625 (0.055)	0.1875 (0.255)	0.0625 (0.055)	0
2	0.2375 (0.1)	0.0125 (0)	0.2375 (0.3)	0.0125 (0)	0.2375 (0.1)	0.0125 (0)	0
3	0.1875 (0.169)	0.0625 (0.085)	0.1875 (0.186)	0.0625 (0.085)	0.1875 (0.203)	0.0625 (0.034)	0
4	0.2375 (0)	0.0125 (0)	0.2375 (0)	0.0125 (0)	0.2375 (0.286)	0.0125 (0)	0
5	0 (0)	0 (0)	0 (0)	0 (0)	0.1875 (0.162)	0.0625 (0.051)	0
6	0 (0)	0 (0)	0 (0)	0 (0)	0.2375 (0.219)	0.0125 (0)	0
7	0 (0)	0 (0)	0 (0)	0 (0)	0.1875 (0.203)	0.0625 (0.053)	0
8	0 (0)	0 (0)	0 (0)	0 (0)	0.2375 (0.227)	0.0125 (0.013)	0
9	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
10	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
11	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
12	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
13	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
14	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
15	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
16	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
17	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
18	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
19	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0
20	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0

Table 2: Transition Probability Matrix M_1