Dynamic problem

Value function

$$V(s_t) = \max_{a_t \in A} \left\{ U(a_t, s_t) + \beta \int V(s_{t+1}|a_t, s_t) ds_{t+1} \right\}$$

$$V(s_t) = \max_{a_t \in A} \{v(a_t, s_t)\}$$

In our context, the agent i chooses the optimal action a_t that maximizes the utilty function $v(a_t, s_t)$ given the state s_t. But he or she has uncertainties about the future state s_{t+1}.

Now we specify the form of $v(a_t, s_t)$ which consists of two parts. One is the current utility and the other is the continuation value.

$$v(a_t, s_t) = U(a_t, s_t) + \beta \int V(s_{t+1}|a_t, s_t) ds_{t+1}$$

Current value

The **current value** is

$$U(a_t, s_t) = u(a_t, x_t) + \epsilon_t$$

Continuation value

The **continuation value** is

$$\int V(s_{t+1}|a_t, s_t) ds_{t+1} = \int \max_{a_{t+1} \in A} \{v(a_{t+1}, s_{t+1})\} \, ds_{t+1}$$

$$= \int V_{t+1}(s_{t+1}) f(s_{t+1}|a_t, s_t) ds_{t+1}$$

We need to assume that the state $s_{t+1}|a_t, s_t$ follows a certain distribution.

$$f(s_{t+1}|a_t, s_t) = f(x_{t+1}, \epsilon_{t+1}|a_t, x_t, \epsilon_t) = f(x_{t+1}|x_t, a_t, \epsilon_t) g(\epsilon_{t+1}|x_t, a_t, \epsilon_t) = f(x_{t+1}|x_t, a_t) g(\epsilon_{t+1})$$

Then we have the continuation value as

$$\int V_{t+1}(s_{t+1}) f(s_{t+1}|a_t, s_t) ds_{t+1} = \int V_{t+1}(s_{t+1}) f(x_{t+1}|x_t, a_t) g(\epsilon_{t+1}) d\epsilon_{t+1} dx_{t+1}$$

Intermediate value function

We define as the intermediate value function as the expectation of value function w.r.t. the ϵ_t.

$$\bar{V}(x_t) = \int V_t(x_t, \epsilon_t) g(\epsilon_t) d\epsilon_t$$

Therefore, we can rewrite the original value function as

$$V(s_t) = \max_{a_t \in A} v(a_t, s_t)$$

$$= \max_{a_t \in A} u(a_t, x_t) + \epsilon_t + \beta \int \bar{V}_{t+1}(x_{t+1}) f(x_{t+1}|x_t, a_t) dx_{t+1}$$

$$= \max_{a_t \in A} \tilde{u}(a_t, x_t) + \epsilon_t$$

If we had known the continuation value given the current state x_t, then we know the $\tilde{u}(a_t, x_t)$. Then, - If we also know the current ϵ_t, then the value function $V_t(s_t)$ is directly found. - If we don't know the current ϵ_t, the intermediate value function can be found by the logsum formula

$$\bar{V}(x_t) = \gamma + \ln \sum_{a_t \in A} \exp\{\tilde{u}(x_t, a_t)\}$$

The next section presents the static case.

Parameter estimation

1. For each parameter vector θ, we can derive the (intermediate) value function and the policy function as previously shown. (Inner step)
2. Then we want to find the best parameter vector θ that fits what we observe in the data. (Outer step)

Static problem

Recall that in the discrete choice problem. - if we had known the current state $s_t = (x_t, \epsilon_t)$, the value function will be

$$V(x_t, \epsilon_t) = \max_a v(s_t, a_t) = \max_a u(x_t, a_t) + \epsilon_t$$

- However, if we only know x_t, the intermediate value function is

$$\bar{V}(x_t) = \int V(x_t, \epsilon_t) g(\epsilon_t) d\epsilon_t$$

$$\bar{V}(x_t) = \gamma + \ln \sum_{a_t \in A} \exp\{u(x_t, a_t)\}$$

if assuming the ϵ_t follows the EVT1 distribution.

(Wait, how do we come up with this logsum expression?)

Example: Rust bus engine replacement problem

Static setup

The setup is as follows: - Each i is a bus. - The decision a_t is whether to replace the engine or not. - The state $s_t = (x_t, \epsilon_t)$ is the mileage of the engine and the shock. - The utility/profit function takes the following form

$$u(a_t, x_t) = y_t(a_t, x_t, \epsilon_t) - rc(a_t, x_t)$$

where y_t is the maintenance cost and c_t is the replacement cost.

We put some structure on the costs:

$$y_t(a_t, x_t, \epsilon_t) = -c((1 - a_t)x_t, \epsilon_t)$$

$$rc_t(a_t, x_t) = a_t(\theta_0 + \theta_1 x_t)$$

Then the profit function can be written as

$$\pi(a_t, x_t, \epsilon_t) = \begin{cases} -c(0, \epsilon_t) - (\theta_0 + \theta_1 x_t) & \text{if } a_t = 1 \\ -c(x_t, \epsilon_t) & \text{if } a_t = 0 \end{cases}$$

We assume additive separaability of the maintenance cost $y_t = (1 - a_t)(\theta_{c0} + \theta_{c1} x_t) + \epsilon_t$.

The profit function is in the following linear form

$$\pi(a_t, x_t, \epsilon_t) = \begin{cases} -(\theta_{r0} + \theta_{r1} x_t) + \epsilon_t & \text{if } a_t = 1 \\ -(\theta_{c0} + \theta_{c1} x_t) + \epsilon_t & \text{if } a_t = 0 \end{cases}$$

Therefore, the parameter to estimate (and can identify) in the static setup is $\theta_s = (\theta_{c0} - \theta_{r0}, \theta_{c1} - \theta_{r1})$.

The discount factor is β and can not be identified here.

Dynamic setup

We discretize the mileage x_t into N bins. The value function is

$$\begin{aligned} V(x_t, \epsilon_t) &= \max_a \left\{ \pi(a_t, x_t, \epsilon_t) + \beta \int \bar{V}(x_{t+1}) f(x_{t+1}|x_t, a_t) dx_{t+1} \right\} \\ &= \max_a \left\{ \pi(a_t, x_t, \epsilon_t) + \beta \sum_{x_{t+1}} \bar{V}(x_{t+1}) f(x_{t+1}|x_t, a_t) \right\} \end{aligned}$$

Therefore, in addition to the static parameters θ, we need to estimate the transition matrix $f(x_{t+1}|x_t, a_t)$. We **impose** the following structure on the evolution of x_t (transition probability):

$$f(x_{t+1}|x_t, a_t) = \begin{cases} \theta_t \exp(\theta_t x_{t+1}) & \text{if } a_t = 1 \\ \theta_t \exp(\theta_t (x_{t+1} - x_t)) & \text{if } a_t = 0 \end{cases}$$

Therefore, in total, we have three parameters to estimate: $\theta = (\theta_{c0} - \theta_{r0}, \theta_{c1} - \theta_{r1}, \theta_t)$.