Dynamic Discrete Choice

Problem Set 2
Empirical Industrial Organization

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Value function Consider the following dynamic model of (consumer's) inventory control. The value function is given by

$$\begin{aligned} V(i,c,p,\epsilon_t) &= \max_{x\in\{0,1\}} \{u(i,c,p,x)+\epsilon(x)+\beta\mathbb{E}[V(i',c',p',\epsilon_{t+1})|i,c,p,x]\} \\ &= \max_{x\in\{0,1\}} \Big\{u(i,c,p,x)+\epsilon(x) \\ &\quad + \beta\sum_{i',c',p'}\mathbb{E}_{\epsilon_{t+1}}[V(i',c',p',\epsilon_{t+1})|i,c,p,x]\Pr(i',c',p'|i,c,p,x)\Big\} \end{aligned}$$

Utility The consumer's utility function $u(i,c,p,x)$ is given by

$$u(i,c,p,x) = -\lambda\mathbb{1}(c>0)\mathbb{1}(i=0)+\alpha c - xp$$

Variables In terms of the variables (data that we have),

- i is the inventory level.
- c is the consumption (not a choice variable, but a state variable).
- p is the price.
- x is the consumer's purchase decision.
- $\epsilon(x)$ is choice specific random utility shock

Parameters In terms of the parameters (to be estimated, but actually given in this problem),

- $\lambda = 3$ is the penalty of stocking out (I want to consume but unfortunately I have nothing left).
- $\alpha = 2$ is the marginal utility of consuming the good.
- $\beta = 0.99$ is the discount factor.

State transition The variables follow a certain process. Here, we assume that the variables follow discrete Markov process. The variables in the next period:

- Inventory i' will be the current level + consumer's purchase x - consumption c:

$$i' = \min\{\bar{i} = 4, i + x - c\}$$

- Consumer's purchase decision c' (firm's sales):

$$c' = \begin{cases} 0 & \text{with probability } \gamma = 0.5 \\ 1 & \text{with probability } 1 - \gamma = 0.5 \end{cases}$$

- Price p' with two discrete states $p_s = 0.5$ and $p_r = 2$:

$$\Pi = \begin{pmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{pmatrix}$$

Question 1 Transition Probability

Since we have discrete state variables (i, c, p), the transition probability can be expressed in matrix form. Moreover, the transition of c and p are independent of each other, i and x. We only specify the transition of i here which takes values from 0 to 4.
When $x = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix} \tag{1}$$

When $x = 1$,

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Then the transition probability matrix for state s is given by the Kronecker product of the transition matrices of i, c, and p, which is $P_s(x) = P_i(x) \otimes P_c \otimes P_p$.

Question 2 Expected Value Function

2.2 Expected/Intermediate Value Function $\bar{V}(i, c, p)$

We denote $\bar{V}(s) = \mathbb{E}_\epsilon[V(s, \epsilon)]$ as the expected value function (I used to call it intermediate value function).

$$\begin{aligned}\bar{V}(s) &= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \mathbb{E}_{\epsilon_{t+1}}[V(s', \epsilon_{t+1})|s,x] \Pr(s'|s,x)\right\} f(\epsilon) d\epsilon \\ &= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\} f(\epsilon) d\epsilon\end{aligned} \tag{3}$$

Since ϵ is assumed to follow Type 1 Extreme Value distribution (which allows the integral over the maximum to collapse into a log-sum-exp form[1]), we have

$$\bar{V}(s) = \gamma + \ln\left(\sum_{x\in\{0,1\}} \exp\left\{u(s,x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\}\right) \tag{4}$$

2.2 Numerical solution of $\bar{V}(i, c, p)$

We use the equation 4 to solve for $\bar{V}(i, c, p)$. I rewrite the equations for all (i, c, p) in matrix form. Since we have a total of $20 = 5 \times 2 \times 2$ discrete state s, denote

- $\bar{V}$ as a vector of length 20.
- u_0 as a vector of length 20 where the i-th element is $u(s, 0)$. Similarly for u_1.
- M_0 as a matrix of size 20×20 where the i-th row is the vector of $\Pr(s'|s, 0)$. Similarly for M_1.

Then we have

$$\bar{V} = \gamma + \ln\left(\exp\left(u_0 + \beta M_0 \bar{V}\right) + \exp\left(u_1 + \beta M_1 \bar{V}\right)\right) \tag{5}$$

The goal is to numerically solve for equation 5 for $\bar{V}$. The result is shown in table 1.

[1] I am not sure when and whether to include the Euler-Mascheroni constant γ

	Inventory	Consumer purchase	Price	Expected value function
1	0	0	0.50	172.41
2	0	0	2.00	171.53
3	0	1	0.50	171.41
4	0	1	2.00	170.53
5	1	0	0.50	175.39
6	1	0	2.00	174.29
7	1	1	0.50	177.39
8	1	1	2.00	176.29
9	2	0	0.50	177.07
10	2	0	2.00	176.41
11	2	1	0.50	179.07
12	2	1	2.00	178.41
13	3	0	0.50	178.09
14	3	0	2.00	177.60
15	3	1	0.50	180.09
16	3	1	2.00	179.60
17	4	0	0.50	178.64
18	4	0	2.00	178.29
19	4	1	0.50	180.64
20	4	1	2.00	180.29

Table 1: Expected value function $\bar{V}(i, c, p)$ for each state $s = (i, c, p)$

Question 3 Simulation

3.3 Simulation results

The simulation follows the following steps.

1. At period $t = 0$, simulate state s as well as the shock $\epsilon \sim \text{EV}(1)$.

2. Find the optimal choice x given current s and ϵ by the following

$$x^* = \underset{x \in \{0,1\}}{\arg\max} \left\{ u(s, x) + \epsilon + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x) \right\}$$

3. Given x^*, simulate a new state s' from the transition matrix M_{x^*}.

4. Repeat step 2 and 3 for $T = 10000$ periods.

3.3 Summary statistics

The summary statistics of simulation are shown in the table 2. Moreover, it would be interesting to compute each **summary statistics** from the given parameters. To be specific,

	statistic	value
1	Frequency of purchase	0.549
2	Probability of purchase when sales	0.616
3	Average duration between sales	1.268
4	Average duration between purchases	1.821

Table 2: Simulation

	statistic	value
1	Frequency of purchase	0.552
2	Probability of purchase when sales	0.620
3	Average duration between sales	1.263
4	Average duration between purchases	1.813

Table 3: Theoretical steady state

1. Frequency of positive purchases: $\mathbb{E}(x) = \sum_s \Pr(x = 1|s = s) \Pr(s = s)$.
2. Probability of purchasing on sale: $\Pr(x = 1|p = 0.5) = \sum_{s \in (i,c,p=0.5)} \Pr(x = 1|s = s) \Pr(s = s)$
3. Average duration between sales: $\frac{1}{\Pr(p=0.5)}$
4. Average duration between purchases: $\frac{1}{\mathbb{E}(x)}$

The above statistics are straightforward once we have the following.

- **Conditional choice probability vector** $\Pr(x = 1|s)$ of length 20 (the number of states), which can be directly computed by

$$\Pr(x = 1|s) = \frac{\exp\left\{u(s, x = 1) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x)\right\}}{\sum_{x \in \{0,1\}} \exp\left\{u(s, x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x)\right\}}$$

- **Stationary distribution vector** p which is found by solving $\Pi' p = p$, where

$$\begin{aligned}\Pi = &\underbrace{M_1}_{\text{trans prob when x=1}} \underbrace{\circ}_{\text{element-wise product}} \underbrace{(\Pr(x = 1|s) \ldots \Pr(x = 1|s))}_{\text{repeat 20 times}} \\ &+ M_0 \circ (\Pr(x = 0|s), \ldots, \Pr(x = 0|s))\end{aligned}$$

The results are shown in table 3. The simulation results are quite close to the theoretical steady state values. It seems that with 10000 periods, the simulation has (almost) converged to the steady state.

Question 4 Estimate $\bar{V}(i, c, p)$ using CCP method

In this question, we **rewrite** equation 3 in terms of the choice probability $\Pr(x|s)$.

$$\begin{aligned}\bar{V}(s) = \sum_{x \in \{0,1\}} &\Pr(x|s)\big\{u(s, x) + \mathbb{E}[\epsilon(x)|s, x] \\ &+ \beta \sum \bar{V}(s') \Pr(s'|s, x)\big\}\end{aligned} \tag{6}$$

Note that the terms that are known are

- $u(s,x)$ is the **utility function** which is explicitly given.
- $\Pr(s'|s,x)$ is the **transition probability** of s given s,x (see equation 1 and 2).

The unknown terms are

- $\Pr(x|s)$ is the **choice probability**.
- $E(\epsilon(x)|s,x)$ is the expectation of $\epsilon(x)$ conditional on s and x being the optimal choice. Under the assumption of T1EV, we have

$$E(\epsilon(x)|s,x) = \gamma - \ln(\Pr(x|s))$$

- $\bar{V}(s)$ is the **expected value function**.

In our binary choice case (with the usual assumption on ϵ), we have

$$\begin{aligned}\bar{V}(s) &= \sum_{x\in\{0,1\}} \Pr(x|s)\big\{u(s,x) + \gamma - \ln(\Pr(x|s)) + \beta \sum \bar{V}(s')\Pr(s'|s,x)\big\} \\ &= \gamma + \Pr(x=0|s)\{u(s,0) + \beta \sum_{s'} \bar{V}(s')\Pr(s'|s,0)\} \\ &\quad + \Pr(x=1|s)\{u(s,1) + \beta \sum_{s'} \bar{V}(s')\Pr(s'|s,1)\}\end{aligned} \tag{7}$$

The next steps are

1. Estimate the choice probability $\Pr(x|s)$.
2. Given the known $\Pr(s'|s,x)$ and $u(s,x)$, along with the estimated $\Pr(x|s)$, solve equation 7 for $\bar{V}(s)$.

The results are tabulated against the $\bar{V}(s)$ estimated in question 2 in table 4.

	Inventory	Consumer purchase	Price	$\bar{V}$	$\bar{V}_{ccp}$
1	0	0	0.50	172.41	184.37
2	0	0	2.00	171.53	183.79
3	0	1	0.50	171.41	183.18
4	0	1	2.00	170.53	182.88
5	1	0	0.50	175.39	186.51
6	1	0	2.00	174.29	185.62
7	1	1	0.50	177.39	188.65
8	1	1	2.00	176.29	187.65
9	2	0	0.50	177.07	188.11
10	2	0	2.00	176.41	187.42
11	2	1	0.50	179.07	190.09
12	2	1	2.00	178.41	189.40
13	3	0	0.50	178.09	188.87
14	3	0	2.00	177.60	188.40
15	3	1	0.50	180.09	190.95
16	3	1	2.00	179.60	190.46
17	4	0	0.50	178.64	189.23
18	4	0	2.00	178.29	188.93
19	4	1	0.50	180.64	191.24
20	4	1	2.00	180.29	190.91

Table 4: Comparison of $\bar{V}(s)$ estimated by CCP method with the true value

The reason that second estimate is different/imprecise lies in the fact that the choice probability $\Pr(x|s)$ is not precisely estimated. I have (naively) applied the frequency estimator. It may be improved by applying some other non-parametric estimator (along with some smoothing). There are no other sources of differences because all the $u(s,x)$ and transition probabilities are taken from the true model.