Dynamic Discrete Choice

Problem Set 2
Empirical Industrial Organization

Zixuan, Anoosheh, Shuo

December 11, 2024

The value function is given by

$$\begin{aligned}
V(i,c,p,\epsilon_t) &= \max_{x\in\{0,1\}} \{u(i,c,p,x) + \epsilon(x) + \beta\mathbb{E}[V(i',c',p',\epsilon_{t+1})|i,c,p,x]\} \\
&= \max_{x\in\{0,1\}} \Big\{u(i,c,p,x) + \epsilon(x) \\
&\quad + \beta \sum_{i',c',p'} \mathbb{E}_{\epsilon_{t+1}}[V(i',c',p',\epsilon_{t+1})|i,c,p,x] \Pr(i',c',p'|i,c,p,x)\Big\}
\end{aligned}$$

The utility function $u(i,c,p,x)$ is given by

$$u(i,c,p,x) = -\lambda\mathbb{1}(c>0)\mathbb{1}(i=0) + \alpha c - xp$$

In terms of the variables (data that we have),

- i is the inventory level.
- c is the consumer's purchase decision (firm's sales).
- p is the price.
- x is the firm's purchase decision.
- $\epsilon(x)$ is choice specific random utility shock

In terms of the parameters (to be estimated, but actually given in this problem),

- $\lambda = 3$ is the penalty of stocking out (the consumer wants to buy, but the firm does not have the product).
- $\alpha = 2$ is the marginal utility of selling the product.
- $\beta = 0.99$ is the discount factor.

The variables follow a certain process. Here, we assume that the variables follow discrete Markov process. The variables in the next period:

- Inventory i will be the current level + firm's purchase - sales:

$$i' = \min\{\bar{i} = 4, i + x - c\}$$

- Consumer's purchase decision c (firm's sales):

$$c' = \begin{cases} 0 & \text{with probability } \gamma = 0.5 \\ 1 & \text{with probability } 1 - \gamma = 0.5 \end{cases}$$

- Price p with two discrete states $p_s = 0.5$ and $p_r = 2$:

$$\Pi = \begin{pmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{pmatrix}$$

1 Question 1: Transition Probability

Since we have discrete state variables (i, c, p), the transition probability can be expressed in matrix form. Moreover, the transition of c and p are independent of each other, i and x. We only specify the transition of i here which takes values from 0 to 4.
When $x = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix} \tag{1}$$

When $x = 1$,

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Then the transition probability matrix for state s is given by the Kronecker product of the transition matrices of i, c, and p, which is $P_s(x) = P_i(x) \otimes P_c \otimes P_p$.

2 Question 2: Expected Value Function

2.1 Expected/Intermediate Value Function $\bar{V}(i,c,p)$

We denote $\bar{V}(s) = \mathbb{E}_\epsilon[V(s,\epsilon)]$ as the expected value function (I used to call it intermediate value function).

$$\begin{aligned}\bar{V}(s) &= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \mathbb{E}_{\epsilon_{t+1}}[V(s',\epsilon_{t+1})|s,x] \Pr(s'|s,x)\right\} f(\epsilon) d\epsilon \\ &= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\} f(\epsilon) d\epsilon\end{aligned} \tag{3}$$

Since ϵ is assumed to follow Type 1 Extreme Value distribution (which allows the integral over the maximum to collapse into a log-sum-exp form[1]), we have

$$\bar{V}(s) = \gamma + \ln\left(\sum_{x\in\{0,1\}} \exp\left\{u(s,x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\}\right) \tag{4}$$

2.2 Numerical solution of $\bar{V}(i,c,p)$

We use the equation 4 to solve for $\bar{V}(i,c,p)$. I rewrite the equations for all s in matrix form. Since we have a total of $20 = 5 \times 2 \times 2$ discrete state s, denote

- $\bar{V}$ as a vector of length 20.
- u_0 as a vector of length 20 where the i-th element is $u(s,0)$.
- M_0 as a matrix of size 20×20 where the i-th row is the vector of $\Pr(s'|s,0)$

Then we have

$$\bar{V} = \gamma + \ln\left(\exp\left(u_0 + \beta M_0 \bar{V}\right) + \exp\left(u_1 + \beta M_1 \bar{V}\right)\right) \tag{5}$$

The goal is to numerically solve for equation 5 for $\bar{V}$. The result is shown in table 1.

[1] I am not sure when and whether to include the Euler-Mascheroni constant γ

	Inventory	Consumer purchase	Price	Expected value function
1	0	0	0.50	172.41
2	0	0	2.00	171.53
3	0	1	0.50	171.41
4	0	1	2.00	170.53
5	1	0	0.50	175.39
6	1	0	2.00	174.29
7	1	1	0.50	177.39
8	1	1	2.00	176.29
9	2	0	0.50	177.07
10	2	0	2.00	176.41
11	2	1	0.50	179.07
12	2	1	2.00	178.41
13	3	0	0.50	178.09
14	3	0	2.00	177.60
15	3	1	0.50	180.09
16	3	1	2.00	179.60
17	4	0	0.50	178.64
18	4	0	2.00	178.29
19	4	1	0.50	180.64
20	4	1	2.00	180.29

Table 1: Expected value function $\bar{V}(i, c, p)$ for each state $s = (i, c, p)$

3 Question 3: Simulation

1. At period $t = 0$, simulate state s as well as the shock $\epsilon \sim \text{EV}(1)$.
2. Find the optimal choice x given current s and ϵ by the following

$$x^* = \arg\max_{x \in \{0,1\}} \left\{ u(s, x) + \epsilon + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x) \right\}$$

3. Given x^*, simulate a new state s' from the transition matrix M_{x^*}.
4. Repeat step 2 and 3 for T periods.

The summary statistics are shown in the table 2.

1. frequency of positive purchases,
2. probability of purchasing on sale,
3. average duration between sales,
4. average duration between purchases

	statistic	value
1	Frequency of purchase	0.55
2	Probability of purchase when sales	0.62
3	Average duration between sales	1.27
4	Average duration between purchases	1.82

Table 2: Summary statistics of the simulation

4 Question 4: Estimate $\bar{V}(i, c, p)$ using CCP method

In this question, we **rewrite** equation 3 in terms of the choice probability $\Pr(x|s)$.

$$\begin{aligned} \bar{V}(s) = \sum_{x \in \{0,1\}} & \Pr(x|s) \big\{ u(s,x) + \mathbb{E}[\epsilon(x)|s,x] \\ & + \beta \sum \bar{V}(s') \Pr(s'|s,x) \big\} \end{aligned} \tag{6}$$

Note that the terms that are known are

- $u(s,x)$ is the **utility function** which is explicitly given.
- $\Pr(s'|s,x)$ is the **transition probability** of s given s, x (see equation 1 and 2).

The unknown terms are

- $\Pr(x|s)$ is the **choice probability**.
- $E(\epsilon(x)|s,x)$ is the expectation of $\epsilon(x)$ conditional on s and x being the optimal choice. Under the assumption of T1EV, we have

$$E(\epsilon(x)|s,x) = \gamma - \ln(\Pr(x|s))$$

- $\bar{V}(s)$ is the **expected value function**.

In our binary choice case (with the usual assumption on ϵ), we have

$$\begin{aligned} \bar{V}(s) &= \sum_{x \in \{0,1\}} \Pr(x|s) \big\{ u(s,x) + \gamma - \ln(\Pr(x|s)) + \beta \sum \bar{V}(s') \Pr(s'|s,x) \big\} \\ &= \gamma + \Pr(x=0|s)\{u(s,0) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,0)\} \\ &\quad + \Pr(x=1|s)\{u(s,1) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,1)\} \end{aligned} \tag{7}$$

The next steps are

1. Estimate the choice probability $\Pr(x|s)$.

	Inventory	Consumer purchase	Price	$\bar{V}$	$\bar{V}_{ccp}$
1	0	0	0.50	172.41	184.37
2	0	0	2.00	171.53	183.79
3	0	1	0.50	171.41	183.18
4	0	1	2.00	170.53	182.88
5	1	0	0.50	175.39	186.51
6	1	0	2.00	174.29	185.62
7	1	1	0.50	177.39	188.65
8	1	1	2.00	176.29	187.65
9	2	0	0.50	177.07	188.11
10	2	0	2.00	176.41	187.42
11	2	1	0.50	179.07	190.09
12	2	1	2.00	178.41	189.40
13	3	0	0.50	178.09	188.87
14	3	0	2.00	177.60	188.40
15	3	1	0.50	180.09	190.95
16	3	1	2.00	179.60	190.46
17	4	0	0.50	178.64	189.23
18	4	0	2.00	178.29	188.93
19	4	1	0.50	180.64	191.24
20	4	1	2.00	180.29	190.91

Table 3: Comparison of $\bar{V}(s)$ estimated by CCP method with the true value

2. Given the known $\Pr(s'|s,x)$ and $u(s,x)$, along with the estimated $\Pr(x|s)$, solve equation 7 for $\bar{V}(s)$.

The results are tabulated against the $\bar{V}(s)$ estimated in question 2 in table 3. The reason that second estimate is different/imprecise is lies in the fact that the choice probability $\Pr(x|s)$ is not precisely estimated. I have (naively) applied the frequency estimator. It may be improved by applying some other non-parametric estimator (along with some smoothing). There are no other sources of differences because all the $u(s,x)$ and transition probabilities are taken from the true model.