Dynamic Discrete Choice

Problem Set 2
Empirical Industrial Organization

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December 12, 2024

Value function The value function is given by

$$
\begin{aligned}
V(i, c, p, \epsilon_t) &= \max_{x \in \{0,1\}} \{u(i, c, p, x) + \epsilon(x) + \beta \mathbb{E}[V(i', c', p', \epsilon_{t+1}) | i, c, p, x]\} \\
&= \max_{x \in \{0,1\}} \Big\{u(i, c, p, x) + \epsilon(x) \\
&\quad + \beta \sum_{i', c', p'} \mathbb{E}_{\epsilon_{t+1}}[V(i', c', p', \epsilon_{t+1}) | i, c, p, x] \Pr(i', c', p' | i, c, p, x)\Big\}
\end{aligned}
$$

Utility The utility function $u(i, c, p, x)$ is given by

$$
u(i, c, p, x) = -\lambda \mathbb{1}(c > 0) \mathbb{1}(i = 0) + \alpha c - xp
$$

Variables In terms of the variables (data that we have),

- i is the inventory level.
- c is the consumer's purchase decision (firm's sales).
- p is the price.
- x is the firm's purchase decision.
- $\epsilon(x)$ is choice specific random utility shock

Parameters In terms of the parameters (to be estimated, but actually given in this problem),

- $\lambda = 3$ is the penalty of stocking out (the consumer wants to buy, but the firm does not have the product).
- $\alpha = 2$ is the marginal utility of selling the product.
- $\beta = 0.99$ is the discount factor.

State transition The variables follow a certain process. Here, we assume that the variables follow discrete Markov process. The variables in the next period:

- Inventory i will be the current level + firm's purchase - sales:

$$i' = \min\left\{\bar{i} = 4, i + x - c\right\}$$

- Consumer's purchase decision c (firm's sales):

$$c' = \begin{cases} 0 & \text{with probability } \gamma = 0.5 \\ 1 & \text{with probability } 1 - \gamma = 0.5 \end{cases}$$

- Price p with two discrete states $p_s = 0.5$ and $p_r = 2$:

$$\Pi = \begin{pmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{pmatrix}$$

Question 1 Transition Probability

Since we have discrete state variables (i, c, p), the transition probability can be expressed in matrix form. Moreover, the transition of c and p are independent of each other, i and x. We only specify the transition of i here which takes values from 0 to 4.
When $x = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix} \tag{1}$$

When $x = 1$,

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Then the transition probability matrix for state s is given by the Kronecker product of the transition matrices of i, c, and p, which is $P_s(x) = P_i(x) \otimes P_c \otimes P_p$.

Question 2 Expected Value Function

2.2 Expected/Intermediate Value Function $\bar{V}(i,c,p)$

We denote $\bar{V}(s) = \mathbb{E}_\epsilon[V(s,\epsilon)]$ as the expected value function (I used to call it intermediate value function).

$$\begin{aligned}
\bar{V}(s) &= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \mathbb{E}_{\epsilon_{t+1}}[V(s',\epsilon_{t+1})|s,x] \Pr(s'|s,x)\right\} f(\epsilon)d\epsilon \\
&= \int \max_{x\in\{0,1\}} \left\{u(s,x) + \epsilon(x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\} f(\epsilon)d\epsilon
\end{aligned} \tag{3}$$

Since ϵ is assumed to follow Type 1 Extreme Value distribution (which allows the integral over the maximum to collapse into a log-sum-exp form[1]), we have

$$\bar{V}(s) = \gamma + \ln\left(\sum_{x\in\{0,1\}} \exp\left\{u(s,x) + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s,x)\right\}\right) \tag{4}$$

2.2 Numerical solution of $\bar{V}(i,c,p)$

We use the equation 4 to solve for $\bar{V}(i,c,p)$. I rewrite the equations for all (i,c,p) in matrix form. Since we have a total of $20 = 5 \times 2 \times 2$ discrete state s, denote

- $\bar{V}$ as a vector of length 20.
- u_0 as a vector of length 20 where the i-th element is $u(s,0)$. Similarly for u_1.
- M_0 as a matrix of size 20×20 where the i-th row is the vector of $\Pr(s'|s,0)$. Similarly for M_1.

Then we have

$$\bar{V} = \gamma + \ln\left(\exp\left(u_0 + \beta M_0 \bar{V}\right) + \exp\left(u_1 + \beta M_1 \bar{V}\right)\right) \tag{5}$$

The goal is to numerically solve for equation 5 for $\bar{V}$. The result is shown in table 1.

[1] I am not sure when and whether to include the Euler-Mascheroni constant γ

	Inventory	Consumer purchase	Price	Expected value function
1	0	0	0.50	172.41
2	0	0	2.00	171.53
3	0	1	0.50	171.41
4	0	1	2.00	170.53
5	1	0	0.50	175.39
6	1	0	2.00	174.29
7	1	1	0.50	177.39
8	1	1	2.00	176.29
9	2	0	0.50	177.07
10	2	0	2.00	176.41
11	2	1	0.50	179.07
12	2	1	2.00	178.41
13	3	0	0.50	178.09
14	3	0	2.00	177.60
15	3	1	0.50	180.09
16	3	1	2.00	179.60
17	4	0	0.50	178.64
18	4	0	2.00	178.29
19	4	1	0.50	180.64
20	4	1	2.00	180.29

Table 1: Expected value function $\bar{V}(i, c, p)$ for each state $s = (i, c, p)$

Question 3 Simulation

The simulation follows the following steps.

1. At period $t = 0$, simulate state s as well as the shock $\epsilon \sim \text{EV}(1)$.
2. Find the optimal choice x given current s and ϵ by the following

$$x^* = \underset{x \in \{0,1\}}{\arg\max} \left\{ u(s, x) + \epsilon + \beta \sum_{s'} \bar{V}(s') \Pr(s'|s, x) \right\}$$

3. Given x^*, simulate a new state s' from the transition matrix M_{x^*}.
4. Repeat step 2 and 3 for $T = 10000$ periods.

The summary statistics of simulation are shown in the table 2, which includes

1. frequency of positive purchases,
2. probability of purchasing on sale,

	statistic	value
1	Frequency of purchase	0.55
2	Probability of purchase when sales	0.62
3	Average duration between sales	1.27
4	Average duration between purchases	1.82

Table 2: Summary statistics of the simulation

3. average duration between sales,

4. average duration between purchases

It would be interesting to compute the **true statistics** from the given parameters and compare them with the **simulated ones**.

Question 4 Estimate $\bar{V}(i, c, p)$ using CCP method

In this question, we **rewrite** equation 3 in terms of the choice probability $\Pr(x|s)$.

$$\begin{aligned}\bar{V}(s) = \sum_{x\in\{0,1\}} &\Pr(x|s)\big\{u(s,x) + \mathbb{E}[\epsilon(x)|s,x] \\ &+ \beta \sum \bar{V}(s')\Pr(s'|s,x)\big\}\end{aligned} \tag{6}$$

Note that the terms that are known are

- $u(s,x)$ is the **utility function** which is explicitly given.
- $\Pr(s'|s,x)$ is the **transition probability** of s given s, x (see equation 1 and 2).

The unknown terms are

- $\Pr(x|s)$ is the **choice probability**.
- $E(\epsilon(x)|s,x)$ is the expectation of $\epsilon(x)$ conditional on s and x being the optimal choice. Under the assumption of T1EV, we have
$$E(\epsilon(x)|s,x) = \gamma - \ln(\Pr(x|s))$$
- $\bar{V}(s)$ is the **expected value function**.

In our binary choice case (with the usual assumption on ϵ), we have

$$\begin{aligned}\bar{V}(s) &= \sum_{x\in\{0,1\}} \Pr(x|s)\big\{u(s,x) + \gamma - \ln(\Pr(x|s)) + \beta\sum \bar{V}(s')\Pr(s'|s,x)\big\} \\ &= \gamma + \Pr(x=0|s)\{u(s,0) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,0)\} \\ &\quad + \Pr(x=1|s)\{u(s,1) + \beta\sum_{s'}\bar{V}(s')\Pr(s'|s,1)\}\end{aligned} \tag{7}$$

The next steps are

1. Estimate the choice probability $\Pr(x|s)$.
2. Given the known $\Pr(s'|s,x)$ and $u(s,x)$, along with the estimated $\Pr(x|s)$, solve equation 7 for $\bar{V}(s)$.

The results are tabulated against the $\bar{V}(s)$ estimated in question 2 in table 3.

	Inventory	Consumer purchase	Price	$\bar{V}$	$\bar{V}_{ccp}$
1	0	0	0.50	172.41	184.37
2	0	0	2.00	171.53	183.79
3	0	1	0.50	171.41	183.18
4	0	1	2.00	170.53	182.88
5	1	0	0.50	175.39	186.51
6	1	0	2.00	174.29	185.62
7	1	1	0.50	177.39	188.65
8	1	1	2.00	176.29	187.65
9	2	0	0.50	177.07	188.11
10	2	0	2.00	176.41	187.42
11	2	1	0.50	179.07	190.09
12	2	1	2.00	178.41	189.40
13	3	0	0.50	178.09	188.87
14	3	0	2.00	177.60	188.40
15	3	1	0.50	180.09	190.95
16	3	1	2.00	179.60	190.46
17	4	0	0.50	178.64	189.23
18	4	0	2.00	178.29	188.93
19	4	1	0.50	180.64	191.24
20	4	1	2.00	180.29	190.91

Table 3: Comparison of $\bar{V}(s)$ estimated by CCP method with the true value

The reason that second estimate is different/imprecise lies in the fact that the choice probability $\Pr(x|s)$ is not precisely estimated. I have (naively) applied the frequency estimator. It may be improved by applying some other non-parametric estimator (along with some smoothing). There are no other sources of differences because all the $u(s, x)$ and transition probabilities are taken from the true model.