Problem 1. $\quad Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i , \quad \varepsilon_i \sim N(0, \sigma^2)$

(a) Least square Estimators of β_0, β_1 are the following:

$$\begin{cases} \hat{\beta_1} = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \\ \\ \hat{\beta_0} = \bar{y} - \hat{\beta_1}\bar{x} \end{cases}$$

Show that they are unbiased estimators :
in other words, to prove ① $E(\hat{\beta_1}) = \beta_1$ ② $E(\hat{\beta_0}) = \beta_0$

① First prove $E(\hat{\beta_1}) = \beta_1$

Given that $\hat{\beta_1} = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \dfrac{S_{xy}}{S_{xx}}$

let $k_i = \dfrac{x_i - \bar{x}}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \dfrac{x_i - \bar{x}}{S_{xx}}$, $i = 1, 2 \dots n$

so k_i is constant for each level of X at $i = 1, 2, \dots n$.

k_i has a property that $\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \dfrac{x_i - \bar{x}}{S_{xx}} = \dfrac{\sum_{i=1}^{n} x_i - \bar{x}}{S_{xx}} = 0$

Rewrite $\hat{\beta_1}$ as a linear combination of y_i observations

$$\hat{\beta_1} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{S_{xx}} = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{S_{xx}}\right)(y_i - \bar{y}) = \sum_{i=1}^{n} (k_i)(y_i - \bar{y})$$

$$= \sum_{i=1}^{n} k_i y_i - \underbrace{\bar{y} \sum_{i=1}^{n} k_i}_{0}$$

$$= \sum_{i=1}^{n} k_i y_i$$

because $\sum_{i=1}^{n} k_i = 0$
as proven above

then $E(\hat{\beta_1}) = E(\sum_{i=1}^{n} k_i y_i) = \sum_{i=1}^{n} E(k_i y_i)$ by linear property of expectation.

$$= \sum_{i=1}^{n} k_i E(y_i)$$

we need to find $E(y_i)$

our model is $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma^2)$

then

$$E(y_i) = E(\beta_0+\beta_1 x_i+\varepsilon_i) = E(\beta_0+\beta_1 x_i)+E(\varepsilon_i) = \beta_0+\beta_1 x_i$$

thus $$E(\hat{\beta_1}) = \sum_{i=1}^{n} k_i(\beta_0+\beta_1 x_i) = \underbrace{\beta_0 \sum_{i=1}^{n} k_i}_{=0} + \beta_1 \sum_{i=1}^{n} k_i x_i = \beta_1 \sum_{i=1}^{n} k_i x_i$$

work on $$\sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} \left(\frac{x_i-\bar{x}}{S_{xx}}\right) x_i$$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x}) x_i$$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x})(x_i-\bar{x})$$

$$\sum (x_i-\bar{x})(x_i-\bar{x}) = \sum x_i(x_i-\bar{x}) - \underbrace{\sum \bar{x}(x_i-\bar{x})}_{=0} = \sum x_i(x_i-\bar{x})$$

so equivalent

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i-\bar{x})^2$$

$$= \frac{1}{S_{xx}} \cdot S_{xx} = 1$$

therefore $E(\hat{\beta_1}) = \beta_1 \sum_{i=1}^{n} k_i x_i = \beta_1$ (unbiased estimater)

② prove $E(\hat{\beta_0}) = \beta_0$

Given that $\hat{\beta_0} = \bar{y} - \hat{\beta_1}\bar{x}$

$$E(\hat{\beta_0}) = E(\bar{y} - \hat{\beta_1}\bar{x}) = E(\bar{y}) - \bar{x}\,E(\hat{\beta_1}) = E(\bar{y}) - \bar{x}\beta_1$$

Find $$E(\bar{y}) = E\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(y_i) = \frac{1}{n}\sum_{i=1}^{n}(\beta_0+\beta_1 x_i)$$

$$= \frac{1}{n}\left(n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i\right)$$

$$= \beta_0 + \beta_1 \bar{x}$$

then $E(\hat{\beta_0}) = \beta_0 + \beta_1\bar{x} - \bar{x}\beta_1 = \beta_0$ (unbiased estimater)

Hence, Least square estimators $\hat{\beta_0}$, $\hat{\beta_1}$ are unbiased estimater of true β_0, β_1.

(b) fitted regression line: $\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1} X_i$

$$\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1}(X_i - \bar{X}) + \hat{\beta_1}\bar{X}$$
$$= \hat{\beta_0} + \hat{\beta_1}\bar{X} + \hat{\beta_1}(X_i - \bar{X})$$
$$= \bar{Y} + \hat{\beta_1}(X_i - \bar{X})$$

because $\hat{\beta_0} = \bar{Y} - \hat{\beta_1}\bar{X}$

then $\bar{Y} = \hat{\beta_0} + \hat{\beta_1}\bar{X}$

this is an alternative form of fitted line:

$$\hat{Y_i} = \bar{Y} + \hat{\beta_1}(X_i - \bar{X})$$

when $X_i = \bar{X}$, $\hat{Y_i} = \bar{Y} + 0 = \bar{Y}$

So this regression line always goes through $(\bar{X}, \bar{Y})$

(c) use MLE to derive estimator for σ^2

Model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma^2)$

Here we assume that ε_i is normaly distributed.

So the pdf of Y: $f(Y_i | \beta_0, \beta_1, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(Y_i - \beta_0 - \beta_1 X_i)^2}{2\sigma^2}\right]$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma}\right)^2\right]$$

The likelihood function becomes:

$$L(\sigma^2 | Y_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma}\right)^2\right]$$
$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \cdot \prod_{i=1}^{n} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma}\right)^2\right]$$

Take log-likelihood function:

$$\log L(\sigma^2 | Y_i) = \log f(\sigma^2 | Y_i) = -n \log(\sqrt{2\pi}\sigma) + \log \prod_{i=1}^{n} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma}\right)^2\right]$$
$$= -n\log(\sqrt{2\pi}\sigma) + \sum_{i=1}^{n} -\frac{1}{2}\left(\frac{Y_i - \beta_0 - \beta_1 X_i}{\sigma}\right)^2$$

we need to find $\hat{\sigma}^2$ that maximize this $\log L(\sigma^2)$.

$$\frac{d\log L(\sigma^2)}{d\sigma} = -\frac{n\cdot\sqrt{2\pi}}{\sqrt{2\pi}\sigma} + \left(-\frac{1}{2}\right)\sum_{i=1}^{n} 2\left(\frac{Y_i-\hat{\beta_0}-\hat{\beta_1}X_i}{\sigma}\right)\cdot(Y_i-\hat{\beta_0}-\hat{\beta_1}X_i)\left(-\frac{1}{\sigma^2}\right)$$

$$= -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(Y_i-\hat{\beta_0}-\hat{\beta_1}X_i)^2}{\sigma^3}$$

Set $\frac{d\operatorname{Log} L(\sigma^2)}{d\sigma} = 0$

$$-\frac{n}{\sigma}+\sum_{i=1}^{n}\frac{(Y_i-\hat{\beta_0}-\hat{\beta_1}x_i)^2}{\sigma^3} = 0$$

$$\frac{1}{\sigma^2}\Sigma(Y_i-\hat{\beta_0}-\hat{\beta_1}x_i)^2 = n$$

then $\hat{\sigma}^2 = \frac{\Sigma(Y_i-\hat{\beta_0}-\hat{\beta_1}X_i)^2}{n} = \frac{\Sigma(Y_i-\hat{Y_i})^2}{n} = \frac{SSE}{n} = MSE$

thus MSE is the MLE estimater of σ^2

Then we need to find $E(MSE)$

Since $MSE = \frac{SSE}{n}$, we can first find $E(SSE)$

$$E(SSE) = E\left(\sum_{i=1}^{n}(Y_i-\hat{Y_i})^2\right)$$

$$= E\left(\sum_{i=1}^{n} e_i^2\right)$$

$$= E\left(\Sigma(e_i-\bar{e_i})^2\right) \quad \text{because } \bar{e_i}=0$$

$$= \Sigma E(e_i-\bar{e_i})^2 = \sum_{i=1}^{n}\operatorname{Var}(e_i)$$

Rewrite e_i :

$$e_i = Y_i-\hat{Y_i} = Y_i-(\hat{\beta_0}+\hat{\beta_1}x_i) = Y_i-(\bar{Y}-\hat{\beta_1}\bar{x}+\hat{\beta_1}x_i)$$

$$= (Y_i-\bar{Y})-(x_i-\bar{x})\hat{\beta_1}$$

then $\sum_{i=1}^{n}\operatorname{Var}(e_i) = \sum_{i=1}^{n}\operatorname{Var}[(Y_i-\bar{Y})-(x_i-\bar{x})\hat{\beta_1}]$

$$= \sum_{i=1}^{n}\operatorname{Var}(Y_i-\bar{Y})+\operatorname{Var}[(x_i-\bar{x})\hat{\beta_1}]-2\operatorname{cov}[(Y_i-\bar{Y}),(x_i-\bar{x})\hat{\beta_1}]$$

$$= (n-1)\sigma^2+\Sigma\operatorname{Var}[(x_i-\bar{x})\hat{\beta_1}]-\underbrace{\Sigma 2\operatorname{cov}[(Y_i-\bar{Y}),(x_i-\bar{x})\hat{\beta_1}]}$$

$$= \Sigma 2\operatorname{cov}(\hat{\beta_1},\hat{\beta_1})\cdot(x_i-\bar{x})^2$$

$$= \Sigma 2\operatorname{Var}(\hat{\beta_1})\cdot(x_i-\bar{x})^2$$

$$= (n-1)\sigma^2 + \sum_{i=1}^{n} (x_i - \bar{x})^2 Var(\hat{\beta_1}) - \sum 2(x_i - \bar{x})^2 Var(\hat{\beta_1})$$

$$= (n-1)\sigma^2 - \sum_{i=1}^{n} (x_i - \bar{x})^2 Var(\hat{\beta_1})$$

$$= (n-1)\sigma^2 - \sum (x_i - \bar{x})^2 \cdot \frac{\sigma^2}{\sum (x_i - \bar{x})^2}$$

$$= (n-1)\sigma^2 - \sigma^2 = (n-2)\sigma^2$$

therefore $E(SSE) = (n-2)\sigma^2$

then $E(\frac{SSE}{n-2}) = E(MSE) = \sigma^2$

therefore MSE is an unbiased estimator of σ^2 in any situation.

p8130 HW4 Regression

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Problem 2 Heart disease

We are interested in if there is an association between **total cost** in dollars diagnosed with heart disease and the **number of ER visits**. Other factors will be adjusted later on.

a) short description of data and look at the data

```
heart_disease <- read_csv("./data/HeartDisease.csv") %>%
  mutate(gender = as.factor(gender),
         complications = as.factor(complications))
```

Overview:

In this dataset, there are 788 observations of patients with 10 variables:

- **id**: subscriber id
- **totalcost**: total cost ($) of claims by subscriber
- **age**: age of subscribers
- **gender**: gender of patient (1 = male, 0 = otherwise)
- **interventions**: total number of interventions or procedures carried out
- **drugs**: number of tracked drugs prescribed
- **ERvisits**: number of ER visits
- **complications**: number of complications that arose during heart disease treatment
- **comorbidities**: number of co-presence of other diseases
- **duration**: duration of treament condition (in days)

Based our investigation interest, the main outcome is **total cost** of subscribers with heart disease and the main predictor is **ERvisits** (number of ER visits). Other important covariates also need to be considered because they could be confounders or have modifier effects on the association relationship between our main predictor and main outcome, including age, interventions, drugs used, complications, and duration of disease. We will first take a look at the availible variables:

i) First we took a look at the distribution of each variable of interest

```
variable_set1 <- dplyr::select(heart_disease, totalcost, ERvisits, everything(),
                               -c(id, gender, complications))
variable_set2 <- dplyr::select(heart_disease, gender, complications)
knitr::kable(summary(variable_set1))
```

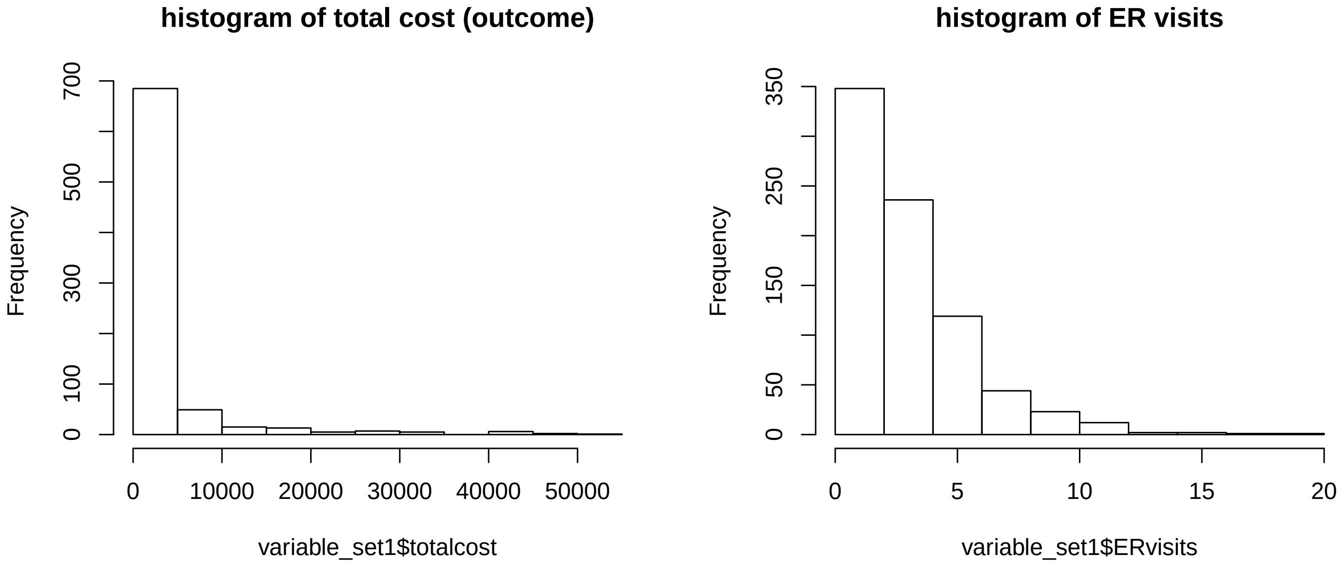
totalcost	ERvisits	age	interventions	drugs	comorbidities	duration
Min. : 0.0	Min. : 0.000	Min. :24.00	Min. : 0.000	Min. :0.0000	Min. : 0.000	Min. : 0.00
1st Qu.: 161.1	1st Qu.: 2.000	1st Qu.:55.00	1st Qu.: 1.000	1st Qu.:0.0000	1st Qu.: 0.000	1st Qu.: 41.75
Median : 507.2	Median : 3.000	Median :60.00	Median : 3.000	Median :0.0000	Median : 1.000	Median :165.50
Mean : 2800.0	Mean : 3.425	Mean :58.72	Mean : 4.707	Mean :0.4467	Mean : 3.767	Mean :164.03
3rd Qu.: 1905.5	3rd Qu.: 5.000	3rd Qu.:64.00	3rd Qu.: 6.000	3rd Qu.:0.0000	3rd Qu.: 5.000	3rd Qu.:281.00
Max. :52664.9	Max. :20.000	Max. :70.00	Max. :47.000	Max. :9.0000	Max. :60.000	Max. :372.00

```
knitr::kable(summary(variable_set2))
```

gender	complications
0:608	0:745
1:180	1: 42
NA	3: 1

Visualize the distribution of variables:

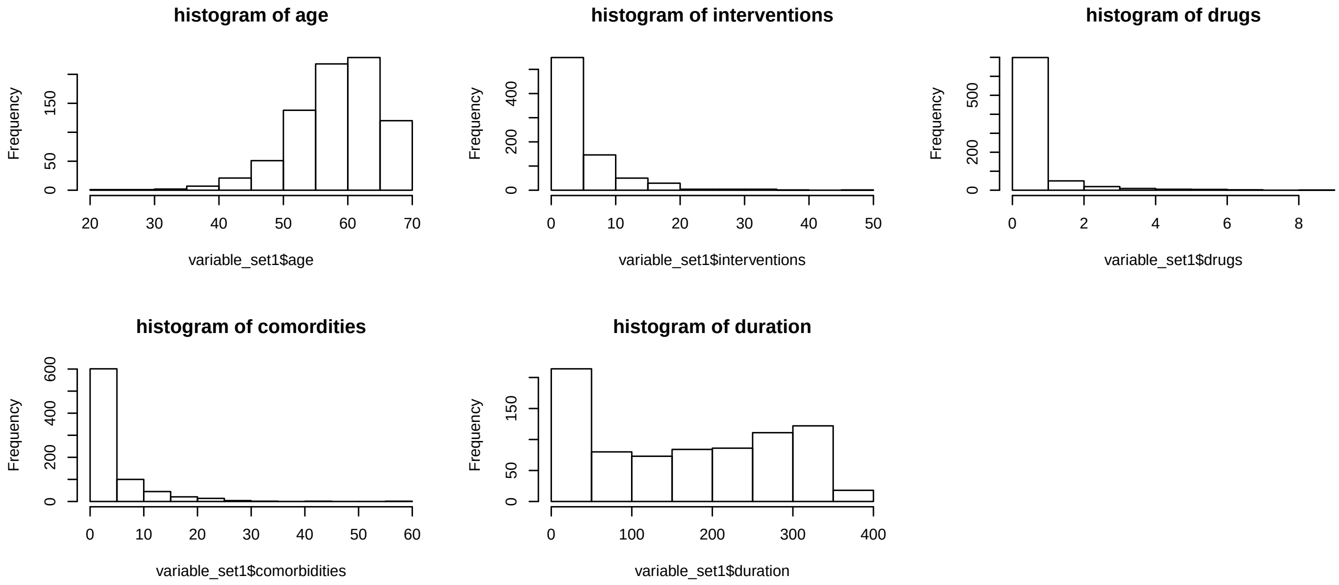
```r
par(mfrow = c(1,2))
hist(variable_set1$totalcost, main = "histogram of total cost (outcome)")
hist(variable_set1$ERvisits, main = "histogram of ER visits")
```



Describe the main outcome and main predictor:

Since total cost and ER visits are both heavily right skewed on the histograms, we better use median and IQR in the summay table to describe them. Especially for total cost, there are many extreme values at the right tail end which needed to be investigated further in the following analysis. We categorized two other remaining variables gender and complications as categorical variables. From the summary table, we saw

```r
par(mfrow = c(2,3))
hist(variable_set1$age, main = "histogram of age")
hist(variable_set1$interventions, main = "histogram of interventions")
hist(variable_set1$drugs, main = "histogram of drugs")
hist(variable_set1$comorbidities, main = "histogram of comordities")
hist(variable_set1$duration, main = "histogram of duration")
```



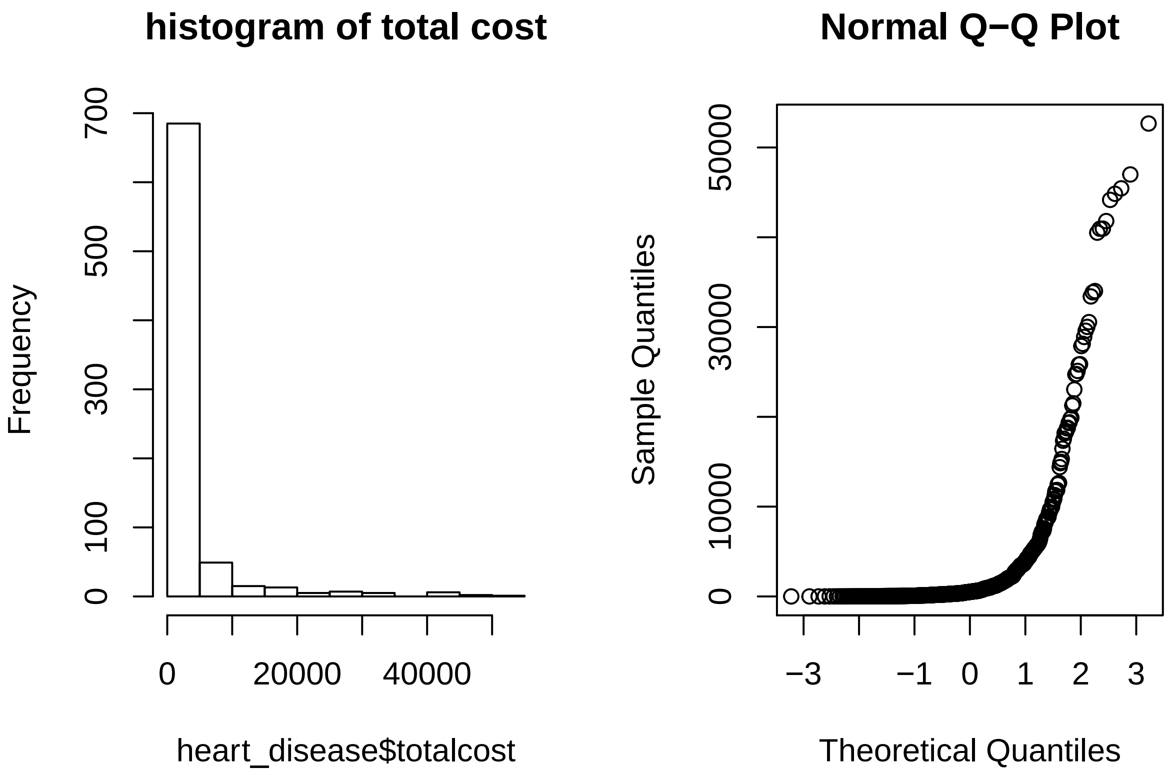
Describe other covariables:

Age is slightly left skewed which means elder people have been overly sampled. The median of intervention is about 5 with large IQR = 5. Number of tracked drugs are right skewed so we better make it a categorical variable. Commordities have median of 3.7 with large IQR = 5. Duration of heart disease is roughly uniformly distributed from 50 to 350 days with median 165 days and IQR 240 days. Therefore, these co-variables are not normally distributed in the sample. We have categorized two other remaining variables gender and complications as categorical variables. From the summary table, we saw males are undersampled and other sexs are oversampled (608). The majority of patients do not have any complications.

b) investigate the shape of distribution for total cost

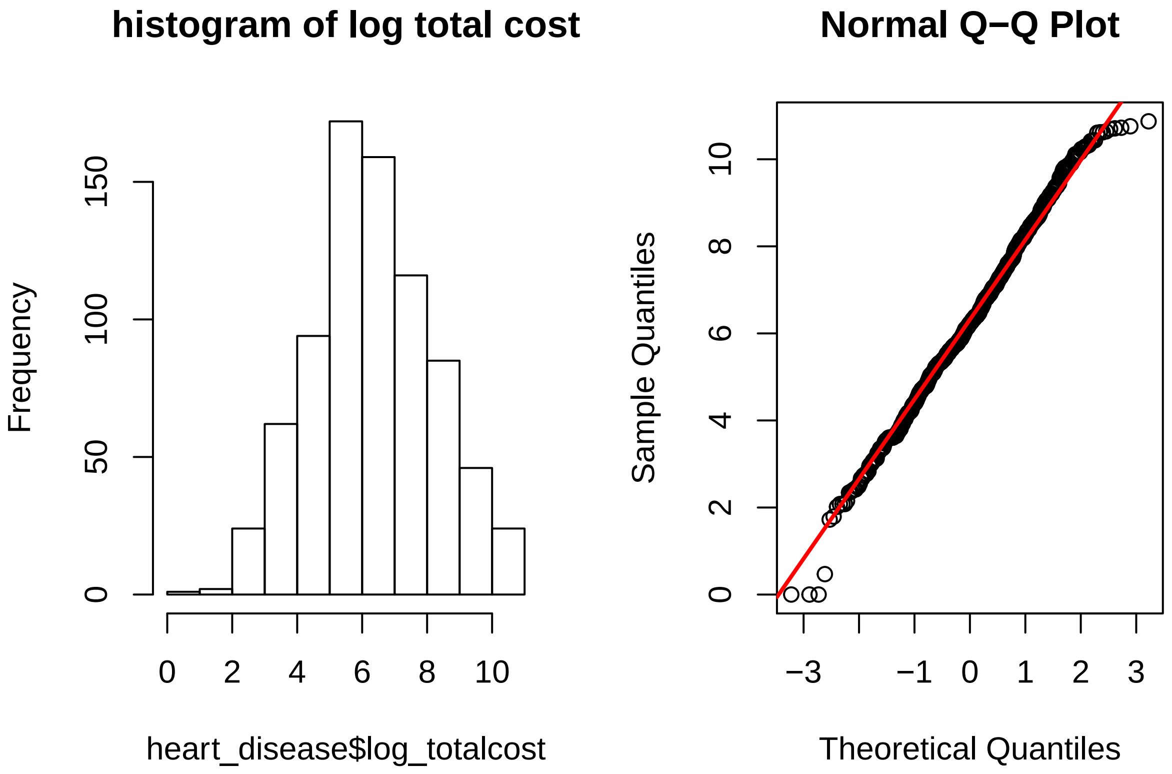
First we examined the distribution of raw data of total cost and check its normality:

```
par(mfrow = c(1,2))
hist(heart_disease$totalcost, main = "histogram of total cost")
qqnorm(heart_disease$totalcost)
```



Then we try **log transformation** on totalcost to see if this will improve the normality.

```
heart_disease <- mutate(heart_disease, log_totalcost = log(totalcost))
par(mfrow = c(1,2))
hist(heart_disease$log_totalcost, main = "histogram of log total cost")
heart_disease$log_totalcost[is.infinite(heart_disease$log_totalcost)] = 0.001
qqnorm(heart_disease$log_totalcost)
qqline(heart_disease$log_totalcost, col = "red", lwd = 2)
```

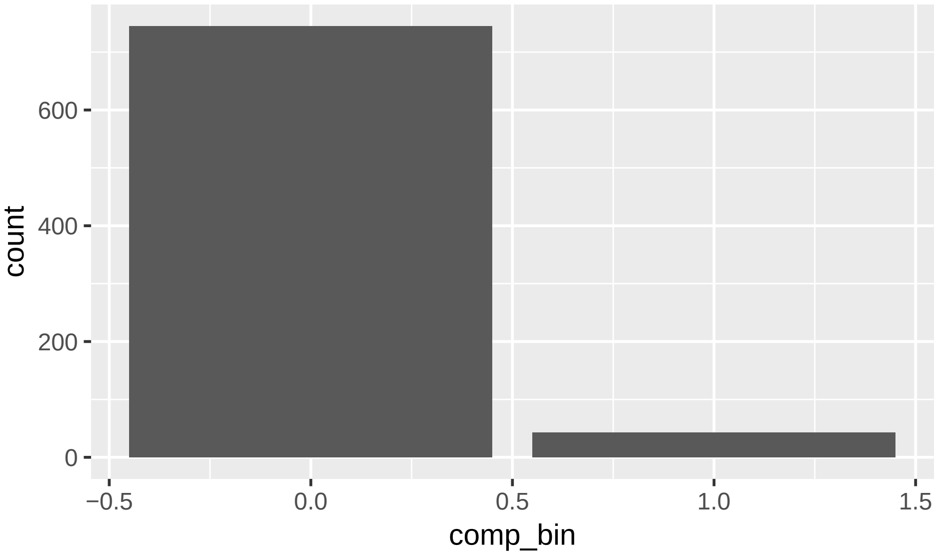


Comment: After log transformation, we saw a pretty good bell shaped ditribution. So we will use this transformed data in the linear model fitting and interpretation.

c) dichotomize complications

0 represents no complications; 1 represents having complications

```
heart_disease <- heart_disease %>%
  mutate(comp_bin = ifelse(complications == 0, 0, 1))
heart_disease %>% ggplot(aes(x = comp_bin)) + geom_bar()
```



d) fit linear model SLR

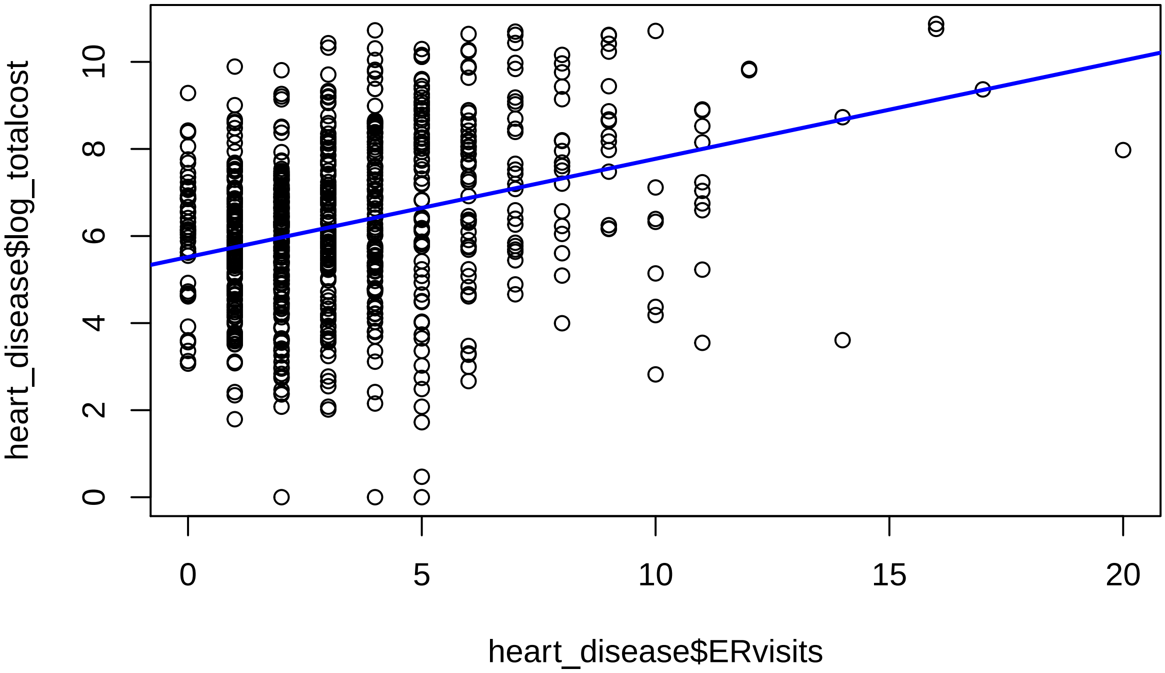
From part (b), we saw the transformed data look better in normal shape, we will use the transformed data to fit SLR. So we fit a simple linear regression model between outcome **log_totalcost** and predictor **ERvisits**. Let Y_i = response(total cost), X_i = predictor (ERvisits).

Then our model is $logY_i = \beta_0 + \beta_1 X_i + \epsilon_i$. Here we assume the error is normally distribued. then $\epsilon_i \sim N(0, \sigma^2)$

```
SLR <- lm(log_totalcost ~ ERvisits, data = heart_disease)
summary(SLR)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits, data = heart_disease)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.6444 -1.1195  0.0371  1.2872  4.3046
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.51701    0.10585  52.119   <2e-16 ***
## ERvisits     0.22569    0.02449   9.215   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.812 on 786 degrees of freedom
## Multiple R-squared:  0.09749,    Adjusted R-squared:  0.09635
## F-statistic: 84.91 on 1 and 786 DF,  p-value: < 2.2e-16
```

```
plot(heart_disease$ERvisits, heart_disease$log_totalcost)
abline(SLR, col = "blue", lwd = 2)
```



The result of regression tells that the fitted model is :

$$\hat{logY_i} = 5.517 + 0.23X_i$$

Results and Interpretation: In order to interpret the slope coefficient, we need to transform the response back to its original scale and interpret. After transformation, the mean reponse Yi will multiply by 1.26 for every increase in ER visit. When the ER visit is 0, the expected total cost in dollar *on logarithm scale* will be 249 dollars on original scale. The p value for two estimators β_0 and β_1 are well below 0.001. So we are very confident that there is a strong association between total cost and ER visits, and our simple regression model describes their relationship.

e) fit MLR with comp_bin and ERvisits

i) test if **comp_bin** is an effect modifier of the relationship between **totalcost** and **ERvisits**

Let Y_i = response(total cost), X_{i1} = predictor (ERvisits), X_{i2} = comp_bin(factor with two levels)

The full model is :

$logY_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i$

Now add a potential modifier (interaction):

$logY_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} X_{i3} + \epsilon_i$

Our hypothesis statement is: $H_0 : \beta_3 = 0$ vs. $H_a : \beta_3 \neq 0$

```
MLR_comp <- lm(log_totalcost ~ ERvisits + comp_bin, data = heart_disease)
MLR_comp_inter <- lm(log_totalcost ~ ERvisits + comp_bin + ERvisits*comp_bin, data = heart_disease)
summary(MLR_comp)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits + comp_bin, data = heart_disease)
```

```
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.5156 -1.0745 -0.0009  1.1931  4.4110
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.50040    0.10354  53.125  < 2e-16 ***
## ERvisits     0.20324    0.02423   8.388 2.29e-16 ***
## comp_bin     1.71352    0.28117   6.094 1.72e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.772 on 785 degrees of freedom
## Multiple R-squared:  0.1383, Adjusted R-squared:  0.1361
## F-statistic: 62.98 on 2 and 785 DF,  p-value: < 2.2e-16
```

```
summary(MLR_comp_inter)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits + comp_bin + ERvisits *
##     comp_bin, data = heart_disease)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.5265 -1.0797  0.0104  1.2075  4.4066
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)        5.47863    0.10576  51.801  < 2e-16 ***
## ERvisits           0.20978    0.02508   8.363 2.77e-16 ***
## comp_bin           2.20005    0.55850   3.939 8.90e-05 ***
## ERvisits:comp_bin -0.09779    0.09700  -1.008    0.314
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.772 on 784 degrees of freedom
## Multiple R-squared:  0.1394, Adjusted R-squared:  0.1361
## F-statistic: 42.32 on 3 and 784 DF,  p-value: < 2.2e-16
```

```
anova(MLR_comp, MLR_comp_inter) %>% tidy
```

```
## Warning: Unknown or uninitialised column: 'term'.
```

```
## # A tibble: 2 x 6
##   res.df   rss    df sumsq statistic p.value
## *  <dbl> <dbl> <dbl> <dbl>     <dbl>   <dbl>
## 1    785 2465.    NA NA         NA     NA
## 2    784 2461.     1  3.19       1.02   0.314
```

Comment: Based on the regression summary and anova result comparing two models, p value for the interaction coefficient β_3 is 0.314, which is quite large. Anova F test for comparing two models indicate adding the interaction term does not increase SSR by significant amount. The adjusted R^2 is the same for these nested models. Therefore at 0.95 significance level, we do not have evidence to reject the null. Hence there is no significant interaction or modifier effect of complications in the relationship between total cost and ER visits.

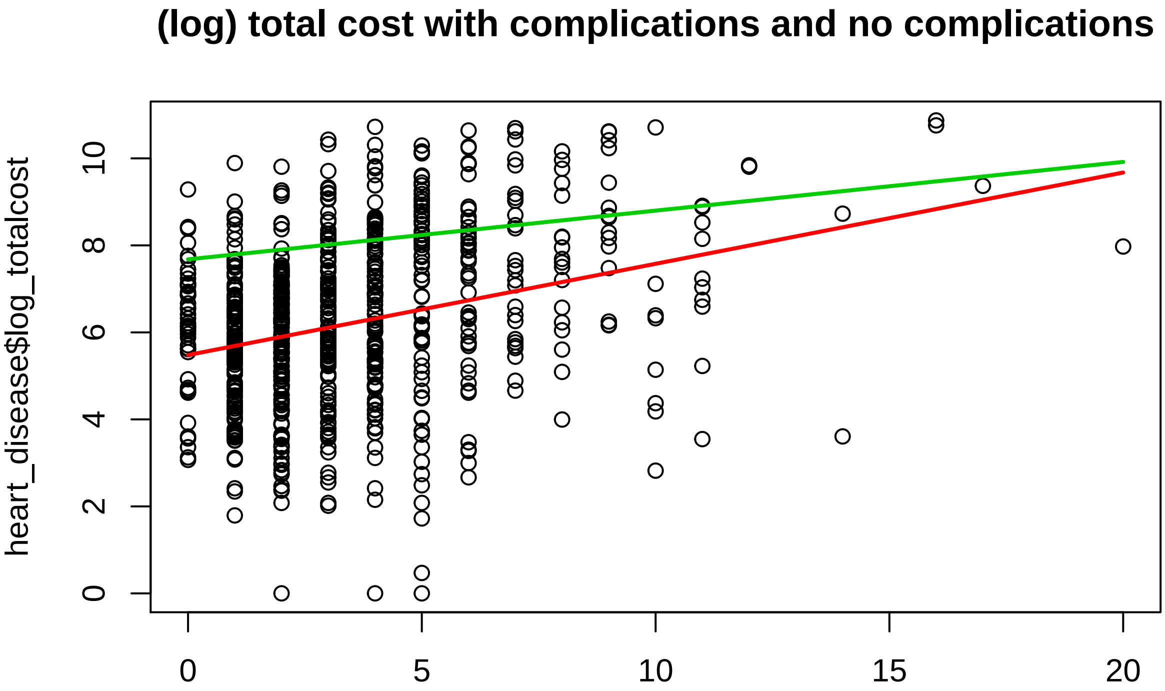
We can also visualize this interaction model:

```
range(heart_disease$ERvisits)
```

```
## [1]  0 20
```

```
ER <- seq(0,20,0.5)
beta <- MLR_comp_inter$coefficients
# comp_bin = 0
yhat1 <- beta[1] + beta[2]*ER
# comp_bin = 1
yhat2 <- beta[1] + beta[3] + (beta[2] + beta[4])*ER

plot(heart_disease$ERvisits, heart_disease$log_totalcost,
     main = "(log) total cost with complications and no complications",
     xlab = "")
lines(ER, yhat1, col = 2, lwd = 2) # total cost of comp_bin = 0 with fixed ER
lines(ER, yhat2, col = 3, lwd = 2) # total cost of comp_bin greater than 0 with fixed ER
```



model-1.bb

Comment: although we expect two parallel lines on the plot if there is truly no interaction effect. However, the statistical test we conducted above indicate that although there is some interaction effect, the effect is not significant. As a conclusion, we will not consider this mediator effect.

ii) test if **comp_bin** is a confounder of relationship between total cost and ERvisits

Model 1 without comp_bin: $logY_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i$

Model 2 with comp_bin: $logY_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i$

```
SLR <- lm(log_totalcost ~ ERvisits, data = heart_disease)
MLR_comp <- lm(log_totalcost ~ ERvisits + factor(comp_bin), data = heart_disease)
summary(SLR)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits, data = heart_disease)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.6444 -1.1195  0.0371  1.2872  4.3046
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.51701    0.10585  52.119   <2e-16 ***
## ERvisits     0.22569    0.02449   9.215   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.812 on 786 degrees of freedom
## Multiple R-squared:  0.09749,    Adjusted R-squared:  0.09635
## F-statistic: 84.91 on 1 and 786 DF,  p-value: < 2.2e-16
```

```
summary(MLR_comp)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits + factor(comp_bin), data = heart_disease)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.5156 -1.0745 -0.0009  1.1931  4.4110
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)        5.50040    0.10354  53.125  < 2e-16 ***
## ERvisits           0.20324    0.02423   8.388 2.29e-16 ***
## factor(comp_bin)1  1.71352    0.28117   6.094 1.72e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.772 on 785 degrees of freedom
## Multiple R-squared:  0.1383, Adjusted R-squared:  0.1361
## F-statistic: 62.98 on 2 and 785 DF,  p-value: < 2.2e-16
```

```
anova(SLR, MLR_comp) %>% tidy
```

```
## Warning: Unknown or uninitialised column: 'term'.
```

```
## # A tibble: 2 x 6
##   res.df   rss    df sumsq statistic  p.value
## *  <dbl> <dbl> <dbl> <dbl>     <dbl>    <dbl>
## 1    786 2581.    NA   NA        NA   NA
## 2    785 2465.     1  117.      37.1  1.72e-9
```

Comment: From the regression result, we saw the coefficient of **comp_bin** is quite significant with p value well below 0.001. If we calculate the F statistics for this nested model: $F = \frac{(SSR_{large}-SSR_{small})/1}{SSE_{large}/785} = \frac{116.6}{2464.668/785} = 37.14 \sim F(1,785)$, then we will reject the null at 0.05 level since this test statistics is very large. In addition, the adjusted R^2 increased when adjusting for **comp_bin**. The slope of ERvisits changes from 0.225 to 0.2 (change by 11%), which is significant. The anova result for two model comparison shows that adding complication variable greatly reduce the overall SSTO while increasing SSR, with p value well below 0.001. So we should include complications in our linear model.

iii) decide if comp_bin should be included along with ERvisits

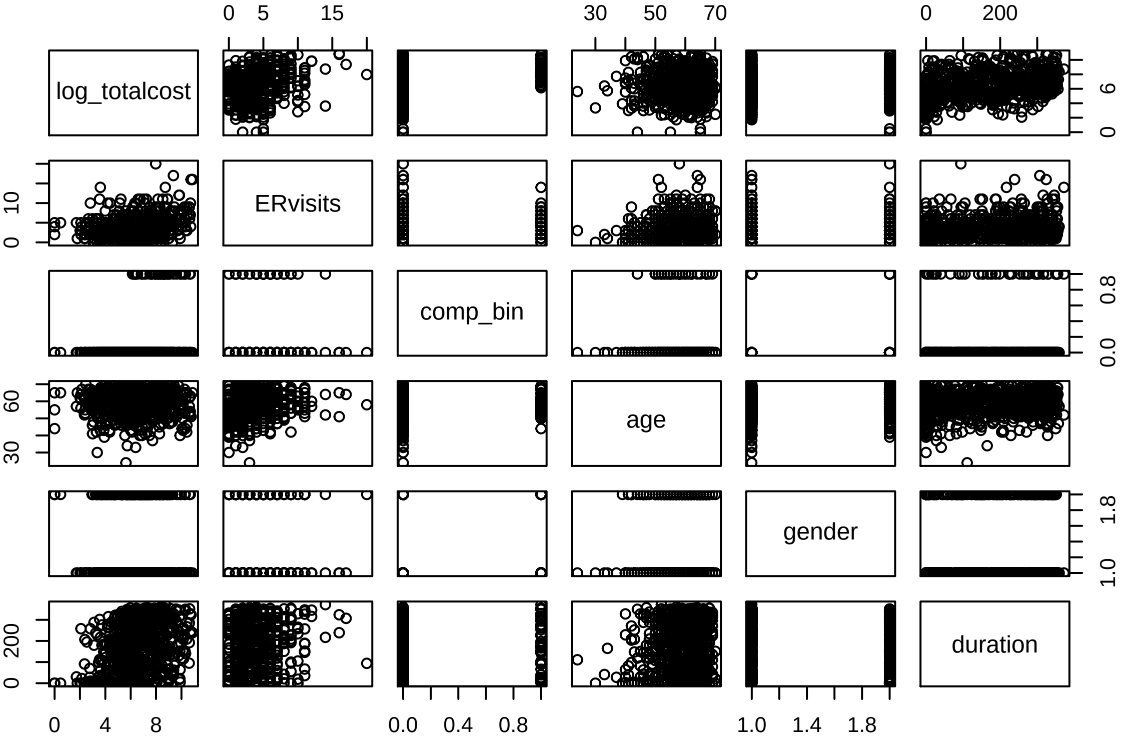
From above test, we should include **comp_bin** as a predictor in our additive linear model. Adding **com_bin** in the model increase SSR significantly. The coefficient of **comp_bin** is also significant in the linear model from above discussion.

f) examine additional covariates

(i) fit a MLR

We start with screening for any colineariltity of variables among **ERvisits**, **comp_bin**, **age**, **gender**, and **duration**

```
heart_disease %>% dplyr::select(log_totalcost, ERvisits, comp_bin, age, gender, duration) %>% pairs()
```



Then we fit a MLR with all variables of interests:

```
fit_all <- lm(log_totalcost ~ ERvisits + comp_bin + age + factor(gender) + duration,
              data = heart_disease)
vif(fit_all)
```

```
##        ERvisits        comp_bin             age factor(gender)1
##        1.057863        1.030044        1.024444        1.013165
##        duration
##        1.042700
```

```
summary(fit_all)
```

```
##
## Call:
## lm(formula = log_totalcost ~ ERvisits + comp_bin + age + factor(gender) +
##     duration, data = heart_disease)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.4529 -1.0367 -0.1108  0.9507  4.3478
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)      5.9377052  0.5139206  11.554  < 2e-16 ***
## ERvisits         0.1746113  0.0227290   7.682 4.68e-14 ***
## comp_bin         1.5103177  0.2602679   5.803 9.46e-09 ***
## age             -0.0208968  0.0087343  -2.392    0.017 *
## factor(gender)1 -0.2075121  0.1396551  -1.486    0.138
## duration         0.0057688  0.0004922  11.720  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.635 on 782 degrees of freedom
## Multiple R-squared:  0.269,  Adjusted R-squared:  0.2643
## F-statistic: 57.56 on 5 and 782 DF,  p-value: < 2.2e-16
```

```
anova(fit_all)
```

```
## Analysis of Variance Table
##
## Response: log_totalcost
##                  Df  Sum Sq Mean Sq  F value    Pr(>F)
## ERvisits          1  278.84  278.84 104.2978 < 2.2e-16 ***
## comp_bin          1  116.61  116.61  43.6154  7.38e-11 ***
## age               1    1.83    1.83   0.6828    0.4089
## factor(gender)    1    4.91    4.91   1.8368    0.1757
## duration          1  367.24  367.24 137.3612 < 2.2e-16 ***
## Residuals       782 2090.69    2.67
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Comment: From the VIF test, there is no significant colinearity between predictors. Based on the t test statistics for each regression coefficients along with their p values, we observed linear relationship between gender and total cost is weak. The adjusted R^2 for this full model is about 0.26. Then we performed ANOVA test. From F test for each nested model indicate that we should definitely include **duration** in our model and better exclude age and gender since they do not much additional information in the exisiting model. But we still need to decide which combinations of predictors will provide the best association between predictors and outcomes. So we run into the stage of model selection:

(ii) compare SLR and MLR

Here we construct several nested MLR to determine if we want to include the predicor in the model or not.

```
SLR <- lm(log_totalcost ~ ERvisits, data = heart_disease) # start from here
heart <- heart_disease %>% dplyr::select(ERvisits, comp_bin, duration, age, gender,log_totalcost)
```

Find the best model using both adjusted R^2 and Cp criterion for each size of predictors

```
MLRs <- regsubsets(log_totalcost ~ ., data=heart)
summary(MLRs)
```

```
## Subset selection object
## Call: regsubsets.formula(log_totalcost ~ ., data = heart)
## 5 Variables  (and intercept)
##          Forced in Forced out
## ERvisits     FALSE      FALSE
## comp_bin     FALSE      FALSE
## duration     FALSE      FALSE
## age          FALSE      FALSE
## gender1      FALSE      FALSE
## 1 subsets of each size up to 5
## Selection Algorithm: exhaustive
##          ERvisits comp_bin duration age gender1
## 1  ( 1 ) " "      " "      "*"      " " " "
## 2  ( 1 ) "*"      " "      "*"      " " " "
## 3  ( 1 ) "*"      "*"      "*"      " " " "
## 4  ( 1 ) "*"      "*"      "*"      "*" " "
## 5  ( 1 ) "*"      "*"      "*"      "*" "*"
```

So we have the best models for each size of predictors in terms of Cp and adjusted R^2. Then we can build models for each of them:

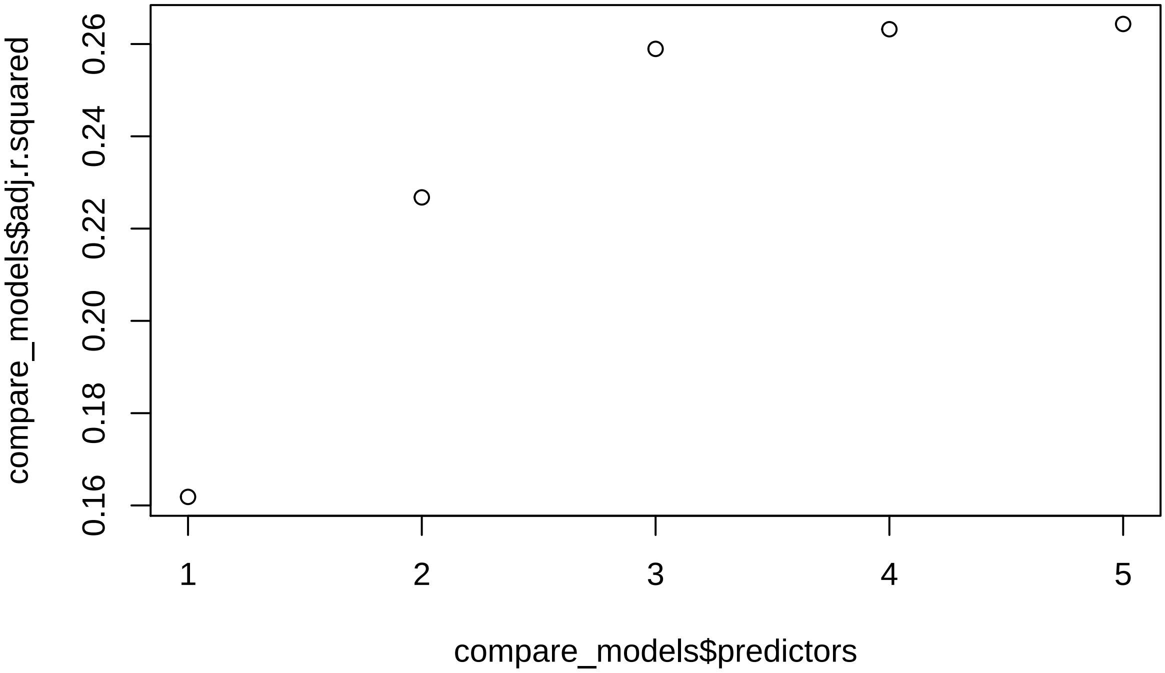
```
modelfit_1 <- lm(log_totalcost ~ duration, data = heart)
modelfit_2 <- lm(log_totalcost ~ ERvisits + duration, data = heart)
modelfit_3 <- lm(log_totalcost ~ ERvisits + comp_bin + duration, data = heart)
modelfit_4 <- lm(log_totalcost ~ ERvisits + comp_bin + duration + age, data = heart)
modelfit_5 <- lm(log_totalcost ~ ERvisits + comp_bin + duration + age + factor(gender), data = heart)
model_result <- tibble(predictors = c(1,2,3,4,5),
       data = list(modelfit_1, modelfit_2, modelfit_3, modelfit_4, modelfit_5))
```

them compare their AIC, BIC, and adjusted R^2

```
compare_models <- model_result %>% mutate(glance_result = map(data, glance)) %>%
  dplyr::select(-data) %>%
  unnest() %>%
  dplyr::select(predictors, AIC, BIC, adj.r.squared)
knitr::kable(compare_models)
```

predictors	AIC	BIC	adj.r.squared
1	3117.940	3131.949	0.1618537
2	3055.420	3074.098	0.2267613
3	3022.911	3046.258	0.2589496
4	3019.362	3047.379	0.2632095
5	3019.141	3051.827	0.2643443

```
plot(compare_models$predictors, compare_models$adj.r.squared)
```



```
summary(modelfit_3) %>% tidy
```

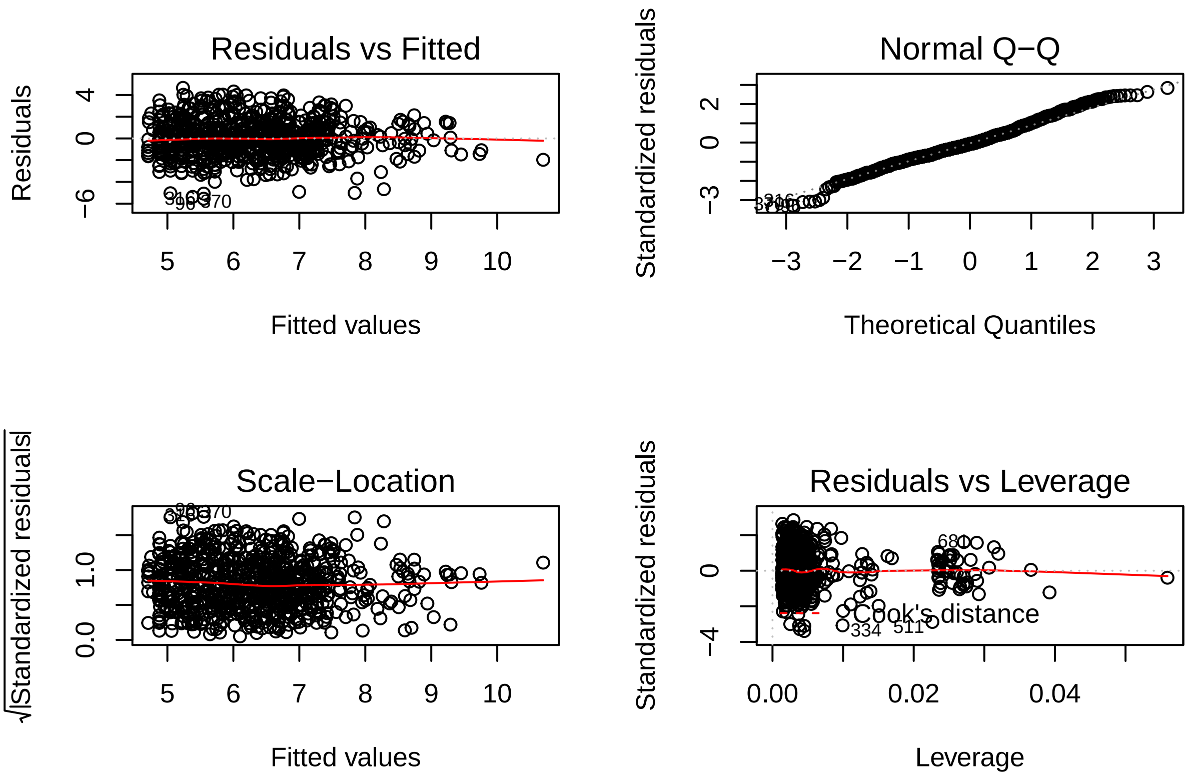
```
## # A tibble: 4 x 5
##   term        estimate std.error statistic   p.value
##   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept)  4.71     0.118        39.9  1.11e-190
## 2 ERvisits      0.168     0.0226         7.43 2.88e- 13
## 3 comp_bin      1.55      0.261          5.92 4.69e-  9
## 4 duration      0.00561   0.000490      11.5  3.44e- 28
```

Comment: Since the investigators are primarily interested in the assoication between total cost and ER visits while adjusting for other covariates, our model choice should better rely on adjusted coefficient of determination which describes the goodness of fit of the model. The adjusted R^2 does not change much after the 3 predictors model. so we will use the three predictors: **ERvisit**, **comp_bin** and **duration**. So our model becomes:

total cost (log) ~ **0.17ERvisits + 1.545comp_bin + 0.0056duration**

then we check the assumption again:

```
par(mfrow=c(2,2))
plot(modelfit_3)
```



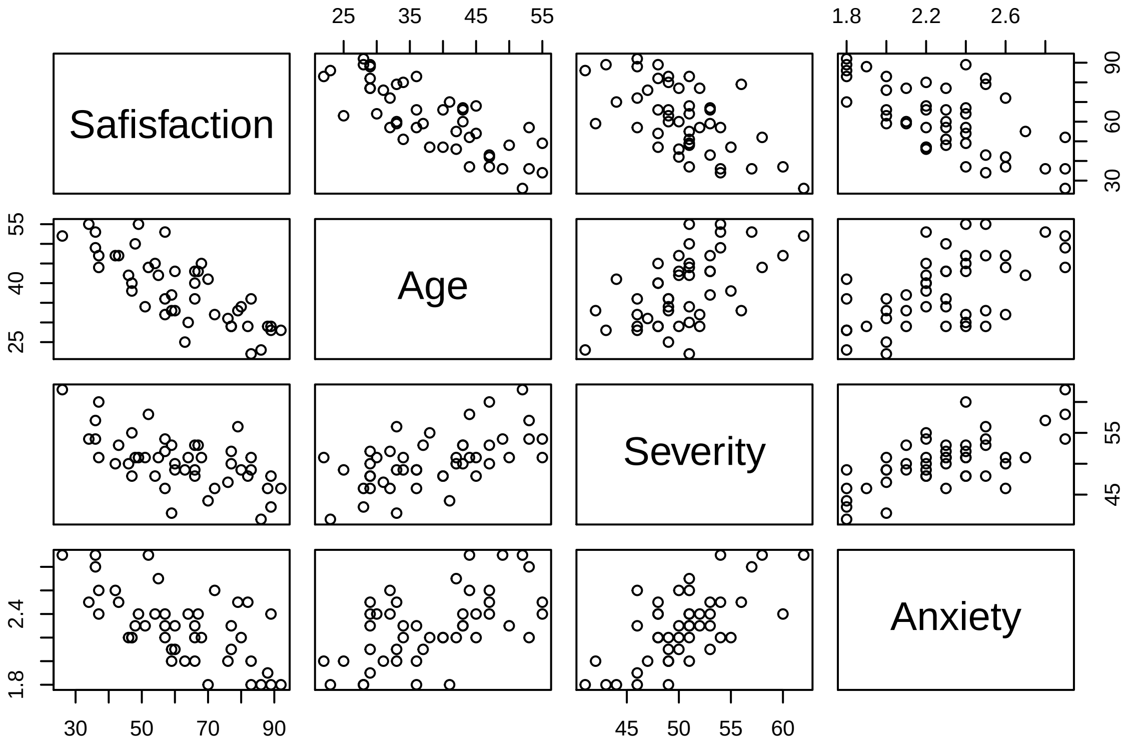
Comment: The assumption seems to be satisfied in our 3 predictor model stated above. So it is valid to use that model to descibe the association between total cost and predictors.

Problem 3

The investigators wants to test the relationship between patient's satisfaction (Y) and age, severity of illness, and anxiety level. The dataset contains 46 patients observations

a) correlation matrix

```
pat_sat <- readxl::read_excel("./data/PatSatisfaction.xlsx")
pairs(pat_sat)
```



```
cor(pat_sat) %>% knitr::kable()
```

	Safisfaction	Age	Severity	Anxiety
Safisfaction	1.0000000	-0.7867555	-0.6029417	-0.6445910
Age	-0.7867555	1.0000000	0.5679505	0.5696775
Severity	-0.6029417	0.5679505	1.0000000	0.6705287
Anxiety	-0.6445910	0.5696775	0.6705287	1.0000000

Comment: the correlation matrix shows that age, severity of illness and anxiety level are consistently negatively correlated with satisfaction score. Age seems to have the strongest correlation with satisfaction score while the other variables also have significant coefficient of correlations. However, covariates are positively correlated with each other significantly as well. So we should keep this in mind.

b) fit a MLR and test whether there is a regression relation

In this MLR model, we will use the satisfaction as response while all other three variables as predictors. Let $Y_i =$ satisfaction (outcome), $X_{i1} =$ age, $X_{i2} =$ severity of illness, $X_{i3} =$ anxiety level

Full Model: $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_2 X_{i2} + \epsilon_i$

```
MLR_all <- lm(Safisfaction ~ Age + Severity + Anxiety, data = pat_sat)
summary(MLR_all)
```

```
##
## Call:
## lm(formula = Safisfaction ~ Age + Severity + Anxiety, data = pat_sat)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -18.3524  -6.4230   0.5196   8.3715  17.1601
##
## Coefficients:
```

```
##              Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 158.4913    18.1259   8.744 5.26e-11 ***
## Age          -1.1416     0.2148  -5.315 3.81e-06 ***
## Severity     -0.4420     0.4920  -0.898   0.3741    
## Anxiety     -13.4702     7.0997  -1.897   0.0647 .  
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 10.06 on 42 degrees of freedom
## Multiple R-squared:  0.6822, Adjusted R-squared:  0.6595 
## F-statistic: 30.05 on 3 and 42 DF,  p-value: 1.542e-10
```

First We need to do an overall F test for the three predictors:

State the hypothesis:

$$H_0 : \beta_1 = \beta_2 = \beta_3 = 0 H_a : \text{at least one of the coefficient is nonzero}$$

Test Statistic: $F_{test} = \frac{MSR}{MSE} = \frac{9120.5/3}{4248.8/42} = 30.05 \sim F(3,42)$

Decision Rule: at $\alpha = 0.05$, we will reject the null if $F_{test} > F(0.95, 3, 42) = 2.83$. Here we have $F_{test} = 30.05 > 2.83$, so we should reject the null and conclude that there is at least one linear association among these predictors with the outcome satisfaction level.

c) compute 95% CI for estimated coefficients

create a table with estimator and 95% Confidence Interval:

```
summary(MLR_all) %>% 
  tidy %>% 
  mutate(lower_bound = estimate - qt(0.975, 42) * std.error,
         upper_bound = estimate + qt(0.975, 42) * std.error) %>% 
  dplyr::select(term, estimate, std.error, lower_bound, upper_bound)
```

```
## # A tibble: 4 x 5
##   term        estimate std.error lower_bound upper_bound
##   <chr>          <dbl>     <dbl>       <dbl>       <dbl>
## 1 (Intercept)  158.       18.1        122.       195.   
## 2 Age           -1.14      0.215       -1.58      -0.708
## 3 Severity      -0.442     0.492       -1.43       0.551
## 4 Anxiety      -13.5       7.10       -27.8        0.858
```

Interpret severity of illness:

While holding age and anxiety level constant, the expected **decrease** of satisfaction score with an unit increase in severity of illness is 0.442. We are 95% confident that the true mean change of satisfaction score with one unit increase in severity of illness is between -1.43 to 0.551.

d) Obtain interval estimate for a new patient

```
new_data <- tibble(Age = 35,
                   Severity = 42,
                   Anxiety = 2.1)
predict.lm(MLR_all, new_data, interval="prediction", conf.level = 0.95)
```

```
##        fit      lwr      upr
## 1 71.68332 50.06237 93.30426
```

Comment: The point estimator for this new patient's satisfaction score is 71.7. We are 95% confident that the predicted satisfaction score for this new patient will be between 50.1 to 93.3

e) test whether anxiety level can be dropped from the MLR

State hypothesis: $H_0 : \beta_3 = 0$ vs. $H_a : \beta_3 \neq 0$

Test statistic: $F_{test} = \frac{SSR_{X3|X1,X2}/1}{SSE_{X1,X2}/43} = \frac{364.16}{4613/43} = 3.6 \sim F(1, 43)$

Rejection rule: at $\alpha = 0.05$, we should reject null if $F_{test} > F(0.95, 1, 43) = 4.07$. However, we obtained $F_{test} = 3.6 < 4.07$, so we do not have evidence to reject the null. Therefore we should not include anxiety level as one of the explaintary variable since it does not reduce SSTO significantly in a model with exisiting variables age and Severity of illness.

Perform test in R:

```
MLR_Age_Sev <- lm(Safisfaction ~ Age + Severity, data = pat_sat)
MLR_all <- lm(Safisfaction ~ Age + Severity + Anxiety, data = pat_sat)
anova(MLR_Age_Sev) %>% tidy
```

```
## # A tibble: 3 x 6
##   term          df sumsq meansq statistic   p.value
##   <chr>      <int> <dbl>  <dbl>     <dbl>     <dbl>
## 1 Age            1 8275.  8275.     77.1   3.80e-11
## 2 Severity       1  481.   481.      4.48  4.01e- 2
## 3 Residuals     43 4613.   107.     NA    NA
```

```
anova(MLR_all) %>% tidy
```

```
## # A tibble: 4 x 6
##   term          df sumsq meansq statistic   p.value
##   <chr>      <int> <dbl>  <dbl>     <dbl>     <dbl>
## 1 Age            1 8275.  8275.     81.8   2.06e-11
## 2 Severity       1  481.   481.      4.75  3.49e- 2
## 3 Anxiety        1  364.   364.      3.60  6.47e- 2
## 4 Residuals     42 4249.   101.     NA    NA
```

```
anova(MLR_Age_Sev, MLR_all) %>% tidy
```

```
## Warning: Unknown or uninitialised column: 'term'.

## # A tibble: 2 x 6
##   res.df   rss    df sumsq statistic p.value
## *  <dbl> <dbl> <dbl> <dbl>     <dbl>   <dbl>
## 1     43 4613.    NA    NA        NA    NA
## 2     42 4249.     1   364.      3.60   0.0647
```