# 19

# *Multidimensional Numerical Methods*

In the remainder of the book we concentrate on finite volume methods for multidimensional hyperbolic equations. We will begin by considering uniform Cartesian grids in two dimensions, using the notation illustrated in Figure 19.1(a). The value $Q_{ij}^n$ represents a cell average over the $(i, j)$ grid cell at time $t_n$,

$$Q_{ij}^n \approx \frac{1}{\Delta x\, \Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, y, t_n)\, dx\, dy. \tag{19.1}$$

As in one dimension, we can use the integral form of the equations to determine how this cell average varies with time, and develop finite volume methods based on numerical approximations to the fluxes at each cell edge. Various approaches to doing this are summarized starting in Section 19.2.

## 19.1 Finite Difference Methods

Rather than working with the cell averages and the integral form of the equations, one could instead view $Q_{ij}^n$ as a pointwise approximation to the value of $q(x_i, y_j, t_n)$ at the point indicated in Figure 19.1(b). Discretizing the differential equations by finite differences then gives a finite difference method. As in one dimension, this approach often gives methods that look very similar to related finite volume methods. We will concentrate on finite volume methods, since this viewpoint allows the derivation of methods that are more robust when discontinuities are present, as well as being exactly conservative. However, it is sometimes useful to think of the methods in terms of their finite difference interpretation, in particular when computing the local truncation error by comparing with a Taylor series expansion of the true solution at the point $(x_i, y_j)$. The flux differences arising in a finite volume method are often seen to give approximations to terms in this Taylor expansion, at least when applied to smooth solutions.

### *19.1.1 Taylor Series Expansion of the Exact Solution*

We develop the Taylor series expansion of the exact solution at a point after a single time step for the constant-coefficient linear hyperbolic system

$$q_t + Aq_x + Bq_y = 0. \tag{19.2}$$

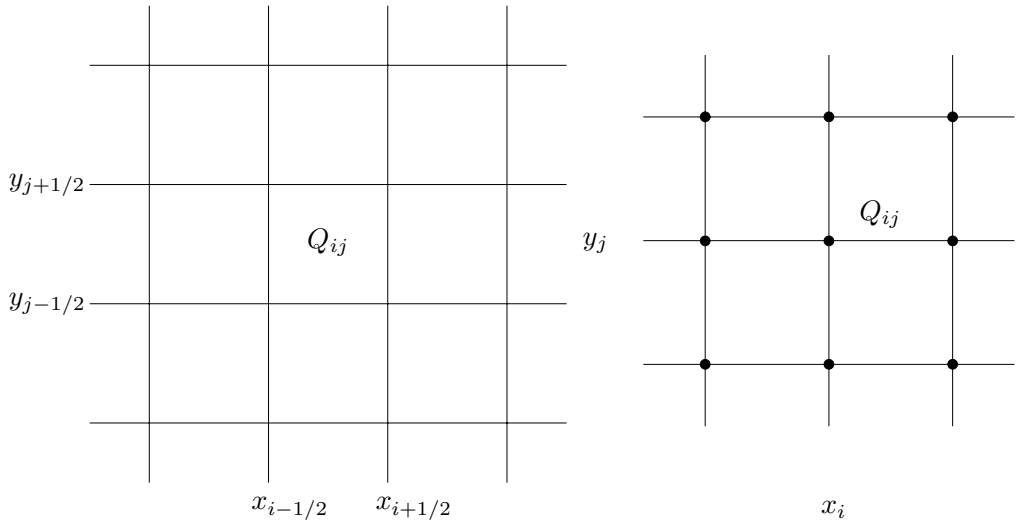


Fig. 19.1. (a) Finite volume grid in two space dimensions, where $Q_{ij}$ represents a cell average. (b) Finite difference grid in two dimensions, where $Q_{ij}$ represents a pointwise value.

We will require higher-order time derivatives in this expansion, which are most easily found from the expression

$$\partial_t^j q = [-(A\partial_x + B\partial_y)]^j q. \tag{19.3}$$

So, in particular,

$$q_{tt} = A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}. \tag{19.4}$$

Note that although $q_{yx} = q_{xy}$, in general $AB \neq BA$, and so we cannot combine the middle two terms.

The Taylor series expansion at $(x_i, y_j)$ after time $\Delta t$ can be written as

$$\begin{aligned} q(x_i, y_j, t_n + \Delta t) &= q + \Delta t\, q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots \\ &= q - \Delta t\,(Aq_x + Bq_y) \\ &\quad + \frac{1}{2}\Delta t^2 (A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}) + \cdots . \end{aligned} \tag{19.5}$$

Note that if $A$ and $B$ vary with $x$ and $y$, then (19.3) still holds, but (19.4) becomes

$$q_{tt} = A(Aq_x)_x + A(Bq_y)_x + B(Aq_x)_y + B(Bq_y)_y. \tag{19.6}$$

The expansion becomes somewhat more complicated, as would the Lax–Wendroff method developed in the next subsection.

### *19.1.2 The Lax–Wendroff Method*

The Lax–Wendroff method for the equation (19.2) is obtained by keeping only the terms shown in (19.5) and replacing the derivatives by centered finite differences, just as in

one space dimension. The purely one-dimensional derivatives are approximated as in one dimension, e.g.,

$$q_{yy} = \frac{1}{\Delta y^2}\bigl(Q^n_{i,j-1} - 2Q^n_{ij} + Q^n_{i,j+1}\bigr).$$

In addition, there are now cross-derivative terms, which can also be approximated to second-order with centered differences,

$$q_{xy} = q_{yx} \approx \frac{1}{4\,\Delta x\,\Delta y}\bigl[\bigl(Q^n_{i+1,j+1} - Q^n_{i-1,j+1}\bigr) - \bigl(Q^n_{i+1,j-1} - Q^n_{i-1,j-1}\bigr)\bigr]. \quad (19.7)$$

Using these approximations in (19.5) gives

$$\begin{aligned} Q^{n+1}_{ij} = {} & Q^n_{ij} - \frac{\Delta t}{2\,\Delta x}A\bigl(Q^n_{i+1,j} - Q^n_{i-1,j}\bigr) - \frac{\Delta t}{2\,\Delta y}B\bigl(Q^n_{i,j+1} - Q^n_{i,j-1}\bigr) \\ & + \frac{\Delta t^2}{2\,\Delta x^2}A^2\bigl(Q^n_{i+1,j} - 2Q^n_{ij} + Q^n_{i-1,j}\bigr) + \frac{\Delta t^2}{2\,\Delta y^2}B^2\bigl(Q^n_{i,j+1} - 2Q^n_{ij} + Q^n_{i,j-1}\bigr) \\ & + \frac{\Delta t^2}{8\,\Delta x\,\Delta y}(AB + BA)\bigl[\bigl(Q^n_{i+1,j+1} - Q^n_{i-1,j+1}\bigr) - \bigl(Q^n_{i+1,j-1} - Q^n_{i-1,j-1}\bigr)\bigr]. \end{aligned} \quad (19.8)$$

This has a nine-point stencil involving all nine of the points in Figure 19.1(b) in the update for $Q_{ij}$. In Section 19.3.1 we will see that this same method can be viewed as a finite volume method for updating the cell average $Q_{ij}$ shown in Figure 19.1(a), resulting from defining numerical fluxes at the four edges of the cell in a natural way, based on the nine nearby cell values.

## 19.2 Finite Volume Methods and Approaches to Discretization

As in one space dimension, the two-dimensional Lax–Wendroff method suffers from problems with numerical dispersion, leading to phase errors and to unphysical oscillations in problems with discontinuities or steep gradients. By reinterpreting this method as a finite volume method and introducing upwind biasing and flux limiting, we will see that it can be greatly improved. We will also write the resulting methods in a more general form that applies also to nonlinear conservation laws and to variable-coefficient problems, by using the waves resulting from solving Riemann problems at each cell edge as the basis for upwinding and limiting.

This goal is pursued starting in the next chapter, where we begin to focus on this approach to developing multidimensional high-resolution methods. This is certainly not the only approach, however, and a wide variety of other multidimensional algorithms have been developed and successfully used. The remainder of this chapter is devoted to introducing some general notions of multidimensional finite volume methods.

Three of the most popular general approaches to obtaining multidimensional methods are:

- *Fully discrete flux-differencing methods.* In Section 19.3 we will see that the Lax–Wendroff method can be rewritten as a flux-differencing method. A numerical flux at each edge of the grid cell is defined, based on the data at the beginning of the time step. Differencing these fluxes gives the update to the cell average over a time step. To obtain better than first-order accuracy it is necessary to use the Taylor series expansion developed above in defining these fluxes. To obtain high-resolution nonoscillatory results it is also necessary to introduce limiters. There are many ways in which this can be done, and one particular method of this form is developed in Chapters 20 and 21.
- *Semidiscrete methods with Runge–Kutta time stepping.* Rather than using the Taylor series to replace time derivatives by spatial derivatives, we can focus on obtaining good accuracy of the flux at one instant in time and then use a Runge–Kutta method to perform the time stepping. This approach was introduced in one space dimension in Section 10.4. The two-dimensional extension is briefly discussed in Section 19.4.
- *Dimensional splitting.* By far the simplest approach to obtaining a multidimensional method is to apply a fractional-step method to split a multidimensional problem up into a sequence of one-dimensional problems. To solve $q_t + f(q)_x + g(q)_y = 0$, for example, we might alternate between solving $q_t + f(q)_x = 0$ and $q_t + g(q)_y = 0$, similarly to the way fractional-step methods are used for handling source terms, as discussed in Chapter 17. This approach, which is often surprisingly effective, is discussed in Section 19.5.

We will concentrate on methods that use one-dimensional Riemann solvers as a basic tool in the determination of interface fluxes. This is possible because the local problem at the edge of a grid cell is essentially one-dimensional in the direction normal to the edge. In order to obtain better than first-order accuracy it is necessary to bring in more multidimensional information, as is clear from the Taylor series expansion (19.5), but this can be done in various ways while still only using one-dimensional Riemann solutions.

One particular approach is developed starting in Chapter 20, in the form of the wave-propagation algorithms developed in [282], [283], and [257]. These algorithms take the form of fully discrete flux-differencing methods and are implemented in CLAWPACK in such a way that they can be applied to a wide variety of hyperbolic problems. Some related methods of this form can be found, for example, in [22], [24], [80], [100], [389].

Other methods have been developed that are based on a full decomposition of the data into multidimensional waves, rather than relying on one-dimensional Riemann solvers. For some examples, see [3], [52], [105], [106], [134], [135], [199], [342], [414], [415], [428].

## 19.3 Fully Discrete Flux-Differencing Methods

In deriving the two-dimensional conservation law $q_t + f(q)_x + g(q)_y = 0$ in Section 18.1, we considered an arbitrary region $\Omega$. Now consider the special case where $\Omega$ is a rectangular grid cell of the form $\mathcal{C}_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, as shown in Figure 19.1, where $x_{i+1/2} - x_{i-1/2} = \Delta x$ and $y_{j+1/2} - y_{j-1/2} = \Delta y$. In this special case, the formula (18.7) simplifies, since the normal vector always points in either the $x$- or the $y$-direction. The normal flux is given by $f(q)$ along the left and right edges and by $g(q)$ along the top and

bottom. Integrating around the edges as required in (18.7) then gives

$$\frac{d}{dt}\iint_{\mathcal{C}_{ij}} q(x,y,t)\,dx\,dy = \int_{y_{j-1/2}}^{y_{j+1/2}} f\big(q\big(x_{i+1/2},y,t\big)\big)\,dy - \int_{y_{j-1/2}}^{y_{j+1/2}} f\big(q\big(x_{i-1/2},y,t\big)\big)\,dy + \int_{x_{i-1/2}}^{x_{i+1/2}} g\big(q\big(x,y_{j+1/2},t\big)\big)\,dx - \int_{x_{i-1/2}}^{x_{i+1/2}} g\big(q\big(x,y_{j-1/2},t\big)\big)\,dx. \tag{19.9}$$

If we integrate this expression from $t_n$ to $t_{n+1}$ and divide by the cell area $\Delta x\,\Delta y$, we are led to a fully discrete flux-differencing method of the form

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x}\big[F_{i+1/2,j}^n - F_{i-1/2,j}^n\big] - \frac{\Delta t}{\Delta y}\big[G_{i,j+1/2}^n - G_{i,j-1/2}^n\big], \tag{19.10}$$

where

$$\begin{aligned} F_{i-1/2,j}^n &\approx \frac{1}{\Delta t\,\Delta y}\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} f\big(q\big(x_{i-1/2},y,t\big)\big)\,dy\,dt, \\ G_{i,j-1/2}^n &\approx \frac{1}{\Delta t\,\Delta x}\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} g\big(q\big(x,y_{j-1/2},t\big)\big)\,dx\,dt. \end{aligned} \tag{19.11}$$

The numerical fluxes $F^n$ and $G^n$ at each edge are typically computed from the data $Q^n$ at the initial time. (As in one dimension, these methods can also be extended to nonconservative hyperbolic systems, see Section 19.3.3.)

For the linear system $q_t + Aq_x + Bq_y = 0$ we can obtain approximations to these interface fluxes by using the Taylor expansion (19.5), which can be rewritten as

$$q(x,y,t_n+\Delta t) = q - \Delta t\left(Aq - \frac{\Delta t}{2}A^2q_x - \frac{\Delta t}{2}ABq_y\right)_x - \Delta t\left(Bq - \frac{\Delta t}{2}B^2q_y - \frac{\Delta t}{2}BAq_x\right)_y + \cdots. \tag{19.12}$$

This suggests that we need

$$\begin{aligned} F_{i-1/2,j} &\approx Aq\big(x_{i-1/2},y_j,t_n\big) - \frac{\Delta t}{2}A^2q_x\big(x_{i-1/2},y_j,t_n\big) - \frac{\Delta t}{2}ABq_y\big(x_{i-1/2},y_j,t_n\big), \\ G_{i,j-1/2} &\approx Bq\big(x_i,y_{j-1/2},t_n\big) - \frac{\Delta t}{2}B^2q_y\big(x_i,y_{j-1/2},t_n\big) - \frac{\Delta t}{2}BAq_x\big(x_i,y_{j-1/2},t_n\big). \end{aligned} \tag{19.13}$$

It can be shown that for this problem these expressions agree with the integrals in (19.11) to $\mathcal{O}(\Delta t^2)$.

### 19.3.1 Flux-Differencing Form of the Lax–Wendroff Method

The Lax–Wendroff method (19.8) for the constant-coefficient linear system $q_t + Aq_x + Bq_y = 0$ can be interpreted as a method of the form (19.10), where the fluxes are given by

$$\begin{aligned} F_{i-1/2,j} &= \frac{1}{2}A(Q_{i-1,j} + Q_{ij}) - \frac{\Delta t}{2\,\Delta x}A^2(Q_{ij} - Q_{i-1,j}) \\ &\quad - \frac{\Delta t}{8\,\Delta y}AB[(Q_{i,j+1} - Q_{ij}) + (Q_{i-1,j+1} - Q_{i-1,j}) \\ &\qquad + (Q_{ij} - Q_{i,j-1}) + (Q_{i-1,j} - Q_{i-1,j-1})], \\ G_{i,j-1/2} &= \frac{1}{2}B(Q_{i,j-1} + Q_{ij}) - \frac{\Delta t}{2\,\Delta y}B^2(Q_{ij} - Q_{i,j-1}) \\ &\quad - \frac{\Delta t}{8\,\Delta x}BA[(Q_{i+1,j} - Q_{ij}) + (Q_{i+1,j-1} - Q_{i,j-1}) \\ &\qquad + (Q_{ij} - Q_{i-1,j}) + (Q_{i,j-1} - Q_{i-1,j-1})]. \end{aligned} \tag{19.14}$$

These fluxes relate directly to (19.13). Note in particular that the expression $ABq_y$ in (19.13), for example, is approximated by

$$\begin{aligned} ABq_y(x_{i-1/2}, y_j, t_n) \approx \frac{1}{4\,\Delta y}[&AB(Q_{i,j+1} - Q_{ij}) + AB(Q_{i-1,j+1} - Q_{i-1,j}) \\ &+ AB(Q_{ij} - Q_{i,j-1}) + AB(Q_{i-1,j} - Q_{i-1,j-1})]. \end{aligned} \tag{19.15}$$

In Chapters 20 and 21 we will see how this method can be greatly improved by introducing an upwind bias and flux limiting into the formulas.

### 19.3.2 Godunov's Method

For a general conservation law, the simplest flux-differencing method of the form (19.10) is Godunov's method. A natural two-dimensional generalization of the method developed in Section 15.1 is obtained by simply solving the normal Riemann problem at each cell edge to find the value $Q^{\Downarrow}$ that propagates with speed 0, and then evaluating the appropriate flux function at this value to obtain the numerical flux at this edge. This gives

$$\begin{aligned} F_{i-1/2,j} &= f\big(Q^{\Downarrow}_{i-1/2,j}\big), \\ G_{i,j-1/2} &= g\big(Q^{\Downarrow}_{i,j-1/2}\big), \end{aligned} \tag{19.16}$$

where $Q^{\Downarrow}_{i-1/2,j}$ is obtained by solving the Riemann problem for $q_t + f(q)_x = 0$ with data $Q_{i-1,j}$ and $Q_{ij}$, while $Q^{\Downarrow}_{i,j-1/2}$ is obtained by solving the Riemann problem for $q_t + g(q)_y = 0$ with data $Q_{i,j-1}$ and $Q_{ij}$. As in one dimension, approximate Riemann solvers can be used in place of the exact Riemann solution.

For a linear system of equations with $f(q) = Aq$ and $g(q) = Bq$, we denote the eigenvector matrices for $A$ and $B$ by $R^x$ and $R^y$, respectively, and the eigenvalue matrices by $\Lambda^x$ and $\Lambda^y$, as in Section 18.5. We can then define matrices $A^{\pm}$ and $B^{\pm}$ analogous to

(4.45) by

$$A^{\pm} = R^x(\Lambda^x)^{\pm}(R^x)^{-1}, \qquad B^{\pm} = R^y(\Lambda^y)^{\pm}(R^y)^{-1}. \tag{19.17}$$

In terms of this notation, we find that the Godunov fluxes for a linear problem are the natural generalization of the one-dimensional flux (4.56),

$$\begin{aligned} F_{i-1/2,j} &= A^+ Q_{i-1,j} + A^- Q_{ij}, \\ G_{i,j-1/2} &= B^+ Q_{i,j-1} + B^- Q_{ij}. \end{aligned} \tag{19.18}$$

This amounts to using only the first terms in the fluxes (19.13), and an upwind approximation to these. Of course, this method is only first-order accurate and moreover is typically stable only for Courant number up to $1/2$ in two dimensions. This is illustrated for the advection equation in Section 20.4.

### *19.3.3 Fluctuation Form*

As in one space dimension, we will develop finite volume methods in a more general form than the flux-differencing formula (19.10), so that they are also applicable to hyperbolic equations that are not in conservation form. To make this extension we will rewrite the method as follows, motivated by the one-dimensional method (15.62):

$$\begin{aligned} Q_{ij}^{n+1} = Q_{ij} &- \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+ \Delta Q_{i-1/2,j} + \mathcal{A}^- \Delta Q_{i+1/2,j}\right) \\ &- \frac{\Delta t}{\Delta y}\left(\mathcal{B}^+ \Delta Q_{i,j-1/2} + \mathcal{B}^- \Delta Q_{i,j+1/2}\right) \\ &- \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}\right) - \frac{\Delta t}{\Delta y}\left(\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}\right). \end{aligned} \tag{19.19}$$

The term $\mathcal{A}^+ \Delta Q_{i-1/2,j}$, for example, represents the first-order Godunov update to the cell value $Q_{ij}$ resulting from the Riemann problem at the edge $(i - 1/2, j)$. The other three similar terms are the Godunov updates resulting from the Riemann problems at the other three edges. For the linear system discussed above, these fluctuations are simply given by

$$\begin{aligned} \mathcal{A}^{\pm} \Delta Q_{i-1/2,j} &= A^{\pm}(Q_{ij} - Q_{i-1,j}), \\ \mathcal{B}^{\pm} \Delta Q_{i,j-1/2} &= B^{\pm}(Q_{ij} - Q_{i,j-1}). \end{aligned} \tag{19.20}$$

For Godunov's method we take $\tilde{F} = \tilde{G} = 0$ everywhere. Later the fluxes $\tilde{F}$ and $\tilde{G}$ will be used for correction terms, both those arising from introducing slopes as in one dimension to model the $A^2$ and $B^2$ terms in (19.13), and also new ones modeling the cross-derivative terms involving $AB$ and $BA$ in (19.13).

For a general nonlinear conservation law where the Godunov fluxes are defined by (19.16), we can set

$$\begin{aligned}
\mathcal{A}^+\Delta Q_{i-1/2,j} &= f(Q_{ij}) - f\bigl(Q^{\Downarrow}_{i-1/2,j}\bigr),\\
\mathcal{A}^-\Delta Q_{i-1/2,j} &= f\bigl(Q^{\Downarrow}_{i-1/2,j}\bigr) - f(Q_{i-1,j}),\\
\mathcal{B}^+\Delta Q_{i,j-1/2} &= g(Q_{ij}) - g\bigl(Q^{\Downarrow}_{i,j-1/2}\bigr),\\
\mathcal{B}^-\Delta Q_{i,j-1/2} &= g\bigl(Q^{\Downarrow}_{i,j-1/2}\bigr) - g(Q_{i,j-1}).
\end{aligned} \tag{19.21}$$

Godunov's method results from using these formulas in (19.19) and setting all $\tilde{F} = \tilde{G} = 0$. As in one dimension, the fluctuations $\mathcal{A}^{\pm}\Delta Q$ and $\mathcal{B}^{\pm}\Delta Q$ can also be computed in terms of the waves and speeds arising in the Riemann solution, using formulas analogous to (4.42).

## 19.4 Semidiscrete Methods with Runge–Kutta Time Stepping

As we see already from the Lax–Wendroff method with fluxes (19.14), obtaining even second-order accuracy with a flux-differencing method based on the time-integrated fluxes (19.11) can lead to complicated formulas. As discussed in Section 10.3 for one-dimensional problems, this Taylor series approach is not easily extended to obtain higher-order methods. For this reason, another popular approach is to proceed as in Section 10.4 and use the expression (19.9) to derive ordinary differential equations for the evolution of the cell averages

$$Q_{ij}(t) = \iint_{\mathcal{C}_{ij}} q(x, y, t)\, dx\, dy. \tag{19.22}$$

This is accomplished by defining numerical flux functions

$$\begin{aligned}
F_{i-1/2,j}(Q(t)) &\approx \frac{1}{\Delta y}\int_{y_{j-1/2}}^{y_{j+1/2}} f\bigl(q\bigl(x_{i-1/2}, y, t\bigr)\bigr)\, dy,\\
G_{i,j-1/2}(Q(t)) &\approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} g\bigl(q\bigl(x, y_{j-1/2}, t\bigr)\bigr)\, dx
\end{aligned} \tag{19.23}$$

by some procedure based on the nearby cell averages at this instant in time. Then the system of ODEs

$$\begin{aligned}
\frac{d}{dt}Q_{ij}(t) = &-\frac{1}{\Delta x}\bigl[F_{i+1/2,j}(Q(t)) - F_{i-1/2,j}(Q(t))\bigr]\\
&-\frac{1}{\Delta y}\bigl[G_{i,j+1/2}(Q(t)) - G_{i,j-1/2}(Q(t))\bigr]
\end{aligned} \tag{19.24}$$

is solved by an ODE method, typically a multistage Runge–Kutta method. In order to achieve high-order accuracy it is still necessary to use information from several grid cells nearby in defining the fluxes (19.23), typically by some multidimensional interpolation method. However, since we do not attempt to also approximate the time-derivative terms,

this may be relatively simple. To obtain a high-resolution method, it is necessary to include some form of upwinding and/or limiting in the process of approximating the flux, e.g., by a multidimensional version of the ENO method described in Section 10.4.4.

## 19.5 Dimensional Splitting

The easiest way to extend one-dimensional numerical methods to more space dimensions is to use *dimensional splitting*, an application of the fractional-step procedure discussed in Chapter 17. A multidimensional problem is simply split into a sequence of one-dimensional problems. This is easy to apply on a Cartesian grid aligned with the coordinate axes as shown in Figure 19.1.

For example, the two-dimensional linear problem

$$q_t + Aq_x + Bq_y = 0$$

can be split into

$$x\text{-sweeps}: \quad q_t + Aq_x = 0, \tag{19.25}$$

$$y\text{-sweeps}: \quad q_t + Bq_y = 0. \tag{19.26}$$

In the $x$-sweeps we start with cell averages $Q_{ij}^n$ at time $t_n$ and solve one-dimensional problems $q_t + Aq_x = 0$ along each row of cells $\mathcal{C}_{ij}$ with $j$ fixed, updating $Q_{ij}^n$ to $Q_{ij}^*$:

$$Q_{ij}^* = Q_{ij}^n - \frac{\Delta t}{\Delta x}\big(F_{i+1/2,j}^n - F_{i-1/2,j}^n\big), \tag{19.27}$$

where $F_{i-1/2,j}^n$ is an appropriate numerical flux for the one-dimensional problem between cells $\mathcal{C}_{i-1,j}$ and $\mathcal{C}_{ij}$. In the $y$-sweeps we then use the $Q_{ij}^*$ values as data for solving $q_t + Bq_y = 0$ along each column of cells with $i$ fixed, which results in $Q_{ij}^{n+1}$:

$$Q_{ij}^{n+1} = Q_{ij}^* - \frac{\Delta t}{\Delta x}\big(G_{i,j+1/2}^* - G_{i,j-1/2}^*\big). \tag{19.28}$$

Note that there will generally be a splitting error (see Section 17.3) unless the operators $\mathcal{A} = A\partial_x$ and $\mathcal{B} = B\partial_y$ commute, i.e., unless $AB = BA$. Only in the case where the multidimensional problem decouples into scalar advection equations can we use dimensional splitting with no splitting error. Even in this case we must be careful with boundary conditions (see Section 17.9).

However, the splitting error is often no worse than the errors introduced by the numerical methods in each sweep, and dimensional splitting can be a very effective approach. It gives a simple and relatively inexpensive way to extend one-dimensional high-resolution methods to two or three dimensions.

Note that with the dimensional-splitting approach we do not explicitly model the cross-derivative terms involving $q_{xy}$ in the Taylor series expansion (19.5). In each sweep we only model second derivatives in each coordinate direction, $q_{xx}$ and $q_{yy}$, which appear in the one-dimensional algorithm. The $q_{xy}$ term arises automatically through the fractional-step

procedure. The intermediate solution $q^*$ resulting from $x$-sweeps involves terms modeling $Aq_x$. In the $y$-sweeps we compute terms modeling $Bq_y^*$, which thus model $B(Aq_x)_y$.

Instead of this *Godunov splitting*, one might instead use the *Strang splitting*

$$\begin{aligned} Q_{ij}^* &= Q_{ij}^n - \frac{\Delta t}{2\,\Delta x}\bigl(F_{i+1/2,j}^n - F_{i-1/2,j}^n\bigr), \\ Q_{ij}^{**} &= Q_{ij}^* - \frac{\Delta t}{\Delta x}\bigl(G_{i,j+1/2}^* - G_{i,j-1/2}^*\bigr), \\ Q_{ij}^{n+1} &= Q_{ij}^n - \frac{\Delta t}{2\,\Delta x}\bigl(F_{i+1/2,j}^{**} - F_{i-1/2,j}^{**}\bigr), \end{aligned} \tag{19.29}$$

as discussed in Section 17.4. With the Strang splitting we also obtain terms modeling $A(Bq_y)_x$ from the second $x$-sweep, which are also needed in the Taylor series expansion. Only in the constant-coefficient case with $AB = BA$ does the Godunov splitting give a fully second-order accurate method. However, in practice there is often very little difference in results obtained with the two approaches, as is also the case for other fractional-step methods as discussed in Section 17.5.

In fact, if the Strang splitting is implemented as in (19.29), it may give worse results than the Godunov splitting, because the $x$-sweeps are taken with time step $\Delta t/2$ and hence will typically have Courant number less than $1/2$. This introduces more numerical smearing as well as more work. If instead the Strang splitting is implemented by simply alternating the order in which $x$-sweeps and $y$-sweeps are performed, then this is avoided (see Section 17.4). However, this is somewhat harder to implement in connection with variable-size time steps, and in CLAWPACK the form (19.29) is implemented as the Strang splitting, though the Godunov splitting is generally recommended instead.

### 19.5.1 CLAWPACK Implementation

In order to apply the fractional-step approach, we need to be able to solve each of the one-dimensional equations (19.25) and (19.26), and hence must have two different Riemann solvers available. For many physical systems the equations and solution procedure are essentially the same in each direction. For example, the acoustics equations (18.25) in an isotropic medium have exactly the same form when restricted to plane waves in $x$ or $y$, but with the roles of $u$ and $v$ reversed. The same is true for the shallow water equations (18.38) and the Euler equations (18.45). For this reason it is often simplest to write a single Riemann solver with a flag indicating the desired direction. This convention is used in CLAWPACK, where a single subroutine `rpn2` must be provided that solves the Riemann problem normal to edges of grid cells along one slice of the domain. The flag `ixy` indicates whether the slice is in the $x$-direction or the $y$-direction. The other parameters of this routine are identical to the parameters appearing in the one-dimensional Riemann solver `rp1`.

The two-dimensional CLAWPACK code also allows a second Riemann solver `rpt2` to be provided. This must solve a different sort of Riemann problem in the transverse direction, as described in Section 21.3, and is not used in the dimensional-splitting algorithm.

Dimensional splitting is invoked in CLAWPACK by setting `method(3)=-1` for the Godunov splitting, which is generally recommended, or `method(3)=-2` for Strang

splitting. Positive values of `method(3)` instead invoke the unsplit methods described in later chapters. (See Section 21.3.)

### Exercise

19.1. The system given in Exercise 18.3 is hyperbolic in $x$ and $y$ separately, and so we can apply dimensional splitting to attempt to solve this nonhyperbolic system. Implement this in CLAWPACK, and analyze what happens.