Part three

Multidimensional Problems

18

Multidimensional Hyperbolic Problems

Practical problems involving conservation laws and hyperbolic systems must frequently be solved in more than one space dimension. Most of this book has been devoted to the one-dimensional theory and algorithms, but in the remaining chapters we will see that this forms the basis for understanding and solving multidimensional problems as well.

In two dimensions a conservation law takes the form

$$q_t + f(q)_x + g(q)_y = 0, \tag{18.1}$$

where $q(x, y, t)$ is a vector of m conserved quantities, and $f(q)$ and $g(q)$ are flux functions in the x- and y-directions, as described below. More generally, a quasilinear hyperbolic system has the form

$$q_t + A(q, x, y, t)q_x + B(q, x, y, t)q_y = 0, \tag{18.2}$$

where the matrices A and B satisfy certain conditions given below in Section 18.5. In three dimensions a third term would be added to each of these equations:

$$q_t + f(q)_x + g(q)_y + h(q)_z = 0, \tag{18.3}$$

and

$$q_t + A(q, x, y, z, t)q_x + B(q, x, y, z, t)q_y + C(q, x, y, z, t)q_z = 0, \tag{18.4}$$

respectively.

18.1 Derivation of Conservation Laws

We begin by deriving the conservation law (18.1) in two dimensions from the more fundamental integral form. Again the integral form can be used directly in the development of finite volume methods, as we will see beginning in Chapter 19.

As in one dimension (see Chapter 2), we derive the conservation law by considering an arbitrary spatial domain Ω over which q is assumed to be conserved, so that the integral of q over Ω varies only due to flux across the boundary of Ω. This boundary is denoted by

$\partial\Omega$. We thus have

$$\frac{d}{dt}\iint_\Omega q(x, y, t)\,dx\,dy = \text{net flux across } \partial\Omega. \tag{18.5}$$

The net flux is determined by integrating the flux of q normal to $\partial\Omega$ around this boundary.

Let $f(q)$ represent the flux of q in the x-direction (per unit length in y, per unit time). This means that the total flux through an interval from (x_0, y_0) to $(x_0, y_0 + \Delta y)$ over time Δt is roughly $\Delta t\,\Delta y\, f(q(x_0, y_0))$, for Δt and Δy sufficiently small.

Similarly, let $g(q)$ be the flux in the y-direction, and let $\vec{f}(q) = (f(q),\ g(q))$ be the flux vector. Finally, let $\vec{n}(s) = (n^x(s), n^y(s))$ be the outward-pointing unit normal vector to $\partial\Omega$ at a point $(x(s), y(s))$ on $\partial\Omega$, where s is the arclength parameterization of $\partial\Omega$. Then the flux at $\vec{x}(s) = (x(s), y(s))$ in the direction $\vec{n}(s)$ is

$$\vec{n}(s)\cdot\vec{f}(q(x(s), y(s), t)) = n^x(s) f(q) + n^y(s) g(q), \tag{18.6}$$

and (18.5) becomes

$$\frac{d}{dt}\iint_\Omega q(x, y, t)\,dx\,dy = -\int_{\partial\Omega} \vec{n}\cdot\vec{f}(q)\,ds. \tag{18.7}$$

If q is smooth then we can use the divergence theorem to rewrite this as

$$\frac{d}{dt}\iint_\Omega q(x, y, t)\,dx\,dy = -\iint_\Omega \vec{\nabla}\cdot\vec{f}(q)\,dx\,dy, \tag{18.8}$$

where the divergence of $\vec{f}$ is

$$\vec{\nabla}\cdot\vec{f}(q) = f(q)_x + g(q)_y.$$

This leads to

$$\iint_\Omega [q_t + \vec{\nabla}\cdot\vec{f}(q)]\,dx\,dy = 0. \tag{18.9}$$

Since this must hold over any arbitrary region Ω, the integrand must be zero, and we obtain the conservation law (18.1) in differential form. Note that this argument is exactly analogous to what we did in Chapter 2 in one dimension. As in the one-dimensional case, this derivation assumes q is smooth and so the differential form holds only for smooth solutions. To properly compute discontinuous solutions we will again use finite volume methods based on the integral form.

The same argument extends to three dimensions with the flux vector

$$\vec{f}(q) = [f(q), g(q), h(q)]. \tag{18.10}$$

If we integrate over an arbitrary volume Ω, so that $\partial\Omega$ is the surface bounding this volume, then we obtain (18.3) for smooth solutions.

In the notation above we have assumed q is a scalar. If $q,\ f(q), g(q), h(q) \in \mathbb{R}^m$, then the vector $\vec{f}$ of (18.10) is a vector in $\mathbb{R}^{3m}$, while $\vec{n}(s) = (n^x, n^y, n^z)$ is in $\mathbb{R}^3$, and we interpret dot products by the formula

$$\breve{f}(q) \equiv \vec{n} \cdot \vec{f}(q) = n^x f(q) + n^y g(q) + n^z h(q). \tag{18.11}$$

In general we use an arrow on a symbol to denote a spatial vector with components corresponding to each spatial dimension. In addition to $\vec{n}$ and $\vec{f}$, we also use $\vec{u} = (u, v, w)$ for the velocity vector and $\vec{A} = (A, B, C)$ as a vector of matrices. We will see below that we need to investigate the linear combination

$$\breve{A} \equiv \vec{n} \cdot \vec{A} = n^x A + n^y B + n^z C \tag{18.12}$$

to determine whether the system (18.4) is hyperbolic. We use the breve accent $\breve{\cdot}$ to denote a quantity that has been restricted to a particular direction specified by $\vec{n}$. As a mnemonic device, the circular arc of the breve accent can be thought of as indicating rotation to the desired direction. This will be heavily used in Chapter 23, where we discuss numerical methods on general quadrilateral grids. One-dimensional Riemann problems will be solved in the direction normal to each cell edge in order to compute the normal fluxes, and doing so requires rotating the flux function to that direction.

For simplicity we will mostly restrict our attention to the case of two dimensions, but the essential ideas extend directly to three dimensions, and this case is briefly discussed as we go along.

18.2 Advection

As a simple example, suppose a fluid is flowing with a known velocity $\vec{u} = (u(x, y, t), v(x, y, t))$ in the plane, and let the scalar $q(x, y, t)$ represent the concentration of a tracer, measured in units of mass per unit area in the plane (see Section 9.1). Then the flux functions are

$$\begin{aligned} f &= u(x, y, t)\, q(x, y, t), \\ g &= v(x, y, t)\, q(x, y, t), \end{aligned} \tag{18.13}$$

so that $\vec{f}(q) = \vec{u}q$ and we obtain the conservation law

$$q_t + (uq)_x + (vq)_y = 0. \tag{18.14}$$

Note that in this case f and g may depend explicitly on (x, y, t) as well as on the value of q, and that the derivation above carries over to this situation. If in fact $(u(x, y, t), v(x, y, t)) = (\bar{u}, \bar{v})$ is constant in space and time, so the fluid is simply translating at constant speed in a fixed direction, then (18.14) reduces to

$$q_t + \bar{u}q_x + \bar{v}q_y = 0.$$

The solution is then easily seen to be

$$q(x, y, t) = \overset{\circ}{q}(x - \bar{u}t, y - \bar{v}t), \tag{18.15}$$

so that the initial density simply translates at this velocity. The solution to the more general variable-coefficient advection equation is discussed in Section 20.5.

18.3 Compressible Flow

In Section 2.6 we saw the equations of compressible gas dynamics in one space dimension, in the simplest case where the equation of state relates the pressure directly to the density,

$$p = P(\rho). \tag{18.16}$$

These equations are easily extended to two space dimensions. The conservation-of-mass equation (continuity equation) for the density $\rho(x, y, t)$ is identical to the advection equation (18.14) derived in the previous section:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0. \tag{18.17}$$

But now the velocities (u, v) are not known *a priori*. Instead the continuity equation must be coupled with equations for the conservation of x-momentum ρu and y-momentum ρv. Each of these momenta advects with the fluid motion, giving fluxes analogous to (18.14) with q replaced by ρu or ρv respectively. In addition, pressure variations lead to acceleration of the fluid. Variation in the x-direction, measured by p_x, accelerates the fluid in that direction and appears in the equation for $(\rho u)_t$, while p_y appears in the equation for $(\rho v)_t$. The conservation-of-momentum equations are

$$\begin{aligned} &(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ &(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0. \end{aligned} \tag{18.18}$$

These equations (18.17) and (18.18), together with the equation of state (18.16), give a closed system of three conservation laws for mass and momentum. If the equation of state is more complicated, then these equations will generally also have to be coupled with the equation for the conservation of energy, as discussed in Chapter 14. The simple case will suffice for our purposes now. In particular, from these equations we can derive the linearized equations of acoustics just as we did in Section 2.8 in one space dimension.

These gas dynamics equations can be written as a system of conservation laws of the form (18.1) with

$$\begin{aligned} q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \end{bmatrix}, \qquad f(q) &= \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)^2/q^1 + P(q^1) \\ q^2 q^3/q^1 \end{bmatrix}, \\ g(q) &= \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \end{bmatrix} = \begin{bmatrix} q^3 \\ q^2 q^3/q^1 \\ (q^3)^2/q^1 + P(q^1) \end{bmatrix}. \end{aligned} \tag{18.19}$$

These equations can also be written in quasilinear form

$$q_t + f'(q) q_x + g'(q) q_y = 0, \tag{18.20}$$

in terms of the Jacobian matrices

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + P'(\rho) & 2u & 0 \\ -uv & v & u \end{bmatrix}, \qquad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + P'(\rho) & 0 & 2v \end{bmatrix}. \tag{18.21}$$

18.4 Acoustics

Linearizing the equations derived in the previous section about a constant state $q_0 = (\rho_0,\ u_0,\ v_0)$ gives

$$q_t + f'(q_0)q_x + g'(q_0)q_y = 0,$$

where q now represents perturbations from the constant state q_0. In particular, if we wish to study acoustics in a stationary gas, then we can take $u_0 = v_0 = 0$ and the Jacobian matrices simplify considerably:

$$f'(q_0) = \begin{bmatrix} 0 & 1 & 0 \\ P'(\rho_0) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad g'(q_0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ P'(\rho_0) & 0 & 0 \end{bmatrix}. \tag{18.22}$$

As in one dimension, we can now manipulate these equations to derive an equivalent linear system in terms of perturbations in pressure and velocity, a linear system of the form

$$q_t + Aq_x + Bq_y = 0, \tag{18.23}$$

where (again for $u_0 = v_0 = 0$)

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix}. \tag{18.24}$$

These are the *equations of acoustics* in two space dimensions, where again $K_0 = \rho_0 P'(\rho_0)$ is the bulk modulus of compressibility. More generally, for acoustics against a background flow with constant velocity $\vec{u}_0 = (u_0, v_0)$, the coefficient matrices are

$$A = \begin{bmatrix} u_0 & K_0 & 0 \\ 1/\rho_0 & u_0 & 0 \\ 0 & 0 & u_0 \end{bmatrix}, \qquad B = \begin{bmatrix} v_0 & 0 & K_0 \\ 0 & v_0 & 0 \\ 1/\rho_0 & 0 & v_0 \end{bmatrix}. \tag{18.25}$$

18.5 Hyperbolicity

Recall that in one space dimension the linear system $q_t + Aq_x = 0$ is said to be *hyperbolic* if the matrix A is diagonalizable with real eigenvalues. In two space dimensions we need this condition to hold for each of the coefficient matrices A and B, but we also need something more: the same property should hold for any linear combination of these matrices. This is formalized in Definition 18.1 below after some motivation.

The essence of hyperbolicity is that wavelike solutions should exist. In one space dimension a linear system of m equations generally gives rise to m waves moving at constant speeds and unchanged shape. In two dimensions we should see this same behavior if we take special initial data that varies only in one direction – not just the x- or y-direction, but any arbitrary direction specified by a unit vector $\vec{n} = (n^x, n^y)$, so that the data has the form

$$q(x, y, 0) = \mathring{q}(\vec{n} \cdot \vec{x}) = \mathring{q}(n^x x + n^y y). \tag{18.26}$$

The contour lines of $q(x, y, 0)$ are straight lines, and we expect wave motion in the direction normal to these lines, which is the direction given by $\vec{n}$.

In particular, there should be special initial data of this form that yields a *single* wave propagating at some constant speed s, a *plane wave* of the form

$$q(x, y, t) = \breve{q}(\vec{n} \cdot \vec{x} - st).$$

This is the multidimensional analogue of the simple wave discussed in Section 3.5. If we compute q_t, q_x, and q_y for this *Ansatz* and insert them into the equation $q_t + Aq_x + Bq_y = 0$, we find that

$$\breve{A}\,\breve{q}'(\vec{n} \cdot \vec{x} - st) = s\,\breve{q}'(\vec{n} \cdot \vec{x} - st),$$

where

$$\breve{A} = \vec{n} \cdot \vec{A} = n^x A + n^y B. \tag{18.27}$$

Except for the trivial case $\breve{q} \equiv$ constant, (18.27) can only hold if s is an eigenvalue of the matrix $\breve{A}$, with $\breve{q}'(\xi)$ a corresponding eigenvector of this matrix for each value of ξ. This leads to our definition of hyperbolicity in two space dimensions.

Definition 18.1. *The constant-coefficient system $q_t + Aq_x + Bq_y = 0$ is (strongly)* hyperbolic *if, for every choice of $\vec{n}$, the matrix $\breve{A} = \vec{n} \cdot \vec{A}$ is diagonalizable with real eigenvalues. The quasilinear system (18.20) is hyperbolic in some region of state space if the Jacobian matrix $\breve{f}'(q) = \vec{n} \cdot \vec{f}'(q) = n^x f'(q) + n^y g'(q)$ is diagonalizable with real eigenvalues for every $\vec{n}$, for all q in this region.*

Note in particular that for $\vec{n} = (1, 0)$ or $\vec{n} = (0, 1)$ we have propagation in the x- or y-direction respectively. In these cases we obtain the usual one-dimensional conditions on the matrices A and B separately. The obvious three-dimensional extension of this definition is given in Section 18.6.

For the acoustics equations with (18.24), we have

$$\breve{A} = \begin{bmatrix} 0 & n^x K_0 & n^y K_0 \\ n^x/\rho_0 & 0 & 0 \\ n^y/\rho_0 & 0 & 0 \end{bmatrix}. \tag{18.28}$$

This matrix has eigenvalues that are independent of $\vec{n}$:

$$\breve{\lambda}^1 = -c_0, \qquad \breve{\lambda}^2 = 0, \qquad \breve{\lambda}^3 = +c_0,$$

where $c_0 = \sqrt{K_0/\rho_0}$ is the speed of sound. This is exactly what we should expect, since sound waves can propagate in any direction at the same speed (for the uniform isotropic medium we are considering here, with ρ_0 and K_0 constant).

For acoustics against a constant background flow, we expect sound waves to propagate at speed c_0 relative to the moving fluid. For the matrices A and B of (18.25) we have

$$\breve{A} = \begin{bmatrix} \breve{u}_0 & n^x K_0 & n^y K_0 \\ n^x/\rho_0 & \breve{u}_0 & 0 \\ n^y/\rho_0 & 0 & \breve{u}_0 \end{bmatrix}, \tag{18.29}$$

where $\breve{u}_0 = \vec{n} \cdot \vec{u}_0$ is the fluid velocity in the $\vec{n}$-direction. Since this differs from (18.28) only by a multiple of the identity matrix, it has the same eigenvectors (given below in (18.33)), and the eigenvalues are simply shifted by $\breve{u}_0$:

$$\breve{\lambda}^1 = \breve{u}_0 - c_0, \qquad \breve{\lambda}^2 = \breve{u}_0, \qquad \breve{\lambda}^3 = \breve{u}_0 + c_0, \tag{18.30}$$

exactly as we expected.

In one space dimension we can diagonalize a general linear hyperbolic equation using the matrix of eigenvectors, decoupling it into independent scalar advection equations for each characteristic variable. For a linear system in more dimensions, we can do this in general only for the special case of a plane-wave solution. The full system $q_t + Aq_x + Bq_y = 0$ with arbitrary data can be diagonalized only if the coefficient matrices commute, e.g., if $AB = BA$ in (18.23), in which case the matrices have the *same* eigenvectors. Then A and B can be *simultaneously diagonalized* by a common eigenvector matrix R:

$$A = R\Lambda^x R^{-1}, \qquad B = R\Lambda^y R^{-1},$$

where $\Lambda^x = \operatorname{diag}(\lambda^{x1}, \ldots, \lambda^{xm})$ and $\Lambda^y = \operatorname{diag}(\lambda^{y1}, \ldots, \lambda^{ym})$ contain the eigenvalues, which may be different. The system in (18.23) can then be diagonalized by setting $w = R^{-1}q$ to obtain

$$w_t + \Lambda^x w_x + \Lambda^y w_y = 0,$$

yielding m independent advection equations. Note that in this case there are only m distinct directions in which information propagates. The pth characteristic variable w^p propagates with velocity $(\lambda^{xp}, \lambda^{yp})$. This is not what we would expect in acoustics, for example, since sound waves can propagate in any direction.

If $AB \neq BA$, then there is no single transformation that will simultaneously diagonalize A and B. If the system is hyperbolic, then we can diagonalize each matrix separately,

$$A = R^x\Lambda^x (R^x)^{-1}, \qquad B = R^y\Lambda^y (R^y)^{-1},$$

but the two matrices have different eigenvectors R^x and R^y, respectively. In this case the equations are more intricately coupled. This is the usual situation physically. In the case of acoustics, for example, the matrices A and B of (18.24) or (18.25) are not simultaneously diagonalizable. The matrix A of (18.25) has right eigenvectors

$$r^{x1} = \begin{bmatrix} -Z_0 \\ 1 \\ 0 \end{bmatrix}, \quad r^{x2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^{x3} = \begin{bmatrix} Z_0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{so that} \quad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{18.31}$$

while the matrix B has right eigenvectors

$$r^{y1} = \begin{bmatrix} -Z_0 \\ 0 \\ 1 \end{bmatrix}, \quad r^{y2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad r^{y3} = \begin{bmatrix} Z_0 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{so that} \quad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{18.32}$$

where $Z_0 = \rho_0 c_0$ is again the impedance.

Note that for a plane wave moving in the x-direction these acoustics equations reduce to $q_t + Aq_x = 0$ with A given by (18.25). This system has exactly the same eigenstructure as the coupled acoustics–advection system of Section 3.10. In this case the y-component of the velocity perturbation, v, is simply advected with the background velocity u_0 and does not affect the acoustics. (Recall that in such a plane wave we assume the variables only vary with x. Variations of v in the y-direction would of course generate acoustic signals.)

A plane wave in the y-direction gives a similar structure, but now the eigenvectors r^{y1} and r^{y3} corresponding to acoustic waves involving the pressure p and vertical velocity perturbation v, while the x-component of the velocity perturbations, u, is simply advected at the background speed v_0.

The more general matrix $\breve{A}$ from (18.29) has eigenvalues (18.30) and the eigenvectors

$$\breve{r}^1 = \begin{bmatrix} -Z_0 \\ n^x \\ n^y \end{bmatrix}, \qquad \breve{r}^2 = \begin{bmatrix} 0 \\ -n^y \\ n^x \end{bmatrix}, \qquad \breve{r}^3 = \begin{bmatrix} Z_0 \\ n^x \\ n^y \end{bmatrix}, \tag{18.33}$$

which reduce to (18.31) or (18.32) when $\vec{n}$ is in the x- or y-direction. Note that more generally the acoustic waves $\breve{r}^1$ and $\breve{r}^3$ have velocity components in the $\vec{n}$-direction, as we expect for these compressional waves. The 2-wave carries velocity perturbations in the orthogonal direction (a shear wave), which are simply advected with the flow. In Section 22.1 we consider elastic waves in a solid that resists shear motion, in which case shear waves have more interesting structure.

In one space dimension we could diagonalize the acoustics equations to obtain a coupled pair of advection equations. Solutions consist simply of two waves advecting with velocities $-c_0$ and $+c_0$ in the two possible directions. In two dimensions, even though the structure of each matrix is that of one-dimensional acoustics, the nondiagonalizable coupling between them leads to a much richer structure. In general we obtain waves propagating in all of the infinitely many possible directions in the plane.

18.6 Three-Dimensional Systems

The three-dimensional linear system

$$q_t + Aq_x + Bq_y + Cq_z = 0 \tag{18.34}$$

is hyperbolic provided that, for any direction defined by the unit vector $\vec{n} = (n^x, n^y, n^z)$, the matrix $\breve{A}$ given by (18.12) is diagonalizable with real eigenvalues and a complete set of eigenvectors. The eigenvalues have the interpretation of physical propagation velocities for plane waves in this direction.

The acoustics equations in three dimensions are a linear hyperbolic system for perturbations in the pressure and three velocity components (u, v, w). Rather than displaying the coefficient matrices A, B, and C separately, it is more compact to just display $\breve{A}$ in an arbitrary direction $\vec{n}$. We have

$$q = \begin{bmatrix} p \\ u \\ v \\ w \end{bmatrix}, \qquad \breve{A} = \begin{bmatrix} \breve{u}_0 & n^x K_0 & n^y K_0 & n^z K_0 \\ n^x/\rho_0 & \breve{u}_0 & 0 & 0 \\ n^y/\rho_0 & 0 & \breve{u}_0 & 0 \\ n^z/\rho_0 & 0 & 0 & \breve{u}_0 \end{bmatrix}, \tag{18.35}$$

where

$$\breve{u}_0 = n^x u_0 + n^y v_0 + n^z w_0 \tag{18.36}$$

is the component of the background velocity in the direction $\vec{n}$. For any choice of direction, the eigenvalues of $\breve{A}$ are

$$\breve{\lambda}^1 = \breve{u}_0 - c_0, \qquad \breve{\lambda}^2 = \breve{\lambda}^3 = \breve{u}_0, \qquad \breve{\lambda}^4 = \breve{u}_0 + c_0. \tag{18.37}$$

Note that there is a two-dimensional eigenspace corresponding to the eigenvalue $\breve{\lambda}^2 = \breve{\lambda}^3 = \breve{u}_0$, since shear waves can now carry an arbitrary jump in each of the two velocity components orthogonal to $\vec{n}$.

18.7 Shallow Water Equations

In two space dimensions the shallow water equations take the form

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0, \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0, \end{aligned} \tag{18.38}$$

where h is the depth and (u, v) the velocity vector, so that hu and hv are the momenta in the two directions. These are a natural generalization of the one-dimensional equations (13.5) and are identical to the two-dimensional compressible flow equations derived in Section 18.3 if we replace ρ by h there and use the hydrostatic equation of state

$$p = P(h) = \frac{1}{2}gh^2 \tag{18.39}$$

as derived in (13.3) (taking $\bar{\rho} = 1$). From (18.21), the flux Jacobian matrices are thus

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}, \qquad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix}. \tag{18.40}$$

Let $c = \sqrt{gh}$ be the speed of gravity waves. Then the matrix $f'(q)$ has eigenvalues and eigenvectors

$$\begin{gathered}\lambda^{x1} = u - c, \qquad \lambda^{x2} = u, \qquad \lambda^{x3} = u + c, \\ r^{x1} = \begin{bmatrix} 1 \\ u - c \\ v \end{bmatrix}, \quad r^{x2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^{x3} = \begin{bmatrix} 1 \\ u + c \\ v \end{bmatrix}.\end{gathered} \tag{18.41}$$

The Jacobian $g'(q)$ has a similar set of eigenvalues and eigenvectors,

$$\begin{gathered}\lambda^{y1} = v - c, \qquad \lambda^{y2} = v, \qquad \lambda^{y3} = v + c, \\ r^{y1} = \begin{bmatrix} 1 \\ u \\ v - c \end{bmatrix}, \quad r^{y2} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad r^{y3} = \begin{bmatrix} 1 \\ u \\ v + c \end{bmatrix}\end{gathered} \tag{18.42}$$

in which the roles of u and v are switched along with x and y.

In each case the 1-wave and 3-wave are nonlinear gravity waves, while the 2-wave is linearly degenerate. Compare these with (13.64) and (13.65), the Jacobian matrix and eigenstructure for the one-dimensional shallow water equations augmented by a passive tracer.

18.7.1 The Plane-Wave Riemann Problem

Consider a two-dimensional Riemann problem for the shallow water equations with variation only in the x-direction. In this case the velocity v plays no dynamic role in the gravity waves, and any jump in v is simply carried along passively at the fluid velocity u_m that arises between the two nonlinear waves. This is again a contact discontinuity that lies at the interface between the two original fluids. The fluid to the left always has y-velocity v_l while the one to the right has y-velocity v_r. Figure 13.20 illustrates this, if we let the dark and light regions now represent different velocities v. Figure 18.1 gives another illustration of this, showing a top view of a two-dimensional version of Figure 13.20. The contact discontinuity is also called a *shear wave* in this context.

One should recall that we are ignoring fluid viscosity with this hyperbolic model. In reality a jump discontinuity in shear velocity would be smeared out due to frictional forces (diffusion of the momentum hv in the x-direction) and may lead to Kelvin–Helmholtz instabilities along such an interface.

The true solution to this Riemann problem is easily computed using the one-dimensional theory. We simply solve the one-dimensional problem ignoring v, and then introduce a jump in v at the contact surface.

To verify that the two-dimensional shallow water equations are hyperbolic, we compute the Jacobian matrix $\breve{f}'(q) = \vec{n} \cdot \vec{f}'(q)$ in an arbitrary direction $\vec{n}$,

$$\breve{f}'(q) = \begin{bmatrix} 0 & n^x & n^y \\ n^x gh - u\breve{u} & \breve{u} + n^x u & n^y u \\ n^y gh - v\breve{u} & n^x v & \breve{u} + v n^y \end{bmatrix}, \tag{18.43}$$

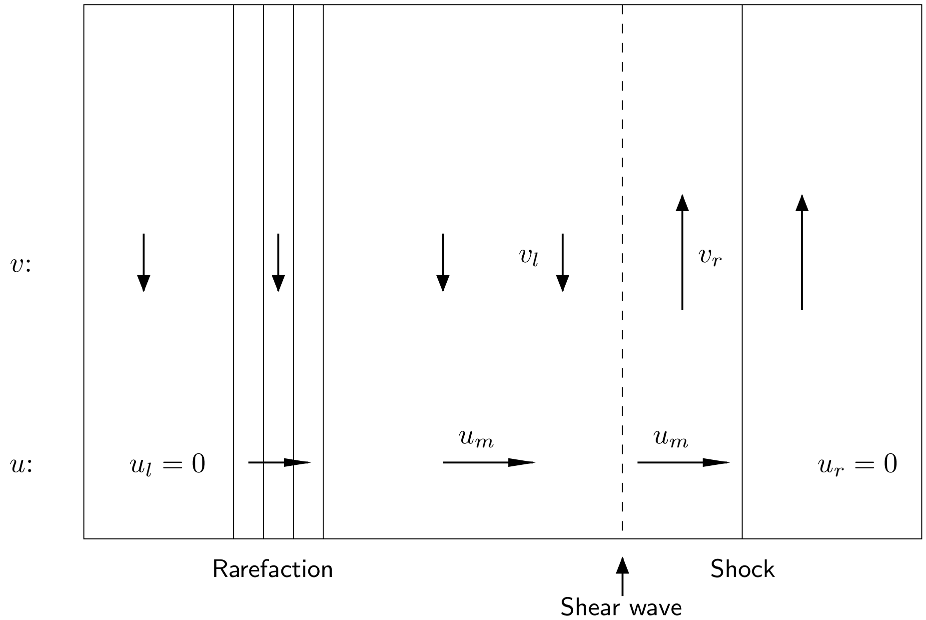

Fig. 18.1. Solution to a Riemann problem in the x-direction for the two-dimensional shallow water equations. The depth h and normal velocity u are as shown in Figure 13.20, and the shading in that figure represents v.

where $\breve{u} = \vec{n} \cdot \vec{u}$. This matrix has eigenvalues and eigenvectors given by

$$\breve{\lambda}^1 = \breve{u} - c, \qquad \breve{\lambda}^2 = \breve{u}, \qquad \breve{\lambda}^3 = \breve{u} + c,$$
$$\breve{r}^1 = \begin{bmatrix} 1 \\ u - n^x c \\ v - n^y c \end{bmatrix}, \quad \breve{r}^2 = \begin{bmatrix} 0 \\ -n^y \\ n^x \end{bmatrix}, \quad \breve{r}^3 = \begin{bmatrix} 1 \\ u + n^x c \\ v + n^y c \end{bmatrix}. \tag{18.44}$$

The expressions (18.41) and (18.42) are special cases of this. For any direction $\vec{n}$ the eigenvalues are real and correspond to wave speeds 0, $\pm c$ relative to the moving fluid.

18.8 Euler Equations

The two-dimensional Euler equations have the same form as the compressible flow equations presented in Section 18.3, but with the addition of an energy equation for the general case where the equation of state is more complicated than (18.16):

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E + p)u \end{bmatrix}, \qquad g(q) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E + p)v \end{bmatrix}. \tag{18.45}$$

The equation of state for a γ-law polytropic gas is the obvious extension of (14.23). The total energy is the sum of internal and kinetic energy,

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2). \tag{18.46}$$

The Jacobian matrix $f'(q)$ has the four eigenvalues

$$\lambda^1 = u - c, \quad \lambda^2 = u, \quad \lambda^3 = u, \quad \lambda^4 = u + c. \tag{18.47}$$

As in one dimension, the sound speed is $c = \sqrt{\gamma p/\rho}$. The eigenvectors are

$$r^{x1} = \begin{bmatrix} 1 \\ u - c \\ v \\ H - uc \end{bmatrix}, \quad r^{x2} = \begin{bmatrix} 1 \\ u \\ v \\ \frac{1}{2}(u^2 + v^2) \end{bmatrix}, \quad r^{x3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ v \end{bmatrix}, \quad r^{x4} = \begin{bmatrix} 1 \\ u + c \\ v \\ H + uc \end{bmatrix}. \tag{18.48}$$

The eigenvalues and eigenvectors in the y-direction are similar, with the roles of u and v reversed.

18.8.1 The Plane-Wave Riemann Problem

Consider a Riemann problem in which the data varies only in x. For the one-dimensional Euler equations the density can be discontinuous across the contact discontinuity, as illustrated in Figure 14.1. In the two-dimensional extension, there can also be a jump in the transverse velocity v across the contact discontinuity, exactly as was illustrated for the two-dimensional shallow water equations in Figure 18.1. The jump in density and the jump in shear velocity are carried by two independent linearly degenerate waves that both travel at the same velocity. These two waves correspond to the two eigenvalues $\lambda^2 = \lambda^3 = u$ of the Jacobian matrix. (The two vectors r^{x2} and r^{x3} in (18.48) are just one possible basis for this two-diemensional eigenspace.)

This two-dimensional Riemann problem is easily solved based on the one-dimensional theory, just as in the case of the shallow water equations. We can follow the procedure of Section 14.11, ignoring the transverse velocity v, since the primitive variables u and p are still continuous across the contact discontinuity. We then introduce a jump in v from v_l to v_r at the contact discontinuity. Note that this also gives a jump in E in the eigenvector r^{x3}, since v comes into the equation of state (18.46).

18.8.2 Three-Dimensional Euler Equations

In three space dimensions the Euler equations are similar, but with the addition of a fifth equation for the conservation of momentum ρw in the z-direction, where w is the z-component of velocity. The conserved quantities and fluxes are then

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \quad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (E+p)u \end{bmatrix}, \quad g(q) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ (E+p)v \end{bmatrix}, \quad h(q) = \begin{bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (E+p)w \end{bmatrix}. \tag{18.49}$$

The equation of state now includes the kinetic energy $\frac{1}{2}\rho(u^2 + v^2 + w^2) = \frac{1}{2}\rho\vec{u}\cdot\vec{u}$, where $\vec{u} = (u, v, w)$ is the velocity vector. In any arbitrary direction $\vec{n}$ there are two nonlinear acoustic fields with eigenvalues $(\vec{n}\cdot\vec{u})\pm c$, and three linearly degenerate fields with eigenvalue $\vec{n}\cdot\vec{u}$. These three fields correspond to jumps in the density (entropy waves), and jumps in the two transverse velocities (shear waves). For example, if $\vec{n} = (1, 0, 0)$, then we are looking in the x-direction and arbitrary jumps in ρ, v, and w across the contact discontinuity can all propagate with speed u.

18.9 Symmetry and Reduction of Dimension

For some problems we may be able to reduce the complexity of the numerical problem substantially by taking advantage of symmetry. For example, if we are solving a problem where the solution is known to be radially symmetric, then we should be able to rewrite the equations in polar or spherical coordinates, obtaining a system that reduces to a problem in the single space variable r. The transformed equations will typically involve *geometric source terms*.

For example, when rewritten in polar r–θ coordinates, the compressible flow equations (18.19) take the form

$$\frac{\partial}{\partial t}\begin{bmatrix} r\rho \\ r\rho U \\ r\rho V \end{bmatrix} + \frac{\partial}{\partial r}\begin{bmatrix} r\rho U \\ r\rho U^2 + p \\ r\rho U V \end{bmatrix} + \frac{1}{r}\frac{\partial}{\partial \theta}\begin{bmatrix} r\rho V \\ r\rho U V \\ r\rho V^2 + p \end{bmatrix} = 0, \tag{18.50}$$

where $U(r, \theta, t)$ is the velocity in the radial direction and $V(r, \theta, t)$ is the velocity in the θ-direction. If we assume that $V(r, \theta, t) \equiv 0$ and there is no variation in the θ-direction, then these equations reduce to the two equations

$$\begin{aligned} (r\rho)_t + (r\rho U)_r &= 0, \\ (r\rho U)_t + (r\rho U^2 + p)_r &= 0. \end{aligned} \tag{18.51}$$

This system can be rewritten as

$$\begin{aligned} \rho_t + (\rho U)_r &= -(\rho U)/r, \\ (\rho U)_t + (\rho U^2 + p)_r &= -(\rho U^2)/r, \end{aligned} \tag{18.52}$$

which has exactly the same form as the one-dimensional system of equations (2.38), but with the addition of a geometric source term on the right-hand side.

The full two- or three-dimensional Euler equations with radial symmetry yield

$$\begin{aligned} \rho_t + (\rho U)_r &= -\frac{\alpha}{r}(\rho U), \\ (\rho U)_t + (\rho U^2 + p)_r &= -\frac{\alpha}{r}(\rho U^2), \\ E_t + ((E+p)U)_r &= -\frac{\alpha}{r}((E+p)U), \end{aligned} \tag{18.53}$$

where $\alpha = 1$ in two dimensions and $\alpha = 2$ in three dimensions.

Even if the real problems of interest must be studied multidimensionally, radially symmetric solutions are very valuable in testing and validating numerical codes. A highly accurate solution to the one-dimensional problem can be computed on a fine grid and used to test solutions computed with the multidimensional solver. This is useful not only in checking that the code gives essentially the correct answer in at least some special cases, but also in determining whether the numerical method is isotropic or suffers from *grid-orientation effects* that lead to the results being better resolved in some directions than in others. See Section 21.7.1 for one such example.

The D-dimensional acoustics equations with radial symmetry reduce to

$$\begin{aligned} &p_t + K_0 U_r = -\frac{\alpha}{r}(K_0 U), \\ &\rho_0 U_t + p_r = 0, \end{aligned} \tag{18.54}$$

where again $\alpha = D - 1$.

Exercises

18.1. Consider the system $q_t + Aq_x + Bq_y = 0$ with

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Show that these matrices are simultaneously diagonalizable, and determine the general solution to this system with arbitrary initial data. In particular, sketch how the solution evolves in the x–y plane with data

$$q^1(x, y, 0) = \begin{cases} 1 & \text{if } x^2 + y^2 \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad q^2(x, y, 0) \equiv 0.$$

18.2. (a) Suppose that A and B are both symmetric matrices. Show that the system $q_t + Aq_x + Bq_y = 0$ must then be hyperbolic.

(b) The matrices A and B are *simultaneously symmetrizable* if there is an invertible matrix M such that $M^{-1}AM$ and $M^{-1}BM$ are both symmetric. Show that in this case the system $q_t + Aq_x + Bq_y = 0$ must be hyperbolic.

(c) Show that the matrices in (18.25) for the linearized acoustics equations are simultaneously symmetrizable with a matrix M of the form $M = \mathrm{diag}(d_1, 1, 1)$.

18.3. Consider the two-dimensional system $q_t + Aq_x + Bq_y = 0$ with matrices

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 10 & 1 \end{bmatrix},$$

each of which is diagonalizable with real eigenvalues. Show, however, that this system is not hyperbolic. (See also Exercise 19.1.)

18.4. Determine the eigenvectors of the three-dimensional acoustics matrix $\breve{A}$ from (18.35).

18.5. Show that the three-dimensional system (2.115) of Maxwell's equations is hyperbolic.