

## **Part one**

### Linear Equations



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## *Conservation Laws and Differential Equations*

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To see how conservation laws arise from physical principles, we will begin by considering the simplest possible fluid dynamics problem, in which a gas or liquid is flowing through a one-dimensional pipe with some known velocity  $u(x, t)$ , which is assumed to vary only with  $x$ , the distance along the pipe, and time  $t$ . Typically in fluid dynamics problems we must determine the motion of the fluid, i.e., the velocity function  $u(x, t)$ , as part of the solution, but let's assume this is already known and we wish to simply model the concentration or density of some chemical present in this fluid (in very small quantities that do not affect the fluid dynamics). Let  $q(x, t)$  be the density of this chemical *tracer*, the function that we wish to determine.

In general the density should be measured in units of mass per unit volume, e.g., grams per cubic meter, but in studying the one-dimensional pipe with variations only in  $x$ , it is more natural to assume that  $q$  is measured in units of mass per unit length, e.g., grams per meter. This density (which is what is denoted by  $q$  here) can be obtained by multiplying the three-dimensional density function by the cross-sectional area of the pipe (which has units of square meters). Then

$$\int_{x_1}^{x_2} q(x, t) dx \tag{2.1}$$

represents the total mass of the tracer in the section of pipe between  $x_1$  and  $x_2$  at the particular time  $t$ , and has the units of mass. In problems where chemical kinetics is involved, it is often necessary to measure the “mass” in terms of moles rather than grams, and the density in moles per meter or moles per cubic meter, since the important consideration is not the mass of the chemical but the number of molecules present. For simplicity we will speak in terms of mass, but the conservation laws still hold in these other units.

Now consider a section of the pipe  $x_1 < x < x_2$  and the manner in which the integral (2.1) changes with time. If we are studying a substance that is neither created nor destroyed within this section, then the total mass within this section can change only due to the *flux* or flow of particles through the endpoints of the section at  $x_1$  and  $x_2$ . Let  $F_i(t)$  be the rate at which the tracer flows past the fixed point  $x_i$  for  $i = 1, 2$  (measured in grams per second, say). We use the convention that  $F_i(t) > 0$  corresponds to flow to the right, while  $F_i(t) < 0$  means a leftward flux, of  $|F_i(t)|$  grams per second. Since the total mass in the section  $[x_1, x_2]$

changes only due to fluxes at the endpoints, we have

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = F_1(t) - F_2(t). \quad (2.2)$$

Note that  $+F_1(t)$  and  $-F_2(t)$  both represent fluxes *into* this section.

The equation (2.2) is the basic *integral form* of a conservation law, and equations of this type form the basis for much of what we will study. The rate of change of the total mass is due only to fluxes through the endpoints – this is the basis of *conservation*. To proceed further, we need to determine how the flux functions  $F_j(t)$  are related to  $q(x, t)$ , so that we can obtain an equation that might be solvable for  $q$ . In the case of fluid flow as described above, the flux at any point  $x$  at time  $t$  is simply given by the product of the density  $q(x, t)$  and the velocity  $u(x, t)$ :

$$\text{flux at } (x, t) = u(x, t)q(x, t). \quad (2.3)$$

The velocity tells how rapidly particles are moving past the point  $x$  (in meters per second, say), and the density  $q$  tells how many grams of chemical a meter of fluid contains, so the product, measured in grams per second, is indeed the rate at which chemical is passing this point.

Since  $u(x, t)$  is a known function, we can write this flux function as

$$\text{flux} = f(q, x, t) = u(x, t)q. \quad (2.4)$$

In particular, if the velocity is independent of  $x$  and  $t$ , so  $u(x, t) = \bar{u}$  is some constant, then we can write

$$\text{flux} = f(q) = \bar{u}q. \quad (2.5)$$

In this case the flux at any point and time can be determined directly from the value of the conserved quantity at that point, and does not depend at all on the location of the point in space–time. In this case the equation is called *autonomous*. Autonomous equations will occupy much of our attention because they arise in many applications and are simpler to deal with than nonautonomous or *variable-coefficient* equations, though the latter will also be studied.

For a general autonomous flux  $f(q)$  that depends only on the value of  $q$ , we can rewrite the conservation law (2.2) as

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = f(q(x_1, t)) - f(q(x_2, t)). \quad (2.6)$$

The right-hand side of this equation can be rewritten using standard notation from calculus:

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = -f(q(x, t)) \Big|_{x_1}^{x_2}. \quad (2.7)$$

This shorthand will be useful in cases where the flux has a complicated form, and also suggests the manipulations performed below, leading to the differential equation for  $q$ .

Once the flux function  $f(q)$  is specified, e.g., by (2.5) for the simplest case considered above, we have an equation for  $q$  that we might hope to solve. This equation should hold over every interval  $[x_1, x_2]$  for arbitrary values of  $x_1$  and  $x_2$ . It is not clear how to go about finding a function  $q(x, t)$  that satisfies such a condition. Instead of attacking this problem directly, we generally transform it into a partial differential equation that can be handled by standard techniques. To do so, we must assume that the functions  $q(x, t)$  and  $f(q)$  are sufficiently smooth that the manipulations below are valid. This is very important to keep in mind when we begin to discuss nonsmooth solutions to these equations.

If we **assume that  $q$  and  $f$  are smooth functions**, then this equation can be rewritten as

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x, t)) dx, \quad (2.8)$$

or, with some further modification, as

$$\int_{x_1}^{x_2} \left[ \frac{\partial}{\partial t} q(x, t) + \frac{\partial}{\partial x} f(q(x, t)) \right] dx = 0. \quad (2.9)$$

Since this integral must be zero for all values of  $x_1$  and  $x_2$ , it follows that the integrand must be identically zero. This gives, finally, the differential equation

$$\frac{\partial}{\partial t} q(x, t) + \frac{\partial}{\partial x} f(q(x, t)) = 0. \quad (2.10)$$

This is called the **differential form of the conservation laws**. Partial differential equations (PDEs) of this type will be our main focus. Partial derivatives will usually be denoted by subscripts, so this will be written as

$$q_t(x, t) + f(q(x, t))_x = 0. \quad (2.11)$$

## 2.1 The Advection Equation

For the flux function (2.5), the conservation law (2.10) becomes

$$q_t + \bar{u} q_x = 0. \quad (2.12)$$

This is called the **advection equation**, since it models the advection of a tracer along with the fluid. By a *tracer* we mean a substance that is present in very small concentrations within the fluid, so that the magnitude of the concentration has essentially no effect on the fluid dynamics. For this one-dimensional problem the concentration (or density)  $q$  can be measured in units such as grams per meter along the length of the pipe, so that  $\int_{x_1}^{x_2} q(x, t) dx$  measures the total mass (in grams) within this section of pipe. In Section 9.1 we will consider more carefully the manner in which this is measured and the form of the resulting advection equation in more complicated cases where the diameter of the pipe and the fluid velocity need not be constant.

**Equation (2.12) is a scalar, linear, constant-coefficient PDE of hyperbolic type.** The general solution of this equation is very easy to determine. Any smooth function of the

form

$$q(x, t) = \tilde{q}(x - \bar{u}t) \quad (2.13)$$

satisfies the differential equation (2.12), as is easily verified, and in fact any solution to (2.12) is of this form for some  $\tilde{q}$ . Note that  $q(x, t)$  is constant along any ray in space–time for which  $x - \bar{u}t = \text{constant}$ . For example, all along the ray  $X(t) = x_0 + \bar{u}t$  the value of  $q(X(t), t)$  is equal to  $\tilde{q}(x_0)$ . Values of  $q$  simply advect (i.e., translate) with constant velocity  $\bar{u}$ , as we would expect physically, since the fluid in the pipe (and hence the density of tracer moving with the fluid) is simply advecting with constant speed. These rays  $X(t)$  are called the **characteristics** of the equation. More generally, characteristic curves for a PDE are curves along which the equation simplifies in some particular manner. For the equation (2.12), we see that along  $X(t)$  the time derivative of  $q(X(t), t)$  is

$$\begin{aligned} \frac{d}{dt}q(X(t), t) &= q_t(X(t), t) + X'(t)q_x(X(t), t) \\ &= q_t + \bar{u}q_x \\ &= 0. \end{aligned} \quad (2.14)$$

and the equation (2.12) reduces to a trivial ordinary differential equation  $\frac{d}{dt}Q = 0$ , where  $Q(t) = q(X(t), t)$ . This again leads to the conclusion that  $q$  is constant along the characteristic.

To find the particular solution to (2.12) of interest in a practical problem, we need more information in order to determine the particular function  $\tilde{q}$  in (2.13): *initial conditions* and perhaps *boundary conditions* for the equation. First consider the case of an infinitely long pipe with no boundaries, so that (2.12) holds for  $-\infty < x < \infty$ . Then to determine  $q(x, t)$  uniquely for all times  $t > t_0$  we need to know the initial condition at time  $t_0$ , i.e., the initial density distribution at this particular time. Suppose we know

$$q(x, t_0) = \overset{\circ}{q}(x), \quad (2.15)$$

where  $\overset{\circ}{q}(x)$  is a given function. Then since the value of  $q$  must be constant on each characteristic, we can conclude that

$$q(x, t) = \overset{\circ}{q}(x - \bar{u}(t - t_0))$$

for  $t \geq t_0$ . The initial profile  $\overset{\circ}{q}$  simply translates with speed  $\bar{u}$ .

If the pipe has finite length,  $a < x < b$ , then we must also specify the density of tracer entering the pipe as a function of time, at the inflow end. For example, if  $\bar{u} > 0$  then we must specify a boundary condition at  $x = a$ , say

$$q(a, t) = g_0(t) \quad \text{for } t \geq t_0$$

in addition to the initial condition

$$q(x, t) = \overset{\circ}{q}(x) \quad \text{for } a < x < b.$$

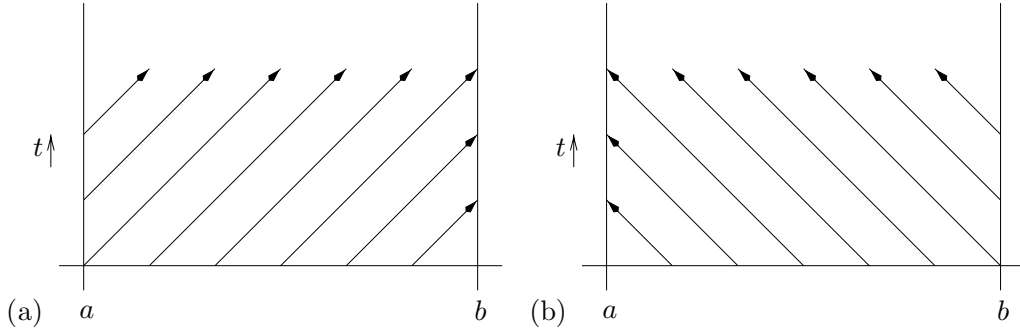


Fig. 2.1. The solution to the advection equation is constant along characteristics. When solving this equation on the interval  $[a, b]$ , we need boundary conditions at  $x = a$  if  $\bar{u} > 0$  as shown in (a), or at  $x = b$  if  $\bar{u} < 0$  as shown in (b).

The solution is then

$$q(x, t) = \begin{cases} g_0(t - (x - a)/\bar{u}) & \text{if } a < x < a + \bar{u}(t - t_0), \\ \bar{q}(x - \bar{u}(t - t_0)) & \text{if } a + \bar{u}(t - t_0) < x < b. \end{cases}$$

Note that we do not need to specify a boundary condition at the outflow boundary  $x = b$  (and in fact cannot, since the density there is entirely determined by the data given already).

If on the other hand  $\bar{u} < 0$ , then flow is to the left and we would need a boundary condition at  $x = b$  rather than at  $x = a$ . Figure 2.1 indicates the flow of information along characteristics for the two different cases. **The proper specification of boundary conditions is always an important part of the setup of a problem.**

From now on, we will generally take the initial time to be  $t = 0$  to simplify notation, but everything extends easily to general  $t_0$ .

### 2.1.1 Variable Coefficients

If the fluid velocity  $u$  varies with  $x$ , then the flux (2.4) leads to the conservation law

$$q_t + (u(x)q)_x = 0. \quad (2.16)$$

In this case the **characteristic curves  $X(t)$**  are solutions to the ordinary differential equations

$$X'(t) = u(X(t)). \quad (2.17)$$

Starting from an arbitrary initial point  $x_0$ , we can solve the equation (2.17) with initial condition  $X(0) = x_0$  to obtain a particular characteristic curve  $X(t)$ . Note that these curves track the motion of particular material particles carried along by the fluid, since their velocity at any time matches the fluid velocity. Along a characteristic curve we find that the advection

equation (2.16) simplifies:

$$\begin{aligned}
 \frac{d}{dt}q(X(t), t) &= q_t(X(t), t) + X'(t)q_x(X(t), t) \\
 &= q_t + u(X(t))q_x \\
 &= q_t + (u(X(t))q)_x - u'(X(t))q \\
 &= -u'(X(t))q(X(t), t).
 \end{aligned} \tag{2.18}$$

Note that when  $u$  is not constant, the curves are no longer straight lines and the solution  $q$  is no longer constant along the curves, but still the original partial differential equation has been reduced to solving sets of ordinary differential equations.

The operator  $\partial_t + u\partial_x$  is often called the *material derivative*, since it represents differentiation along the characteristic curve, and hence computes the rate of change observed by a material particle moving with the fluid.

The equation (2.16) is an advection equation in conservation form. In some applications it is more natural to derive a nonconservative advection equation of the form

$$q_t + u(x)q_x = 0. \tag{2.19}$$

Again the characteristic curves satisfy (2.17) and track the motion of material points. For this equation the second line of the right-hand side of (2.18) reduces to zero, so that  $q$  is now constant along characteristic curves. Which form (2.16) or (2.19) arises often depends simply on what units are used to measure physical quantities, e.g., whether we measure concentration in grams per meter as was assumed above (giving (2.16)), or whether we use grams per cubic meter, as might seem to be a more reasonable definition of concentration in a physical fluid. The latter choice leads to (2.19), as is discussed in detail in Chapter 9, and further treatment of variable-coefficient problems is deferred until that point.

## 2.2 Diffusion and the Advection–Diffusion Equation

Now suppose that the fluid in the pipe is not flowing, and has zero velocity. Then according to the advection equation,  $q_t = 0$  and the initial profile  $\hat{q}(x)$  does not change with time. However, if  $\hat{q}$  is not constant in space, then in fact it should still tend to slowly change due to molecular diffusion. The velocity  $\bar{u}$  should really be thought of as a *mean velocity*, the average velocity that the roughly  $10^{23}$  molecules in a given drop of water have. But individual molecules are bouncing around in different directions, and so molecules of the substance we are tracking will tend to get spread around in the water, as a drop of ink spreads. There will tend to be a net motion from regions where the density is large to regions where it is smaller. *Fick's law of diffusion* states that the net flux is proportional to the *gradient* of  $q$ , which in one space dimension is simply the derivative  $q_x$ . The flux at a point  $x$  now depends on the value of  $q_x$  at this point, rather than on the value of  $q$ , so we write

$$\text{flux of } q = f(q_x) = -\beta q_x, \tag{2.20}$$

where  $\beta$  is the *diffusion coefficient*. Using this flux in (2.10) gives

$$q_t = \beta q_{xx}, \tag{2.21}$$

which is known as the *diffusion equation*.



In some problems the diffusion coefficient may vary with  $x$ . Then  $f = -\beta(x)q_x$  and the equation becomes

$$q_t = (\beta(x)q_x)_x. \quad (2.22)$$

Returning to the example of fluid flow, more generally there would be both advection and diffusion occurring simultaneously. Then the flux is  $f(q, q_x) = \bar{u}q - \beta q_x$ , giving the *advection–diffusion* equation

$$q_t + \bar{u}q_x = \beta q_{xx}. \quad (2.23)$$

The diffusion and advection–diffusion equations are examples of the general class of PDEs called *parabolic*.

### 2.3 The Heat Equation

The equation (2.21) (or more generally (2.22)) is often called the *heat equation*, for heat diffuses in much the same way as a chemical concentration. In the case of heat, there may be no net motion of the material, but thermal vibration of molecules causes neighboring molecules to vibrate and this internal energy diffuses through the material. Let  $q(x, t)$  now be the temperature of the material at point  $x$  (e.g., a metal rod, since we are in one space dimension). The density of internal energy at point  $x$  is then given by

$$E(x, t) = \kappa(x)q(x, t),$$

where  $\kappa(x)$  is the *heat capacity* of the material at this point. It is this energy that is conserved, and hence varies in a test section  $[x_1, x_2]$  only due to the flux of energy past the endpoints. The heat flux is given by *Fourier's law of heat conduction*,

$$\text{flux} = -\beta q_x,$$

where  $\beta$  is the *coefficient of thermal conductivity*. This looks identical to Fick's law for diffusion, but note that Fourier's law says that the *energy* flux is proportional to the *temperature* gradient. If the heat capacity is identically constant, say  $\kappa \equiv 1$ , then this is identical to Fick's law, but there is a fundamental difference if  $\kappa$  varies. Equation (2.22) is the heat equation when  $\kappa \equiv 1$ . More generally the heat equation is derived from the conservation law

$$\frac{d}{dt} \int_{x_1}^{x_2} \kappa(x)q(x, t) dx = -\beta(x)q_x(x, t) \Big|_{x_1}^{x_2}, \quad (2.24)$$

and has the differential form

$$(\kappa q)_t = (\beta q_x)_x. \quad (2.25)$$

Typically  $\kappa$  does not vary with time and so this can be written as

$$\kappa q_t = (\beta q_x)_x. \quad (2.26)$$

### 2.4 Capacity Functions

In the previous section we saw how the heat capacity comes into the conservation law for heat conduction. There are also other situations where a “capacity” function naturally arises in the derivation of a conservation law, where again the flux of a quantity is naturally defined in terms of one variable  $q$ , whereas it is a different quantity  $\kappa q$  that is conserved. If the flux function is  $f(q)$ , then the obvious generalization of (2.24) yields the conservation law

$$\kappa q_t + f(q)_x = 0. \quad (2.27)$$

While it may be possible to incorporate  $\kappa$  into the definition of  $f(q)$ , it is often preferable numerically to work directly with the form (2.27). This is discussed in Section 6.16 and is useful in many applications. In fluid flow problems,  $\kappa$  might represent the capacity of the medium to hold fluid. For flow through a pipe with a varying diameter,  $\kappa(x)$  might be the cross-sectional area, for example (see Section 9.1). For flow in porous media,  $\kappa$  would be the porosity, the fraction of the medium available to fluid. On a nonuniform grid a capacity  $\kappa$  appears in the numerical method that is related to the size of a physical grid cell; see Section 6.17 and Chapter 23.

### 2.5 Source Terms

In some situations  $\int_{x_1}^{x_2} q(x, t) dx$  changes due to effects other than flux through the endpoints of the section, if there is some source or sink of the substance within the section. Denote the density function for such a source by  $\psi(q, x, t)$ . (Negative values of  $\psi$  correspond to a sink rather than a source.) Then the equation becomes

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = \int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x, t)) dx + \int_{x_1}^{x_2} \psi(q(x, t), x, t) dx.$$

This leads to the PDE

$$q_t(x, t) + f(q(x, t))_x = \psi(q(x, t), x, t). \quad (2.28)$$

In this section we mention only a few effects that lead to source terms. Conservation laws with source terms are more fully discussed in Chapter 17.

#### 2.5.1 External Heat Sources

As one example, consider heat conduction in a rod as in Section 2.3, with  $\kappa \equiv 1$  and  $\beta \equiv$  constant, but now suppose there is also an external energy source distributed along the rod with density  $\psi$ . Then we obtain the equation

$$q_t(x, t) = \beta q_{xx}(x, t) + \psi(x, t).$$

This assumes the heat source is independent of the current temperature. In some cases the strength of the source may depend on the value of  $q$ . For example, if the rod is immersed in a liquid that is held at constant temperature  $q_0$ , then the flux of heat into the rod at the

point  $(x, t)$  is proportional to  $q_0 - q(x, t)$  and the equation becomes

$$q_t(x, t) = \beta q_{xx}(x, t) + D(q_0 - q(x, t)),$$

where  $D$  is the conductivity coefficient between the rod and the bath.

### 2.5.2 Reacting Flow

As another example, consider a fluid flowing through a pipe at constant velocity as in Section 2.1, but now suppose there are several different chemical species being advected in this flow (in minute quantities compared to the bulk fluid). If these chemicals react with one another, then the mass of each species individually will not be conserved, since it is used up or produced by the chemical reactions. We will have an advection equation for each species, but these will include source terms arising from the chemical kinetics.

As an extremely simple example, consider the advection of a radioactive isotope with concentration measured by  $q^1$ , which decays spontaneously at some rate  $\alpha$  into a different isotope with concentration  $q^2$ . If this decay is taking place in a fluid moving with velocity  $\bar{u}$ , then we have a system of two advection equations with source terms:

$$\begin{aligned} q_t^1 + \bar{u}q_x^1 &= -\alpha q^1, \\ q_t^2 + \bar{u}q_x^2 &= +\alpha q^1. \end{aligned} \tag{2.29}$$

This has the form  $q_t + Aq_x = \psi(q)$ , in which the coefficient matrix  $A$  is diagonal with both diagonal elements equal to  $\bar{u}$ . This is a hyperbolic system, with a source term. More generally we might have  $m$  species with various chemical reactions occurring simultaneously between them. Then we would have a system of  $m$  advection equations (with diagonal coefficient matrix  $A = \bar{u}I$ ) and source terms given by the standard kinetics equations of mass action.

If there are spatial variations in concentrations, then these equations may be augmented with diffusion terms for each species. This would lead to a system of **reaction–advection–diffusion equations** of the form

$$q_t + Aq_x = \beta q_{xx} + \psi(q). \tag{2.30}$$

The diffusion coefficient could be different for each species, in which case  $\beta$  would be a diagonal matrix instead of a scalar.

Other types of source terms arise from external forces such as gravity or from geometric transformations used to simplify the equations. See Chapter 17 for some other examples.

## 2.6 Nonlinear Equations in Fluid Dynamics

In the pipe-flow model discussed above, the function  $q(x, t)$  represented the density of some tracer that was carried along with the fluid but was present in such small quantities that the distribution of  $q$  has no effect on the fluid velocity. Now let's consider the density of the fluid itself, again in grams per meter, say, for this one-dimensional problem. We will denote the fluid density by the standard symbol  $\rho(x, t)$ . If the fluid is incompressible (as most liquids can be assumed to be for most purposes), then  $\rho(x, t)$  is constant and this

one-dimensional problem is not very interesting. If we consider a gas, however, then the molecules are far enough apart that compression or expansion is possible and the density may vary from point to point.

If we again assume that the velocity  $\bar{u}$  is constant, then the density  $\rho$  will satisfy the same advection equation as before (since the flux is simply  $\bar{u}\rho$  and  $\bar{u}$  is constant),

$$\rho_t + \bar{u}\rho_x = 0, \quad (2.31)$$

and any initial variation in density will simply translate at speed  $\bar{u}$ . However, this is not what we would expect to happen physically. If the gas is compressed in some region (i.e., the density is higher here than nearby) then we would expect that the gas would tend to push into the neighboring gas, spreading out, and lowering the density in this region while raising the density nearby. (This does in fact happen provided that the *pressure* is also higher in this region; see below.) In order for the gas to spread out it must move relative to the neighboring gas, and hence we expect the velocity to change as a result of the variation in density.

While previously we assumed the tracer density  $q$  had no effect on the velocity, this is no longer the case. Instead we must view the velocity  $u(x, t)$  as another unknown to be determined along with  $\rho(x, t)$ . The density flux still takes the form (2.3), and so the conservation law for  $\rho$  has the form

$$\rho_t + (\rho u)_x = 0, \quad (2.32)$$

which agrees with (2.31) only if  $u$  is constant. **This equation is generally called the *continuity equation* in fluid dynamics, and models the *conservation of mass*.**

In addition to this equation we now need a second equation for the velocity. The velocity itself is not a conserved quantity, but the momentum is. The product  $\rho(x, t)u(x, t)$  gives the density of momentum, in the sense that the integral of  $\rho u$  between any two points  $x_1$  and  $x_2$  yields the total momentum in this interval, and this can change only due to the flux of momentum through the endpoints of the interval. The momentum flux past any point  $x$  consists of two parts. First there is momentum carried past this point along with the moving fluid. For any density function  $q$  this flux has the form  $qu$ , as we have already seen at the beginning of this chapter, and so for the momentum  $q = \rho u$  this contribution to the flux is  $(\rho u)u = \rho u^2$ . This is essentially an advective flux, although in the case where the quantity being advected is the velocity or momentum of the fluid itself, the phenomenon is often referred to as *convection* rather than advection.

In addition to this macroscopic convective flux, there is also a microscopic momentum flux due to the *pressure* of the fluid, as described in Section 14.1. This enters into the momentum flux, which now becomes

$$\text{momentum flux} = \rho u^2 + p.$$

The integral form of the conservation law (2.7) is then

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t)u(x, t) dx = -[\rho u^2 + p]_{x_1}^{x_2}. \quad (2.33)$$

Note that it is only a *difference* in pressure between the two ends of the interval that will cause a change in the net momentum, as we would expect. We can think of this pressure difference as a net force that causes an acceleration of the fluid, though this isn't strictly correct and a better interpretation is given in Section 14.1.

If we assume that  $\rho$ ,  $u$ , and  $p$  are all smooth, then we obtain the differential equation

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \quad (2.34)$$

modeling **conservation of momentum**. Combining this with the continuity equation (2.32), we have a system of two conservation laws for the conservation of mass and momentum. These are coupled equations, since  $\rho$  and  $\rho u$  appear in both. They are also clearly nonlinear, since products of the unknowns appear.

In developing the conservation law for  $\rho u$  we have introduced a new unknown, the pressure  $p(x, t)$ . It appears that we need a third differential equation for this. Pressure is not a conserved quantity, however, and so instead we introduce a fourth variable, the *energy*, and an additional equation for the **conservation of energy**. The density of energy will be denoted by  $E(x, t)$ . This still does not determine the pressure, and to close the system we must add an *equation of state*, an algebraic equation that determines the pressure at any point in terms of the mass, momentum, and energy at the point. The energy equation and equations of state will be discussed in detail in Chapter 14, where we will derive the full system of three conservation laws.

**For the time being we consider special types of flow where we can drop the conservation-of-energy equation and use a simpler equation of state that determines  $p$  from  $\rho$  alone.** For example, if no shock waves are present, then it is often correct to assume that the *entropy* of the gas is constant. Such a flow is called *isentropic*. This is discussed further in Chapter 14. This assumption is reasonable in particular if we wish to derive the equations of *linear acoustics*, which we will do in the next section. In this case we look at very small-amplitude motions (sound waves) and the flow remains isentropic. In the isentropic case the equation of state is simply

$$p = \hat{\kappa} \rho^\gamma \equiv P(\rho), \quad (2.35)$$

where  $\hat{\kappa}$  and  $\gamma$  are two constants (with  $\gamma \approx 1.4$  for air).

More generally we could **assume an equation of state of the form**

$$p = P(\rho), \quad (2.36)$$

where  $P(\rho)$  is a given function specifying the pressure in terms of density. To be physically realistic we can generally assume that

$$P'(\rho) > 0 \quad \text{for } \rho > 0. \quad (2.37)$$

This matches our intuition (already used above) that increasing the density of the gas will cause a corresponding increase in pressure. Note that the isentropic equation of state (2.35) has this property. We will see below that the assumption (2.37) is necessary in order to obtain a hyperbolic system.

Using the equation of state (2.36) in (2.34), together with the continuity equation (2.32), gives a closed system of two equations:

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0.\end{aligned}\tag{2.38}$$

This is a coupled system of two nonlinear conservation laws, which we can write in the form

$$q_t + f(q)_x = 0\tag{2.39}$$

if we define

$$q \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \equiv \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}, \quad f(q) \equiv \begin{bmatrix} \rho u \\ \rho u^2 + P(\rho) \end{bmatrix} \equiv \begin{bmatrix} q^2 \\ (q^2)^2/q^1 + P(q^1) \end{bmatrix}.\tag{2.40}$$

More generally, a system of  $m$  conservation laws takes the form (2.39) with  $q \in \mathbb{R}^m$  and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . The components of  $f$  are the fluxes of the respective components of  $q$ , and in general each flux may depend on the values of any or all of the conserved quantities at that point.

Again it should be stressed that this differential form of the conservation law is derived under the assumption that  $q$  is smooth, from the more fundamental integral form. Note that when  $q$  is smooth, we can also rewrite (2.39) as

$$q_t + f'(q)q_x = 0,\tag{2.41}$$

where  $f'(q)$  is the Jacobian matrix with  $(i, j)$  entry given by  $\partial f_i / \partial q_j$ . The form (2.41) is called the *quasilinear form* of the equation, because it resembles the linear system

$$q_t + Aq_x = 0,\tag{2.42}$$

where  $A$  is a given  $m \times m$  matrix. In the linear case this matrix does not depend on  $q$ , while in the quasilinear equation (2.41) it does. A thorough understanding of linear systems of the form (2.42) is required before tackling nonlinear systems, and **the first 10 chapters concern only linear problems**. There is a close connection between these theories, and the Jacobian matrix  $f'(q)$  plays an important role in the nonlinear theory.

## 2.7 Linear Acoustics

In general one can always obtain a linear system from a nonlinear problem by *linearizing about some state*. This amounts to defining  $A \equiv f'(q_0)$  for some fixed state  $q_0$  in the linear system (2.42), and gives a mathematically simpler problem that is useful in some situations, particularly when the interest is in studying small perturbations about some constant state.

To see how this comes about, suppose we wish to model the **propagation of sound waves** in a one-dimensional tube of gas. An acoustic wave is a very small pressure disturbance that propagates through the compressible gas, causing infinitesimal changes in the density and pressure of the gas via small motions of the gas with infinitesimal values of the velocity  $u$ .

Our eardrums are extremely sensitive to small changes in pressure and translate small oscillations in the pressure into nerve impulses that we interpret as sound. Consequently, **most sound waves are essentially linear phenomena**: the magnitudes of disturbances from the background state are so small that products or powers of the perturbation amplitude can be ignored. **As linear phenomena, they also do not involve shock waves, and so a linearization of the isentropic equations introduced above is suitable.** (An exception is the “sonic boom” caused by supersonic aircraft – this is a nonlinear shock wave, or at least originates as such.)

To perform the linearization of (2.40), let

$$q(x, t) = q_0 + \tilde{q}(x, t), \quad (2.43)$$

where  $q_0 = (\rho_0, \rho_0 u_0)$  is the background state we are linearizing about and  $\tilde{q}$  is the perturbation we wish to determine. Typically  $u_0 = 0$ , but it can be nonzero if we wish to study the propagation of sound in a constant-strength wind, for example. Using (2.43) in (2.11) and discarding any terms that involve powers or products of the  $\tilde{q}$  variables, **we obtain the linearized equations**

$$\tilde{q}_t + f'(q_0)\tilde{q}_x = 0 \quad (2.44)$$

**This is a constant-coefficient linear system modeling the evolution of small disturbances.**

To obtain the acoustics equations, we compute the Jacobian matrix for the simplified system of gas dynamics (2.38). Differentiating the flux function from (2.40) gives

$$\begin{aligned} f'(q) &= \begin{bmatrix} \partial f^1 / \partial q^1 & \partial f^1 / \partial q^2 \\ \partial f^2 / \partial q^1 & \partial f^2 / \partial q^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -(q^2)^2 / (q^1)^2 + P'(q^1) & 2q^2 / q^1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -u^2 + P'(\rho) & 2u \end{bmatrix}. \end{aligned} \quad (2.45)$$

**The equations of linear acoustics thus take the form of a constant-coefficient linear system (2.44) with**

$$A \equiv f'(q_0) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix}. \quad (2.46)$$

Note that the vector  $\tilde{q}$  in the system (2.44) has components  $\tilde{\rho}$  and  $\tilde{\rho}u$ , the perturbation of density and momentum. When written out in terms of its components, the system is

$$\begin{aligned} \tilde{\rho}_t + (\tilde{\rho}u)_x &= 0 \\ (\tilde{\rho}u)_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\tilde{\rho}u)_x &= 0. \end{aligned} \quad (2.47)$$

Physically it is often more natural to model perturbations  $\tilde{u}$  and  $\tilde{p}$  in velocity and pressure, since these can often be measured directly. To obtain such equations, first note that pressure perturbations can be related to density perturbations through the equation of state,

$$p_0 + \tilde{p} = P(\rho_0 + \tilde{\rho}) = P(\rho_0) + P'(\rho_0)\tilde{\rho} + \cdots,$$

and since  $p_0 = P(\rho_0)$ , we obtain

$$\tilde{p} \approx P'(\rho_0)\tilde{\rho}.$$

Also we have

$$\rho u = (\rho_0 + \tilde{\rho})(u_0 + \tilde{u}) = \rho_0 u_0 + \tilde{\rho} u_0 + \rho_0 \tilde{u} + \tilde{\rho} \tilde{u},$$

and so

$$\tilde{\rho} \tilde{u} \approx u_0 \tilde{\rho} + \rho_0 \tilde{u}.$$

Using these expressions in the equations (2.47) and performing some manipulations (Exercise 2.1) leads to the alternative form of the linear acoustics equations

$$\begin{aligned} \tilde{p}_t + u_0 \tilde{p}_x + K_0 \tilde{u}_x &= 0, \\ \rho_0 \tilde{u}_t + \tilde{p}_x + \rho_0 u_0 \tilde{u}_x &= 0, \end{aligned} \quad (2.48)$$

where

$$K_0 = \rho_0 P'(\rho_0). \quad (2.49)$$

The equations (2.48) can be written as a linear system

$$\begin{bmatrix} \tilde{p} \\ \tilde{u} \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{u} \end{bmatrix}_x = 0. \quad (2.50)$$

Here and from now on we will generally drop the tilde on  $p$  and  $u$  and use

$$q(x, t) = \begin{bmatrix} p(x, t) \\ u(x, t) \end{bmatrix}$$

to denote the pressure and velocity perturbations in acoustics.

The system (2.50) can also be derived by first rewriting the conservation laws (2.38) as a nonconservative set of equations for  $u$  and  $p$ , which is valid only for smooth solutions, and then linearizing this system; see Exercise 2.2.

An important special case of these equations is obtained by setting  $u_0 = 0$ , so that we are linearizing about the motionless state. In this case the coefficient matrix  $A$  appearing in the system (2.50) is

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \quad (2.51)$$

and the equations reduce to

$$\begin{aligned} p_t + K_0 u_x &= 0, \\ \rho_0 u_t + p_x &= 0. \end{aligned} \quad (2.52)$$

In Section 2.12 we will see that essentially the same set of equations can be derived for one-dimensional acoustics in an elastic solid. The parameter  $K_0$  is called the *bulk modulus of compressibility* of the material; see Section 22.1.2 for more about this parameter.



### 2.8 Sound Waves

If we solve the equations just obtained for linear acoustics in a stationary gas, we expect the solution to consist of sound waves propagating to the left and right. Since the equations are linear, we should expect that the general solution consists of a linear superposition of waves moving in each direction, and that each wave propagates at constant speed (the speed of sound) with its shape unchanged. This suggests looking for solutions to the system (2.52) of the form

$$q(x, t) = \bar{q}(x - st)$$

for some speed  $s$ , where  $\bar{q}(\xi)$  is some function of one variable. With this *Ansatz* we compute that

$$q_t(x, t) = -s\bar{q}'(x - st), \quad q_x(x, t) = \bar{q}'(x - st),$$

and so the equation  $q_t + Aq_x = 0$  reduces to

$$A\bar{q}'(x - st) = s\bar{q}'(x - st). \quad (2.53)$$

Since  $s$  is a scalar while  $A$  is a matrix, this is only possible if  $s$  is an eigenvalue of the matrix  $A$ , and  $\bar{q}'(\xi)$  must also be a corresponding eigenvector of  $A$  for each value of  $\xi$ . Make sure you understand why this is so, as this is a key concept in understanding the structure of hyperbolic systems.

For the matrix  $A$  in (2.51) we easily compute that the eigenvalues are

$$\lambda^1 = -c_0 \quad \text{and} \quad \lambda^2 = +c_0, \quad (2.54)$$

where

$$c_0 = \sqrt{K_0/\rho_0}, \quad (2.55)$$

which must be the *speed of sound* in the gas. As expected, waves can propagate in either direction with this speed. Recalling (2.49), we see that

$$c_0 = \sqrt{P'(\rho_0)}. \quad (2.56)$$

The intuitively obvious assumption (2.37) (that pressure increases with density so that  $P'(\rho) > 0$ ) turns out to be important mathematically in order for the speed of sound  $c_0$  to be a real number.

For the more general coefficient matrix  $A$  of (2.50) with  $u_0 \neq 0$ , the eigenvalues are found to be

$$\lambda^1 = u_0 - c_0 \quad \text{and} \quad \lambda^2 = u_0 + c_0. \quad (2.57)$$

When the fluid is moving with velocity  $u_0$ , sound waves still propagate at speed  $\pm c_0$  relative to the fluid, and at velocities  $\lambda^1$  and  $\lambda^2$  relative to a fixed observer. (See Figure 3.7.)

Regardless of the value of  $u_0$ , the **eigenvectors of the coefficient matrix** are

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \quad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}. \quad (2.58)$$

Any scalar multiple of each vector would still be an eigenvector. We choose the particular normalization of (2.58) because the quantity

$$Z_0 \equiv \rho_0 c_0 \quad (2.59)$$

is an important parameter in acoustics, called the **impedance of the medium**.

A sound wave propagating to the left with velocity  $-c_0$  must have the general form

$$q(x, t) = \bar{w}^1(x + c_0 t) r^1 \quad (2.60)$$

for some scalar function  $\bar{w}^1(\xi)$ , so that

$$q(x, t) = \bar{w}^1(x + c_0 t) r^1 \equiv \bar{q}(x + c_0 t)$$

and hence  $\bar{q}'(\xi)$  is a scalar multiple of  $r^1$  as required by (2.53) for  $s = -c_0$ . In terms of the components of  $q$  this means that

$$\begin{aligned} p(x, t) &= -Z_0 \bar{w}^1(x + c_0 t), \\ u(x, t) &= \bar{w}^1(x + c_0 t). \end{aligned} \quad (2.61)$$

We see that in a left-going sound wave the pressure and velocity perturbations are always related by  $p = -Z_0 u$ . Analogously, in a right-going sound wave  $p = +Z_0 u$  everywhere and  $q(x, t) = \bar{w}^2(x - c_0 t) r^2$  for some scalar function  $\bar{w}^2(\xi)$ . (See Figure 3.1.)

The general solution to the acoustic equations consists of a superposition of left-going and right-going waves, and has

$$q(x, t) = \bar{w}^1(x + c_0 t) r^1 + \bar{w}^2(x - c_0 t) r^2 \quad (2.62)$$

for some scalar functions  $\bar{w}^1(\xi)$  and  $\bar{w}^2(\xi)$ . Exactly what these functions are will depend on the *initial data* given for the problem. Let

$$q(x, 0) = \bar{q}(x) = \begin{bmatrix} \bar{p}(x) \\ \bar{u}(x) \end{bmatrix}$$

be the pressure and velocity perturbation at time  $t = 0$ . To compute the resulting solution  $q(x, t)$  we need to determine the scalar functions  $\bar{w}^1$  and  $\bar{w}^2$  in (2.62). To do so we can evaluate (2.62) at time  $t = 0$  and set this equal to the given data  $\bar{q}$ , obtaining

$$\bar{w}^1(x) r^1 + \bar{w}^2(x) r^2 = \bar{q}(x).$$

At each point  $x$  this gives a  $2 \times 2$  linear system of equations to solve for  $\bar{w}^1(x)$  and  $\bar{w}^2(x)$  at this particular point (since the vectors  $r^1$ ,  $r^2$ , and  $\bar{q}(x)$  are all known). Let

$$R = [r^1 | r^2] \quad (2.63)$$

be the  $2 \times 2$  matrix with columns  $r^1$  and  $r^2$ . Then this system of equations can be written as

$$R\bar{w}(x) = \bar{q}(x), \quad (2.64)$$

where  $\bar{w}(x)$  is the vector with components  $\bar{w}^1(x)$  and  $\bar{w}^2(x)$ . For acoustics the matrix  $R$  is

$$R = \begin{bmatrix} -Z_0 & Z_0 \\ 1 & 1 \end{bmatrix}, \quad (2.65)$$

which is a nonsingular matrix provided  $Z_0 > 0$  as it will be in practice. The solution to (2.64) can be found in terms of the inverse matrix

$$R^{-1} = \frac{1}{2Z_0} \begin{bmatrix} -1 & Z_0 \\ 1 & Z_0 \end{bmatrix}. \quad (2.66)$$

We find that

$$\begin{aligned} \bar{w}^1(x) &= \frac{1}{2Z_0} [-\bar{p}(x) + Z_0 \bar{u}(x)], \\ \bar{w}^2(x) &= \frac{1}{2Z_0} [\bar{p}(x) + Z_0 \bar{u}(x)]. \end{aligned} \quad (2.67)$$

The solution (2.62) then becomes

$$\begin{aligned} p(x, t) &= \frac{1}{2} [\bar{p}(x + c_0 t) + \bar{p}(x - c_0 t)] - \frac{Z_0}{2} [\bar{u}(x + c_0 t) - \bar{u}(x - c_0 t)], \\ u(x, t) &= -\frac{1}{2Z_0} [\bar{p}(x + c_0 t) - \bar{p}(x - c_0 t)] + \frac{1}{2} [\bar{u}(x + c_0 t) + \bar{u}(x - c_0 t)]. \end{aligned} \quad (2.68)$$

## 2.9 Hyperbolicity of Linear Systems

The process we have just gone through to solve the acoustics equations motivates the definition of **a first-order hyperbolic system of partial differential equations**. This process generalizes to solve any linear constant-coefficient hyperbolic system.

**Definition 2.1.** *A linear system of the form*

$$q_t + Aq_x = 0 \quad (2.69)$$

*is called hyperbolic if the  $m \times m$  matrix  $A$  is diagonalizable with real eigenvalues.*

We denote the eigenvalues by

$$\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m.$$

The matrix is diagonalizable if there is a *complete* set of eigenvectors, i.e., if there are nonzero vectors  $r^1, r^2, \dots, r^m \in \mathbb{R}^m$  such that

$$Ar^p = \lambda^p r^p \quad \text{for } p = 1, 2, \dots, m, \quad (2.70)$$

and these vectors are linearly independent. In this case the matrix

$$R = [r^1 | r^2 | \cdots | r^m], \quad (2.71)$$

formed by collecting the vectors  $r^1, r^2, \dots, r^m$  together, is nonsingular and has an inverse  $R^{-1}$ . We then have

$$R^{-1}AR = \Lambda \quad \text{and} \quad A = R\Lambda R^{-1}, \quad (2.72)$$

where

$$\Lambda = \begin{bmatrix} \lambda^1 & & \\ & \lambda^2 & \\ & & \ddots \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \dots, \lambda^m).$$

Hence we can bring  $A$  to diagonal form by a similarity transformation, as displayed in (2.72). The importance of this from the standpoint of the PDE is that we can then rewrite the linear system (2.69) as

$$R^{-1}q_t + R^{-1}ARR^{-1}q_x = 0. \quad (2.73)$$

If we define  $w(x, t) \equiv R^{-1}q(x, t)$ , then this takes the form

$$w_t + \Lambda w_x = 0. \quad (2.74)$$

Since  $\Lambda$  is diagonal, this system decouples into  $m$  independent advection equations for the components  $w^p$  of  $w$ :

$$w_t^p + \lambda^p w_x^p = 0 \quad \text{for } p = 1, 2, \dots, m. \quad (2.75)$$

Since each  $\lambda^p$  is real, these advection equations make sense physically and can be used to solve the original system of equations (2.69). Complete details are given in the next chapter, but clearly the solution will consist of a linear combination of  $m$  “waves” traveling at the characteristic speeds  $\lambda^1, \lambda^2, \dots, \lambda^m$ . (Recall that eigenvalues are also sometimes called “characteristic values.”) These values define the characteristic curves  $X(t) = x_0 + \lambda^p t$  along which information propagates in the decoupled advection equations. The functions  $w^p(x, t)$  are called the characteristic variables; see Section 3.2.

There are some special classes of matrices  $A$  for which the system is certainly hyperbolic. If  $A$  is a symmetric matrix ( $A = A^T$ ), then  $A$  is always diagonalizable with real eigenvalues and the system is said to be symmetric hyperbolic. Also, if  $A$  has distinct real eigenvalues  $\lambda^1 < \lambda^2 < \cdots < \lambda^m$ , then the eigenvectors must be linearly independent and the system is hyperbolic. Such a system is called strictly hyperbolic. The equations of linear acoustics are strictly hyperbolic, for example. The homogeneous part of the system (2.29) (i.e., setting  $\alpha = 0$ ) is symmetric hyperbolic but not strictly hyperbolic. Difficulties can arise in studying certain nonstrictly hyperbolic equations, as discussed briefly in Section 16.2. If  $A$  has real eigenvalues but is not diagonalizable, then the system is weakly hyperbolic; see Section 16.3.

### 2.9.1 Second-Order Wave Equations

From the acoustics equations (2.52) we can eliminate the velocity  $u$  and obtain a second-order equation for the pressure. Differentiating the pressure equation with respect to  $t$  and the velocity equation with respect to  $x$  and then combining the results gives

$$p_{tt} = -K_0 u_{xt} = -K_0 u_{tx} = K_0 \left( \frac{1}{\rho_0} p_x \right)_x = c_0^2 p_{xx}.$$

This yields the *second-order wave equation* of the classical form

$$p_{tt} = c_0^2 p_{xx} \quad (c_0 \equiv \text{constant}). \quad (2.76)$$

This is also a *hyperbolic* equation according to the standard classification of second-order differential equations. **In this book, however, we concentrate almost entirely on first-order hyperbolic systems** as defined at the start of Section 2.9. There is a certain equivalence as suggested by the above transformation for acoustics. Conversely, given a second-order equation of the type (2.76), we can derive a first-order hyperbolic system by defining new variables

$$q^1 = p_t, \quad q^2 = -p_x,$$

so that (2.76) becomes  $q_t^1 + c_0^2 q_x^2 = 0$ , while the equality of mixed partial derivatives gives  $q_t^2 + q_x^1 = 0$ . These two equations taken together give a system  $q_t + \tilde{A} q_x = 0$ , with the coefficient matrix

$$\tilde{A} = \begin{bmatrix} 0 & c_0^2 \\ 1 & 0 \end{bmatrix}. \quad (2.77)$$

This matrix is *similar* to the matrix  $A$  of (2.51), meaning that there is a similarity transformation  $\tilde{A} = SAS^{-1}$  relating the two matrices. The matrix  $S$  relates the two sets of variables and leads to a corresponding change in the eigenvector matrix, while the eigenvalues of the two matrices are the same,  $\pm c_0$ .

Many books take the viewpoint that the equation (2.76) is the fundamental wave equation and a first-order system can be derived from it by introducing “artificial” variables such as  $p_t$  and  $p_x$ . In fact, however, it is the first-order system that follows directly from the physics, as we have seen. Since effective numerical methods are more easily derived for the first-order system than for the second-order scalar equation, there is no need for us to consider the second-order equation further.

## 2.10 Variable-Coefficient Hyperbolic Systems

**A variable-coefficient linear system of PDEs** might take the form

$$q_t + A(x)q_x = 0. \quad (2.78)$$

This system is hyperbolic at any point  $x$  where the coefficient matrix satisfies the conditions laid out in Section 2.9. In Section 9.6, for example, we will see that the equations of acoustics in a heterogeneous medium (where the density and bulk modulus vary with  $x$ ) can be written

as such a system which is hyperbolic everywhere, with eigenvalues given by  $\pm c(x)$ , where the sound speed  $c(x)$  varies with position depending on the material parameters.

In some cases we might have a conservative system of linear equations of the form

$$q_t + (A(x)q)_x = 0, \quad (2.79)$$

in which the flux function  $f(q, x) = A(x)q$  depends explicitly on  $x$ . This system could be rewritten as

$$q_t + A(x)q_x = -A'(x)q \quad (2.80)$$

as a system of the form (2.78) with the addition of a *source term*. Again the problem is hyperbolic at any point where  $A(x)$  is diagonalizable with real eigenvalues. Such problems are discussed further in Chapter 9.

### 2.11 Hyperbolicity of Quasilinear and Nonlinear Systems

A *quasilinear* system

$$q_t + A(q, x, t)q_x = 0 \quad (2.81)$$

is said to be *hyperbolic* at a point  $(q, x, t)$  if the matrix  $A(q, x, t)$  satisfies the hyperbolicity condition (diagonalizable with real eigenvalues) at this point.

The nonlinear conservation law (2.11) is *hyperbolic* if the Jacobian matrix  $f'(q)$  appearing in the quasilinear form (2.41) satisfies the hyperbolicity condition for each physically relevant value of  $q$ .

**Example 2.1.** The nonlinear equations of isentropic gas dynamics (2.38) have the Jacobian matrix (2.45). The eigenvalues are

$$\lambda^1 = u - c, \quad \lambda^2 = u + c,$$

where the velocity  $u$  may now vary from point to point, as does the sound speed

$$c = \sqrt{P'(\rho)}. \quad (2.82)$$

However, since  $P'(\rho) > 0$  at all points in the gas, this nonlinear system is strictly hyperbolic. (Provided we stay away from the “vacuum state” where  $\rho$  and  $p$  go to zero. For the equation of state (2.35),  $c \rightarrow 0$  as well in this case, and the nonstrict hyperbolicity at this point causes additional difficulties in the nonlinear analysis.)

Solutions to nonlinear hyperbolic systems also involve wave propagation, and for a system of  $m$  equations we will often be able to find  $m$  independent waves at each point. However, since the wave speeds depend on the value of the solution  $q$ , wave shapes will typically deform, and the solution procedure is greatly complicated by this nonlinear structure. Nonlinear conservation laws are discussed starting in Chapter 11.

In the remainder of this chapter some other hyperbolic systems are introduced. These sections can be skipped at this point without loss of continuity.

## 2.12 Solid Mechanics and Elastic Waves

The equations of linear acoustics were derived in Section 2.8 by linearizing the equations of isentropic gas dynamics. Essentially the same system of equations can be derived from elasticity theory to obtain the equations modeling a one-dimensional acoustic wave in a solid, which again is a small-amplitude compressional disturbance in which the material moves back and forth in the same direction as the wave propagates, leading to small-scale changes in density and pressure. Unlike a gas or liquid, however, a solid also supports a second distinct type of small-amplitude waves called *shear waves*, in which the motion of the material is orthogonal to the direction of wave propagation. These two types of waves travel at distinct speeds, as illustrated in Figure 2.2. In general these two types of waves are coupled together and the equations of linear elasticity are a single set of hyperbolic equations that must be solved for all motions of the solid, which are coupled together. However, if we restrict our attention to one-dimensional plane waves, in which all quantities vary only in one direction, then these equations decouple into two independent hyperbolic systems of two equations each. Mathematically these linear systems are not very interesting, since each has the same structure as the acoustics equations we have already studied in detail. Because of this, however, some of the basic concepts of wave propagation in solids can be most easily introduced in this context, and this foundation will be useful when we develop the multidimensional equations.

Figure 2.2 shows the two distinct types of plane-wave motion in an elastic solid. Other types of waves can also be observed in solids, such as *surface waves* at a free surface or interface between different solids, but these have a truly multidimensional structure. The strips shown in Figure 2.2 should be viewed as taken from an infinite three-dimensional solid in which all quantities vary only with  $x$ , so the motion shown extends infinitely far in the

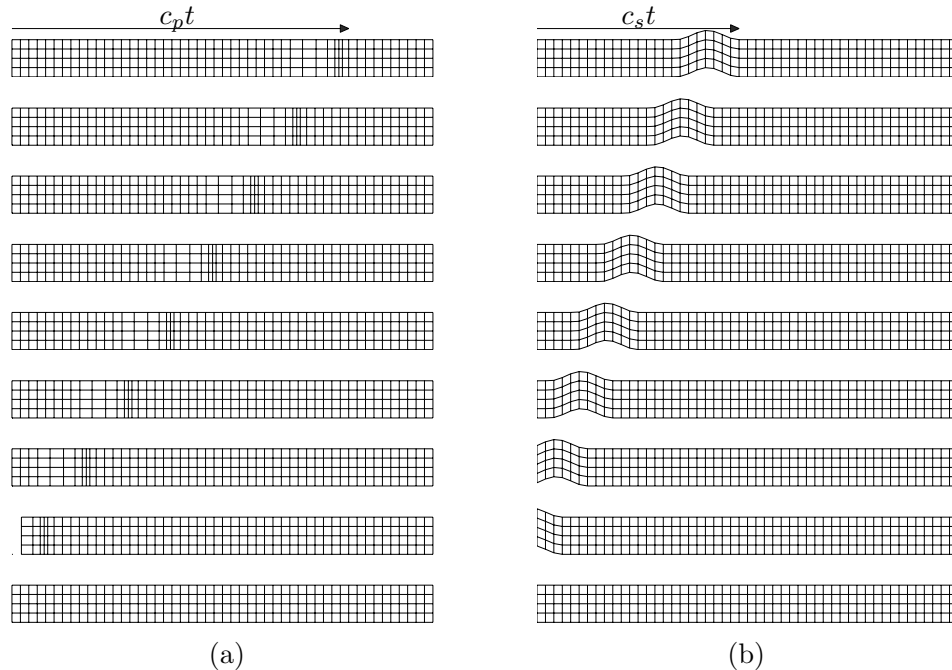


Fig. 2.2. Illustration of (a) P-waves and (b) S-waves in elastic solids. Time advances going upwards.

$y$  (upward) direction and in  $z$  (normal to the page). Related one-dimensional equations can also be used to study elastic motion in a finite elastic bar, but then additional complications arise in that a compression in the  $x$ -direction will typically result in some spreading in the  $y$ - and  $z$ -directions, whereas in the infinite plane-wave case there is no opportunity for such spreading; see Chapter 22.

In Figure 2.2(a) the material is compressed at the left edge by a motion in the  $x$ -direction confined to a small region in  $x$ . This compressional wave moves in the  $x$ -direction at some speed  $c_p$  and is analogous to an acoustic wave in a gas. Compressing the material leads to an increase in stress and hence to acceleration in the  $x$ -direction. These changes in stress and velocity are coupled together to result in wave motion.

Figure 2.2(b) shows a different type of wave, a shear wave in which the material is displaced in the  $y$ -direction over a small region in  $x$ . In a gas or liquid, a shear displacement of this type would not result in any restoring force or wave motion. There is no compression or expansion of the material, and hence no stress results. Molecules of a gas or liquid may be freely rearranged as long as there is no change in the pressure and there will be no restoring force. Of course frictional (viscous) forces will arise during a rearrangement as the molecules move past one another, but once rearranged they are not pulled back towards their original locations. A solid is fundamentally different in that the constituent molecules are attached to one another by chemical bonds that resist any deformations. The bonds will stretch slightly to allow small elastic deformations, but like tiny springs they exert a restoring force that typically grows with the magnitude of any deformation. This operates like pressure in the case of compressional waves, but these bonds also resist shear deformations, and the restoring forces result in shear waves as illustrated in Figure 2.2(b). These waves move at a speed  $c_s$  that we will see is always smaller than the speed  $c_p$  of compressional waves. The two types of waves are often called *P-waves* and *S-waves*, with “P” and “S” having two possible interpretations: “pressure” and “shear” waves, or alternatively “primary” and “secondary” waves in view of the fact that  $c_p > c_s$  and so the P-wave arising from some disturbance always arrives at a distant observer before the S-wave.

The theory of *linear elasticity* results from assuming that the deformations are small enough that the restoring force is linearly related to an appropriate measure of the deformation of the solid. For larger deformations the response may be nonlinear. The material is still elastic if the deformation is sufficiently small that the material can be expected to eventually return to its original configuration if all external forces are removed. If the deformation is too extreme, however, the material may simply fail (fracture or break), if enough bonds are irreparably broken, or it may undergo a *plastic deformation*, in which bonds are broken and reformed in a new configuration so that the resulting solid has a different resting configuration from that of the original solid. The theory of *plasticity* then applies.

### 2.12.1 Elastic Deformations

The mathematical notation of solid mechanics is somewhat different from that of fluid dynamics. For an elastic body we are typically concerned with small displacements about some *reference configuration*, the location of the body at rest, for example, and so it makes sense to consider the actual location of material points at some time as a function of their reference location. For example, in two space dimensions we can let  $(X(x, y), Y(x, y))$



represent the location at time  $t$  of the material whose reference location is  $(x, y)$ . The *displacement vector*  $\vec{\delta}$  is then defined to be

$$\vec{\delta}(x, y, t) = \begin{bmatrix} \delta^1(x, y, t) \\ \delta^2(x, y, t) \end{bmatrix} = \begin{bmatrix} X(x, y, t) \\ Y(x, y, t) \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}. \quad (2.83)$$

The symbol  $\vec{u}$  is often used for the displacement vector, but we reserve this for the velocity vector, which is the time derivative of the displacement,

$$\vec{u}(x, y, t) = \begin{bmatrix} u(x, y, t) \\ v(x, y, t) \end{bmatrix} = \begin{bmatrix} \delta_t^1(x, y, t) \\ \delta_t^2(x, y, t) \end{bmatrix}. \quad (2.84)$$

Displacements of the body often lead to *strains* within the body. A strain is a deformation that results in changes of length or shape within the body. These strains in turn lead to *stress*, the interior forces due to the stretching or compression of atomic bonds. These forces result in acceleration of the material, affecting the motion and hence the evolution of the strains. The equations of elasticity consist of Newton's law relating force to acceleration together with *stress-strain relations* describing the force that results from a given strain. This *constitutive relation* depends on the particular material (similarly to the *equation of state* for a gas). For sufficiently small strains the stress may be assumed to vary linearly with strain, resulting in the equations of linear elasticity.

### 2.12.2 Strain

Not all deformations result in a strain. Rigid-body motions (translations and rotations) in which the body is simply moved as a rigid entity do not lead to any internal strain or stress. Rigid translations correspond to a displacement vector  $\vec{\delta}(x, y, t)$  that varies only with  $t$  and is independent of spatial position. Clearly there will be a strain in the material only if  $\vec{\delta}$  varies in space, so that some points are displaced relative to other points in the body. Hence the strain depends only on the *displacement gradient*

$$\nabla \vec{\delta} = \begin{bmatrix} \delta_x^1 & \delta_y^1 \\ \delta_x^2 & \delta_y^2 \end{bmatrix} = \begin{bmatrix} X_x - 1 & X_y \\ Y_x & Y_y - 1 \end{bmatrix}, \quad (2.85)$$

where the subscripts denote partial derivatives. Note that for a rigid translation  $\nabla \vec{\delta} = 0$ .

We still need to eliminate solid-body rotations, which can be done by splitting  $\nabla \vec{\delta}$  into the sum of a symmetric and a skew-symmetric matrix,

$$\nabla \vec{\delta} = \epsilon + \Omega, \quad (2.86)$$

with

$$\epsilon = \frac{1}{2}[\nabla \vec{\delta} + (\nabla \vec{\delta})^T] = \begin{bmatrix} \delta_x^1 & \frac{1}{2}(\delta_y^1 + \delta_x^2) \\ \frac{1}{2}(\delta_y^1 + \delta_x^2) & \delta_y^2 \end{bmatrix} \quad (2.87)$$

and

$$\Omega = \frac{1}{2}[\nabla \vec{\delta} - (\nabla \vec{\delta})^T] = \begin{bmatrix} 0 & \frac{1}{2}(\delta_y^1 - \delta_x^2) \\ -\frac{1}{2}(\delta_y^1 - \delta_x^2) & 0 \end{bmatrix}. \quad (2.88)$$

The *rotation matrix*  $\Omega$  measures rigid rotations, whereas the symmetric matrix  $\epsilon$  is the desired *strain matrix*, which will also be written as

$$\epsilon = \begin{bmatrix} \epsilon^{11} & \epsilon^{12} \\ \epsilon^{21} & \epsilon^{22} \end{bmatrix}.$$

The diagonal elements  $\epsilon^{11}$  and  $\epsilon^{22}$  measure *extensional strains* in the  $x$ - and  $y$ -directions, whereas  $\epsilon^{12} = \epsilon^{21}$  is the *shear strain*.

**Example 2.2.** The P-wave shown in Figure 2.2 has a displacement of the form

$$\vec{\delta}(x, y, t) = \begin{bmatrix} w(x - c_p t) \\ 0 \end{bmatrix}$$

for some wave form  $w$ , and hence

$$\epsilon = \begin{bmatrix} w'(x - c_p t) & 0 \\ 0 & 0 \end{bmatrix}$$

with only  $\epsilon^{11}$  nonzero.

The S-wave shown in Figure 2.2 has a displacement of the form

$$\vec{\delta}(x, y, t) = \begin{bmatrix} 0 \\ w(x - c_s t) \end{bmatrix}$$

for some waveform  $w$ , and hence

$$\epsilon = \begin{bmatrix} 0 & \frac{1}{2}w'(x - c_s t) \\ \frac{1}{2}w'(x - c_s t) & 0 \end{bmatrix}$$

with only the shear strain nonzero.

To study one-dimensional elastic waves of the sort shown in Figure 2.2, we need only consider the components  $\epsilon^{11}$  and  $\epsilon^{12}$  of the strain and must assume that these are functions of  $(x, t)$  alone, independent of  $y$  and  $z$ . For two-dimensional elasticity we must consider  $\epsilon^{22}$  as well, with all three variables being functions of  $(x, y, t)$ . For full three-dimensional elasticity the displacement vector and strain matrix must be extended to three dimensions. The formula (2.87) still holds, and  $\epsilon$  is now a  $3 \times 3$  symmetric matrix with six independent elements, three extensional strains on the diagonal, and three shear strains off the diagonal. See Chapter 22 for more discussion of these equations and their proper relation to three-dimensional elasticity.

### 2.12.3 Stress

A strain in an elastic body typically results in a restoring force called the *stress*. For one-dimensional elasticity as described above, we need only be concerned with two components of the stress:  $\sigma^{11}(x, t)$ , the force in the  $x$ -direction (the *normal stress*), and  $\sigma^{12}(x, t)$ , the force in the  $y$ -direction (the *shear stress*).

In one-dimensional linear elasticity there is a complete decoupling of compressional and shear effects. The normal stress  $\sigma^{11}$  depends only on the strain  $\epsilon^{11}$ , while the shear stress  $\sigma^{12}$  depends only on the shear strain  $\epsilon^{12}$ , and these constitutive relations are linear:

$$\sigma^{11} = (\lambda + 2\mu)\epsilon^{11} \quad \text{with } \lambda + 2\mu > 0, \quad (2.89)$$

$$\sigma^{12} = 2\mu\epsilon^{12} \quad \text{with } \mu > 0. \quad (2.90)$$

Here  $\lambda$  and  $\mu$  are the *Lamé parameters* characterizing the material. The parameter  $\mu$  is also called the *shear modulus*. The parameter  $\lambda$  does not have a simple physical meaning, but is related to other properties of the material in Section 22.1. It is unfortunate that the symbol  $\lambda$  is standard for this parameter, which should not be confused with an eigenvalue.

#### 2.12.4 The Equations of Motion

We are now ready to write down **the equations of motion for one-dimensional elastic waves**. P-waves are governed by the system of equations

$$\begin{aligned} \epsilon_t^{11} - u_x &= 0, \\ \rho u_t - \sigma_x^{11} &= 0, \end{aligned} \quad (2.91)$$

where  $\rho > 0$  is the density of the material. The first equation follows from the equality  $X_{xt} = X_{tx}$ , since

$$\begin{aligned} \epsilon^{11}(x, t) = X_x(x, t) - 1 &\implies \epsilon_t^{11} = X_{xt}, \\ u(x, t) = X_t(x, t) &\implies u_x = X_{tx}. \end{aligned} \quad (2.92)$$

The second equation of (2.91) is Newton's second law since  $u_t$  is the acceleration.

The system (2.91) involves both  $\epsilon^{11}$  and  $\sigma^{11}$ , and one of these must be eliminated using the constitutive relation (2.89). If we eliminate  $\sigma^{11}$ , we obtain

$$\begin{bmatrix} \epsilon^{11} \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -(\lambda + 2\mu)/\rho & 0 \end{bmatrix} \begin{bmatrix} \epsilon^{11} \\ u \end{bmatrix}_x = 0. \quad (2.93)$$

This is a hyperbolic system, since the matrix has eigenvalues  $\lambda = \pm c_p$  with

$$c_p = \sqrt{(\lambda + 2\mu)/\rho}. \quad (2.94)$$

If we instead eliminate  $\epsilon^{11}$  we obtain

$$\begin{bmatrix} \sigma^{11} \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -(\lambda + 2\mu) \\ -1/\rho & 0 \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ u \end{bmatrix}_x = 0. \quad (2.95)$$

Again the coefficient matrix has eigenvalues  $\pm c_p$ . Note that this form is essentially equivalent to the acoustic equations derived in Section 2.8 if we identify

$$p(x, t) = -\sigma^{11}(x, t). \quad (2.96)$$

Since  $\epsilon^{11}$  measures the extensional stress (positive when the material is stretched, negative when compressed), a positive pressure corresponds to a negative value of  $\sigma^{11}$ . Note that

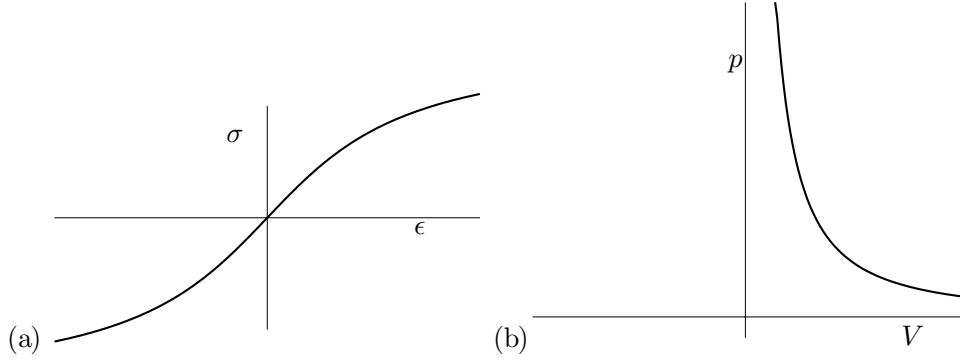


Fig. 2.3. (a) A typical stress–strain relation  $\sigma = \sigma(\epsilon)$  for the nonlinear elasticity equation (2.97). (b) The equation of state  $p = p(V)$  for isentropic gas dynamics in a Lagrangian frame using the  $p$ -system (2.108).

the stress  $\sigma^{11}$  can have either sign, depending on whether the material is compressed or stretched, while the pressure in a gas can only be positive. A gas that is “stretched” by allowing it to expand to a larger volume will not attempt to contract back to its original volume the way a solid will. This is another consequence of the fact that there are no intermolecular bonds between the gas molecules.

One-dimensional *nonlinear P-waves* can be modeled by the more general form of (2.91) given by

$$\begin{aligned}\epsilon_t - u_x &= 0, \\ \rho u_t - \sigma(\epsilon)_x &= 0,\end{aligned}\tag{2.97}$$

where  $\epsilon$  is the extensional strain  $\epsilon^{11}$  and  $\sigma^{11} = \sigma(\epsilon)$  is a more general nonlinear constitutive relation between stress and strain. A typical stress–strain relation might look something like what is shown in Figure 2.3(a). In the case shown the derivative of the stress with respect to strain decreases as the magnitude of the strain is increased. This is shown for small values of  $|\epsilon|$ , in particular for  $-\epsilon \ll 1$ , since  $\epsilon = -1$  corresponds to a state of complete compression,  $X_x = 0$ . Elasticity theory typically breaks down long before this. For very small deformations  $\epsilon$ , this nonlinear function can generally be replaced by a linearization  $\sigma = (\lambda + 2\mu)\epsilon$ , where  $(\lambda + 2\mu) \equiv \sigma'(0)$ . This is the relation (2.90) used in linear elasticity.

The equations for a linear S-wave are essentially identical to (2.91) but involve the shear strain, shear stress, and vertical velocity:

$$\begin{aligned}\epsilon_t^{12} - \frac{1}{2}v_x &= 0, \\ \rho v_t - \sigma_x^{12} &= 0.\end{aligned}\tag{2.98}$$

The relationship (2.90) is now used to eliminate either  $\epsilon^{12}$  or  $\sigma^{12}$ , resulting in a closed system of two equations, either

$$\begin{bmatrix} \epsilon^{12} \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -1/2 \\ -2\mu/\rho & 0 \end{bmatrix} \begin{bmatrix} \epsilon^{12} \\ v \end{bmatrix}_x = 0\tag{2.99}$$

if  $\epsilon^{12}$  is used, or

$$\begin{bmatrix} \sigma^{12} \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -\mu \\ -1/\rho & 0 \end{bmatrix} \begin{bmatrix} \sigma^{12} \\ v \end{bmatrix}_x = 0 \quad (2.100)$$

if  $\sigma^{12}$  is used. In either case, the eigenvalues of the coefficient matrix are  $\lambda = \pm c_s$ , with the wave speed

$$c_s = \sqrt{\mu/\rho}. \quad (2.101)$$

In general,  $\mu < \lambda + 2\mu$  and so  $c_s < c_p$ .

We have assumed that shear-wave motion is in the  $y$ -direction. In a three-dimensional body one could also observe a plane shear wave propagating in the  $x$ -direction for which the shear motion is in the  $z$ -direction. These are governed by a set of equations identical to (2.100) but involving  $\epsilon^{13}$  and  $\sigma^{13}$  in place of  $\epsilon^{12}$  and  $\sigma^{12}$ , and the  $z$ -component of velocity  $w$  in place of  $v$ . Shear motion need not be aligned with either the  $y$ - or the  $z$ -axis, but can occur in any direction perpendicular to  $x$ . Motion in any other direction is simply a linear combination of these two, however, so that there are really two decoupled systems of equations for S-waves, along with the system of equations of P-waves, needed to describe all *plane waves* in  $x$ . Note that the systems (2.95) and (2.100) both have the same mathematical structure as the acoustics equations studied previously.

For a general two- or three-dimensional motion of an elastic solid it is not possible to decompose the resulting equations into independent sets of equations for P-waves and S-waves. Instead one obtains a single coupled hyperbolic system. For motions that are fully two-dimensional (but independent of the third direction), one obtains a system of five equations for the velocities  $u, v$  and the components of the stress tensor  $\sigma^{11}, \sigma^{12}$ , and  $\sigma^{22}$  (or alternatively the three components of the strain tensor). Only in the case of purely one-dimensional motions do these equations decouple into independent sets. These decoupled systems are related to the full three-dimensional equations in Chapter 22.

### 2.13 Lagrangian Gas Dynamics and the $p$ -System

The fluid dynamics equations derived in Section 2.6 are in *Eulerian form*, meaning that  $x$  represents a fixed location in space, and quantities such as the velocity  $u(x, t)$  refer to the velocity of whatever fluid particle happens to be at the point  $x$  at time  $t$ . Alternatively, the equations can be written in *Lagrangian form*, where fixing the coordinate  $\xi$  corresponds to tracking a particular fluid particle. The Lagrangian velocity  $U(\xi, t)$  then gives the velocity of this particle at time  $t$ . We must then determine the mapping  $X(\xi, t)$  that gives the physical location of the particle labeled  $\xi$  at time  $t$ . This is more like the approach used in elasticity, as described in Section 2.12, and in one dimension a system of equations very similar to (2.97) results. (The term *fluid particle* refers to an infinitesimally small volume of fluid, but one that still contains a huge number of molecules so that the small-scale random variations in velocity can be ignored.)

To set up the labeling of points initially, we take an arbitrary physical location  $x_0$  (say  $x_0 = 0$ ) and then at each point  $x$  assign the label

$$\xi = \int_{x_0}^x \bar{\rho}(s) ds \quad (2.102)$$

to the particle initially located at  $x$ , where  $\bar{\rho}$  is the initial data for the density. If the density is positive everywhere, then this gives a one-one map. Note that  $\xi$  has units of mass and the label  $\xi$  gives the total mass between  $x_0$  and  $X(\xi, t)$ . Moreover  $\xi_2 - \xi_1$  is the total mass of all particles between those labeled  $\xi_1$  and  $\xi_2$  (at any time  $t$ , since particles cannot cross in this one-dimensional model).

The Lagrangian velocity is related to the Eulerian velocity by

$$U(\xi, t) = u(X(\xi, t), t).$$

Since  $X(\xi, t)$  tracks the location of this particle, we must have

$$X_t(\xi, t) = U(\xi, t).$$

We could define a Lagrangian density function similarly, but the conservation of mass equation in the Lagrangian framework is more naturally written in terms of the *specific volume*

$$V(\xi, t) = \frac{1}{\rho(X(\xi, t), t)}.$$

This has units of volume/mass (which is just length/mass in one dimension), so it makes sense to integrate this over  $\xi$ . Since integrating the specific volume over a fixed set of particles gives the volume occupied by these particles at time  $t$ , we must have

$$\int_{\xi_1}^{\xi_2} V(\xi, t) d\xi = X(\xi_2, t) - X(\xi_1, t). \quad (2.103)$$

Differentiating this with respect to  $t$  gives

$$\begin{aligned} \frac{d}{dt} \int_{\xi_1}^{\xi_2} V(\xi, t) d\xi &= U(\xi_2, t) - U(\xi_1, t) \\ &= \int_{\xi_1}^{\xi_2} \frac{\partial}{\partial t} V(\xi, t) d\xi. \end{aligned} \quad (2.104)$$

Rearranging this and using the fact that it must hold for all choices of  $\xi_1$  and  $\xi_2$  gives the differential form of the conservation law,

$$V_t - U_\xi V = 0. \quad (2.105)$$

Now consider the conservation of momentum. In Eulerian form  $\rho u$  is the density of momentum in units of momentum/volume. In Lagrangian form we instead consider  $U(\xi, t)$ , which can be interpreted as the momentum per unit mass, with

$$\int_{\xi_1}^{\xi_2} U(\xi, t) d\xi$$

being the total momentum of all particles between  $\xi_1$  and  $\xi_2$ . By conservation of momentum, this integral changes only due to flux at the endpoints. Since in the Lagrangian framework the endpoints are moving with the fluid, there is no “advective flux” and the only change in momentum comes from the pressure difference between the two endpoints, so

$$\frac{d}{dt} \int_{\xi_1}^{\xi_2} U(\xi, t) d\xi = p(\xi_1, t) - p(\xi_2, t),$$

which leads to the conservation law

$$U_t + p_\xi = 0. \quad (2.106)$$

If we consider isentropic or isothermal flow, then we have only these two conservation laws and the equation of state gives  $p$  in terms of  $V$  alone. Then (2.105) and (2.106) give the system of conservation laws known as the  $p$ -system,

$$\begin{aligned} V_t - U_\xi &= 0, \\ U_t + p(V)_\xi &= 0. \end{aligned} \quad (2.107)$$

This is another simple system of two equations that is useful in understanding conservation laws. It is slightly simpler than the corresponding Eulerian equations (2.38) in that the only nonlinearity is in the function  $p(V)$ . This system is hyperbolic if  $p'(V) < 0$  (see Exercise 2.7). Note that for isentropic flow we have  $p(V) = \hat{k} V^{-\gamma}$ , corresponding to the equation of state (2.35), with the shape shown in Figure 2.3(b).

Frequently the  $p$ -system is written using lowercase symbols as

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned} \quad (2.108)$$

and we will generally use this notation when the  $p$ -system is used as a generic example of a hyperbolic system. To relate this system to the Eulerian gas dynamics equations, however, it is important to use distinct notation as derived above.

The  $p$ -system (2.108) has a very similar structure to the nonlinear elasticity equation (2.97) if we equate  $p$  with the negative stress  $-\sigma$  as discussed in Section 2.12. Note, however, that in the gas dynamics case we must have  $V > 0$  and  $p > 0$ , whereas in elasticity the stress and strain can each be either positive or negative, corresponding to extension and compression respectively (recall Figure 2.3(a)).

## 2.14 Electromagnetic Waves

Electromagnetic waves are governed by Maxwell’s equations. In the simplest case this is a hyperbolic system of equations, though in materials where waves are attenuated due to induced electric currents these are modified by additional source terms. If we assume there is no net electric charge or current in the material through which the wave is propagating,

then Maxwell's equations reduce to

$$\vec{D}_t - \nabla \times \vec{H} = 0, \quad (2.109)$$

$$\vec{B}_t + \nabla \times \vec{E} = 0, \quad (2.110)$$

$$\nabla \cdot \vec{D} = 0, \quad (2.111)$$

$$\nabla \cdot \vec{B} = 0. \quad (2.112)$$

Here  $\vec{E}$ ,  $\vec{D}$ ,  $\vec{B}$ , and  $\vec{H}$  are all vectors with three spatial components. The electric field  $\vec{E}$  and the magnetic field  $\vec{B}$  are related to the two other fields  $\vec{D}$  and  $\vec{H}$  via *constitutive relations* that characterize the medium in which the wave is propagating. These are similar to the stress–strain relations needed in elasticity theory. In general they take the form

$$\vec{D} = \epsilon \vec{E}, \quad (2.113)$$

$$\vec{B} = \mu \vec{H}, \quad (2.114)$$

where  $\epsilon$  is the *permittivity* and  $\mu$  is the *magnetic permeability* of the medium. In a homogeneous isotropic material these are both scalar constants. More generally they could be  $3 \times 3$  matrices and also vary in space.

If the initial data satisfies the divergence-free conditions (2.111) and (2.112), then it can be shown that these will hold for all time, and so equations (2.109) and (2.110) for  $\vec{D}_t$  and  $\vec{B}_t$  can be taken as the time evolution equations for electromagnetic waves.

If  $\epsilon$  and  $\mu$  are scalar constants, then we can eliminate  $\vec{D}$  and  $\vec{H}$  and rewrite the wave-propagation equations as

$$\begin{aligned} \vec{E}_t - \frac{1}{\epsilon\mu} \nabla \times \vec{B} &= 0, \\ \vec{B}_t + \nabla \times \vec{E} &= 0. \end{aligned} \quad (2.115)$$

This is a linear hyperbolic system of equations in three dimensions.

In this chapter we consider only the simplest case of a plane wave propagating in the  $x$ -direction. The  $B$ - and  $E$ -fields then oscillate in the  $y$ – $z$  plane, so that electromagnetic waves are somewhat like shear waves, with the oscillations orthogonal to the direction of propagation. However, there are now two fields, and there is an additional relationship that the  $B$ -field oscillations for a given wave are orthogonal to the  $E$ -field oscillations. Figure 2.4 illustrates a wave in which the  $E$ -field oscillates in  $y$  while the  $B$ -field oscillates in  $z$ . In this case only the  $y$ -component of  $\vec{E}$  and the  $z$ -component of  $\vec{B}$  are nonzero, and both vary only with  $x$  and  $t$ . Then  $\vec{E}$  and  $\vec{B}$  have the form

$$\vec{E} = \begin{bmatrix} 0 \\ E^2(x, t) \\ 0 \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} 0 \\ 0 \\ B^3(x, t) \end{bmatrix}. \quad (2.116)$$



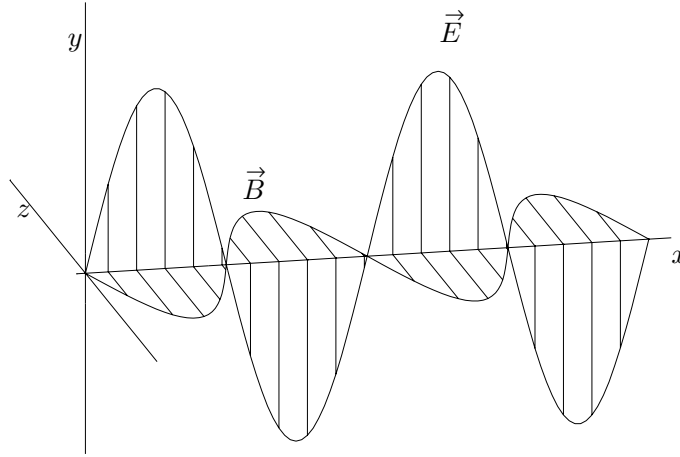


Fig. 2.4. The  $\vec{E}$  and  $\vec{B}$  fields for an electromagnetic plane wave propagating in the  $x$ -direction.

Maxwell's equations (2.115) then reduce to

$$\begin{aligned} E_t^2 + \frac{1}{\epsilon\mu} B_x^3 &= 0, \\ B_t^3 + E_x^2 &= 0. \end{aligned} \quad (2.117)$$

This has exactly the same structure as the one-dimensional linear acoustics and elasticity equations considered previously, with the coefficient matrix

$$A = \begin{bmatrix} 0 & 1/\epsilon\mu \\ 1 & 0 \end{bmatrix}. \quad (2.118)$$

The eigenvalues are  $\lambda^{1,2} = \pm c$ , where

$$c = \frac{1}{\sqrt{\epsilon\mu}} \quad (2.119)$$

is the *speed of light* in the medium. In a vacuum the parameters  $\epsilon$  and  $\mu$  take particular constant values  $\epsilon_0$  and  $\mu_0$  known as the *permittivity and permeability of free space*, and

$$c_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}} \quad (2.120)$$

is the *speed of light in a vacuum*. For any other medium we have  $c < c_0$ .

In a heterogeneous medium that consists of one-dimensional layers of isotropic material,  $\epsilon$  and  $\mu$  would be scalar but vary with  $x$  and we would obtain the variable-coefficient hyperbolic system

$$\begin{aligned} \epsilon(x)E_t^2(x, t) + \left( \frac{1}{\mu(x)} B^3(x, t) \right)_x &= 0, \\ B_t^3(x, t) + E_x^2(x, t) &= 0. \end{aligned} \quad (2.121)$$

The methods discussed in Section 9.6 for variable-coefficient acoustics could also be applied to this system.

In some media  $\epsilon$  and/or  $\mu$  may depend on the strength of the electric or magnetic field, and hence will vary as a wave passes through. In this case the constitutive relations become nonlinear, and Maxwell's equations yield a nonlinear hyperbolic system of equations. Actually, in most materials  $\epsilon$  and  $\mu$  do vary with the field strength, but normally the fields associated with electromagnetic waves are so weak that the linearized theory is perfectly adequate. However, in some problems with very strong fields or special materials it is necessary to consider nonlinear effects. The field of *nonlinear optics* is important, for example, in the design and study of fiber-optic cables used to transmit pulses of light over thousands of kilometers in the telecommunications industry.

### Exercises

- 2.1. Derive the equations (2.48) of linear acoustics from the linearized system (2.47).
- 2.2. (a) Show that for smooth solutions the conservation laws (2.38) can be manipulated into the following set of nonconservative nonlinear equations for the pressure and velocity:

$$\begin{aligned} p_t + up_x + \rho P'(\rho)u_x &= 0, \\ u_t + (1/\rho)p_x + uu_x &= 0, \end{aligned} \tag{2.122}$$

where we assume that the equation of state can be inverted to define  $\rho$  as a function of  $p$  to complete this system. Note that linearizing this nonlinear system about some state  $(\rho_0, u_0, p_0 = P(\rho_0))$  again gives the acoustics system (2.47).

- (b) Show that the nonlinear system (2.122) is hyperbolic provided  $P'(\rho) > 0$ , and has the same characteristic speeds as the conservative version (2.38).
- 2.3. Determine the eigenvalues and eigenvectors the matrix  $\tilde{A}$  in (2.77) and also the similarity transformation relating this to  $A$  from (2.51) when  $u_0 = 0$ .
- 2.4. Determine the eigenvalues and eigenvectors the matrix  $A$  from (2.46), and show that these agree with (2.57). Determine the similarity transformation relating this matrix to  $A$  from (2.51).
- 2.5. Determine the condition on the function  $\sigma(\epsilon)$  that is required in order for the nonlinear elasticity equation (2.91) to be hyperbolic.
- 2.6. Show that  $X_\xi(\xi, t) = V(\xi, t)$  and hence (2.105) is simply the statement that  $X_{\xi t} = X_{t\xi}$ .
- 2.7. Show that the  $p$ -system (2.108) is hyperbolic provided the function  $p(V)$  satisfies  $p'(V) < 0$  for all  $V$ .
- 2.8. Isothermal flow is modeled by the system (2.38) with  $P(\rho) = a^2\rho$ , where  $a$  is constant; see Section 14.6.
  - (a) Determine the wave speeds of the linearized equations (2.50) in this case.
  - (b) The Lagrangian form of the isothermal equations have  $p(V) = a^2/V$ . Linearize the  $p$ -system (2.107) in this case about  $V_0, U_0$ , and compute the wave speeds for Lagrangian acoustics. Verify that these are what you expect in relation to the Eulerian acoustic wave speeds.