3

Characteristics and Riemann Problems for Linear Hyperbolic Equations

In this chapter we will further explore the characteristic structure of linear hyperbolic systems of equations. In particular, we will study solutions to the *Riemann problem*, which is simply the given equation together with very special initial data consisting of a piecewise constant function with a single jump discontinuity. This problem and its solution are discussed starting in Section 3.8, after laying some more groundwork. This simple problem plays a very important role in understanding the structure of more general solutions. It is also a fundamental building block for the finite volume methods discussed in this book.

Linear hyperbolic systems of the form

$$q_t + Aq_x = 0 \tag{3.1}$$

were introduced in the last chapter. Recall that the problem is hyperbolic if $A \in \mathbb{R}^{m\times m}$ is diagonalizable with real eigenvalues, so that we can write

$$A = R\Lambda R^{-1}, \tag{3.2}$$

where R is the matrix of right eigenvectors. Then introducing the new variables

$$w = R^{-1}q$$

allows us to reduce the system (3.1) to

$$w_t + \Lambda w_x = 0, \tag{3.3}$$

which is a set of m decoupled advection equations. Note that this assumes A is constant. If A varies with x and/or t, then the problem is still linear, but R and Λ will typically depend on x and t as well and the manipulations used to obtain (3.3) are no longer valid. See Chapter 9 for discussion of variable-coefficient problems.

3.1 Solution to the Cauchy Problem

Consider the Cauchy problem for the constant-coefficient system (3.1), in which we are given data

$$q(x,0) = \overset{\circ}{q}(x) \quad \text{for } -\infty < x < \infty.$$

From this data we can compute data

$$\overset{\circ}{w}(x) \equiv R^{-1}\overset{\circ}{q}(x)$$

for the system (3.3). The pth equation of (3.3) is the advection equation

$$w_t^p + \lambda^p w_x^p = 0 \tag{3.4}$$

with solution

$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}{}^p(x - \lambda^p t).$$

Having computed all components $w^p(x,t)$ we can combine these into the vector $w(x,t)$, and then

$$q(x,t) = Rw(x,t) \tag{3.5}$$

gives the solution to the original problem. This is exactly the process we used to obtain the solution (2.68) to the acoustics equations in the previous chapter.

3.2 Superposition of Waves and Characteristic Variables

Note that we can write (3.5) as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p, \tag{3.6}$$

so that we can view the vector $q(x,t)$ as being some linear combination of the right eigenvectors $r^1, \ldots, r^m$ at each point in space–time, and hence as a superposition of waves propagating at different velocities λ^p. The scalar values $w^p(x,t)$ for $p = 1, \ldots, m$ give the coefficients of these eigenvectors at each point, and hence the *strength* of each wave. The requirements of hyperbolicity insure that these m vectors are linearly independent and hence every vector q has a unique representation in this form. The manipulations resulting in (3.4) show that the eigencoefficient $\overset{\circ}{w}{}^p(x) = w^p(x,0)$ is simply advected at constant speed λ^p as time evolves, i.e., $w^p(x,t) \equiv \overset{\circ}{w}{}^p(x_0)$ all along the curve $X(t) = x_0 + \lambda^p t$. These curves are called *characteristics of the pth family*, or simply *p-characteristics*. These are straight lines in the case of a constant-coefficient system. Note that for a strictly hyperbolic system, m distinct characteristic curves pass through each point in the x–t plane.

The coefficient $w^p(x,t)$ of the eigenvector r^p in the eigenvector expansion (3.6) of $q(x,t)$ is constant along any p-characteristic. The functions $w^p(x,t)$ are called the *characteristic variables*.

As an example, for the acoustics equations with A given by (2.51), we found in Section 2.8 that the characteristic variables are $-p + Z_0 u$ and $p + Z_0 u$ (or any scalar multiples of these functions), where Z_0 is the impedance; see (2.67).

3.3 Left Eigenvectors

Let $L = R^{-1}$, and denote the rows of the matrix L by $\ell^1,\ \ell^2, \ldots,\ \ell^m$. These row vectors are the left eigenvectors of the matrix A,

$$\ell^p A = \lambda^p \ell^p,$$

whereas the r^p are the right eigenvectors. For example, the left eigenvectors for acoustics are given by the rows of the matrix R^{-1} in (2.66).

We can write the characteristic variable $w^p(x, t)$, which is the pth component of $R^{-1}q(x, t) = Lq(x, t)$, simply as

$$w^p(x, t) = \ell^p q(x, t). \tag{3.7}$$

We can then rewrite the solution $q(x, t)$ from (3.6) succinctly in terms of the initial data $\overset{\circ}{q}$ as

$$q(x, t) = \sum_{p=1}^{m} [\ell^p \overset{\circ}{q}(x - \lambda^p t)]r^p. \tag{3.8}$$

3.4 Simple Waves

We can view the solution $q(x, t)$ as being the superposition of m waves, each of which is advected independently with no change in shape. The pth wave has shape $\overset{\circ}{w}{}^p(x)r^p$ and propagates with speed λ^p. This solution has a particularly simple form if $w^p(x, 0)$ is constant in x for all but one value of p, say $\overset{\circ}{w}{}^p(x) \equiv \bar{w}^p$ for $p \neq i$. Then the solution has the form

$$\begin{aligned} q(x, t) &= \overset{\circ}{w}{}^i(x - \lambda^i t)r^i + \sum_{p \neq i} \bar{w}^p r^p \\ &= \overset{\circ}{q}(x - \lambda^i t) \end{aligned} \tag{3.9}$$

and the initial data simply propagates with speed λ^i. Since $m - 1$ of the characteristic variables are constant, the equation essentially reduces to $q_t + \lambda^i q_x = 0$, which governs the behavior of the ith family. Nonlinear equations have analogous solutions, called *simple waves*, in which variations occur only in one characteristic family; see Section 13.8.

3.5 Acoustics

An arbitrary solution to the acoustics equations, as derived in Section 2.8, can be decomposed as in (3.6),

$$\begin{bmatrix} p(x, t) \\ u(x, t) \end{bmatrix} = w^1(x, t)\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + w^2(x, t)\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}, \tag{3.10}$$

where $w^1 = [-p + Z_0 u]/2Z_0$ is the strength of the left-going 1-wave, and $w^2 = [p + Z_0 u]/2Z_0$ is the strength of the right-going 2-wave. The functions $w^1(x, t)$ and $w^2(x, t)$

satisfy scalar advection equations,

$$w_t^1 - c_0 w_x^1 = 0 \quad \text{and} \quad w_t^2 + c_0 w_x^2 = 0, \tag{3.11}$$

so from arbitrary initial data we can compute

$$\begin{aligned} w^1(x,t) &= w^1(x + c_0 t, 0) = \overset{\circ}{w}{}^1(x + c_0 t), \\ w^2(x,t) &= w^2(x - c_0 t, 0) = \overset{\circ}{w}{}^2(x - c_0 t), \end{aligned} \tag{3.12}$$

where $\overset{\circ}{w}(x) = R^{-1}\overset{\circ}{q}(x)$ is the initial data for w, and (3.6) agrees with (2.68).

The advection equations (3.11) are often called the *one-way wave equations*, since each one models the strength of an acoustic wave going in only one direction.

If one of the characteristic variables w^1 or w^2 is identically constant, then the solution (3.10) is a *simple wave* as defined in Section 3.4. Suppose, for example, that $w^1 \equiv \bar{w}^1 =$ constant, in which case

$$q(x,t) = \bar{w}^1 r^1 + \overset{\circ}{w}{}^2(x - c_0 t) r^2.$$

In this case it is also easy to check that the full solution q satisfies the one-way wave equation $q_t + c_0 q_x = 0$.

Simple waves often arise in physical problems. Suppose for example that we take initial data in which $p = u = 0$ everywhere except in some small region near the origin. If we choose p and u as arbitrary functions in this region, unrelated to one another, then the solution will typically involve a superposition of a left-going and a right-going wave. Figure 3.1 shows the time evolution in a case where

$$\begin{aligned} p(x,0) &= \frac{1}{2}\exp(-80x^2) + S(x), \\ u(x,0) &= 0, \end{aligned} \tag{3.13}$$

with

$$S(x) = \begin{cases} 1 & \text{if } -0.3 < x < -0.1, \\ 0 & \text{otherwise.} \end{cases}$$

For small time the solution changes in a seemingly haphazard way as the left-going and right-going waves superpose. But observe that eventually the two waves separate and for larger t their individual forms are easy to distinguish. Once they have separated, each wave is a simple wave that propagates at constant velocity with its shape unchanged. In this example $\rho_0 = 1$ and $K_0 = 0.25$, so that $c_0 = Z_0 = 1/2$. Notice that the left-going wave has $p = -u/2$ while the right-going wave has $p = u/2$, as expected from the form of the eigenvectors.

3.6 Domain of Dependence and Range of Influence

Let (X, T) be some fixed point in space–time. We see from (3.8) that the solution $q(X, T)$ depends only on the data $\overset{\circ}{q}$ at m particular points $X - \lambda^p T$ for $p = 1, 2, \ldots, m$. This set

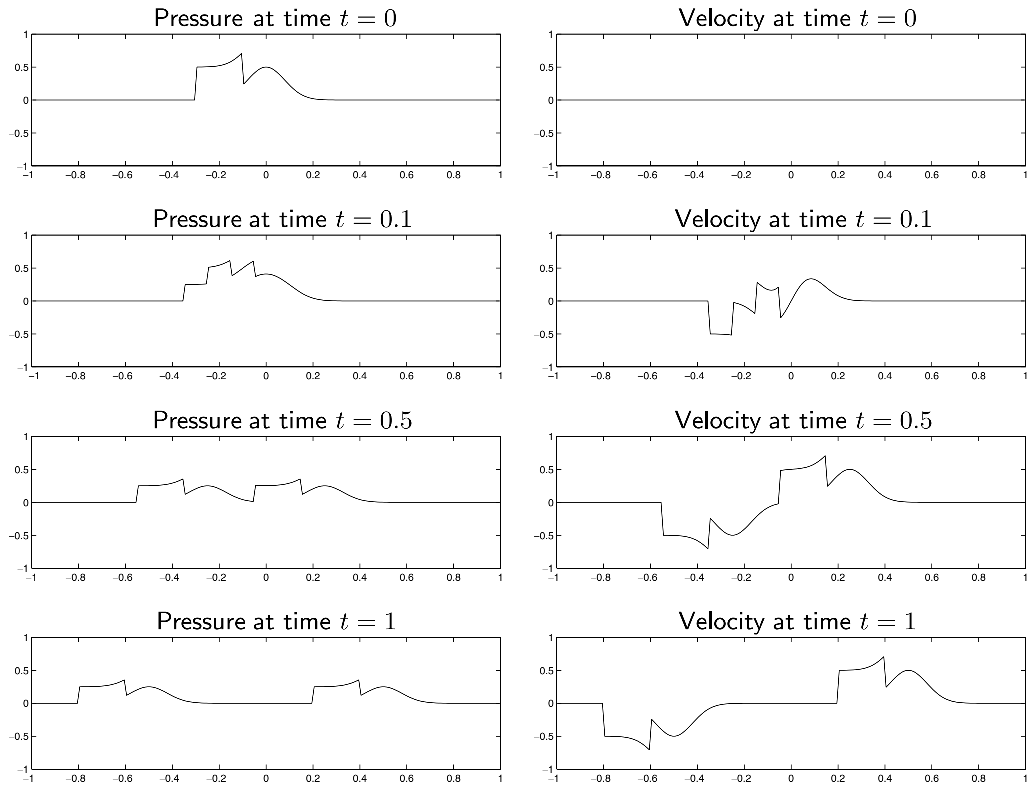

Fig. 3.1. Evolution of an initial pressure perturbation, concentrated near the origin, into distinct simple waves propagating with velocities $-c_0$ and c_0. The left column shows the pressure perturbation $q^1 = p$, and the right column shows the velocity $q^2 = u$. (Time increases going downwards.) [`claw/book/chap3/acousimple`]

of points,

$$\mathcal{D}(X, T) = \{X - \lambda^p T \colon p = 1, 2, \ldots, m\}, \tag{3.14}$$

is called the *domain of dependence* of the point (X, T). See Figure 3.2(a). The value of the initial data at other points has no influence on the value of q at (X, T).

For hyperbolic equations more generally, the domain of dependence is always a *bounded set*, though for nonlinear equations the solution may depend on data over a whole interval rather than at only a finite number of distinct points. The bounded domain of dependence results from the fact that information propagates at finite speed in a hyperbolic equation, as we expect from wave motion or advection. This has important consequences in the design of numerical methods, and means that explicit methods can often be efficiently used.

By contrast, for the heat equation $q_t = \beta q_{xx}$, the domain of dependence of any point (X, T) is the entire real line. Changing the data anywhere would in principle change the value of the solution at (X, T), though the contribution dies away exponentially fast, so data at points far away may have little effect. Nonetheless, this means that *implicit* numerical methods are often needed in solving parabolic equations. This is discussed further in Section 4.4 in relation to the *CFL condition*.

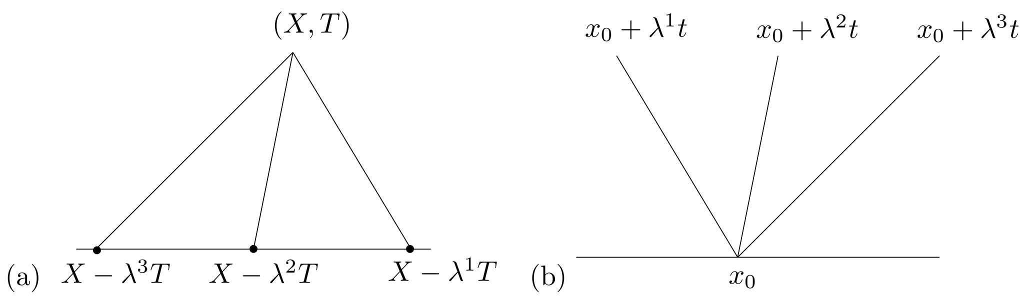

Fig. 3.2. For a typical hyperbolic system of three equations with $\lambda^1 < 0 < \lambda^2 < \lambda^3$, (a) shows the domain of dependence of the point (X, T), and (b) shows the range of influence of the point x_0.

Rather than looking at which initial data affects the solution at (X, T), we can turn things around and focus on a single point x_0 at time $t = 0$, and ask what influence the data $\overset{\circ}{q}(x_0)$ has on the solution $q(x, t)$. Clearly the choice of data at this point will only affect the solution along the characteristic rays $x_0 + \lambda^p t$ for $p = 1, 2, \ldots, m$. This set of points is called the *range of influence* of the point x_0. The range of influence is illustrated in Figure 3.2(b).

3.7 Discontinuous Solutions

While classical solutions of differential equations must be smooth (sufficiently differentiable) functions, the formula (3.6) can be used even if the initial data $\overset{\circ}{q}(x)$ is not smooth, or is even discontinuous, at some points. If the data has a singularity (a discontinuity in some derivative) at some point x_0, then one or more of the characteristic variables $w^p(x, 0)$ will also have a singularity at this point. Such singularities in the initial data can then propagate along the characteristics and lead to singularities in the solution $q(x, t)$ at some or all of the points $x_0 + \lambda^p t$.

Conversely, if the initial data is smooth in a neighborhood of all the points $\bar{x} - \lambda^p \bar{t}$, then the solution $q(x, t)$ must be smooth in a neighborhood of the point $(\bar{x}, \bar{t})$. This means that singularities can *only* propagate along characteristics for a linear system.

3.8 The Riemann Problem for a Linear System

The *Riemann problem* consists of the hyperbolic equation together with special initial data that is piecewise constant with a single jump discontinuity,

$$\overset{\circ}{q}(x) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0. \end{cases}$$

By the remarks in Section 3.7, we expect this discontinuity to propagate along the characteristic curves.

For the scalar advection equation $q_t + \bar{u} q_x = 0$, the coefficient "matrix" is the 1×1 scalar value $\bar{u}$. The single eigenvalue is $\lambda^1 = \bar{u}$, and we can choose the eigenvector to be $r^1 = 1$. The solution to the Riemann problem consists of the discontinuity $q_r - q_l$ propagating at speed $\bar{u}$, along the characteristic, and the solution is $q(x, t) = \overset{\circ}{q}(x - \bar{u}t)$.

For a general $m \times m$ linear system we can solve the Riemann problem explicitly using the information we have obtained above. It is very important to understand the structure of this solution, since we will see later that Riemann solutions for nonlinear conservation laws have a similar structure. Moreover, many of the numerical methods we will discuss (beginning in Chapter 4) are based on using solutions to the Riemann problem to construct approximate solutions with more general data.

For the Riemann problem we can simplify the notation if we decompose q_l and q_r as

$$q_l = \sum_{p=1}^{m} w_l^p r^p \quad \text{and} \quad q_r = \sum_{p=1}^{m} w_r^p r^p. \tag{3.15}$$

Then the pth advection equation (3.4) has Riemann data

$$\overset{\circ}{w}{}^p(x) = \begin{cases} w_l^p & \text{if } x < 0, \\ w_r^p & \text{if } x > 0, \end{cases} \tag{3.16}$$

and this discontinuity simply propagates with speed λ^p, so

$$w^p(x,t) = \begin{cases} w_l^p & \text{if } x - \lambda^p t < 0, \\ w_r^p & \text{if } x - \lambda^p t > 0. \end{cases} \tag{3.17}$$

If we let $P(x,t)$ be the maximum value of p for which $x - \lambda^p t > 0$, then

$$q(x,t) = \sum_{p=1}^{P(x,t)} w_r^p r^p + \sum_{p=P(x,t)+1}^{m} w_l^p r^p, \tag{3.18}$$

which we will write more concisely as

$$q(x,t) = \sum_{p:\lambda^p < x/t} w_r^p r^p + \sum_{p:\lambda^p > x/t} w_l^p r^p. \tag{3.19}$$

The determination of $q(x,t)$ at a given point (X,T) is illustrated in Figure 3.3. In the case shown, $w^1 = w_r^1$ while $w^2 = w_l^2$ and $w^3 = w_l^3$. The solution at the point illustrated is thus

$$q(X,T) = w_r^1 r^1 + w_l^2 r^2 + w_l^3 r^3. \tag{3.20}$$

Note that the solution is the same at any point in the wedge between the $x = \lambda^1 t$ and $x = \lambda^2 t$ characteristics. As we cross the pth characteristic, the value of $x - \lambda^p t$ passes through 0 and the corresponding w^p jumps from w_l^p to w_r^p. The other coefficients w^i $(i \neq p)$ remain constant.

The solution is constant in each of the wedges as shown in Figure 3.3. Across the pth characteristic the solution jumps with the jump in q given by

$$\left(w_r^p - w_l^p\right) r^p \equiv \alpha^p r^p. \tag{3.21}$$

Note that this jump in q is an eigenvector of the matrix A (being a scalar multiple of r^p). This is an extremely important fact, and a generalization of this statement is what will allow us to solve the Riemann problem for nonlinear systems of equations. This condition,

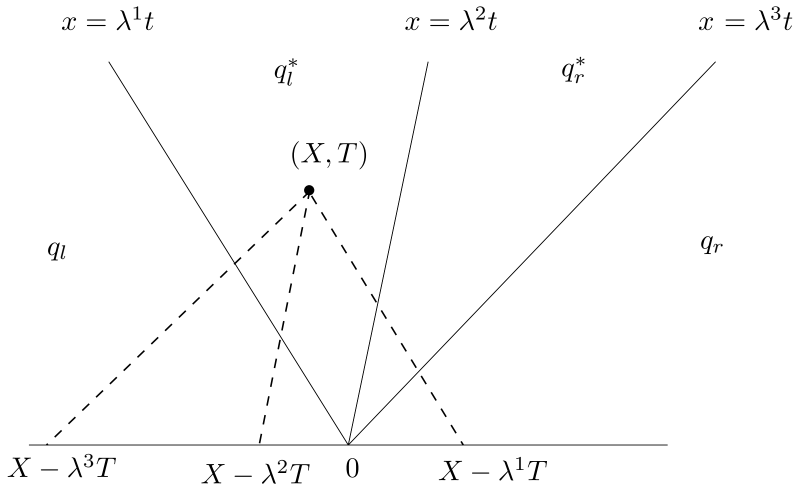

Fig. 3.3. Construction of the solution to the Riemann problem at (X, T). We trace back along the pth characteristic to determine the value of w^p from the initial data. The value of q is constant in each wedge of the x–t plane: $q_l = w_l^1 r^1 + w_l^2 r^2 + w_l^3 r^3$ $q_l^* = w_r^1 r^1 + w_l^2 r^2 + w_l^3 r^3$ $q_r^* = w_r^1 r^1 + w_r^2 r^2 + w_l^3 r^3$ $q_r = w_r^1 r^1 + w_r^2 r^2 + w_r^3 r^3$. Note that the jump across each discontinuity in the solution is an eigenvector of A.

called the *Rankine–Hugoniot jump condition*, will be derived from the integral form of the conservation law and seen to hold across any propagating discontinuity; see Section 11.8. Typically the given data (q_l, q_r) will not satisfy this condition, and the process of solving the Riemann problem can be viewed as an attempt to split up the jump $q_r - q_l$ into a series of jumps, defining the different waves, each of which does satisfy this condition.

For the case of a linear system, solving the Riemann problem consists of taking the initial data (q_l, q_r) and decomposing the jump $q_r - q_l$ into eigenvectors of A:

$$q_r - q_l = \alpha^1 r^1 + \cdots + \alpha^m r^m. \tag{3.22}$$

This requires solving the linear system of equations

$$R\alpha = q_r - q_l \tag{3.23}$$

for the vector α, and so $\alpha = R^{-1}(q_r - q_l)$. The vector α has components $\alpha^p = \ell^p(q_r - q_l)$, where ℓ^p is the left eigenvector defined in Section 3.3, and $\alpha^p = w_r^p - w_l^p$. Since $\alpha^p r^p$ is the jump in q across the pth wave in the solution to the Riemann problem, we introduce the notation

$$\mathcal{W}^p = \alpha^p r^p \tag{3.24}$$

for these waves.

The solution $q(x, t)$ from (3.8) can be written in terms of the waves in two different forms:

$$q(x, t) = q_l + \sum_{p:\lambda^p < x/t} \mathcal{W}^p \tag{3.25}$$

$$= q_r - \sum_{p:\lambda^p \ge x/t} \mathcal{W}^p. \tag{3.26}$$

This can also be written as

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t)\mathcal{W}^p, \tag{3.27}$$

where $H(x)$ is the *Heaviside function*

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{3.28}$$

3.9 The Phase Plane for Systems of Two Equations

It is illuminating to view the splitting of $q_r - q_l$ in *state space*, often called the *phase plane* for systems of two equations. This is simply the q^1–q^2 plane, where $q = (q^1, q^2)$. Each vector $q(x,t)$ is represented by a point in this plane. In particular, q_l and q_r are points in this plane, and a discontinuity with left and right states q_l and q_r can propagate as a single discontinuity only if $q_r - q_l$ is an eigenvector of A, which means that the line segment from q_l to q_r must be parallel to the eigenvector r^1 or r^2. Figure 3.4 shows an example. For the state q_l illustrated there, the jump from q_l to q_r can propagate as a single discontinuity if and only if q_r lies on one of the two lines drawn through q_l in the directions r^1 and r^2. These lines give the locus of all points that can be connected to q_l by a 1-wave or a 2-wave. This set of states is called the *Hugoniot locus*. We will see that there is a direct generalization of this to nonlinear systems in Chapter 13.

Similarly, there is a Hugoniot locus through any point q_r that gives the set of all points q_l that can be connected to q_r by an elementary p-wave. These curves are again in the directions r^1 and r^2.

For a general Riemann problem with arbitrary q_l and q_r, the solution consists of two discontinuities traveling with speeds λ^1 and λ^2, with a new constant state in between that we will call q_m. By the discussion above,

$$q_m = w_r^1 r^1 + w_l^2 r^2, \tag{3.29}$$

so that $q_m - q_l = (w_r^1 - w_l^1)r^1$ and $q_r - q_m = (w_r^2 - w_l^2)r^2$. The location of q_m in the phase

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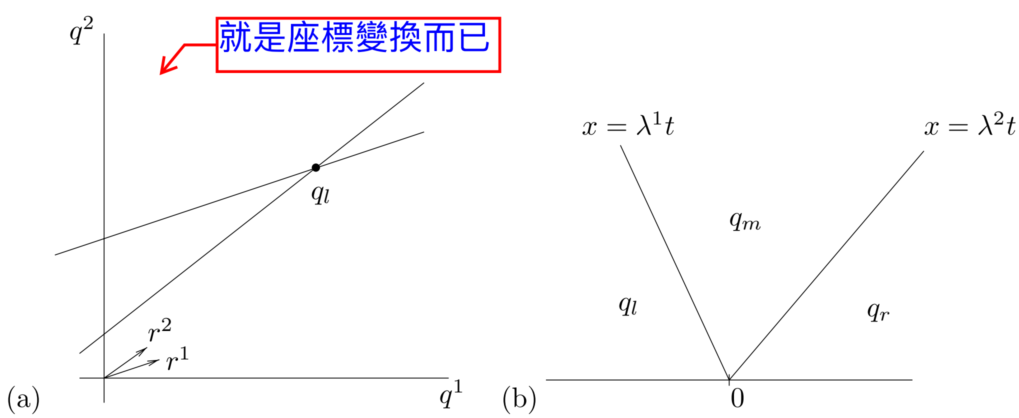

Fig. 3.4. (a) The Hugoniot locus of the state q_l consists of all states that differ from q_l by a scalar multiple of r^1 or r^2. (b) Solution to the Riemann problem in the x–t plane.

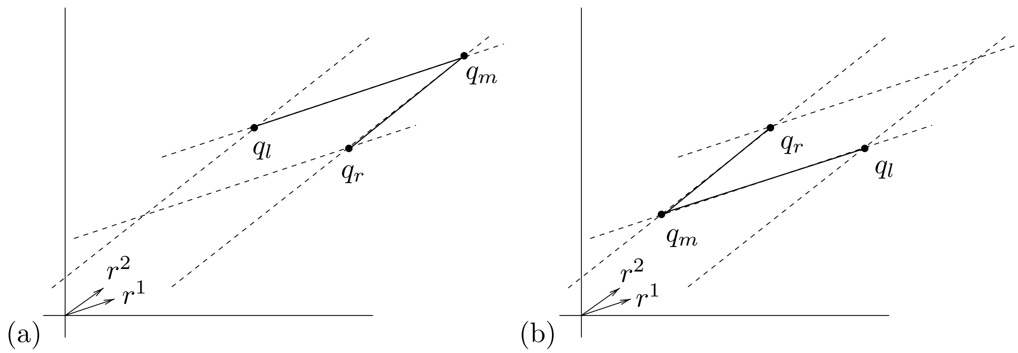

Fig. 3.5. The new state q_m arising in the solution to the Riemann problem for two different choices of q_l and q_r. In each case the jump from q_l to q_m lies in the direction of the eigenvector r^1 corresponding to the lower speed, while the jump from q_m to q_r lies in the direction of the eigenvector r^2.

plane must be where the 1-wave locus through q_l intersects the 2-wave locus through q_r. This is illustrated in Figure 3.5(a).

Note that if we interchange q_r and q_l in this picture, the location of q_m changes as illustrated in Figure 3.5(b). In each case we travel from q_l to q_r by first going in the direction r^1 and then in the direction r^2. This is required by the fact that $\lambda^1 < \lambda^2$, since clearly the jump between q_l and q_m must travel slower than the jump between q_m and q_r (see Figure 3.4(b)) if we are to obtain a single-valued solution.

For systems with more than two equations, the same interpretation is possible but becomes harder to draw, since the state space is now m-dimensional. Since the m eigenvectors r^p are linearly independent, we can decompose any jump $q_r - q_l$ into the sum of jumps in these directions via (3.22), obtaining a piecewise linear path from q_l to q_r in m-dimensional space.

3.9.1 Acoustics

As a specific example, consider the acoustics equations discussed in Sections 2.7–2.8 with $u_0 = 0$,

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0. \tag{3.30}$$

The eigenvalues and eigenvectors of A are given by (2.54) and (2.58). The phase plane is the p–u plane, and the eigenvectors are symmetric about the u-axis as indicated in Figure 3.6(a). Solving the general Riemann problem gives $\alpha = R^{-1}(q_r - q_l)$ with components

$$\begin{aligned} \alpha^1 &= \ell^1(q_r - q_l) = \frac{-(p_r - p_l) + Z_0(u_r - u_l)}{2Z_0}, \\ \alpha^2 &= \ell^2(q_r - q_l) = \frac{(p_r - p_l) + Z_0(u_r - u_l)}{2Z_0}, \end{aligned} \tag{3.31}$$

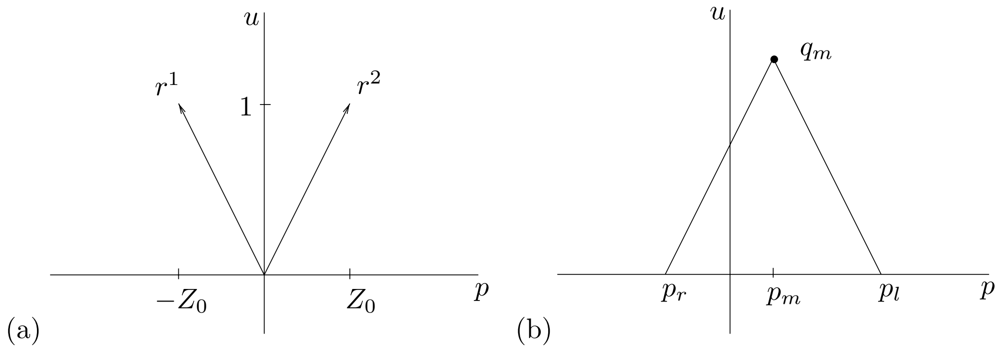

Fig. 3.6. (a) Eigenvectors for the acoustics equations in the p–u phase plane, where Z_0 is the impedance. (b) Solution to a Riemann problem in which $u_l = u_r = 0$ and $p_r < p_l$.

and the waves are $\mathcal{W}^1 = \alpha^1 r^1$ and $\mathcal{W}^2 = \alpha^2 r^2$. The intermediate state is

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) - Z_0(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z_0 \end{bmatrix}. \tag{3.32}$$

Example 3.1. Consider a Riemann problem in which $u_l = u_r = 0$ and there is only a jump in pressure with $p_r < p_l$. The phase-plane solution to the Riemann problem is sketched in Figure 3.6(b), and we compute that

$$\alpha^1 = \frac{p_l - p_r}{2Z_0}, \qquad \alpha^2 = \frac{p_r - p_l}{2Z_0},$$

so that the intermediate state is

$$q_m = q_l + \alpha^1 r^1 = q_r - \alpha^2 r^2 = \frac{1}{2}\begin{bmatrix} p_l + p_r \\ -(p_r - p_l)/Z_0 \end{bmatrix}.$$

(Recall that p represents the perturbation of pressure from the constant state p_0, so it is fine for it to be negative.)

3.10 Coupled Acoustics and Advection

Now consider acoustics in a fluid moving at constant speed $u_0 > 0$, and to make the problem more interesting suppose that there is also a passive tracer being advected in this fluid, with density denoted by $\phi(x, t)$. Then we can solve the acoustics and advection equation together as a system of three equations,

$$\begin{bmatrix} p \\ u \\ \phi \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 & 0 \\ 1/\rho_0 & u_0 & 0 \\ 0 & 0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}_x. \tag{3.33}$$

Of course the acoustics and advection could be decoupled into two separate problems, but it is illuminating to solve the Riemann problem for this full system, since its structure is closely related to what is seen in the nonlinear Euler equations of gas dynamics studied later, and is also important in solving two-dimensional acoustics (Section 18.4).

The coefficient matrix in (3.33) has eigenvalues

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0, \qquad \lambda^3 = u_0 + c_0, \tag{3.34}$$

and corresponding eigenvectors

$$r^1 = \begin{bmatrix} -Z_0 \\ 1 \\ 0 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad r^1 = \begin{bmatrix} Z_0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.35}$$

The solution to the Riemann problem is easily determined:

$$\begin{bmatrix} p_r - p_l \\ u_r - u_l \\ \phi_r - \phi_l \end{bmatrix} = \alpha^1 \begin{bmatrix} -Z_0 \\ 1 \\ 0 \end{bmatrix} + \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha^3 \begin{bmatrix} Z_0 \\ 1 \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} \alpha^1 &= \frac{1}{2Z_0}[-(p_r - p_l) + Z_0(u_r - u_l)], \\ \alpha^2 &= \phi_r - \phi_l, \\ \alpha^3 &= \frac{1}{2Z_0}[(p_r - p_l) + Z_0(u_r - u_l)]. \end{aligned} \tag{3.36}$$

Note that the 1-wave and 3-wave are standard acoustic waves independent of ϕ, while the 2-wave gives the advection of ϕ.

Suppose ϕ measures the concentration of a dye in the fluid and that $\phi_r > \phi_l$, so that at time $t = 0$ the fluid to the left is dark while the fluid to the right is light. Then the 2-wave marks the interface between the dark and light fluids as time evolves, as indicated in Figure 3.7. The two fluids remain in contact across this discontinuity in ϕ, which has no dynamic effect, since this tracer does not affect the fluid dynamics and the pressure and velocity are both constant across the 2-wave. This wave is called a *contact discontinuity*.

Within each of the two fluids there is an acoustic wave moving at speed c_0 (relative to the fluid) away from the origin. The jump in pressure and/or velocity in the original Riemann data creates a “noise,” that moves through the fluids at the speed of sound.

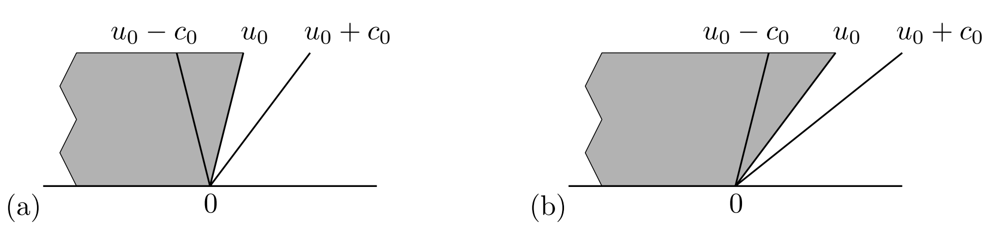

Fig. 3.7. Solution to the Riemann problem for the coupled acoustics and advection problem. The interface between dark and light fluid advects at the fluid velocity u_0, and acoustic waves move at speed c_0 relative to the fluid. The speed of each wave is indicated. (a) A subsonic case. (b) A supersonic case.

Figure 3.7 shows two different situations. In Figure 3.7(a) the fluid velocity u_0 is positive but *subsonic* ($u_0 < c_0$), and so the left-going acoustic wave (the 1-wave) has a negative velocity $u_0 - c_0 < 0$ relative to a fixed observer. Figure 3.7(b) illustrates a *supersonic* flow, where $u_0 > c_0$ and so $u_0 - c_0 > 0$. In this case all three waves propagate to the right and no information can propagate upstream from the observer. This distinction is not very important in this linear example. In nonlinear gas dynamics the distinction can be very important. The ratio $M = |u_0|/c_0$ is called the *Mach number* of the flow.

3.11 Initial–Boundary-Value Problems

Now consider a hyperbolic system on a bounded interval $a \le x \le b$. This is called the *initial–boundary-value problem*, or IBVP for short, since it is a time-dependent problem for which we need both initial data and boundary data. For a system of m equations we need a total of m boundary conditions. Typically some conditions must be prescribed at the left boundary $x = a$ and some at the right boundary $x = b$. How many are required at each boundary depends on the number of eigenvalues of A that are positive and negative, respectively.

We considered the IBVP for the advection equation in Section 2.1 and saw that we need a boundary condition only at $x = a$ if $\bar{u} > 0$ and only at $x = b$ if $\bar{u} < 0$. So if we diagonalize a general linear system to obtain a decoupled set of advection equations

$$w_t^p + \lambda^p w_x^p = 0,$$

then we need to specify boundary data on $w^p(x, t)$ at $x = a$ if $\lambda^p > 0$ and at $x = b$ if $\lambda^p < 0$. (For now assume all eigenvalues are nonzero, i.e., that the boundary is *noncharacteristic*.)

So if the system of m equations has $n \le m$ negative eigenvalues and $m - n$ positive eigenvalues, i.e.,

$$\lambda^1 \le \lambda^2 \le \cdots \le \lambda^n < 0 < \lambda^{n+1} \le \cdots \le \lambda^m,$$

then we need to specify $m - n$ boundary conditions at $x = a$ and n boundary conditions at $x = b$. What sort of boundary data should we impose? Partition the vector w as

$$w = \begin{bmatrix} w^{\mathrm{I}} \\ w^{\mathrm{II}} \end{bmatrix}, \tag{3.37}$$

where $w^{\mathrm{I}} \in \mathbb{R}^n$ and $w^{\mathrm{II}} \in \mathbb{R}^{m-n}$. Then at the left boundary $x = a$, for example, we must specify the components of w^{II}, while w^{I} are outflow variables. It is valid to specify w^{II} in terms of w^{I}. For example, we might use a linear boundary condition of the form

$$w^{\mathrm{II}}(a, t) = B_1 w^{\mathrm{I}}(a, t) + g_1(t), \tag{3.38}$$

where $B_1 \in \mathbb{R}^{(m-n)\times n}$ and $g_1 \in \mathbb{R}^{m-n}$. If $B_1 = 0$, then we are simply specifying given values for the inflow variables. But at a physical boundary there is often some *reflection* of outgoing waves, and this requires a nonzero B_1.

Boundary conditions should be specified as part of the problem and are determined by the physical setup – generally not in terms of the characteristic variables, unfortunately. It is not always easy to see what the correct conditions are to impose on the mathematical equation. We may have several pieces of information about what is happening at the boundary. Which are the correct ones to specify at the boundary? If we specify too few or too many conditions, or inappropriate conditions (such as trying to specify the value of an outflow characteristic variable), then the mathematical problem is ill posed and will have no solution, or perhaps many solutions. It often helps greatly to know what the characteristic structure is, which reveals how many boundary conditions we need and allows us to check that we are imposing appropriate conditions for a well-posed problem. In Chapter 7 boundary conditions are discussed further, and we will see how to impose such boundary conditions numerically.

Example 3.2. Consider the acoustics problem (2.50) in a closed tube of gas, $a \le x \le b$. We expect an acoustic wave hitting either closed end to be reflected. Since the system has eigenvalues $-c_0$ and $+c_0$, we need to specify one condition at each end ($n = m - n = 1$ and $w^{\mathrm{I}} = w^1$, $w^{\mathrm{II}} = w^2$). We do not have any information on values of the pressure at the boundary *a priori*, but we do know that the velocity must be zero at each end at all times, since the gas cannot flow through the solid walls (and shouldn't flow away from the walls or a vacuum would appear). This suggests that we should set

$$u(a, t) = u(b, t) = 0 \tag{3.39}$$

as our two boundary conditions, and this is correct. Note that we are specifying the same thing at each end, although the ingoing characteristic variable is different at the two ends. From Section 2.8 we know that the characteristic variables are

$$w^1 = -p + Z_0 u, \qquad w^2 = p + Z_0 u. \tag{3.40}$$

We can combine w^1 and w^2 to see that specifying $u = 0$ amounts to requiring that $w^1 + w^2 = 0$ at each end. At $x = a$ we can write this as

$$w^2(a, t) = -w^1(a, t),$$

which has the form (3.38) with $B_1 = 1$ and $g_1 = 0$. The outgoing wave is completely reflected and feeds back into the incoming wave. Conversely, at $x = b$ we can interpret the boundary condition $u = 0$ as

$$w^1(b, t) = -w^2(b, t),$$

which sets the incoming variable at this boundary, again by complete reflection of the outgoing variable.

Example 3.3. Suppose we set $B_1 = 0$ and $g_1 = 0$ in (3.38), so that this becomes $w^{\mathrm{II}}(a, t) = 0$. Then there is nothing flowing into the domain at the left boundary, and any left-going waves will simply leave the domain with no reflection. These are called *outflow boundary conditions*.

Figure 3.8 shows a continuation of the example shown in Figure 3.1 to later times, with a solid wall at the left and outflow boundary conditions imposed at the right, which amount to setting $w^1(b, t) = 0$ and hence $p(b, t) = Z_0 u(b, t)$.

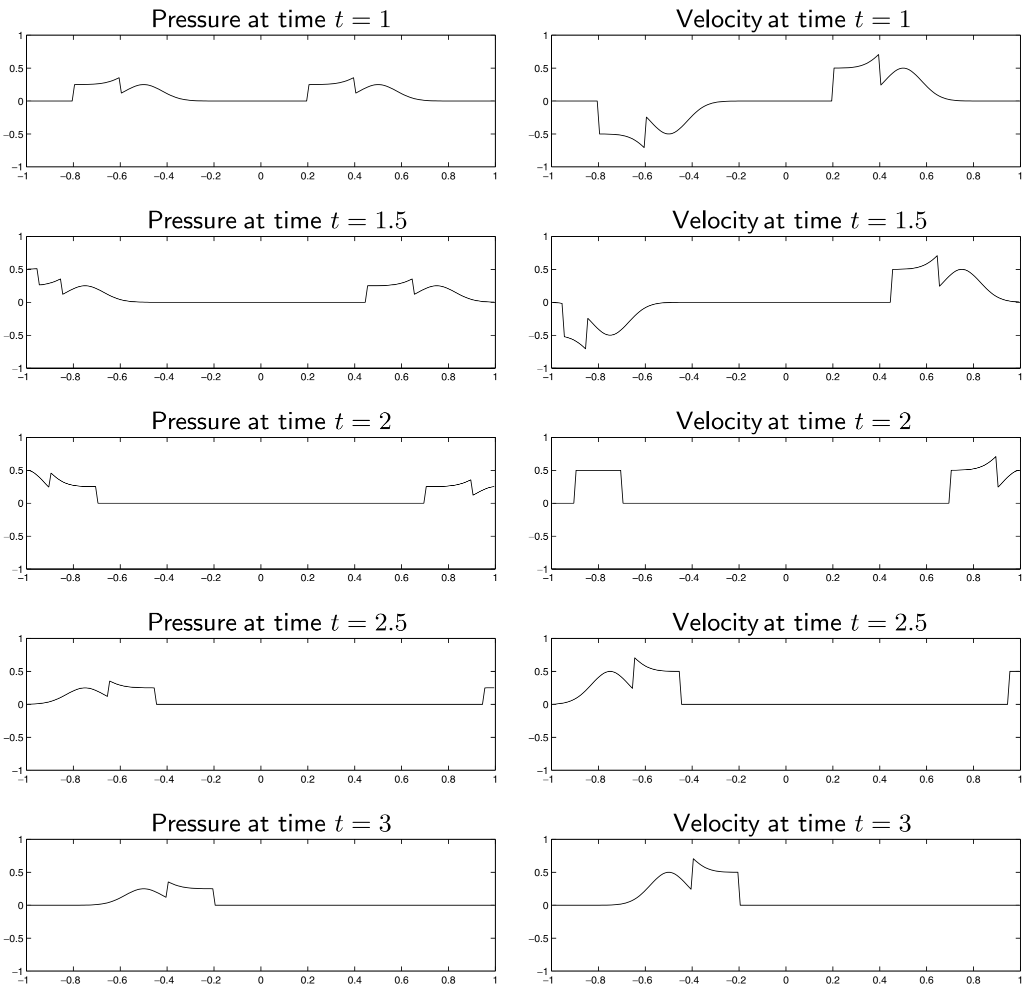

Fig. 3.8. Continuation of the example shown in Figure 3.1 with a solid wall at the left and outflow boundary conditions imposed at the right. Note that the wave that strikes the left boundary is a 1-wave with $p = -u/2$, while the reflected wave is a 2-wave with $p = u/2$. [claw/book/chap3/acousimple]

Example 3.4. A set of boundary conditions that is often useful mathematically is the *periodic boundary conditions*

$$q(a, t) = q(b, t). \tag{3.41}$$

This set of boundary conditions couples information at the two boundaries, and the idea is that waves going out one end should reenter at the other end. Solving the IBVP with periodic boundary conditions is equivalent to solving a Cauchy problem with periodic initial data, where the data given in $a \le x \le b$ is periodically extended to the whole real line.

We are specifying m coupled boundary conditions rather than $m - n$ at one end and n at the other, but we can reinterpret (3.41) in terms of the characteristic variables as

$$\begin{aligned} w^{\mathrm{II}}(a, t) &= w^{\mathrm{II}}(b, t), \\ w^{\mathrm{I}}(b, t) &= w^{\mathrm{I}}(a, t). \end{aligned} \tag{3.42}$$

The $m - n$ incoming values w^{II} at $x = a$ are specified using the outgoing values at $x = b$, while the n incoming values w^{I} at $x = b$ are specified using the outgoing values at $x = a$.

Exercises

3.1. For each of the Riemann problems below, sketch the solution in the phase plane, and sketch $q^1(x, t)$ and $q^2(x, t)$ as functions of x at some fixed time t:

$$\text{(a)}\ A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\text{(b)}\ A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\text{(c)}\ A = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

$$\text{(d)}\ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

$$\text{(e)}\ A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\text{(f)}\ A = \begin{bmatrix} 2 & 1 \\ 10^{-4} & 2 \end{bmatrix}, \quad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

3.2. Write a script in Matlab or other convenient language that, given any 2×2 matrix A and states q_l and q_r, solves the Riemann problem and produces the plots required for Exercise 3.1. Test it out on the problems of Exercise 3.1 and others.

3.3. Solve each of the Riemann problems below. In each case sketch a figure in the x–t plane similar to Figure 3.3, indicating the solution in each wedge.

$$\text{(a)}\ A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

$$\text{(b)}\ A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

3.4. Consider the acoustics equations (3.30) with

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}, \qquad \overset{\circ}{p}(x) = \begin{cases} 1 & \text{if } 1 \le x \le 2, \\ 0 & \text{otherwise}, \end{cases} \qquad \overset{\circ}{u}(x) \equiv 0.$$

Find the solution for $t > 0$. This might model a popping balloon, for example (in one dimension).

3.5. Solve the IBVP for the acoustics equations from Exercise 3.4 on the finite domain $0 \le x \le 4$ with boundary conditions $u(0, t) = u(4, t) = 0$ (solid walls). Sketch the solution (u and p as functions of x) at times $t = 0, 0.5, 1, 1.5, 2, 3$.

3.6. In the problem of Exercise 3.5, what is the domain of dependence of the point $X = 1,\ T = 10$? In this case the domain of dependence should be defined to include not only the set of points x where the initial data affects the solution, but also the set of times on each boundary where the boundary conditions can affect the solution at the point (X, T).

3.7. Suppose a tube of gas is bounded by a piston at $x = 0$ and a solid wall at $x = 1$, and that the piston is very slowly pushed into the tube with constant speed $\epsilon \ll c$, where c is the speed of sound. Then we might expect the gas in the tube to be simply compressed slowly with the pressure essentially uniform through the tube and increasing in time like $p = p_0 + \epsilon t K_0$, where K_0 is the bulk modulus. The velocity should be roughly linear in x, varying from $u = \epsilon$ at the piston to $u = 0$ at the solid wall. For very small ϵ we can model this using linear acoustics on the fixed interval $0 \le x \le 1$ with initial data

$$\overset{\circ}{u}(x) = 0, \qquad \overset{\circ}{p}(x) = p_0,$$

and boundary conditions

$$u(0, t) = \epsilon, \qquad u(1, t) = 0.$$

The solution consists of a single acoustic wave bouncing back and forth between the piston and solid wall (very rapidly relative to the wall motion), with p and u piecewise constant. Determine this solution, and show that by appropriately averaging this rapidly varying solution one observes the expected behavior described above. This illustrates the fact that slow-scale motion is sometimes mediated by high-speed waves.

3.8. Consider a general hyperbolic system $q_t + Aq_x = 0$ in which $\lambda = 0$ is a simple or multiple eigenvalue. How many boundary conditions do we need to impose at each boundary in this case? As a specific example consider the system (3.33) in the case $u_0 = 0$.