# HOW TO DRAW A GRAPH By W.T. TUTTE

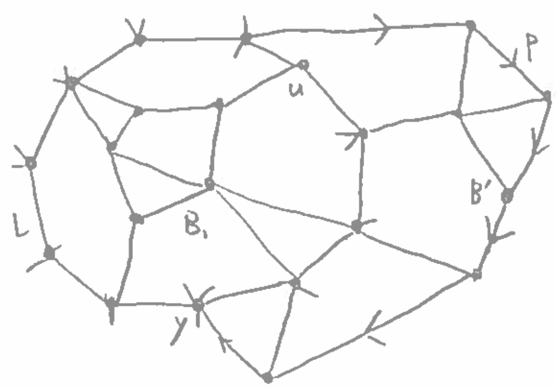
<u>3-connected</u>: A graph $G$ is 3-connected (nodally 3-connected) if it is simple and non-separable and satisfies the following condition; if $G$ is the union of two proper subgraphs $H$ and $K$ such that $H \cap K$ consists solely of two vertices $u$ and $v$, then one of $H$ and $K$ is a link-graph (arc-graph) with ends $u$ and $v$.

<u>peripheral polygon</u>: Let $J$ be a polygon of $G$ and let $\beta(J)$ denote the number of bridges of $J$ in $G$. If $\beta(J) \leq 1$ we call $J$ a peripheral polygon of $G$.

(2.1) Let $G$ be a nodally 3-connected graph. Let $J$ be a polygon of $G$ and $B$ any bridge of $J$ in $G$. Then either $J$ is a peripheral or $J$ has another bridge $B'$ which does not avoid $B$.

<u>avoid</u>: If one of residual arc-graphs of $B$ in $J$ includes all the vertices of attachment of a second bridge $B'$ of $J$ in $G$ we say that $B'$ avoids $B$. Then $B$ avoids $B'$.

(2.2) Let $G$ be a nodally 3-connected graph. Let $K_1$ be a polygon of $G$, $B_1$ a bridge of $K_1$ in $G$, $C$ a subgraph of $B_1$, and $L$ a branch of $G$ in $K_1$. Then we can find a peripheral polygon $J$ of $G$ such that $L \subset J$ and $J \cap C \subseteq K_1 \cap C$.




(2.3) Let $G$ be a nodally 3-connected graph which is not a polygon or a link-graph, and let $L$ be a branch of $G$. Then we can find two peripheral polygons $J_1$ and $J_2$ of $G$ such that $J_1 \cap J_2 = L$.

(2.4) Let $G$ be a nodally 3-connected graph, $K$ a polygon of $G$, $B$ a bridge of $K$ in $G$, and $L$ a branch of $G$ contained in $K$. Let $J_1$ and $J_2$ be peripheral polygons of $G$ such that $L \subseteq J_1 \cap J_2$ and neither $B \cap J_1$ nor $B \cap J_2$ is a subgraph of $K$. Then we can find a peripheral polygon $J_3$, distinct from $J_1$ and $J_2$, such that $L \subset J_3$.

(2.5) Let $G$ be a nodally 3-connected non-null graph Then we can find a set of $p_1(G)$ independent peripheral cycles of $G$.

<u>peripheral cycle</u>: Consider the set of cycles of a connected graph $G$ The rank of this set, the maximum number of cycles independent under mod-2 addition, is

$$p_1(G) = \alpha_1(G) - \alpha_0(G) + 1$$

We refer to the elementary cycles associated with a peripheral polygon as a peripheral cycle.

(2.6) Let $G$ be a nodally 3-connected non-null graph. with at least two edges, which is not a polygon. Suppose that no edge of $G$ belongs to more than two distinct peripheral polygons. Then $G$ has just $p_1(G)+1$ distinct peripheral cycles. and they constitute a planar mesh of $G$.

(2.7) A peripheral polygon $K$ of a non-seperable graph $G$ belongs to every planar mesh of $G$.

(2.8) If $M$ is a planar mesh of a nodally 3-connected graph $G$, then each member of $M$ is peripheral.

(2.7) + (2.8) show that a nodally 3-connected graph has at most one planar mesh.

<u>representation</u>: We call $H$ a representation of $G$ in $\Pi$ (closed plane) if it satifies the following conditions

(i) No edge of $H$ contains any vertex of $H$.

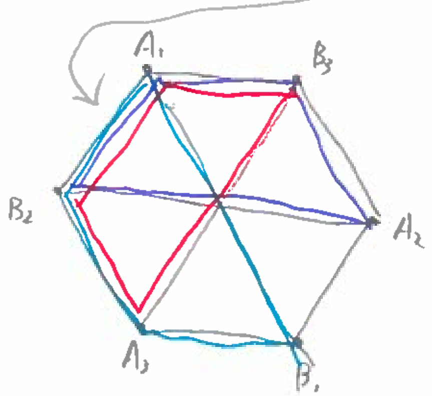
(ii) If $e$ and $e'$ are distinct edges of $G$, then $f(e)$ and $f(e')$ are disjoint.

A graph $G$ is said to be planar if it has a representation in $\Pi$

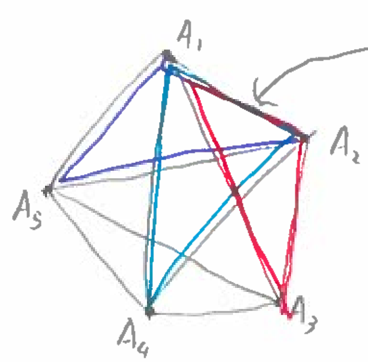
(3.1) Each peripheral polygon of $H$ bounds a face of $H$.

(3.2) If a graph $G$ has three distinct peripheral polygons with a <u>common edge</u>, then $G$ is non-planar.

Kuratowski graph of Type I.




Kuratowski graph of Type II

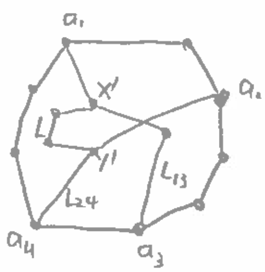



(4.1) Every Kuratowski graph is non-planar.

COROLLARY. Any graph having a Kuratowski subgraph is non-planar.

Crossing diagonals: Let $J$ be a polygon of a graph $G$. Let $a_1, a_2, a_3, a_4$ be distinct vertices of $J$ such that $a_1$ and $a_3$ separate $a_2$ from $a_4$ on $J$. Let $L_{13}$ and $L_{24}$ be disjoint arc-graphs of $G$ spanning $J$. Then we say that $L_{13}$ and $L_{24}$ are crossing diagonals of $J$.

(5.1) Given a peripheral polygon of $G$ with a pair of crossing diagonals we can find a Kuratowski subgraph of $G$ of Type I.




($a_1, a_2, a_3, a_4, x', y'$ form a Kuratowski graph of Type I).

(5.2) Let $J$ be a peripheral polygon of a graph $G$. Let $a$, $b$, and $c$ be distinct vertices of $J$. Let $Y_1, Y_2$ be $Y$-graphs of $G$, each with ends $a$, $b$, and $c$, which spans $J$. Suppose further that $Y_1 \cap Y_2$ consists solely of the vertices $a$, $b$, and $c$. Then we can find a Kuratowski subgraph of $G$.

## Barycentric mappings

If $n < i \leq m$ let $A(i)$ be the set of all vertices of $G$ adjacent to $v_i$, that is joined to $v_i$ by an edge. For each $v_j$ in $A(i)$ let a unit mass $m_j$ be placed at point $f(v_j)$. Then $f(v_i)$ is required to be the centroid of the masses $m_j$.

Denoting the coordinates of $f(v_i)$, $1 \leq i \leq m$, by $(x_i, y_i)$.
Define a matrix $K(G) = \{C_{ij}\}$, $1 \leq (i, j) \leq m$, as follows.
If $i \neq j$ then $C_{ij}$ is minus the number of edges joining $v_i$ and $v_j$.
If $i = j$ then $C_{ij}$ is the degree of $v_i$.

Then the foregoing requirement specifies the coordinates $x_j$ and $y_j$, for $n < j \leq m$, as the solutions of the equations

$$(5) \qquad \sum_{j=1}^{m} C_{ij} x_j = 0$$

$$(6) \qquad \sum_{j=1}^{m} C_{ij} y_j = 0.$$

where $n < i < m$.

$\varphi(i)$: Choose a line $l$ in the plane and define $\varphi(i)$, $1 \le i \le m$, as the perpendicular distance of $f(v_i)$ from $l$, counted positive on one side of $l$ and negative on the other.

$\varphi$-active: We call $v_i$ $\varphi$-active if there is an adjacent vertex $v_j$ of $G$ such that $\varphi(j) \neq \varphi(i)$

positive $\varphi$-poles: The nodes $v_i$ of $J$ with the greatest value of $\varphi(i)$ are the positive $\varphi$-poles of $G$. The number of positive $\varphi$-poles is either 1 or 2.

rising (falling) $\varphi$-path: Let $P$ be a simple path in $G$. We call $P$ a rising (falling) $\varphi$-path if each vertex of $P$ other than the last corresponds to a smaller (greater) value of the function $\varphi(i)$ than does the immediately succeeding vertex.

(6.1) Suppose that $v_i$, where $n < i \le m$, is a $\varphi$-active vertex. Then it has adjacent vertices $v_j$ and $v_k$ such that $\varphi(v_j) < \varphi(v_i) < \varphi(v_k)$

(6.2) Let $v_i$ be a $\varphi$-active vertex. Then we can find a rising $\varphi$-path $P$ from $v_i$ to a positive $\varphi$-pole, and a falling $\varphi$-path $P'$ from $v_i$ to a negative $\varphi$-pole.

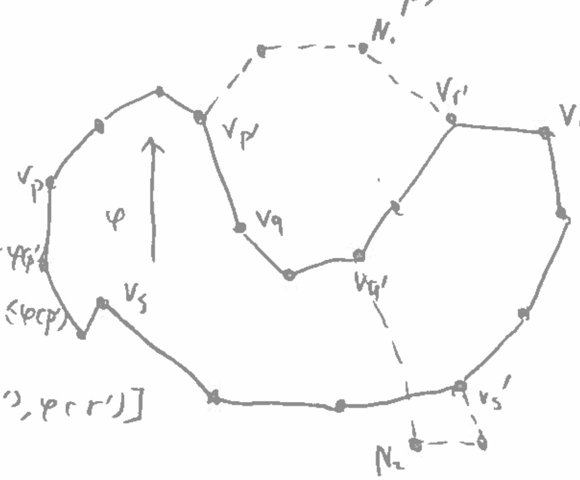
(7.1) Every node of $G$ is $\varphi$-active.

(7.2) Suppose $v_i \notin V(J)$. Then $f(v_i)$ is in the interior of $Q$.

(8.1) Let $K$ be a peripheral polygon of $G$ such that $V(K)$ includes just three nodes $x$, $y$, and $z$ of $G$. Then $f(x)$, $f(y)$, and $f(z)$ are not collinear.

(8.2) Let $K$ be a peripheral polygon of $G$. Let $v_p$, $v_q$, $v_r$, and $v_s$ be nodes of $G$ in $V(K)$ such that $v_p$ and $v_r$ separate $v_q$ and $v_s$ in $K$. Then it is not true that

(7) $$\varphi(p) \ge \varphi(q) \le \varphi(r) \ge \varphi(s) \le \varphi(p)$$


✓ (8) $\varphi(p') \ge \varphi(q) \le \varphi(r') \ge \varphi(s) \le \varphi(p')$

✓ (9) $\varphi(p') \ge \varphi(q') \le \varphi(r') \ge \varphi(s') \le \varphi(p')$

$\varphi(j) < \max[\varphi(q'), \varphi(s')] \le \min[\varphi(p'), \varphi(r')]$

(8.3) The nodes of any peripheral polygon $K$ of $G$ are mapped by $f$ onto distinct points of the plane, no three of which are collinear.

(8.4) Let $L$ be a branch of $G$ having just $t \geq 1$ internal vertices, and let its ends be $a$ and $b$. Then $f(a)$ and $f(b)$ are distinct and $f$ maps the internal vertices onto $t$ distinct points of the segment $f(a)f(b)$ subdividing it into $t+1$ equal parts. Moreover, the order of the vertices from $a$ to $b$ in $L$ agrees with that of their images in $f(a)f(b)$.

(8.5) Let $K$ be any peripheral polygon of $G$. Then $f$ maps the nodes of $G$ in $K$ onto the vertices of a (geometrical) convex polygon $Q_K$ so that the cyclic order of nodes in $K$ agrees with that of vertices in $Q_K$.

(8.6) Let $e$ be any edge of $R$. Then just two distinct peripheral polygons of $G$ pass through $e$, and the two corresponding regions $R_K$ lie on opposite sides of the segment $f(e)$.

**Barycentric representations.**

(9.1) $\underline{\delta(A)} = 1$ for each $A$ in $S$.

$S$ ↳ $|H|$'s complentary set in the plane.

$A$: point

$\delta(A)$: number of distinct peripheral polygons $K$ of $G$ such that $A \in R_K$.

(9.2) Let $G$ be a nodally 3-connected graph having no Kuratowski subgraph. Let $J$ be a peripheral polygon of $G$ which includes just $n \geq 3$ nodes of $G$. Let $Q$ be an $n$-sided convex polygon in the Euclidean plane. Then there is a unique barycentric representation of $G$ on $Q$ mapping the nodes of $G$ occuring on $J$ onto the vertices of $Q$ in any arbitrary specified way preserving the cyclic order.

(9.3) Let $G$ be a nodally 3-connected graph having at least one polygon. Then if $G$ has no Kuratowski subgraph we can construct a convex representation of $G$.

**Straight representations**

(10.1) Let $G$ be the union of two proper subgraphs $H$ and $K$ such that $H \cap K$ is either null or a vertex-graph. Let $M_H$ and $M_K$ be planar meshes of $H$ and $K$ respectively. Then $M_H \cup M_K$ is a planar mesh of $G$. Moreover, any planar mesh of $G$ can be represented in this form.

(10.6) Let $G$ be a graph having a planar mesh $M$. Then each subgraph of $G$ has a planar mesh.

(10.7) If a graph has a planar mesh it has no Kuratowski subgraph.

(10.8) Let $G$ be any simple graph having a planar mesh. Then by adding new links to $G$, with ends in $V(G)$, we can construct a nodally 3-connected graph $T$ having a planar mesh.

(10.9) If $G$ is a simple graph having a planar mesh we can find a straight representation of $G$ in this plan

{MIT 6.889} by Erik Demaine, Shay Mozes, Christian Sommer, Siamak Tazari

"Solve your favourite problems faster for graphs that matter!"

<u>Survey</u>:

| Problems | general | vs. planar |
| --- | --- | --- |
| - single-source shortest paths (arbitrary weights) | $O(nm)$ [Bellman-Ford] | $O(n \frac{\lg^2 n}{\lg\lg n})$ [Mozes & Wulff-Nilsen - ESA 2010] |
| - nonnegative weights | $O(n \lg n + m)$ [Dijkstra] + [Fredman & Tarjan] - JACM 1987 | $O(n)$ [Henzinger, Klein, Rao, Subramanian - JCSS 1997] |
| - maximum flow | $O(nm \lg n)$ [Goldberg & Tarjan 1986] $O(m^{3/2} \lg n \lg u)$ [Goldberg & Rao 1997] | $O(n \lg n)$ [Borradaile & Klein - JACM 2009] |
| - undirected | | $O(n \lg\lg n)$ [Italiano, Nussbaum, Sankowski, Wulff-Nilsen - STOC 2011] |
| - multiterminal | | $O(n \lg^3 n)$ [Borradaile, Klein, Mozes, Nussbaum, Wulff-Nilsen - FOCS 2011] |
| - min spanning tree | $O(n)$ rand. [Karger, Klein, Tarjan 1985] | $O(n)$ det |