HOW TO DRAW A PLANAR GRAPH ON A GRID

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The paper shows that **Theorem 1** every plane graph with n vertices has a Fáry embedding (i.e., straight-line embedding) on the $2n-4$ by $n-2$ grid and provides an $O(n)$ space, $O(n \log n)$ time algorithm to effect this embedding.

It also shows that any set F, which can support a Fáry embedding of every planar graph of size n, has cardinality at least $n + (1-o(1))\sqrt{n}$.

① Run Hopcroft–Tarjan planarity testing algorithm
outputs a topological embedding of a planar graph.

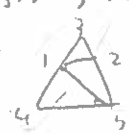
② maximal plane graph – triangulated (all faces are triangles).

Proposition 1

Given a maximal planar graph G and a face uvw, there is a labelling of the vertices, $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$ and a Fáry embedding such that the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ is the same as the convex hull of $\{f(v_1), f(v_2), f(v_k)\}$ for $k = 4, \ldots, n$

Proposition 2

Given a maximal planar graph G and a face uvw, there is a labelling of the vertices, $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$ and a Fáry embedding f such that the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ is the same as the convex hull of $\{f(v_{k-2}), f(v_{k-1}), f(v_k)\}$, for $k = 4, \ldots, n$.



Proposition 3

Given a maximal planar graph G and a face uvw, there is a labelling of the vertices $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$ and a Fáry embedding f such that the boundary of the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ is a cycle in G and $f(v_{k+1})$ is not contained in the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$.

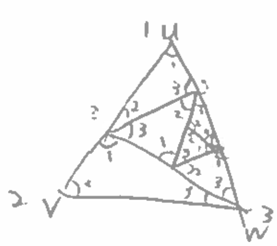
Proposition 4 Schnyder

Given a maximal planar graph G with exterior face uvw, there is a labelling of the angles of the internal triangles with labels 1, 2 and 3 such that

(i) each triangle has labels 1, 2 and 3 in counterclockwise order.

(ii) all angles at u, v and w are labelled 1, 2 and 3, respectively.

(iii) around each internal vertex the angles of each label appear in a single block.



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Theorem 2 If F is universal for planar graphs with n vertices then
$$|F| > n + (1 - o(1))\sqrt{n}$$

outerplanar A planar graph which can be obtained from a simple cycle by adding some of its internal diagonals is called outerplanar.

Proposition 5 Every set of n points in the plane, in general position, supports every outerplanar graph with n vertices. Moreover, this property characterizes the outerplanar graphs.

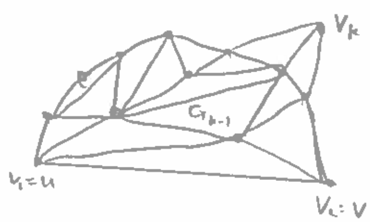
Lemma Let G be a simple planar graph embedded in the plane and $u = u_1, u_2, \dots, u_k = v$ be a cycle of G. Then there exists a vertex w' (resp. w'') on the cycle, different from u and v and not adjacent to any inside chord (resp. outside chord).

Canonical representation lemma for plane graphs

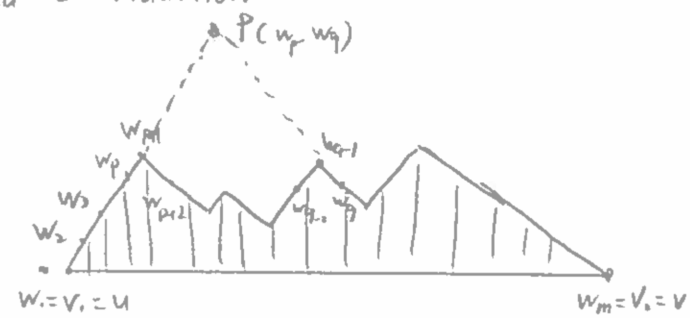
Let G be a maximal planar graph embedded in the plane with exterior face u, v, w. Then there exists a labelling of the vertices $v_1 = u, v_2 = v, v_3, \dots, v_n = w$ meeting the following requirements for every $4 \le k \le n$.

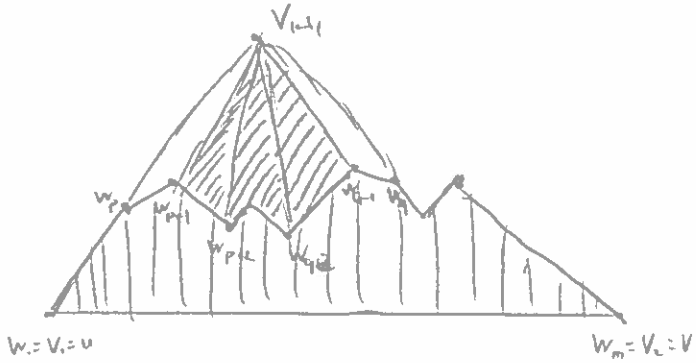
(i) The subgraph $G_{k-1} \subseteq G$ induced by $v_1, v_2, \dots, v_{k-1}$ is 2-connected, and the boundary of its exterior face is a cycle C_{k-1} containing the edge uv;

(ii) v_k is in the exterior face of G_{k-1}, and its neighbours in G_{k-1} form an (at-least 2-element) subinterval of the path $C_{k-1} - uv$.



Key idea of induction





To realize this goal, assume that for each vertex w_i on the exterior face of G_k we have already defined a subset $M(k, w_i) \subseteq V(G_k)$ such that

(a) $w_j \in M(k, w_i)$ iff $j \ge i$

(b) $M(k, w_1) \supset M(k, w_2) \supset \dots \supset M(k, w_m)$

(c) For any nonnegative numbers $\alpha_1, \alpha_2, \dots, \alpha_m$, if we sequentially translate all vertices in $M(k, w_i)$ with distance α_i to the right ($i = 1, 2, \dots, m$), then the embedding of G_k remains a Fáry embedding.

Embedding Planar Graphs on the Grid by Walter Schnyder

This paper shows that each plane graph of order $n \geq 3$ has a straight line embedding on the $n-2$ by $n-2$ grid. This embedding is computable in time $O(n)$.

cf. I. Fáry, On straight line representation of planar graphs, Acta Sci. Math. Szeged 11 (1948), 229-233.

Characterization: planar graphs are graphs whose incidence relation is the intersection of three total orders.

THEOREM 1.1 Let $\lambda_1, \lambda_2, \lambda_3$ be three pairwise non parallel straight lines in the plane. Then each plane graph has a straight line embedding in which any two disjoint edges are separated by a straight line parallel to λ_1, λ_2, or λ_3.

THEOREM 1.2 Each plane graph with $n \geq 3$ vertices has a straight-line embedding on the $n-2$ by $n-2$ grid.

Barycentric representations

A barycentric representation of a graph G is an injective function $v \in V(G) \longrightarrow (v_1, v_2, v_3) \in \mathbb{R}^3$ that satisfies the conditions:

(1) $v_1 + v_2 + v_3 = 1$ for all vertices v

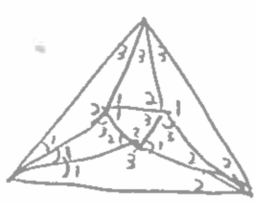
(2) For each edge $\{x, y\}$ and each vertex $z \notin \{x, y\}$, there is some $k \in \{1,2,3\}$ such that $x_k < z_k$ and $y_k < z_k$

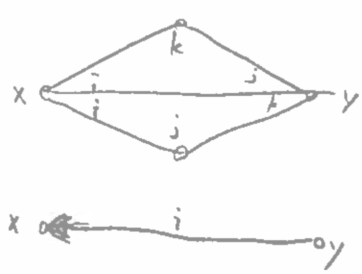
normal labeling

A normal labeling of a triangular graph G is a labeling of the angles of G with the labels 1, 2, 3 satisfying the conditions

(1) Each elementary triangle of G has an angle labeled 1, an angle labeled 2, and an angle labeled 3. The corresponding vertices appear in counterclockwise order.

(2) The labels of the angles of each interior vertex v of G form, in counterclockwise order, a nonempty interval of 1's followed by a nonempty interval of 2's by a nonempty interval of 3's.



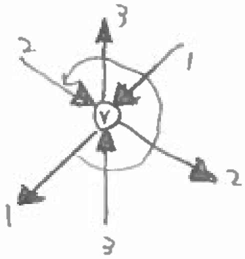


THEOREM 4.2. Each triangular graph has a normal labeling.

realizer A realizer of a triangular graph G is a partition of the interior edges of G in three sets T_1, T_2, T_3 of directed edges such that for each interior vertex v there holds:

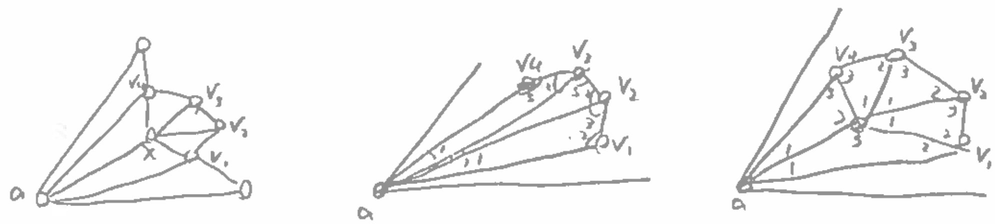
(1) v has outdegree one in each of T_1, T_2, T_3

(2) The counterclockwise order of the edges incident on v is: leaving T_1, entering T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.



The edge $\{x,y\}$ is contractible if x and y have exactly two common neigbours

Prove THEOREM 4.2 by induction.



THEOREM 4.5

Let G be a triangular graph with at least four vertices and let T_1, T_2, T_3 be a realizer of G. Then each T_i is a tree including all interior vertices and exactly one exterior vertex and all edges of T_i are directed toward this exterior vertex The exterior vertices belonging to T_1, T_2, T_3 are distinct and appear in counterclockwise order.

THEOREM 4.6

If T_1, T_2, T_3 is a realizer of a triangular graph, then for $i = 1,2,3$ the relation $T_i \cup T_{i+1}^{-1} \cup T_{i+2}^{-1}$ has no directed cycle (indices are modulo 3).

THEOREM 6.1

The function $f: v \in V(G) \longrightarrow \frac{1}{2n-5}(v_1, v_2, v_3)$ is a barycentric representation of G and the labeling of G that is induced by f is identical to the given labeling of G

COROLLARY 6.2

Let a, b and c denote the roots of T_1, T_2, T_3. Then for any choice of noncolinear positions of a, b and c the mapping

$$f: v \rightarrow \frac{1}{2n-5}(v_1 a + v_2 b + v_3 c)$$

is a straight line embedding of G in the plane spanned by a, b, c.

PROPOSITION 6.3

The mapping $v \in V(G) \longrightarrow (v_1, v_2)$ is a straight line embedding of G on the $2n-5$ by $2n-5$ grid.