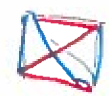
Graph Theory MATH 442 Term 2 [2018-09-11]
CPSC 536E ~~topics~~

matrix representation of graph ① computing path
② Google's page rank how connected is the graph?
③ useful for graph drawing eigenvalue

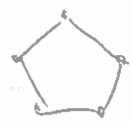
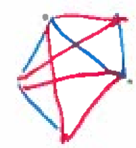
major topics:
TODO: ① spectral graph theory
~~② Ramsey's theory~~ Complete graph: color edge red or blue,
smallest number of vertices.

clique: → 

$R(3,3)=3$ $R(4,4)=18$
= size of smallest complete graph so that for all red/blue colorings the exists either a red 3-clique or a blue 3-clique.

$R(5,5)=43$

③ Random walks on graphs

Give a graph, how many perfect matchings are there?

① ④ planar graphs

characterization drawn on a plane without edges crossing.

Pointers for things to look at:

To get started with planar graphs

① Things for next time:
things about planar graphs
how to tell?
algorithms for planar detection

② graph drawing

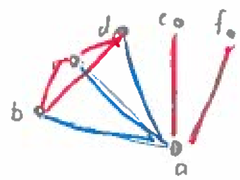
② GRAPH DRAWING / 4 things interested in
ALGORITHMS FOR THE VISUALIZ ATION OF GRAPH,
italian GIUSEPPE DI BATTISTA
Austr. PETER EADES
italian ROBERTO TAMASSIA
Greek IOANNIS G. TOLLIS

③ Graph Theory with Applications
J.A. Bondy U.S.R. Murty.
read chapter on planar graphs

Ramsey number

Proof that $R(3,3) = 6$

In a complete graph (K_6) for node a adjacent to all other 5 nodes, color them red or blue.



No matter how you color it, at least 3 edges are colored with the same color (either red or blue).

WLOG, let's say a is connected with b, c, d colored blue.

To avoid a clique with a, bc, cd, bd has to be colored red. In this way, b, c, d form a red 3-clique.

Thus, K_6 must have either a red 3-clique or a blue 3-clique. □

Bondy and Murty

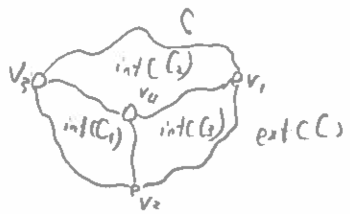
[1.1] P5 : <u>planar graph</u>: A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a <u>planar graph</u>, and such a drawing is called a <u>planar embedding</u> of the graph.

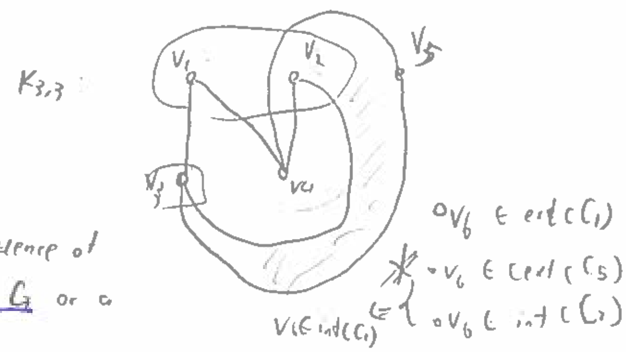
[10.1] P243 <u>planar graph</u>: A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends.

[10.1] P244 <u>plane graph</u>: we often refer to a planar embedding $\tilde{G}$ of a planar graph G as a plane graph; and we call the vertices of $\tilde{G}$ points and its edges lines.

[10.1] P245 <u>THE JORDAN CURVE THEOREM</u>: Any simple closed curve C in the plane partitions the rest of the plane into two <u>disjoint arcwise-connected open sets</u>.
↳ called interior $int(C)$ and exterior $ext(C)$.

[10.1] P245 <u>K_5 is nonplanar</u>:





[10.1] P246 <u>subdivision</u>: Any graph derived from a graph G by a sequence of edge subdivisions is called a <u>subdivision of G</u> or a <u>G-subdivision</u>.

[10.1] P246. Proposition 10.3: A graph G is planar if and only if every subdivision of G is planar.

[10.1] P247 Theorem 10.4: A graph G is embeddable on the plane if and only if it is embeddable on the sphere.

stereographic projection from z

[10.2] P250 Proposition 10.5: Let G be a planar graph, and let f be a face in some planar embedding of G. Then G admits a planar embedding whose outer face has the same boundary as f.
Proof idea - consider an embedding $\tilde{G}$ of G on the sphere.

[10.2] P250 Theorem 10.6: <u>THE JORDAN-SCHÖNFLIESS THEOREM</u> ?
Any homeomorphism of a simple closed curve in the plane onto another simple closed curve can be extended to a homeomorphism of the plane.

[10.2] P251 Theorem 10.7: In a nonseparable plane graph other than K_1 or K_2, each face is bounded by a cycle. ? ?

[10.2] P252 <u>dual</u>: Given a plane graph G, one can define a second graph G^* as follows. Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e there is an edge e^* of G^*. Two vertices f^* and g^* are joined by the edge e^* in G^* iff their corresponding faces f and g are separated by the edge e in G. Observe that if e is a cut edge of G, then $f=g$, so e^* is a loop of G^*; conversely, if e is a loop of G, the edge e^* is a cut edge of G^*. The graph G^* is called the dual of G.

[10.3] P359 EULER'S FORMULA:

For a connected plane graph G. Proof by induction

$$v(G) - e(G) + f(G) = 2$$

[10.3] P359 Corollary 10.20:

All planar embeddings of a connected planar graph have the same number of faces.

[10.3] P359 Corollary 10.21:

Let G be a simple planar graph on at least three vertices. Then $m \le 3n-6$. Furthermore, $m = 3n-6$ if and only if every planar embedding of G is a triangulation.

[10.3] P359 Corollary 10.22:

Every simple planar graph has a vertex of degress at most five.

[10.3] P359 Corollary 10.23:

K_5 is nonplanar proof: If K_5 were planar, corollary 10.21 would give $10 = e(K_5) \le 3v(K_5) - 6 =$

[10.3] P359 Corollary 10.24:

$K_{3,3}$ is nonplanar. proof: If $K_{3,3}$ were planar, $K_{3,3}$ has no cycle of length < 4, so every face of G has degree ≥ 4. By Theorem 10.10, $4f(G) \le \sum_{f \in F} d(f) = 2e(G) = 18$

$\therefore \underbrace{f(G)}_{\text{integer}} \le 4$

Euler's formula implies that

$$\underset{6}{v(G)} - \underset{9}{e(G)} + f(G) = 2$$

$6 - 9 + f(G) = 2 \Rightarrow f(G) = 5$

10.2] P353 Theorem 10.10: If G is a plane graph

$$\sum_{f \in F} d(f) = 2m$$

10.5] P368 Theorem 10.30: KURATOWSKI'S THEOREM

A graph is planar iff it contains no subdivision of either K_5 or $K_{3,3}$.

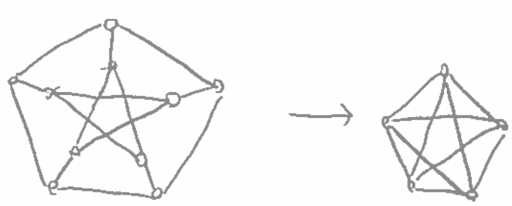
A subdivision of K_5 or $K_{3,3}$ is consequently called a Kuratowski subdivision.

10.5] P368 minor: A minor of a graph is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions ?

Alternatively, consider a partition $(V_0, V_1, \ldots, V_k)$ of V such that $G[V_i]$ is connected, $1 \le i \le k$, and let H be the graph obtained from G by deleting V_0 and shrinking each induced subgraph $G[V_i]$, $1 \le i \le k$, to a single vertex. The any spanning? subgraph F of H is a minor of G.

e.g. K_5 is a minor of the Petersen graph because it can be obtained by contracting the five 'spoke' edges of the latter graph.

Petersen graph



important:

1) any graph which contains an F-subdivision also has an F-minor

2) provided that F is a graph of maximum degree three or less, any graph which has an F-minor also contains an F-subdivision.

If F is a minor of G, we write $F \preceq G$.

Exe. find a $K_{3,3}$ minor from Petersen graph.

[10.5] P369 WAGNER'S THEOREM

A graph is planar iff it has no Kuratowski minor

[1.1] P_3 loop: an edge with identical ends is called a loop
link: an edge with distinct ends is called a link
parallel edges: two or more links with the same pair of ends are said to be parallel edges
simple: a graph is simple if it has no loops or parallel edges.

[D: Battista et al.]

{1} P_6 drawing: a drawing Γ of a graph (digraph) G is a function which maps each vertex v to a distinct point $\Gamma(v)$ and each edge (u,v) to a simple open Jordan curve $\Gamma(u,v)$, with endpoints $\Gamma(u)$ and $\Gamma(v)$.

{1} P_7 drawing planar: a drawing Γ is planar if no two distinct edges intersect.
A graph is planar if it admits a planar drawing

{1} P_8 connected: a graph is connected if there is a path between u and v for each pair (u,v) of vertices.

cut vertex: A cut vertex in graph G is a vertex whose removal disconnects G.
biconnected: A connected graph with no cutvertices is biconnected.
blocks: The maximal biconnected subgraphs of a graph are its blocks. (sometimes called biconnected components).
A graph is planar iff its blocks are planar.

{1} P_9 important: ?1) The skeleton of a convex polyhedron is a planar triconnected graph.
?2) A planar triconnected graph has a unique embedding, up to a reversal of the circular ordering of the neighbors of each vertex

{2.1} P_{12} drawing convention: A drawing convention is a basic rule that the drawing must satisfy to be admissible.

- Polyline drawing
- Straight-line drawing
- Orthogonal drawing
- Grid Drawing
- Planar Drawing
- Upward (resp. downward) Drawing

{2.1} P_{14} aesthetics: aesthetics specify graph properties of the drawing that we would like to apply as much as possible, to achieve readability.

- crossings: minimize
- area: minimize (grid drawing. straight-line drawing where distance$(u,v) \geq 1$)
  convex hull
- total edge length: minimize
- maximum edge length: minimize.
- uniform edge length minimize variance of lengths of the edges
- total bends: minimize. (important for orthogonal drawings)
- maximum bends: minimize
- uniform bends: minimize
- angular resolution: maximize (straight-line drawing)
- aspect ratio: minimize longest-side : shortest side
- symmetry

constraints: refer to specific subgraphs or sub drawings

efficiency: interactive applications