

matrix representation of graph

① computing path

② Google's page rank how connected is the graph?

③ useful for graph drawing eigenvalue

major topics:

TODO: ① spectral graph theory

② ~~Ramsey's theory~~Complete graph: color <sup>edge</sup> red or blue,  
smallest number of  
vertices.

clique:



$$R(3,3)=3$$

= size of

smallest complete  
graph so that for  
all red/blue colorings  
there exists either  
a red 3-clique  
or a blue 3-clique.

$$R(4,4)=18$$

$$R(5,5)=43$$

③ Random walks on graphsGive a graph, how many perfect matchings are there?① ④ planar graphs

characterization drawn on a plane  
without edges crossing.

Pointers for things to look at:

To get started, with planar graphs

① Try for next time:

things about planar  
graphs

how to tell?  
algorithms for  
planar detection

② graph drawing

② GRAPH DRAWING / 4 things interested in  
ALGORITHMS FOR THE VISUALIZATION OF GRAPHS

italian: GIUSEPPE DI BATTISTA  
german: PETER EADES  
italian: ROBERTO TAMASSIA  
greek: IOANNIS G. TOLLIS

③ Graph Theory

with Applications

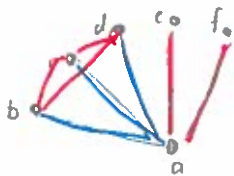
J.A. Bondy

U.S.R. Murty

read chapters on planar graphs

Proof that  $R(3,3) = 6$

In a complete graph  $K_6$  for node  $a$  adjacent to all other 5 nodes, color them red or blue.



No matter how you color it, at least 3 edges are colored with the same color (either red or blue).

WLOG, let's say  $a$  is connected with  $b, c, d$  colored blue.

To avoid a clique with  $a$ ,  $bc, cd, bd$  has to be colored red. In this way,  $b, c, d$  form a red 3-clique.

Thus,  $K_6$  must have either a red 3-clique or a blue 3-clique.

□

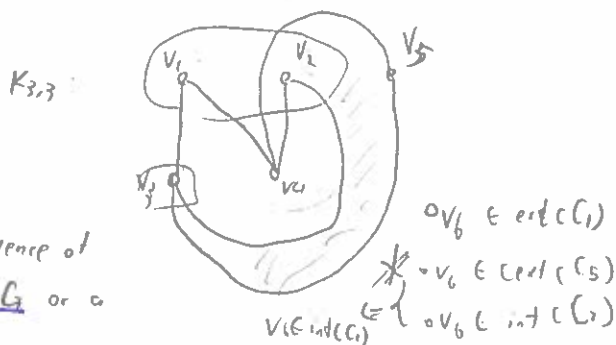
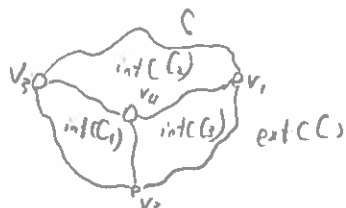
[1.1] P5: planar graph: A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a planar graph, and such a drawing is called a planar embedding of the graph.

[10.1] P243: planar graph: A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends.

[10.1] P244: plane graph: we often refer to a planar embedding  $\tilde{G}$  of a planar graph  $G$  as a plane graph; and we call the vertices of  $\tilde{G}$  points and its edges lines.

[10.1] P245: THE JORDAN CURVE THEOREM: Any simple closed curve  $C$  in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets.

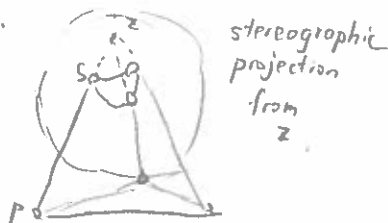
[10.1] P245:  $K_5$  is nonplanar:  $\hookrightarrow$  called interior  $\text{int}(C)$  and exterior  $\text{ext}(C)$ .



[10.1] P246: subdivision: Any graph derived from a graph  $G$  by a sequence of edge subdivisions is called a subdivision of  $G$  or a  $G$ -subdivision.

[10.1] P246: Proposition 10.3: A graph  $G$  is planar if and only if every subdivision of  $G$  is planar.

[10.1] P247: Theorem 10.4: A graph  $G$  is embeddable on the plane if and only if it is embeddable on the sphere.



[10.2] P250: Proposition 10.5: Let  $G$  be a planar graph, and let  $f$  be a face in some planar embedding of  $G$ . Then  $G$  admits a planar embedding whose outer face has the same boundary as  $f$ .

Proof idea - consider an embedding  $\tilde{G}$  of  $G$  on the sphere.

[10.2] P250: Theorem 10.6: THE JORDAN-SCHÖNFLIUS THEOREM?

Any homeomorphism of a simple closed curve in the plane onto another simple closed curve can be extended to a homeomorphism of the plane.

[10.2] P251: Theorem 10.7: In a nonseparable plane graph other than  $K_1$  or  $K_2$ , each face is bounded by a cycle.

[10.2] P252: dual: Given a plane graph  $G$ , one can define a second graph  $G^*$  as follows. Corresponding to each face  $f$  of  $G$  there is a vertex  $f^*$  of  $G^*$ , and corresponding to each edge  $e$  there is an edge  $e^*$  of  $G^*$ . Two vertices  $f^*$  and  $g^*$  are joined by the edge  $e^*$  in  $G^*$  iff their corresponding faces  $f$  and  $g$  are separated by the edge  $e$  in  $G$ . Observe that if  $e$  is a cut edge of  $G$ , then  $f=g$ , so  $e^*$  is a loop of  $G^*$ ; conversely, if  $e$  is a loop of  $G$ , the edge  $e^*$  is a cut edge of  $G^*$ . The graph  $G^*$  is called the dual of  $G$ .

# [10.3] P59 EULER'S FORMULA:

For a connected plane graph  $G$ .

Proof by induction

$$v(G) - e(G) + f(G) = 2$$

## [10.3] P59 Corollary 10.20:

All planar embeddings of a connected planar graph have the same number of faces.

## [10.3] P59 Corollary 10.21:

Let  $G$  be a simple planar graph on at least three vertices. Then  $m \leq 3n - 6$ . Furthermore,  $m = 3n - 6$  if and only if every planar embedding of  $G$  is a triangulation.

## [10.3] P59 Corollary 10.22:

Every simple planar graph has a vertex of degree at most five.

## [10.3] P59 Corollary 10.23:

$K_5$  is nonplanar

proof: If  $K_5$  were planar, Corollary 10.21 would give  $10 = e(K_5) \leq 3v(K_5) - 6 = 9$ .

## [10.3] P59 Corollary 10.24:

$K_{3,3}$  is nonplanar

proof: If  $K_{3,3}$  were planar,  $K_{3,3}$  has no cycle of length  $< 4$ , so every face of  $G$  has degree  $\geq 4$ . By Theorem 10.10,  $4f(G) \leq \sum_{f \in F} d(f) = 2e(G) = 18$

## [10.2] P53 Theorem 10.10: If $G$ is a plane graph

$$\sum_{f \in F} d(f) = 2m$$

Euler's formula implies that  $\therefore \frac{f(G)}{\text{integer}} \leq 4$

$$\frac{v(G) - e(G) + f(G)}{6 - 9 + f(G)} = 2 \Rightarrow f(G) = 5$$

## [10.5] P68 Theorem 10.30: KURATOWSKI'S THEOREM

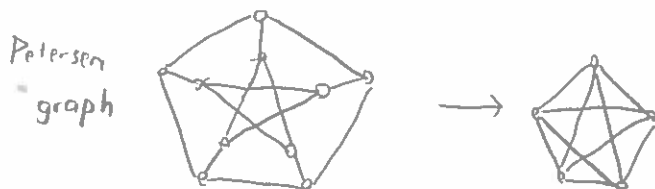
A graph is planar iff it contains no subdivision of either  $K_5$  or  $K_{3,3}$ .

A subdivision of  $K_5$  or  $K_{3,3}$  is consequently called a Kuratowski subdivision.

## [10.5] P68 minor: A minor of a graph is any graph obtainable from $G$ by means of a sequence of vertex and edge deletions and edge contractions?

Alternatively, consider a partition  $(V_0, V_1, \dots, V_k)$  of  $V$  such that  $G[V_i]$  is connected,  $1 \leq i \leq k$ , and let  $H$  be the graph obtained from  $G$  by deleting  $V_0$  and shrinking each induced subgraph  $G[V_i]$ ,  $1 \leq i \leq k$ , to a single vertex. The any spanning subgraph  $F$  of  $H$  is a minor of  $G$ .

e.g.  $K_5$  is a minor of the Petersen graph because it can be obtained by contracting the five 'spoke' edges of the latter graph.



important:

- 1) any graph which contains an  $F$ -subdivision also has an  $F$ -minor
- 2) provided that  $F$  is a graph of maximum degree three or less, any graph which has an  $F$ -minor also contains an  $F$ -subdivision.

If  $F$  is a minor of  $G$ , we write  $F \leq G$ .

Ex. find a  $K_{3,3}$  minor from Petersen graph.

## [10.5] P69 WAGNER'S THEOREM

A graph is planar iff it has no Kuratowski minor.

[1.1]  $P_3$  loop: an edge with identical ends is called a loop

link: an edge with distinct ends is called a link

parallel edges: two or more links with the same pair of ends are said to be parallel edges

simple: a graph is simple if it has no loops or parallel edges.

[Di Battista et al.]

{1}  $P_6$  drawing: a drawing  $\Gamma$  of a graph (digraph)  $G$  is a function which maps each vertex  $v$  to a distinct point  $\Gamma(v)$  and each edge  $(u,v)$  to a simple open Jordan curve  $\Gamma(u,v)$  with endpoints  $\Gamma(u)$  and  $\Gamma(v)$ .

{1}  $P_7$  drawing planar: a drawing  $\Gamma$  is planar if no two distinct edges intersect.

A graph is planar if it admits a planar drawing

{1}  $P_8$  connected: a graph is connected if there is a path between  $u$  and  $v$  for each pair  $(u,v)$  of vertices

cut vertex: A cut vertex in graph  $G$  is a vertex whose removal disconnects  $G$ .

biconnected: A connected graph with no cut vertices is biconnected.

blocks: The maximal biconnected subgraphs of a graph are its blocks. (sometimes called biconnected components).  
A graph is planar iff its blocks are planar.

{1}  $P_9$  important: 1) The skeleton of a convex polyhedron is a planar triconnected graph.

2) A planar triconnected graph has a unique embedding, up to a reversal of the circular ordering of the neighbors of each vertex

{2.1}  $P_{12}$  drawing convention: A drawing convention is a basic rule that the drawing must satisfy to be admissible.

- Poly line drawing
- Straight-line drawing
- Orthogonal drawing
- Grid Drawing
- Planar Drawing
- Upward (resp. downward) Drawing

{2.1}  $P_{14}$  aesthetics: aesthetics specify graph properties of the drawing that we would like to apply as much as possible, to achieve readability.

constraints:

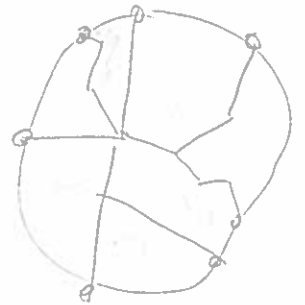
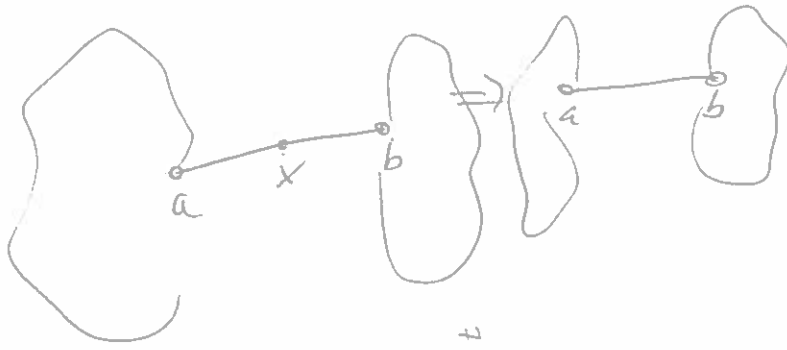
refer to specific subgraphs or sub drawings

efficiency:

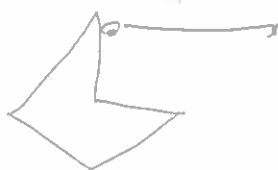
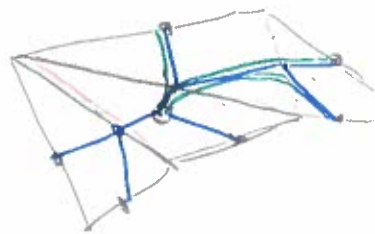
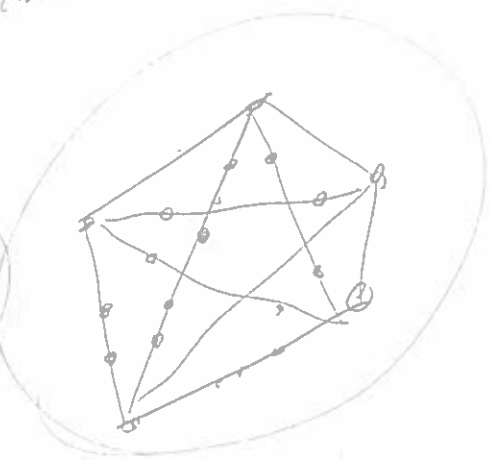
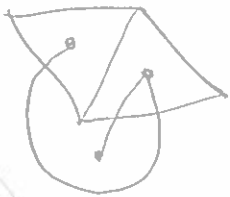
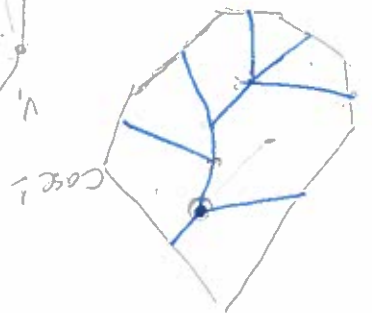
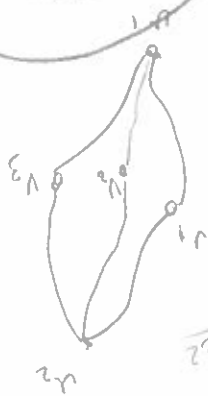
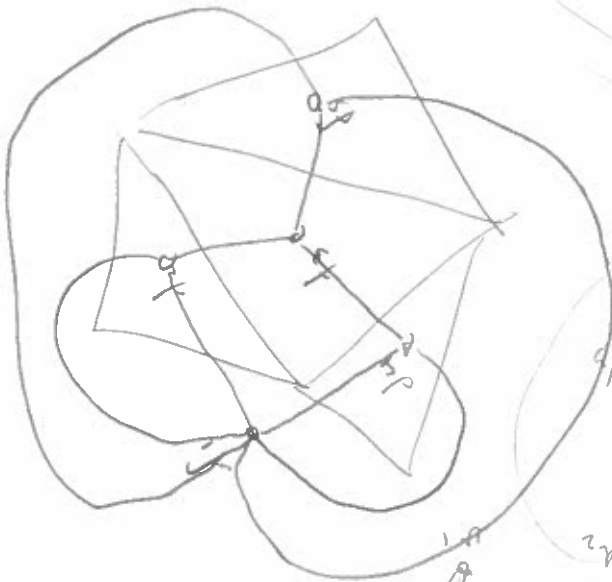
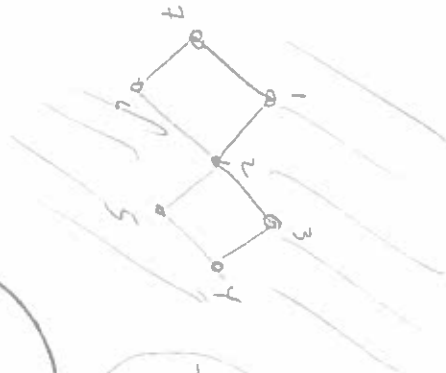
interactive applications

- crossings: minimize
- area: minimize (good drawing: straight-line drawing where  $\text{distance}(u,v) \geq 1$ )  
convex hull
- total edge length: minimize
- maximum edge length: minimize
- uniform edge length: minimize variance of lengths of the edges
- total bends: minimize (important for orthogonal drawings)
- maximum bends: minimize
- uniform bends: minimize
- angular resolution: maximize (straight-line drawing)
- aspect ratio: minimize longest-side: shortest side  $\square$

- symmetry



1 2 3 4 5 6 7



# [The Left-Right Planarity Test] by Ulrik Brandes

- Efficient Planarity Testing by John Hopcroft & Robert Tarjan (1974)
- A Depth-first-search Characterization of Planarity by H. De Fraysseix & P. Rosenstiehl (1982)
- On the Realization of Complexes in Euclidean Spaces 吴文俊 Wen-Tsun Wu (1955)

{P<sub>4</sub>} There are only two significant ways to draw a simple cycle planarly, namely clockwise and counterclockwise.

{P<sub>4</sub>} Testing planarity amounts to deciding whether there is a consistent simultaneous orientation of all cycles.

{P<sub>5</sub>} In any planar drawing the back edges can be partitioned into left and right depending on whether their fundamental cycle is counterclockwise or clockwise.

{P<sub>6</sub>} In the oriented graph, we denote by  $E^+(v) = \{(v, w) \in E : w \in V\}$  the set of all outgoing edges of  $v \in V$ , so that  $E = \bigcup_{v \in V} E^+(v)$ .

{P<sub>6</sub>} A DFS traversal yields a bipartition  $E = T \cup B$  of the edges, where those in  $T$  are called tree edges and the non-tree edges in  $B$  are called back edges. We write  $u \rightarrow v$  for  $(u, v) \in T$  and  $v \hookrightarrow w$  for  $(v, w) \in B$ .

{P<sub>6</sub>} fundamental cycle:  $C(v \hookrightarrow w) = w \xrightarrow{+} v \hookrightarrow w$   
overlapping. Two cycles are called overlapping, if they share an edge.

{P<sub>6</sub>} Lemma 3 Let  $G = (V, T \cup B)$  be a DFS-oriented graph.

(1) The fundamental cycles are exactly the simple directed cycles of  $G$ .

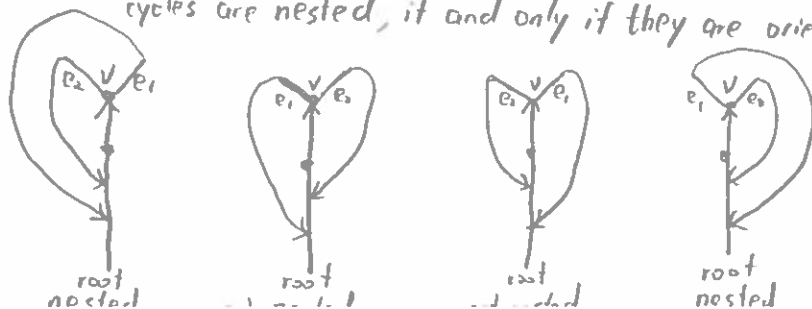
(2) Two distinct fundamental cycles are either disjoint, or their intersection forms a tree path

{P<sub>7</sub>} fork: For two overlapping cycles, the last edge  $u \rightarrow v$  on the shared tree path together with succeeding edges  $e_1 = (v, w_1)$ ,  $e_2 = (v, w_2)$  on each cycle is called their fork, and  $v$  its branching point

{P<sub>7</sub>} a linearization of the cycle order: It is defined by splitting the clockwise order restricted to outgoing edges at the incoming tree edge, or between any two consecutive outgoing edges if  $v$  is the root of a DFS tree.

{P<sub>7</sub>} nested: Two overlapping fundamental cycles are called nested, if the part of one cycle that is not common to both is drawn completely inside the other cycle.

{P<sub>8</sub>} Observation 1: In a planar drawing of a DFS-oriented graph  $G = (V, T \cup B)$ , two overlapping cycles are nested, if and only if they are oriented alike.



{P<sub>6</sub>} return points: The return points of a tree edge  $v \rightarrow w \in T$  are the ancestors  $u$  of  $v$  with  $u \xrightarrow{+} v \rightarrow w \xrightarrow{*} x \hookrightarrow u$  for some descendant  $x$  of  $w$ .  
The return points of a vertex  $v \in V$  are formed by the union of all return points of outgoing edges  $(v, w) \in E^+(v) \subseteq T \cup B$ .

{P<sub>8</sub>} lowpoint: The lowpoint of an edge is its lowest return point, if any, or its source if none exists.

{P<sub>9</sub>} Observation 2: In a planar drawing of a connected DFS-oriented graph  $G = (V, T \cup B)$  with the root of the DFS tree on the outer face, overlapping fundamental cycles are nested according to their lowpoint order.

{P<sub>3</sub>} left, right: the side of a back edge in a planar drawing is right, if its fundamental cycle is oriented clockwise, and left otherwise.

{P<sub>9</sub>} LR partition: Let  $G = (V, T \cup B)$  be a DFS-oriented graph. A partition  $B = L \cup R$  of its back edges into two classes, referred to as left and right, is called left-right partition, or LR partition for short, if every fork consisting of  $u \rightarrow v \in T$  and  $e_1, e_2 \in E^+(v)$   
 (1) all return edges of  $e_1$  ending strictly higher than  $\text{lowpt}(e_2)$  belong to one class  
 and  
 (2) all return edges of  $e_2$  ending strictly higher than  $\text{lowpt}(e_1)$  belong to the other.

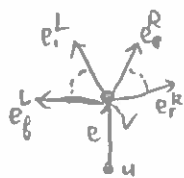
{P<sub>10</sub>} Left-Right Planarity Criterion: A graph is planar if and only if it admits an LR partition.

{P<sub>10</sub>} aligned: An LR partition is called aligned, if all return edges of a tree edge  $e$  that return to  $\text{lowpt}(e)$  are on the same side.

{P<sub>11</sub>} Lemma 6: Any LR partition can be turned into an aligned LR partition.

{P<sub>12</sub>}  $e_1 \prec e_2$ : we have to define  $e_1 \prec e_2$  if and only if the lowpoint of  $e_1$  is strictly lower than that of  $e_2$ . If both have the same lowpoint, but say, only  $e_2$  has another return point, we say that  $e_2$  is chordal and let  $e_1 \prec e_2$ .

{P<sub>12</sub>} Definition 1 LR Ordering: Given an LR partition, let  $e_1^L \prec \dots \prec e_\ell^L$  be the left outgoing edges of a vertex  $v$  and  $e_1^R \prec \dots \prec e_r^R$  its right outgoing edges. If  $v$  is not the root, let  $u$  be its parent. The clockwise left-right ordering, or LR ordering for short, of the edges around  $v$  is defined as follows:



$(u, v),$   
 $L(e_1^L), e_1^L, R(e_1^L), \dots, L(e_\ell^L), e_\ell^L, R(e_\ell^L),$   
 $L(e_1^R), e_1^R, R(e_1^R), \dots, L(e_r^R), e_r^R, R(e_r^R)$



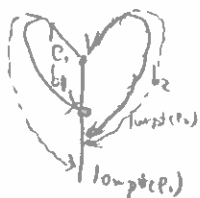
where  $(u, v)$  is absent if  $v$  is the root, and  $L(e)$  and  $R(e)$  denote the left and right incoming back edges whose cycles share  $e$ . For two back edges  $b_1 = x_1 \hookrightarrow v, b_2 = x_2 \hookrightarrow v \in R(e)$  let  $z \rightarrow x, (x, y_1), (x, y_2)$  be the fork of  $C(b_1), C(b_2)$ . Then,  $b_1$  comes after  $b_2$  in  $R(e)$  if and only if  $(x, y_1) \prec (x, y_2)$ . If  $b_1, b_2 \in L(e)$ , the order is reversed.



{P12} Lemma 8. Given an LR partition, LR ordering yields a planar embedding.

{P14} Corollary 9. Let  $G = (V, T \cup B)$  be a DFS-oriented graph. For a pair of back edges  $b_1, b_2 \in B$  with overlapping fundamental cycles let  $v_i \rightarrow \dots \rightarrow v_k$  be the tree path of the intersection and  $(v_{k-1}, v_k), e_i, e_i$  the corresponding fork with  $e_i \xrightarrow{*} b_1$  and  $e_i \xrightarrow{*} b_2$ . Then,  $b_1$  and  $b_2$  are subject to

- a different-constraint, iff  $\text{lowpoint}(e_2) < \text{lowpoint}(b_1)$  and  $\text{lowpoint}(e_1) < \text{lowpoint}(b_2)$
- a same-constraint, iff  $\text{lowpoint}(e') < \min\{\text{lowpoint}(b_1), \text{lowpoint}(b_2)\}$  for some  $e' = (v_i, w) \in T \cup B, 1 \leq i \leq k, w = v_{k+1}$



(a) different-constraint



{P16} Definition 10. Let  $G = (V, T \cup B)$  be a DFS-oriented graph such that each pair of back edges  $b_1, b_2 \in B$  is subject to at most one type of constraint. The signed graph  $CC(G) = (B, E(C); \sigma: E(C) \rightarrow \{-1, +1\})$  with

$$\sigma(b_1, b_2) = \begin{cases} -1 & \text{if } b_1, b_2 \in B \text{ are subject to a different constraint} \\ +1 & \text{if } b_1, b_2 \in B \text{ are subject to a same constraint} \end{cases}$$

is called constraint graph of  $G$ .

{P16} If any pair of back edges is subject to both a same-constraint and a different-constraint, no LR partition can exist and hence the graph is non-planar.

{P18} Algorithm 1: Left-Right Planarity Algorithm

input: simple, undirected graph  $G = (V, E)$

output: planar embedding Challs if graph is not planar)

if  $|V| \geq 2$  and  $|E| > 3|V| - 6$  then HALT: not planar

▼ orientation

for  $s \in V$  do

if  $\text{height}[s] = \infty$  then

$\text{height}[s] \leftarrow 0$ ; append  $\text{Roots} \leftarrow s$

DFS1(s)

▼ testing

sort adjacency lists according to non-decreasing nesting\_depth

for  $s \in \text{Roots}$  do DFS2(s)

▼ embedding

for  $e \in E$  do  $\text{nesting\_depth}[e] = \text{sign}(e) \cdot \text{nesting\_depth}[e]$

sort adjacency lists according to non-decreasing nesting\_depth

for  $s \in \text{Roots}$  do DFS3(s)

where

integer  $\text{sign}(edge\ e)$

if  $\text{ref}[e] \neq 1$  then

$\text{side}[e] \leftarrow \text{side}[e] \cdot \text{sign}(\text{ref}[e])$

$\text{ref}[e] \leftarrow 1$

# HOW TO DRAW A PLANAR GRAPH ON A GRID

by H. DE FRAYSSEIX, J. PACH and R. POLLACK.

## Theorem 1

The paper shows that every plane graph with  $n$  vertices has a Fáry embedding (i.e., straight-line embedding) on the  $2n-4$  by  $n-2$  grid and provides an  $O(n)$  space,  $O(n \log n)$  time algorithm to effect this embedding.

It also shows that any set  $F$ , which can support a Fáry embedding of every planar graph of size  $n$ , has cardinality at least  $n + (1 - o(1)) \sqrt{n}$ .

- ① Run Hopcroft-Tarjan planarity testing algorithm  
outputs a topological embedding of a planar graph.
- ② maximal plane graph - triangulated (all faces are triangles).

## Proposition 1

Given a maximal planar graph  $G$  and a face  $uvw$ , there is a labelling of the vertices,  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  and a Fáry embedding such that the convex hull of  $\{f(v_1), f(v_2), f(v_3), \dots, f(v_k)\}$  is the same as the convex hull of  $\{f(v_1), f(v_2), f(v_k)\}$  for  $k = 4, \dots, n$ .



## Proposition 2

Given a maximal planar graph  $G$  and a face  $uvw$ , there is a labelling of the vertices,  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  and a Fáry embedding  $f$  such that the convex hull of  $\{f(v_1), f(v_2), f(v_3), \dots, f(v_k)\}$  is the same as the convex hull of  $\{f(v_{k-1}), f(v_{k-1}), f(v_k)\}$  for  $k = 4, \dots, n$ .



## Proposition 3

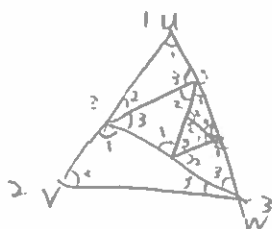
Given a maximal planar graph  $G$  and a face  $uvw$ , there is a labelling of the vertices,  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  and a Fáry embedding  $f$  such that the boundary of the convex hull of  $\{f(v_1), f(v_2), f(v_3), \dots, f(v_k)\}$  is a cycle in  $G$  and  $f(v_{k+1})$  is not contained in the convex hull of  $\{f(v_1), f(v_2), f(v_3), \dots, f(v_k)\}$ .

## Proposition 4

Schnyder

Given a maximal planar graph  $G$  with exterior face  $uvw$ , there is a labelling of the angles of the internal triangles with labels 1, 2 and 3 such that

- (i) each triangle has labels 1, 2 and 3 in counterclockwise order.
- (ii) all angles at  $u, v$  and  $w$  are labelled 1, 2 and 3, respectively.
- (iii) around each internal vertex the angles of each label appear in a single block.



Theorem 2 If  $F$  is universal for planar graphs with  $n$  vertices then

$$|F| > n + (1 - o(1))\sqrt{n}$$

outerplanar A planar graph which can be obtained from a simple cycle by adding some of its internal diagonals is called outerplanar.



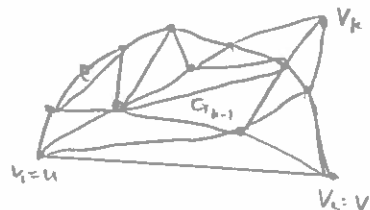
Proposition 5 Every set of  $n$  points in the plane, in general position, supports every outerplanar graph with  $n$  vertices. Moreover, this property characterizes the outerplanar graphs.

Lemma Let  $G$  be a simple planar graph embedded in the plane and  $u = u_1, u_2, \dots, u_k = v$  be a cycle of  $G$ . Then there exists a vertex  $w'$  (resp.  $w''$ ) on the cycle, different from  $u$  and  $v$  and not adjacent to any inside chord (resp. outside chord).

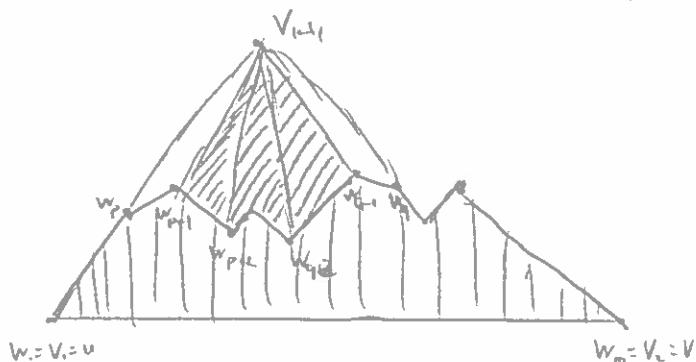
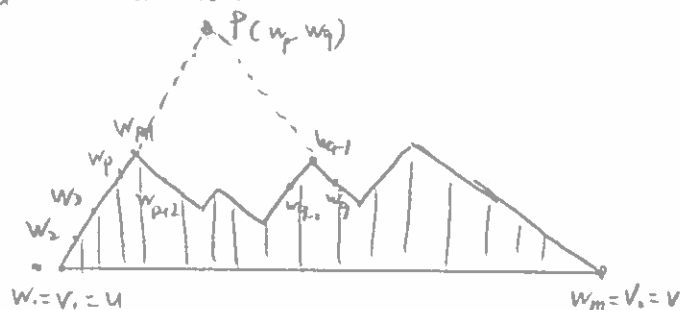
Canonical representation lemma for plane graphs

Let  $G$  be a maximal planar graph embedded in the plane with exterior face  $u, v, w$ . Then there exists a labelling of the vertices  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  meeting the following requirements for every  $4 \leq k \leq n$ .

- (i) The subgraph  $G_{k-1} \subseteq G$  induced by  $v_1, v_2, \dots, v_{k-1}$  is 2-connected, and the boundary of its exterior face is a cycle  $C_{k-1}$  containing the edge  $uv$ .
- (ii)  $v_k$  is in the exterior face of  $G_{k-1}$ , and its neighbours in  $G_{k-1}$  form an (at least 2-element) subinterval of the path  $C_{k-1} - uv$ .



Key idea of induction



To realize this goal, assume that for each vertex  $w_i$  on the exterior face of  $G_k$  we have already defined a subset  $M(k, w_i) \subseteq V(G_k)$  such that

(a)  $w_j \in M(k, w_i)$  iff  $j \geq i$

(b)  $M(k, w_1) \supset M(k, w_2) \supset \dots \supset M(k, w_m)$

(c) For any nonnegative numbers  $\alpha_1, \alpha_2, \dots, \alpha_m$ , if we sequentially translate all vertices in  $M(k, w_i)$  with distance  $\alpha_i$  to the right ( $i = 1, 2, \dots, m$ ), then the embedding of  $G_k$  remains a Fáry embedding.

## Embedding Planar Graphs on the Grid by Walter Schnyder

This paper shows that each plane graph of order  $n \geq 3$  has a straight line embedding on the  $n-2$  by  $n-2$  grid. This embedding is computable in time  $O(n)$ .

cf. J. F  ry, On straight line representation of planar graphs, Acta Sci. Math. Szeged 11 (1948), 229-233.

Characterization: planar graphs are graphs whose incidence relation is the intersection of three total orders.

THEOREM 1.1 Let  $\lambda_1, \lambda_2, \lambda_3$  be three pairwise non parallel straight lines in the plane. Then each plane graph has a straight line embedding in which any two disjoint edges are separated by a straight line parallel to  $\lambda_1, \lambda_2$ , or  $\lambda_3$ .

THEOREM 1.2 Each plane graph with  $n \geq 3$  vertices has a straight-line embedding on the  $n-2$  by  $n-2$  grid.

### Barycentric representations

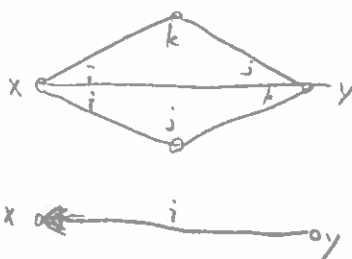
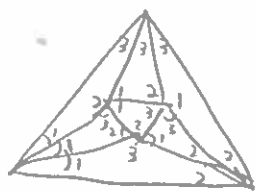
A barycentric representation of a graph  $G$  is an injective function  $v \in V(G) \rightarrow (v_1, v_2, v_3) \in \mathbb{R}^3$  that satisfies the conditions:

- (1)  $v_1 + v_2 + v_3 = 1$  for all vertices  $v$
- (2) For each edge  $\{x, y\}$  and each vertex  $z \in \{x, y\}$ , there is some  $k \in \{1, 2, 3\}$  such that  $x_k < z_k$  and  $y_k < z_k$ .

### normal labeling

A normal labeling of a triangular graph  $G$  is a labeling of the angles of  $G$  with the labels 1, 2, 3 satisfying the conditions

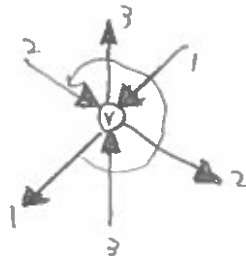
- (1) Each elementary triangle of  $G$  has an angle labeled 1, an angle labeled 2, and an angle labeled 3. The corresponding vertices appear in counterclockwise order.
- (2) The labels of the angles of each interior vertex  $v$  of  $G$  form, in counterclockwise order, a nonempty interval of 1's followed by a nonempty interval of 2's by a nonempty interval of 3's.



THEOREM 4.2. Each triangulated graph has a normal labeling.

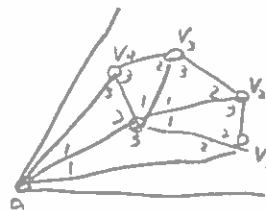
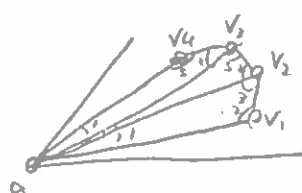
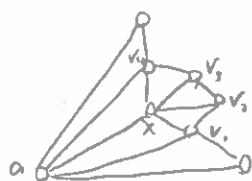
realizer A realizer of a triangular graph  $G$  is a partition of the interior edges of  $G$  in three sets  $T_1, T_2, T_3$  of directed edges such that for each interior vertex  $v$  there holds:

- (1)  $v$  has outdegree one in each of  $T_1, T_2, T_3$ .
- (2) The counterclockwise order of the edges incident on  $v$  is: leaving  $T_1$ , entering  $T_3$ , leaving in  $T_2$ , entering in  $T_1$ , leaving in  $T_3$ , entering in  $T_2$ .



The edge  $\{x, y\}$  is contractible if  $x$  and  $y$  have exactly two common neighbours

Prove THEOREM 4.2 by induction.



### THEOREM 4.5

Let  $G$  be a triangular graph with at least four vertices and let  $T_1, T_2, T_3$  be a realiser of  $G$ . Then each  $T_i$  is a tree including all interior vertices and exactly one exterior vertex and all edges of  $T_i$  are directed toward this exterior vertex. The exterior vertices belonging to  $T_1, T_2, T_3$  are distinct and appear in counterclockwise order.

### THEOREM 4.6

If  $T_1, T_2, T_3$  is a realization of a triangular graph, then for  $i=1,2,3$  the relation  $T_i \cup T_{i+1}' \cup T_{i+2}'$  has no directed cycle (indices are modulo 3).

### THEOREM 6.1

The function  $f: v \in V(G) \rightarrow \frac{1}{2n-5} (v_1, v_2, v_3)$  is a barycentric representation of  $G$  and the labeling of  $G$  that is induced by  $f$  is identical to the given labeling of  $G$

RY 6.2

### COROLLARY 6.2

Let  $a, b$  and  $c$  denote the roots of  $T_1, T_2, T_3$ . Then for any choice of non zero linear positions of  $a, b$  and  $c$  the mapping

$$f: v \rightarrow \frac{1}{2n-5} (v_1 a + v_2 b + v_3 c)$$

is a straight line embedding of  $G$  in the plane spanned by  $c, b, c$ .

### PROPOSITION 6.3

The mapping  $v \in V(G) \rightarrow (v_1, v_2)$  is a straight line embedding of  $G$  on the  $2n-5$  by  $2n-5$  grid.

# HOW TO DRAW A GRAPH By W.T. TUTTE

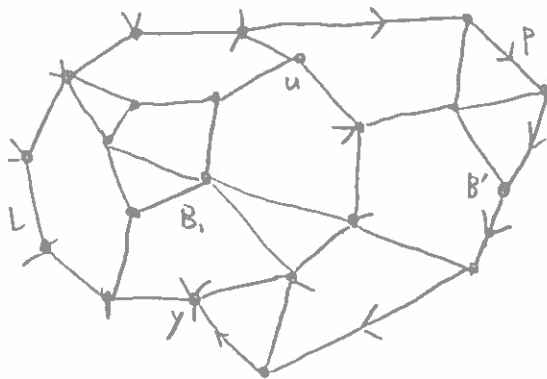
3-connected: A graph  $G$  is 3-connected (nodally 3-connected) if it is simple and non-separable and satisfies the following condition; if  $G$  is the union of two proper subgraphs  $H$  and  $K$  such that  $H \cap K$  consists solely of two vertices  $u$  and  $v$ , then one of  $H$  and  $K$  is a link-graph (arc-graph) with ends  $u$  and  $v$ .

peripheral polygon: Let  $J$  be a polygon of  $G$  and let  $\beta(J)$  denote the number of bridges of  $J$  in  $G$ . If  $\beta(J) \leq 1$  we call  $J$  a peripheral polygon of  $G$ .

(2.1) Let  $G$  be a nodally 3-connected graph. Let  $J$  be a polygon of  $G$  and  $B$  any bridge of  $J$  in  $G$ . Then either  $J$  is a peripheral or  $J$  has another bridge  $B'$  which does not avoid  $B$ .

avoid: If one of residual arc-graphs of  $B$  in  $J$  includes all the vertices of attachment of a second bridge  $B'$  of  $J$  in  $G$  we say that  $B'$  avoids  $B$ . Then  $B$  avoids  $B'$ .

(2.2) Let  $G$  be a nodally 3-connected graph. Let  $K$  be a polygon of  $G$ ,  $B$  a bridge of  $K$  in  $G$ ,  $C$  a subgraph of  $B$ , and  $L$  a branch of  $G$  in  $K$ . Then we can find a peripheral polygon  $J$  of  $G$  such that  $L \subset J$  and  $J \cap C \subseteq K \cap C$ .



(2.3) Let  $G$  be a nodally 3-connected graph which is not a polygon or a link-graph, and let  $L$  be a branch of  $G$ . Then we can find two peripheral polygons  $J_1$  and  $J_2$  of  $G$  such that  $J_1 \cap J_2 = L$ .

(2.4) Let  $G$  be a nodally 3-connected graph,  $K$  a polygon of  $G$ ,  $B$  a bridge of  $K$  in  $G$ , and  $L$  a branch of  $G$  contained in  $K$ . Let  $J_1$  and  $J_2$  be peripheral polygons of  $G$  such that  $L \subseteq J_1 \cap J_2$  and neither  $B \cap J_1$  nor  $B \cap J_2$  is a subgraph of  $K$ . Then we can find a peripheral polygon  $J_3$ , distinct from  $J_1$  and  $J_2$ , such that  $L \subset J_3$ .

(2.5) Let  $G$  be a nodally 3-connected non-null graph. Then we can find a set of  $p_1(G)$  independent peripheral cycles of  $G$ .

peripheral cycle: Consider the set of cycles of a connected graph  $G$ . The rank of this set, the maximum number of cycles independent under mod-2 addition, is

$$p_1(G) = \alpha_1(G) - \alpha_0(G) + 1$$

We refer to the elementary cycles associated with a peripheral polygon as a peripheral cycle.

(2.6) Let  $G$  be a nodally 3-connected non-null graph, with at least two edges, which is not a polygon. Suppose that no edge of  $G$  belongs to more than two distinct peripheral polygons. Then  $G$  has just  $p_1(G) + 1$  distinct peripheral cycles, and they constitute a planar mesh of  $G$ .

(2.7) A peripheral polygon  $K$  of a non-separable graph  $G$  belongs to every planar mesh of  $G$ .

(2.8) If  $M$  is a planar mesh of a nodally 3-connected graph  $G$ , then each member of  $M$  is peripheral.

(2.7) + (2.8) show that a nodally 3-connected graph has at most one planar mesh.

representation: We call  $H$  a representation of  $G$  in  $\Pi$  if it satisfies the following conditions

(i) No edge of  $H$  contains any vertex of  $H$ .

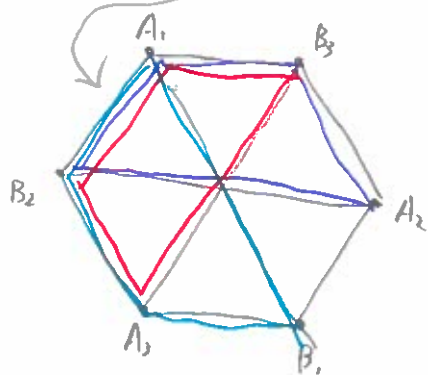
(ii) If  $e$  and  $e'$  are distinct edges of  $G$ , then  $f(e)$  and  $f(e')$  are disjoint.

A graph  $G$  is said to be planar if it has a representation in  $\Pi$ .

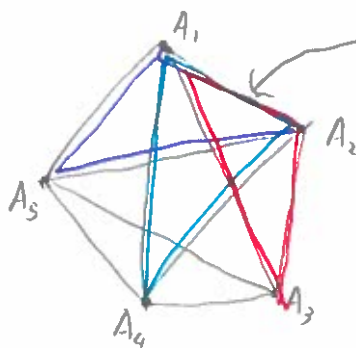
(3.1) Each peripheral polygon of  $H$  bounds a face of  $H$ .

(3.2) If a graph  $G$  has three distinct peripheral polygons with a common edge, then  $G$  is non-planar.

Kuratowski graph of Type I.



Kuratowski graph of Type II

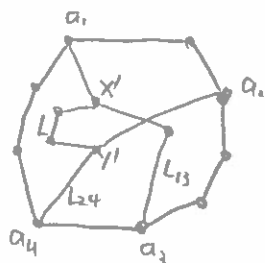


(4.1) Every Kuratowski graph is non-planar.

COROLLARY. Any graph having a Kuratowski subgraph is non-planar.

Crossing diagonals: Let  $J$  be a polygon of a graph  $G$ . Let  $a_1, a_2, a_3, a_4$  be distinct vertices of  $J$  such that  $a_1$  and  $a_3$  separate  $a_2$  from  $a_4$  on  $J$ . Let  $L_{13}$  and  $L_{24}$  be disjoint arc-graphs of  $G$  spanning  $J$ . Then we say that  $L_{13}$  and  $L_{24}$  are crossing diagonals of  $J$ .

(5.1) Given a peripheral polygon of  $G$  with a pair of crossing diagonals we can find a Kuratowski subgraph of  $G$  of Type I.



$(a_1, a_2, a_3, a_4, x', y')$  form a Kuratowski graph of Type I.

(5.2) Let  $J$  be a peripheral polygon of a graph  $G$ . Let  $a, b$ , and  $c$  be distinct vertices of  $J$ . Let  $\gamma_1, \gamma_2$  be  $\gamma$ -graphs of  $G$ , each with ends  $a, b$ , and  $c$ , which spans  $J$ . Suppose further that  $\gamma_1 \cap \gamma_2$  consists solely of the vertices  $a, b$ , and  $c$ . Then we can find a Kuratowski subgraph of  $G$ .

### Barycentric mappings

If  $n < i \leq m$  let  $A(i)$  be the set of all vertices of  $G$  adjacent to  $v_i$ , that is joined to  $v_i$  by an edge. For each  $v_j$  in  $A(i)$  let a unit mass  $m_j$  be placed at point  $f(v_j)$ . Then  $f(v_i)$  is required to be the centroid of the masses  $m_j$ .

Denoting the coordinates of  $f(v_i)$ ,  $1 \leq i \leq m$ , by  $(x_i, y_i)$ .

Define a matrix  $K(G) = \{C_{ij}\}$ ,  $1 \leq i, j \leq m$ , as follows.

If  $i \neq j$  then  $C_{ij}$  is minus the number of edges joining  $v_i$  and  $v_j$ .

If  $i = j$  then  $C_{ij}$  is the degree of  $v_i$ .

Then the foregoing requirement specifies the coordinates  $x_j$  and  $y_j$ , for  $n < j \leq m$ , as the solutions of the equations

$$(5) \quad \sum_{j=1}^m C_{ij} x_j = 0$$

$$(6) \quad \sum_{j=1}^m C_{ij} y_j = 0,$$

where  $n < i < m$ .



$\varphi(i)$ : Choose a line  $l$  in the plane and define  $\varphi(i)$ ,  $1 \leq i \leq m$ , as the perpendicular distance of  $f(v_i)$  from  $l$ , counted positive on one side of  $l$  and negative on the other.

$\varphi$ -active: We call  $v_i$   $\varphi$ -active if there is an adjacent vertex  $v_j$  of  $G$  such that  $\varphi(j) \neq \varphi(i)$ .

positive  $\varphi$ -poles: The nodes  $v_i$  of  $J$  with the greatest value of  $\varphi(i)$  are the positive  $\varphi$ -poles of  $G$ .

The number of positive  $\varphi$ -poles is either 1 or 2.

rising (falling)  $\varphi$ -path: Let  $P$  be a simple path in  $G$ . We call  $P$  a rising (falling)  $\varphi$ -path if each vertex of  $P$  other than the last corresponds to a smaller (greater) value of the function  $\varphi(i)$  than does the immediately succeeding vertex.

(6.1) Suppose that  $v_i$ , where  $n < i \leq m$ , is a  $\varphi$ -active vertex. Then it has adjacent vertices  $v_j$  and  $v_k$  such that  $\varphi(v_j) < \varphi(v_i) < \varphi(v_k)$ .

(6.2) Let  $v_i$  be a  $\varphi$ -active vertex. Then we can find a rising  $\varphi$ -path  $P$  from  $v_i$  to a positive  $\varphi$ -pole, and a falling  $\varphi$ -path  $P'$  from  $v_i$  to a negative  $\varphi$ -pole.

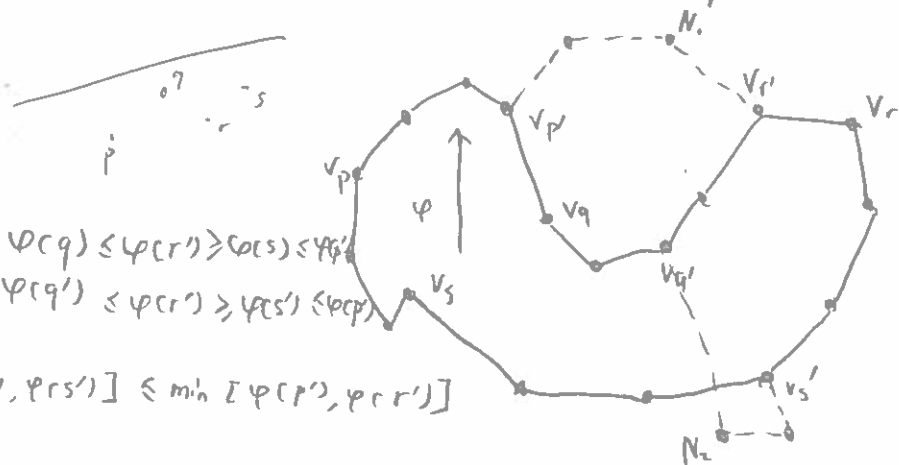
(7.1) Every node of  $G$  is  $\varphi$ -active.

(7.2) Suppose  $v_i \notin V(J)$ . Then  $f(v_i)$  is in the interior of  $Q$ .

(8.1) Let  $K$  be a peripheral polygon of  $G$  such that  $V(K)$  includes just three nodes  $x$ ,  $y$ , and  $z$  of  $G$ . Then  $f(x)$ ,  $f(y)$ , and  $f(z)$  are not collinear.

(8.2) Let  $K$  be a peripheral polygon of  $G$ . Let  $v_p, v_q, v_r$ , and  $v_s$  be nodes of  $G$  in  $V(K)$  such that  $v_p$  and  $v_r$  separate  $v_q$  and  $v_s$  in  $K$ . Then it is not true that

$$(7) \quad \varphi(p) \geq \varphi(q) \leq \varphi(r) \geq \varphi(s) \leq \varphi(p)$$



$$(8) \quad \varphi(p') \geq \varphi(q) \leq \varphi(r') \geq \varphi(s) \leq \varphi(p')$$

$$(9) \quad \varphi(p') \geq \varphi(q') \leq \varphi(r') \geq \varphi(s') \leq \varphi(p')$$

$$\varphi(j) < \max[\varphi(q'), \varphi(s')] \leq \min[\varphi(p'), \varphi(r')]$$

(8.3) The nodes of any peripheral polygon  $K$  of  $G$  are mapped by  $f$  onto distinct points of the plane, no three of which are collinear.

(8.4) Let  $L$  be a branch of  $G$  having just  $t \geq 1$  internal vertices, and let its ends be  $a$  and  $b$ . Then  $f(a)$  and  $f(b)$  are distinct and  $f$  maps the internal vertices onto  $t$  distinct points of the segment  $f(a)f(b)$  subdividing it into  $t+1$  equal parts. Moreover, the order of the vertices from  $a$  to  $b$  in  $L$  agrees with that of their images in  $f(a)f(b)$ .

(8.5) Let  $K$  be any peripheral polygon of  $G$ . Then  $f$  maps the nodes of  $G$  in  $K$  onto the vertices of a (geometrical) convex polygon  $Q_K$  so that the cyclic order of nodes in  $K$  agrees with that of vertices in  $Q_K$ .

(8.6) Let  $e$  be any edge of  $R$ . Then just two distinct peripheral polygons of  $G$  pass through  $e$ , and the two corresponding regions  $R_K$  lie on opposite sides of the segment  $f(e)$ .

### Barycentric representations.

(9.1)  $\underbrace{\delta(A)}_{\substack{\text{number of distinct} \\ \text{peripheral polygons} \\ \text{K of G such that } A \in R_K}} = 1 \text{ for each } A \text{ in } \underbrace{S}_{\substack{\uparrow \\ \text{point}}}$   $\hookrightarrow H$ 's complementary set in the plane.

(9.2) Let  $G$  be a nodally 3-connected graph having no Kuratowski subgraph. Let  $J$  be a peripheral polygon of  $G$  which includes just  $n \geq 3$  nodes of  $G$ . Let  $Q$  be an  $n$ -sided convex polygon in the Euclidean plane. Then there is a unique barycentric representation of  $G$  on  $Q$  mapping the nodes of  $G$  occurring on  $J$  into the vertices of  $Q$  in any arbitrary specified way preserving the cyclic order.

(9.3) Let  $G$  be a nodally 3-connected graph having at least one polygon. Then if  $G$  has no Kuratowski subgraph we can construct a convex representation of  $G$ .

### Straight representations

(10.1) Let  $G$  be the union of two proper subgraphs  $H$  and  $K$  such that  $H \cap K$  is either null or a vertex-graph. Let  $M_H$  and  $M_K$  be planar meshes of  $H$  and  $K$  respectively. Then  $M_H \cup M_K$  is a planar mesh of  $G$ . Moreover, any planar mesh of  $G$  can be represented in this form.

(10.6) Let  $G$  be a graph having a planar mesh  $M$ . Then each subgraph of  $G$  has a planar mesh.

(10.7) If a graph has a planar mesh it has no Kuratowski subgraph.

(10.8) Let  $G$  be any simple graph having a planar mesh. Then by adding new links to  $G$ , with ends in  $V(G)$ , we can construct a nodally 3-connected graph  $T$  having a planar mesh.

(10.9) If  $G$  is a simple graph having a planar mesh we can find a straight representation of  $G$  in this plan

{MIT 6.889} by Erik Demaine, Shay Mozes, Christian Sommer, Ziomak Lazari

"Solve your favourite problems faster for graphs that matter!"

Survey: Problems general vs. planar

- single-source shortest paths (arbitrary weights)  $O(nm)$  [Bellman-Ford]  $O(n \frac{\lg^2 n}{\lg \lg n})$  [Mozes & Wulff-Nilsen - ESA 2010]

- nonnegative weights  $O(n \lg n + m)$  [Dijkstra] + [Fredman & Tarjan - JACM 1987]  $O(n)$  [Henzinger, Klein, Rao, Subramanian - JCSS 1997]

- maximum flow  $O(nm \lg n)$  [Goldberg & Tarjan 1986]  $O(n \lg n)$  [Borradale & Klein - JACM 2009]  
 $O(m^{\frac{2}{3}} \lg n \lg u)$  [Goldberg & Rao 1997]

- undirected  $O(n \lg \lg n)$  [Italiano, Nussbaum, Samkowski, Wulff-Nilsen - STOC 2011]

- multi-terminal  $O(n \lg^3 n)$  [Borradale, Klein, Mozes, Nussbaum, Wulff-Nilsen - FOCS 2011]

- min. spanning tree  $O(n)$  rand.  $O(n)$  det  
[Karger, Klein, Tarjan 1985]

# << Algorithm Design >> by Jon Kleinberg and Éva Tardos

## § Chapter 13: Randomized Algorithms

When one thinks about random process, it is usually in one of two distinct ways.

- 1) average-case analysis (randomly generated input)
- 2) consider algorithms that behave randomly  $\rightarrow$  randomized algorithm

### 13.1 A First Application: Contention Resolution

Suppose  $n$  processes  $P_1, P_2, \dots, P_n$ , each competing for access to a single shared database. Time is being divided into discrete rounds. Database can be accessed by at most one process in a single round. Processes can't communicate with one another at all.

#### Algorithm

each process will attempt to access the database in each round with probability  $p$ , independently of the decisions of the other processes.

#### Analyzing the Algorithm

let  $A[i, t]$  denote the event that  $P_i$  attempts to access database in round  $t$ .  $\Pr[A[i, t]] = p$   
let  $S[i, t]$  denote the event that  $P_i$  succeeds in accessing the database in round  $t$ .  $\Pr[\overline{A[i, t]}] = 1 - p$

$$\Pr[S[i, t]] = \Pr[A[i, t]] \cdot \prod_{j \neq i} \Pr[\overline{A[j, t]}] = p (1-p)^{n-1}$$

$$f(p) = p (1-p)^{n-1}$$

$$f'(p) = (1-p)^{n-1} - p \cdot (n-1) (1-p)^{n-2}$$

$$(1-p)^{n-1} - (n-1) \cdot p (1-p)^{n-2} \geq 0$$

$$(1-p) \geq (n-1)p$$

$$\frac{1}{p} - 1 \geq n-1$$

$$p \leq \frac{1}{n}$$

$$\text{we set } p = \frac{1}{n}, \quad \Pr[S[i, t]] = \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n-1}$$

(13.1) (a) The function  $\left(1 - \frac{1}{n}\right)^n$  converges monotonically from  $\frac{1}{4}$  up to  $\frac{1}{e}$  as  $n$  increases from 2.

(b) The function  $\left(1 - \frac{1}{n}\right)^{n-1}$  converges monotonically from  $\frac{1}{2}$  down to  $\frac{1}{e}$  as  $n$  increases from 2.

$$\frac{1}{en} \leq \Pr[S[i, t]] \leq \frac{1}{2n}$$

$\hookrightarrow \Theta\left(\frac{1}{n}\right)$

let  $F[i, t]$  denote the "failure event" that process  $P_i$  does not succeed in any of the rounds 1 through  $t$ .

$$\Pr[F[i, t]] = \Pr\left[\bigcap_{r=1}^t \overline{S[i, r]}\right] = \left[1 - \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n-1}\right]^t \leq \left(1 - \frac{1}{en}\right)^t \stackrel{\text{set } t = \lceil en \rceil}{\leq} \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$$

$$t = \lceil \epsilon n \rceil \cdot c \ln n$$

$$\Pr[F(i, t)] \leq \left(1 - \frac{1}{\epsilon n}\right)^t = \left(\left(1 - \frac{1}{\epsilon n}\right)^{\lceil \epsilon n \rceil}\right)^{c \ln n} \leq e^{-c \ln n} = n^{-c}$$

Conclusion: After  $\Theta(n)$  rounds ( $t = \lceil \epsilon n \rceil$ ), the probability that  $P_i$  has not succeeded is bounded by a constant ( $\frac{1}{e}$ ); and between then and  $\Theta(n \ln n)$ , this probability drops to a quantity that is quite small, bounded by an inverse polynomial in  $n$ .

Waiting for All Processes to Get Through

$$F_i = \bigcup_{i=1}^n F(i, t)$$

(13.2) [The Union Bound] Given events  $E_1, E_2, \dots, E_n$ , we have

$$\Pr\left[\bigcup_{i=1}^n E_i\right] \leq \sum_{i=1}^n \Pr[E_i].$$

$$\therefore \Pr[F_i] \leq \sum_{i=1}^n \Pr[F(i, t)]$$

choose  $t = \lceil \epsilon n \rceil (c \ln n)$ ,  $\Pr[F(i, t)] \leq n^{-c}$

choose  $t = 2 \lceil \epsilon n \rceil \ln n$

$$\Pr[F_i] \leq \sum_{i=1}^n \Pr[F(i, t)] \leq n \cdot n^{-2} = n^{-1}$$

(13.3) With probability at least  $1 - n^{-1}$ , all processes succeed in accessing the database at least once within  $t = 2 \lceil \epsilon n \rceil \ln n$  rounds.

## 13.2 Finding the Global Minimum Cut

undirected graph  $G = (V, E)$ , define a cut of  $G$  to be a partition of  $V$  into two nonempty sets  $A$  and  $B$ . For a cut  $(A, B)$ , the size of  $(A, B)$  is the number of edges with one end in  $A$  and the other in  $B$ . A global minimum cut is a cut of minimum size.

(13.4) There is a polynomial-time algorithm to find a global min-cut in an undirected graph  $G$ .  
 convert to directed graph  
 fix  $s$ . for every  $t \in V - \{s\}$ , run push-relabel.  
 the best among these will be a global min-cut of  $G$ .

David Karger 1992.

Algorithm. (Contraction Algorithm)

↳ works with connected multigraph.

The Contraction Algorithm applied to a multigraph  $G = (V, E)$ :

For each node  $v$ , we will record the set  $S(v)$  of the nodes that have been contracted into  $v$ .

Initially  $S(v) = \{v\}$  for each  $v$

If  $G$  has two nodes  $v_1$  and  $v_2$ , then return the cut  $(S(v_1), S(v_2))$ .

Else choose an edge  $e = (u, v)$  of  $G$  uniformly at random

Let  $G'$  be the graph resulting from the contraction of  $e$ ,  
with a new node  $z_{uv}$  replacing  $u$  and  $v$ .

Define  $S(z_{uv}) = S(u) \cup S(v)$

Apply the Contraction Algorithm recursively to  $G'$

Endif

### Analyzing the Algorithm

(13.5) The Contraction Algorithm returns a global min-cut of  $G$  with probability at least  $\frac{1}{\binom{n}{2}}$

Suppose the global min-cut has size  $k$ , a set  $F$  of  $k$  edges with one end in  $A$  and the other in  $B$ .  $\hookrightarrow$  every node in  $G$  has degree at least  $k$ .

We want an upper bound on the probability that an edge in  $F$  is contracted, and for this we need a lower bound on the size of  $E$ .

$$|E| \geq \frac{1}{2} kn$$

Hence the probability that an edge in  $F$  is contracted is at most

$$\frac{k}{\frac{1}{2} kn} = \frac{2}{n}.$$

Consider the situation after  $j$  iterations, there are  $n-j$  supernodes in  $G'$ .

Thus  $G'$  has at least  $\frac{1}{2} k(n-j)$  edges. So the probability that an edge of  $F$  is contracted in the next iteration  $j+1$  is at most.

$$\frac{k}{\frac{1}{2} k(n-j)} = \frac{2}{n-j}$$

We write  $E_j$  for the event that an edge of  $F$  is not contracted in iteration  $j$ , then we have

shown  $\Pr[E_1] \geq 1 - \frac{2}{n}$  and  $\Pr[E_{j+1} | E_1 \cap E_2 \cap \dots \cap E_j] \geq 1 - \frac{2}{n-j}$

We are interested in lower-bounding the quantity  $\Pr[E_1 \cap E_2 \dots \cap E_{n-2}]$ .

$$\begin{aligned} & \Pr[E_1] \cdot \Pr[E_2 | E_1] \cdots \Pr[E_{j+1} | E_1 \cap E_2 \cdots \cap E_j] \cdots \Pr[E_{n-2} | E_1 \cap E_2 \cdots \cap E_{n-3}] \\ & \geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{n-j}\right) \cdots \left(1 - \frac{2}{3}\right) \\ & = \left(\frac{n-2}{n}\right) \cdot \left(\frac{n-3}{n-1}\right) \left(\frac{n-4}{n-2}\right) \cdots \left(\frac{2}{5}\right) \left(\frac{2}{4}\right) \left(\frac{1}{3}\right) \\ & = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \end{aligned}$$

So we know that a single run of the Contraction Algorithm fails to find a global min-cut with probability at most  $(1 - 1/\binom{n}{2})$ .

If we run the algorithm  $\binom{n}{2}$  times,

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}} \leq \frac{1}{e}$$

If we run the algorithm  $\binom{n}{2} \ln n$  times

$$\left[\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}}\right]^{\ln n} \leq e^{-\ln n} = n^{-1}$$

### Further Analysis: The Number of Global Minimum Cuts

Given an undirected graph  $G = (V, E)$ , what is the maximum number of global min-cuts it can have (as a function of  $n$ )?

undirected graph has  $\binom{n}{2}$  global min-cuts.

(13.6) An undirected graph  $G = (V, E)$  on  $n$  nodes has at most  $\binom{n}{2}$  global min-cuts.

Let  $G$  be a graph, and let  $C_1, \dots, C_r$  denote all its global min-cuts.

Let  $E_i$  denote the event that  $C_i$  is returned by the Contraction Algorithm,

let  $E = \bigcup_{i=1}^r E_i$  denote the event that the algorithm returns any global min-cut.

$$\Pr[E] = \Pr\left[\bigcup_{i=1}^r E_i\right] = \sum_{i=1}^r \Pr[E_i] \geq r / \binom{n}{2}$$

$$\Pr[E] \leq 1 \Rightarrow r \leq \binom{n}{2}$$

### 13.3 Random Variables and Their Expectations

random variable: Given a probability space, a random variable  $X$  is a function from the underlying sample space to the natural numbers, such that for each natural number  $j$ , the set  $X^{-1}(j)$  of all sample points taking value  $j$  is an event.

$\Pr[X=j]$  as loose shorthand for  $\Pr[X^{-1}(j)]$ .

expectation: the "average value" assumed by  $X$ .

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X=j]$$

$$S = \sum_{j=1}^{\infty} j (1-p)^j = 1(1-p)^1 + 2(1-p)^2 + 3(1-p)^3 + \dots$$

$$(1-p)S = 1(1-p)^2 + 2(1-p)^3 + 3(1-p)^4 + \dots$$

$$S - (1-p)S = (1-p)^1 + (1-p)^2 + (1-p)^3 + \dots = \frac{(1-p)}{p}$$

$$pS = \frac{(1-p)}{p} \Rightarrow S = \frac{(1-p)}{p^2}$$

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X=j] = \sum_{j=1}^{\infty} j (1-p)^{j-1} p = \frac{p}{1-p} \sum_{j=1}^{\infty} j (1-p)^{j-1} = \frac{p}{1-p} \cdot \frac{(1-p)}{p^2} = \frac{1}{p}$$

(13.8) Linearity of Expectation. Given two random variables  $X$  and  $Y$  defined over the same probability space, we can define  $X+Y$  to be the random variable equal to  $X(\omega) + Y(\omega)$  on a sample point  $\omega$ . For any  $X$  and  $Y$ , we have

$$E[X+Y] = E[X] + E[Y].$$

(13.10)  $H(n) = \sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  harmonic number  $H(n)$ .

$$\int_1^{n+1} \frac{1}{x} dx = \ln(n+1)$$

$$\ln(n+1) < H(n) < 1 + \ln n, \text{ and more loosely } H(n) = \Theta(\log n).$$

### Conditional Expectation

Suppose we have a random variable  $X$  and an event  $E$  of positive probability. Then we define the conditional expectation of  $X$ , given  $E$ , to be the expected value of  $X$  computed only over the part of sample space corresponding to  $E$ .

$$E[X|E] = \sum_{j=0}^{\infty} j \cdot P[X=j|E].$$

### 13.4 A Randomized Approximation Algorithm for MAX 3-SAT

Algorithm: set each variable  $x_1, \dots, x_n$  independently to 0 or 1 with probability  $\frac{1}{2}$  each. Let  $Z_i = 1$  if clause  $C_i$  is satisfied, and 0 otherwise.

Thus  $Z = Z_1 + Z_2 + \dots + Z_k$ .  $E[Z_i]$  is equal to the probability that  $C_i$  is satisfied.

In order for  $C_i$  not to be satisfied, each of its three variables must be assigned the value that fails to make it true,  $(\frac{1}{2})^3 = \frac{1}{8}$ , so  $E[Z_i] = \frac{7}{8}$ .

$$E[Z] = E[Z_1] + E[Z_2] + \dots + E[Z_k] = \frac{7}{8}k.$$

(13.14) Consider a 3-SAT formula, where each clause has three different variables. The expected number of clauses satisfied by a random assignment is within an approximation factor  $\frac{7}{8}$  of optimal.

(13.15) For every instance of 3-SAT, there is a truth assignment that satisfies at least a  $\frac{7}{8}$  fraction of all clauses.

Corollary: Every instance of 3-SAT with at most seven clauses is satisfiable.

(13.16) There is a randomized algorithm with polynomial expected running time that is guaranteed to produce a truth assignment satisfying at least  $\frac{7}{8}$  fraction of all clauses.  $P \geq \frac{1}{8k}$   $n \leq 8/k$