

matrix representation of graph

① computing path

② Google's page rank

how connected is the graph?

③ useful for graph drawing eigenvalue

major topics:

TODO: ① spectral graph theory

② Ramsey's theory

Complete graph: color ^{edge} red or blue,

smallest number of

vertices.

clique:



$$R(3,3)=3$$

= size of

$$R(4,4)=18$$

$$R(5,5)=43$$

smallest complete graph so that for all red/blue colorings there exists either a red 3-clique or a blue 3-clique.



③ Random walks on graphs

Give a graph, how many perfect matchings are there?① ④ planar graphs

characterization

drawn on a plane

without edges crossing.

Pointers for things to look at:

To get started, with planar graphs

① Try for next time:

things about planar graphs

how to tell?

algorithms for planar detection

② graph drawing② GRAPH DRAWING

ALGORITHMS FOR THE VISUALIZATION OF GRAPHS

italian: GIUSEPPE DI BATTISTA

german: PETER EADES

italian: ROBERTO TAMASSIA

greek: IOANNIS G. TOLLIS

③ Graph Theory

with Applications

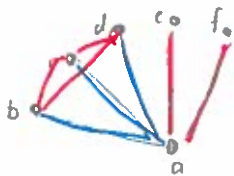
J.A. Bondy

U.S.R. Murty

read chapters on planar graphs

Proof that $R(3,3) = 6$

In a complete graph K_6 for node a adjacent to all other 5 nodes, color them red or blue.



No matter how you color it, at least 3 edges are colored with the same color (either red or blue).

WLOG, let's say a is connected with b, c, d colored blue.

To avoid a clique with a , bc, cd, bd has to be colored red. In this way, b, c, d form a red 3-clique.

Thus, K_6 must have either a red 3-clique or a blue 3-clique.

□

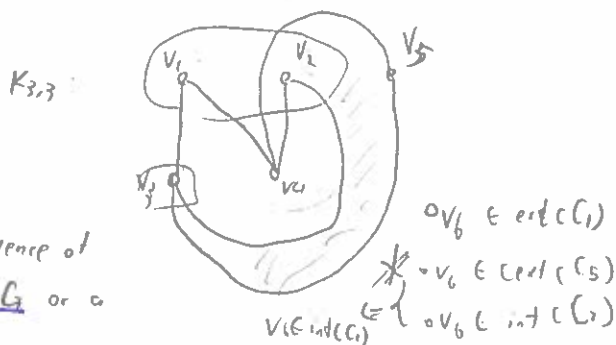
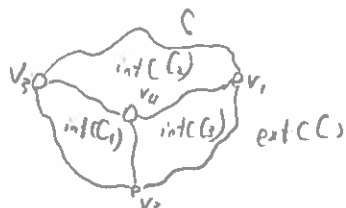
[1.1] P5: planar graph: A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a planar graph, and such a drawing is called a planar embedding of the graph.

[10.1] P243: planar graph: A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends.

[10.1] P244: plane graph: we often refer to a planar embedding \tilde{G} of a planar graph G as a plane graph; and we call the vertices of \tilde{G} points and its edges lines.

[10.1] P245: THE JORDAN CURVE THEOREM: Any simple closed curve C in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets.

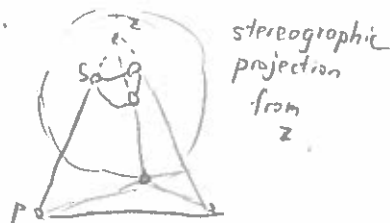
[10.1] P245: K_5 is nonplanar: \hookrightarrow called interior $\text{int}(C)$ and exterior $\text{ext}(C)$.



[10.1] P246: subdivision: Any graph derived from a graph G by a sequence of edge subdivisions is called a subdivision of G or a G -subdivision.

[10.1] P246: Proposition 10.3: A graph G is planar if and only if every subdivision of G is planar.

[10.1] P247: Theorem 10.4: A graph G is embeddable on the plane if and only if it is embeddable on the sphere.



[10.2] P250: Proposition 10.5: Let G be a planar graph, and let f be a face in some planar embedding of G . Then G admits a planar embedding whose outer face has the same boundary as f .

Proof idea - consider an embedding \tilde{G} of G on the sphere.

[10.2] P250: Theorem 10.6: THE JORDAN-SCHÖNFLIUS THEOREM?

Any homeomorphism of a simple closed curve in the plane onto another simple closed curve can be extended to a homeomorphism of the plane.

[10.2] P251: Theorem 10.7: In a nonseparable plane graph other than K_1 or K_2 , each face is bounded by a cycle.

[10.2] P252: dual: Given a plane graph G , one can define a second graph G^* as follows. Corresponding to each face f of G there is a vertex f^* of G^* , and corresponding to each edge e there is an edge e^* of G^* . Two vertices f^* and g^* are joined by the edge e^* in G^* iff their corresponding faces f and g are separated by the edge e in G . Observe that if e is a cut edge of G , then $f=g$, so e^* is a loop of G^* ; conversely, if e is a loop of G , the edge e^* is a cut edge of G^* . The graph G^* is called the dual of G .

[10.3] P59 EULER'S FORMULA:

For a connected plane graph G .

Proof by induction

$$v(G) - e(G) + f(G) = 2$$

[10.3] P59 Corollary 10.20:

All planar embeddings of a connected planar graph have the same number of faces.

[10.3] P59 Corollary 10.21:

Let G be a simple planar graph on at least three vertices. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of G is a triangulation.

[10.3] P59 Corollary 10.22:

Every simple planar graph has a vertex of degree at most five.

[10.3] P59 Corollary 10.23:

K_5 is nonplanar

proof: If K_5 were planar, Corollary 10.21 would give $10 = e(K_5) \leq 3v(K_5) - 6 = 9$.

[10.3] P59 Corollary 10.24:

$K_{3,3}$ is nonplanar

proof: If $K_{3,3}$ were planar, $K_{3,3}$ has no cycle of length ≤ 4 , so every face of G has degree ≥ 4 . By Theorem 10.10, $4f(G) \leq \sum_{f \in F} d(f) = 2e(G) = 18$

[10.2] P53 Theorem 10.10: If G is a plane graph

$$\sum_{f \in F} d(f) = 2m$$

Euler's formula implies that $\therefore \frac{f(G)}{\text{integer}} \leq 4$

$$\frac{v(G) - e(G) + f(G)}{6 - 9 + f(G)} = 2 \Rightarrow f(G) = 5$$

[10.5] P68 Theorem 10.30: KURATOWSKI'S THEOREM

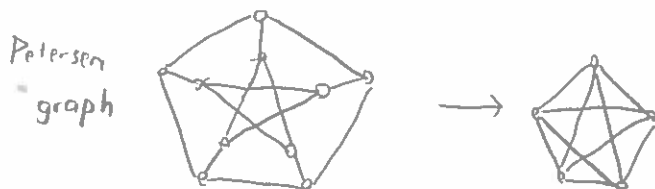
A graph is planar iff it contains no subdivision of either K_5 or $K_{3,3}$.

A subdivision of K_5 or $K_{3,3}$ is consequently called a Kuratowski subdivision.

[10.5] P68 minor: A minor of a graph is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions?

Alternatively, consider a partition (V_0, V_1, \dots, V_k) of V such that $G[V_i]$ is connected, $1 \leq i \leq k$, and let H be the graph obtained from G by deleting V_0 and shrinking each induced subgraph $G[V_i]$, $1 \leq i \leq k$, to a single vertex. The any spanning subgraph F of H is a minor of G .

e.g. K_5 is a minor of the Petersen graph because it can be obtained by contracting the five 'spoke' edges of the latter graph.



important:

- 1) any graph which contains an F -subdivision also has an F -minor
- 2) provided that F is a graph of maximum degree three or less, any graph which has an F -minor also contains an F -subdivision.

If F is a minor of G , we write $F \leq G$.

Ex. find a $K_{3,3}$ minor from Petersen graph.

[10.5] P69 WAGNER'S THEOREM

A graph is planar iff it has no Kuratowski minor.

[1.1] P_3 loop: an edge with identical ends is called a loop

link: an edge with distinct ends is called a link

parallel edges: two or more links with the same pair of ends are said to be parallel edges

simple: a graph is simple if it has no loops or parallel edges.

[Di Battista et al.]

{1} P_6 drawing: a drawing Γ of a graph (digraph) G is a function which maps each vertex v to a distinct point $\Gamma(v)$ and each edge (u, v) to a simple open Jordan curve $\Gamma(u, v)$ with endpoints $\Gamma(u)$ and $\Gamma(v)$.

{1} P_7 drawing planar: a drawing Γ is planar if no two distinct edges intersect.

A graph is planar if it admits a planar drawing

{1} P_8 connected: a graph is connected if there is a path between u and v for each pair (u, v) of vertices

cut vertex: A cut vertex in graph G is a vertex whose removal disconnects G .

biconnected: A connected graph with no cut vertices is biconnected.

blocks: The maximal biconnected subgraphs of a graph are its blocks. (sometimes called biconnected components).
A graph is planar iff its blocks are planar.

{1} P_9 important: ?1) The skeleton of a convex polyhedron is a planar triconnected graph.

?2) A planar triconnected graph has a unique embedding, up to a reversal of the circular ordering of the neighbors of each vertex

{2.1} P_{12} drawing convention: A drawing convention is a basic rule that the drawing must satisfy to be admissible.

- Poly line drawing
- Straight-line drawing
- Orthogonal drawing
- Grid Drawing
- Planar Drawing
- Upward (resp. downward) Drawing

{2.1} P_{14} aesthetics: aesthetics specify graph properties of the drawing that we would like to apply as much as possible, to achieve readability.

constraints:

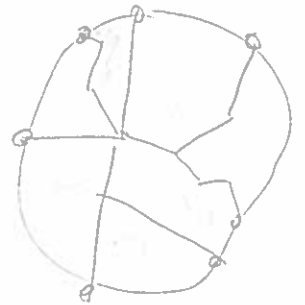
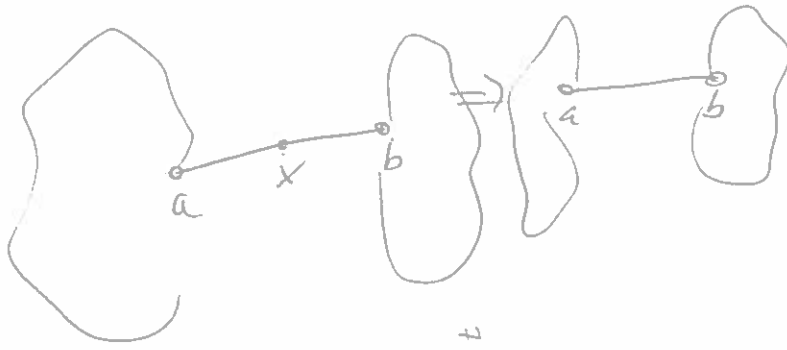
refer to specific subgraphs or sub drawings

efficiency:

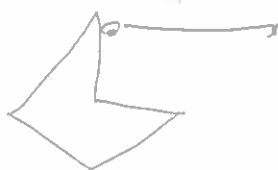
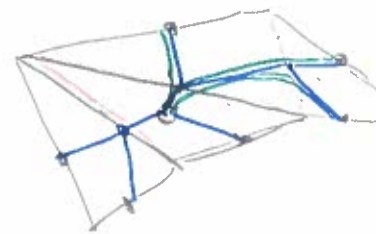
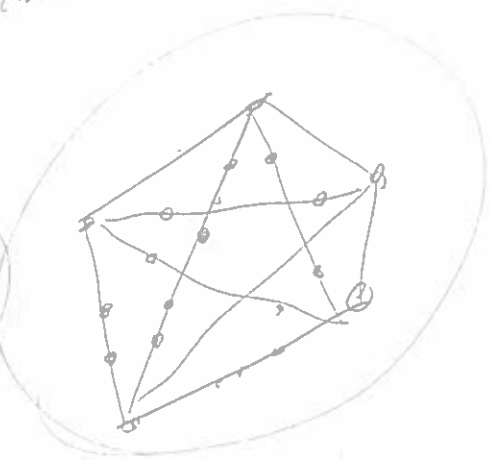
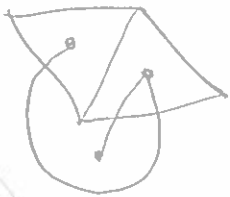
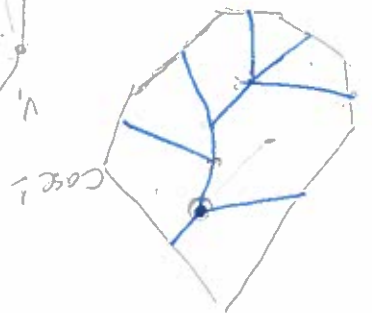
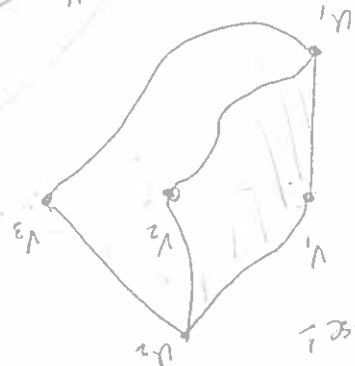
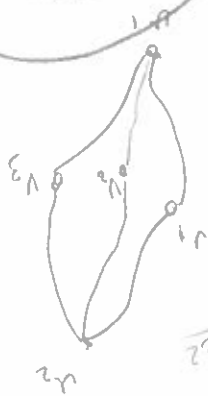
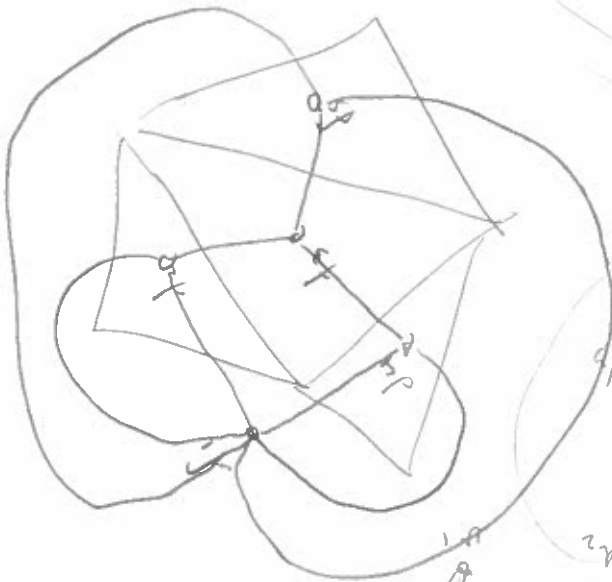
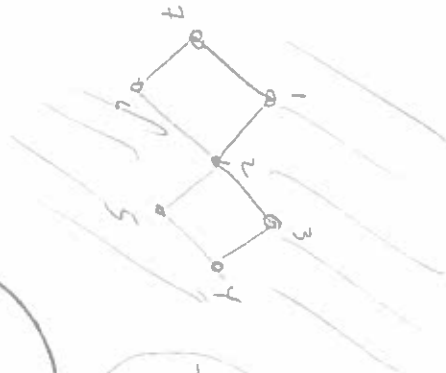
interactive applications

- crossings: minimize
- area: minimize (good drawing: straight-line drawing where $\text{distance}(u, v) \geq 1$)
convex hull
- total edge length: minimize
- maximum edge length: minimize
- uniform edge length: minimize variance of lengths of the edges
- total bends: minimize (important for orthogonal drawings)
- maximum bends: minimize
- uniform bends: minimize
- angular resolution: maximize (straight-line drawing)
- aspect ratio: minimize longest-side: shortest side \square

- symmetry



1 2 3 4 5 6 7



[The Left-Right Planarity Test] by Ulrik Brandes

- Efficient Planarity Testing by John Hopcroft & Robert Tarjan (1974)
- A Depth-first-search Characterization of Planarity by H. De Fraysseix & P. Rosenstiehl (1982)
- On the Realization of Complexes in Euclidean Spaces 吴文俊 Wen-Tsun Wu (1955)

{P₄} There are only two significant ways to draw a simple cycle planarly, namely clockwise and counterclockwise.

{P₄} Testing planarity amounts to deciding whether there is a consistent simultaneous orientation of all cycles.

{P₅} In any planar drawing the back edges can be partitioned into left and right depending on whether their fundamental cycle is counterclockwise or clockwise.

{P₆} In the oriented graph, we denote by $E^+(v) = \{(v, w) \in E : w \in V\}$ the set of all outgoing edges of $v \in V$, so that $E = \bigcup_{v \in V} E^+(v)$.

{P₆} A DFS traversal yields a bipartition $E = T \cup B$ of the edges, where those in T are called tree edges and the non-tree edges in B are called back edges. We write $u \rightarrow v$ for $(u, v) \in T$ and $v \hookrightarrow w$ for $(v, w) \in B$.

{P₆} fundamental cycle: $C(v \hookrightarrow w) = w \xrightarrow{+} v \hookrightarrow w$
overlapping. Two cycles are called overlapping, if they share an edge.

{P₆} Lemma 3 Let $G = (V, T \cup B)$ be a DFS-oriented graph.

(1) The fundamental cycles are exactly the simple directed cycles of G .

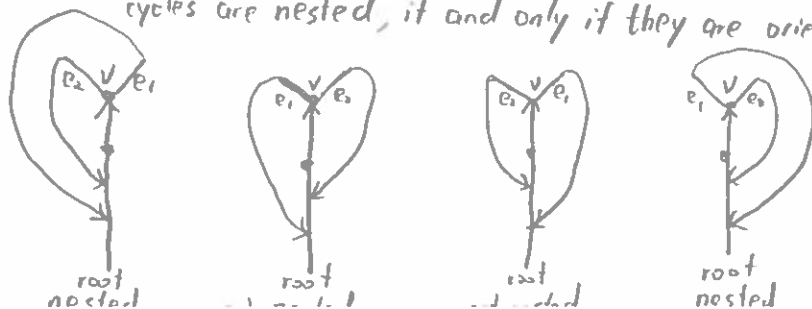
(2) Two distinct fundamental cycles are either disjoint, or their intersection forms a tree path

{P₇} fork: For two overlapping cycles, the last edge $u \rightarrow v$ on the shared tree path together with succeeding edges $e_1 = (v, w_1)$, $e_2 = (v, w_2)$ on each cycle is called their fork, and v its branching point

{P₇} a linearization of the cycle order: It is defined by splitting the clockwise order restricted to outgoing edges at the incoming tree edge, or between any two consecutive outgoing edges if v is the root of a DFS tree.

{P₇} nested: Two overlapping fundamental cycles are called nested, if the part of one cycle that is not common to both is drawn completely inside the other cycle.

{P₈} Observation 1: In a planar drawing of a DFS-oriented graph $G = (V, T \cup B)$, two overlapping cycles are nested, if and only if they are oriented alike.



{P₆} return points: The return points of a tree edge $v \rightarrow w \in T$ are the ancestors u of v with $u \xrightarrow{+} v \rightarrow w \xrightarrow{*} x \hookrightarrow u$ for some descendant x of w .
The return points of a vertex $v \in V$ are formed by the union of all return points of outgoing edges $(v, w) \in E^+(v) \subseteq T \cup B$.

{P₈} lowpoint: The lowpoint of an edge is its lowest return point, if any, or its source if none exists.

{P₉} Observation 2: In a planar drawing of a connected DFS-oriented graph $G = (V, T \cup B)$ with the root of the DFS tree on the outer face, overlapping fundamental cycles are nested according to their lowpoint order.

{P₃} left, right: the side of a back edge in a planar drawing is right, if its fundamental cycle is oriented clockwise, and left otherwise.

{P₉} LR partition: Let $G = (V, T \cup B)$ be a DFS-oriented graph. A partition $B = L \cup R$ of its back edges into two classes, referred to as left and right, is called left-right partition, or LR partition for short, if every fork consisting of $u \rightarrow v \in T$ and $e_1, e_2 \in E^+(v)$
 (1) all return edges of e_1 ending strictly higher than $\text{lowpt}(e_2)$ belong to one class
 and
 (2) all return edges of e_2 ending strictly higher than $\text{lowpt}(e_1)$ belong to the other.

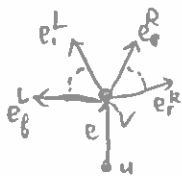
{P₁₀} Left-Right Planarity Criterion: A graph is planar if and only if it admits an LR partition.

{P₁₀} aligned: An LR partition is called aligned, if all return edges of a tree edge e that return to $\text{lowpt}(e)$ are on the same side.

{P₁₁} Lemma 6: Any LR partition can be turned into an aligned LR partition.

{P₁₂} $e_1 \prec e_2$: we have to define $e_1 \prec e_2$ if and only if the lowpoint of e_1 is strictly lower than that of e_2 . If both have the same lowpoint, but say, only e_2 has another return point, we say that e_2 is chordal and let $e_1 \prec e_2$.

{P₁₂} Definition 1 LR Ordering: Given an LR partition, let $e_1^L \prec \dots \prec e_\ell^L$ be the left outgoing edges of a vertex v and $e_1^R \prec \dots \prec e_r^R$ its right outgoing edges. If v is not the root, let u be its parent. The clockwise left-right ordering, or LR ordering for short, of the edges around v is defined as follows:



$(u, v),$
 $L(e_1^L), e_1^L, R(e_1^L), \dots, L(e_\ell^L), e_\ell^L, R(e_\ell^L),$
 $L(e_1^R), e_1^R, R(e_1^R), \dots, L(e_r^R), e_r^R, R(e_r^R)$

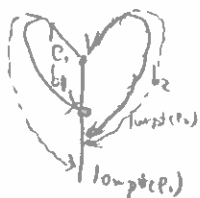


where (u, v) is absent if v is the root, and $L(e)$ and $R(e)$ denote the left and right incoming back edges whose cycles share e . For two back edges $b_1 = x_1 \hookrightarrow v, b_2 = x_2 \hookrightarrow v \in R(e)$ let $z \rightarrow x, (x, y_1), (x, y_2)$ be the fork of $C(b_1), C(b_2)$. Then, b_1 comes after b_2 in $R(e)$ if and only if $(x, y_1) \prec (x, y_2)$. If $b_1, b_2 \in L(e)$, the order is reversed.

{P12} Lemma 8. Given an LR partition, LR ordering yields a planar embedding.

{P14} Corollary 9. Let $G = (V, T \cup B)$ be a DFS-oriented graph. For a pair of back edges $b_1, b_2 \in B$ with overlapping fundamental cycles let $v_i \rightarrow \dots \rightarrow v_k$ be the tree path of the intersection and $(v_{k-1}, v_k), e_i, e_i$ the corresponding fork with $e_i \xrightarrow{*} b_1$ and $e_i \xrightarrow{*} b_2$. Then, b_1 and b_2 are subject to

- a different-constraint, iff $\text{lowpoint}(e_2) < \text{lowpoint}(b_1)$ and $\text{lowpoint}(e_1) < \text{lowpoint}(b_2)$
- a same-constraint, iff $\text{lowpoint}(e') < \min\{\text{lowpoint}(b_1), \text{lowpoint}(b_2)\}$ for some $e' = (v_i, w) \in T \cup B, 1 \leq i \leq k, w = v_{k+1}$



(a) different-constraint



{P16} Definition 10. Let $G = (V, T \cup B)$ be a DFS-oriented graph such that each pair of back edges $b_1, b_2 \in B$ is subject to at most one type of constraint. The signed graph $CC(G) = (B, E(C); \sigma: E(C) \rightarrow \{-1, +1\})$ with

$$\sigma(b_1, b_2) = \begin{cases} -1 & \text{if } b_1, b_2 \in B \text{ are subject to a different constraint} \\ +1 & \text{if } b_1, b_2 \in B \text{ are subject to a same constraint} \end{cases}$$

is called constraint graph of G .

{P16} If any pair of back edges is subject to both a same-constraint and a different-constraint, no LR partition can exist and hence the graph is non-planar.

{P18} Algorithm 1: Left-Right Planarity Algorithm

input: simple, undirected graph $G = (V, E)$

output: planar embedding Challs if graph is not planar)

if $|V| \geq 2$ and $|E| > 3|V| - 6$ then HALT: not planar

▼ orientation

for $s \in V$ do

if $\text{height}[s] = \infty$ then

$\text{height}[s] \leftarrow 0$; append $\text{Roots} \leftarrow s$

DFS1(s)

▼ testing

sort adjacency lists according to non-decreasing nesting_depth

for $s \in \text{Roots}$ do DFS2(s)

▼ embedding

for $e \in E$ do $\text{nesting_depth}[e] = \text{sign}(e) \cdot \text{nesting_depth}[e]$

sort adjacency lists according to non-decreasing nesting_depth

for $s \in \text{Roots}$ do DFS3(s)

where

integer $\text{sign}(edge\ e)$

if $\text{ref}[e] \neq 1$ then

$\text{side}[e] \leftarrow \text{side}[e] \cdot \text{sign}(\text{ref}[e])$

$\text{ref}[e] \leftarrow 1$