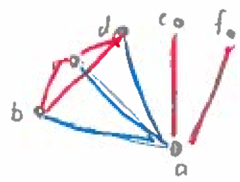
Ramsey number

Proof that $R(3,3) = 6$

In a complete graph (K_6) for node a adjacent to all other 5 nodes, color them red or blue.



No matter how you color it, at least 3 edges are colored with the same color (either red or blue).

WLOG, let's say a is connected with b, c, d colored blue.

To avoid a clique with a, bc, cd, bd has to be colored red. In this way, b, c, d form a red 3-clique.

Thus, K_6 must have either a red 3-clique or a blue 3-clique. $\square$

Bondy and Murty

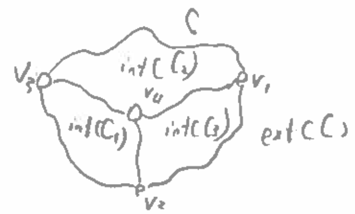
[1.1] P5 : <u>planar graph</u>: A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a <u>planar graph</u>, and such a drawing is called a <u>planar embedding</u> of the graph.

[10.1] P243 <u>planar graph</u>: A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends.

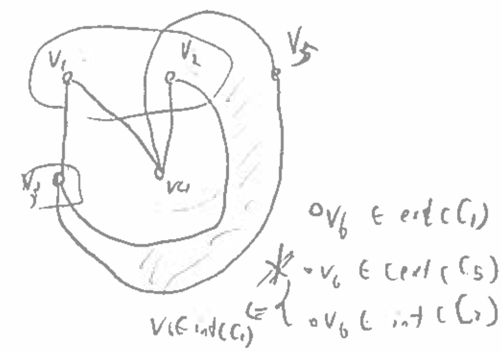
[10.1] P244 <u>plane graph</u>: we often refer to a planar embedding $\widetilde{G}$ of a planar graph G as a plane graph; and we call the vertices of $\widetilde{G}$ points and its edges lines.

[10.1] P245 <u>THE JORDAN CURVE THEOREM</u>: Any simple closed curve C in the plane partitions the rest of the plane into two <u>disjoint arcwise-connected open sets</u>.
└ called interior $int(C)$ and exterior $ext(C)$.

[10.1] P245 <u>K_5 is nonplanar</u>:



$K_{3,3}$:



[10.1] P246 <u>subdivision</u>: Any graph derived from a graph G by a sequence of edge subdivisions is called a <u>subdivision of G</u> or a <u>G-subdivision</u>.

[10.1] P246. Proposition 10.3: A graph G is planar if and only if every subdivision of G is planar.

[10.1] P247 Theorem 10.4: A graph G is embeddable on the plane if and only if it is embeddable on the sphere.

stereographic projection from z

[10.2] P250 Proposition 10.5: Let G be a planar graph, and let f be a face in some planar embedding of G. Then G admits a planar embedding whose outer face has the same boundary as f.
Proof idea – consider an embedding $\widetilde{G}$ of G on the sphere.

[10.2] P250 Theorem 10.6: <u>THE JORDAN-SCHÖNFLIESS THEOREM</u> ?
Any homeomorphism of a simple closed curve in the plane onto another simple closed curve can be extended to a homeomorphism of the plane.

[10.2] P251 Theorem 10.7: In a nonseparable plane graph other than K_1 or K_2, each face is bounded by a cycle.
? ?

[10.2] P252 <u>dual</u>: Given a plane graph G, one can define a second graph G^* as follows. Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e there is an edge e^* of G^*. Two vertices f^* and g^* are joined by the edge e^* in G^* iff their corresponding faces f and g are separated by the edge e in G. Observe that if e is a cut edge of G, then $f = g$, so e^* is a loop of G^*; conversely, if e is a loop of G, the edge e^* is a cut edge of G^*. The graph G^* is called the dual of G.

[10.3] P259 **EULER'S FORMULA:**

For a connected plane graph G.

$$v(G) - e(G) + f(G) = 2$$

Proof by induction

[10.3] P259 Corollary 10.20:

All planar embeddings of a connected planar graph have the same number of faces.

[10.3] P259 Corollary 10.21:

Let G be a simple planar graph on at least three vertices. Then $m \leq 3n-6$. Furthermore, $m = 3n-6$ if and only if every planar embedding of G is a triangulation.

[10.3] P259 Corollary 10.22:

Every simple planar graph has a vertex of degress at most five.

[10.3] P259 Corollary 10.23:

K_5 is nonplanar

proof: If K_5 were planar, corollary 10.21 would give $10 = e(K_5) \leq 3v(K_5) - 6 = 9$

[10.3] P259 Corollary 10.24:

$K_{3,3}$ is nonplanar.

proof: If $K_{3,3}$ were planar, $K_{3,3}$ has no cycle of length < 4, so every face of G has degree ≥ 4. By Theorem 10.10, $4f(G) \leq \sum_{f \in F} d(f) = 2e(G) = 18$

$\therefore \underbrace{f(G)}_{\text{integer}} \leq 4$

Euler's formula implies that

$\underline{v(G)} - e(G) + f(G) = 2$
$6 - 9 + f(G) = 2 \Rightarrow f(G) = 5$

[10.2] P253 Theorem 10.10: If G is a plane graph

$$\sum_{f \in F} d(f) = 2m$$

[10.5] P268 Theorem 10.30: **KURATOWSKI'S THEOREM**

A graph is planar iff it contains no subdivision of either K_5 or $K_{3,3}$.

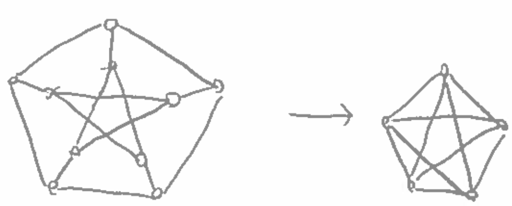
A subdivision of K_5 or $K_{3,3}$ is consequently called a Kuratowski subdivision.

[10.5] P268 minor: A minor of a graph is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions?

Alternatively, consider a partition $(V_0, V_1, \ldots, V_k)$ of V such that $G[V_i]$ is connected, $1 \leq i \leq k$, and let H be the graph obtained from G by deleting V_0 and shrinking each induced subgraph $G[V_i]$, $1 \leq i \leq k$, to a single vertex. The any spanning? subgraph F of H is a minor of G.

e.g. K_5 is a minor of the Petersen graph because it can be obtained by contracting the five 'spoke' edges of the latter graph.

Petersen graph



important:

1) any graph which contains an F-subdivision also has an F-minor
2) provided that F is a graph of maximum degree three or less, any graph which has an F-minor also contains an F-subdivision.

If F is a minor of G, we write $F \preceq G$.

Exe. find a $K_{3,3}$ minor from Petersen graph.

[10.5] P269 **WAGNER'S THEOREM**

A graph is planar iff it has no Kuratowski minor

[1.1] P_3 loop: an edge with identical ends is called a loop
link: an edge with distinct ends is called a link
parallel edges: two or more links with the same pair of ends are said to be parallel edges
simple: a graph is simple if it has no loops or parallel edges.

[D; Battista et al.]

{1} P_6 drawing: a drawing Γ of a graph (digraph) G is a function which maps each vertex v to a distinct point $\Gamma(v)$ and each edge (u,v) to a simple open Jordan curve $\Gamma(u,v)$, with endpoints $\Gamma(u)$ and $\Gamma(v)$.

{1} P_7 drawing planar: a drawing Γ is planar if no two distinct edges intersect.
A graph is planar if it admits a planar drawing

{1} P_8 connected: a graph is connected if there is a path between u and v for each pair (u,v) of vertices.

cut vertex: A cut vertex in graph G is a vertex whose removal disconnects G.
biconnected: A connected graph with no cutvertices is biconnected.
blocks: The maximal biconnected subgraphs of a graph are its blocks. (sometimes called biconnected components).
A graph is planar iff its blocks are planar.

{1} P_9 important: ?1) The skeleton of a convex polyhedron is a planar triconnected graph.
?2) A planar triconnected graph has a unique embedding, up to a reversal of the circular ordering of the neighbors of each vertex

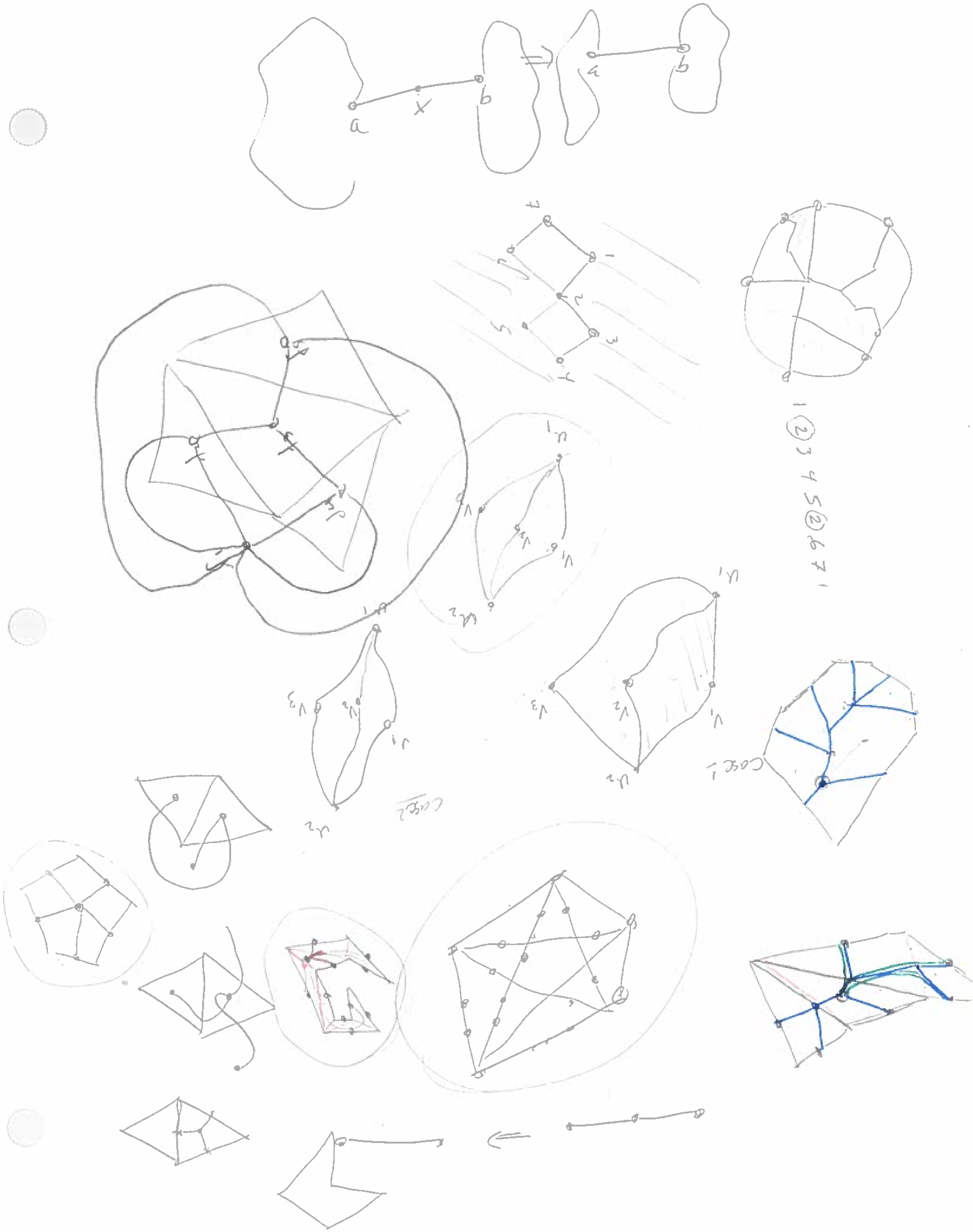
{2.1} P_{12} drawing convention: A drawing convention is a basic rule that the drawing must satisfy to be admissible.
- Polyline drawing
- Straight-line drawing
- Orthogonal drawing
- Grid Drawing
- Planar Drawing
- Upward (resp. downward) Drawing

{2.1} P_{14} aesthetics: aesthetics specify graph properties of the drawing that we would like to apply, as much as possible, to achieve readability.

constraints: refer to specific subgraphs or sub drawings

efficiency: interactive applications

- crossings: minimize
- area: minimize (grid drawing, straight-line drawing where distance(u,v) ≥ 1)
  convex hull
- total edge length: minimize
- maximum edge length: minimize
- uniform edge length: minimize variance of lengths of the edges
- total bends: minimize (important for orthogonal drawings)
- maximum bends: minimize
- uniform bends: minimize
- angular resolution: maximize (straight-line drawing)
- aspect ratio: minimize longest-side : shortest side
- symmetry


a
x
b
a
b
case 1
case 2
1 2 3 4 5 2 6 7 1

[The Left-Right Planarity Test] by Ulrik Brandes

- Efficient Planarity Testing by John Hopcroft & Robert Tarjan (1974)
- A Depth-first-search Characterization of Planarity by H. De Fraysseix & P. Rosenstiehl (1982)
- On the Realization of Complexes in Euclidean Spaces 吴文俊 Wen-Tsun Wu (1955)

{P4} There are only two significant ways to draw a simple cycle planarly, namely clockwise and counterclockwise.

{P4} Testing planarity amounts to deciding whether there is a consistent simultaneous orientation of all cycles.

{P5} In any planar drawing the back edges can be partitioned into left and right depending on whether their fundamental cycle is counterclockwise or clockwise.

{P6} In the oriented graph, we denote by $E^+(v) = \{(v,w) \in E : w \in V\}$ the set of all outgoing edges of $v \in V$, so that $E = \bigcup_{v \in V} E^+(v)$.

{P6} A DFS traversal yields a bipartition $E = T \uplus B$ of the edges, where those in T are called tree edges, and the non-tree edges in B are called back edges. We write $u \to v$ for $(u,v) \in T$ and $v \hookrightarrow w$ for $(v,w) \in B$.

{P6} fundamental cycle: $C(v \hookrightarrow w) = w \xrightarrow{+} v \hookrightarrow w$

overlapping. Two cycles are called overlapping, if they share an edge.

{P6} Lemma 3 Let $G = (V, T \uplus B)$ be a DFS-oriented graph.

(1) The fundamental cycles are exactly the simple directed cycles of G.

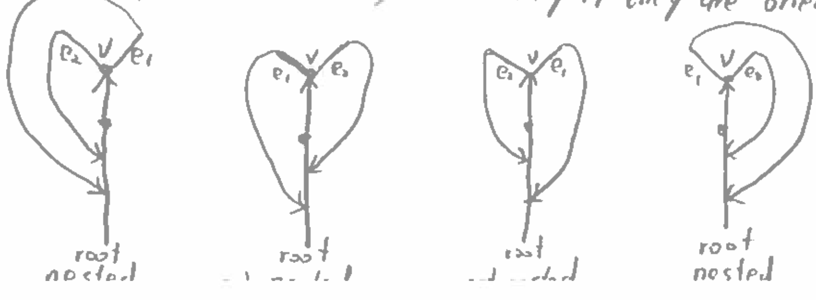
(2) Two distinct fundamental cycles are either disjoint, or their intersection forms a tree path

{P7} fork: For two overlapping cycles, the last edge $u \to v$ on the shared tree path together with succeeding edges $e_1 = (v, w_1)$, $e_2 = (v, w_2)$ on each cycle is called their fork, and v its branching point

{P7} a linearization of the cyclic ordering: It is defined by splitting the clockwise order restricted to outgoing edges at the incoming tree edge, or between any two consecutive outgoing edges if v is the root of a DFS tree.

{P7} nested: Two overlapping fundamental cycles are called nested, if the part of one cycle that is not common to both is drawn completely inside the other cycle.

{P8} Observation 1: In a planar drawing of a DFS-oriented graph $G = (V, T \uplus B)$, two overlapping cycles are nested, if and only if they are oriented alike.



{P6} <u>return points</u>: The return points of a tree edge $v \rightarrow w \in T$ are the ancestors u of v with $u \xrightarrow{+} v \rightarrow w \xrightarrow{*} x \hookrightarrow u$ for some descendant x of w.
The return points of a vertex $v \in V$ are formed by the union of all return points of outgoing edges $(v,w) \in E^{+}(v) \subseteq T \uplus B$.

{P8} <u>lowpoint</u>: The lowpoint of an edge is its lowest return point, if any, or its source if none exists.

{P9} <u>Observation 2</u>: In a planar drawing of a connected DFS-oriented graph $G = (V, T \uplus B)$ with the root of the DFS tree on the outer face, overlapping fundamental cycles are nested according to their lowpoint order.

{P9} <u>left, right</u>: the side of a back edge in a planar drawing is right, if its fundamental cycle is oriented clockwise, and left otherwise.

{P9} <u>LR partition</u>: Let $G = (V, T \uplus B)$ be a DFS-oriented graph. A partition $B = L \uplus R$ of its back edges into two classes, referred to as left and right, is called left-right partition, or LR partition for short, if every fork consisting of $u \rightarrow v \in T$ and $e_1, e_2 \in E^{+}(v)$
(1) all return edges of e_1 ending strictly higher than $lowpt(e_2)$ belong to one class
and
(2) all return edges of e_2 ending strictly higher than $lowpt(e_1)$ to the other.

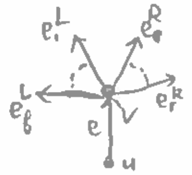
{P10} <u>Left-Right Planarity Criterion</u>: A graph is planar if and only if it admits an LR partition.

{P10} <u>aligned</u>: An LR partition is called aligned, if all return edges of a tree edge e that return to $lowpt(e)$ are on the same side.

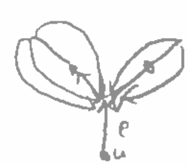
{P11} <u>Lemma 6</u>: Any LR partition can be turned into an aligned LR partition.

{P11} <u>$e_1 \prec e_2$</u>: we have to define $e_1 \prec e_2$ if and only if the lowpoint of e_1 is strictly lower than that of e_2. If both have the same lowpoint, but say, only e_2 has another return point, we say that e_2 is <u>chordal</u> and let $e_1 \prec e_2$.

{P12} <u>Definition 7 LR Ordering</u>: Given an LR partition, let $e_1^L \prec \cdots \prec e_\ell^L$ be the left outgoing edges of a vertex v and $e_1^R \prec \cdots \prec e_r^R$ its right outgoing edges. If v is not the root, let u be its parent. The clockwise leftright ordering, or LR ordering for short, of the edges around v is defined as follows:



(u,v),
$L(e_\ell^L), e_\ell^L, R(e_\ell^L), \ldots, L(e_1^L), e_1^L, R(e_1^L)$,
$L(e_1^R), e_1^R, R(e_1^R), \ldots, L(e_r^R), e_r^R, R(e_r^R)$

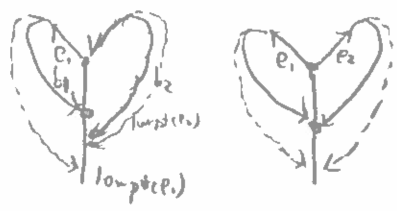


where (u,v) is absent if v is the root, and $L(e)$ and $R(e)$ denote the left and right incoming back edges whose cycles share e. For two back edges $b_1 = x_1 \hookrightarrow v$, $b_2 = x_2 \hookrightarrow v \in R(e)$ let $z \rightarrow x, (x,y_1), (x,y_2)$ be the fork of $C(b_1), C(b_2)$. Then, b_1 comes after b_2 in $R(e)$ if and only if $(x,y_1) \prec (x,y_2)$. If $b_1, b_2 \in L(e)$, the order is reversed.

{P12} Lemma 8. Given an LR partition, LR ordering yields a planar embedding.

{P14} Corollary 9. Let $G = (V, T \uplus B)$ be a DFS-oriented graph. For a pair of back edges $b_1, b_2 \in B$ with overlapping fundamental cycles let $v_1 \to \cdots \to v_k$ be the tree path of the intersection and $(v_{k-1}, v_k), e_1, e_2$ the corresponding fork with $e_1 \xrightarrow{*} b_1$ and $e_2 \xrightarrow{*} b_2$. Then, b_1 and b_2 are subject to

- a different-constraint, iff $\text{lowpoint}(e_2) < \text{lowpoint}(b_1)$ and $\text{lowpoint}(e_1) < \text{lowpoint}(b_2)$
- a same-constraint, iff $\text{lowpoint}(e') < \min\{\text{lowpoint}(b_1), \text{lowpoint}(b_2)\}$ for some $e' = (v_i, w) \in T \uplus B$, $1 < i < k$, $w = v_{i+1}$



(a) different-constraint

{P16} Definition 10: Let $G = (V, T \uplus B)$ be a DFS-oriented graph such that each pair of back edges $b_1, b_2 \in B$ is subject to at most one type of constraint. The signed graph $C(G) = (B, E(C); \sigma : E(C) \to \{-1, +1\})$ with

$$\sigma(b_1, b_2) = \begin{cases} -1 & \text{if } b_1, b_2 \in B \text{ are subject to a different constraint} \\ +1 & \text{if } b_1, b_2 \in B \text{ are subject to a same constraint} \end{cases}$$

is called constraint graph of G.

{P16} If any pair of back edges is subject to both a same-constraint and a different-constraint, no LR partition can exist and hence the graph is non-planar.

{P18} Algorithm 1: Left-Right Planarity Algorithm

```
input: simple, undirected graph G=(V,E)
output: planar embedding (halts if graph is not planar)
if |V| > 2 and |E| > 3|V| - 6 then HALT: not planar

▼ orientation
  for s ∈ V do
    if height[s] = ∞ then
      height[s] ← 0; append Roots ← s
      DFS1(s)

▼ testing
  sort adjacency lists according to non-descending nesting_depth
  for s ∈ Roots do DFS2(s)

▼ embedding
  for e ∈ E do nesting_depth[e] = sign(e) · nesting_depth[e]
  sort adjacency lists according to non-decreasing nesting_depth
  for s ∈ Roots do DFS3(s)

where
integer sign(edge e)
  if ref[e] ≠ ⊥ then
    side[e] ← side[e] · sign(ref[e])
    ref[e] ← ⊥
```