1 Polymerase chain reaction

1.1 PCR

Let K denote the number of successes in copying DNA, then we have $K \sim Bin(n, p), p = 0.5$

1.1.1 what is the probability that there are exactly 8 billion copies immediately after 9 rounds?

In this particular problem, $n = 9$, so $K \sim Bin(9, 0.5)$
We start from 1 billion, and each success doubles the amount of copies. In the end, there are exactly 8 billion copies. So the number of successes is $k = \log_2 8 = 3$.

So the probability is

$$Pr(K = 3) = \binom{9}{3} \cdot 0.5^3 \cdot (1 - 0.5)^{(9-3)} = \binom{9}{3} \cdot (1/2)^9 = 21/128 = 0.1640625$$

1.1.2 Find a general formula for the probability that there are exactly x billion copies after y rounds.

The number of successes after y rounds is $k = \log_2 x$.
So the probability is $Pr(K = k) = \binom{y}{k} \cdot 0.5^k \cdot (1 - 0.5)^{y-k} = \binom{y}{k} \cdot 0.5^y = \binom{y}{\log_2 x} \cdot 0.5^y$

k needs to satisfy $0 \leq k \leq y$, so the generic formula is:

$$p = f(x, y) = \begin{cases} \binom{y}{\log_2 x} \cdot 0.5^y & x = 2^k, 0 \leq k \leq y, k \in \mathbb{N} \\ 0 & otherwise \end{cases}$$

1.2 Optional

1.2.1 What is the probability that there are exactly 2 billion copies immediately after nine rounds?

Each successful round the amount doubles, and each failing round the amount is cut by half. In the end, there are exactly 2 billion copies. So we have equation

$$2^k + (1/2)^(9 - k) = 2^1$$

Solving it we get $k = 5$. So the probability is

$$Pr(K = 5) = \binom{9}{5} \cdot 0.5^5 \cdot (1 - 0.5)^{9-5} = \binom{9}{5} \cdot 0.5^9 = 63/256 = 0.24609375$$

1.2.2 Find a general formula for the probability that there are exactly x copies after y rounds.

Following the same logic as in 1.2.1, we have formula:

$$2^k \cdot (1/2)^{(y-k)} = 2^{\log_2 x}$$

Solving it we get $k = (\log_2 x + y)/2$. So the probability is:

$$Pr(K = k) = \binom{y}{k} \cdot 0.5^k \cdot (1 - 0.5)^{y-k} = \binom{y}{k} \cdot 0.5^y = \binom{y}{(\log_2 x + y)/2} \cdot 0.5^y$$

k needs to satisfy $0 \le k \le y, (0 \le (\log_2 x + y)/2 \le y)$, so the generic formula is:

$$p = f(x, y) = \begin{cases} \binom{y}{(\log_2 x+y)/2} \cdot 0.5^y & x = 2^j, -y \le j \le y, j \in \mathbb{N} \\ 0 & otherwise \end{cases}$$

2 Hypothesis testing

If we randomly choose 4 cups to be red wine, there are in total $\binom{8}{4}$ combinations. To identify at least 3 of the red wines, it needs to be either choose 3 red wines and 1 white wine $\binom{4}{3} \cdot \binom{4}{1}$, or choose all 4 red wines $\binom{4}{4}$. The probability is:

$$p = \frac{\binom{4}{3} \cdot \binom{4}{1} + \binom{4}{4}}{\binom{8}{4}} = \frac{17}{70}$$

3 Gift exchange problem

3.1 Suppose there are four people at his party (including himself). In how many ways can the boxes be distributed so that exactly one person is assigned his/her own gift?

Suppose we have A, B, C, D four people. We randomly choose 1 person to have his/her own gift, so $\binom{4}{1}$. Without missing of generality, we look at the case where A gets his/her own gift. Then B, C, D has to work with a set that each of them does not get his/her own gift, and there are only 2 ways to do that.

	B	C	D
(1)	C	D	B
(2)	D	B	C

So in total there are $\binom{4}{1} \cdot 2 = 8$ ways.

3.2 In how many ways can the boxes be distributed so that no one is assigned his/her own gift? Do not use explicit enumeration in this question. Instead, introduce the number of ways, N_i, of having i people getting their own gift. What is $\sum_{i=0}^{5} N_i$?

Let N_i be having i people getting their own gift. Then N_0 means nobody gets his/her own gift, and N_1 means exactly 1 person gets his/her own gift, ..., etc. So $\sum_{i=0}^{5} N_i$ should include all the permutations of distributing 5 gifts, so

$$\sum_{i=0}^{5} N_i = 5!$$

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Let $f(x)$ denote the case that in a total of x people, each of them not getting his/her own gift. From the previous question, we know that:

$$\begin{aligned}
&f(0) = 1, \texttt{by default}\\
&f(1) = 0, \texttt{no chance for 1 person not getting his/her own gift}\\
&f(2) = 1, \texttt{having 2 people and each gets the other's}\\
&f(3) = 2, \texttt{from the previous question we know there are 2 ways}
\end{aligned}$$

Consider 4 people and each not getting his/her own gift, we have

$$\begin{aligned}
&f(4) = 4! - \binom{4}{1} \cdot f(3) - \binom{4}{2} \cdot f(2) - \binom{4}{3} \cdot f(1) - \binom{4}{4} \cdot f(0)\\
&f(4) = 24 - 4 \times 2 - 6 \times 1 - 4 \times 0 - 1 \times 1\\
&f(4) = 9
\end{aligned}$$

Getting no one is assigned his/her own gift is equivalent to getting $f(5) = N_0 = 5! - \sum_{i=1}^{5} N_i = 5! - N_1 - N_2 - N_3 - N_4 - N_5$.

$$\begin{aligned}
&f(5) = 5! - \binom{5}{1} \cdot f(5-1) - \binom{5}{2} \cdot f(5-2) - \binom{5}{3} \cdot f(5-3) - \binom{5}{4} \cdot f(5-4) - \binom{5}{5} \cdot f(5-5)\\
&f(5) = 5! - \binom{5}{1} \cdot f(4) - \binom{5}{2} \cdot f(3) - \binom{5}{3} \cdot f(2) - \binom{5}{4} \cdot f(1) - \binom{5}{5} \cdot f(0)\\
&f(5) = 120 - 5 \times 9 - 10 \times 2 - 10 \times 1 - 5 \times 0 - 1 \times 1\\
&f(5) = 44
\end{aligned}$$

Therefore, there are 44 ways to be distributed so that no one is assigned his/her own gift.

3.3 If the boxes are randomly distributed in a party of 5, what is the probability that no one is assigned his/her own gift?

There are in total 5! ways of permutation, and 44 ways to satisfy the goal, so the probability is:

$$p = \frac{44}{5!} = \frac{11}{30}$$

3.4 What is the probability that Tom ends up receiving Finn's gift, but Finn does not receive Tom's gift?

There are $1 + 5 = 6$ people in total in the party, so the total number of permutations are 6!
Tom gets Finn's (**1**). In the remaining 5 gifts, **4** of them are not from Tom for Finn to choose. The rest of permutations are **4!**
So the probability is:

$$p = \frac{1 \times 4 \times 4!}{6!} = \frac{2}{15}$$

4 Guessing answers on a test