1 Polymerase chain reaction

1.1 PCR

Let K denote the number of successes in copying DNA, then we have $K \sim Bin(n, p), p = 0.5$

1.1.1 what is the probability that there are exactly 8 billion copies immediately after 9 rounds?

In this particular problem, $n = 9$, so $K \sim Bin(9, 0.5)$
We start from 1 billion, and each success doubles the amount of copies. In the end, there are exactly 8 billion copies. So the number of successes is $k = \log_2 8 = 3$.

So the probability is

$$Pr(K = 3) = \binom{9}{3} \cdot 0.5^3 \cdot (1 - 0.5)^{(9-3)} = \binom{9}{3} \cdot (1/2)^9 = 21/128 = 0.1640625$$

1.1.2 Find a general formula for the probability that there are exactly x billion copies after y rounds.

The number of successes after y rounds is $k = \log_2 x$.
So the probability is $Pr(K = k) = \binom{y}{k} \cdot 0.5^k \cdot (1 - 0.5)^{y-k} = \binom{y}{k} \cdot 0.5^y = \binom{y}{\log_2 x} \cdot 0.5^y$

k needs to satisfy $0 \leq k \leq y$, so the generic formula is:

$$p = f(x, y) = \begin{cases} \binom{y}{\log_2 x} \cdot 0.5^y & x = 2^k, 0 \leq k \leq y, k \in \mathbb{N} \\ 0 & otherwise \end{cases}$$

1.2 Optional

1.2.1 What is the probability that there are exactly 2 billion copies immediately after nine rounds?

Each successful round the amount doubles, and each failing round the amount is cut by half. In the end, there are exactly 2 billion copies. So we have equation

$$2^k + (1/2)^(9 - k) = 2^1$$

Solving it we get $k = 5$. So the probability is

$$Pr(K = 5) = \binom{9}{5} \cdot 0.5^5 \cdot (1 - 0.5)^{9-5} = \binom{9}{5} \cdot 0.5^9 = 63/256 = 0.24609375$$

1.2.2 Find a general formula for the probability that there are exactly x copies after y rounds.

Following the same logic as in 1.2.1, we have formula:

$$2^k \cdot (1/2)^{(y-k)} = 2^{\log_2 x}$$

Solving it we get $k = (\log_2 x + y)/2$. So the probability is:

$$Pr(K = k) = \binom{y}{k} \cdot 0.5^k \cdot (1 - 0.5)^{y-k} = \binom{y}{k} \cdot 0.5^y = \binom{y}{(\log_2 x + y)/2} \cdot 0.5^y$$

k needs to satisfy $0 \leq k \leq y, (0 \leq (\log_2 x + y)/2 \leq y)$, so the generic formula is:

$$p = f(x, y) = \begin{cases} \binom{y}{(\log_2 x+y)/2} \cdot 0.5^y & x = 2^j, -y \leq j \leq y, j \in \mathbb{N} \\ 0 & otherwise \end{cases}$$

2 Hypothesis testing

If we randomly choose 4 cups to be red wine, there are in total $\binom{8}{4}$ combinations. To identify at least 3 of the red wines, it needs to be either choose 3 red wines and 1 white wine $\binom{4}{3} \cdot \binom{4}{1}$, or choose all 4 red wines $\binom{4}{4}$. The probability is:

$$p = \frac{\binom{4}{3} \cdot \binom{4}{1} + \binom{4}{4}}{\binom{8}{4}} = \frac{17}{70}$$

3 Gift exchange problem

4 Guessing answers on a test