习题6

1、用追赶法求解下列方程组:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 3 & -2 & \\ & -1 & 2 & -1 \\ & & -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

解: 1° 消元过程:

将三对角线系数阵的方程组加工为单位上二对角方程组:

由 $\begin{cases} x_i + u_i x_{i+1} = y_i, & i=1,2,3,4 \\ x_4 = y_4 \end{cases}$

得: $\begin{cases} x_1 + (\frac{-1}{2})x_2 = 3 \\ x_2 + (\frac{-4}{5})x_3 = \frac{8}{5} \\ x_3 + (\frac{-5}{6})x_4 = \frac{4}{3} \\ x_4 = 2 \end{cases}$

其中 u_i, y_i 是由以下运算得到的:

$$\begin{cases} u_1 = \frac{-1}{2}, & y_1 = \frac{6}{2} = 3 \\ u_2 = \frac{-2}{3-(-1)\times(-\frac{1}{2})} = -\frac{4}{5}, & y_2 = \frac{1-(-1)\times 3}{3-(-1)\times(-\frac{1}{2})} = \frac{8}{5} \\ u_3 = \frac{-1}{2-(-1)\times(-\frac{4}{5})} = -\frac{5}{6}, & y_3 = \frac{0-(-1)\times\frac{8}{5}}{2-(-1)\times(-\frac{4}{5})} = \frac{4}{3} \\ y_4 = \frac{1-(-3)\times\frac{4}{3}}{5-(-3)\times(-\frac{5}{6})} = 2 \end{cases}$$

2° 回代过程:

$x_4 = y_4 = 2$, $x_3 = \frac{4}{3} + \frac{5}{6}\times x_4 = 3$

$x_2 = \frac{8}{5} + \frac{4}{5}\times x_3 = 4$, $x_1 = 3 + \frac{1}{2}x_2 = 5$

∴方程组的解为 $x_1 = 5$, $x_2 = 4$, $x_3 = 3$, $x_4 = 2$.

5. 将下列矩阵A分解为LL^T:

$$A=\begin{bmatrix}3&2&3\\2&2&0\\3&0&12\end{bmatrix}$$ 这里L为对角线元素为正的下三角阵.

解:
$$\begin{bmatrix}3&2&3\\2&2&0\\3&0&12\end{bmatrix} \quad A=LL^T$$

$l_{11}=\sqrt{a_{11}}=\sqrt{3}$ $\qquad$ $l_{22}=\sqrt{a_{22}-l_{21}^2}=\sqrt{2-\frac{4}{3}}=\sqrt{\frac{2}{3}}$

$l_{21}=\frac{a_{21}}{l_{11}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$ $\qquad$ $l_{32}=\frac{a_{32}-l_{21}l_{31}}{l_{22}}=\frac{0-\frac{2}{\sqrt{3}}\times\sqrt{3}}{\sqrt{\frac{2}{3}}}=-\sqrt{6}$

$l_{31}=\frac{a_{31}}{l_{11}}=\frac{3}{\sqrt{3}}=\sqrt{3}$ $\qquad$ $l_{33}=\sqrt{a_{33}-l_{31}^2-l_{32}^2}=\sqrt{12-3-6}=\sqrt{3}$

即
$$\begin{bmatrix}\sqrt{3}&&\\&&\\&&\end{bmatrix}\to\begin{bmatrix}\sqrt{3}&&\\\frac{2\sqrt{3}}{3}&&\\\sqrt{3}&&\end{bmatrix}\to\begin{bmatrix}\sqrt{3}&&\\\frac{2\sqrt{3}}{3}&\sqrt{\frac{2}{3}}&\\\sqrt{3}&-\sqrt{6}&\end{bmatrix}\to\begin{bmatrix}\sqrt{3}&&\\\frac{2\sqrt{3}}{3}&\frac{\sqrt{6}}{3}&\\\sqrt{3}&-\sqrt{6}&\sqrt{3}\end{bmatrix}$$

∴得

$$A=LL^T$$
$$=\begin{bmatrix}\sqrt{3}&&\\\frac{2\sqrt{3}}{3}&\frac{\sqrt{6}}{3}&\\\sqrt{3}&-\sqrt{6}&\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{3}&\frac{2\sqrt{3}}{3}&\sqrt{3}\\&\frac{\sqrt{6}}{3}&-\sqrt{6}\\&&\sqrt{3}\end{bmatrix}$$

7. 用矩阵分解方法求解方程组

$$\begin{bmatrix} 5 & 7 & 9 & 10 \\ 6 & 8 & 10 & 9 \\ 7 & 10 & 8 & 7 \\ 5 & 7 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

解：利用 DooLittle 分解法，$A=L_1U$

采用增广矩阵的形式：

$$\bar{A} = \begin{bmatrix} 5 & 7 & 9 & 10 & 1 \\ 6 & 8 & 10 & 9 & 1 \\ 7 & 10 & 8 & 7 & 1 \\ 5 & 7 & 6 & 5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 7 & 9 & 10 & 1 \\ \frac{6}{5} & -\frac{2}{5} & -\frac{4}{5} & -3 & -\frac{1}{5} \\ \frac{7}{5} & -\frac{1}{2} & -5 & -\frac{17}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{3}{5} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

$$\therefore L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{6}{5} & 1 & 0 & 0 \\ \frac{7}{5} & -\frac{1}{2} & 1 & 0 \\ 1 & 0 & \frac{3}{5} & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -\frac{1}{5} \\ -\frac{1}{2} \\ \frac{3}{10} \end{bmatrix}, \quad U = \begin{bmatrix} 5 & 7 & 9 & 10 \\ 0 & -\frac{2}{5} & -\frac{4}{5} & -3 \\ 0 & 0 & -5 & -\frac{17}{2} \\ 0 & 0 & 0 & \frac{1}{10} \end{bmatrix}$$

解方程组：

$$Ux = \begin{bmatrix} 5 & 7 & 9 & 10 \\ 0 & -\frac{2}{5} & -\frac{4}{5} & -3 \\ 0 & 0 & -5 & -\frac{17}{2} \\ 0 & 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{5} \\ -\frac{1}{2} \\ \frac{3}{10} \end{bmatrix}$$

得原方程组的解：$x_4=3$，$x_3=-5$，$x_2=-12$，$x_1=20$。