Rigid analytic p-adic Simpson correspondence for line bundles

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Overview

1. Background: Simpson correspondence
2. p-adic Simpson correspondence: Faltings local construction
3. Rank one case: the corrspondence is rigid analytic
4. Further discussion

Background: Simpson correspondence

In the complex case, Simpson correspondence generalizes the classical Narasimhan-Seshadri correspondence on compact Riemann surface.

Let X be a smooth projective variety over $\mathbb{C}$.

Definition

A *Higgs bundle* is a pair consisting of a holomorphic vector bundle E and a holomorphic map $\theta : E \to E \otimes \Omega_X^1$ such that $\theta \wedge \theta = 0$.

Based on the work of Donaldson, Uhlenbeck-Yau, Corlette, Simpson established the following result:

Theorem

There is an equivalence of categories between finite-dimensional complex representations of $\pi_1(X^{an})$ and semi-stable Higgs bundles on X with vanishing Chern classes.

Simpson's rank one construction for moduli

- $\mathbf{M}_{Dol}$: moduli space whose points parametrizes direct sums of stable Higgs bundles on X with vanishing Chern classes.
- $\mathbf{M}_B$: moduli space whose points parametrizes semi-simple representations of the fundamental group.

Suppose X is a curve of genus g. We aim to parametrize representations of rank one, then

$$\mathbf{M}_{Dol} = Jac(X) \times H^0(X, \Omega^1_X) \longrightarrow \mathbf{M}_B = (\mathbb{C}^*)^{2g}$$

is a real analytic group homeomorphism.

Simpson's rank one construction for moduli

- $Jac(X) \longrightarrow (S^1)^{2g}$

  By the following commutative diagram and note that $Jac(X)$ is compact

$$\begin{tikzcd} H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, "exp"] & H^1(X^{an}, \mathbb{C}) \simeq H^1(X^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d, "id \otimes exp"] \\ Jac(X) \arrow[r] & H^1(X^{an}, \mathbb{C}^*) \simeq H^1(X^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \end{tikzcd}$$

- $H^0(X, \Omega^1_X) \longrightarrow (\mathbb{R}^*_+)^{2g}$

  For $\theta \in H^0(X, \Omega^1_X)$, θ maps to $(\cdots, exp(-\int_{\gamma_i}(\theta + \bar{\theta})), \cdots)$ is also a real analytic group homeomorphism, where γ_i form the basis for the first homology $H_1(X, \mathbb{Z})$.

Faltings p-adic Simpson correspondence: local small case

- $\mathbf{o}$: ring of integers in $\mathbb{C}_p$
- V: complete DVR with perfect residue field k of char p and fraction field K of char 0
- R: étale over $V[T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}]$
- $\overline{R}$: maximal étale extension of R in char 0
- $R_1 = R \otimes_V \overline{V}$, $R_\infty = \oplus_\alpha R_1 T^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\cup_{n \geq 0} \frac{\frac{1}{p^n}\mathbb{Z}}{\mathbb{Z}})^d$, $T^\alpha = T_1^{\alpha_1} \ldots T_d^{\alpha_d}$.

Definition

A *generalized representation* on R is a pair $(\overline{M}, \rho)$ such that $\overline{M}$ is a finitely generated free $\widehat{\overline{R}}$-module and $\rho \in H^1_{cont}(Gal(\overline{R}/R_1), GL(\overline{M}))$,equivalently, it is an $\widehat{\overline{R}}$-module with a continuous semilinear action of $Gal(\overline{R}/R_1)$. A generalized representation is called *small* if $\bar{\rho} = 0 \in H^1_{cont}(Gal(\overline{R}/R_1), GL(\overline{M}/p^{2\beta}\overline{M}))$ for some $\beta > \frac{1}{p-1}$.

Faltings p-adic Simpson correspondence: local small case

Definition

A *Higgs module* on $R_1 = R \otimes_V \overline{V}$ is a pair (M, θ) where M is a finitely generated free R_1-module and $\theta : M \longrightarrow M \otimes_{R_1} \Omega^1_{R_1/\overline{V}} \otimes_{\overline{V}} \widehat{\overline{V}}\xi^{-1}$ such that $\theta \wedge \theta = 0$. The Higgs bundle (E, θ) is called *small* if $p^\alpha | \theta$ for some $\alpha > \frac{1}{p-1}$.

Theorem (p-adic analogue)

There exists an equivalence of categories between local small Higgs bundles and local small generalized representations.

Key ingredient of small case: descent of small representations

Denote $\Delta = Gal(\overline{R}/R_1)$ and $\Delta_\infty = Gal(R_\infty/R_1)$. The main idea of the proof states as follows:

(1) Descent part:

$$Rep_\Delta^{small}(\widehat{\overline{R}}) \longrightarrow Higgs^{small}(R_1),\ (\overline{M}, \rho) \mapsto (M, \theta)$$

Key ingredient: using the descent of small representations, there is an equivalence of categories

$$Rep_{\Delta_\infty}^{small}(R_1) \longrightarrow Rep_\Delta^{small}(\widehat{\overline{R}})$$

(2) Sen theory part:

$$Higgs^{small}(R_1) \longrightarrow Rep_\Delta^{small}(\widehat{\overline{R}}),\ (M, \theta) \mapsto M \otimes_{R_1} \widehat{\overline{R}}$$

Key ingredient: using Sen operator to recover the representation.

Main result

My results

The p-adic Simpson correspondence for line bundles

$$Pic^0_{X/\overline{\mathbb{Q}_p}}(\mathbb{C}_p) \times (\Gamma(X, \Omega^1_X) \otimes \mathbb{C}_p(-1))_{small} \longrightarrow Hom(\pi_1^{ab}(X), \mathbb{C}_p^*)_{small}$$

can be locally enhanced to the rigid analytic morphism

$$(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \times (\Gamma(X, \Omega^1_X) \otimes \overline{\mathbb{Q}_p}(-1) \otimes \mathbb{G}_a)_{small} \longrightarrow (\mathbb{G}_m^{2g})^{an}$$

under small conditions.

Main idea of the proof

- vector bundle case
- Higgs bundle case

Rigid geometry: Maximum principle for affinoid K-algebra

Let K be a field with a complete nonarchimedean absolute value that is nontrivial.

Definition

The K-algebra $T_n = K\langle \xi_1, \ldots, \xi_n \rangle$ of all formal power series

$$\sum_{\nu \in \mathbb{N}^n} c_\nu \xi^\nu \in K[[\xi_1, \ldots, \xi_n]], \quad c_\nu \in K, \quad \lim_{|\nu| \to \infty} |c_\nu| = 0.$$

i.e, converging on $\mathbb{B}^n(\overline{K}) := \{(x_1, \ldots, x_n) \in \overline{K}^n : |x_i| \leq 1\}$, is called the *Tate algebra* of restricted power series.

Definition

A K-algebra A is called an *affinoid K-algebra* if there is an epimorphism of K-algebra $\alpha : T_n \longrightarrow A$ for some $n \in \mathbb{N}$.

Rigid geometry: Maximum principle for affinoid K-algebra

For elements $f \in A$, set

$$|f|_{sup} = \sup_{x \in Max\, A} |f(x)|$$

where $Max\, A$ is the spectrum of maximal ideals in A and for any $x \in Max\, A$, write $f(x)$ for the residue class of f in A/x.

Theorem

For any affinoid K-algebra A and for any $f \in A$, there exists a point $x \in Max\, A$ such that $|f(x)| = |f|_{sup}$.

Rigid geometry: Canonical topology on affinoid K-space

For an affinoid K-space $X = Sp\,A$, set

$$X(f, \varepsilon) = \{x \in X : |f(x)| \leq \varepsilon\}$$

for $f \in A$ and $\varepsilon \in \mathbb{R}_{>0}$.

Definition

For any affinoid K-space $X = Sp\,A$, the topology generated by all sets of type $X(f; \varepsilon)$ with $f \in A$ and $\varepsilon \in \mathbb{R}_{>0}$ is called the canonical topology of X. Define $X(f) := X(f; 1)$.

Proposition

Let $\varphi^* : A \longrightarrow B$ be a morphism of affinoid K-algebras, and let $\varphi : Sp\,B \longrightarrow Sp\,A$ be the associated morphism of affinoid K-spaces. Then for $f_1, \ldots, f_r \in A$, we have

$$\varphi^{-1}((Sp\,A)(f_1, \ldots, f_r)) = (Sp\,B)(\varphi^*(f_1), \ldots, \varphi^*(f_r)).$$

In particular, φ is continuous with respect to the canonical topology.

Rigid geometry: GAGA functor on rigid analytic space

Let X be an affinoid K-space. For any affinoid subdomain $U \subset X$ we denote the affinoid K-algebra corresponding to U by $\mathcal{O}_X(U)$. $\mathcal{O}_X$ is a presheaf of affinoid K-algebras on the category of affinoid subdomains of X. Moreover, $\mathcal{O}_X$ is a sheaf by Tate's acyclicity Theorem.

Definition

A *rigid analytic K-space* is a locally G-ringed space $(X, \mathcal{O}_X)$ such that
(i) the Grothendieck topology of X satisfies $(G_0), (G_1)$ and (G_2).
(ii) X admits an admissible covering $(X_i)_{i\in I}$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K-space for all $i \in I$.

A *morphism of rigid K-spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism in the sense of locally G-ringed K-spaces. As usual, global rigid K-spaces can be constructed by gluing local ones.

Rigid geometry: GAGA functor on rigid analytic space

Now we construct a functor that associates any K-scheme Z of locally of finite type to a rigid K-space Z^{an}, called the *rigid analytification* of Z.

Proposition

Every K-scheme Z of locally finite type admits an analytification $Z^{an} \longrightarrow Z$. Furthermore, the underlying map of sets identifies the points of Z^{an} with the closed points of Z.

p-adic Simpson correspondence: Deninger-Werner construction

Aim: give the construction of the map

$$\alpha : Pic^0_{X/\overline{\mathbb{Q}_p}}(\mathbb{C}_p) \longrightarrow Hom_c(\pi_1^{ab}(X), \mathbb{C}_p^*)$$

The map α can be rephrased in terms of the Albanese variety A of X as a continuous, Galois equivariant homomorphism

$$\alpha : \widehat{A}(\mathbb{C}_p) \longrightarrow Hom_c(TA, \mathbb{C}_p^*)$$

The above map can be reduced from an additive exact functor

$$\rho : \mathcal{B}_{A_{\mathbb{C}_p}} \longrightarrow Rep_{TA}(\mathbb{C}_p),\ F \mapsto \rho_F : TA \longrightarrow Aut_{\mathbb{C}_p}(F_x)$$

Vector bundle case

Suppose that X is a curve with good reduction over $\overline{\mathbb{Q}_p}$.

Theorem

Let $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}$ *denote the set of topologically p-torsion elements of* $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an}$. *Then we may locally enhance the set-theoretical map*

$$\alpha : Pic^0_{X/\overline{\mathbb{Q}_p}}(\mathbb{C}_p) \longrightarrow Hom_c(\pi_1^{ab}(X), \mathbb{C}_p^*)$$

to the rigid analytic morphism

$$\alpha^{an} : (Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \longrightarrow (\mathbb{G}_m^{2g})^{an}.$$

Main idea

The main idea is to regard the morphism α^{an} as gluing by the rigid analytic morphism on the affinoid neighborhood from the viewpoint of the Lie algebra map. The motivation comes from the Taylor expansion and then consider the tangent space at the origin. The key ingredient is the following commutative diagram

$$\begin{tikzcd}
(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \arrow[r, "log"] \arrow[d, "\alpha^{an}"] & H^1(A,\mathcal{O}) \otimes \mathbb{G}_a \arrow[d, "I"] \\
(\mathbb{G}_m^{2g})^{an,tt} \arrow[r, "log"] & (\mathbb{G}_a^{2g})^{an}
\end{tikzcd}$$

Reduction to affinoid neighborhood

The morphism

$$\alpha^{an} : (Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \longrightarrow (\mathbb{G}_m^{2g})^{an,tt}$$

maps 0 to 1. On the point level, we have $\alpha^{an}(\mathbb{C}_p) = \alpha|_{(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}}$. Fix an element $x_0 \in |(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}|$ and define $y_0 = \alpha^{an}(x_0)$. Observe that both sides have rigid group structure. Consider the translation maps

$$T_{x_0}(x) = x + x_0, \quad T_{y_0}(y) = y \cdot y_0$$

and the following diagram

$$\begin{tikzcd} U_0 \arrow[r, "\alpha^{an}|_{U_0}"] \arrow[d, "T_{x_0}"] & U_1 \arrow[d, "T_{y_0}"] \\ U_{x_0} \arrow[r, "\alpha^{an}|_{U_{x_0}}"] & U_{y_0} \end{tikzcd} \tag{0.1}$$

Reduction to affinoid neighborhood

We can check that

$$\alpha^{an}|_{U_{x_0}} \circ T_{x_0} = T_{y_0} \circ \alpha^{an}|_{U_0}.$$

Namely, the above diagram is commutative.

It suffices to prove that $\alpha^{an}|_{U_0}$ is rigid analytic. Then $\alpha^{an}|_{U_{x_0}} = T_{y_0} \circ \alpha^{an}|_{U_0} \circ T_{-x_0}$ is rigid analytic. It yields that α^{an} is rigid analytic by gluing the rigid analytic morphisms.

The techinique step is to shrink U_0 to an open affinoid neighborhood in order to give the natural transformation of the sheaf of rigid analytic functions.

logarithm

let us construct logarithm on rigid analytic space in the following diagram.

$$\begin{tikzcd}
(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \arrow[r, "log"] \arrow[d, "\alpha^{an}"] & H^1(A,\mathcal{O}) \otimes \mathbb{G}_a \arrow[d, "I"] \\
(\mathbb{G}_m^{2g})^{an,tt} \arrow[r, "log"] & (\mathbb{G}_a^{2g})^{an}
\end{tikzcd}$$

Let $K/\mathbb{Q}_p$ be a complete valued field.

Definition

A commutative rigid analytic K-group G is said to be *rigid analytic p-divisible group* if
(i) (Topologically p-torsion)
For any $g \in G^{Berk}$, $\lim_{n\to\infty} p^n g = 0$ with respect to the topology of the Berkovich space $|G^{Berk}|$.
(ii) The morphism $\times p : G \longrightarrow G$ is finite and surjective.

logarithm

Proposition (Fargues)

For a rigid analytic p-divisible group G, there is a functorial morphism of rigid analytic spaces

$$log_G : G \longrightarrow Lie(G) \otimes \mathbb{G}_a.$$

Let $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}$ denote the set of topologically p-torsion elements of $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an}$, then $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}$ is the open subspace of $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an}$.

Notice that the morphism $\times p : (Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt} \longrightarrow (Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}$ is finite and surjective, we deduce that $(Pic^0_{X/\overline{\mathbb{Q}_p}})^{an,tt}$ is a rigid analytic p-divisible group.

Higgs bundle case

Theorem

One can enhance the map

$$(\Gamma(X, \Omega^1_X) \otimes \mathbb{C}_p(-1))_{small} \longrightarrow Hom(\pi_1^{ab}(X), \mathbb{C}_p^*)$$

to the morphism of rigid analytic spaces

$$(\Gamma(X, \Omega^1_X) \otimes \overline{\mathbb{Q}_p}(-1) \otimes \mathbb{G}_a)_{small} \longrightarrow (\mathbb{G}_m^{2g})^{an}$$

with small conditions.

Main idea

Now we present the construction of the map by Faltings

$$\Gamma(X, \Omega^1_X) \otimes_{\overline{\mathbb{Q}_p}} \mathbb{C}_p(-1) \longrightarrow Hom(\pi_1^{ab}(X), \mathbb{C}_p^*)$$

The above map is the exponential of a $\mathbb{C}_p$-linear map into the Lie algebra of the group of representations.

Hodge-Tate decomposition

More precisely, the $\mathbb{C}_p$-linear map is

$$\Gamma(X, \Omega_X^1) \otimes_{\overline{\mathbb{Q}_p}} \mathbb{C}_p(-1) \longrightarrow (\Gamma(X, \Omega_X^1) \otimes \mathbb{C}_p(-1)) \oplus (H^1(X, \mathcal{O}_X) \otimes \mathbb{C}_p)$$

where the first component is the identity, and the second one depends on the choice of lifting of X to $A_2(V)$. Using the Hodge-Tate decomposition, we have the isomorphism

$$H^1_{ét}(X, \overline{\mathbb{Q}_p}) \otimes_{\overline{\mathbb{Q}_p}} \mathbb{C}_p \simeq (\Gamma(X, \Omega_X^1) \otimes \mathbb{C}_p(-1)) \oplus (H^1(X, \mathcal{O}_X) \otimes \mathbb{C}_p).$$

Note that $Hom(\pi_1^{ab}(X), \mathbb{C}_p) \simeq H^1_{ét}(X, \overline{\mathbb{Q}_p}) \otimes_{\overline{\mathbb{Q}_p}} \mathbb{C}_p$, then an exponential map from $\mathbb{C}_p$ to $\mathbb{C}_p^*$ induces the map

$$Hom(\pi_1^{ab}(X), \mathbb{C}_p) \longrightarrow Hom(\pi_1^{ab}(X), \mathbb{C}_p^*).$$

Composing the maps together, we get the desired map.

Expontential map

Proposition

Any exponential map Exp is not rigid analytic.

The main idea of the proof is to illustrate that we can not extend *exp* to *Exp* rigid analytically by progressively increasing balls.

Since any exponential map *Exp* is not rigid analytic as shown in Proposition, we would like to add small conditions, which is the restriction of *Exp* on the open ball $\{x \in \mathbb{C}_p : |x| < r_p = |p|^{\frac{p}{p-1}}\}$. Under the small condition, the exponential map is rigid analytic.

Further discussion: Ben Heuer's work

Ben Heuer also gives an alternative approach to the problem as follows:

- *Construction*: Up to a splitting of Hodge-Tate logarithm sequence, v-line bundles can be interpreted as Higgs bundles. For proper X, we use this to construct the p-adic Simpson correspondence of rank one.
- *Upgraded*: Determine which line bundles are trivialized by pro-finite-étale covers, use this to show that there is an isomorphism of rigid analytic group varieties between the moduli space of continuous characters of $\pi_1(X, x)$ and that of pro-finite étale Higgs line bundles on X.

Four topologies on smooth rigid analytic space

Suppose X is a smooth rigid analytic space

- Consider X as an adic space, we have the inclusion of sites;

$$X_{an} \subset X_{\acute{e}t} \subset X_{pro\acute{e}t}$$

- Via the diamondification functor $X \mapsto X^{\diamond}$, we have the inclusion of sites:

$$X^{\diamond}_{\acute{e}t} \subset X^{\diamond}_{qpro\acute{e}t} \subset X^{\diamond}_{v}$$

- Combine the above, it turns out that

$$X_{an} \subset X_{\acute{e}t} = X^{\diamond}_{\acute{e}t} \subset X_{pro\acute{e}t} \subset X^{\diamond}_{qpro\acute{e}t} \subset X^{\diamond}_{v}$$

p-adic Simpson correspondence for line bundles

Let K be a perfectoid field over $\mathbb{Q}_p$ and X be a smooth proper rigid analytic space over K. Then we have the following exact sequence

$$0 \longrightarrow Pic_{an}(X) \longrightarrow Pic_v(X) \xrightarrow{HTlog} H^0(X, \Omega^1_X\{-1\}) \longrightarrow 0$$

it induces that

$$0 \longrightarrow Pic^{tt}_{an}(X) \longrightarrow Hom(\pi_1(X,x), K^\times) \longrightarrow H^0(X, \Omega^1_X\{-1\}) \longrightarrow 0$$

is a splitting exact sequence.

Let $s : H^0(X, \Omega^1_X\{-1\}) \longrightarrow Hom(\pi_1(X,x), K^\times)$ be the splitting, for $\rho \in Hom(\pi_1(X,x), K^\times)$, define $\theta_\rho = HTlog(\rho)$ and $L_\rho = \rho/s(\theta_\rho)$, then $\rho \mapsto (L_\rho, \theta_\rho)$ is an equivalence of categories from representations to Higgs bundles of rank one.

Upgrade to isomorphism of moduli spaces

Theorem

Let X be a connected smooth proper rigid analytic space over K and fix $x \in X(K)$, then there is a natural short exact sequence of rigid group varieties

$$0 \longrightarrow \underline{Pic}^{tt}_{X,\acute{e}t} \longrightarrow \underline{Hom}(\pi_1(X,x),\mathbb{G}_m) \xrightarrow{HTlog} H^0(X,\Omega^1_X)\otimes_K \mathbb{G}_a \longrightarrow 0$$

where $\pi_1(X,x)$ is considered as a profinite adic group. Any choice of an exponential function on K and a Hodge-Tate splitting for X induces a splitting of this sequence.


Thanks