Outline

Motivation. Weight-monodromy conjecture X: sm. proj. var. $/\mathbb{Q}_p$

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright V = H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$$

- $\Phi \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ geometric Frobenius

$\rightsquigarrow V = \bigoplus_{j=0}^{2i} V_j$, where Φ acts on V_j through Weil number of weight j.

- monodromy functor $N: V \to V$

$$N(V_j) = V_{j-2}$$

Conjecture: $N^j : V_{i+j} \xrightarrow{\simeq} V_{i-j}$, $\forall j = 0, 1, \dots, i$

Deligne's result: Known in equal char p case. i.e, $\mathbb{F}_p((t))$

Idea: $\mathbb{Q}_p$ v.s. $\mathbb{F}_p((t))$ global field

$\uparrow$ mixed char. $(0,p)$ $\uparrow$ equal char p

添 $p^{\frac{1}{p^\infty}}$

$\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ v.s. $\mathbb{F}_p((t))$

(arithmetic)

Thm. (Fontaine-Wintenberger)
The absolute Galois group of $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ and $\mathbb{F}_p((t))$ are isomorphic.

(geometric counterpart)

$$\{\text{perfectoid spaces}/K\} \xrightarrow{\simeq} \{\text{perfectoid spaces}/K^\flat\}$$

$$X \longmapsto X^\flat$$

In detail. $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(p^{\frac{1}{p^\infty}})) \underset{\uparrow}{\simeq} \mathrm{Gal}(\mathbb{F}_p((t))^{sep}/\mathbb{F}_p((t)))$

canonical

Following proof: $\{$finite étale $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$-alg.$\}$

$$\simeq \{\text{finite étale } \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})\text{-alg.}\}$$

Introduce the notion "perfectoid" in relative case

(inspired by the vanishing of the cotangent complex)

$\{$perfectoid $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$-alg.$\}$ $\xrightarrow{\simeq}$ $\{$perfectoid $\mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$-alg.$\}$

$$R \longmapsto R^\flat = (\varprojlim_{\mathrm{Frob}} R^\circ/p) \otimes_{\mathbb{F}_p[[t]][t^{\frac{1}{p^\infty}}]} \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$$

<u>Pass to geometry</u> $(\mathbb{P}^n_{\mathbb{F}_p((t))})_{\text{ét}} = \varprojlim_{\varphi} (\mathbb{P}^n_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})_{\text{ét}}$

$$\varphi(x_0 : x_1 : \cdots : x_n) = (x_0^p : x_1^p : \cdots : x_n^p)$$

$X \subset \mathbb{P}^n$ sm. proj. var.

$$\begin{array}{ccc} \mathbb{P}^n_{\mathbb{F}_p((t))} & \xrightarrow{\pi} & \mathbb{P}^n_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \\ \cup & & \cup \\ \pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}) & \longrightarrow & X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \end{array}$$

Apply Deligne's result $\to$ $\pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})$ not algebraic

How far does it away from algebraic?

<u>Easy case</u>: If X is complete intersection, then in any ε-neighborhood of $\pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})$ are alg. varieties of same dim.

Enough to conclude.

Motivation on "Perfectoid Spaces"

Weight monodromy conjecture

monodromy operator

Weight decomposition

(About ℓ-adic cohomology of projective smooth varieties over a p-adic field K)

Let X be a projective smooth scheme $/\mathbb{Q}_p$

Fix $i \geq 0$, $\ell \neq p$ prime

Consider the ~~Galois~~ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations of $V = H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright V = H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$$

- known

(Weight decomposition of V): ~~There~~

If $\Phi \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is geometric Frobenius,

$\Phi : \overline{\mathbb{Q}}_p \longrightarrow \overline{\mathbb{Q}}_p$ project to residue field (char p) is Frob.

then $V = \bigoplus_{j=0}^{2i} V_j$ $\quad \Phi \curvearrowright V_j \quad v \in V_j,\ \Phi \circ v \in V_j$

where Φ acts through the Weil number of weight j on V_j.

(Rapport - Zink : If X has semistable reduction)

de Jong : In general (Reduction to semistable reduction)

- There is a monodromy operator

$N: V \to V$ coming from action of inertia subgroup

In particular, $N: V_j \longrightarrow V_{j-2}$

(Weight-monodromy conjecture)

$\forall j = 0, 1, \dots, i, \quad N^j : V_{i+j} \xrightarrow{\simeq} V_{i-j}$

Weight decomposition is start from semistable reduction

$i+j-2j = i-j$

<u>Example</u>. (1) X has good reduction

(2) E elliptic curve with multiplicative reduction

<u>Remark</u> (1) Weight-monodromy conjecture is true for $i = 0, 1, 2$

$i=1$. Reduction to abelian varieties / curves

$H^1_{ét}(X_{\bar{\mathbb{Q}}_p}, \bar{\mathbb{Q}}_\ell)$ Using Neron model / semistable model

$i=2$ Rapoport - Zink & de Jong

(2). Deligne. Known in equal char. p. i.e, over $\mathbb{F}_p((t))$

Proved by Deligne's Weil 2 paper, using L-functions over function fields have good properties

(3). Weight-monodromy conj. is critical to understanding local factors of Hasse-Weil Zeta functions at places of bad reduction

Introduction to spectral sequence

Compute the homology or cohomology of a chain complex.
(arrow 反过来)

Motivation: Computing the homology of the total chain complex T_* of a first quadrant double complex E_{**}.
$T_* = \mathrm{Tot}(E_{**})$

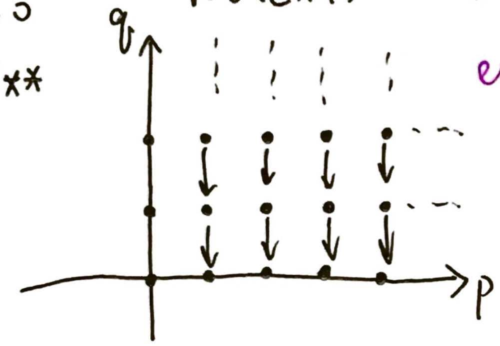
The algorithm for computing $H_*(T)$ up to extension 问题 called a spectral sequence.

$H_*(\mathrm{Tot}(E_{**}))$

$H_*(E_{**})$

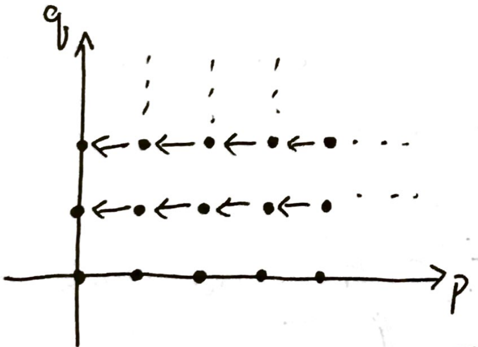
逼近

E^0_{**}



$E^1_{pq} = H_q(E^0_{p*})$ at (p, q) spot

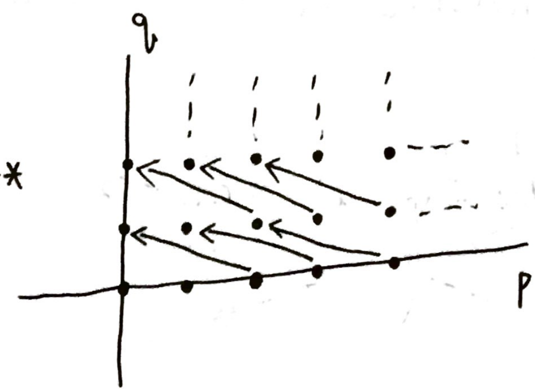
E^1_{**}



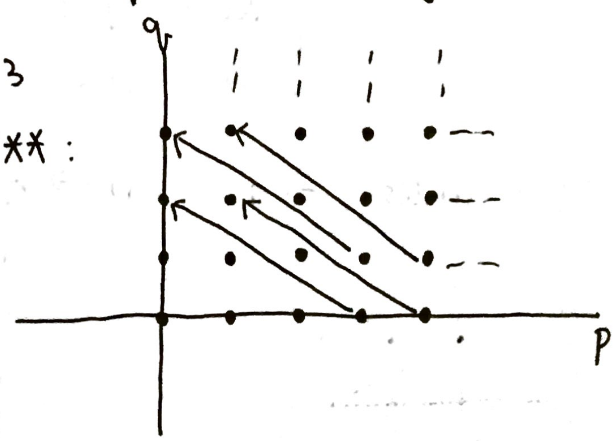
$E^2_{pq} = H_p(E^1_{*q})$

The elements of E^2_{pq} are a second-order approximation of the homology of $T_* = \mathrm{Tot}(E_{**})$

E^2_{**}



E^3_{**}:



$$E^a_{pq} \Longrightarrow H_{p+q}$$

(2 columns). Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $p = 0, 1$

Show that there are exact sequences 上下标换

$$0 \longrightarrow E^2_{1,n-1} \longrightarrow H_n \longrightarrow E^2_{0n} \longrightarrow 0$$

Example of weight monodromy conjecture

E : elliptic curve with multiplicative reduction $\quad E = \mathbb{G}_m / q^{\mathbb{Z}}$ rigid analytic space

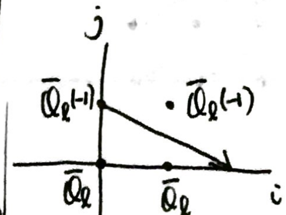
$V = H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) = H^1_{\acute{e}t}(\underset{\substack{\uparrow \\ \mathbb{G}_m}}{\mathbb{G}_m} / q^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell) \qquad 0 \neq q \in \mathbb{Q}_p,\ |q| < 1$

By Hochsohild - Serre spectral sequence. $E_2^{ij} = \underset{\text{gp. coho.}}{H^i(\mathbb{Z}}, H^j_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)) \Rightarrow H^{i+j}_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}/q^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell)$

$$H^j_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) = \begin{cases} \overline{\mathbb{Q}}_\ell & j = 0 \\ \overline{\mathbb{Q}}_\ell(-1) & j = 1 \\ 0 & \text{else} \end{cases}$$

$\mathbb{Z} \curvearrowright H^j_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$ trivial action

$E_2^{ij} = 0$ unless $j = 0, 1$



$$\cancel{0 \to E^2_{01} \to H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) \to E^2_{10} \to 0}$$

$$E^2_{01} = \overline{\mathbb{Q}}_\ell(-1) = V_2, \quad H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) = V, \quad E^2_{10} = \overline{\mathbb{Q}}_\ell = V_0$$

$$
\begin{tikzcd}
0 \arrow[r] & E_2^{10} \arrow[r] \arrow[d, phantom, "E_2^{10}"] & H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) \arrow[r] & E_2^{01} \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
& {} \arrow[d, phantom, "\shortparallel"] & & \overline{\mathbb{Q}}_\ell(-1) \arrow[d, phantom, "\shortparallel"] & \\
& \overline{\mathbb{Q}}_\ell \arrow[d, phantom, "\shortparallel"] & & V_2 & \\
& V_0 & & &
\end{tikzcd}
$$

$$V = V_0 \oplus V_2$$

上下左右标互换 $\quad H_n \quad H^n$

Weight-Monodromy predicts $N : V_2 \xrightarrow{\simeq} V_0$

Check this.

Combine objects in mixed char. $(0,p)$ with objects in equal char. p

<u>Thm</u>. (Fontaine - Wintenberger)
(arithmetic)
The absolute Galois group of $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ and $\mathbb{F}_p((t))$ are isomorphic.

(Geometric counterpart)

Generalize to $K = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$, $K^\flat = \mathbb{F}_p((t^{\frac{1}{p^\infty}}))$

$$\{\text{perfectoid spaces}/K\} \xrightarrow[\text{equiv.}]{\simeq} \{\text{perfectoid spaces}/K^\flat\}$$

$$X \longmapsto X^\flat$$

<u>tilt functor</u>

- perfectoid field. <u>Def</u>. A NA field of residue field char p is perfectoid if (1) the value group $|K^\times| \subset \mathbb{R}_{>0}$ is nondiscrete

  (2) K°/p is semiperfect. i.e. $K^\circ/p \twoheadrightarrow K^\circ/p$
  $x \longmapsto x^p$

$\mathbb{Q}_p$ not satisfied (1)

$|\mathbb{Q}_p^\times| = |p|^{\mathbb{Z}} \subset \mathbb{R}_{>0}$ is discrete.

<u>Example</u>. (1) toy example.

$K = \widehat{\overline{\mathbb{Q}_p}} = \mathbb{C}_p$ $\quad K = \overline{K} \supset K^\circ \ni t$

$x^p - t = 0$ has roots in K

$|K^*| = |p|^{\mathbb{Q}} \subset \mathbb{R}_{>0}$ nondiscrete

(2) $K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$

(completion. Do analysis (take limits))

$$\mathbb{Q}_p(p^{\frac{1}{p^\infty}}) = \bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}}) = \varinjlim_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})$$

- $\varinjlim$, completion 保 valuation ring 结构

- perfection of a ring R

R, char $R = p$ is perfect if $\begin{array}{c} R \to R \\ x \mapsto x^p \end{array}$ is an isomorphism.

$R^\flat := \varprojlim_{\phi} (R/p)$ is a perfect ring of char p

$$\varprojlim_{x \mapsto x^p} R \xrightarrow{\sim} \varprojlim_{\phi} (R/p) = R^\flat$$

$$(a_n) \longmapsto (a_n \bmod p)$$

$b_n = \lim_{k \to \infty} a_{n+k}^{p^k}$ lifting $\bar{a}_n$

- tilt functor of a perfectoid field

K: perfectoid field

Define $K^\flat = K^{\circ\flat}[\frac{1}{t}] = (\varprojlim_{x \mapsto x^p} K^\circ/p)[\frac{1}{t}]$, where $t = (p^{\frac{1}{p^n}})_{n \in \mathbb{N}}$ p-power inverse system.

is a perfectoid field of char p

(In fact, $K^\flat$ is a perfect field)

Example.

(1). $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$

$K^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$

(2) $K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$

$K^\flat = \widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$

<u>Remark</u>. $K^\flat \neq \varprojlim_{x \mapsto x^p} K$ if char $K = 0$

To show this, let us make the following observation

Suppose K is NA field, K is perfect field of char $p \iff K$ is perfectoid field of char p

pf: "$\Rightarrow$" $\begin{array}{c} K \to K \\ x \mapsto x^p \end{array}$ is isom. $\Rightarrow |K^\times|$ is nondiscrete

char $K = p \Rightarrow$ char $K^\circ = p \Rightarrow K^\circ/p = K^\circ$

$$\begin{array}{ccc} K & \to & K \\ \cup & & \cup \\ K^\circ & \to & K^\circ \end{array}$$

inj is obvious.

surj. $\forall y \in K^\circ \subset K$, $\exists x \in K$ s.t. $y = x^p$, $1 \geq |y| = |x|^p$. $|x| \leq 1$, $x \in K^\circ$

$\Leftarrow$ $K^\circ \twoheadrightarrow K^\circ$ surj.
$x \mapsto x^p$

$|p|<1$. $\forall x \in K$, $\exists n \in \mathbb{N}$ s.t. $|p^n x| \leq 1$

$x \in K^\circ[\frac{1}{p}]$

$K = K^\circ[\frac{1}{p}]$ localization

$K^\circ \twoheadrightarrow K^\circ \rightsquigarrow \underset{K=}{K^\circ[\frac{1}{p}]} \twoheadrightarrow K^\circ[\frac{1}{p}] = K$ $\quad$ $K \to K$
$x \mapsto x^p$ $\quad$ $x \mapsto x^p$

$x^p = 0 \Rightarrow x = 0$ inj.

<u>char $K = p$</u>. $\varprojlim_{x \mapsto x^p} K \simeq K$

K is perfectoid of char p
$\Rightarrow K$ is perfect
$\Rightarrow K \to K$ isom.
$x \mapsto x^p$

$$\left(\varprojlim_{x\mapsto x^p} (K^\circ/p)\right)[\tfrac{1}{t}] = \left(\varprojlim_{x\mapsto x^p} K^\circ\right)[\tfrac{1}{t}] \xrightarrow[pr_0]{\simeq} K^\circ[\tfrac{1}{p}] = K$$

char $K^\circ = p$, perfect

But if char $K=0$, $$\left(\varprojlim_{x\mapsto x^p} (K^\circ/p)\right)[\tfrac{1}{t}] \simeq \left(\varprojlim_{x\mapsto x^p} K^\circ\right)[\tfrac{1}{t}] \not\simeq \varprojlim_{x\mapsto x^p} K^\circ[\tfrac{1}{p^{\frac{1}{p^n}}}] = \varprojlim_{x\mapsto x^p} K$$

localization and <u>lim</u> doesn't commute

- tilt of perfectoid alg. over perfectoid field

A: perfectoid algebra over a perfectoid ~~alg~~ field K

$A^\flat := \left(\varprojlim_{x\mapsto x^p} (A^\circ/p)\right)[\tfrac{1}{t}]$ perfectoid alg. over $K^\flat$

/ Back to weight-monodromy conjecture.

Rapoport's suggestion:

Try to reduce to the equal char. p case after base change to some very ramified $K/\mathbb{Q}_p$. $X \times_{\mathbb{Q}_p} K$

model

Attempt:

Idea: If $\mathfrak{X}$ is integral model (say, semistable) of $X \times_{\mathbb{Q}_p} K$,

then its special fiber $\mathfrak{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}(\mathcal{O}_K/p)$ lives over

$\mathcal{O}_K/p \simeq \cancel{\mathbb{F}_q[t]}\ \mathbb{F}_q[t]/t^e$, e = ramification index of $K/\mathbb{Q}_p$

$\mathcal{O}_K/p \simeq \mathbb{F}_q[t]/t^e$

If $e \gg 0$, this is almost $\mathbb{F}_q[[t]]$

Of course, this does really work.

If e is large, need to deform $\mathfrak{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}(\mathcal{O}_K/p)$

from $\mathcal{O}_K/p \simeq \mathbb{F}_q[t]/t^e$ to $\mathbb{F}_q[[t]]$

Usually, there are a lot of deformations.

deformation the special fiber of model from $\mathbb{F}_q[t]/t^e$ to

$\mathbb{F}_q[[t]]$, $\mathbb{F}_q[[t]] \underset{\text{fraction}}{\overset{\text{completion}}{\rightsquigarrow}} \mathbb{F}_q((t))$, $X'/_{\mathbb{F}_p((t))}$ generic fiber of deformatio

Also relate (in the end)

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell) = V$$

to $$\mathrm{Gal}(\mathbb{F}_p((t))^{sep}/\mathbb{F}_p((t))) \curvearrowright H^i_{\acute{e}t}(X'_{\overline{\mathbb{F}_p((t))}}, \overline{\mathbb{Q}}_\ell)$$

where $X'/_{\mathbb{F}_p((t))}$ is the generic fiber of the deformation

Q: In the semistable case, can we use log geometry to do this?
(relate to the isomorphism of the tame quotients of
$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathrm{Gal}(\mathbb{F}_p((t))^{sep}/\mathbb{F}_p((t)))$)

Hard way

Scholze's way: (in detail of arithmetic and its geometric counterpart before)

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$

Theorem (Fontaine-Wintenberger)

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(p^{\frac{1}{p^\infty}})) \underset{\substack{\uparrow \\ \text{canonical}}}{\simeq} \mathrm{Gal}(\mathbb{F}_p((t))^{sep}/\mathbb{F}_p((t))) \qquad \mathbb{Q}_p(p^{\frac{1}{p^\infty}}) = \bigcup_n \mathbb{Q}_p(p^{\frac{1}{p^n}}) \subset \overline{\mathbb{Q}}_p$$

Faltings. $$\pi_1^{\acute{e}t}(\mathrm{Spec}\, \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\langle T^{\pm\frac{1}{p^\infty}}\rangle) \simeq \pi_1^{\acute{e}t}(\mathrm{Spec}\, \mathbb{F}_p((t))\langle T^{\pm 1}\rangle)$$

$H^i_{\acute{e}t}$ like this.

Pf of Theorem: Involves Fontaine's construction like $\varprojlim_{\mathrm{Frob}} \mathcal{O}_{\overline{\mathbb{Q}}_p}/p$

Things started to resolve after I realized the following proof of

Fontaine - Wintenberger's Theorem.

$\boxed{\text{finite étale } K\text{-alg. } A \text{ is } A \simeq \prod_{i=1}^{n} L_i/K \text{ for some } n \in \mathbb{N},\ L_i/K \text{ finite separable}}$

$\{\text{finite étale } \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\text{-alg.}\}$

$\wr\!\!\mid$ ↗ Tate's "p-divisible group"

technical condition ↙

$\{\text{almost finite étale } \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]\text{-alg.}\}$ integral $\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} = \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$

See Appendix

unique lifting of finite étale alg. $\to\ \wr\!\!\mid$

$\{\text{almost finite étale } \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]/p\text{-alg.}\}$

$\wr\!\!\mid$

$\{\text{almost finite étale } \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t\text{-alg.}\}$

$\wr\!\!\mid$

$\{\text{almost finite étale } \mathbb{F}_p[[t]][t^{\frac{1}{p^\infty}}]\text{-alg.}\}$

$\wr\!\!\mid$

$\{\text{finite étale } \mathbb{F}_p((t))\text{-alg.}\} \simeq \{\text{finite étale } \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})\text{-alg.}\}$

This suggests to do in the relative case.

Introduce the notion "perfectoid" (Inspired by the vanishing of cotangent complex)

$\{\text{perfectoid } \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\text{-alg.}\} \simeq \{\text{almost perfectoid } \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]\text{-alg.}\}$

$\wr\!\!\mid$

$\{\text{almost perfectoid } \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]/p\text{-alg.}\}$

$\wr\!\!\mid$

$\vdots$

$\{\text{perfectoid } \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})\text{-alg.}\}$

If R is almost perfectoid $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]/p$-alg. then the cotangent complex $\mathbb{L}_{R/\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]/p} = 0$

relate to Ω (differential)

Another reason.

Lemma: If $S \to R$ ~~is~~ a map of $\mathbb{F}_p$-algebras that is "relatively perfect"

i.e, relative Frobenius $\Phi_{R/S} : R \otimes_{S,\Phi} S \to R$ is an isomorphism.

then $\mathbb{L}_{R/S} = 0$

$r \otimes s \mapsto r^p s$

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "s \mapsto f(s)\otimes 1"'] \arrow[dr, phantom, "\square"] & S \arrow[d, "f"] \\ R \otimes_{S,\Phi} S \arrow[r] & R \end{tikzcd}$$

$r \otimes s \mapsto r^p \cdot s$

Pf: $\Phi_{R/S}$ is an isomorphism of $\mathbb{L}_{R/S}$

but also equal to 0 as $d(x^p) = p\,\cancel{dx}\, p x^{p-1} dx = 0$

Definition A perfectoid $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$-algebra is a uniform Banach ~~algebra~~ $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$-alg. s.t. $R^\circ/p \big/ \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]/p$ is relatively perfect.

over

where $R^\circ = \{$ power bounded elements in $R \}$ $\quad \sup_n \|x^n\| < \infty$

Equivalently, $\Phi : R^\circ/p \twoheadrightarrow R^\circ/p$ Frobenius is surj.

Corollary. $\{\text{perfectoid } \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\text{-alg.}\} \xrightarrow{\cong} \{\text{perfectoid } \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})\text{-alg.}\}$

$$R \longmapsto R^\flat$$

This can be written explicitly in terms of Fontaine functor as follows: $R^\flat = \left(\varprojlim_{\text{Frob}} (R^\circ/p)\right) \otimes_{\mathbb{F}_p[[t]][t^{\frac{1}{p^\infty}}]} \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$

Pass to geometry

$$(\mathbb{P}^n_{\mathbb{F}_p((t))})_{ét} = \varprojlim_{\varphi} \left(\cancel{\mathbb{P}^n_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}}\right) (\mathbb{P}^n_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})_{ét}$$

étale topos

where $\varphi(x_0 : x_1 : \cdots : x_n) = (x_0^p : x_1^p : \cdots : x_n^p)$

$X \subset \mathbb{P}^n$ is your smooth projective variety.

$$\begin{array}{ccc} \mathbb{P}^n_{\mathbb{F}_p((t))} & \xrightarrow{\pi} & \mathbb{P}^n_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \\ \cup & & \cup \\ \pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}) & \longrightarrow & X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \end{array}$$

Apply Deligne's result

But the problem is that $\pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})$ is not algebraic! (adic space)

Easy case: Suppose X is complete intersection. then in any ε-neighborhood of $\pi^{-1}(X_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})})$ are algebraic varieties of same dim.

Enough to conclude.

Appendix $\widehat{K} = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \Rightarrow K^\circ = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$

Proof: Step 1: completion of valuation rings are valuation rings

i.e, $\mathcal{O}_{\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}} = \widehat{\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}}$

Step 2: filtered colimit of valuation rings are valuation rings

i.e, $\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} = \mathcal{O}_{\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})} = \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})}$

Step 1. Denote $L = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$

$\mathcal{O}_{\hat{L}} = \hat{\mathcal{O}}_L$

"$\subseteq$" $\forall \alpha \in \mathcal{O}_{\hat{L}}$, $\exists \{\alpha_n\} \subseteq L$ s.t. $\alpha = \lim\limits_{n\to\infty} \alpha_n$

$\mathcal{O}_{\hat{L}} \subseteq \hat{L}$

$1 \geq |\alpha| = \lim\limits_{n\to\infty} |\alpha_n|$

$\exists n_0 \in \mathbb{N}$ s.t. $|\alpha_n| \leq 1$, $\forall n \geq n_0$

i.e, $\alpha_n \in \mathcal{O}_L$, $\forall n \geq n_0$

Hence $\alpha \in \widehat{\mathcal{O}_L}$

"$\supseteq$" $\forall \beta \in \widehat{\mathcal{O}_L} \subseteq \hat{L}$, $\exists \{\beta_n\} \subseteq \mathcal{O}_L$ s.t. $\beta = \lim\limits_{n\to\infty} \beta_n$

$|\beta| = \lim\limits_{n\to\infty} |\beta_n| \leq 1 \Rightarrow \beta \in \mathcal{O}_{\hat{L}}$

Step 2. $\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} = \mathcal{O}_{\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})} = \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})}$

"$\subseteq$" $\forall \alpha \in \mathcal{O}_{\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})}$

$\exists n_0 \in \mathbb{N}$ st. $\alpha \in \mathbb{Q}_p(p^{\frac{1}{p^{n_0}}})$

$$|\alpha| \leq 1 \Longrightarrow \alpha \in \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})}$$

"$\supseteq$" $\mathbb{Q}_p(p^{\frac{1}{p^n}}) \subseteq \bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})$, $\forall n$

$$\Rightarrow \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})} \subseteq \mathcal{O}_{\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})} \Rightarrow \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})} \subseteq \mathcal{O}_{\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{\frac{1}{p^n}})}$$

$$\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})} = \mathbb{Z}_p[p^{\frac{1}{p^n}}]$$

In fact, $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$ (Laurent series p^{-n}).

$|p^{\frac{1}{p^n}}| = |p|^{\frac{1}{p^n}} < 1$

$\overset{L}{\cancel{K(\alpha)}}/K$ $\quad |\alpha|' = |N_{K(\alpha)/K}(\alpha)|^{\frac{1}{d}}$

$|\cdot|'$ $\;|\cdot|$ $\quad \alpha \in L$

Hence $\mathcal{O}_{\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}} = \widehat{\mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}}$

$$= \widehat{\bigcup_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{Q}_p(p^{\frac{1}{p^n}})}}$$

$$= \widehat{\bigcup_{n\in\mathbb{N}} \mathbb{Z}_p[p^{\frac{1}{p^n}}]}$$

$$= \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$$