# Single-Step ODE solvers (PRU00)

**Submission deadline: 14.11.2022, 23:59**

Edilbert Christhuraj

November 1, 2022

## 1 Problem Description

### 1.1 Abstract problem

We consider an Ordinary Differential Equation (ODE) of the following form

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = F(t, X(t)), \tag{1}$$

where $X(t) \in \mathbb{R}^n$, where $n \in \mathbb{R}$, and $t \in [t_{\text{initial}}, T]$, where $t_{\text{initial}}$ and $T \in \mathbb{R}^+$, with a given initial condition as

$$X(t_{\text{initial}}) = X_{\text{initial}}. \tag{2}$$

**Test your understanding**

Look at Equations (1) and (2) and ask yourself

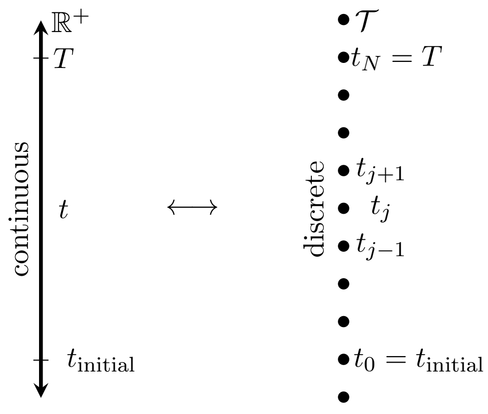
- What quantity am I supposed to find ?
- What is/(are) the dependent and independent variable(s)?
- What are the dimensions of $X(t)$, $t$, and $F(t, X(t))$?

### 1.2 Discrete problem

Since we want to use computer algorithms to solve our abstract problem (1) and (2), we must first convert the continuous abstract problem into a discrete one. Consider the continuous variable $t$ and discretise it by $t_j \in \mathcal{T}$ where $\mathcal{T}$ given by

$$\mathcal{T} := \{t_j \mid t_j \in \mathbb{R}^+; t_0 = t_{\text{initial}} < t_{j-1} < t_j < t_{j+1} < t_N = T; j = 0, 1, \cdots, N; N \in \mathbb{N}\}. \tag{3}$$

The above fuzzy-looking mathematical cypher can be decyphered into the following picture:



For convenience you can imagine the variable $t$ as time but it does not necessarily have to be time and it could be anything. Then, $t_0$ is the initial time which is equal to given initial time $t_{\text{inital}}$ which could be zero or any value in $\mathbb{R}^+$. At this point, we introduce the quantities

$$h_j = t_{j+1} - t_j \quad \forall j \in \{0, 1, \cdots, N\}, \tag{4}$$

which are the time-steps between two successive points in the discrete time axis. For this programming exercise (PRU00) we consider $h_j = h$.

**Test your understanding**

- Can you recall $\mathbb{R}^+$?
- Can you describe what you see in the diagram in your own words?
- What is $N$ above? Can you picture the discrete setup for $N = 5$?
- What happens to the discrete setup if $N \to \infty$?

Having discretised the independent variable $t$, we introduce the following notation for the dependent variable $X(t_j) := X_j$. You may understand $X_j$ as the function $X(t_j)$ evaluated at time $t_j$. We apply these notations in the Equation (1) to approximate the time derivative of $X(t)$ which gives

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \approx \frac{X_{j+1} - X_j}{h_j}. \tag{5}$$

But what about $F(t, X(t))$? We have several choices to approximate it. Based on our choices we have difference numerical schemes which we compiled below:

$$\begin{aligned}
\text{Explicit Euler Method} \quad & X_{j+1} = X_j + h_j F(t_j, X_j), && (6)\\
\text{Implicit Euler Method} \quad & X_{j+1} = X_j + h_j F(t_{j+1}, X_{j+1}), && (7)\\
\text{Implicit Midpoint rule Method} \quad & X_{j+1} = X_j + h_j F\left(t_j + \frac{h_j}{2}, \frac{X_j + X_{j+1}}{2}\right), && (8)\\
\text{Explicit Modified Euler Method} \quad & X_{j+1} = X_j + h_j F\left(t_j + \frac{h_j}{2}, X_j + \frac{h}{2} F(t_j, X_j)\right), && (9)\\
\text{Trapezoidal Method} \quad & X_{j+1} = X_j + h_j \frac{F(t_j, X_j) + F(t_{j+1}, X_{t+1})}{2}. && (10)
\end{aligned}$$

## 2 Test cases - Initial Value Problems (IVP)

### 2.1 IVP1

Consider an ODE

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \exp(-t), \tag{11}$$

where $X : \mathbb{R} \to \mathbb{R}$ and $t \in [0, 1.3]$, with initial the condition

$$X(0) = -1. \tag{12}$$

### 2.2 IVP2

Consider an ODE

$$\frac{\mathrm{d}^2 X(t)}{\mathrm{d}t^2} = -wX(t), \tag{13}$$

where $X : \mathbb{R} \to \mathbb{R}$, a constant $w \in \mathbb{R}^+$ (for e.g. you may take $w = 16$ ) and $t \in [0, 4]$, with the initial condition

$$X(0) = 1. \tag{14}$$

**Tip:**

Convert the problem into a system of first-order ODEs.

## 3 Tasks

1. Implement the numerical schemes Eqns. (6) - (10) in Python.
2. Derive the exact solutions for IVP-1 and IVP-2.
3. Compare the numerical solutions of Eqns. (6) - (10) with the derived exact solutions by plotting it.
4. Provide a convergence plot for Eq. (6) - (10). You may consider the following formula to calculate the $L_2$ errors

$$\frac{\|f^{\text{exact}} - f^{\text{approx.}}\|_2}{\|f^{\text{exact}}\|_2}. \tag{15}$$

20 **Points**