Stat 545 Part II Homework # 1

Fall 2022 Rice University

Date of Assignment:	October 19, 2022 (Wednesday)
Due Date:	October 28, 2022 (Friday)

Instructions:

- All problems can be answered by hand-writing but Word or LaTex is preferred.
- The total points of all the problems equal to 90, but if you prepare the answer sheet with Word or LaTex, an extra 10 points will be added to your score.
- Please staple all the answer sheets together in the right order and be sure to write your name on the first page.

Problem # 1. (15 points; 4 points for each of (a)-(e)) Let Y_i's be independent Poisson(λ_i) random variables ($i = 1, \cdots, n$) with density

$$\Pr(Y_i = y_i) = \frac{exp(-\lambda_i)\lambda_i^{y_i}}{y_i!} \text{ for } y_i = 0, 1, 2, \cdots .$$

(a) What are the natural parameter θ_i and the scale parameter ϕ?

(b) What is $b(\theta_i)$?

(c) What is $c(Y_i, \phi)$?

(d) Find/verify the mean and variance of Y_i based on θ_i and ϕ using the formulas presented in class notes.

(e) What is the "canonical" link function for the distribution?

Problem # 2. (15 points) Let Y_i's be indepdendent *Exponential*(λ_i) random variables ($i = 1, \cdots, n$) with density

$$f_{Y_i}(y_i; \lambda_i) = \lambda_i exp(-\lambda_i y_i) \text{ for } y_i > 0.$$

Repeat parts (a)-(e) of Question 1.

Problem # 3. (15 points) Let Y_i's be indepdendent $Gamma(\mu_i, \alpha)$ random variables $(i = 1, \cdots, n)$ with density

$$f_{Y_i}(y_i; \mu_i, \alpha) = \frac{1}{\Gamma(\alpha)(\mu_i/\alpha)^\alpha} y_i^{\alpha-1} exp(-\alpha y_i/\mu_i) \text{ for } y_i > 0.$$

where α is a fixed constant and μ_i is the parameter of interest. Repeat parts (a)-(e) of Question 1.

Problem # 4. (30 points; 10 points for each of (a)-(c)) What are the forms of the "Deviance" for the following three distributions:

(a) The inverse Gaussian distribution. The density function is

$$f_{Y_i}(y_i; \mu_i, \sigma) = \frac{1}{\sqrt{2\pi y_i^3 \sigma^2}} exp(\frac{-(y_i - \mu_i)^2}{2(\mu_i \sigma)^2 y_i}) \text{ for } y_i > 0 \ (i = 1, 2, \cdots, n),$$

where σ is a fixed constant and μ_i is the parameter of interest

(b) The Gamma(μ_i, α) distribution. The density function is shown in Problem 3.

(c) The Geometric distribution. The probability mass function is

$$f(Y_i = y_i) = (1 - p_i)^{y_i} p_i \text{ for } y_i \geq 0 \ (i = 1, 2, \cdots, n),$$

where p_i is the parameter of interest.

Problem # 5. (15 points) We consider a generalized linear model with $g(\mu_i) = \boldsymbol{x}_i^T \boldsymbol{\beta}$ $(i = 1, \cdots, n)$ and its score function $\boldsymbol{U}(\boldsymbol{\beta})$ where $\boldsymbol{\beta}$ is a vector of p parameters. Let $\boldsymbol{\beta} = (\beta_1^T, \beta_2^T)^T$ where $\boldsymbol{\beta}_1$ is the vector of first p_1 parameters, $\boldsymbol{\beta}_2$ is the vector of last $p - p_1$ parameters $(p > p_1)$; correspondingly $\boldsymbol{U}(\boldsymbol{\beta}) = (\boldsymbol{U}_1^T(\boldsymbol{\beta}), \boldsymbol{U}_2^T(\boldsymbol{\beta}))^T$ where $\boldsymbol{U}_1^T(\boldsymbol{\beta})$ is the vector of first p_1 equations, $\boldsymbol{U}_2^T(\boldsymbol{\beta})$ is the vector of last $p - p_1$ equations. Consider a composite null hypothesis $H_0 : \boldsymbol{\beta}_1 = \mathbf{0}$. Let $\widehat{\boldsymbol{\beta}}^0$ denote the maximum likelihood estimate of $\boldsymbol{\beta}$ under H_0. Find the log-likelihood and score equations under H_0, and show that $\boldsymbol{U}_2(\widehat{\boldsymbol{\beta}}^0) = \mathbf{0}_{p-p_1}$ where $\mathbf{0}_{p-p_1}$ is a vector of $p - p_1$ with each element equal to 0.