# The *Spot It!/Dobble* Problem

For context, *Spot It!* (also known as *Dobble* overseas) is a party game that is also frequently used in classrooms for educational purposes. The basic structure of the game is as follows: the deck features 55 cards, each of which has eight symbols printed on it in such a way that any two cards have exactly one symbol in common. At first, it may seem magical that the *Spot It!* cards match up so perfectly. It turns out that the math behind the game is arguably even more magical.

**Lemma 1.** *Given a finite field $k = F_q$ with $q$ elements and the three-dimensional vector space $V = k^3$, each two-dimensional subspace (plane through the origin) contains $q + 1$ one-dimensional subspaces (lines through the origin).*

**Proof:** Within a plane, every line through the origin can be defined by connecting the origin with some point on the plane distinct from the origin. Since there are $q^2$ points in each plane, we know that there are $q^2 - 1$ points in the plane distinct from the origin. Suppose we consider one such point and the line defined by connecting this point and the origin. This line will contain $q - 1$ points in the plane distinct from the origin. Therefore, if we were to define a line in a plane through the origin as a connection between the origin and a point in the place distinct from the origin, we would overcount by a factor of $q - 1$. Hence, every plane contains $\frac{q^2-1}{q-1} = q + 1$ lines through the origin.

**Lemma 2.** *Given a finite field $k = F_q$ with $q$ elements and the three-dimensional vector space $V = k^3$, $V$ contains $q^2 + q + 1$ two-dimensional subspaces (planes through the origin).*

**Proof:** We will first show that $V$ contains $q^2 + q + 1$ lines through the origin. Every line through the origin can be defined by connecting the origin with some point distinct from the origin. Since that are $q^3$ points in $V$, we know that there are $q^3 - 1$ points in the field distinct from the origin. If we were to define a line as such, we would also overcount be a factor of $q - 1$ because every line through the origin contains $q - 1$ points distinct from the origin. Hence, $V$ contains $\frac{q^3-1}{q-1} = q^2 + q + 1$ lines through the origin. Every plane can be defined by two distinct through the origin. If we were to simply take $\binom{q^2+q+1}{2}$, we would overcount by a factor of $q + 12$ because each plane contains $q+1$ lines through the origin. Hence, $V$ contains $\frac{\binom{q^2+q+1}{2}}{\binom{q1}{2}} = q^2+q+1$ planes through the origin.

We can use geometry over finite fields to construct decks of cards with the desired property. Given a finite field $k = F_q$ with $q$ elements and the three-dimensional vector space $V = k^3$, each two-dimensional subspace (plane passing through the origin) represent a card and each one-dimensional subspace (line passing through the origin) represent a symbol. We desire that the planes intersect exactly once at some line. Given that there are 8 symbols printed on each card, there must be 8 lines passing through the contained in each plane. Lemma 1 tells us that the size of the field $k$ must equal $q = 8 - 1 = 7$. Using this information, Lemma 2 tells us that the optimal number of cards, which is equivalent to the number of planes passing through the origin in $V$, is equal to $q^2+q+1 = 57$.

There's an elegance to this solution that I have yet to completely appreciate. Through its application of finite-dimensional geometry, this solution allows one to visualize abstract concepts in number theory and combinatorics. This approach, which connects two seemingly distinct topics, is also applicable to various other problems and areas of thinking. Most notably, one solution to the Kirkman's school girl problem (*Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast*) use projective geometry over the space PG(3,2). This idea of matching individuals or parties without repeats can be seen in cooperative learning strategy, speed networking or dating events, sports competitions, and progressive dinner party designs.