Coin Problems

Problem 1:
A coin that lands head with probability p is tossed repeatedly until n consecutive heads occur. What is the expected number of coin tosses?

Solution:
Let x represent the expected number of coin tosses. If we get a tail immediately with probability $(1-p)$, then the expected number of tosses is $x+1$. If we get a head and then a tail with probability $p(1-p)$, then the expected number of tosses is $x+2$. Continuing with this logic, if we get $n-1$ heads and then a tail with probability $p^{n-1}(1-p)$, then the expected number of tosses is $x+n$. Finally, if we get n consecutive heads with probability p^n, then the expected number of tosses is n. This gives us the relation

$$\begin{aligned} x &= (1-p)(x+1) + p(1-p)(x+2) + p^2(1-p)(x+3) + \cdots + p^{n-1}(1-p)(x+n) + np^n \\ &= np^n + (1-p)\sum_{i=1}^{n} p^{i-1}(x+i) \\ &= np^n + (1-p)\cdot[x\sum_{i=1}^{n}(p^{i-1}) + \sum_{i=1}^{n}(p^{i-1}i)] \end{aligned} \tag{1}$$

The first sum is the sum of a finite geometric series.

$$x\sum_{i=1}^{n}(p^{i-1}) = x\cdot\frac{1-p^n}{1-p} \tag{2}$$

We can use a bit of clever manipulation to solve for the second sum.

$$S(p) = \sum_{i=1}^{n}(p^{i-1}i) = 1 + 2p + 3p^2 + \cdots + np^{n-1} \tag{3}$$

$$pS(p) = p + 2p^2 + 3p^3 + \cdots + (n-1)p^{n-1} + np^n \tag{4}$$

$$(1-p)S(p) = (1 + p + p^2 + \cdots + p^{n-1}) - np^n = \frac{1-p^n}{1-p} - np^n \tag{5}$$

$$S(p) = \sum_{i=1}^{n}(p^{i-1}i) = \frac{1-p^n}{(1-p)^2} - \frac{np^n}{1-p} \tag{6}$$

Substituting these two sums back in, we get

$$\begin{aligned} x &= np^n + (1-p)\cdot[x\cdot\frac{1-p^n}{1-p} + \frac{1-p^n}{(1-p)^2} - \frac{np^n}{1-p}] \\ &= np^n + (1-p^n)x + \frac{1-p^n}{1-p} - np^n \\ &= (1-p^n)x + \frac{1-p^n}{1-p} \end{aligned} \tag{7}$$

Rearranging terms, we arrive at the answer

$$\boxed{x = \frac{1-p^n}{p^n(1-p)}} \tag{8}$$