Short Brain Teasers

Problem 1: Ants on a Stick

A horizontal stick is one metre long. One hundred ants are placed in random positions on the stick (using a Uniform distribution), facing random directions. The ants crawl head first along the stick, moving at one metre per minute. If an ant reaches the end of the stick, it falls off. If two ants meet, they both change direction. How long would you have to wait until you are sure that all ants have fallen off? What is the expected time for all the ants to fall off of the string?

Solution 1:

Imagine we give each ant a slip of paper labelled with a number from 1 to 100. Whenever two ants meet, have them switch their slips of paper. This way, each slip of paper will move in a straight line at constant speed and thus will have fallen off within a minute.

Symmetry tells us that the direction that the ant is traveling does not matter either. If an ant is placed x meters from the end its traveling to, the ant will fall off in exactly x minutes. If this is facing the opposite direction, we can switch x with $1 - x$. The time for all the ants to fall off is equivalent to the maximum of 100 i.i.d $\text{Unif}(0, 1)$ distributions. The expectation of this is $\boxed{\frac{100}{101}}$

Problem 2: Prisoners and Hats

Every day, the prison warden takes out his 100 prisoners to play an evil game. He places them in a circle where everyone can see each other. He then places a hat on each prisoner's head. The hat is either red or white and he always gives the same hat to the same person. The prisoner cannot see the color of his own hat. Then, on his command, all prisoners with a white hat have to step forward. If one too few or one too many steps forward, they will all be executed. If they do it right, they all go free. If they all do nothing, they all go back to their cells and the game continues the next day. If all prisoners know there is at least one person with a white hat, how many days did it take for the prisoners to get out, and how did they do it? (Note: they are not allowed to communicate in any way about the color of their hats.)

Solution 2:

Let N denote the number of prisoners who are assigned white hats. If $N = 1$, then the prisoner with the white hat will notice that the 99 other prisoners are wearing red hats. This allows him to deduce that he is wearing a white hat and step out on the first day. If $N = 2$, let us denote the two prisoners with white hats as A and B. On the first day A will notice that B has a white hat while everyone else has red hats and will expect B to step out on the first day. When B does not, A recognizes that they are wearing a white hat as well and will step out on the second day. The same occurs for B. If $N = 3$, we will denote the three prisoners as A, B, and C. On the first day A will notice that B and C have white hats while everyone else has black hats and will expect that they will step out on the second day. When they do not do so, A will recognize that they are wearing a white hat as well and will step out on the third day. The same occurs for B and C. This line of thinking can be generalized for N prisoners, leading us to the conclusion that if there are N prisoners with white hats, then they will step out on the Nth day.