Basketball Scores Problem

Problem Statement (PGTFI):
A basketball player is taking 100 free throws. She scores one point if the ball passes through the hoop and zero points if she misses. She has scored on her first throw and missed on her second. For each of the following throws, the probability of her scoring is the fraction of throws she has made so far. For example, if she has scored 23 points after the 40th throw, the probability that she will score in the 41st throw is 23/40. After 100 throws (including the first and the second), what is the probability that she scores exactly 50 baskets?

Solution 1 (Ziyong):
The probability the player misses a throw is simply the fraction of throws she has missed so far. This means that the probability of the player making OR missing the ith throw is some fraction of the throws she has attempted and thus the denominator would be $i-1$. Since shots 3-100 are yet to be determined, the probability of a sequence of consisting of exactly 50 makes can be expressed as

$$\frac{?}{2 \cdot 3 \cdot \cdots \cdot 98 \cdot 99} \tag{1}$$

For makes $2 \to 50$, the numerator of the probability is the number of shots the player has made beforehand, giving us numerator values $1, 2, \cdots, 49$. For misses $2 \to 50$, the numerator of the probability is the number of shots the player has missed beforehand, giving us numerator values $1, 2, \cdots, 49$ as well. Thus, the probability of a sequence of consisting of exactly 50 makes is

$$\frac{1 \cdot 2 \cdot \cdots \cdot 49 \cdot 1 \cdot 2 \cdot \cdots \cdot 49}{2 \cdot 3 \cdot \cdots \cdot 98 \cdot 99} \tag{2}$$

Since the player has already made the first shot and missed the second shot, there are $\binom{98}{49}$ sequences of misses and makes that will give the player exactly 50 points after 100 shots. This means that the probability our players scores exactly 50 baskets is

$$\begin{aligned} P_{100,50} &= \binom{98}{49} \cdot \frac{1 \cdot 2 \cdot \cdots \cdot 49 \cdot 1 \cdot 2 \cdot \cdots \cdot 49}{2 \cdot 3 \cdot \cdots \cdot 98 \cdot 99} \\ &= \frac{98!}{49! \cdot 49!} \cdot \frac{49! \cdot 49!}{99!} \\ &= \boxed{\frac{1}{99}} \end{aligned} \tag{3}$$

We can generalize this for n free throws and k makes where $n \geq 3$. The denominator of one possible sequence can be expressed as

$$\frac{?}{(n-1)!} \tag{4}$$

For makes $2 \to k$ (the $k = 1$ case is trivial), the numerator values would be $1, 2, \cdots, k-1$. For misses $2 \to (n-k)$ (the $n-k = 1$ case is trivial), the numerator values would be $1, 2, \cdots, n-k-1$. There are

$\binom{n-2}{k-1}$ such sequences, giving us a final probability of

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\begin{aligned}
P_{n,k} &= \binom{n-2}{k-1} \cdot \frac{1 \cdot 2 \cdot \dots \cdot (k-1) \cdot 1 \cdot 2 \cdot \dots \cdot (n-k-1)}{2 \cdot 3 \cdot \dots \cdot 98 \cdot n-1} \\
&= \frac{(n-2)!}{(k-1)! \cdot (n-k-1)!} \cdot \frac{(k-1)! \cdot (n-k-1)!}{(n-1)!} \\
&= \boxed{\frac{1}{n-1}}
\end{aligned} \tag{5}
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References

[1] Xinfeng Zhou. *A Practical Guide to Quantitative Finance Interviews*. Xinfeng Zhou, 2020.