Gambler's Ruin Problem

Problem Statement:
The Gambler's Ruin problem, as stated in Example 2.7.3 of the Stat 110 textbook, begins with two gamblers, A and B, who make a series of \$1 bets. In each bet, gambler A has a probability p of winning, and gambler B has a probability $q = 1 - p$ of winning. Gambler A starts with i dollars and gambler B starts with $N - i$ dollars; the total wealth between the two remains constant since every time A loses a dollar, the dollar goes to B, and vice versa. The game ends when either A or B is ruined (when one gambler reaches 0 or N). What is the probability that A wins the game?

Solution:
We will denote p_i as the probability that A wins the game given that A starts with i dollars and W as the event that A wins the game. By the Law of Total Probability, we can condition on the outcome of the first round to see that

$$\begin{aligned} p_i &= P(W|A \text{ starts at } i, \text{ wins round } 1) \cdot p + P(W|A \text{ starts at } i, \text{ loses round } 1) \cdot q \\ &= P(W|A \text{ starts at } i+1) \cdot p + P(W|A \text{ starts at } i-1) \cdot q \\ &= p_{i+1} \cdot p + p_{i-1} \cdot q \end{aligned} \tag{1}$$

This holds true for all $1 \leq i \leq N-1$ while the boundary probabilities are $p_0 = 0$ and $p_N = 1$. We can rearrange the terms in Equation 1 to see that

$$\begin{aligned} p_{i+1} - p_i &= \frac{q}{p}(p_i - p_{i-1}) \\ &= \left(\frac{q}{p}\right)^2 (p_{i-1} - p_{i-2}) \\ &= \cdots \\ &= \left(\frac{q}{p}\right)^i (p_1 - p_0) \end{aligned} \tag{2}$$

We can use this to solve for p_i in terms of p_{i-1},p_1, and p_0.

$$p_i = p_{i-1} + \left(\frac{q}{p}\right)^{i-1} (p_1 - p_0) = p_{i-1} + \left(\frac{q}{p}\right)^{i-1} p_1 \tag{3}$$

This holds true for all $2 \leq i \leq N$. This means that we can start with $p_2 = \left(1 + \frac{q}{p}\right) p_1$ and successively evaluate p_i as an expression of p_1.

$$p_i = \left(1 + \frac{q}{p} + \cdot + \left(\frac{q}{p}\right)^{i-1}\right) p_1 \tag{4}$$

We can extend this to p_N and by finding the sum of a finite geometric series.

$$p_N = \left(1 + \frac{q}{p} + \cdot + \left(\frac{q}{p}\right)^{N-1}\right) p_1 = \begin{cases} \frac{1-(q/p)^N}{1-q/p} p_1, & \text{if } q/p \neq 1 \\ N p_1, & \text{if } q/p = 1 \end{cases} \tag{5}$$

Since $p_N = 1$, we can solve for p_1.

$$p_1 = \begin{cases} \frac{1-q/p}{1-(q/p)^N}, & \text{if } q/p \neq 1 \\ 1/N, & \text{if } q/p = 1 \end{cases} \tag{6}$$

We can plug Equation 6 into Equation 4 to solve for p_i.

$$\begin{aligned} p_i &= \left(1 + \frac{q}{p} + \cdot + \left(\frac{q}{p}\right)^{i-1}\right) p_1 \\ &= \begin{cases} \frac{1-(q/p)^i}{1-q/p} p_1, & \text{if } q/p \neq 1 \\ i p_1, & \text{if } q/p = 1 \end{cases} \\ &= \begin{cases} \frac{1-(q/p)^i}{1-q/p} \cdot \frac{1-q/p}{1-(q/p)^N}, & \text{if } q/p \neq 1 \\ i \cdot 1/N, & \text{if } q/p = 1 \end{cases} \\ &= \boxed{\begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } q/p \neq 1 \\ i/N, & \text{if } q/p = 1 \end{cases}} \end{aligned} \tag{7}$$

References

[1] Xinfeng Zhou. *A Practical Guide to Quantitative Finance Interviews*. Xinfeng Zhou, 2020.

[2] Joseph K. Blitzstein, Jessica Hwang. *Introduction to Probability*. Chapman and Hall/CRC, 2014.