Mass Spring Model with LQR Control

Xinyu Li

July, 2019

1 Model and Cost Rate

Model:

$$X_{n+1} = AX_n + BU_n + Wn$$

Cost rate:

$$J_n(G) = \mathbb{E}[|X_{n+1}|^2 + r|U_n|^2]$$

We are looking for the control in the form of $U_n = GX_n$, where $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, our aim is to find G such that the cost rate is minimized in the steady state.

2 Covariance matrix S of X

- Let S_n be the covariance matrix of X_n, R be the covariance matrix of noise W_n, then

$$S_{n+1} = (A + BG)S_n(A + BG)^T + R \tag{*}$$

- In the steady state, let $M = A + BG$, we will have

$$S = (A + BG)S(A + BG)^T + R = MSM^T + R$$

- As S is a symmetric matrix, we can rewrite the above equation as

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix} = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & M_{22}M_{12} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix} + \begin{pmatrix} R_{11} \\ R_{12} \\ R_{22} \end{pmatrix}$$

- Let $\xi_S = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix}$, $D = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & M_{22}M_{12} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix}$, $\xi_R = \begin{pmatrix} R_{11} \\ R_{12} \\ R_{22} \end{pmatrix}$, then

$$\xi_S = D\xi_S + \xi_R, (I - D)\xi_S = \xi_R$$

2.1 Differentiate S w.r.t G

- If differentiating S with respect to parameter θ in the steady state, according to chain rule we will get
$$\dot{S} = M\dot{S}M^T + \dot{M}SM^T + MS\dot{M}^T$$
- Let $u = \dot{M}SM^T + MS\dot{M}^T$, then by the similar method above, we will have
$$(I - D)\xi_{\dot{S}} = \xi_u$$

2.2 Use S to represent the cost rate

- $\mathbb{E}[X^T X] = \text{tr}(S)$
- $\mathbb{E}[U^T rU] = \text{tr}(X^T G^T rGX) = \text{tr}(G^T rGXX^T) = \text{tr}(G^T rGS) = \text{tr}(rGSG^T)$(cyclic propery of trace)
- Then the cost rate can be represented as
$$J_n(G) = \text{tr}(S) + \text{tr}(rGSG^T) \qquad (**)$$
- Differentiate J_n w.r.t θ gives
$$\begin{aligned} 0 &= \frac{\partial J}{\partial \theta} \\ &= \text{tr}(\dot{S}) + \text{tr}(r(\dot{G}SG^T + G\dot{S}G^T + GS\dot{G}^T)) \\ &= \text{tr}(\dot{S}) + \text{tr}(r(2GS\dot{G}^T + G\dot{S}G^T)) \end{aligned}$$
where θ is G_1 or G_2

Then, we want to use G to represent S and $\dot{S}$ in the above equation:

- $\xi_S = E\xi_R$
- $\xi_{\dot{S}} = E\dot{D}E\xi_R$
- $\text{tr}(\dot{S}) = \begin{pmatrix}1 & 0 & 1\end{pmatrix} \xi_{\dot{S}} := q\xi_{\dot{S}}$
- $\text{tr}(r(2GS\dot{G}^T + G\dot{S}G^T)) = r(2GS\dot{G}^T + G\dot{S}G^T)$
- $G\dot{S}G^T = \begin{pmatrix}G_1^2 & 2G_1G_2 & G_2^2\end{pmatrix} \xi_{\dot{S}} := H\xi_{\dot{S}}$
- Let $\xi_G = \begin{pmatrix} G_1 & G_2 & 0 \\ 0 & G_1 & G_2 \end{pmatrix}$, then $S\dot{G}^T = \xi_{\dot{G}}\xi_S$
- Plugging them in the derivative of cost rate function:
$$0 = q\xi_{\dot{S}} + r(H\xi_{\dot{S}} + 2G\xi_{\dot{G}}\xi_S)$$
- Plugging in the formulas for $\xi_{\dot{S}}$ and ξ_S:
$$0 = [qE\dot{D} + r(HE\dot{D} + 2G\xi_{\dot{G}})]E\xi_R$$

The equation is only about G_1 and G_2, in the form of $f(G_1, G_2) = 0$, so we can use Newton method to find out the value of G_1 and G_2.
Remark: Equation (*) and (**) are verified in the Python Code.

2.3 Appendix

- $\frac{\partial D}{\partial G_1} = \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & 0 \\ 2(a_{21}+G_1) & 2(a_{22}+G_2) & 0 \end{pmatrix}$,
  $\frac{\partial D}{\partial G_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & 2(a_{21}+G_1) & 2(a_{22}+G_2) \end{pmatrix}$
- $D = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & M_{22}M_{12} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix}$
  with $M_{11} = a_{11}, M_{12} = a_{12}, M_{21} = a_{21}+G_1, M_{22} = a_{22}+G_2$
- $\frac{\partial \xi_G}{\partial G_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\frac{\partial \xi_G}{\partial G_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

3 Find U_n Based on Cost Rate

Consider the general case

$$\begin{aligned} J_n &= \mathbb{E}[U_n^T r U_n + X_{n+1}^T X_{n+1}] \\ &= \mathbb{E}[U_n^T r U_n + (AX_n + BU_n + W_n)^T (AX_n + BU_n + W_n)] \\ &= \mathbb{E}[U_n^T r U_n + X_n^T A^T A X_n + X_n^T A^T B U_n + U_n^T B^T A X_n + U_n^T B^T B U_n] + R \end{aligned}$$

We want to find U_n which is the optimal control for minimizing J_n. Since J_n is quadratic, we can find U_n by letting $\frac{\partial}{\partial U_n} J_n = 0$. Thus,

$$0 = \frac{\partial}{\partial U_n} J_n = \mathbb{E}[2U_n^T r^T + 2X_n^T A^T B + 2U_n^T B^T B]$$

Therefore,

$$U_n = -(r + B^T B)^{-1} B^T A X_n$$

4 Dynamic Programming

4.1 Deterministic LQR

First, let's consider the deterministic model without noise

$$X_{n+1} = AX_n + BU_n$$

Consider the cost-to-go function V_t for $t = 0, 1, 2...$,

$$V_t(z) = \min_{U_t,\ldots,U_{N-1}} \left[\sum_{n=t}^{N-1} (X_n^T X_n + U_n^T r U_n) + X_N^T X_N \right]$$

subject to $X_t = z$, and $X_{n+1} = AX_n + BU_n$, for $n = t, t+1, ...N$. Since V_t is a quadratic function, we can write

$$V_t(z) = z^T S_t z$$

where $S_t = S_t^T$.

Now suppose we know $V_{t+1}(z)$, we want to find optimal U_t. We know that if the cost at time t subject to $U_t = w$, then

$$\begin{aligned} V_t(z) &= \min_w \left[z^T z + w^T r w + V_{t+1}(Az + Bw) \right] \\ &= \min_w \left[z^T z + w^T r w + (Az + Bw)^T S_{t+1}(Az + Bw) \right] \end{aligned}$$

We differentiate V_t w.r.t w to solve for optimal w^*:

$$\begin{aligned} 0 &= \frac{\partial V_t}{\partial w} \\ &= 2w^T r + 2(Az + Bw)^T S_{t+1} \end{aligned}$$

Therefore, the optimal w^* is

$$w^* = -(r + B^T S_{t+1} B)^{-1} B^T S_{t+1} A z$$

Now, let plug w^* back into V_t:

$$\begin{aligned} V_t(z) &= z^T z + w^{*T} r w^* + (Az + Bw^*)^T S_{t+1}(Az + Bw^*) \\ &= z^T z + w^{*T} r w^* + z^T A^T S A z + z^T A^T S B w^* + w^{*T} B^T S A z + w^{*T} B^T S B w^* \\ &= z^T (I + A^T S A) z + w^{*T}(r + B^T S B) w^* - 2z^T A^T S B (r + B^T S_{t+1} B)^{-1} B^T S_{t+1} A z \\ &= z^T (I + A^T S A) z + z^T A^T S B (r + B^T S B)^{-1} (r + B^T S B)(r + B^T S B)^{-1} B^T S A z \\ &- 2z^T A^T S B (r + B^T S_{t+1} B)^{-1} B^T S_{t+1} A z \\ &= z^T \left[I + A^T S_{t+1} A - A^T S_{t+1} B (r + B^T S_{t+1} B)^{-1} B^T S_{t+1} A \right] z \end{aligned}$$

Therefore, we find find the backward relation of S_t:

$$S_t = I + A^T \left[S_{t+1} - S_{t+1} B (r + B^T S_{t+1} B)^{-1} B^T S_{t+1} \right] A$$

4.2 Stochastic LQR

First, let's consider the deterministic model without noise

$$X_{n+1} = AX_n + BU_n + W_n$$

Now consider the cost-to-go function V_t for $t = 0, 1, 2...$ has the form

$$V_t(z) = z^T S_t z + q_t$$

subject to $X_t = z$, and $X_{n+1} = AX_n + BU_n + W_n$, for $n = t, t+1, ...N$. Now suppose we know $V_{t+1}(z)$, we want to find optimal U_t. We know that if the cost at time t subject to $U_t = w$, then

$$\begin{aligned}
V_t(z) &= \min_w \left[z^T z + w^T r w + V_{t+1}(Az + Bw + W_t) \right] \\
&= \min_w \left[z^T z + w^T r w + \mathbb{E}(Az + Bw + W_t)^T S_{t+1}(Az + Bw + W_t) + q_{t+1} \right] \\
&= z^T z + \text{tr}(R S_{t+1}) + q_{t+1} + \min_w \left[w^T r w + (Az + Bw)^T S_{t+1}(Az + Bw) \right]
\end{aligned}$$

Then,

$$\frac{\partial V_t}{\partial w} = 2w^T r + 2(Az + Bw)^T S_{t+1}$$

which is same to the deterministic case. So the formulas of optimal w and S_n are same as those of deterministic case.