1 Kalman filter formulas

Dynamics:

$$X_{n+1} = AX_n + W_n$$

Observation (measurement):

$$Z_{n+1} = CX_{n+1} + V_{n+1} \ .$$

Noise model:

$$\begin{aligned} W_n \text{ and } V_n \text{ independent Gaussian} \\ \mathrm{E}[\, W_n] &= 0 \\ \mathrm{E}[\, V_n] &= 0 \\ \mathrm{E}\big[\, W_n W_n^t\big] &= R \\ \mathrm{E}\big[\, V_n V_n^t\big] &= Q \end{aligned}$$

Estimate with update:

$$\begin{aligned} \widehat{X}_n &= \widehat{X}_n(Z_1, \ldots, Z_n) \ , \quad \text{estimate of } X_n \text{ using the measurements} \\ Y_n &= X_n - \widehat{X}_n = \text{ prediction error} \\ \widehat{Z}_{n+1} &= CA\widehat{X}_n = \text{ predicted observation using old data.} \\ \widehat{X}_{n+1} &= A\widehat{X}_n + K_{n+1}\Big(Z_{n+1} - \widehat{Z}_{n+1}\Big) = \text{ updated prediction using dynamics and new measurement} \end{aligned}$$

Optimal filter: find K to minimize prediction error Y. Three equivalent formulations (check this):

1. Minimize

$$\mathrm{E}\Big[\, \|Y_n\|_n^2\Big] = \mathrm{E}\big[\, Y_n Y_n^t\big]$$

2. Make Y_n independent of measurements $Z_1, \ldots, Z_n$. This is equivalent (because everything is Gaussian) to making Y_n uncorrelated with $Z_1, \ldots, Z_n$

$$\mathrm{E}\big[\, Y_n Z_j^t\big] = 0 \ , \quad j = 1, \ldots, n$$

3. Take the conditional expectation of X_n given the measurements:

$$\widehat{X}_n = \mathrm{E}[\, X_n \mid Z_1, \ldots, Z_n]$$

Solution:

$$T_n = \mathrm{E}\big[\, Y_n Y_n^t\big] = \text{ covariance matrix of prediction error.}$$

$$T_{n+1} = (A - K_nCA)\, T_n\, (A - K_nCA)^t + (I - K_nC)\, R\, (I - K_nC)^t + K_nQK_n^t \ .$$

$$K_n = \big(AT_nA^t + R\big)\, C^t\, \big(CAT_nA^tC^t + CRC^t + Q\big)^{-1} \ .$$