# Model Spring Model with Control

**Model and Cost Rate**

- Model:

$$X_{n+1} = AX_n + BU_n + Wn$$

- Cost rate:

$$J_n(G) = \mathbb{E}[|X_{n+1}|^2 + r|U_n|^2]$$

We are looking for the control in the form of $U_n = GX_n$, where $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Therefore, our aim is to find $G$ such that the cost rate is minimized in the steady state.

**Covariance matrix $S$ of $X$**

- Let $S_n$ be the covariance matrix of $X_n$, $R$ be the covariance matrix of noise $W_n$, then

$$S_{n+1} = (A+BG)S_n(A+BG)^T + R$$

- In the steady state, if let $M = A + BG$, we will have

$$S = (A+BG)S(A+BG)^T + R = MSM^T + R$$

- As $S$ is a symmetric matrix, we can rewrite the above equation as

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix} = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{12} & M_{11}M_{22} + M_{12}M_{21} & M_{22}M_{12} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix} + \begin{pmatrix} R_{11} \\ R_{12} \\ R_{22} \end{pmatrix}$$

- Let $\xi = \begin{pmatrix} S_{11} \\ S_{12} \\ S_{22} \end{pmatrix}$, $D = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{12} & M_{11}M_{22} + M_{12}M_{21} & M_{22}M_{12} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix}$, $\tilde{R} = \begin{pmatrix} R_{11} \\ R_{12} \\ R_{22} \end{pmatrix}$, then

$$\xi = D\xi + \tilde{R},$$
$$\xi = (I - D)^{-1}\tilde{R}$$

Notice, $\frac{\partial M_{21}}{\partial G_1} = 1$, $\frac{\partial M_{22}}{\partial G_2} = 1$, other partial derivatives are 0. Therefore,

$$\frac{\partial \xi}{\partial G_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{12} & 0 \\ 2(a_{21} + G_1) & 2(a_{22} + G_2) & 0 \end{pmatrix} \xi + D\frac{\partial \xi}{\partial G_1} + \tilde{R}$$

$$\frac{\partial \xi}{\partial G_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 2(a_{21} + G_1) & 2(a_{22} + G_2) \end{pmatrix} \xi + D\frac{\partial \xi}{\partial G_2} + \tilde{R}$$

Let $E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{12} & 0 \\ 2(a_{21} + G_1) & 2(a_{22} + G_2) & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 2(a_{21} + G_1) & 2(a_{22} + G_2) \end{pmatrix}$,

then,

$$\frac{\partial \xi}{\partial G_1} = E_1 \xi + D\frac{\partial \xi}{\partial G_1} + \tilde{R}$$
$$\frac{\partial \xi}{\partial G_2} = E_2 \xi + D\frac{\partial \xi}{\partial G_2} + \tilde{R}$$

Let $E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 & a_{11} & 0 \\ 2(a_{21}+G_1) & 2(a_{22}+G_2) & 0 & 0 & 2(a_{21}+G_1) & 2(a_{22}+G_2) \end{pmatrix}$,

$\tilde{R}_2 = \begin{pmatrix} R_{11} & R_{11} \\ R_{12} & R_{12} \\ R_{22} & R_{22} \end{pmatrix} \tilde{\xi} = \begin{pmatrix} (0) & \xi \\ \xi & (0) \end{pmatrix}$

then we can rewrite the equations as:

$$(I-D)\frac{\partial \xi}{\partial G} = E\tilde{\xi} + \tilde{R}_2$$

$$\frac{\partial \xi}{\partial G} = (I-D)^{-1}(E\tilde{\xi} + \tilde{R}_2)$$

***Remark:***

Since $G$ is a row vector, $\frac{\partial \xi}{\partial G} = \begin{pmatrix} \frac{\partial S_{11}}{\partial G1} & \frac{\partial S_{11}}{\partial G2} \\ \frac{\partial S_{12}}{\partial G1} & \frac{\partial S_{12}}{\partial G2} \\ \frac{\partial S_{22}}{\partial G1} & \frac{\partial S_{22}}{\partial G2} \end{pmatrix}$

**Use $S$ to represent the cost rate**

- $\mathbb{E}[X^T X] = \mathrm{tr}(S)$,$\mathbb{E}[U^T U] = \mathrm{tr}(G^T G S)$
- $J_n(G) = \mathrm{tr}[(I + rG^T G)S] = (1 + rG_1^2)S_{11} + 2(rG_1G_2)S_{12} + (1 + rG_2^2)S_{22}$
- $\frac{\partial}{\partial G^T} J_n = (S + S^T)G^T + GG^T : \frac{\partial S}{\partial G^T}$

Writing explicitly, $\frac{\partial}{\partial G^T} J_n = 0$ implies:

$$0 = 2rG_1S_{11} + 2rG_2S_{12} + (1 + rG_1^2)\frac{\partial S_{11}}{\partial G_1} + 2rG_1G_2\frac{\partial S_{12}}{\partial G_1} + (1 + rG_2^2)\frac{\partial S_{11}}{\partial G_1}$$

$$0 = 2rG_1S_{12} + 2rG_2S_{22} + (1 + rG_1^2)\frac{\partial S_{11}}{\partial G_2} + 2rG_1G_2\frac{\partial S_{12}}{\partial G_2} + (1 + rG_2^2)\frac{\partial S_{11}}{\partial G_2}$$

Let $\tilde{G} = \begin{pmatrix} 1 + rG_1^2 & 2rG_1G_2 & 1 + rG_2^2 \end{pmatrix}$, then

$$\tilde{G}\frac{\partial \xi}{\partial G} = 2GS$$
$$\frac{\partial \xi}{\partial G} = (\tilde{G}^T\tilde{G})^{-1}\tilde{G}^T 2GS$$

By combining the above equation, we find the relation between the entries in $S$ and the entries in $G$:

$$(\tilde{G}^T\tilde{G})^{-1}\tilde{G}^T 2GS = (I-D)^{-1}(E\tilde{\xi} + \tilde{R}_2)$$

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Remark: I am not sure how to continue here...
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