Kalman filter formulas

Dynamics:

$$X_{n+1} = AX_n + W_n$$

Observation (measurement):

$$Z_{n+1} = CX_{n+1} + V_{n+1} \ .$$

Noise model:

$$\begin{aligned} W_n \text{ and } V_n &\text{ independent Gaussian} \\ \mathrm{E}[\,W_n] &= 0 \\ \mathrm{E}[\,V_n] &= 0 \\ \mathrm{E}\big[\,W_n W_n^t\big] &= R \\ \mathrm{E}\big[\,V_n V_n^t\big] &= Q \end{aligned}$$

Estimate with update:

$$\begin{aligned} \widehat{X}_n &= \widehat{X}_n(Z_1, \ldots, Z_n) \ , \quad \text{estimate of } X_n \text{ using the measurements} \\ Y_n &= X_n - \widehat{X}_n = \text{ prediction error} \\ \widehat{Z}_{n+1} &= CA\widehat{X}_n = \text{ predicted observation using old data.} \\ \widehat{X}_{n+1} &= A\widehat{X}_n + K_n\Big(Z_n - \widehat{Z}_n\Big) = \text{ updated prediction using dynamics and new measurement} \end{aligned}$$

Optimal filter: find K to minimize prediction error Y. Three equivalent formulations (check this):

1. Minimize
$$\mathrm{E}\Big[\,\|Y_n\|_n^2\Big] = \mathrm{E}\big[\,Y_n Y_n^t\big]$$

2. Make Y_n independent of measurements $Z_1, \ldots, Z_n$. This is equivalent (because everything is Gaussian) to making Y_n uncorrelated with $Z_1, \ldots, Z_n$
$$\mathrm{E}\big[\,Y_n Z_j^t\big] = 0 \ , \ \ j = 1, \ldots, n$$

3. Take the conditional expectation of X_n given the measurements:
$$\widehat{X}_n = \mathrm{E}[\,X_n \mid Z_1, \ldots, Z_n]$$

Solution:

$$T_n = \mathrm{E}\big[\,Y_n Y_n^t\big] = \text{ covariance matrix of prediction error.}$$

$$\begin{aligned} T_{n+1} &= (A - K_nC)\,T_n\,(A - K_nC)^t + (I - K_nC)\,R\,(I - K_nC)^t + KQK^t \ . \\ K_n &= \big(AT_nA^t + R\big)\,C^t\,\big(CAT_nA^tC^t + CTC^t + Q\big)^{-1} \ . \end{aligned}$$