§ Formula of K_n §

$Y_n = X_n - \hat{X}_n$ $\quad$ $\mathbb{E}\,YY^T = T$

$\mathbb{E}\,WW = R$

Y_{n+1} independent of $z_1, \cdots, z_{n+1}$. Therefore, Y_{n+1} independent of $\hat{X}_{n+1}$

$X_{n+1} = AX_n + W_n$

$\hat{X}_{n+1} = A\hat{X}_n + K_{n+1}(Z_{n+1} - \hat{Z}_{n+1})$

$$0 = \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})(CX_{n+1} + V_{n+1})^T]$$

$$= \mathbb{E}[((AX_n + W_n) - A\hat{X}_n - K_{n+1}(Z_{n+1} - \hat{Z}_{n+1}))(CAX_n + CW_n + V_{n+1})^T]$$

$Y_n = X_n - \hat{X}_n$

$$= \mathbb{E}[(AY_n + W_n - K_{n+1}(Z_{n+1} - \hat{Z}_{n+1}))(CA(Y_n + \hat{X}_n) + CW_n + V_{n+1})^T]$$

$$= AT_nA^TC^T + RC^T - K_{n+1}\underbrace{\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})(Y_n + \hat{X}_n)^T A^T C^T]}_{①} - K_{n+1}\underbrace{\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})W_n^T]\,C^T}_{②}$$

$$- K_{n+1}\underbrace{\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})V_{n+1}^T]}_{③} \quad (*)$$

Remark: • $\mathbb{E}[Y_n\hat{X}_n] = \mathbb{E}[Y_n]\,\mathbb{E}[\hat{X}_n] = 0$, since there are independent

• V_n, W_n are independent of X_n, Y_n. $\mathbb{E}[X_nW_n] = \mathbb{E}[X_nV_n] = \mathbb{E}[Y_nV_n]$
$= \mathbb{E}[Y_nW_n] = 0$.

Plug in $\begin{cases} Z_{n+1} = CX_{n+1} + V_{n+1} = C(AX_n + W_n) + V_{n+1} \\ \hat{Z}_{n+1} = CA\hat{X}_n, \quad Z_{n+1} - \hat{Z}_{n+1} = CAY_n + CW_n + V_{n+1} \end{cases}$ to ① ② ③ in (*):

① • $\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})(Y_n + \hat{X}_n)^T A^T C^T] = \mathbb{E}[(CAY_n + CW_n + V_{n+1})(Y_n + \hat{X}_n)^T A^T C^T] = CAT_nA^TC^T$

② • $\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})W_n^T]\,C^T = \mathbb{E}[(CAY_n + CW_n + V_{n+1})W_n^T C^T] = CRC^T$

③ • $\mathbb{E}[(Z_{n+1} - \hat{Z}_{n+1})V_{n+1}^T] = \mathbb{E}[(CAY_n + CW_n + V_{n+1})V_{n+1}^T] = Q$

(*): $AT_nA^TC^T + RC^T - K_{n+1}(CAT_nA^TC^T + CRC^T + Q) = 0$

$$K_{n+1} = (AT_nA^TC^T + RC^T)(CAT_nA^TC^T + CRC^T + Q)^{-1}$$

§ Formula of T_n §

By def of $T_n = \mathbb{E}[Y_n Y_n^T]$

$$T_{n+1} = \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})(X_{n+1} - \hat{X}_{n+1})^T]$$

$$= \mathbb{E}\{[AX_n + W_n - (A\hat{X}_n + K_{n+1}(Z_{n+1} - \hat{Z}_{n+1}))][AX_n + W_n - (A\hat{X}_n + K_{n+1}(Z_{n+1} - \hat{Z}_{n+1}))]^T\}$$

$$= \mathbb{E}\{[AY_n + W_n - K_{n+1}(Z_{n+1} - \hat{Z}_{n+1})][AY_n + W_n - K_{n+1}(Z_{n+1} - \hat{Z}_{n+1})]^T\}$$

$$= \mathbb{E}\{[AY_n + W_n - K_{n+1}(CX_{n+1} + V_n - CA\hat{X}_n)][AY_n + W_n - K_{n+1}(CX_{n+1} + V_n - CA\hat{X}_n)]^T\}$$

$$= \mathbb{E}\{[AY_n + W_n - K_{n+1}(CAX_n + CW_n + V_n - CA\hat{X}_n][AY_n + W_n - K_{n+1}(CAX_n + CW_n + V_n - CA\hat{X}_n]^T\}$$

$$= \mathbb{E}\{[AY_n + W_n - K_{n+1}(CAY_n + CW_n + V_n)][AY_n + W_n - K_{n+1}(CAY_n + CW_n + V_n)]^T\}$$

$$= \mathbb{E}\{[(A - K_{n+1}CA)Y_n + (I - K_{n+1}C)W_n - K_{n+1}V_n][(A - K_{n+1}CA)Y_n + (I - K_{n+1}C)W_n - K_{n+1}V_n]^T\}$$

$$= (A - K_{n+1}CA)T_n(A - K_{n+1}CA)^T + (I - K_{n+1}C)R(I - K_{n+1}C)^T + K_{n+1}QK_{n+1}^T$$