Algorithms and Data Structures

COMP261
Articulation Points 1: Idea

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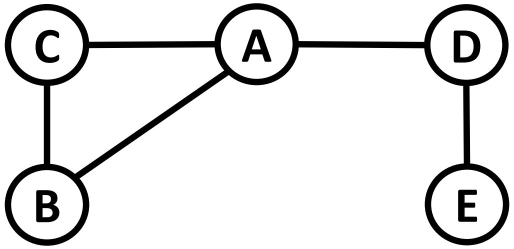
Outline

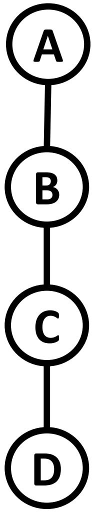
- Articulation Points
- How to find articulation points
  - A bad algorithm
- A good (faster) algorithm

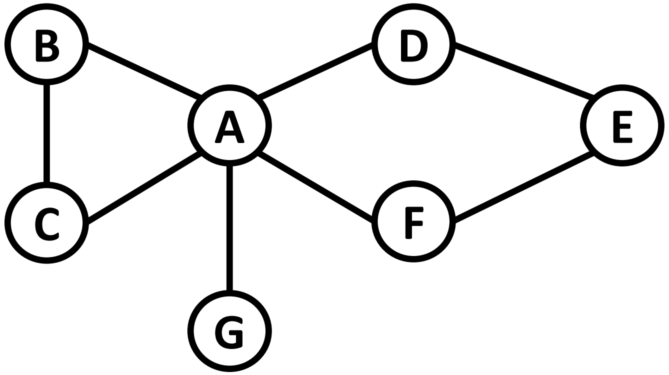
Articulation Point

- In a connected graph, an articulation point is such a point that the graph will become disconnected (split into 2 separate pieces) when it is removed from the graph (along with the edges associated to it)
- Represents the vulnerability of a connected graph

- Which are articulation points?







Why Articulation Points

- Many real-world applications
  - Social network
  - Wireless sensor network
  - Road network
  - Military
  - …

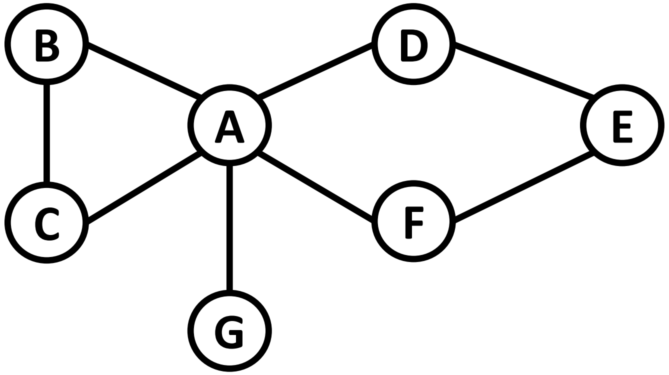
- How to find all the articulation points?

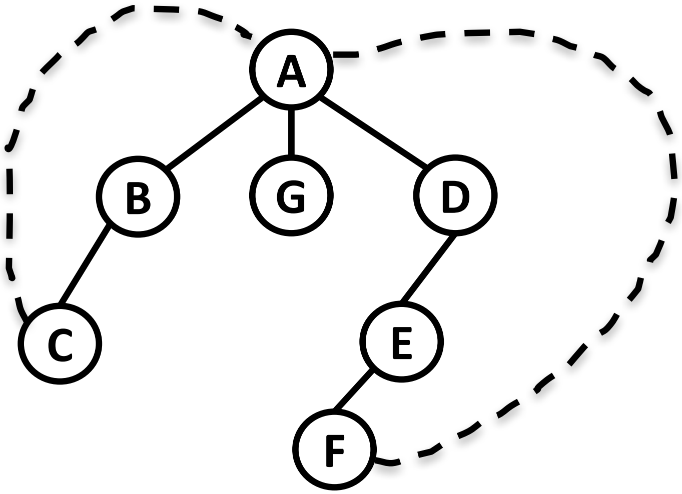
Finding Articulation Points

- **A simple way**
  - For each node, remove it from the graph, and test whether the graph will become disconnected
  - Depth-First Search (DFS) rooted from the node to be removed
    - If the graph will become disconnected, then the DFS tree will have more than one branches

```
All nodes are unvisited, A is visited;
recDFS(first neighbour of A);
if (there exists a neighbour of A unvisited)
        A is an articulation point;

recDFS(node) {
   if (node is unvisited) {
      set node to visited;
      for (each neighbour of node) {
         if (neighbour is unvisited)
            redDFS(neighbour);
      }
   }
}
```

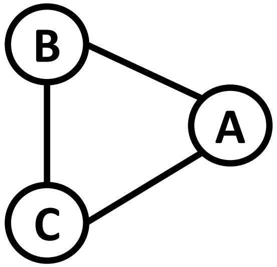




Finding Articulation Points

- But this way is very inefficient
  - Cost of DFS: O(e) or O(n^2) for very dense graphs
  - Cost of algorithm: O(ne) or O(n^3) for very dense graphs

- Many checks are repeated many times unnecessarily
  - 2 checks for A, B and C as root, 6 checks in total
  - But 2 checks are enough to tell none of A, B and C is articulation point

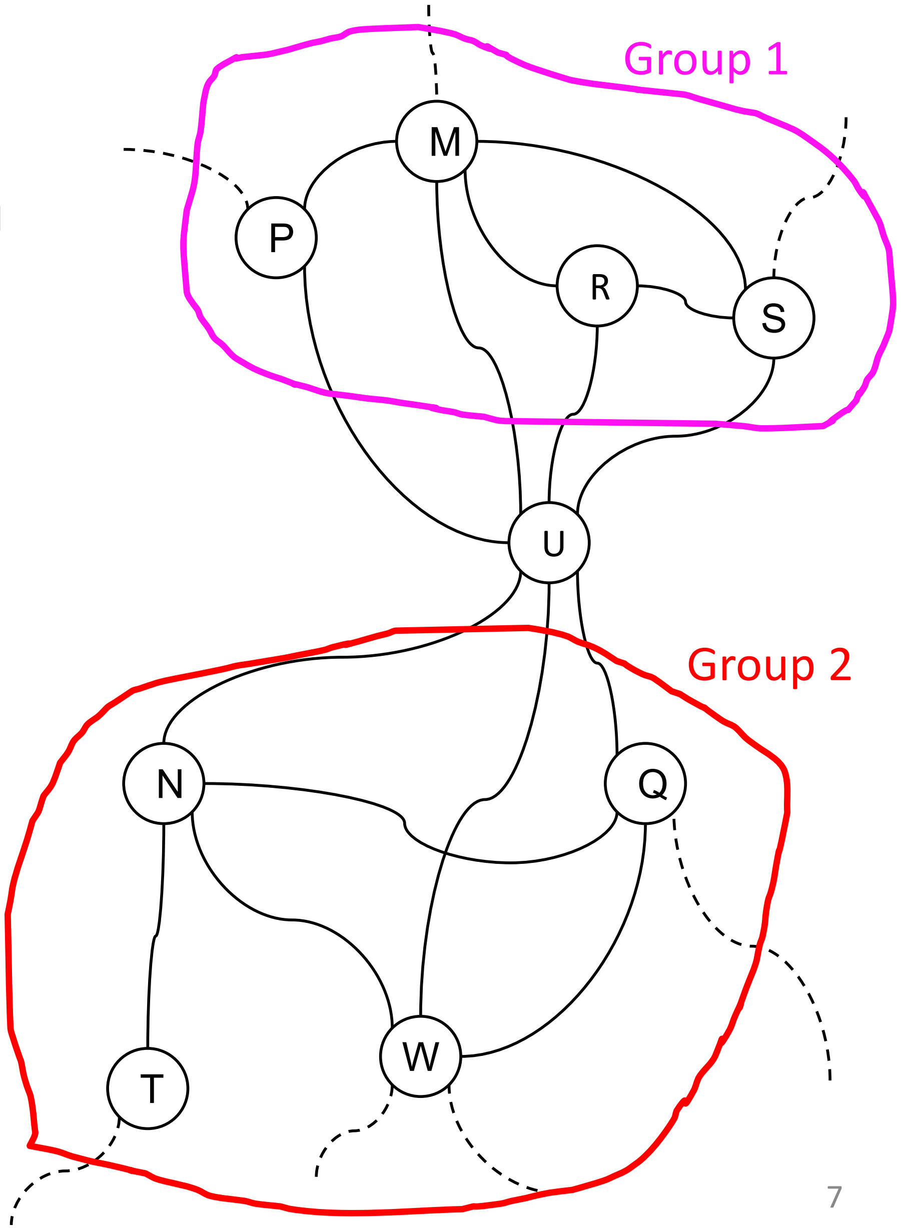


- A new efficient algorithm that finds all the articulation points in a single DFS?

A New Efficient Algorithm

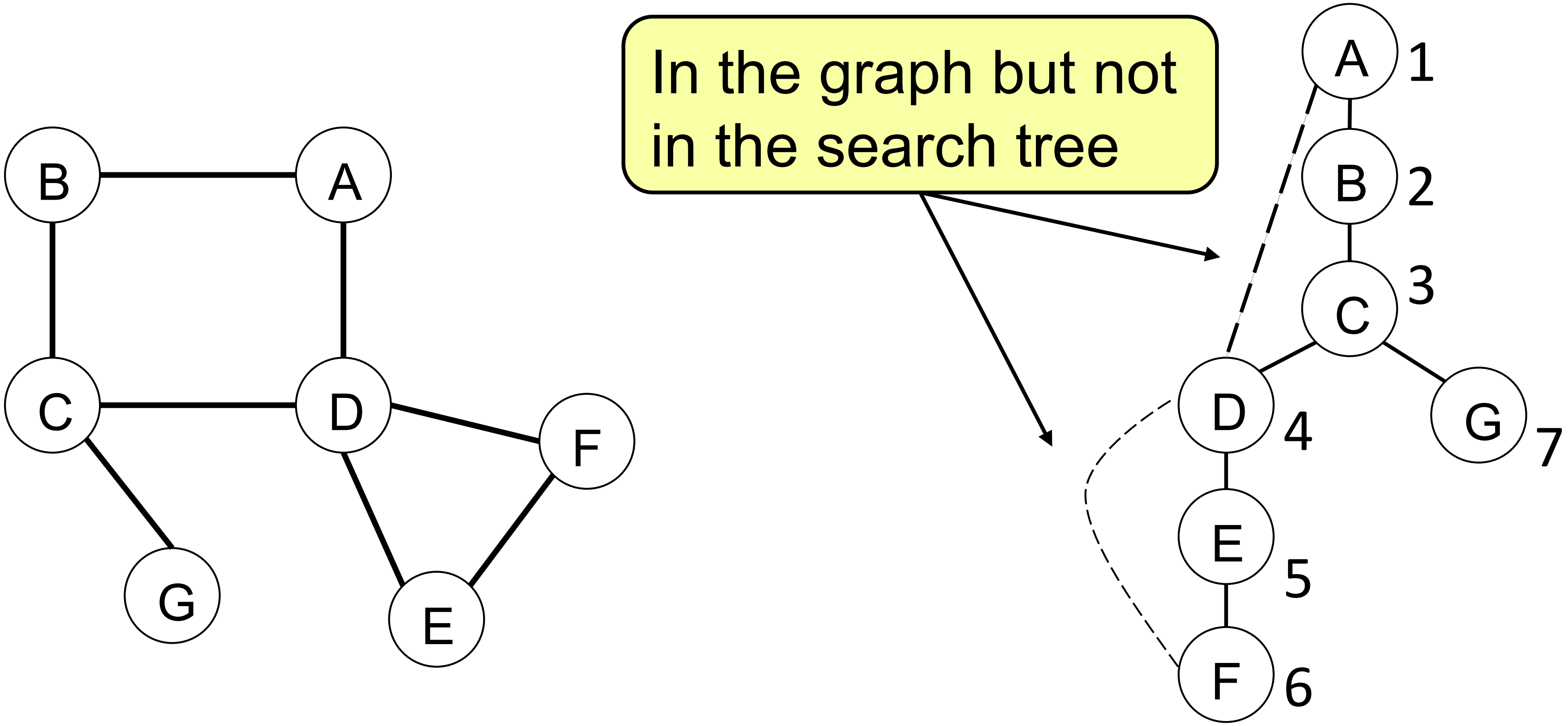
- **Idea**: an articulation point separates the graph into two groups, so that all paths from nodes in one group to nodes in the other group MUST go through the node.

- Example:
  - node U is an articulation point, since all paths between any node in group 1 and any node in group 2 must go through U.



Articulation Points Algorithm

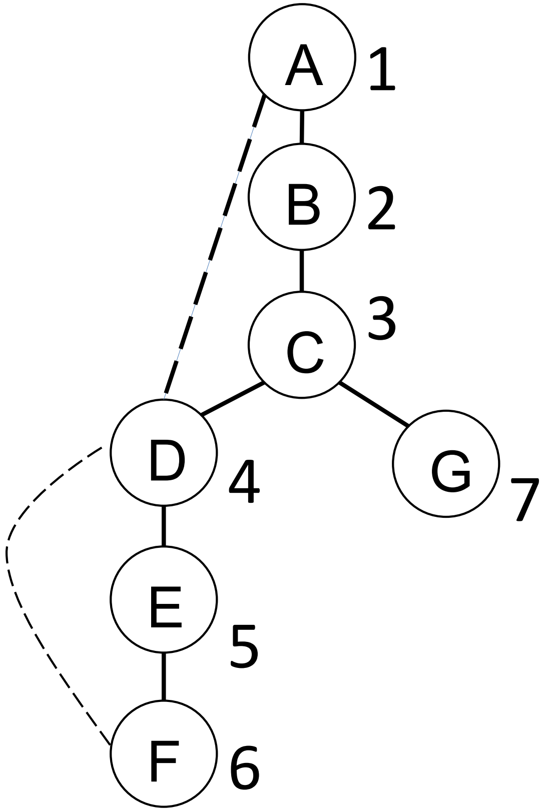
- Traverse the graph using DFS, and index the nodes through the search
  - Assign a count number to each node



In the graph but not in the search tree

Articulation Points Algorithm

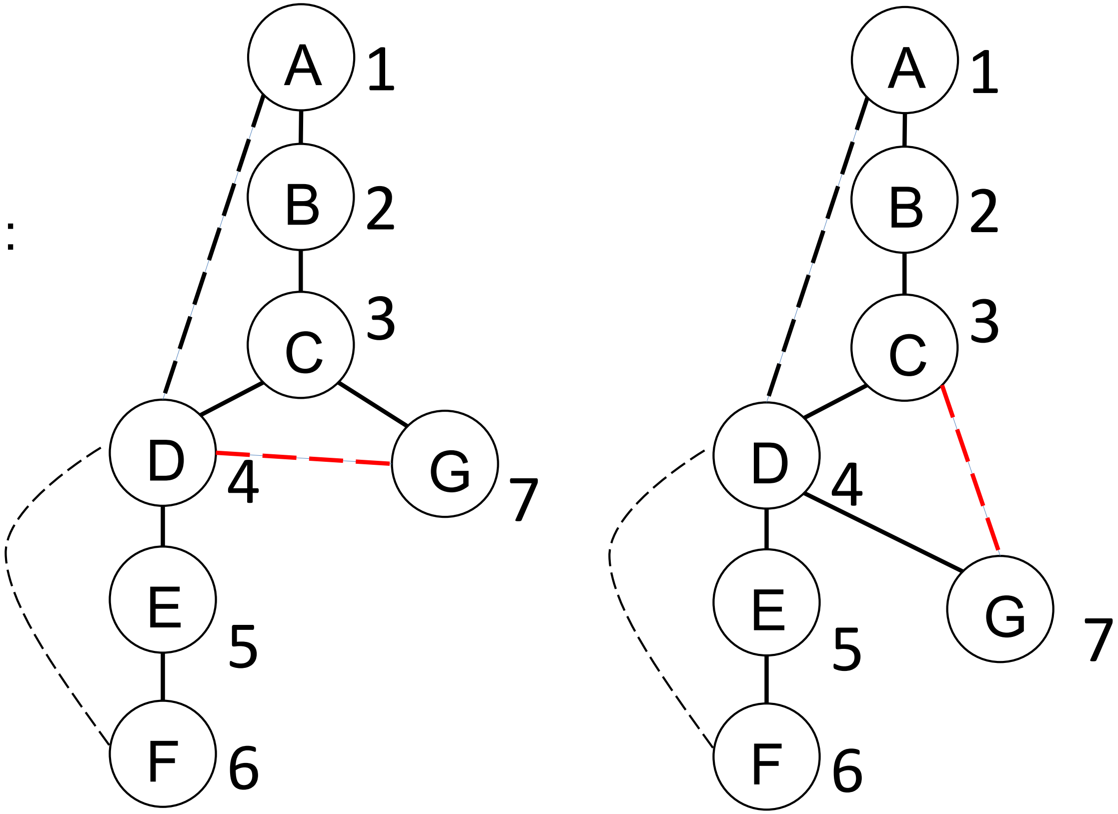
- In the search tree, each node separates the nodes into two subsets
  - **Children set**: Nodes in its subtree
  - **Parents set**: Nodes not in its subtree
- Example:
  - For node D: {E, F} and {A, B, C, G}
  - For node C: {D, E, F, G} and {A, B}
  - For node A: {} and {B, C, D, E, F, G}
- Check if **Children** an **Parents** are separated after removing the node
  - The node is an articulation point if at least one **child** node and **parents** are separated after removing it
  - There is no alternative path
- Should we check nodes in different subtrees? E.g. {D,E,F} and {G} for C?



Articulation Points Algorithm

- **Theorem**: no matter which root note the DFS starts from, and in which order the neighbours are checked, there is **no** alternative path between subtrees of a node
  - **Proof**: (easy) if there is an alternative path between subtrees, then the sub-trees should have been in the same subtree in the first place due to DFS

- Example:
  - If alternative path (D,G):
  - G is a child of D, not C



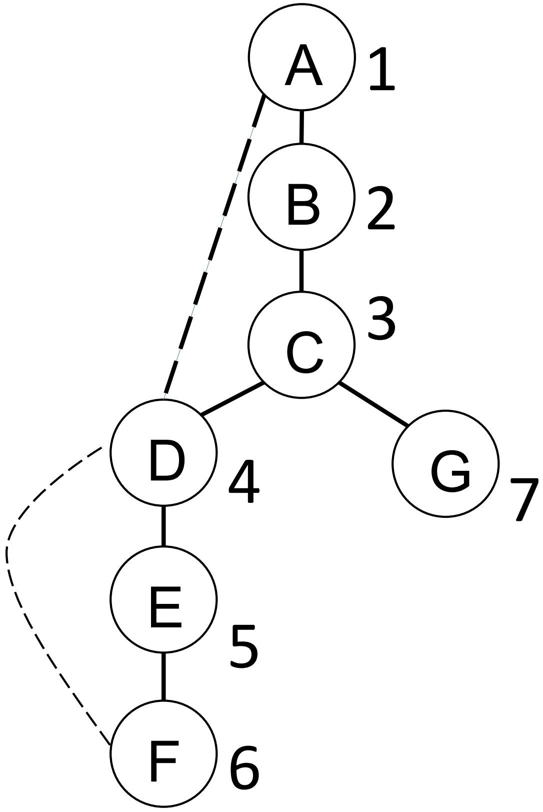
Articulation Points Algorithm

- **Theorem**: a node A is an articulation point, **if and only if** there exists a child node B, for which there is no alternative path from B to any of the **parents**
  - Removing node A will separate B and the parents
  - This is independent of the root node of the DFS and order that the neighbours are checked

- Checking alternative paths for a child node B of node A
  - An edge in the graph, but not in the DFS tree
  - Directly link node B to a parent node
  - Link another child node C in the same subtree of B to a parent node
    - All the child node in the same subtree are connected without node A

Articulation Points Algorithm

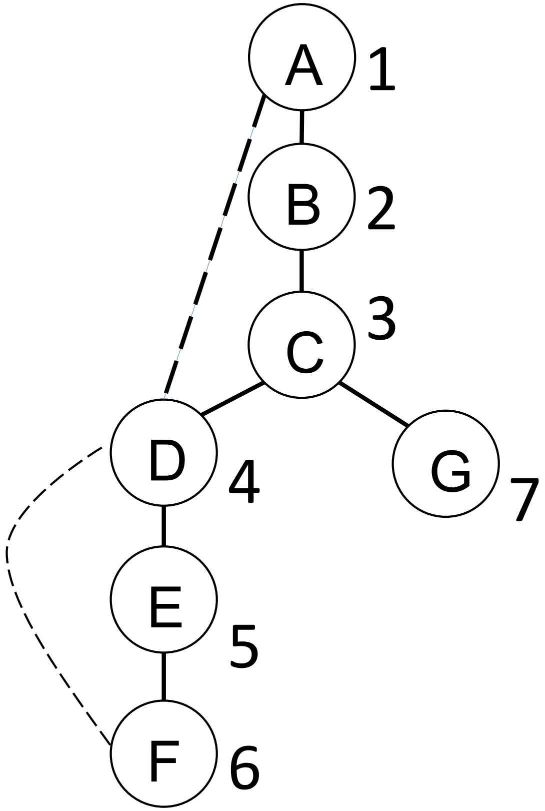
- Example:
  - 2 alternative paths (A, D) and (D, F)
  - C is an articulation point, because there is no alternative path from the child G to any of its parents {A, B}
  - B is not an articulation point, because the edge (A, D) offers an alternative path of all its children {C, D, E, F, G}
  - D? E?
- How about the root node?
  - Parents is empty
  - Checking alternative path is meaningless (always no alternative path)
  - Need to check the root node separately



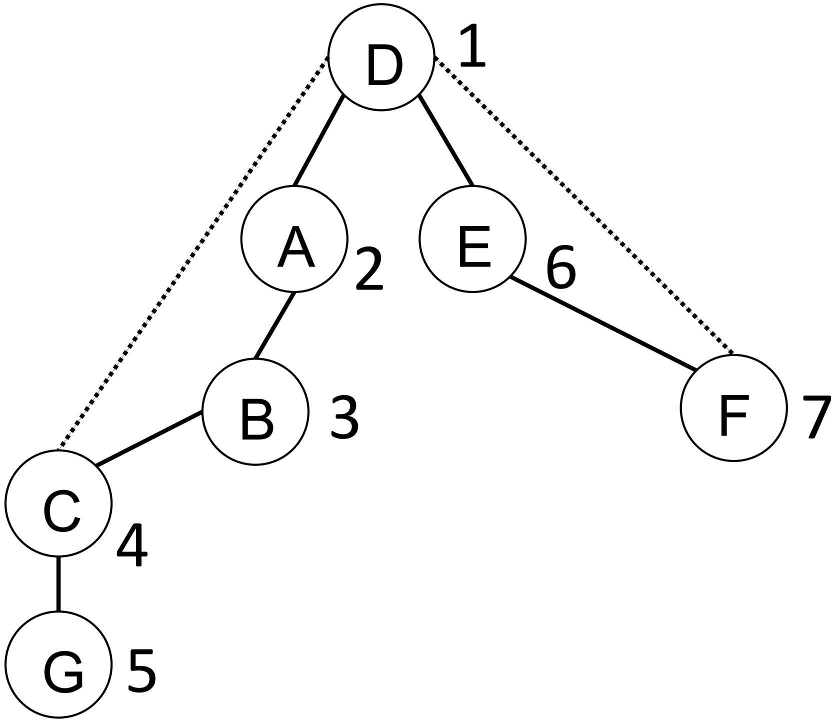
Articulation Points Algorithm

- **Theorem**: a **root node** is an articulation point **if and only if** it has multiple sub-trees in the DFS
  - Proof is easy (no alternative path between sub-trees)

A is not an articulation point (one sub-tree)



D is an articulation point (two sub-trees)



Summary

- Finding articulation points is important in many applications
  - Social network
  - Cyber-security
  - Wireless sensor network
  - …
- Brute force search is time consuming and can be much improved
- A new efficient algorithm with a single DFS
  - Assign count numbers to each node during DFS (larger count numbers are children, smaller are parents)
  - Check for root node: number of sub-trees
  - Check for other nodes: alternative path from children to parents?
- Next lecture: implementation