Verifying the Non-Positive Curvature of the Product Metric on $\mathbb{S}^2 \times \mathbb{S}^2$

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Abstract

The Hopf conjecture is an open problem in Riemmanian Geometry. Here we present the basic ingredients to formulate the question, and proceed through the calculation, by hand and through code, of the sectional curvature of a product metric of $\mathbb{S}^2 \times \mathbb{S}^2$.

1 Introduction

We will first present the basics of differential geometry. This includes both manifolds and smooth manifolds, metrics, and connections. We then introduce Riemannian metrics and focus in on Riemannian Geometry. We will then present the natural product metric on $\mathbb{S}^2 \times \mathbb{S}^2$ and ensure the Hopf Conjecture holds in this case.

Theorem 1 (The Hopf Conjecture)**.** $\mathbb{S}^2 \times \mathbb{S}^2$ *Admits no Riemannian metric with positive sectional curvature [3].*

This conjecture, the first of Prof. Shing-Tung Yau's famous problem selections has stood at the forefront of Riemannian geometry. It remains interesting because the sign of the sectional curvature of a surface fully characterizes the surface in a sense as a result of both (3) and (4). These two theorems allow us to understand surfaces very well. In a sense, we can see the Hopf Conjecture as attempting to classify the differences between $\mathbb{S}^4$ and $\mathbb{S}^2 \times \mathbb{S}^2$. We understand $\mathbb{S}^4$ to have a metric with positive sectional curvature, as spheres are typified by this fact. With this in mind we imagine that there could be no such metric for $\mathbb{S}^2 \times \mathbb{S}^2$ otherwise the two manifolds might not be distinguishable in some sense.

2 Riemannian Geometry

We begin with the standard ingredients of Riemannian Geometry. Directly from [2], also see [1].

"Smooth" means "infinitely differentiable", that is, C^∞.

Definition 1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

1. $U_\alpha \subseteq M; V_\alpha \subseteq \mathbb{R}^n$ is open $\forall \alpha \in I$
2. $\bigcup_{\alpha \in I} U_\alpha = M$
3. Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection
4. For every $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the composition $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$: $\varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or changes of coordinates.

Definition 2. Let M have a smooth atlas. A set $A \subseteq M$ is open if for every $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbb{R}^n$. If $A \subset M$ is open and $x \in A, A$ is called an open neighborhood of x.

Definition 3. M is called Hausdorff if for each $x, y \in M, x \neq y$, there exist open sets $A_x \ni x$ and $A_y \ni y$ such that $A_x \cap A_y = \emptyset$.

Definition 4. M is called a smooth n-dimensional manifold if M has a countable n-dimensional smooth atlas and M is Hausdorff.

Definition 5. Let $f : M^m \to N^n$ be a map of smooth manifolds with atlases $(U_i, \varphi_i(U_i), \varphi_i)_{i \in I}$ and $(W_j, \psi_j(W_j), \psi_j)_{j \in J}$. The map f is smooth if it induces smooth maps between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, i.e. if $\psi_j \circ f \circ \varphi_i^{-1}\big|_{\varphi_i(U_i \cap f^{-1}(W_j \cap f(U_i)))}$ is smooth for all $i \in I, j \in J$.

If f is a bijection and both f and f^{-1} are smooth then f is called a diffeomorphism.

Definition 6. A derivation on the set $C^\infty(M,p)$ of all smooth functions on M defined in a neighborhood of p is a linear map $\delta : C^\infty(M,p) \to \mathbb{R}$, s.t. for all $f, g \in C^\infty(M,p)$ holds $\quad \delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p)$ (the Leibniz rule).

The set of all derivations is denoted by $\mathcal{D}^\infty(M,p)$. This is a real vector space (exercise).

Definition 7. The space $\mathcal{D}^\infty(M,p)$ is called the tangent space of M at p, denoted T_pM. Derivations are tangent vectors.

Definition 8. Let $\gamma : (a,b) \to M$ be a smooth curve in $M, t_0 \in (a,b), \gamma(t_0) = p$ and $f \in C^\infty(M,p)$. Define the directional derivative $\gamma'(t_0)(f) \in \mathbb{R}$ of f at p along γ by

$$\gamma'(t_0)(f) = \lim_{s \to 0} \frac{f(\gamma(t_0+s)) - f(\gamma(t_0))}{s} = (f \circ \gamma)'(t_0) = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \gamma)$$

Definition 9. Define $\frac{\partial}{\partial x_i}\Big|_p = \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\Big|_p (f) = (f \circ \gamma_i)'(0) = \frac{d}{dt}\left(f \circ \varphi^{-1}\right)(\varphi(p) + te_i)\Big|_{t=0} = \frac{\partial}{\partial x_i}\left(f \circ \varphi^{-1}\right)(\varphi(p)),$$

where $\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative.

By definition, we have

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\} \subseteq \{\text{ Directional derivatives }\} \subseteq \mathcal{D}^\infty(M, p)$$

Corollary 1.

$$\{\text{ \textit{Directional derivatives} }\} = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle = \mathcal{D}^\infty(M, p)$$

Proposition 1. *If an n-manifold M is embedded into $\mathbb{R}^N$, then every tangent vector at $p \in M$ can be identified with vector $(\gamma_1'(0), \dots, \gamma_N'(0)) \in \mathbb{R}^N$, where $\gamma : (-\varepsilon, \varepsilon) \to M$ is some smooth curve with $\gamma(0) = p$.*

Definition 10. Let M be a smooth manifold. A disjoint union $TM = \cup_{p \in M} T_pM$ of tangent spaces to each $p \in M$ is called a tangent bundle.

There is a canonical projection $\Pi : TM \to M, \Pi(v) = p$ for every $v \in T_pM$.

Definition 11. A vector field X on a smooth manifold M is a smooth map $X : M \to TM$ such that $\forall p \in M X(p) \in T_pM$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Proposition 2. *Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.*

This vector field $Z = XY - YX$ is denoted by $[X, Y]$ and called the Lie bracket of X and Y.

Definition 12. Some properties of the lie bracket

1. $[X, Y] = -[Y, X]$
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $a, b \in \mathbb{R}$
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)
4. $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for $f, g \in C^\infty(M)$

Definition 13. Let M be a smooth manifold. A Riemannian metric $g_p(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ depending smoothly on $p \in M$. A smooth manifold M with a Riemannian metric g is called a Riemannian manifold (M, g).

Definition 14. Given two vector spaces V_1, V_2 with real inner products $(V_i, \langle\cdot,\cdot\rangle_i)$, an isomorphism $T : V_1 \to V_2$ of vector spaces is a linear isometry if $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_1$.

Definition 15. A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an isometry if $Df(p) : T_pM \to T_{f(p)}N$ is a linear isometry for all $p \in M$.

Definition 16. (M, g) is a Riemannian manifold, $c : [a, b] \to M$ is a smooth curve. The length $L(c)$ of c is defined by $L(c) = \int_a^b \|c'(t)\|\, dt$, where $\|c'(t)\| = \langle c'(t), c'(t)\rangle_{c(t)}^{1/2}$. The length of a piecewise-smooth curve is defined as the sum of lengths of its smooth pieces.

Definition 17. Definition 4.3. Let M be a smooth manifold. A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), (X, Y) \mapsto \nabla_X Y$ is affine connection if for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ holds

1. $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$
2. $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$
3. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

Theorem 2 (Levi-Civita, Fundamental Theorem of Riemannian Geometry)**.** *Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇ on M with the additional properties for all $X, Y, Z \in \mathfrak{X}(M)$*

1. $Z(\langle X, Y\rangle) = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ *(Riemannian property)*
2. $\nabla_X Y - \nabla_Y X = [X, Y]$ *(∇ is torsion-free)*

This connection is called Levi-Civita connection of (M, g).

Definition 18. Let ∇ be the Levi-Civita connection on (M, g), and let $\varphi : U \to V$ be a coordinate chart with coordinates $\varphi = (x_1, \ldots, x_n)$. Since $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) \in T_pM$, there exists a uniquely determined collection of functions $\Gamma_{ij}^k \in C^\infty(U)$ s.t. $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma_{ij}^k(p) \frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of ∇ with respect to the chart φ.

Proposition 3. $\Gamma_{ij}^k = \frac{1}{2} g^{km} \left[\frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} + \frac{\partial g_{mj}}{\partial x_i}\right]$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Definition 19. Let (M, g) be a Riemannian manifold, let $\mathfrak{X}(M)$ be the space of vector fields on M, and let ∇ be the Levi-Civita connection. Define a map (Riemann curvature tensor)$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Definition 20. Define components of Riemann curvature tensor
$R_{ijkl} = \left\langle R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right\rangle$, and R_{ijk}^l

Then $R_{ijk\ell} = R_{ijk}^{\ell} g_{ml}$ and $R_{ijk}^{l} = R_{ijkm} g^{ml}$.

Definition 21. Let (M, g) be a Riemannian manifold, $p \in M, v_1, v_2 \in T_pM$, and let $\Pi \subset T_pM$ be the 2-plane spanned by v_1, v_2

The sectional curvature of Π at p is $K(\Pi) = K(v_1, v_2) = \frac{\langle R(v_1,v_2)v_2,v_1\rangle}{\|v_1\|^2\|v_2\|^2 - \langle v_1,v_2\rangle^2}$.

Theorem 3 (Gauss-Bonnet)**.** *Given a compact, boundaryless, two-dimensional Riemannian manifold*

$$\int\int_M K \; dA = 2\pi\chi(M)$$

where χ is defined as the euler characteristic.

Theorem 4 (Uniformization)**.** *Let M be a complete simply-connected Riemannian manifold of constant sectional curvature K. Then*

1. *if $K > 0$ then M is isometric to S^n (assuming $K = 1$)*
2. *if $K = 0$ then M is isometric to $\mathbb{E}^n$*
3. *if $K < 0$ then M is isometric to $\mathbb{H}^n$ (assuming $K = -1$)*

3 The Sectional Curvature of the Product Matrix of $\mathbb{S}^2 \times \mathbb{S}^2$, Who should I cite for formulas

We will now proceed through the process of calculating the sectional curvature of $\mathbb{S}^2 \times \mathbb{S}^2$ under the following metric:

Proposition 4. *The Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$ becomes*

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^2(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2(\theta_2) \end{bmatrix}.$$

Proof. Using the standard parameterization of $\mathbb{S}^2$ in R^3

$$\sigma(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \tag{1}$$

for $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ Then, taking partial derivatives with respect to both ϕ and θ we find

$$\sigma_\phi = (-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0)$$
$$\sigma_\theta = (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta))$$

From here we can determine our first fundamental form, which is

$$g = \begin{bmatrix} \langle\sigma_\phi, \sigma_\phi\rangle & \langle\sigma_\theta, \sigma_\phi\rangle \\ \langle\sigma_\phi, \sigma_\theta\rangle & \langle\sigma_\theta, \sigma_\theta\rangle \end{bmatrix} \tag{2}$$

So, calculating these values we find

$$\begin{aligned}
\langle \sigma_\theta, \sigma_\theta \rangle &= \cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta) = 1 \\
\langle \sigma_\theta, \sigma_\phi \rangle &= -\cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi) + \cos(\theta)\sin(\phi)\sin(\theta)\cos(\phi) = 0 \\
\langle \sigma_\phi, \sigma_\theta \rangle &= -\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi) + \sin(\theta)\cos(\phi)\cos(\theta)\sin(\phi) = 0 \\
\langle \sigma_\phi, \sigma_\phi \rangle &= \sin^2(\theta)\sin^2(\phi) + \sin^2(\theta)\cos^2(\phi) = \sin^2(\theta)
\end{aligned}$$

and so

$$g_{\mathbb{S}^2} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{bmatrix}. \tag{3}$$

We can formulate the product metric of two manifolds with metric g_1, g_2 as

$$g_{1\times 2} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \tag{4}$$

Therefore, for two spheres parameterized with variables $\theta_1, \phi_1, \theta_2, \phi_2$

$$g_{\mathbb{S}^2\times\mathbb{S}^2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^2(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2(\theta_2) \end{bmatrix}$$

□

Next, we show

Proposition 5. *The Christoffel symbols under this metric are*

$$\begin{aligned}
\Gamma^2_{21} = \Gamma^2_{12} &= \cot\theta_1 & \Gamma^4_{43} = \Gamma^4_{34} &= \cot\theta_2 \\
\Gamma^1_{22} &= -\cos(\theta_1)\sin(\theta_1) & \Gamma^3_{44} &= -\cos(\theta_2)\sin(\theta_2),
\end{aligned}$$

with all other Christoffel symbols equal to 0.

Proof. We calculate using the standard formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left[\frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} + \frac{\partial g_{mj}}{\partial x_i} \right]. \tag{5}$$

and note that g^{ij} is the i,jth value of the inverse of the Riemannian metric, which is simply

$$g^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^{-2}(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^{-2}(\theta_2) \end{bmatrix}.$$

We find

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1}\right) \\
&+ \frac{1}{2}g^{12}\left(\frac{\partial g_{21}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_1}\right) \\
&+ \frac{1}{2}g^{13}\left(\frac{\partial g_{31}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_3} + \frac{\partial g_{31}}{\partial x_1}\right) \\
&+ \frac{1}{2}g^{14}\left(\frac{\partial g_{41}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_4} + \frac{\partial g_{41}}{\partial x_1}\right) \\
&= 0
\end{aligned}
$$

Thus, we notice some patterns we can make use of, namely, only one value of the inverse metric matters per Christoffel symbol since all off diagonal values are zero. Now, we choose another symbol to calculate using this

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left[-\frac{\partial g_{22}}{\partial x_1}\right] \\
&= -\frac{1}{2}\left[\frac{\partial \sin^2(\theta_1)}{\partial \theta_1}\right] \\
&= -\cos(\theta_1)\sin(\theta_1)
\end{aligned}
$$

and so, we proceed verifying the Christoffel symbols mentioned in (5) are non-zero, and all others vanish. □

Proposition 6. *The only non-zero values of the curvature tensor are*

$$
\begin{aligned}
R^2_{121} &= 1 & R^4_{343} &= 1 \\
R^1_{122} &= -\sin^2(\theta_1) & R^3_{344} &= -\sin^2(\theta_2) \\
R^2_{211} &= -1 & R^2_{211} &= -1 \\
R^1_{212} &= \sin^2(\theta_1) & R^3_{434} &= \sin^2(\theta_2).
\end{aligned}
$$

Proof. We calculate the curvature 4-tensor using the following formula:

$$
R^\ell_{ijk} = \frac{\partial \Gamma^\ell_{ik}}{\partial x_j} - \frac{\partial \Gamma^\ell_{jk}}{\partial x_i} + \left[\Gamma^p_{ik}\cdot\Gamma^\ell_{jp} - \Gamma^p_{jk}\cdot\Gamma^\ell_{ip}\right].
$$

and so,

$$\begin{aligned}
\mathrm{R}^1_{111} &= \frac{\partial \Gamma^1_{11}}{\partial x_1} - \frac{\partial \Gamma^1_{11}}{\partial x_1} \\
&+ \Gamma^1_{11} \cdot \Gamma^1_{11} - \Gamma^1_{11} \cdot \Gamma^1_{11} \\
&+ \Gamma^2_{11} \cdot \Gamma^1_{12} - \Gamma^2_{11} \cdot \Gamma^1_{12} \\
&+ \Gamma^3_{11} \cdot \Gamma^1_{13} - \Gamma^3_{11} \cdot \Gamma^1_{13} \\
&+ \Gamma^4_{11} \cdot \Gamma^1_{14} - \Gamma^4_{11} \cdot \Gamma^1_{14} \\
&= 0
\end{aligned}$$

Which provides the valuable insight that if the bottom indices are equal the value will always be zero. Now we consider another value

$$\begin{aligned}
\mathrm{R}^1_{212} &= \frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} \\
&+ \Gamma^1_{22} \cdot \Gamma^1_{11} - \Gamma^1_{12} \cdot \Gamma^1_{21} \\
&+ \Gamma^2_{22} \cdot \Gamma^1_{12} - \Gamma^2_{12} \cdot \Gamma^1_{22} \\
&+ \Gamma^3_{22} \cdot \Gamma^1_{13} - \Gamma^3_{12} \cdot \Gamma^1_{23} \\
&+ \Gamma^4_{22} \cdot \Gamma^1_{14} - \Gamma^4_{12} \cdot \Gamma^1_{24} \\
&= \frac{\partial(-\cos(\theta_1)\sin(\theta_1))}{\partial \theta_1} + \cos^2(\theta_1) \\
&= \sin^2(\theta_1)
\end{aligned}$$

and other values follow similarly.

□

Proposition 7. *The following curvature components are non-zero*

$$\begin{aligned}
R_{1212} &= \sin(\theta_1)^2 & R_{3434} &= \sin(\theta_2)^2 \\
R_{1221} &= -\sin(\theta_1)^2 & R_{3443} &= -\sin(\theta_2)^2 \\
R_{2112} &= -\sin(\theta_1)^2 & R_{4334} &= -\sin(\theta_2)^2 \\
R_{2121} &= \sin(\theta_1)^2 & R_{4343} &= \sin(\theta_2)^2.
\end{aligned}$$

all others are zero.

Proof. We calculate the curvature components via the following formula:

$$\mathrm{R}_{ijk\ell} = \mathrm{R}^s_{ijk} g_\ell s.$$

So,

$$\begin{aligned}
\mathrm{R}_{1111} &= \mathrm{R}^1_{111} g_{11} + \mathrm{R}^2_{111} g_{12} \\
&+ \mathrm{R}^3_{111} g_{13} + \mathrm{R}^4_{111} g_{14} \\
&= 0
\end{aligned}$$

Which tells us there will only be one value of g that is non zero, and it corresponds to the value of ℓ. So,

$$\begin{aligned} \mathrm{R}_{1212} &= \mathrm{R}^1_{121} g_{21} + \mathrm{R}^2_{121} g_{22} \\ &+ \mathrm{R}^3_{121} g_{23} + \mathrm{R}^4_{121} g_{24} \\ &= \sin^2(\theta_1) \end{aligned}$$

and if we proceed through the remaining calculations we find the proposition to be true. ☐

Finally,

Proposition 8. *The sectional curvature of $\mathbb{S}^2 \times \mathbb{S}^2$ with respect to the defined matix is greater than or equal to zero, or $\kappa(\mathbb{S}^2 \times \mathbb{S}^2) \geq 0$.*

Proof. We have standard sectional curvature formula

$$\kappa(\sigma) = \frac{v_j u_i \mathrm{R}_{ijk\ell} u_\ell v_k}{g(u,u)g(v,v) - g(u,v)^2}.$$

First, we consider the denominator.

$$\begin{aligned} g(u,u) &= u^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^2(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2(\theta_2) \end{bmatrix} u \\ &= u_1^2 + u_2^2 \sin^2(\theta_1) + u_3^2 + u_4^2 \sin^2(\theta_2) \end{aligned}$$

and,

$$\begin{aligned} g(v,v) &= v_1^2 + v_2^2 \sin^2(\theta_1) + v_3^2 + v_4^2 \sin^2(\theta_2) \\ g(u,v) &= u_1 v_1 + u_2 v_2 \sin^2(\theta_1) + u_3 v_3 + u_4 v_4 \sin^2(\theta_2) \end{aligned}$$

so, the denominator, D is

$$\begin{aligned} D &= (u_1^2 + u_2^2 \sin^2(\theta_1) + u_3^2 + u_4^2 \sin^2(\theta_2) \\ &\times (v_1^2 + v_2^2 \sin^2(\theta_1) + v_3^2 + v_4^2 \sin^2(\theta_2)) \\ &- \left[u_1 v_1 + u_2 v_2 \sin^2(\theta_1) + u_3 v_3 + u_4 v_4 \sin^2(\theta_2)\right]^2 \\ &= (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2 \sin^2(\theta_1) \\ &+ (u_1 v_4 - u_4 v_1)^2 \sin^2(\theta_2) + (u_2 v_3 - u_3 v_2)^2 \sin^2(\theta_1) \\ &+ (u_3 v_4 - u_4 v_3)^2 \sin^2(\theta_2) + (u_2 v_4 - u_4 v_2)^2 \sin^2(\theta_1) \sin^2(\theta_2) \end{aligned}$$

Which we can see is always greater than zero. Now we consider the numerator. We note that each curvature component is used only once, and we have eight non-zero components, therefore the numerator, N, is the sum of only eight values

$$\begin{aligned} N &= v_2u_1R_{1221}u_1v_2 + v_2u_1R_{1212}u_2v_1 + v_1u_2R_{2112}u_2v_1 + v_1u_2R_{2121}u_1v_2 \\ &+ v_4u_3R_{3443}u_3v_4 + v_4u_3R_{3434}u_4v_3 + v_3u_4R_{4334}u_4v_3 + v_3u_4R_{4343}u_3v_4 \\ &= (u_1v_2 - u_2v_1)^2\sin^2(\theta_1) + (u_3v_4 - u_4v_3)^2\sin^2(\theta_2) \end{aligned}$$

so, $N \geq 0$, therefore the value of the curvature is not always positive. □

4 Code

In the calculation of these values I wrote Python code to ensure the validity of my work, here I will present the code I wrote. Note that the Python package SymPy was used in order to do various operations symbolically.

My imports

```
import numpy as np
import sympy as sp
```

Creating the variables I worked with

```
t0 = sp.Symbol('t0', real=True)
t1 = sp.Symbol('t1', positive = True)
t2 = sp.Symbol('t2', real=True)
t3 = sp.Symbol('t3', positive = True)
```

function that calculated the Christoffel Symbols

```
def calculate_chrst_sym_4d(metric, inv_metric, derivs = [t0,t1,t2,t3]):
#Calculate the christoffel symbols for a given metric

#used to take deriv by correct variable
deriv_list = derivs

#empty array to put our christoffel symbols
chrst = np.zeros((4, 4, 4), dtype = 'object')

for k in range(4):
    for i in range(4):
        for j in range(4):
            pre_sum = 0
            for m in range(4):
                pre_sum += .5 * inv_metric[k][m] * (
                  sp.diff(metric[m][i], deriv_list[j]) + sp.diff(metric[m][j
                )
            chrst[i,j,k] = pre_sum
```

```
    return chrst
```

Another that found the curvature tensor

```
def alpha_curvature_tensor(alpha_chrsts):
#finds standard curvature

varis = [t0,t1,t2,t3]

alpha_curve_tensor_arr = np.zeros((4,4,4,4), dtype = 'object')

for l in range(4):
    for i in range(4):
        for j in range(4):
            for k in range(4):
                pre_sum = sp.diff(alpha_chrsts[i][k][l], varis[j]) - sp.diff
                for p in range(4):
                    pre_sum += alpha_chrsts[i][k][p] * alpha_chrsts[j][p][l]
                alpha_curve_tensor_arr[i][j][k][l] = sp.simplify(pre_sum)

return alpha_curve_tensor_arr
```

code to find the curvature components

```
def curvature_components(curv_tens, metric):
#finds curvature components

n = np.shape(curv_tens)[0]

return_arr = np.zeros((n,n,n,n), dtype = 'object')

for i in range(4):
    for j in range(4):
        for k in range(4):
            for l in range(4):
                pre_sum = 0
                for s in range(4):
                    pre_sum += curv_tens[i,j,k,l] * metric[l,s]
                return_arr[i,j,k,l] = pre_sum

return return_arr
```

and finally, the numerator of the curvature

```
def curvature_numerator(comps):
#finds the numerator of the value of the curvature
```

```
#creating vector terms
u0, u1, u2, u3 = sp.symbols('u1 u2 u3 u4')

v0, v1, v2, v3 = sp.symbols('v1:5')

u_list = [u0, u1, u2, u3]
v_list = [v0, v1, v2, v3]

pre_sum = 0

for j in range(4):
    for i in range(4):
        for l in range(4):
            for k in range(4):
                sporting = v_list[j] * u_list[i] * comps[i,j,k,l] * u_list[l
                pre_sum += sporting
                #error could be above, lk are swapped from curvature tensor

return pre_sum
```

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References

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