MAE 271A Project
Calibration of an Accelerometer Using GPS Measurements

Due Date December 4, 2017

Consider that a vehicle accelerates in one dimension in an inertial frame. Assume that the acceleration is a harmonic of the form

$$a(t) = 10sin(2\pi\omega t) \quad \text{meters/sec}^2$$

where $\omega = .1\text{rad/sec}$. Suppose that the acceleration is measured by an accelerometer with a sample rate of 200 Hz at sample times t_j. The accelerometer is modelled with additive white Gaussian noise w with zero mean and variance $V = .0004(\text{meters/sec}^2)^2$. The accelerometer has a bias b_a with *a priori* statistics $b_a \sim N(0, .01(\text{meters/sec}^2)^2)$. The accelerometer a_c is modelled as

$$a_c(t_j) = a(t_j) + b_a + w(t_j)$$

A GPS receiver is used to measure position and velocity in an inertial space. The measurements which are available at a 5 Hz rate (synchronized with the accelerometer) are

$$z_{1i} = x_i + \eta_{1i}$$
$$z_{2i} = v_i + \eta_{2i}$$

where x_i is the position and v_i is the velocity. Their *a priori* statistics are $x_0 \sim N(0\,\text{meters}, (10\,\text{meters})^2)$ and $v_0 \sim N(100\,\text{m/s}, (1\,\text{m/s})^2)$. The additive measurement noises are assumed to be white noise sequences and independent of each other with statistics

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1\text{meter}^2 & 0 \\ 0 & (4\text{cm/sec})^2 \end{bmatrix}\right)$$

- Using the difference between the integration of the accelerometer output and the GPS as a new measurement, determine an associated stochastic discrete time system that is approximately independent of the acceleration profile.
- For that stochastic system implement a minimum variance estimator.
- To test your filter implementation, from the above data simulate the exact stochastic system and do the test over a 30 sec. interval.
  - Show error estimates of position, velocity, and accelerometer bias for one realization. Include the one sigma bound obtained from the error variance computed in the Kalman filter.
  - Show that the error variance obtained from the simulation and averaged over an ensemble of realizations is close to the error variance used in the Kalman filter algorithm.
  - Show that the theoretical orthogonality properties, such as $E[(x_k - \hat{x}_k)\hat{x}_k^T]$ and $E[(x_k - H\bar{x}_k)(x_j - H\bar{x}_j)^T]$ $k > j$, are satisfied for this filter.

In plotting the states, errors, and variances, try to show values over the time interval, i.e. do not plot what looks like zero. Turn in a report having an abstract, and sections such as introduction, theory and algorithm, results and performance, and conclusions.

Approach to the Calibration of an Accelerometer Using GPS

The truth model that is used as the simulation model is

$$a(t) = a\sin(\omega t)$$

where a and ω are given above. By integrating the sin we get the actual velocity v and position p as

$$\begin{aligned} v(t) &= v(0) + \frac{a}{\omega} - \frac{a}{\omega}\cos(\omega t) \\ p(t) &= p(0) + (v(0) + \frac{a}{\omega})t - \frac{a}{\omega^2}\sin(\omega t) \end{aligned} \tag{1}$$

where the statistics for the initial values of velocity v and position p are $v(0) \sim \mathrm{N}(\bar{v}_0,\, M_0^v)$ and $p(0) \sim \mathrm{N}(\bar{p}_0,\, M_0^p)$, respectively, with values of the mean and variance as given above.

Accelerometer Model

From above the accelerometer model is

$$a_c(t_j) = a(t_j) + b + w(t_j) \tag{2}$$

where $t_{j+1} - t_j = \Delta t = 0.005\,\mathrm{sec.}$. The accelerometer integrates by an Euler formula as

$$\begin{aligned} v_c(t_{j+1}) &= v_c(t_j) + a_c(t_j)\Delta t \\ p_c(t_{j+1}) &= p_c(t_j) + v_c(t_j)\Delta t + a_c(t_j)\frac{\Delta t^2}{2} \end{aligned} \tag{3}$$

with initial conditions $v_c(0) = \bar{v}_0$, $p_c(0) = \bar{p}_0 = 0$

Derivation of Dynamic Model

The objective is to make the system model for the Kalman filter independent of the actual acceleration $a(t)$. We make the *assumption* that the actual or true acceleration is integrated by the same Euler integration formula as the accelerometer, i.e.

$$\begin{aligned} v_E(t_{j+1}) &= v_E(t_j) + a(t_j)\Delta t \\ p_E(t_{j+1}) &= p_E(t_j) + v_E(t_j)\Delta t + a(t_j)\frac{\Delta t^2}{2} \end{aligned} \tag{4}$$

with initial statistics $v(0) = v_E(0) \sim \mathrm{N}(\bar{v}_0,\, M_0^v)$ and $p(0) = p_E(0) \sim \mathrm{N}(\bar{p}_0,\, M_0^p)$.

To obtain the dynamics for the Kalman filter, subtract (3) from (4) using (2) to obtain the dynamic equations whose coefficients are independent of the acceleration as

$$\begin{bmatrix} \delta p_E(t_{j+1}) \\ \delta v_E(t_{j+1}) \\ b(t_{j+1}) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & -\frac{\Delta t^2}{2} \\ 0 & 1 & -\Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta p_E(t_j) \\ \delta v_E(t_j) \\ b(t_j) \end{bmatrix} - \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \\ 0 \end{bmatrix} w(t_j) \tag{5}$$

where

$$\begin{aligned}
\delta p_E(t_0) &= p_E(t_0) - p_c(t_0) \sim \mathrm{N}(0,\, M_0^p) \\
\delta v_E(t_0) &= v_E(t_0) - v_c(t_0) \sim \mathrm{N}(0,\, M_0^v) \\
E[w(t_j)] &= 0,\; E[w(t_j)w(t_\ell)^T] = W\delta_{j,\ell} \\
b(0) &\sim \mathrm{N}(0,\, M_0^b)
\end{aligned}$$

The dynamic model coefficients are independent of the acceleration profile.

Measurement Equations

The measurements are corrupted values of $p(t_i)$ and $v(t_i)$ as

$$\begin{aligned}
z^p(t_i) &= p(t_i) + \eta^p(t_i) \\
z^v(t_i) &= v(t_i) + \eta^v(t_i)
\end{aligned}$$

where $p(t_i)$ is the position and $v(t_i)$ is the velocity at the measurement time $t_i = t_{i-1} + 40\Delta t$ and the statistics of $\eta^p(t_i)$ and $\eta^v(t_i)$ are given above. To put the measurement equations in a convenient form to construct the Kalman filter, subtract the computed accelerometer position and velocity from the GPS measurements as

$$\begin{aligned}
\delta z^p(t_i) &= \delta p(t_i) + \eta^p(t_i) \\
\delta z^v(t_i) &= \delta v(t_i) + \eta^v(t_i) \qquad (6)
\end{aligned}$$

where $\delta p(t_i) = p(t_i) - p_c(t_i)$ and $\delta v(t_i) = v(t_i) - v_c(t_i)$. Note that the GPS measures corrupted values of the actual position and velocity.

Kalman Filter

Assume that the approximation can be made that $\delta p(t_i) = \delta p_E(t_i)$ and $\delta v(t_i) = \delta v_E(t_i)$. Based on the measurement equations (6) and the dynamics (5), the approximate *posteriori* conditional mean $\widehat{\delta x}(t_i)$ and the conditional *posteriori* error variance $P(t_i)$ are computed in the Kalman filter algorithm where we define

$$\widehat{\delta x}(t_i) = \begin{bmatrix} \widehat{\delta p}(t_i) \\ \widehat{\delta v}(t_i) \\ \widehat{b}(t_i) \end{bmatrix} = \begin{bmatrix} \widehat{p}(t_i) - p_c(t_i) \\ \widehat{v}(t_i) - v_c(t_i) \\ \widehat{b}(t_i) \end{bmatrix}$$

Note that the state estimate is considered to be the estimate of the actual position and velocity which is what the measurement uses.

Generation of Checks to Validate the Kalman Filter Performance

We first show that the actual error variance obtained directly from a Monte Carlo simulation is close to the error variance $P(t_i)$ computed to form the Kalman gain. The Monte Carlo simulation computes an ensemble of realizations of the state $\delta x(t_j)$ and state estimates $\widehat{\delta x}(t_i)$. Define $\delta x(t_j)$ as

$$\delta x(t_j) = \begin{bmatrix} \delta p(t_j) \\ \delta v(t_j) \\ b(t_j) \end{bmatrix} = \begin{bmatrix} p(t_j) - p_c(t_j) \\ v(t_j) - v_c(t_j) \\ b(t_j) \end{bmatrix}$$

where $p(t_j)$, $v(t_j)$, $b(t_j)$ are from the truth model. The actual *a priori* estimation error is defined as

$$\bar{e}(t_j) = \delta x(t_j) - \overline{\delta x}(t_j) = \begin{bmatrix} p(t_j) - p_c(t_j) \\ v(t_j) - v_c(t_j) \\ b(t_j) \end{bmatrix} - \begin{bmatrix} \overline{p}(t_j) - p_c(t_j) \\ \overline{v}(t_j) - v_c(t_j) \\ \overline{b}(t_j) \end{bmatrix} = \begin{bmatrix} p(t_j) - \overline{p}(t_j) \\ v(t_j) - \overline{v}(t_j) \\ b(t_j) - \overline{b}(t_j) \end{bmatrix}$$

where $\overline{(\cdot)}(t_j)$ denotes the *a priori* estimate or estimation error propagated in the Kalman filter. However, at the measurement time $t_i = t_j$, $\bar{e}(t_i) = \delta x(t_i) - \overline{\delta x}(t_i^-)$ and the *posteriori* estimation error $e(t_i) = \delta x(t_i) - \widehat{\delta x}(t_i^+)$ where t_i^- means before the measurement update and t_i^+ means after the measurement update.

A Monte Carlo simulation is to be constructed to find the ensemble averages over a set of realizations. Let $e^l(t_i)$ represent the actual error for realization l. To obtain this realization, an initial condition for $p(0)$, $v(0)$, b are generated from a Gaussian noise generator. The state estimate is determined from measurements where the measurement noise for $\eta^p(t_i)$ and $\eta^p(t_i)$ are generated at each measurement time t_i from a Gaussian noise generator. The process noise $w(t_j)$ in the accelerometer model is generated at each propagation time t_j.

The ensemble average of $e^l(t_i)$ which produces the actual mean is

$$e^{ave}(t_i) = \frac{1}{N_{ave}} \sum_{l=1}^{N_{ave}} e^l(t_i)$$

where N_{ave}, the number of realizations, should be large enough to produce good approximate statistics. It is expected that $e^{ave}(t_i) \approx 0$ for all $t_i \in [0,\ 30]$.

Next, consider the ensemble average producing the actual error variance P^{ave} as

$$P^{ave}(t_i) = \frac{1}{N_{ave}-1}\sum_{l=1}^{N_{ave}}[e^l(t_i) - e^{ave}(t_i)][e^l(t_i) - e^{ave}(t_i)]^T$$

where $N_{ave} - 1$ is used for an unbiased variance from small sample theory. The matrix $P^{ave}(t_i)$ should be close to $P(t_i)$ computed in the Kalman filter algorithm, i.e. $P^{ave}(t_i) - P(t_i) \approx 0$ for all t_i. This is an important check to verify that the modeling is approximately correct and the Kalman filter has been programmed correctly. Similarly, the orthogonality of the error in estimates with the estimate is checked by the average

$$\frac{1}{N_{ave}}\sum_{l=1}^{N_{ave}}[e^l(t_i) - e^{ave}(t_i)]\hat{x}(t_i)^T \approx 0 \,\forall\, t_i$$

Finally, the independence of the residuals are to be checked. The residual for a realization l is

$$r^l(t_i) = \begin{bmatrix} \delta z^{pl}(t_i) - \overline{\delta z}^{pl}(t_i) \\ \delta z^{vl}(t_i) - \overline{\delta z}^{vl}(t_i) \end{bmatrix}$$

Then, the ensemble average for the correlation of the residuals is

$$\frac{1}{N_{ave}}\sum_{l=1}^{N_{ave}} r^l(t_i) r^l(t_m)^T \approx 0 \,\forall\, t_m < t_i$$

Need only to do this average for only one value of t_m and t_i to satisfy the check for this assignment.