IRS: Final Report

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1 Introduction

IRS is a single player, full knowledge game on a board that contains all integers from 1 to n. The player may claim any number that has at least one unclaimed proper divisor. Upon claiming a number x, all unclaimed proper divisors of x are claimed by the "IRS", which acts as an adversary. If there are no more numbers with unclaimed divisors, the game is over. The sum of the numbers claimed by each player is the final score of that player. The IRS's final score is the difference between the total score and the player's score. The player wins if their score is higher than the IRS score.

Definition 1. *Each number x ($1 \leq x \leq n$) is worth x points. Let R be the ratio of numbers successfully claimed by the player.*

$$R = \frac{\sum numbersClaimed}{\sum_{i=1}^{n} i}.$$

To win this game, R must exceed $\frac{1}{2}$.

This paper applies a combination of strategies to argue that IRS is winnable for all possible n except for 1 and 3. (At $n = 1$, the player loses because there are no proper divisors. At $n = 3$, the player and the IRS tie with 3 points each):

Section 2 details the method "brute force" operated by a modern computer that tries all possible combinations of legal moves. This solves the game for $n = 2$ and $n \in [4, 26]$.

Section 3 solves the games for $n \in [27, 38]$ by performing brute force on several subsets of the board — by considering Factor Counting.

Section 4 introduces a method that secures $\frac{3}{8}$ of the available points by taking top half even numbers.

Section 5 generalizes the $\frac{3}{8}$ method by considering prime factors greater than 2 and solves the game for $n \in [410, \infty)$. We also provide some experimental results of our analysis.

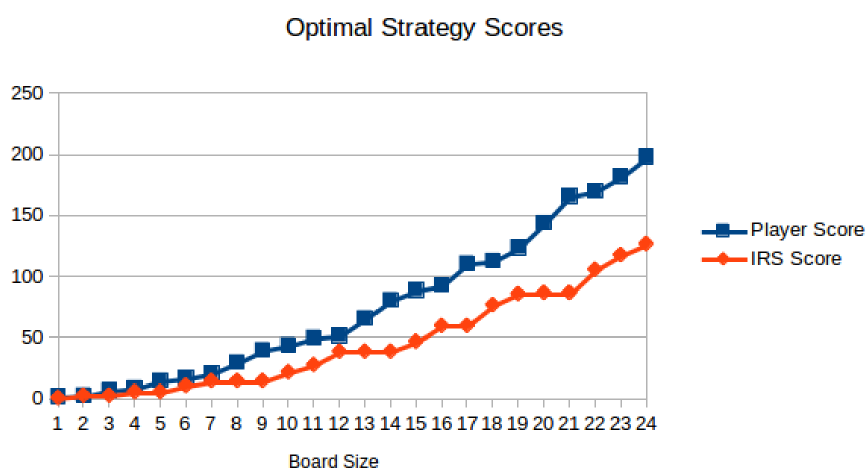
In this paper, we provide an algorithm that converges on a winning ratio of 50.386% of the total score for large game sizes and the findings that led to the discovery of this algorithm. The algorithm is provided in section 5. Moreover, we conjecture that IRS is winnable for all n other than 1 and 3.

2 Brute Force

Combinatorial games with perfect information generally have optimum strategies that can be found by brute force, and IRS is no exception. To start our investigation, we wrote a C++ program to test every possible strategy for game sizes up to $n = 26$. Our program keeps and records the moves for the optimum strategy and reports the score.

At game size $n = 27$, there are $2^{27} = 134\,217\,728$ states for tracking claimed/unclaimed squares. Not all of these states are legal given the rules of IRS, but we know the brute force algorithm is exponential in the number of game moves. Each board size took significantly longer than the previous. Based on the runtime trajectory, brute forcing larger boards seems unfeasible.

The following data from the brute force algorithm gave us hope that the player can beat the IRS for all $n > 3$. (At $n = 1$, the player loses because there are no proper divisors. At $n = 3$, the player and the IRS tie with 3 points each.)



3 Factor Counting

The first direction we explored is an analysis based on what we call "factor counting." Taking the positive integers between 1 and n inclusive we construct a table as follows:

1. The table has ℓ rows where ℓ is the largest power of 2 not exceeding n, that is, $2^{\ell} \leq n < 2^{\ell+1}$.
2. The number of the row denotes the number of prime divisors, counted with multiplicity, which divide any number on the row.
3. The integers in each row are strictly increasing from left to right.
4. The number of the row increases as you ascend the table. So the ℓ^{th} row is the top row and the 1^{st} row is the bottom row.

For example, given $n = 13$ the table looks as follows

3	8	12				
2	4	6	9	10		
1	2	3	5	7	11	13

Now let m_i be the largest number on row i which divides any number on row $i+1$ for $1 \leq i < \ell$. Therefore, in the previous example, $m_1 = 5$ and $m_2 = 6$. It is easy to see that given m_i, any number less than m_i on row i also divides some number on row $i+1$. We will call the set of all integers on row i less than or equal to m_i the *shaded region* and we call the set of all elements on row i greater than m_i the *blank region*. The shaded regions in the previous example are shown below.

3	8	12				
2	4	6	9	10		
1	2	3	5	7	11	13

Factor Counting Strategy

CONJECTURE : We can acquire more than half the total score by, at each row, claiming the highest possible sum of numbers outside the shaded region.

RATIONALE : The numbers outside of the shaded region are larger than the numbers in the shaded region and so are more valuable. Also, by not claiming numbers in the shaded region we gain the opportunity to claim larger numbers at the row above, by how we defined the shaded region.

PROBLEMS :

1. How do we know that we can use all the elements in the shaded region of row i to claim elements in the blank region of row $i+1$?
2. How do we relate the sum of numbers chosen in the blank region of each row to the total sum of numbers?

3. How can we organize the information about numbers in the blank region of row $i+1$ with the information we have about the numbers in the shaded region of row i? That is, how do we formalize how to pick from claimable numbers and in which order?

Problem 1 is of interest because using all of the elements in the shaded region of row i to claim elements in the blank region of row $i+1$ gives us a larger sum than using a subset of the elements in the shaded region of row i.

Problem 2 is of interest because a relationship between the sum of numbers chosen in the blank region of each row to the total sum of numbers makes it easier to test whether we get more than half the total score.

Problem 3 is where we are currently focusing our attention. Since there will, in general, be many numbers in the blank region of row $i+1$ sharing divisors in the shaded region of row i, the number of ways to choose these numbers will be large. Therefore, representing the information of how these numbers interact (which ones share one or more divisors, the value of the numbers sharing divisors, etc.) in a simple way is crucial. Take for example $n = 39$. We have $m_1 = 19, m_2 = 15, m_3 = 12$, and $m_4 = 16$. Then the table looks like

5	32														
4	16	24	36												
3	8	12	18	20	27	28	30								
2	4	6	9	10	14	15	21	22	25	26	33	34	35	38	39
1	2	3	5	7	11	13	17	19	23	29	31	37			

The fact that a choice of numbers exists which wins the game isn't at all obvious from looking at this table. A choice of moves are, by row,

1. 37
2. $38 \to 34 \to 26 \to 39 \to 33 \to 25 \to 35$
3. $27 \to 18 \to 20 \to 28 \to 30$
4. $36 \to 24$
5. 32

The sum of these numbers is 422 and half the total score is $\frac{39 \cdot 40}{4} = 390$. Thus we win. The argument for the choices are, by row,

1. The number chosen from row 1 is always the same. We choose the largest prime.
2. We will for the most part hand-wave the second row and just say that it's an order that works (such an order can be found using a brute force search). Note that 38 and 34 can be chosen at any time because no other

numbers have the divisors 19 and 17, respectively, and will always be available. Every other choice is much more involved and we're not sure an efficient algorithm exists to make these choices "intelligently." This is potentially an area for further research.

3. There are many orders to choose from on row 3. The only orders that must be maintained are:

   (a) 27 must come before 18 if we want to take both. This is because 18 has both 9 and 6 as divisors whereas 27 has only 9. (We are of course only speaking of divisors in the shaded are on row 2.)

   (b) To get 27, 18, and 30 we must claim 18 before 30. This is because if we claim first 30, then by (**a**) we need to claim 27 before 18. But if both 30 and 27 are gone, so are 9 and 6. So 18 cannot be claimed.

   (c) 20 must be claimed before both 28 and 30 are claimed. This is because if both 28 and 30 are gone, then both 10 and 4 are gone and so 20 cannot be claimed.

   With that being said, other equivalent choices for row 3 are

   $$27 \to 18 \to 28 \to 20 \to 30,$$

   $$27 \to 18 \to 30 \to 20 \to 28,$$

   $$20 \to 27 \to 28 \to 18 \to 30,$$

   et cetera.

4. The order in which we choose row 4 is the only order that can used to get both 36 and 24, as can be seen by using the same reasoning for row 3.

5. The choice made on row 5 is clear.

This example shows the importance of Problem 3.

We restate and give a proof of Problem 1 below. Problems 2 and 3 proved to be more difficult to analyze than the first and so we have not been able to construct an algorithm to solve the stated conjecture at this time. Currently we brute force at each level. In particular, we use the integers in the shaded region of row i to claim integers in the blank region of row $i+1$ for all $1 \leq i < n$. Experimentally, this srategy has been tested and verified up to $n = 100$

Theorem 3.1 (Weak form of Bertrand's Postulate)**.** *For every integer $n > 1$ there is at least one prime number p such that*

$$n < p < 2n$$

More information on Bertrand's Postulate can be found at

`http://en.wikipedia.org/wiki/Bertrand's_postulate.`

The proof of Problem 1 depends on the above theorem and is restated and proved below. Define S_i to be the set integers in the shaded region of row i and B_i to be the set of integers in the blank region of row i.

Proposition 3.2. *Given any row i, every integer in S_i divides some integer in B_{i+1}.*

Proof. Suppose not. Then there exists $x \in S_i$ such that x does not divide any element of B_{i+1}. By how S_i is defined, x divides some element in S_{i+1}. Let p_n be the largest prime number such that $p_n \cdot x$ is on in row i. Then p_n is the largest prime multiple of x which is less than or equal to n, by how we define the table. Therefore $p_n \cdot x \in S_{i+1}$ and $p_{n+1} \cdot x > n$. Because $p_n \cdot x \in S_{i+1}$, $p_n \cdot x$ divides some element of row $i+2$. It follows that $2 \cdot p_n \cdot x$ is on row $i+2$. Because $2 \cdot p_n \cdot x$ is on row $i+2$ we have $2 \cdot p_n \cdot x \leq n$. But this implies that $2 \cdot p_n \cdot x \leq n < p_{n+1} \cdot x \implies 2 \cdot p_n < p_{n+1}$ which contradicts Bertrand's Postulate. Therefore the Proposition follows. □

4 Even Numbers in the Top Half: $\frac{3}{8}$ proof

We found that taking large numbers is a good strategy since it allows R to grow faster. Furthermore, by taking the top half of the even numbers, we can secure $\frac{3}{8}$ of the total score.

Theorem 4.1. *For sufficiently large n, we can secure $\frac{3}{8}$ of the available points.*

Proof. The strategy is to take all even numbers x where $x > \frac{n}{2}$ in ascending order. In order to show that this strategy works, we separate the proof into two parts. First, we show that the strategy can legally claim all of even numbers it attempts to claim. Second, we show that claiming these numbers secures $\frac{3}{8}$ of the points (for large n).

Proposition 4.2. *This strategy can legally claim all x where $2|x$ and $x \in (\lfloor \frac{n}{2} \rfloor, n]$.*

Proof. First, every claimed number has a reserved divisor.

$$1 * * * * * \tfrac{n}{4} * * * * * \tfrac{n}{2} * * * * * * * * * * * * n$$

Let $A = (\lfloor \frac{n}{2} \rfloor, n]$ and $B = (\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor]$
$\forall x \in A: \ 2|x \implies \frac{x}{2} \in B$
That is, every even number x in A has a corresponding divisor, $y = \frac{x}{2}$, in B.

Second, we show that claiming the even numbers in A in ascending order is legal. (Each number has divisor, but we need to guarantee that this divisor is always available.) After $x \in A$ is claimed for the player, we assert that none of $\{z \in B : \ z > y\}$ are eliminated. This is because z must differ from x by a factor smaller than 2. The only smaller factor is 1, so $z = x$ and $z \notin B$.

By contradiction, assume $x = 2y$ has a *proper* divisor $z > y$, where $x = kz$.

$$\begin{aligned}
x &= kz \\
2y &= kz \\
2y &> ky \\
2 &> k \implies k = 1 \implies x = z
\end{aligned}$$

This strategy can indeed claim all even numbers larger than $\frac{n}{2}$. $\square$

Proposition 4.3. *Let R be the ratio of points claimed.*

$$\lim_{n\to\infty} R = \frac{3}{8}$$

Proof. First, we calculate the ratio of points available in the top half of the board$(\lfloor \frac{n}{2} \rfloor, n]$. Let this be d. We will use the well-known identity $\sum_{i=m}^{n} = \frac{(n+1-m)(n+m)}{2}$.

When n is even, $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$.

$$\begin{aligned}
d &= \frac{\sum_{i=\frac{n}{2}+1}^{n} i}{\sum_{i=1}^{n} i} \\
&= \frac{(n+1-(\frac{n}{2}+1))(n+(\frac{n}{2}+1))/2}{(n(n+1))/2} \\
&= \frac{\frac{3n}{2}+1}{2(n+1)} \\
\lim_{n\to\infty} d &= \frac{3}{4}
\end{aligned}$$

When n is odd, $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Because of the open interval, we start counting at $\frac{n+1}{2}$.

$$\begin{aligned}
d &= \frac{\sum_{i=\frac{n+1}{2}}^{n} i}{\sum_{i=1}^{n} i} \\
&= \frac{(\frac{n+1}{2}+n)(n-\frac{n+1}{2}+1)/2}{n(n+1)/2} \\
\lim_{n\to\infty} d &= \frac{3}{4}
\end{aligned}$$

We take all the even numbers in $(\frac{n}{2}, n]$, and when n is large, this means we will claim half of the score in that range.

$$\lim_{n\to\infty} R = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$$

$\square$

$\square$

This strategy leaves us needing an additional $\frac{1}{8}$ of the total sum in order to win.

5 Winning All Large n: Generalized "Top Half" Method

We have developed a generalized version of the $\frac{3}{8}$ method that wins the game for all sufficiently large n. We know that this strategy is not optimal, but winning is winning. Experimentally, it starts winning all board sizes for $n \geq 410$. (We were able to test up to board sizes $n \approx 250000$.)

5.1 Looking beyond 2

The $\frac{3}{8}$ strategy focused on numbers divisible by 2. In the window $[\frac{n}{2^2}, \frac{n}{2})$, we could find factors for all numbers x greater than $\frac{n}{2}$ if $2|x$.

A generalization of this strategy works for any prime p less than n. For example, consider the prime number 3. Between $\frac{n}{3^2}$ and $\frac{n}{3}$, we have divisors for all numbers x greater than $\frac{n}{3}$ if $3|x$. This window of divisors has the same properties as the $p = 2$ window used in the $\frac{3}{8}$ proof. Specifically, claiming all the numbers divisible by 3 from the $(\frac{n}{3}, n]$ window in ascending order will work, because a divisor will always be available from the $(\frac{n}{9}, \frac{n}{3}]$ divisor window. [1]

Generalized strategy for a prime number p:

Claim numbers divisible by p in ascending order from the window $(\frac{n}{p}, n]$, using the numbers $(\frac{n}{p^2}, \frac{n}{p}]$ as factors.

5.2 Playing multiple strategies concurrently

Unfortunately, we can't play *both* the $p = 2$ and $p = 3$ strategies completely. Because the $p = 3$ strategy relies on smaller (intuitively more fragile) divisors, playing the $p = 2$ strategy first would hopelessly mangle the $p = 3$ strategy. We will not discuss this ordering further.

Playing the $p = 3$ strategy first is more promising, but there are three distinct problems:

(1) Playing $p = 3$ claims for the player numbers between $\frac{n}{3}$ and $\frac{n}{2}$that are divisible by 3. But those should really be utilized as divisors for the $p = 2$ strategy.

[1] Generalizing the strategy for 4 doesn't make sense, since we already have a way to get all large numbers divisible by 2. For this reason, we only utilize prime numbers.

(2) The strategy for $p = 3$ uses $(\frac{n}{9}, \frac{n}{3}]$ and the strategy for $p = 2$ uses $(\frac{n}{4}, \frac{n}{2}]$. The overlap, that is $(\frac{n}{4}, \frac{n}{3}]$, would be unavailable as divisors for $p = 2$, which needs the entire $(\frac{n}{4}, \frac{n}{2}]$ window. The below image illustrates this overlap for the 2, 3, 5, and 7 factor windows.

(3) Some numbers claimed for the player by $p = 3$ would have divisors *besides* the "reserved" divisors in $(\frac{n}{9}, \frac{n}{3}]$. If a number X is divisible by 3 and divisible by 2, the player loses $\frac{X}{2}$ and $\frac{X}{3}$.

Problem (1) should be avoided altogether, because it greedily sabotages divisors for much larger numbers.

Problems (2) and (3) are more complicated, but we solve them the same way. Let x be a divisor for the $p = 3$ strategy. In general, it is better to use x to claim $3x$ than to claim $2x$. If x is even, we could already get $3x$ with the $p = 2$ strategy using a different divisor y. But if x is odd, we must choose between using x to get $3x$ or $2x$. $3x$ is always better.

That is, we only claim numbers during the $p = 3$ by a smaller strategy (in this case $p = 2$). This leaves some of the numbers in the overlap $(\frac{n}{4}, \frac{n}{3}]$ for $p = 2$, and takes them only if it strictly improves the score. As a side effect, it guarantees that that (3) won't be a problem, because the unintended divisors must be from a larger factor than 3.

Algorithm We describe an algorithm that handles each of these problems. Let P be the set of primes less than (or equal to) the board size n. Sort P in descending order. For each $p \in P$, in descending order, play the generalized strategy for p with two modifications.

First, to solve problem (1), always take the numbers in the window $(\frac{n}{2}, n]$, instead of $(\frac{n}{p}, n]$ (The divisors used will therefore come only from $(\frac{n}{2p}, \frac{n}{p}]$instead of $(\frac{n}{p^2}, \frac{n}{p}]$). This will prevent larger p values from claiming numbers that lower p values can use as divisors for higher numbers.

Second, to solve problems (2) and (3), only claim numbers that are *not* divisible by any of the smaller primes. (Instead, let the smaller prime strategy handle it.)

Input: n
$P \leftarrow \{x \in \mathbb{P} : x \leq n\}$
Sort P in descending order
for $p \in P$ **do**
  dependents $\leftarrow \{x \in P : x < p\}$
  window $\leftarrow \{x \in \mathbb{N} : \frac{n}{p/2} \leq x \leq \frac{n}{p}\}$
  for $q \in window : (\forall y \in dependents : y \nmid q)$ **do**
    claim($p \times q$)
  end
end

It's easy to see that this algorithm gets a strictly better score than the $\frac{3}{8}$

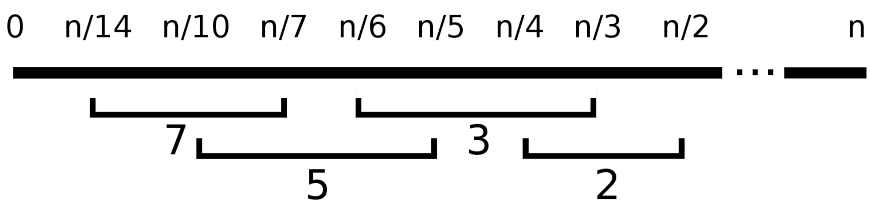


Figure 1: The overlapping windows of divisors (not to scale)

($p = 2$) strategy. The larger p strategies, whenever they interfere with lower p strategies like $p = 2$, only do so to get a larger number with the same factor. Experimentally, this strategy consistently gets 56% of the total score, for large n.

To make the analysis easier, we present a proof that using only the primes 2, 3, 5, and 7, the player will end with 50.386% of the score for sufficiently large n.

5.3 Winning with 2, 3, 5, and 7

In order to analyze the total percentage that the player wins, we will look at the score gained from each strategy individually. Here are the exact steps of the algorithm when performed only on the primes 2, 3, 5, and 7.

Let $A = (\frac{n}{2}, n]$. All of the numbers claimed for the player come from A. Each step claims numbers from A in ascending order.

Step 1: Using $[\frac{n}{14}, \frac{n}{7})$ as divisors, claim $\{x \in A : (7 \mid x) \wedge (5 \nmid x) \wedge (3 \nmid x) \wedge (2 \nmid x)\}$

Step 2: Using $[\frac{n}{10}, \frac{n}{5})$ as divisors, claim $\{x \in A : (5 \mid x) \wedge (3 \nmid x) \wedge (2 \nmid x\}$.

i. Some of divisors used in step 1, specifically in the overlap $[\frac{n}{10}, \frac{n}{7})$, will interfere. This step will fail with some probability for $\frac{n}{2} < x < \frac{5n}{7}$.

ii. None of the divisors in $[\frac{n}{7}, \frac{n}{5})$ were used in step 1, so the this step will succeed for $\frac{5n}{7} \leq x \leq n$.

Step 3: Using $[\frac{n}{6}, \frac{n}{3})$ as divisors, claim $\{x \in A : (3 \mid x) \wedge (2 \nmid x)\}$

i. Some of the divisors used in step 2, specifically in the overlap $[\frac{n}{6}, \frac{n}{5})$, will interfere. This step will fail with some probability for $\frac{n}{2} < x < \frac{3n}{5}$.

ii. None of the divisors in $[\frac{n}{5}, \frac{n}{3})$ were used by steps 1 or 2, so this step will succeed for $\frac{3n}{5} \leq x \leq n$

Step 4: Using $[\frac{n}{4}, \frac{n}{2})$ as divisors, claim $\{x \in A : 2 \nmid x\}$.

i. Some of the divisors used in step 3, specifically in the overlap $[\frac{n}{4}, \frac{n}{3})$, will already be used. This step will fail with some probability for $\frac{n}{2} < x < \frac{2n}{3}$.

ii. None of the divisors in $[\frac{n}{3}, \frac{n}{2})$ were used by steps 1, 2, or 3, so this step will succeed for $\frac{2n}{3} \leq x \leq n$

Example 1. *Consider $N = 34$. Then $A = (17, 34]$. When division is not perfect, we truncate the interval to include valid divisors.*

1. $(p = 7)$ $(\frac{34}{14}, \frac{34}{7}] = [3, 5]$ *Since 2 or 3 or 5 divides all of the numbers in this interval, we claim nothing.*
2. $(p = 5)$ $(\frac{34}{10}, \frac{34}{5}] = [4, 6]$ *We claim 25, using 5 as a divisor*
3. $(p = 3)$ $(\frac{34}{6}, \frac{34}{3}] = [6, 11]$ *We claim 21, 27, then 33*
4. $(p = 2)$ $(\frac{34}{4}, \frac{34}{2}] = [9, 17]$ *We claim 20, 24, 26, 28, 30, 32, 34. Note that we could not claim 18 or 22, because the divisors 9 and 11 were both used in the $p = 3$ strategy when claiming $3 \times 9 = 27$ and $3 \times 11 = 33$ respectively (both larger than the numbers that could be claimed by $p = 2$). (Step 3.)*

This yields a final score of 300.

Theorem 5.1. *Let R be the ratio of the points secured by the player. Using the generalized strategy only with the primes 2, 3, 5, 7:*

$$\lim_{n \to \infty} R = \frac{29627}{58800} \approx 0.50386 \geq 0.5$$

Proof. Our proof will be decomposed into propositis for the marginal score gained by each of seven sub-steps: 1, 2(i), 2(ii), 3(i), 3(ii), 4(i), and 4(ii).

Proposition 5.2. *Let R_1 be the ratio of the points secured in step 1.*

$$\lim_{n \to \infty} R_1 = \frac{1}{7} \times \frac{3}{4} \times (\frac{1}{2}\frac{2}{3}\frac{4}{5}) = \frac{1}{35}$$

Proof. We claim numbers x where $7 \mid x$, so we could potentially claim $\frac{1}{7}$ of the score in $[\frac{n}{2}, n)$. The range $[\frac{n}{2}, n)$ is worth $\frac{3}{4}$ of the total score (similar to the $\frac{3}{4}$ proof discussed prior).

We also require $2 \nmid x$, $3 \nmid x$, and $5 \nmid x$. For numbers divisible by 7, $\frac{1}{2}$ are also divisible by 2, so we multiply by $\frac{1}{2}$. Of these remaining numbers, $\frac{1}{3}$ are divisible by 3, so we multiply by $\frac{2}{3}$. Of these remaining numbers, $\frac{1}{5}$ are divisible by 5, so we multiply by $\frac{4}{5}$.

By only considering numbers in the top half, removing numbers not divisible by 7, and removing numbers that are divisible by 2, 3, or 5, we determine that the $p = 7$ strategy will claim $\frac{1}{35}$ of the total score. □

Proposition 5.3. *Let R_2 be the ratio of the points secured in step 2(i).*

$$\lim_{n \to \infty} R_2 = \frac{1}{5} \times \frac{51}{196} \times (1 - \frac{1}{2}\frac{2}{3}\frac{4}{5}) \times \frac{1}{11} = \frac{17}{4900}$$

Proof. We are claiming numbers where $5 \mid x$, so we can only claim $\frac{1}{5}$of the numbers in the range that we claim from.

For R_2, we only consider divisors within the range $[\frac{n}{10}, \frac{n}{7})$. These divisors allow us to claim a possible $\frac{51}{196}$ of the total score ($\lim_{n\to\infty} \sum_{i=n/2}^{5n/7}(i) = \frac{51}{196}$). This limit can be calculated using the same method discussed in the $\frac{3}{8}$ proof.

Not all of the divisors within this range that satisfy these requirements are still available to use. Some of them will have been utilized as part of the $p = 7$ strategy. The $p = 7$ strategy claimed $(\frac{1}{2}\frac{2}{3}\frac{4}{5})$ of them, so we are left with the ratio $1 - (\frac{1}{2}\frac{2}{3}\frac{4}{5})$.

Among these numbers, ***all*** of them are divisible by 5, 3, or 2. Given this, we want to find the ratio of numbers ***not*** divisible by 3 or 2. We can use probabilistic reasoning to analyze this problem, since we assume n is large. Let D_i denote the event that a given number x in the window is divisible by i.

$$\begin{aligned}
P(\neg D_3 \wedge \neg D_2 \mid D_5 \vee D_3 \vee D_2) &= \frac{P((\neg D_3 \wedge \neg D_2) \wedge (\ D_5 \vee D_3 \vee D_2))}{P(D_5 \vee D_3 \vee D_2)} && \textit{Defn. of conditional expectation} \\
&= \frac{P(D_5 \wedge \neg D_3 \wedge \neg D_2)}{1 - P(\neg D_5 \wedge \neg D_3 \wedge \neg D_2)} && \textit{DeMorgan's laws and simplification} \\
&= \frac{\frac{1}{5}\frac{2}{3}\frac{1}{2}}{1 - (\frac{4}{5}\frac{2}{3}\frac{1}{2})} && \textit{Uniformity of divisors} \\
&= \frac{1}{11}
\end{aligned}$$

$\frac{1}{11}$ of the the remaining divisors satisfy these requirements. ☐

Proposition 5.4. *Let R_3 be the ratio of the points secured in step 2(ii).*

$$\lim_{n\to\infty} R_3 = \frac{1}{5} \times \frac{24}{49} \times (\frac{1}{2}\frac{2}{3}) = \frac{8}{245}$$

Proof. This is similar to R_1. $\frac{1}{5}$ of the numbers can be claimed in the region. This step deals with divisors from $(\frac{n}{7}, \frac{n}{5}]$, so the claimable region is $(\frac{5n}{7}, n]$, and this region has $\frac{24}{49}$ of the available score.

None of these divisors were interfered with by $p = 7$, but we still need to avoid numbers divisible by 2 or 3. ☐

Proposition 5.5. *Let R_4 be the ratio of the points secured in step 3(i).*

$$\lim_{n\to\infty} R_4 = \frac{1}{3} \times \frac{11}{100} \times (1 - \frac{1}{2}\frac{2}{3}) \times \frac{1}{4} = \frac{11}{1800}$$

Proof. This is similar to R_2. Since this is for the $p = 3$ strategy, $\frac{1}{3}$ of the points are available from the claimable region.

The divisors for this step are $(\frac{n}{6}, \frac{n}{5}]$. The claimable region is $(\frac{n}{2}, \frac{3n}{5}]$, which has $\frac{11}{100}$ of the total points.

The $p = 5$ strategy wiped out several of the divisors, leaving only numbers divisible by 2 or 3. (This means a ratio of $(1 - \frac{1}{2}\frac{2}{3})$ remain.)

Using a conditional argument similar to the discussion of R_2, $\frac{1}{4}$ of these remaining divisors are not divisible by 2. Using previous notation, $P(\neg D_2 | D_3 \vee D_2) = \frac{1}{4}$. □

Proposition 5.6. *Let R_5 be the ratio of the points secured in step 3(ii).*

$$\lim_{n\to\infty} R_5 = \frac{1}{3} \times \frac{16}{25} \times (\frac{1}{2}) = \frac{8}{75}$$

Proof. This is still the $p = 3$ strategy (ratio $\frac{1}{3}$), now working with the claimable window $(\frac{3n}{5}, n]$, which has $\frac{16}{25}$ of the total points. $\frac{1}{2}$of the divisors must be left for the $p = 2$ strategy. □

Proposition 5.7. *Let R_6 be the ratio of the points secured in step 4(i).*

$$\lim_{n\to\infty} R_6 = \frac{1}{2} \times \frac{7}{36} \times (1 - \frac{1}{2}) = \frac{7}{144}$$

Proof. This is the $p = 2$ strategy (ratio $\frac{1}{2}$). The divisors are $(\frac{n}{4}, \frac{n}{3}]$, so the claimable window is $(\frac{n}{2}, \frac{2n}{3}]$, with $\frac{7}{36}$of the points. Half of these divisors were used by $p = 3$. □

Proposition 5.8. *Let R_7 be the ratio of the points secured in step 4(ii).*

$$\lim_{n\to\infty} R_7 = \frac{1}{2} \times \frac{5}{9} = \frac{5}{18}$$

Proof. The claimable window $(\frac{2n}{3}, n]$ contains $\frac{5}{9}$ of the total score. This is the $p = 2$ strategy, so we can take only $\frac{1}{2}$ of the numbers (the even numbers). □

Concluding from these seven lemmas, we calculate R.

$$\begin{aligned}
\lim_{n\to\infty} R &= \lim_{n\to\infty} (R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7) \\
&= \frac{1}{35} + \frac{17}{4900} + \frac{8}{245} + \frac{11}{1800} + \frac{8}{75} + \frac{7}{144} + \frac{5}{18} \\
&= \frac{29627}{58800} \\
&\approx 0.50386
\end{aligned}$$

□

5.4 Future work: finding a lower bound on n

It would be useful to find a value x, such that $\forall n > x : \ R > 0.5$. (A lower bound on n, where this strategy provably wins for all boards larger than this lower bound.) However, our analysis relies heavily on the assumption of an asymptotically large value of n, in three distinct ways.

Assumption (1). Suppose we have a divisor x. We assume the ratio of numbers that are divisible by x in a window converges to $\frac{1}{x}$. (To see why this isn't true for small windows, consider the numbers $1, 2, 3, 4, 5, 6$. We would expect $\frac{6}{7}$ of them to be divisible by 7, but none of them are.)

Assumption (2). When claiming numbers divisible by x in a window, such as $(\frac{n}{2}, n]$. We assume the ratio of the *score* in a window derived from numbers divisible by x is $\frac{1}{x}$. (To see why this isn't true for small windows, consider $1, 2, 3, 4, 5, 6, 7$. We would expect $\frac{1}{7} \times 28 = 4$ of the score to be from numbers divisible by 7. But actually $\frac{7}{28}$ of the score comes from such numbers.)

Assumption (3). The score in a window can be analyzed directly with the identity $\sum_{i=m}^{n} i = \frac{(n+m)(n-m+1)}{2}$. We calculated values for this sum by assuming n goes to infinity.

All three of these assumptions are demonstrated in proposition 5.2, where we are using $p = 7$ as our divisor. Applying assumption (2), we multiply by $\frac{1}{7}$ since $\frac{1}{7}$of the score in a window comes from numbers divisible by 7. We apply the assumption (3) to calculate the sum of numbers in our window ($[\frac{n}{2}, n]$), which yields $\frac{3}{4}$. Then we use assumption (1) to get the fraction of numbers that fulfill our requirements (not divisible by 2, 3 or 5), which is $\frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}$.

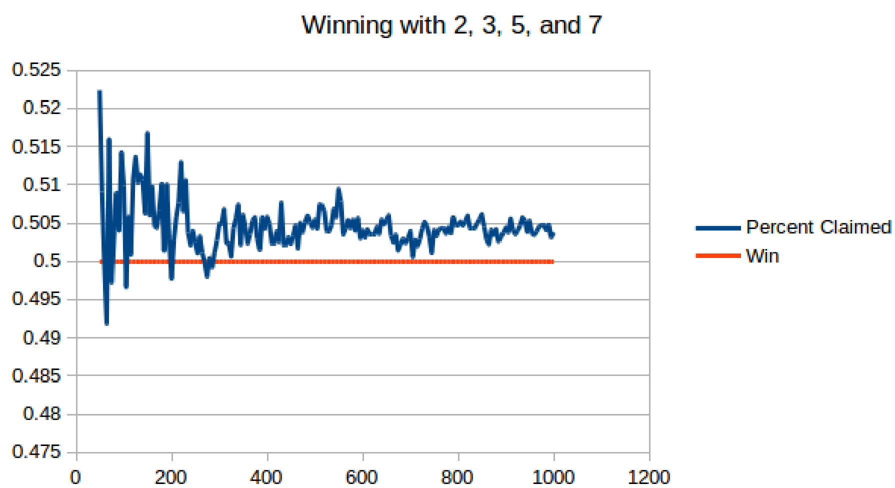
Extracting a lower bound would involve bounding all of these assumptions. Moreover, we often multiply them together, so slight deviations from our expectation would propagate through the analysis. Therefore, we believe finding a lower bound is possible, but it is beyond the scope of our investigation.

5.5 Experimental Validation

Experimentally, we have strong evidence that our analysis is correct. Running our computer implementation of this strategy yielded the following results:

n	R
410	0.50233
500	0.50543
1000	0.50377
5000	0.50392
10000	0.50380
50000	0.50385
100000	0.50385
150000	0.50386
250000	0.50386
∞	0.50386

The strategy converges on taking 50.386% of the total game score. The last (largest) n observed game where this strategy loses is $n = 409$. (This is not perfectly clear in the graph due to image formatting artifacts). We have tested up to $n = 250000$.



6 Conclusions

In this report, we have presented several algorithms for attacking the IRS problem. We now have reason to believe that IRS is solvable for all n (except for 1 and 3). In particular, we completed the following tasks:

- Formulate and describe winning strategies.
- Implement all strategies on a computer.
- Determine the winner for all large n.

6.1 Computer Algorithms

The code for all of the presented methods can be found on GitHub:

- Factor Counting — github.com/VerTiGoEtrex/Math-389-IRS
- Generalized Top-Half Method — github.com/VerTiGoEtrex/Math-389-IRS-Primes

All of these implementations depend on the "primesieve" library, available here — http://primesieve.org/

6.1.1 Division of Labor

- Noah Crocker: Computer algorithms and contributions to the generalized "top half" method.
- Zamar Edwin: Factor Counting strategy.
- Dianna Liu: Analyzed a possible lower bound on the generalized "top half" algorithm.
- Daniel MacLennan: $\frac{3}{8}$ proof, generalized "top half" algorithm/proof/analysis.