Models of (∞, n)-categories I

Jiaheng Zhao

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The notion of a strict n-category is well-defined. One can simply define a strict n-category to be a category enriched over $(n-1)$-categories. However, it is rather difficult to get a satisfying definition of a weak n-category due to the complication of higher coherence data. One possible method is to define the notion of an (∞, n)-category first by using homotopy theories, and then regarding a weak n-category as an (∞, n)-category whose k-morphisms $(k \geq n+1)$ are all identities. In this slides we introduce a model of (∞, n)-categories.

Model categories

Model structure

Let $\mathcal{C}$ be a bicomplete category. Recall that a **model structure** on $\mathcal{C}$ is a choice of three distinguished classes of morphisms: **fibrations** ($\mathcal{F}\mathrm{ib}$), **cofibrations** ($\mathcal{C}\mathrm{of}$) and **weak equivalences** ($\mathcal{W}$) subject to a series of axioms I will list soon. A category equipped with a model structure is called a **model category**.

Let $\mathcal{C}$ be a model category. An object $x \in \mathcal{C}$ is **fibrant** (resp. **cofibrant**) if the unique map $x \to *$ (resp. $\emptyset \to x$)[1] is a fibration (resp. cofibration). We use $\mathcal{C}_f$ (resp. $\mathcal{C}_c$) to denote the full subcategory spanned by all the fibrant (resp. cofibrant) objects. Objects in $\mathcal{C}_{cf} := \mathcal{C}_c \cap \mathcal{C}_f$ are called **bifibrant objects**.

Morphisms in $\mathcal{F}\mathrm{ib} \cap \mathcal{W}$ (resp. $\mathcal{C}\mathrm{of} \cap \mathcal{W}$) are called **trivial/acyclic fibrations** (resp. **trivial/acyclic cofibrations**).

[1] We use $*$ (resp. $\emptyset$) to denote the final (resp. initial) object of $\mathcal{C}$.

The axioms

Let $F \subseteq \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. We use $l(F)$ $(r(F))$ to denote the class of morphisms which have **left lifting property** (**right lifting property**) with respect to F. The axioms of a model structure include:

MC1 $\mathcal{C}$ is bicomplete;

MC2 $\mathcal{W}$ satisfies the 2-out-of-3 property;

MC3 $\mathcal{W}$, $\mathcal{C}\mathrm{of}$ and $\mathcal{F}\mathrm{ib}$ are closed under retracts;

MC4 $\mathcal{C}\mathrm{of} = l(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$, $\mathcal{F}\mathrm{ib} \cap \mathcal{W} = r(\mathcal{C}\mathrm{of})$, $\mathcal{W} \cap \mathcal{C}\mathrm{of} = l(\mathcal{F}\mathrm{ib})$, $\mathcal{F}\mathrm{ib} = r(\mathcal{W} \cap \mathcal{C}\mathrm{of})$.

MC5 Any morphism f can factored **functorially** in two ways

- $f = p \circ i$ where $i \in \mathcal{C}\mathrm{of}$ and $p \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$;
- $f = p \circ i$ where $i \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$ and $p \in \mathcal{F}\mathrm{ib}$.

By MC5, $\mathcal{C}$ is equipped with a fibrant (resp. cofibrant) replacement functor $R : \mathcal{C} \to \mathcal{C}_f$ (resp. $Q : \mathcal{C} \to \mathcal{C}_c$).

Example

We use Top to denote the category of compactly generated Hausdorff spaces. Over Top there are two commonly used model structure:

- The Hurewicz model structure. The weak equivalences are homotopy equivalences. The fibrations are Hurewicz fibrations, i.e. maps with right lifting property w.r.t. the all inclusions $i_0 : A \to A \times I$. Cofibrations are inclusions $A \to X$ such that $X \times 0 \cup A \times I$ is a retract of $X \times I$.
- The Quillen-Serre model structure. This is a cofibrantly generated model structure generated by $I := \{S^n \to D^{n+1}\}_{n \geq 0}$ and $J := \{D^n \to D^n \times I\}_{n \geq 0}$, i.e. $(\mathcal{C}\mathrm{of}, \mathcal{W} \cap \mathcal{F}\mathrm{ib}) = (l(r(I)), r(I))$ and $(\mathcal{C}\mathrm{of} \cap \mathcal{W}, \mathcal{F}\mathrm{ib}) = (l(r(J)), r(J))$. The weak equivalences are given by weak homotopy equivalences. We use Top_Q to denote the Quillen-Serre model category of topological spaces.

Example

Let $\mathcal{A}$ be a category. Define $s\mathcal{A} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$ to be the category of simplicial objects in $\mathcal{A}$. For example, $s\mathrm{Set}$ is the category of simplicial sets. There are two commonly used model structure over $s\mathrm{Set}$:

- The Quillen-Kan model structure. The weak equivalences are weak homotopy equivalences, i.e. morphisms whose geometric realization is a weak homotopy equivalence.This model structure is cofibrantly generated with generating sets $I := \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$ and $J := \{\Lambda^k_n \to \Delta^n, 0 \leq k \leq n\}_{n\geq 0}$. The Quillen-Kan model category $s\mathrm{Set}$ is denoted by $s\mathrm{Set}_Q$.
- The Joyal model structure. The weak equivalences are categorical equivalences. Fibrant objects in Joyal model structure are **quasi-categories**. The Joyal model category $s\mathrm{Set}$ is denoted by $s\mathrm{Set}_J$. The Joyal model structure is also cofibrantly generated but the generating sets are difficult to describe.

Example

$s\mathrm{Cat}$ is the category of all simplicial categories. A simplicial category can also be regarded as a category enriched in $s\mathrm{Set}$. For each simplicial category $\mathcal{C}$ we may associate a usual category $\pi_0(\mathcal{C})$ by defining $\hom_{\pi_0(\mathcal{C})}(x, y) = \pi_0 \hom_{\mathcal{C}}(x, y)$. We usually use the Bergner model structure over $s\mathrm{Cat}$. Weak equivalences in this model structure are given by **Dwyer-Kan equivalences**. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is a Dwyer-Kan equivalence if

- $F_{x,y}$ is a weak equivalence in $s\mathrm{Set}_Q$ for all $x, y \in \mathcal{C}$.
- The induced functor $\pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is essentially surjective.

With respect to this model structure, a simplicial category if fibrant if and only if it is locally fibrant, i.e. if all simplicial mapping spaces are Kan complexes.

Quillen adjunction

We turn to study morphisms between model categories.

Definition

Let $\mathcal{C}, \mathcal{D}$ be model categories. A **Quillen adjunction** is an adjoint pair of functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

such that the left adjoint F preserves cofibrations and the right adjoint G preserves fibrations. A Quillen adjunction is a **Quillen equivalence** if for all cofibrant x in $\mathcal{C}$ and fibrant y in $\mathcal{D}$, a map $f : Fx \to y$ is a weak equivalence in $\mathcal{D}$ if and only if the adjoint map $f^\sharp : x \to Gy$ is a weak equivalence in $\mathcal{C}$.

Model categories are defined to be localized. Let $\mathcal{C}$ be a model category with W the weak equivalences. We use $\mathcal{C}[W^{-1}]$ to denote the localization of $\mathcal{C}$ with respect to W. By doing some abstract homotopy theory we may show that there is an equivalence of categories:

$$h\mathcal{C}_{cf} \simeq \mathrm{Ho}(\mathcal{C}) := \mathcal{C}[W^{-1}]$$

Here $h\mathcal{C}_{cf}$ is a quotient category of $\mathcal{C}_{cf}$ with respect to a well-defined homotopy relation over its hom sets.

Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be a pair of Quillen adjunction, then their **total derived functors** induces a pair of usual adjunction between the localized categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \leftrightarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{L}G$$

Moreover, if $F \dashv G$ is a pair of Quillen equivalence, then $\mathbb{L}F \dashv \mathbb{L}G$ is a pair of adjoint equivalence.

Example

The adjunction $|-| \dashv \mathrm{Sing}$ gives rise to a pair of Quillen equivalence between Top_Q and $s\mathrm{Set}_Q$. Note that, all spaces in Top_Q are fibrant and all simplicial sets in $s\mathrm{Set}_Q$ are cofibrant. Bifibrant objects in Top_Q are cell complexes and their retracts; bibrant objects in $s\mathrm{Set}_Q$ are Kan complexes. We use $\mathcal{C}ell \subseteq \mathrm{Top}$ to denote Top_{cf} and $\mathcal{K}an \subseteq s\mathrm{Set}$ to denote the full subcategory spanned by Kan complexes. Then we have

$$h\mathcal{C}ell = h\mathrm{Top}_{cf} \simeq \mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(s\mathrm{Set}) \simeq hs\mathrm{Set}_{cf} = h\mathcal{K}an$$

The category of CW-complexes (which we denote by $\mathcal{CW}$) is a full subcategory of $\mathcal{C}ell$. It turns out that left hand side of the above equivalence chain can be replaced by $h\mathcal{CW}$. And finally we see that the homotopy theory of CW-complexes is equivalent to the homotopy theory of Kan complexes.

Example

There is a **simplicial nerve functor** $N_\Delta : s\mathrm{Cat} \to s\mathrm{Set}$ defined by the assignment

$$N_\Delta(\mathcal{C})_\bullet = \mathrm{Fun}_{s\mathrm{Cat}}(C[\Delta^\bullet], \mathcal{C})$$

where $C[\Delta^n]$ is a simplicial category which encodes not only the objects and paths in Δ^n but also higher homotopy data. Since $s\mathrm{Cat}$ is cocomplete we can extend the definition of $C[-]$ to any simplicial set X. Then a simple calculation shows that $C[-]$ is left adjoint to the simplicial nerve functor N_Δ. As a matter of fact the adjunction forms a pair of Quillen equivalence:

$$C[-] : s\mathrm{Set}_J \leftrightarrows s\mathrm{Cat} : N_\Delta$$

Reedy model structure

Recall that, the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ inherits many structures and properties from the target category. For example, if $\mathcal{D}$ is additive/abelian/(co)complete/idempotent complete, then so is $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, regardless of what $\mathcal{C}$ is. However, this is not true for model structures. When $\mathcal{D}$ is a model category, there is no general method to construct a model structure for the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, unless we impose some further assumptions on $\mathcal{D}$ or $\mathcal{C}$. In particular, when $\mathcal{C}$ is a **Reedy category**, we can equip $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with a **Reedy model structure**.

We will not give the explicit definition of Reedy categories. The prototypical example of a Reedy category is the simplex category Δ (or its opposite Δ^{op}).

Theorem

Let $\mathcal{C}$ be a Reedy category and $\mathcal{D}$ a model category, then there is a canonical model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ in which the weak equivalences are given by pointwise weak equivalences.

We will consider the functor category $ss\mathrm{Set} := \mathrm{Fun}(\Delta^{\mathrm{op}}, s\mathrm{Set}_Q)$. A functor in $\mathrm{Fun}(\Delta^{\mathrm{op}}, s\mathrm{Set})$ is called a **bisimplicial set** or a **simplicial space**. There are two ways to embed $s\mathrm{Set}$ into $ss\mathrm{Set}$: to view a simplicial set as a **constant simplcial space** or a **discrete simplicial space**. The two embeddings are induced by the two projections $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ respectively. For a simplicial set X, we use $c(X)$ (resp. $d(X)$) to denote the constant (resp. discrete) simplicial space given by X. Denote $\Delta^{m,n} = d(\Delta^m) \times c(\Delta^n)$. This is a presheaf on $\Delta \times \Delta$ represented by the pair $([m], [n])$. Thus we have

$$\hom(\Delta^{m,n}, X) = X_{m,n}.$$

Since $ss\mathrm{Set}$ is a presheaf category it is still Cartesian closed and thus enriched over itself. That is, we have

$$\mathrm{Fun}(X, Y)_{m,n} = \hom(X \times \Delta^{m,n}, Y)$$

Forgetting part of the structure we get a simplicial enrichment as follows

$$\mathrm{Map}(X, Y)_n := \mathrm{Fun}(X, Y)_{0,n} = \hom(X \times c(\Delta^n), Y).$$

Then we have $X_n = \mathrm{Map}(d(\Delta^n), X)$. Taking this into account we may use the notation that $X(S) := \mathrm{Map}(d(S), X)$ for any simpplicial set X. In particular we have $X(\Delta^n) = X_n$.

Now let us describe the Reedy model structure over $ss\mathrm{Set}$:

- The weak equivalences are given by pointwise weak equivalences;
- Cofibrations are given by pointwise cofibrations, i.e. injective maps;
- A map $f : X \to Y$ is a (trivial) fibration iff $f_0 : X_0 \to Y_0$ is a (trivial) Kan fibration and for each $n \geq 0$ the map

$$X_n \to Y_n \times_{Y(\partial\Delta^n)} X(\partial\Delta^n)$$

  is a (trivial) Kan fibration of simplicial sets.

Reedy fibrant simplicial spaces

A **Reedy fibrant** simplicial space X is characterized by the property that the map

$$X_n = X(\Delta^n) \to X(\partial\Delta^n)$$

is a Kan fibration.

Example

- Set $\partial\Delta^0 = \emptyset$. Then X_0 is a Kan complex;
- The map $X_1 \to X_0 \times X_0$ is a Kan fibration. Thus X_1 is a Kan complex;
- Note that fibrations are stable under pullbacks, so by induction we may show that X_n is a Kan complex for each n.

Extra structures

Now we discuss some extra structures over a model category

Definition

A **simplicial model category** is a category $\mathcal{C}$ enriched in $s\mathrm{Set}$ whose underlying category $\mathcal{C}_0$ is equipped with a model structure such that:

* For every cofibration $i : A \to B$ and every fibration $p : X \to Y$ the **pull-back hom** $[B, X] \to [A, X] \times_{[A,Y]} [B, Y]$ is a Kan fibration of simplicial sets. Moreover, this induced map is acyclic when either of i or p is acyclic.

Example

Since $s\mathrm{Set}$ is the category of presheaves over Δ, it is Cartesian closed. So it is enriched over it self. It turns out that $s\mathrm{Set}_Q$ is a simplicial model category.

Definition

A model category $\mathcal{C}$ is **Cartesian** if the underlying category is category closed (so it is enriched over itself) such that

* For every cofibration $i : A \to B$ and every fibration $p : X \to Y$ the pull-back hom $[B, X] \to [A, X] \times_{[A,Y]} [B, Y]$ is fibration. Moreover, this induced fibration is acyclic when either of i or p is acyclic.

Example

The model categories Top_Q and $s\mathrm{Set}_Q$ are both Cartesian.

Complete Segal spaces

$(\infty, 0)$-categories

Homotopy Hypothesis

n-groupoids should be the same as homotopy n-types. In particular we shall identify ∞-groupoids and topological spaces. More explicitly there should be an equivalence:

$$\{\text{topological spaces}\}/\text{weak equivalences} \simeq \{\infty\text{-groupoids}\}/\text{equivalences}$$

When $n = 1$ the Homotopy Hypothesis can be demonstrated as follows: for each groupoid $\mathcal{G}$ we map associate a CW-complex $|N(\mathcal{G})|$ (i.e. the geometric realization of the nerve of $\mathcal{G}$), which can be shown to be a homotopy 1-type; for each topological space X we may associate **fundamental groupoid** $\Pi_1(X)$, which is obviously a groupoid. The pair $|N(-)| \dashv \Pi_1$ gives rise to a pair of (bi-)adjoint equivalences between the 2-categories of homotopy 1-types and the 2-category of groupoids.

Definition

An ∞-groupoid is a topological space.

Similar to the construction of the fundamental groupoid, we may extract higher categorical data from a topological space as follows: given a topological space X, we can think of the points of X as objects, and the paths between points as 1-morphisms. Then a homotopy between two paths with the same endpoints can be thought of as a 2-morphism, and we can continue to take homotopies between homotopies to make sense of i-morphisms for all $i \geq 1$. Since paths and homotopies are invertible up to homotopy, we get an ∞-groupoid. This construction depends only on the weak homotopy type of X so we may also use CW-complexes as models of ∞-groupoids.

$(\infty, 1)$-categories

An $(\infty, 1)$-category should be a category enriched over ∞-groupoids. Thus we use the definition:

Definition

An $(\infty, 1)$-category is a topological category, i.e. a category enriched over topological spaces.

Since the homotopy theory of Top_Q and $s\mathrm{Set}_Q$ are equivalent, we may alternatively define an $(\infty, 1)$-category to be a simplicial category. Many equivalent, but more workable models are developed later. For example: quasi-categories, Segal categories or complete Segal spaces.

I will introduce complete Segal spaces in the final part of the presentation because this notion can be easily generalized to model (∞, n)-categories.

Definition

Given any simplicial space W and any $k \geq 2$, the **Segal map** is the map

$$W_k \to W_1 \times_{W_0} \dots \times_{W_0} W_1$$

induced by $\iota_i : [1] \to [n]$ which sends $0, 1$ to $i, i-1$ respectively.

Definition

A **Segal space** is a Reedy fibrant simplicial space such that the Segal maps are weak equivalences of simplicial sets for all $k \geq 2$.

Taking into account Reedy fibrantness, the Segal maps are actually trivial Kan fibrations.

A Segal space is very close to our desired model of an $(\infty, 1)$-category. Let us fix a Segal space X and look at the categorical structures:

- *Objects*: The set of objects of X is X_{00}.
- *Arrows*: By definition the map $X_1 \to X_0 \times X_0$ is a Kan fibration. Note that fibrations are closed under pull-backs. As a result, the fiber over any pair (x, y) is a Kan complex, which we denote by $\mathrm{Map}(x, y)$.
- *Composition*: The trivial fibration $X_2 \to X_1 \times_{X_0} X_1$ admits a section. Composing it with $d_1 : X_2 \to X_1$, we get a non-unique composition.
- *Identities*: Identities are obtained by applying the degeneracy map $s_0 : X_0 \to X_1$ to objects.
- *Associativity*...

As a matter of fact, the Segal condition allows us to regard X_n as “the space of composable n-tuples of arrows”.

Given a Segal space X we may construct an ordinary category $\mathrm{Ho}(X)$, called the **homotopy category** of X:

- The set of objects of $\mathrm{Ho}(X)$ is X_{00}.
- For $x, y \in X_{00}$ we define $\hom_{\mathrm{Ho}(X)}(x, y) := \pi_0(\mathrm{Map}_X(x, y))$.
- Compositions are associativity are inherited from X. In $\mathrm{Ho}(X)$ two composable arrows have a unique composition.

Definition

We say that an arrow $f \in \mathrm{Map}_X(x, y)$ is an **equivalence** if its image in $\mathrm{Ho}(X)$ is invertible.

Obviously an identity map must be an equivalence. Let $X_{eq} \subseteq X_1$ be the space of equivalences of X, then the degeneracy map s_0 must factor through $s_0 : X_0 \to X_{eq}$

A Segal space has a satisfying categorical structure and a satisfying homotopy structure. However, these two structures are not compatible. For example, let $E(1) := d(N(J))$ where J is the walking isomorphism. Then $E(1)_{00}$ contains two elements which can be connected by a pair of equivalences in $E(1)_1$. However, these two elements do not lie in the same connected components in $E(1)_0$, which is wired. To fix this problem we introduce the following:

Definition

A **complete Segal space** is a Segal space W for which the map $s_0 : X_0 \to X_{eq}$ is a weak homotopy equivalence. An $(\infty, 1)$-category is a complete Segal space

The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. In this sense, one might like to think of complete Segal spaces as a homotopical version of skeletal category.

For simplicity we will take $(\infty, 2)$-categories as an example.

Definition

A Reedy fibrant functor $W : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to s\mathrm{Set}$ if a **double Segal space** if the Segal maps

$$W_{k,\bullet} \to W_{1,\bullet} \times_{W_{0,\bullet}} \cdots \times_{W_{0,\bullet}} W_{1,\bullet}$$

and

$$W_{\bullet,k} \to W_{\bullet,1} \times_{W_{\bullet,0}} \cdots \times_{W_{\bullet,0}} W_{\bullet,1}$$

are weak equivalences for any $k \geq 2$. In other words, the simplicial spaces $W_{k,\bullet}$ and $W_{\bullet,k}$ are Segal spaces for any fixed k.

We do not expect double Segal spaces to model an $(\infty, 2)$-category. As a matter of fact, a double Segal space encodes the information of a homotopical double category. A **double category** is a category internal to categories. More explicitly, a double category has objects, horizontal morphisms and vertical morphisms. The horizontal morphisms commute with vertical morphisms and the commutativity data are encoded in "squares".

Suppose that W is a double Segal space. We can think of $W_{0,0}$ as the space of objects, $W_{0,1}$ as the space of vertical morphisms, $W_{1,0}$ as the space of horizontal morphisms, and $W_{1,1}$ as the space of squares.

A double category degenerate to a bicategory when morphisms in one direction are all identities. For example, we do not want to have nontrivial vertical morphisms, but only horizontal ones. In the case of double Segal categories, this is equivalent to requiring the simplicial space $W_{0,\bullet}$ to be **essentially constant**.

A double category maybe illustrated as follows:

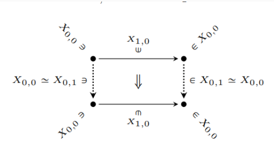

Figure 1: Double category

When the vertical arrows degenerate, the square degenerate to a 2-cell:

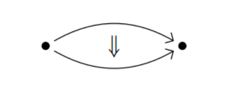

Figure 2: 2-category

In particular, for a double Segal space W with $W_{0,\bullet}$ essentially constant, the categorical data may be described as:

- The space of objects is given by $W_{0,0}$;
- The space of 1-morphisms are given by $W_{1,0}$;
- The space of 2-morphisms are given by $W_{1,1}$;
- The simplicial space $W_{0,1}$, which originally encodes the information of vertical morphisms, is now weak homotopy equivalent to $W_{0,0}$, and so does not contain new information.

Remark

In higher dimensions we do similar things. First we define n-fold Segal spaces and then require part of the simplicial data to be essentially constant, so that the n-tuple category degenerate to an n-category. In particular, if X is an n-fold Segal space, then we expect $X_{0,..,0}$ to be the space of morphisms, $X_{1,0,\dots,0}$ to be the space of 1-morphisms and $X_{1,\dots,1,0,\dots,0}$ to be the space of i-morphisms.

To get a satisfying model of $(\infty, 2)$-categories, we need to further impose completeness conditions. Finally we come to the following definition:

Definition (Calaque, Scheimabauer: 2018)

A 2-fold Segal space $W : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to s\mathrm{Set}$ is a **2-fold complete Segal space** if:

1. for every $k \geq 0$, the simplicial space $W_{k,\bullet}$ is a complete Segal space;
2. The simplicial space $W_{\bullet,0}$ is a complete Segal space;
3. the simplicial space $W_{0,\bullet}$ is essentially constant.

Definition

An $(\infty, 2)$-category is a 2-fold complete Segal space.