“Advances in Quantum Algebra” workshop

Lagrangian algebras in braided fusion 2-categories

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The purpose of this talk is to give a brief introduction to the notion of braided fusion 2-categories and Lagrangian algebras in a braided fusion 2-categories. A braided fusion 2-category is a finite semisimple 2-category that is equipped with a monoidal structure, a braiding structure and has duals. A Lagrangian algebra is a connected étale algebra whose associated category of local modules is trivial. The talk is divided into two parts:

- In the first part I will talk about semisimplicity in the 2-categorical context. In particular, I will focus on the notion of Karoubi completeness.
- In the second part I will talk about monoidal structures, braidings and Lagrangian algebras in 2-categories. I will also introduce some intuitions from quantum physics.

Conventions and notations

- Throughout this presentation, we work over an algebraically closed field $\mathbf{k}$ of characteristic zero. We write Vec for the 1-category of finite dimensional $\mathbf{k}$-vector spaces.
- By a **$\mathbf{k}$-linear category**, we mean a 1-category enriched in Vec. By a **$\mathbf{k}$-linear 2-category**, we mean a 2-category whose hom-categories are $\mathbf{k}$-linear categories and composition maps are $\mathbf{k}$-linear functors.
- For a 2-category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we write $\hom_{\mathcal{C}}(X, Y)$ for the hom-category between X and Y. When $X = Y$, we simply denote this hom-category by $\Omega_X(\mathcal{C})$.
- Let P be a property for 1-categories. We say that a 2-category $\mathcal{C}$ **is locally** P, if every hom-category of $\mathcal{C}$ has property P.

Definition (Douglas-Reutter)

A **semisimple 2-category** over $\mathbf{k}$ is a $\mathbf{k}$-linear 2-category such that it

- is locally semisimple;
- admits adjoints for all 1-morphisms;
- is additive, that is, has a zero object and has finite direct sums;
- is Karoubi complete.

Among the conditions in the above definition, Karoubi completeness (aka idempotent completeness, condensation completeness) is the most important and non-trivial one, and is the main subject we are going to talk about. Karoubi completeness for 2-categories is introduced by C.Douglas and D.Reutter Douglas-Reutter:1812.11933, and is later generalized to higher categories by D.Gaiotto and T.Johnson-Freyd Gaiotto-Johnson-Freyd:1905.09566.

To motivate, we are going to talk about Karoubi completeness and Karoubi completion for 1-categories first.

Let $\mathcal{C}$ be a 1-category, and $X \in \mathcal{C}$ is an object. A morphism $e : X \to X$ is an **idempotent** if $e \circ e = e$. We say that e **splits** if there is an object $Y \in \mathcal{C}$, a morphism $s : X \to Y$ and a morphism $r : Y \to X$ such that $e = r \circ s$ and $\mathbb{1}_Y = s \circ r$:

$$\begin{tikzcd} X \arrow[loop above, "e=r\circ s"] \arrow[r, shift left, "s"] & Y \arrow[loop above, "\mathbb{1}_Y=s\circ r"] \arrow[l, shift left, "r"] \end{tikzcd}$$

We say that $\mathcal{C}$ is **Karoubi complete** if every idempotent in $\mathcal{C}$ splits.

Let e, f and two idempotents in a 1-category $\mathcal{C}$, living on X and Y respectively. An e-f-**bimodule** is a morphism $k : Y \to X$ such that $e \circ k = k$ and $k \circ f = k$, illustrated in the following diagram:

$$\begin{tikzcd} X \arrow[loop above, "e"] & Y \arrow[l, "k"'] \arrow[loop above, "f"] \end{tikzcd}$$

We define a category $\mathrm{Kar}(\mathcal{C})$ as follows:

- An object in $\mathrm{Kar}(\mathcal{C})$ is an idempotent in $\mathcal{C}$;
- A morphism in $\mathrm{Kar}(\mathcal{C})$ is a bimodule between idempotents.

There is a canonical embedding $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ sending each object X to the identity $\mathbb{1}_X \in \mathrm{Kar}(\mathcal{C})$.

We will refer to $\mathrm{Kar}(\mathcal{C})$ as the **Karoubi completion of** $\mathcal{C}$. $\mathrm{Kar}(\mathcal{C})$ has, and is characterized by the following properties:

- $\mathrm{Kar}(\mathcal{C})$ is Karoubi complete;
- Let $\mathcal{D}$ be a Karoubi complete 1-category, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then there exists a unique extension $\tilde{F}$ rendering the following diagram commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, hook] & & \mathcal{D} \\ & \mathrm{Kar}(\mathcal{C}) \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

  justifying the name **Karoubi completion**.

One can think of the construction of $\mathrm{Kar}(\mathcal{C})$ as freely adding splits of idempotents into the category $\mathcal{C}$. Thus we allow every idempotents in $\mathcal{C}$ to split by "brute force".

We now introduce the notion of Karoubi completeness and Karoubi completion in 2-categories. Let $\mathcal{C}$ be a 2-category, $X \in \mathcal{C}$ is an object, so that $\Omega_X(\mathcal{C})$ is a monoidal category.

- A **monad** on X is an algebra in the monoidal category $\Omega_X(\mathcal{C})$. That is, a triple (T, μ, ι) where $T \in \Omega_X(\mathcal{C})$ is an object, $\mu : T \circ T \Rightarrow T$ is an associative multiplication, and $\iota : \mathbb{1}_X \Rightarrow X$ is a unit w.r.t. μ.
- We say that a monad $T \in \Omega_X(\mathcal{C})$ is **separable** if the multiplication $\mu : T \circ T \Rightarrow T$ admits a T-T-bimodule section. That is, there is a T-T-bimodule 2-morphism $s : T \Rightarrow T \circ T$ such that $\mu \circ s = \mathbb{1}_T$. Alternatively, a separable monad over X is a separable algebra in the monoidal category $\Omega_X(\mathcal{C})$.

Example

- For every object X, the identity 1-morphism $\mathbb{1}_X$ is a separable monad in a trivial way.
- Let $\mathcal{C}$ be a monoidal 1-category. We write $B\mathcal{C}$ for the **one-point delooping** of $\mathcal{C}$. More explicitly, $B\mathcal{C}$ is the 2-category with only one object $\bullet$, and $\Omega_\bullet(B\mathcal{C}) = \mathcal{C}$. Then
  - A monad in $B\mathcal{C}$ over the unique object $\bullet$ is an algebra in the monoidal 1-category $\mathcal{C}$.
  - A separable monad in $B\mathcal{C}$ over the unique object $\bullet$ is a separable algebra in $\mathcal{C}$.

Let X, Y be two objects in a 2-category $\mathcal{C}$. A pair of adjunction $X : F \leftrightarrows G : Y$ is **separable** if the counit 2-morphism $\epsilon : F \circ G \Rightarrow \mathbb{1}_Y$ admits a section. That is, there is a 2-morphism $t : \mathbb{1}_Y \Rightarrow F \circ G$ such that $\epsilon \circ t = \mathbb{1}_{\mathbb{1}_Y}$.

A pair of separable adjunction $F \dashv G$ gives rise to a separable monad on the object X. The 1-morphism $T := G \circ F \in \Omega_X(\mathcal{C})$ is equipped with a multiplication

$$\mu : T \circ T = G \circ F \circ G \circ F \Rightarrow G \circ \mathbb{1}_Y \circ F = G \circ F = T.$$

Let t be a section of ϵ, then the 2-morphism $G \circ t \circ F : T \Rightarrow T \circ T$ is a T-T-bimodule section of μ.

Let $T \in \Omega_X(\mathcal{C})$ be a separable monad. We say that T **splits** if there exists of a pair of separable adjunction $F \dashv G$ such that $T = G \circ F$. We say that a 2-category $\mathcal{C}$ is **Karoubi complete** if $\mathcal{C}$ is locally Karoubi complete and every separable monad of $\mathcal{C}$ splits.

For a locally Karoubi complete 2-category $\mathcal{C}$, we can always construct a Karoubi complete 2-category $\mathrm{Kar}(\mathcal{C})$, together with a canonical embedding

$$\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$$

that is universal in a suitable sense. We refer to the 2-category $\mathrm{Kar}(\mathcal{C})$ as the **Karoubi completion** of $\mathcal{C}$. To explain the construction of $\mathrm{Kar}(\mathcal{C})$, we need to introduce the notion of bimodules between monads.

Definition

Let X, Y be two objects in a 2-category $\mathcal{C}$, and $E \in \Omega_X(\mathcal{C})$, $F \in \Omega_Y(\mathcal{C})$ be two monads. An E-F-**bimodule** is a triple (K, α, β) where $K : Y \to X$ is a one morphism and $\alpha : E \circ K \Rightarrow K$ and $K \circ F \Rightarrow K$ are two associative and unital actions. For two E-F-bimodules $K_1, K_2 : Y \to X$, a **bimodule map** is a 2-morphism $\rho : K_1 \Rightarrow K_2$ such that ρ commutes with the bimodule actions.

For a locally Karoubi complete 2-category $\mathcal{C}$, its **Karoubi completion** $\mathrm{Kar}(\mathcal{C})$ is defined as follows:

- An object is a separable monad in $\mathcal{C}$;
- A 1-morphism is a bimodule between separable monads;
- A 2-morphism is a bimodule map.

Moreover, there is a canonical embedding $\iota : \mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ sending each object $X \in \mathcal{C}$ to the trivial separable monad $\mathbb{1}_X$, which exhibits $\mathrm{Kar}(\mathcal{C})$ as the universal Karoubi complete 2-category receiving an arrow from $\mathcal{C}$. Without proof, I give the following facts:

- $\mathrm{Kar}(\mathcal{C})$ is Karoubi complete.
- The functor $\iota : \mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ is an equivalence if and only of $\mathcal{C}$ is Karoubi complete.

Let $\mathcal{C}$ be a multi-fusion 1-category. We define $\Sigma\mathcal{C}$ to be the Karoubi completion of the one-point delooping $B\mathcal{C}$. That is, $\Sigma\mathcal{C} := \mathrm{Kar}(B\mathcal{C})$. By abuse of terminology, we may also refer to $\Sigma\mathcal{C}$ as the Karoubi completion of the multi-fusion 1-category $\mathcal{C}$.

By our construction of the Karoubi completion, $\Sigma\mathcal{C}$ is defined as

- an object of $\Sigma\mathcal{C}$ is a separable monad in $B\mathcal{C}$, which corresponds to a separable algebra in $\mathcal{C}$;
- a 1-morphism $f : A \to B$ between separable algebras in a B-A-bimodule;
- a 2-morphism $\alpha : f_1 \Rightarrow f_2$ between two bimodules is a bimodule map.

As a result, we identify $\Sigma\mathcal{C}$ with the **Morita 2-category of separable algebras in** $\mathcal{C}$.

There is an alternative description of $\Sigma\mathcal{C}$, which is more frequently used in practice.

Construction

Let $\mathcal{C}$ be a multi-fusion category. We use $\mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ to denote the 2-category of finite semisimple right $\mathcal{C}$-modules, constructed as follows:

- an object in $\mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ is a finite semisimple right $\mathcal{C}$-module category;
- a 1-morphism in $\mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ is a $\mathcal{C}$-module functor;
- a 2-morphism in $\mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ is a $\mathcal{C}$-module natural transformation.

There is a canonical 2-functor $F : \Sigma\mathcal{C} \to \mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$, which sends a separable algebra A to the finite semisimple right $\mathcal{C}$-module $\mathrm{LMod}_A(\mathcal{C})$, and sends a bimodule ${}_AM_B$ to the functor $M \otimes_B - : \mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$.

Theorem

The canonical 2-functor $F : \Sigma\mathcal{C} \to \mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ *is fully faithful and essentially surjective, hence an equivalence.*

By our construction of $\Sigma\mathcal{C}$, we can only see that it is a Karoubi complete 2-category. However, via the 2-equivalence $\Sigma\mathcal{C} \simeq \mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$, we see that $\Sigma\mathcal{C}$ is also additive, and that every morphism in $\Sigma\mathcal{C}$ has adjoints.

Theorem (Douglas-Reutter)

Let $\mathcal{C}$ *be a multi-fusion 1-category. Then* $\Sigma\mathcal{C} = \mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$ *is a semisimple 2-category.*

Let us conclude this section by giving some remarks of semisimple 2-categories:

- We say that an object X in a semisimple 2-category $\mathcal{C}$ is **simple** if $\mathbb{1}_X \in \Omega_X(\mathcal{C})$ is a simple object. Similar to a semisimple 1-category, an object in a semisimple 2-category can be decomposed into a finite direct sum of simple objects.
- The 1-categorical Schur's Lemma that "there is no non-trivial morphism between distinct simple objects" cannot be generalized to semisimple 2-categories. In the 2-categorical world, we say that two simple objects X, Y are **connected** if there is a non-zero morphism connecting them. One can show that connectedness defines an equivalence relation on the set of simple objects. We denote the resulting set of equivalence classes by $\pi_0(\mathcal{C})$. We say that $\mathcal{C}$ is **finite** if $\mathcal{C}$ is locally finite semisimple and $\pi_0(\mathcal{C})$ is finite. We say that $\mathcal{C}$ is **indecomposable** if $\pi_0(\mathcal{C})$ is a singleton.
- As an easiest example, for any multi-fusion 1-category $\mathcal{A}$, its Karoubi completion $\Sigma\mathcal{A}$ is finite semisimple.

Monoidal Structures and algebras

Monoidal and braided monoidal 2-categories

A **monoidal 2-category** $\mathcal{C}$ is a 2-category equipped with a monoidal structure. More explicitly, a monoidal structure consists of the following data:

- A multiplication 2-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- A distinguished object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$;
- Adjoint 1-equivalences (α, l, r) in whose components are

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),\ l_A : \mathbb{1}_{\mathcal{C}} \otimes A \to A,\ r_A : A \to A \otimes \mathbb{1}_{\mathcal{C}}$$

- An invertible modification π involving α (which corresponds the the pentagon axiom), together with invertible modifications μ, ρ, λ involving α, l and r;
- The invertible modifications are required to satisfy certain commutative diagrams.

A **braided monoidal 2-category** $\mathcal{C}$ is a monoidal 2-category equipped with a **braiding structure**. A braiding structure on a monoidal 2-category consists of:

- An adjoint 1-equivalence $R : \otimes \to \otimes \circ \tau$ in $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{C})$ (where τ is the 2-functor switching two factors);
- Two invertible modifications $R_{(-|-,-)}$ and $R_{(-,-|-)}$ which correspond to the two commutative diagrams appearing in the hexagon axiom.
- The modifications are required to satisfy some commutative diagrams.

As in the 1-categorical case, we can talk about the duality in a monoidal 2-category. A **left dual** of an object $X \in \mathcal{C}$ consists of a unit morphism $\mathbb{1}_{\mathcal{C}} \to X \otimes X^L$ and a counit morphism $X^L \otimes X \to \mathbb{1}_{\mathcal{C}}$ such that the two zig-zag compositions are isomorphic to the corresponding identities.

Definition

A (braided) **multi-fusion 2-category** is a finite semisimple (braided) monoidal 2-category that has left and right duals. A (braided) **fusion 2-category** is a (braided) multi-fusion 2-category with a simple tensor unit.

One way to construct fusion 2-categories is to take the take Karoubi completion of braided fusion 2-categories: let $\mathcal{C}$ be a braided fusion 1-category, then the braiding structure of $\mathcal{C}$ translates to a monoidal structure over the one-point delooping $B\mathcal{C}$. Then its Karoubi completion $\Sigma\mathcal{C}$ inherits a monoidal structure. As a matter of fact, we have

Theorem (Douglas-Reutter)

Let $\mathcal{C}$ be a braided fusion 1-category. Then $\Sigma\mathcal{C}$ is fusion 2-category.

We can describe the monoidal structure over the fusion 2-category $\Sigma\mathcal{C}$ more explicitly:

- If we exploit the Morita 2-category description, that is, we view objects in $\Sigma\mathcal{C}$ as separable algebras in $\mathcal{C}$, then the monoidal structure is provided tensor product of separable algebras, whose construction depends on the braiding of $\mathcal{C}$.
- It we use the identification $\Sigma\mathcal{C} \simeq \mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$, then the monoidal structure is provided by relative tensor product over $\mathcal{C}$, which also depends on the braiding of $\mathcal{C}$.

Examples

- $2\mathrm{Vec} := \Sigma\mathrm{Vec}$ is a fusion 2-category. An object of $2\mathrm{Vec}$ is a finite semisimple $\mathbf{k}$-linear category. This justifies our notation $\mathrm{RMod}_{\mathcal{C}}(2\mathrm{Vec})$.
- Let G be a finite group and $\omega \in Z^4(G, U(1))$ be a 4-cocycle. We let the $2\mathrm{Vec}_G^\omega$ be the fusion 2-category of G-graded finite semisimple categories, equipped with a ω-twisted monoidal structure:
  - Simple objects in $2\mathrm{Vec}_G^\omega$ are labelled by elements of G, which we denote by δ_g;
  - Multiplications are defined by $\delta_g \otimes \delta_h = \delta_{gh}$; α, l, r are chosen to be identity;
  - Components of the invertible modification π is given by the 4-cocycle ω; the other modifications are chosen to be identities;
  - The constraints imposed on π translate to the 4-cocycle condition.

When ω is the trivial 4-cocycle, we simply denote $2\mathrm{Vec}_G^\omega$ by $2\mathrm{Vec}_G$.

Examples

- Let G be a finite group, we use $\mathrm{Rep}(G)$ to denote the symmetric fusion category of finite-dimensional G-modules. Then $\Sigma\mathrm{Rep}(G)$ is a fusion 2-category. We denote $\Sigma\mathrm{Rep}(G)$ by $2\mathrm{Rep}(G)$. Taking $G = \mathbb{Z}_2$, we can write down the structure of $2\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{RMod}_{\mathrm{Rep}(\mathbb{Z}_2)}(2\mathrm{Vec})$ explicitly:
  - There are two simple objects in $\mathrm{RMod}_{\mathrm{Rep}(\mathbb{Z}_2)}(2\mathrm{Vec})$: the category $\mathrm{Rep}(\mathbb{Z}_2)$ equipped with the regular module structure and the category Vec equipped with the monoidal structure induced by the forgetful functor $\mathrm{Rep}(\mathbb{Z}_2) \to \mathrm{Vec}$.
  - $\Omega_{\mathrm{Rep}(\mathbb{Z}_2)}(2\mathrm{Rep}(\mathbb{Z}_2)) = \mathrm{Rep}(\mathbb{Z}_2)$, $\Omega_{\mathrm{Vec}}(2\mathrm{Rep}(\mathbb{Z}_2)) = \mathrm{Vec}_{\mathbb{Z}_2}$, $\mathrm{Fun}_{\mathrm{Rep}(\mathbb{Z}_2)}(\mathrm{Rep}(\mathbb{Z}_2), \mathrm{Vec}) \simeq \mathrm{Vec} \simeq \mathrm{Fun}_{\mathrm{Rep}(\mathbb{Z}_2)}(\mathrm{Vec}, \mathrm{Rep}(\mathbb{Z}_2))$. That is, the structure of $2\mathrm{Rep}(\mathbb{Z}_2)$ can be written as:

$$\begin{tikzcd}
\mathrm{Rep}(\mathbb{Z}_2) \arrow[loop above, "\mathrm{Rep}(\mathbb{Z}_2)"] \arrow[rr, bend left, "\mathrm{Vec}"] & & \mathrm{Vec} \arrow[loop above, "\mathrm{Vec}_{\mathbb{Z}_2}"] \arrow[ll, bend left, "\mathrm{Vec}"]
\end{tikzcd}$$

Drinfeld center of fusion 2-categories

It is not easy to construct braided fusion 2-categories. One feasible method to produce braided fusion 2-categories take the **Drinfeld center** of a fusion 2-categories.

Let $\mathcal{C}$ be a fusion 2-category, we define a 2-category $\mathfrak{Z}_1(\mathcal{C})$ as follows:

- An object is a triple $(X, R_{X,-}, R_{(X|-,-)})$ where $X \in \mathcal{C}$ is an object, $R_{X,-}$ is a half-braiding, and $R_{(X|-,-)}$ is an invertible modification corresponding to the hexagon axiom;
- A 1-morphism is a pair $(f, R_{f,-})$ where f is a 1-morphism in $\mathcal{C}$ and $R_{f,-}$ is an invertible modification exhibiting the commutativity of f with the half-braiding;
- The modifications are required to satisfy some coherence conditions.

It is shown in that the Drinfeld center of a fusion 2-category is a braided fusion 2-category.

The structure of $\mathfrak{Z}_1(2\mathrm{Vec}_G^\omega)$ has been computed in detail Kong-Tian-Zhou:1905.04644. In particular, $\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2})$ is equivalent to the direct sum of two copies of $2\mathrm{Rep}(\mathbb{Z}_2)$ as semisimple 2-categories, that is

$$\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2}) \simeq 2\mathrm{Rep}(\mathbb{Z}_2) \boxplus 2\mathrm{Rep}(\mathbb{Z}_2).$$

There are four simple objects in $\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2})$, which we denote by $\mathbb{1}, \mathbb{1}_c, m$ and m_c respectively.

$$\begin{tikzcd} \mathbb{1} \arrow[loop above, "\mathrm{Rep}(\mathbb{Z}_2)"] \arrow[r, bend left, "\mathrm{Vec}"] & \mathbb{1}_c \arrow[loop above, "\mathrm{Vec}_{\mathbb{Z}_2}"] \arrow[l, bend left, "\mathrm{Vec}"] \end{tikzcd} \qquad \begin{tikzcd} m \arrow[loop above, "\mathrm{Rep}(\mathbb{Z}_2)"] \arrow[r, bend left, "\mathrm{Vec}"] & m_c \arrow[loop above, "\mathrm{Vec}_{\mathbb{Z}_2}"] \arrow[l, bend left, "\mathrm{Vec}"] \end{tikzcd}$$

The monoidal structure of $\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2})$: $\mathbb{1}$ is the tensor unit, $m \otimes m = \mathbb{1}$, $m_c = m \otimes \mathbb{1}_c$, $\mathbb{1}_c \otimes \mathbb{1}_c = \mathbb{1}_c \oplus \mathbb{1}_c$.

Connection with physics

The language of fusion 2-categories is closely related to the theory of topological order. For example, a (2+1)D topological order can be described by a fusion 2-category Kong-Zhang:2205.05565, a (3+1)D topological order can be described by a braided fusion 2-category Kong-Tian-Zhang:2009.06564.

This correspondence gives us certain intuitions behind (braided) fusion 2-categories. For example, one can think of a braided fusion 2-category as a 3-dimensional topological phases of matter, an object as a string-like defect living in that phase, a 1-morphism as a particle-like defect connecting two string-like defects, and a 2-morphism as a defect in the time direction. Two string-like defects can be fused and braided with each other, which exhibits the monoidal and braiding structure of the 2-category.

Gapped boundaries and Lagrangian algebras

From physicists' viewpoint, it is important to study the gapped boundaries of a topological order. It is known that in the (2+1)D case, any gapped boundary of a topological order can be produced via the process of anyon condensation Kong:1307.8244. The machinery of anyon condensation provides a correspondence between gapped boundaries of a topological order and (Morita classes) of Lagrangian algebras in the associated modular tensor category. As a result, the study of Lagrangian algebras is equivalent to the study of gapped boundaries.

This is proposed that such patterns should be generalized to higher dimensional cases. We now introduce the notion of Lagrangian algebras in braided fusion 2-categories and give explicit examples in $\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2})$. This braided fusion 2-category corresponds to the (3+1)D toric code in physics Kong-Tian-Zhang:2009.06564.

Algebras and modules

There is a natural notion of algebra in a monoidal 2-category, and that of a commutative algebra in a braided monoidal 2-category. An **algebra** in a monoidal 2-category consists of the following data:

- An object $A \in \mathcal{C}$, a multiplication 1-morphism $\mu : A \otimes A \to A$, a unit morphism $\iota : \mathbb{1}_{\mathcal{C}} \to A$;
- An invertible associator 2-morphism α, two invertible unitors l, r;
- α satisfies the pentagon axiom, while α, l, r satisfies the triangle axiom.

Having categorified the notion of an algebra, we can similarly categorify many associated notions, like homomorphisms between algebras, left/right/bi modules over an algebra, etc. We claim that for an algebra A in a monoidal 2-category $\mathcal{C}$, the collection of right A-modules form a 2-category which we denote by $\mathrm{RMod}_A(\mathcal{C})$.

A **commutative algebra** in a braided monoidal 2-category $\mathcal{C}$ is an algebra whose multiplication "commutes with" the braiding of $\mathcal{C}$. More explicitly, a commutative algebra in $\mathcal{C}$ consists of the following data:

- an algebra A in $\mathcal{C}$ with multiplication 1-morphism μ;
- an invertible 2-morphism $\beta : \mu \circ R_{A,A} \Rightarrow \mu$ exhibiting the commutativity of μ and the braiding of $\mathcal{C}$;
- β is required to satisfy (a generalized version of) the hexagon axiom.

Local modules of a commutative algebra

First we recall the notion of local module over a commutative algebra in the 1-categorical case.

Let $\mathcal{C}$ be a braided monoidal 1-category with braiding c, $A \in \mathcal{C}$ be a commutative algebra, and $M \in \mathcal{C}$ be a right A-module. Via the braiding of $\mathcal{C}$, we can induce two left A-module structures over M as follows:

$$\alpha_+ : A \otimes M \xrightarrow{c_{A,M}} M \otimes A \xrightarrow{\mu_M} M$$

$$\alpha_- : A \otimes M \xrightarrow{c^{-1}_{M,A}} M \otimes A \xrightarrow{\mu_M} M$$

We say that M is a **local A-module** if the two induced left A-actions coincide with each other.

Now let $\mathcal{C}$ be a braided monoidal 2-category and $A \in \mathcal{C}$ be a commutative algebra. A **local module** over A consists of the following data Z.-Lou-Zhang-Hung-Kong-Tian:2208.07865:

- A right A-module M;
- An invertible 2-cell $\gamma : \alpha_+ \Rightarrow \alpha_-$, where α_+, α_- are left action 1-morphisms of A over M, induced by the braiding structure of $\mathcal{C}$;
- γ is required to satisfy a series of coherence conditions.

The collection of all local modules over A, together with suitable 1-morphisms and 2-morphisms between local modules, form a braided monoidal 2-category which we denote by $\mathrm{Mod}_A^{loc}(\mathcal{C})$

For an algebra A in a multi-fusion 2-category, the associated 2-category of right A-modules $\mathrm{RMod}_A(\mathcal{C})$ is not a finite semisimple 2-category in general. The notion of separability guarantees the semisimplicity of the category of modules Décoppet:2205.06453. Here we do not list the definition of separability, but only give the following theorem

Theorem (Décoppet)

Let A be a separable algebra in a fusion 2-category $\mathcal{C}$. Then $\mathrm{RMod}_A(\mathcal{C})$ is a finite semisimple 2-category.

An **étale algebra** A in a braided multi-fusion 2-category is a commutative separable algebra. An étale algebra is **connected** if the unit 1-morphism $i : \mathbb{1}_{\mathcal{C}} \to A$ is a simple object in the finite semisimple category $\mathrm{hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, A)$ Décoppet-Xu:2307.02843.

Theorem (Décoppet-Xu)

Let A be an étale algebra in a braided multi-fusion 2-category $\mathcal{C}$, then $\mathrm{Mod}_A^{loc}(\mathcal{C})$ is a braided multi-fusion 2-category. If $\mathcal{C}$ is further a braided fusion 2-category, and $A \in \mathcal{C}$ is connected étale, then $\mathrm{Mod}_A^{loc}(\mathcal{C})$ is a braided fusion 2-category.

Definition

A connected étale algebra A in a braided fusion 2-category $\mathcal{C}$ is **Lagrangian** if $\mathrm{Mod}_A^{loc}(\mathcal{C})$ is equivalent to $2\mathrm{Vec}$ as braided fusion 2-categories Décoppet-Xu:2307.02843.

Example

In my joint work with Jia-qi Lou, Zhi-Hao Zhang, Ling-Yan Hung, Liang Kong and Yin Tian Z.-Lou-Zhang-Hung-Kong-Tian:2208.07865, we give three Lagrangian algebras in the non-degenerate braided fusion 2-category $\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2}) = 2\mathrm{Rep}(\mathbb{Z}_2) \boxplus 2\mathrm{Rep}(\mathbb{Z}_2)$:

The first algebra $A_e = \mathbb{1}_c$. Its multiplication, trivially associative, is defined by

$$\mathbb{1}_c \otimes \mathbb{1}_c = \mathbb{1}_c \oplus \mathbb{1}_c \xrightarrow{\mathrm{Id}_{\mathbb{1}_c} \oplus 0} \mathbb{1}_c.$$

Its unit is defined by the unique simple object in $\hom_{\mathfrak{Z}_1(2\mathrm{Vec}_{\mathbb{Z}_2})}(\mathbb{1}, \mathbb{1}_c) = \mathrm{Vec}$, which we denote by $x : \mathbb{1} \to \mathbb{1}_c$. Higher data are chosen to be identities. This Lagrangian algebra corresponds to the rough boundary of (3+1)D toric code.

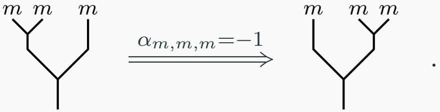
The algebra $A_1 = \mathbb{1} \oplus m$. The multiplication 1-morphism of $\mu_1 \colon A_1 \otimes A_1 \to A_1$ is defined component-wise as follows:

$$\begin{array}{cccc} \mathbb{1} \otimes \mathbb{1} & \mathbb{1} \otimes m & m \otimes \mathbb{1} & m \otimes m \\ \downarrow \mathrm{Id}_{\mathbb{1}} & \downarrow \mathrm{Id}_m & \downarrow \mathrm{Id}_m & \downarrow \mathrm{Id}_{\mathbb{1}} \\ \mathbb{1} & m & m & \mathbb{1} \end{array}$$

The unit morphism u_1 of A_1 is $\mathbb{1} \xrightarrow{1 \oplus 0} \mathbb{1} \oplus m$. The 2-associator and 2-unitors are identity 2-morphisms. The 2-commutator is trivial on all components except $\beta_{m,m} = \pm 1$, which define two commutative algebra structures that are isomorphic to each other. This Lagrangian algebra corresponds to the smooth boundary of (3+1)D toric code.

The algebra $A_2 = (\mathbb{1} \oplus m)_2$ has the same multiplication 1-morphism and unit 1-morphism as those of A_1.

The 2-associator α has only one non-trivial component:



The 2-commutator β is trivial on all components except $\beta_{m,m} = \pm i$. These two choices of $\beta_{m,m}$ define two commutative algebra structures that are isomorphic to each other. This algebra corresponds to the twisted smooth boundary of (3+1)D toric code.

Outlooks

There are many subsequent projects to work on, for example:

- Classify Lagrangian algebras and connected étale algebras in $\mathfrak{Z}_1(2\mathrm{Vec}_G^{\omega})$. The cases where ω is the trivial 4-cocycle have been solved Décoppet-Xu:2307.02843;
- Study higher condensations via the language of higher fusion categories and higher condensable algebras in those categories;
- Define and study higher Morita equivalences (Witt equivalences) of braided fusion 2-categories.

Thank You For Listening!