Thom spectra and Atiyah duality

Jiaheng Zhao

1 Thom spectra revisited

Convention 1.1. *By a vector space, we mean a real vector space. By a vector bundle, we mean a real vector bundle. By a manifold we mean a smooth manifold.*

Definition 1.2. Let X be a space and $p : V \to X$ be a vector bundle over X. Then the **Thom space** of V, $\mathsf{Th}(V)$ is the pointed space given by the cofiber of $V \setminus 0 \to V$, where 0 is the zero section of V.

Remark 1.3. Let $V \to X$ be a vector bundle, then $\mathsf{Th}(V) \simeq D(V)/S(V)$ where $D(V) \subset V$ is the disk bundle and $S(V) \subset D(V)$ is the sphere bundle.

Now let us take a new perspective towards vector bundles and Thom spectra. A rank n vector bundle $V \to X$ is always a Serre fibration (well, probably some extra mild conditions are required), and is a Kan fibration if we use simplicial models. Via straightening-unstraihtening equivalence, we may encode this fibration via a functor $\zeta_V : X \to \mathcal{S}$, where X is the ∞-groupoid corresponding to the space and $\mathcal{S}$ is the ∞-category of spaces. ζ_V sends each point of X to the associated fibre in the original fibration, hence $\zeta_V(x) = \mathbb{R}^n$, for all $x \in X$. Now let $BO(n) \subset \mathcal{S}$ be the (non-full) subcategory of $\mathcal{S}$ consisting of the $\mathbb{R}^n$ and the automorphism subspace $O(n) \subseteq \mathsf{Aut}_{\mathcal{S}}(\mathbb{R}^n)$. Since $V \to X$ is a vector bundle, the functor $\zeta_V : X \to \mathcal{S}$ factors through the subcategory $BO(n) \subseteq \mathcal{S}$. As a matter of fact, the subcategory $BO(n) \subset \mathcal{S}$ is an ∞-groupoid (i.e. a space), and it is a model for the classifying space of the orthogonal group $O(n)$. Then the map $\zeta_V : X \to BO(n)$ is the classical classifying map for the vector bundle $V \to X$. Abusively we will denote the functor $X \to BO(n)$ by ζ_V.

Proposition 1.4. *Let $V \to X$ be a rank n vector bundle, the Thom space of V can be identified with the following colimit*

$$\mathsf{Th}(V) = \varinjlim(X \xrightarrow{\zeta_V} BO(n) \hookrightarrow \mathcal{S}_*).$$

Here $\mathcal{S}_*$ is the ∞-category of pointed spaces. The second arrow is defined as follows: we send the unique object of $BO(n)$ to S^n, and the map on morphism space is given by $O(n) \hookrightarrow \mathsf{Aut}_{\mathcal{S}}(\mathbb{R}^n) \hookrightarrow \mathsf{Aut}_{\mathcal{S}_*}(S^n, S^n)$. We obtain the inclusion $\mathsf{Aut}_{\mathcal{S}}(\mathbb{R}^n) \hookrightarrow \mathsf{Aut}_{\mathcal{S}_*}(S^n, S^n)$ via the identification $S^n \simeq \mathbb{R}^n_+$, where $(-)_+$ means one-point compactification.

Definition 1.5. Let $V \to X$ be a rank n vector bundle. We define the **Thom spectra** of V to be $\Sigma^\infty\mathsf{Th}(V)$ (Note that $\mathsf{Th}(V)$ as a canonical base point). We shall denote the Thom spectra by X^V.

Remark 1.6. Since Σ^∞ preserves colimits, we have

$$X^V \simeq \varinjlim(X \xrightarrow{\zeta_V} BO(n) \hookrightarrow \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}).$$

Notation 1.7. Given a vector bundle $V \to X$, we may abusive denote the associated functors $X \to BO(n)$, $X \to \mathcal{S}$, $X \to \mathcal{S}_*$ and $X \to \mathrm{Sp}$ all by ζ_V.

Notation 1.8. Let $V \to X$ and $W \to Y$ be vector bundles of rank m and n respectively. We have a vector bundle $\pi_X^*(V) \oplus \pi_Y^*(W) \to X \times Y$ of rank $m+n$. For simplicity, we will denote this vector bundle by (V, W).

Lemma 1.9. *Let $V \to X$ and $W \to Y$ be vector bundles of rank m and n respectively. Then the Thom space $\mathsf{Th}((V,W) \to X \times Y$ is equivalent the $\mathsf{Th}(V \to X) \wedge \mathsf{Th}(W \to Y)$. Moreover, we have $(X \times Y)^{(V,W)} \simeq X^V \otimes Y^W$.*

Proof. By inspection, one sees that $(V, W) \to X \times Y$ corresponds to the functor

$$X \times Y \xrightarrow{\zeta_V \times \zeta_W} \mathcal{S}_* \times \mathcal{S}_* \xrightarrow{\wedge} \mathcal{S}_*.$$

Since $\wedge$ commutes with colimits in each variable separably, we have

$$(\varinjlim_{x\in X} \zeta_V(x)) \wedge (\varinjlim_{y\in V} \zeta_W(y)) \simeq \varinjlim_{x\in X} (\zeta_V(x) \wedge (\varinjlim_{y\in Y} \zeta_W(y))) \simeq \varinjlim_{x\in X, y\in Y} \zeta_V(x) \wedge \zeta_W(y).$$

which says that $\mathsf{Th}(V) \wedge \mathsf{Th}(W) \simeq \mathsf{Th}((V,W))$. After applying Σ^∞ we have $(X \times Y)^{(V,W)} \simeq X^V \otimes Y^W$. □

Corollary 1.10. *Let $V \to X$ be a vector bundle. Let $\underline{\mathbb{R}}^n$ be the trivial n-plane bundle on X. We have $\mathsf{Th}(\underline{\mathbb{R}}^n \oplus V) \simeq \Sigma^n \mathsf{Th}(V)$.*

Proof. We identify X with $X \times *$, and we identify $\underline{\mathbb{R}}^n$ with the bundle obtained by pulling back $\mathbb{R}^n$ along the projection $X \to *$. Then apply Lemma **??**. □

Definition 1.11. Inspired by Corollary **??**, we can extend the definition of the Thom spectra (but not a Thom space) to an element in $KO(X)$, i.e. a virtual vector bundle. To do so, note that on a paracompact space X, any virtual bundle E has the form $E = V - \underline{\mathbb{R}}^n$, where V is a vector bundle. We define

$$X^E := \Sigma^{-n} X^V.$$

Remark 1.12. This is an alternative definition of the Thom spectra of a virtual vector bundle. A virtual vector bundle over X is classified by a map

$$X \to \mathbb{Z} \times BO.$$

There is a natural functor sending $\mathbb{Z} \times BO \to \mathrm{Sp}$ which sends $(n, *)$ to $\Sigma^n \mathbb{S}$, and on morphisms is encoded by the J-homomorphism $BO \to \mathsf{Aut}_{\mathrm{Sp}}(\mathbb{S})$. Hence for each virtual bundle $E \in KO(X)$, there is a an associated functor $\zeta_E : X \to \mathrm{Sp}$.

Construction 1.13 (The pushforward map)**.** Let $X \to Y$ be a map and $V \to Y$ be a virtual vector bundle. Note that the pullback bundle f^*V corresponds to the composed functor

$$X \xrightarrow{f} Y \xrightarrow{\zeta_V} \mathrm{Sp}.$$

After taking colimits, there is a canonical comparison map

$$f_* : X^{f^*V} = \varinjlim(X \xrightarrow{f} Y \xrightarrow{\zeta_V} \mathrm{Sp}) \to \varinjlim(Y \xrightarrow{\zeta_V} \mathrm{Sp}) = Y^V,$$

which we will refer to as the **pushforward map**.

Now we turn to the study of smooth manifolds. Let M, N be smooth manifolds. Let $\iota : M \hookrightarrow N$ be a smooth embedding. We shall write $\nu(\iota) \to N$ for the associated normal bundle and simply ν when the embedding is clear from context. Recall that, there is a splitting $TM \oplus \nu \cong \iota^* TN$ of vector bundles over N. In particular, if $N = \mathbb{R}^n$, we obtain an isomorphism $TM \oplus \nu \cong \underline{\mathbb{R}}^n$. Passing to the ring $KO(M)$, we have

$$-TM = \nu - \underline{\mathbb{R}}^n,$$

so that $X^{-TM} = \Sigma^{-n} X^\nu$.

2 Atiyah duality

Our goal in this section is to prove the following:

Theorem 2.1. *Let M be a compact smooth manifold. M^{-TM} is the dual of $\Sigma^\infty_+ M$.*

Construction 2.2 (Pontryagin-Thom collapse map)**.** Let M, N be compact smooth manifolds and $\iota : N \hookrightarrow M$ be an embedding and $\nu \to N$ the associated normal bundle. There is a canonical map $M \to M/M - N \simeq \mathsf{Th}(\nu)$, sending each point inside ν to itself and any other point to ∞. There is an induced map $\Sigma^\infty M \to N^\nu$. More generally, let $V \to M$ be a vector bundle over M, we have a map

$$M^V \to N^{\iota^* V \oplus \nu}$$

To get this map, we consider the zero section map $s : M \to D(V)$. Then we can identify the normal bundle of the composed embedding $N \hookrightarrow M \hookrightarrow D(V)$ with $f^*V \oplus \nu$. This provide us with a collapse map $D(V)/S(V) = \mathsf{Th}(V) \to \mathsf{Th}(\iota^* V \oplus \nu)$. We will denote the resulting map $M^V \to N^{\iota^* V \oplus \nu}$ by $\mathsf{PT}(\iota, V)$.

Similarly, if V is a virtual vector on M we obtain a map

$$\mathsf{PT}(\iota, V) : M^V \to N^{\iota^* V \oplus \nu}.$$

When the virtual bundle V is clear from context, we may simply write $\mathsf{PT}(\iota)$ for the collapse map.

Example 2.3. Let M, N be two closed manifolds. We fix a point $m \in M$, and the associated PT map is given by

$$M \to S^{T_m M},$$

here we are using $S^{T_m M}$ to denote the one-point compactification of $T_m M$. We would like to compute the Pontryagin-Thom map induced by the embedding $N \hookrightarrow M \times N$ sending x to (m, x). Actually this is given by

$$\Sigma^\infty (M \times N)_+ \simeq \Sigma^\infty M_+ \otimes \Sigma^\infty N_+ \to \Sigma^\infty_+ S^{T_m M} \otimes \Sigma^\infty N_+ \simeq \Sigma^n \Sigma^\infty M_+.$$

Construction 2.4. By Whitney's embedding theorem, we can choose an embedding $\iota : M \hookrightarrow \mathbb{R}^n$. We assumr that n is large enough, so that this choice is unique up to isotopy. We may further obtain an embedding $M \hookrightarrow S^n$ considering that M is compact. We have the Pontryagin-Thom collapse map

$$S^n \to \mathsf{Th}(\nu).$$

After applying the functor $\Sigma^{-n}\Sigma^\infty$, we have a map

$$\eta_M : \mathbb{S} \simeq \Sigma^{-n}\Sigma^\infty S^n \to \Sigma^{-n} M^\nu \simeq M^{\nu - \mathbb{R}^n} = M^{-TM}.$$

This provide us with an elment in the set $\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, M^{-TM}) = \pi_0(M^{-TM})$. We shall refer to this element as the **fundamental class** of M.

Now we are ready to prove Atiyah duality. Before doing so we need to construct the evaluation and the coevaluation map which exhibits the duality.

Construction 2.5. We construct a map $\mathsf{ev} : \Sigma^\infty_+ M \otimes M^{-TM} \to \mathbb{S}$ as follows:

$$\mathsf{ev} : \Sigma^\infty_+ M \otimes M^{-TM} = (M \times M)^{(0, -TM)} \xrightarrow{\mathsf{PT}(\Delta, -TM)} M^{\nu(\Delta) - TM} = \Sigma^\infty_+ M \to \mathbb{S},$$

where the last map is induced by the projection $M \to *$. Note that the normal bundle of the diagonal embedding Δ coincide with TM. The coevaluation map is given the composite

$$\mathsf{coev} : \mathbb{S} \to M^{-TM} \xrightarrow{\Delta_*} \Sigma^\infty_+ M \otimes M^{-TM}.$$

where the first map is the map specifying the fundamental class and the second map is the push-forward map.

The following Lemma will be used in our proof of Atiyah duality.

Lemma 2.6. *Let M_1, M_2 be smooth manifolds. Let $N_2 \hookrightarrow M_2$ be a smooth submanifold with normal bundle ν. Let $f : M_1 \to M_2$ be a smooth morphism, transverse to the submanifold $N_2 \subseteq M_2$. Let $N_1 = f^{-1}N_2$. Let $V \to M_2$ be a vector bundle. Then we have a commutative diagram in* Sp*:*

$$\begin{tikzcd}
M_1^{f^*V} \arrow[r] \arrow[d] & M_2^V \arrow[d] \\
N_1^{f^*(\iota^*V\oplus\nu)} \arrow[r] & N_2^{\iota^*V\oplus\nu},
\end{tikzcd}$$

where the horizontal maps are pushforward maps and the vertical maps are Pontryagin-Thom collapse maps.

Proof. We identify ν with a tubular neighbourhood of N_2 in M_2, then $f^{-1}\nu$ is a tubular neighbourhood of N_1 in M_1. Let $D(V)$ be the disk bundle of $V \to M_2$. Then there is a commutative diagram:

$$\begin{tikzcd}
f^*D(V) \arrow[r] \arrow[d] & D(V) \arrow[d] \\
\mathsf{Th}(f^*\iota^*V \oplus f^{-1}(\nu)) \arrow[r] & \mathsf{Th}(\iota^*V \oplus \nu)
\end{tikzcd}$$

To demonstrate its commutativity, let's take a point $(x, v) \in f^*D(V)$ with $x \in X$ and v being point in the fibre of $f(x)$. By inspection, both the two maps in the above diagram will send this point to itself if it lies in the normal bundle $f^*\iota^*V \oplus f^{-1}(v)$ and to ∞ otherwise. Passing to Thom spectra, we get the desired commutative diagram. $\square$

Proof of Atiyah duality. We check that the evaluation and coevaluation maps constructed in Construction **??** satisfies the zig-zag equation, hence exhibiting the desired duality. First we check that the composition

$$\begin{tikzcd}
 & M^{-TM} \otimes \Sigma^\infty M_+ \otimes M^{-TM} \arrow[dr, "\mathsf{Id}\otimes\mathsf{ev}"'] & \\
M^{-TM} \arrow[ur, "\mathsf{coev}\otimes\mathrm{Id}"'] & & M^{-TM}
\end{tikzcd}$$

is homotopic to the identity of M^{-TM}. From the definition of ev and coev, we have a larger diagram

$$\begin{tikzcd}
M^{-TM} \otimes M^{-TM} \arrow[r, "f_*"] & M^{-TM} \otimes \Sigma^\infty M_+ \otimes M^{-TM} \arrow[r, "\mathsf{PT}(g)"] \arrow[drr, "\mathrm{Id}\otimes\mathsf{ev}"'] & M^{-TM} \otimes \Sigma^\infty M_+ \arrow[d, "(\mathsf{pr}_1)_*"] \\
M^{-TM} \arrow[u, "\eta_M\otimes\mathrm{Id}"] \arrow[ur, "\mathsf{coev}\otimes\mathrm{Id}"'] & & M^{-TM}
\end{tikzcd}$$

Here $f : M \times M \to M \times M \times M$ is the map given by sending (x, y) to (x, x, y), and $g : M \times M \to M \times M \times M$ is the closed embedding sending (x, y) to (x, y, y). Our plan is to show that the outer composition is homotopic to identity. The key observation is that f and g are transverse to each other, and they fit into a pullback diagram of smooth manifolds

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M \times M \arrow[d, "g"] \\
M \times M \arrow[r, "f"] & M \times M \times M.
\end{tikzcd}$$

Then applying Lemma **??**, we obtain a commutative diagram as follows:

$$\begin{tikzcd}
(M \times M)^{(-TM,-TM)} \arrow[r, "f_*"] \arrow[d, "\mathsf{PT}(\Delta)"'] & (M \times M \times M)^{(-TM,0,-TM)} \arrow[d, "\mathsf{PT}(g)"] \\
M^0 = \Sigma^\infty M_+ \arrow[r, "\Delta_*"'] & (M \times M)^{(-TM,0)}.
\end{tikzcd}$$

i.e. the composition $\mathsf{PT}(g) \circ f_*$ is homotopic to $\Delta_* \circ \mathsf{PT}(\Delta)$. Now note that $\eta_M \otimes \mathrm{Id} : \mathbb{S} \otimes M^{-TM} \to M^{-TM} \otimes M^{-TM} \simeq (M \times M)^{(-TM,-TM)}$ can be identified with $\Sigma^{-n}\mathsf{PT}(j \times \mathrm{Id})$ where $j \times \mathrm{Id} : M \times M \to S^n \times M$ is the embedding. Now that $\mathsf{PT}(\Delta)\mathsf{PT}(j \times \mathrm{Id}) \simeq \mathsf{PT}((j \times \mathrm{Id}) \circ \Delta) = \mathsf{PT}(j, \mathrm{Id})$. Then note that $(j, \mathrm{Id}) : M \to S^n \times M$ is isotopic to $(0, \mathrm{Id})$, we conclude that $\mathsf{PT}(j, \mathrm{Id})$ is homotopic to identity. It remains to show that $(\mathsf{pr}_1)_* \circ \Delta_*$ is homotpic to identity, which is manifest since $\mathsf{pr}_1 \circ \Delta = \mathrm{Id}$.

The check of the other zig-zag identity is completely parallel to this, and we omit the details. $\square$