An introduction to ∞-categories

Jiaheng Zhao

Preface

This is a note for the seminar of ∞-categories held in AMSS, CAS in Fall 2022. This note is far from complete and is still under construction, hence there might be many typos. Many proofs and materials remain to be added, and the section on factorization homology has not started yet.

Contents

1 Basic constructions

1.1 Simplicial sets

Definition 1.1. Define the **cosimplicial indexing category** Δ to be the following skeletal category:

- An object of Δ is a finite linearly ordered set $[n] = \{0 \to 1 \to ... \to n\}$, where n is a non-negative integer.
- A morphism $[m] \to [n]$ is a non-decreasing map of linearly ordered sets.

Definition 1.2. Fix a positive integer n. Let $d^i : [n-1] \to [n]$ $(0 \leq i \leq n)$ be unique injective map whose image does **not** contain i. Let $s^i : [n+1] \to [n]$ $(0 \leq i \leq n)$ be the unique surjective map such that the preimage of i contains two elements. We will refer to d^i as **the i-th coface map** and to s^i as **the i-th codegeneracy map**.

The following combinatoric fact will be left as an exercise:

Lemma 1.3. *Δ is generated by all the codegeneracy maps and coface maps.*

Definition 1.4. Define **the category of simplicial sets** to be the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. We will refer to this category as sSet.

Taking Lemma 1.3 into consideration, a simplicial set $X \in \mathrm{sSet}$ consists of a collection of sets $\{X_n\}_{n\geq 0}$ together with a collection of face $d_i = X(d^i)$ and degeneracy maps $s_i = X(s^i)$ between them:

$$
\begin{tikzcd}
\cdots & X_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l]
\end{tikzcd}
$$

These face maps and degeneracy maps are required to satisfy a series of simplicial identities.

Definition 1.5. An element $x \in X_n$ is called an **n-simplice**. An n-simplice x is called **degenerate** if there exists some $y \in X_{n-1}$ and some $0 \leq i \leq n-1$ such that $x = s_i(y)$ and is called **non-degenerate** if not. We use $NX_n \subseteq X_n$ to denote the subset of non-degenerate n-simplices.

Example 1.6 (Representable simplicial sets)**.** Let $\mathcal{Y} : \Delta \to \mathrm{sSet}$ be the Yoneda embedding, sending $[n]$ to the representable simplicial set $\hom_\Delta(-, [n])$, which we will denote by Δ^n. A k-simplice of Δ^n is a morphism $f \in \hom_\Delta([k], [n])$. A k-simplice is non-degenerate if and only if f is injective. As a result, Δ^n has a unique non-degenerate n-simplice given by the identity map $\mathrm{Id} : [n] \to [n]$.

Definition 1.7. Let X be a simplicial set. A **subset** $S = \{S_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of X is a choice of subset $S_n \subset X_n$ for each n. A **simplicial subset**, or a **subcomplex** of X is a subset closed under the action of all face maps and degeneracy maps. Let $S \subset X$ be a subset, the **subcomplex generated by** S is the smallest subcomplex of X containing S.

Example 1.8. Δ^n is generated by the unique non-degenerate n-simplice $\mathrm{Id} : [n] \to [n]$ in $N\Delta^n_n$.

Definition 1.9. The **k-th face** of Δ^n is the subcomplex generated the unique injective monotone function $d^k : [n-1] \to [n]$ whose image does not contain k. The **boundary** of Δ^n is the subcomplex generated by $\{d^0, ..., d^n\}$ and is denote by $\partial\Delta^n$. The **k-th horn** is the subcomplex generated by $\{d^0, ..., \hat{d^k}, ..., d^n\}$. Here $\hat{}$ means the element is removed. The k-th horn is denoted by Λ^n_k. Note that we have canonical embedding of simplicial sets $\partial\Delta^n \to \Delta^n$, $\Lambda^n_k \to \Delta^n$.

Definition 1.10. Let e^i denote the monotone functions $e_i : [1] \to [n]$ by sending $\{0 \to 1\}$ to $\{i \to i+1\}$.The n-th **spine** is the subcomplex of Δ^n generated by $\{e^0, ..., e^{n-1}\}$, which is denoted by $\mathrm{Sp}(n)$.

By Yoneda lemma, we can view a simplice $x \in X_n$ as a morphism $x : \Delta^n \to X$. And in fact we can identify the image of the morphism x with the subcomplex generated by the simplice x. In particular, we may view the function d^k as an embedding $\Delta^{n-1} \to \Delta^n$ and identify the k-th face with the image of this map. Similarly, we may view e^i as an embedding $\Delta^1 \to \Delta^n$.

Proposition 1.11. *We may present $\partial\Delta^n$ and $\mathrm{Sp}(n)$ by the following coequalizers*

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

and

$$\coprod_{i=1}^{n-1} \Delta^0 \rightrightarrows \coprod_{i=0}^{n-1} \Delta^1 \to \mathrm{Sp}(n).$$

Definition 1.12. Let X be a simplicial set. Define $\mathrm{sk}_n(X)$ to be the subcomplex of X generated by simplices of degree $\le n$, which is called the **n-th skeleton of** X.

Note that X can be written as a union

$$X = \bigcup_{n\ge 0} \mathrm{sk}_n(X).$$

It turns out that X has a "cell structure", similar to the construction of CW-complexes.

Proposition 1.13 (Cell structure of simplicial sets)**.** *There are pushout diagrams*

$$\begin{tikzcd} \coprod_{x\in NX_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}X \arrow[d] \\ \coprod_{x\in NX_n} \Delta^n \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

1.2 Yoneda extension and nerves

Let $\mathcal{C}$ be a small category. The **category of presheaves** over $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, which we will denote by $\widehat{\mathcal{C}}$. An object $X \in \widehat{\mathcal{C}}$ is called a presheaf over $\mathcal{C}$. Given an object $c \in \mathcal{C}$, an element in the set $X_c = X(c)$ is called a **section** of X over c. By the Yoneda Lemma there is a fully faithful embedding $\mathcal{Y} : \mathcal{C} \to \widehat{\mathcal{C}}$, $c \mapsto \hom_{\mathcal{C}}(-, c)$. For simplicity we denote the functor $\hom_{\mathcal{C}}(-, c)$ by h_c.

Lemma 1.14. *The category $\widehat{\mathcal{C}}$ is complete and cocomplete.*

Proof. Recall that the category Set is complete and cocomplete. Then the limit of a diagram $J \to \widehat{\mathcal{C}}$ can be computed pointwisely. □

Definition 1.15. Let $X \in \widehat{\mathcal{C}}$. The **category of elements**, or **Grothendieck construction** of X is the category whose objects are pairs (a, s), where $a \in \mathcal{C}$ is an object and s is a section of X over a. A morphism between (a, s) and (b, t) is a morphism $u : a \to b$ in $\mathcal{C}$ such that $X(u)(t) = s$. The category of elements of X is denoted by $\int_{\mathcal{C}} X$. Similar, for $Y : \mathcal{C} \to \mathrm{Set}$ a functor, we have a similar Grothendieck construction which we denote by $\int^{\mathcal{C}} Y$

Note that there is a canonical projection functor $\pi_X : \int_{\mathcal{C}} X \to \mathcal{C}$ by sending (a, s) to a.

Remark 1.16. By the Yoneda Lemma a section $s \in X_a$ corresponds to a natural transformation $s \in \mathrm{Nat}(h_a, X)$. Then the requirement that $X(u)(t) = s$ translates to the following diagram:

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

So the category of elements $\int_{\mathcal{C}} X$ is isomorphic to the comma category $(\mathcal{Y}, \downarrow, X)$ where $\mathcal{Y}$ is the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$.

Remark 1.17. The category of elements $\int_{\mathcal{C}} X$ is also isomorphic to the comma category $(\{*\}, \downarrow, X)$ where $\{*\}$ denotes the singleton.

The following theorem says that representable presheaves are dense in the category of presheaves.

Theorem 1.18 (Density theorem)**.** *Consider the faithful functor $\varphi_X := \mathcal{Y} \circ \pi_X : \int_{\mathcal{C}} X \to \widehat{\mathcal{C}}$ such that $\varphi_X(a, s) = h_a$, $\varphi_X(u) = u$. There is an obvious cocone from φ_X to X defined by the following collections of maps:*

$$s : h_a \to X, \quad (a, s) \in \int_{\mathcal{C}} X$$

Then these maps exhibits X as a colimit of φ_X.

Proof. Let $\varphi_X \Rightarrow Y$ be another cocone under φ_X. Consider the following diagram

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"] \arrow[ddr, "p"'] & & h_b \arrow[dl, "t"'] \arrow[ddl, "q"] \\ & X & \\ & Y & \end{tikzcd}$$

The small triangle is equivalent to the data of two sections $s \in X_a$, $t \in X_b$ such that $X(u)(t) = s$. The large triangle is equivalent to the data of $p \in Y_a$, $q \in Y_b$ such that $Y(u)(q) = p$. Then there is exists a unique natural transformation $\alpha : X \Rightarrow Y$ whose components α_a is defined by $\alpha_a(t) = p$.

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"] \arrow[ddr, "p"'] & & h_b \arrow[dl, "t"'] \arrow[ddl, "q"] \\ & X \arrow[d, dashed, "\alpha"] & \\ & Y & \end{tikzcd}$$

∎

The following theorem exhibits $\widehat{\mathcal{C}}$ as the free cocompletion of $\mathcal{C}$.

Theorem 1.19 (Free cocompletion)**.** *Let $\mathcal{D}$ be a cocomplete category and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then there is a unique (up to isomorphism) cocontinuous functor $\tilde{F} : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $\tilde{F} \circ \mathcal{Y} \cong F$. Moreover $\tilde{F}$ is left adjoint to the canonical functor $H : \mathcal{D} \to \widehat{\mathcal{C}}$ defined by $H(d) := \hom_{\mathcal{D}}(F(-), d)$.*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{D} \arrow[dl, bend right, "H"'] \\ & \widehat{\mathcal{C}} \arrow[ur, bend right, "\tilde{F}"'] & \end{tikzcd}$$

Proof. It suffices to show that any such cocontinuous functor gives an adjunction $\tilde{F} \dashv H$. Take $X \in \widehat{\mathcal{C}}$, note that $X \cong \varinjlim_{(a,s)\in \int_{\mathcal{C}} X} h_a$. Since $\tilde{F}$ is cocontinuous, we have

$$\tilde{F}(X) \cong \varinjlim_{(a,s)\in \int_{\mathcal{C}} X} \tilde{F}(h_a) \cong \varinjlim_{(a,s)\in \int_{\mathcal{C}} X} F(a)$$

As a result, we have

$$\begin{aligned}
\hom_{\widehat{\mathcal{C}}}(X, H(d)) &\cong \varprojlim_{(a,s)} \hom_{\widehat{\mathcal{C}}}(h_a, \hom_{\mathcal{D}}(F(-), d)) \\
&\cong \varprojlim_{(a,s)} \hom_{\mathcal{D}}(F(a), d) \\
&\cong \hom_{\mathcal{D}}(\varinjlim_{(a,s)} F(a), d) \\
&\cong \hom_{\mathcal{D}}(\tilde{F}(X), d)
\end{aligned}$$

□

Corollary 1.20. *Let $\mathcal{C}, \mathcal{D}$ be cocomplete categories. If $\mathcal{C}$ is the presheaf category of some small category $\mathcal{A}$, then any cocontinuous functor $\mathcal{C} \to \mathcal{D}$ admits a right adjoint.*

Corollary 1.21. *Let $\mathcal{C} = \hat{\mathcal{A}}$ is the presheaf category of $\mathcal{A}$, then $\mathcal{C}$ is Cartesian closed. Moreover, for $X, Y \in \mathcal{C}$.*

Proof. Let $A \in \mathcal{C}$ be a presheaf over $\mathcal{A}$, then $A \times - : \mathcal{C} \to \mathcal{C}$ preserves colimits. As a result there exists a right adjont $T_A : \mathcal{C} \to \mathcal{C}$ such that

$$\hom_{\mathcal{C}}(A \times X, Y) \cong \hom_{\mathcal{C}}(X, T_A(Y)).$$

Moreover, take $X = h_a$ for $a \in \mathcal{A}$, we see that

$$T_A(Y)(a) := \hom_{\mathcal{C}}(A \times h_a, Y).$$

□

Example 1.22. The category sSet is Cartesian closed. We will denote the internal hom object by

$$\mathrm{Fun}(A, Y) := T_A(Y)$$

and referring to it as **the function complex from A to Y**. Note that $\mathrm{Fun}(A, Y)_n = \hom_{\mathrm{sSet}}(A \times \Delta^n, Y)$.

Example 1.23. Let Δ denote the category of finite ordinals whose presheaf category is the category of simplicial sets. There is a functor $|-| : \Delta \to \mathrm{Top}$ sending $[n]$ to the standard topological n-simplice $|\Delta^n|$. Freely extending this functor to sSet by Theorem 1.19 , we get the famous **geometric realization functor**, which we still denote by $|-| : \mathrm{sSet} \to \mathrm{Top}$. Its right adjoint is called **singular functor**, which we denote by $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$.

$$\begin{tikzcd}
\Delta \arrow[rr] \arrow[dr, "\mathcal{y}"'] & & \mathrm{Top} \arrow[dl, bend right, "\mathrm{Sing}"'] \\
& \mathrm{sSet} \arrow[ur, bend right, "|-|"'] &
\end{tikzcd}$$

Example 1.24. Let Cat denote the category of all categories[1]. Note that this category is cocomplete. Since any partially ordered set can be viewed as a category, there is a functor $i : \Delta \to \mathrm{Cat}$ which sends the totally ordered set $[n]$ to the category $[n]$. Freely extending this functor to sSet by Theorem 1.19 , we get the **fundamental category functor/ categorical realization functor** , which we denote by $\pi : \mathrm{sSet} \to \mathrm{Top}$. Its right adjoint is the famous **nerve functor** $N : \mathrm{Cat} \to \mathrm{sSet}$.

$$\begin{tikzcd}
\Delta \arrow[rr, "i"] \arrow[dr, "\mathcal{Y}"'] & & \mathrm{Cat} \arrow[dl, bend right, "N"'] \\
& \mathrm{sSet} \arrow[ur, bend right, "\pi"'] &
\end{tikzcd}$$

The remaining part of this section is devoted to the construction of the homotopy coherent nerve. First let us introduce of the notion of a simplicial category.

Definition 1.25. A **simplicial category** is a sSet-enriched category. A **simplicial functor** between simplicial categories is a sSet-functor. We use sCat to denote the 1-category of simplicial categories and simplicial functors. For a simplicial category $\mathcal{C}$ and $x, y \in \mathcal{C}$ be two objects, we use $\mathrm{Map}_{\mathcal{C}}(x, y)$ to denote the corresponding mapping simplicial set.

The following proposition will be left as an exercise.

Proposition 1.26. *There is an equivalence of categories*

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{sCat}.$$

Example 1.27. sSet is a simplicial category since it is Cartesian closed.

Our purpose is to define a functor $\Delta \to \mathrm{sCat}$ so that we can freely extend it to sSet.

Definition 1.28. Let $[n] \in \Delta$ be an object. Define $\mathfrak{C}([n])$ to be the following simplicial category:

1. The objects of $\mathfrak{C}([n])$ are the elements of $[n]$.

2. Let $i \leq j$ be two elements of $[n]$. Let $\mathrm{Path}(i, j)$ be the partially ordered set of finitely linearly ordered subsets $\{i = i_0 < i_1 < ... < i_m = j\}$ with least element i and largest element j, ordered by **reverse inclusion**. We define

$$\mathrm{Map}_{\mathfrak{C}([n])}(i, j) := N(\mathrm{Path}(i, j)).$$

3. For $i \leq j \leq k \in [n]$ the composition law

$$\mathrm{Map}_{\mathfrak{C}([n])}(j, k) \times \mathrm{Map}_{\mathfrak{C}([n])}(i, j) \to \mathrm{Map}_{\mathfrak{C}([n])}(i, k)$$

is given on vertices by the composition $(S, T) \mapsto S \cup T$.

Moreover, the assignment $[n] \mapsto \mathfrak{C}([n])$ extends to a functor $\mathfrak{C} : \Delta \to \mathrm{sCat}$. Since sCat is cocomplete, we may freely extend it to a functor $\mathfrak{C} : \mathrm{sSet} \to \mathrm{sCat}$.

Example 1.29. The simplicial category $\mathfrak{C}([0])$ has a unique object 0 with $\mathrm{Map}_{\mathfrak{C}([0])}(0, 0) = \Delta^0$.

Example 1.30. The simplicial category $\mathfrak{C}([1])$ has two objects $0, 1$ with $\mathrm{Map}_{\mathfrak{C}([1])}(0, 1) = \Delta^0$.

[1] Actually the collection of all categories has a structure of a 2-category. Here we view it as a 1-category by discarding all the natural transformations. As a result, it does not make sense to talk about isomorphism of functors.

Example 1.31. The simplicial category $\mathfrak{C}([2])$ has three objects $0, 1, 2$. We see that the mapping space

$$\mathrm{Map}_{\mathfrak{C}([2])}(0, 2) = \Delta^1.$$

The vertex 0 is given by the path $\{0 \to 1 \to 2\}$, while the vertex 1 is given by the path $0 \to 2$.

Example 1.32. The simplicial category $\mathfrak{C}([3])$ has four objects $0, 1, 2, 3$. We see that the mapping space

$$\mathrm{Map}_{\mathfrak{C}([3])}(0, 3) = \Delta^1 \times \Delta^1.$$

The simplicial set $\Delta^1 \times \Delta^1$ can be exhibited by the following square:

$$\begin{tikzcd}
\{0 \to 1 \to 2 \to 3\} \arrow[r] \arrow[d] \arrow[dr] & \{0 \to 2 \to 3\} \arrow[d] \\
\{0 \to 1 \to 3\} \arrow[r] & \{0 \to 3\}
\end{tikzcd}$$

More generally, we could see that for $n \geq 1$, $\mathrm{Map}_{\mathfrak{C}([n])}(0, n) = (\Delta^1)^{\times(n-1)}$, is given by the $(n-1)$-dimensional cube. The 2^{n-1} vertexes of the cube corresponds to 2^{n-1} different subsets of $\{1, ..., n-1\}$.

Definition 1.33. The **homotopy coherent nerve** functor $N : \mathrm{sCat} \to \mathrm{sSet}$ is the right adjoint of $\mathfrak{C} : \mathrm{sSet} \to \mathrm{sCat}$:

$$\begin{tikzcd}
\Delta \arrow[rr, "\mathfrak{C}"] \arrow[dr] & & \mathrm{sCat} \arrow[dl, bend right, "N"'] \\
& \mathrm{sSet} \arrow[ur, bend right] &
\end{tikzcd}$$

In particular, for $\mathcal{C}$ a simplicial category, we can see that $N(\mathcal{C})_n$ is the set of simplicial functors from $\mathfrak{C}([n])$ to $\mathcal{C}$.

Note that we use the same notation for nerve and homotopy coherent nerve. This shall not lead to any confusion due to the following lemma:

Lemma 1.34. *Let $\mathcal{C}$ be a category. We may alternatively view $\mathcal{C}$ as a simplicial category by regarding its hom-sets as discrete simplicial sets. Then the nerve of $\mathcal{C}$ coincide with the homotopy coherent nerve of $\mathcal{C}$.*

Example 1.35. Let $\mathcal{C}$ be a simplicial category and $N(\mathcal{C})$ its homotopy coherent nerve. Then

- An element of $N(\mathcal{C})_0$ is a vertex of $\mathcal{C}$.
- An element of $N(\mathcal{C})_1$ is morphism $f : x \to y$ in $\mathcal{C}$. That is, a vertex in $\mathrm{Map}_{\mathcal{C}}(x, y)$ for some x, y.
- An element of $N(\mathcal{C})_2$ corresponds to three morphisms $f : x \to y$, $g : y \to z$ and $h : x \to z$ together with an edge $\alpha : g \circ f \Rightarrow h$ in $\mathrm{Map}_{\mathcal{C}}(x, z)$.
- An element of $N(\mathcal{C})_3$ corresponds to six morphisms $f_{01} : x \to y$, $f_{12} : y \to z$, $f_{23} : z \to w$, $f_{02} : x \to z$, $f_{13} : y \to w$ and $f_{03} : x \to w$ together with five edges $\alpha : f_{12} \circ f_{01} \Rightarrow f_{02}$, $\beta : f_{23} \circ f_{12} \Rightarrow f_{13}$, $\gamma : f_{13} \circ f_{01} \to f_{03}$, $\delta : f_{23} \circ f_{02} \Rightarrow f_{03}$ and $\theta : f_{23} \circ f_{12} \circ f_{01} \Rightarrow f_{03}$, together with two 2-simplexes represented by the two triangles in the figure:

$$\begin{tikzcd}
f_{23} \circ f_{12} \circ f_{01} \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, "\theta"] & f_{23} \circ f_{02} \arrow[d, "\delta"] \\
f_{13} \circ f_{01} \arrow[r, "\gamma"'] & f_{03}
\end{tikzcd}$$

1.3 Factorisation systems and small object argument

Definition 1.36. Let $i : A \to B$ and $p : X \to Y$ be two morphisms in a category $\mathcal{C}$. We say that i has the **left lifting property** (LLP) with respect to p, or, equivalently, that p has the **right lifting property** (RLP) with respect to i, if any commutative square of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] & Y \end{tikzcd}$$

has a diagonal filler

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

Let F be a class of morphisms in $\mathcal{C}$. A morphism has the left (right) lifting property with respect to F if it has the left (right) lifting property with respect to all elements of F. One denotes by $l(F)$ $(r(F))$ the class of morphisms which has the left lifting property with respect to F. We may also use the notation $f \boxslash g$ to mean that f has LLP w.r.t. g. Equivalently, $f \boxslash g$ if and only if the canonical morphism π

$$\hom(B, X) \xrightarrow{\pi} \hom(A, X) \times_{\hom(A,Y)} \hom(B, Y)$$

is a surjection.

Example 1.37. In Set, a morphism $f : x \to y$ is surjective if and only if f has the RLP with respect to the unique map $\emptyset \to \{*\}$. Thus we have $Epi = r(\emptyset \to \{*\})$ where Epi denotes the set of epimorphisms and in Set they are exactly surjective maps. A morphism f is injective if and only f has the RLP with respect to the unique map $\{**\} \to \{*\}$. Thus we have $Mon = r(\{**\} \to \{*\})$ where Mon denotes the set of monomorphisms and in Set they are exactly injective maps.

Example 1.38. In the category Top_k, the **Hurewicz fibrations** are defined to be $r(\{i_0 : A \to A \times I\})$, where i_0 denotes the inclusion $x \mapsto (x, 0)$. The **Hurewicz cofibrations** are $l(\{p_0 : A^I \to A\})$, where p_0 denotes the evaluation $f \mapsto f(0)$. **Serre fibrations** are defined to be $r(\{i_0 : D^n \to D^n \times I\})$.

Proposition 1.39. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction and F is the left adjoint. Let $f : x \to y$ be a morphism in $\mathcal{C}$ and $g : m \to n$ be a morphism in $\mathcal{D}$. Then $F(f) \boxslash g \Leftrightarrow f \boxslash G(g)$.*

Proof. Suppose that $F(f) \boxslash g$, we show that $f \boxslash G(g)$. Choose $\alpha : x \to G(m)$ and $\beta : y \to G(n)$ such that $G(g) \circ \alpha = \beta \circ f$. Recall that we have the commutative diagram:

$$\begin{tikzcd} \hom(F(x), m) \arrow[r, "\cong"] \arrow[d, "g_*"'] & \hom(x, G(m)) \arrow[d, "G(g)_*"] \\ \hom(F(x), n) \arrow[r, "\cong"] & \hom(x, G(n)) \\ \hom(F(y), n) \arrow[r, "\cong"'] \arrow[u, "F(f)^*"] & \hom(y, G(n)) \arrow[u, "f^*"'] \end{tikzcd}$$

Let $\alpha^\sharp$ be the adjunct of α and $\beta^\sharp$ be the adjunct of β. We have $g \circ \alpha^\sharp = \beta^\sharp \circ F(f)$. That is, we have a commutative diagram as follows:

$$\begin{tikzcd} F(x) \arrow[r, "\alpha^\sharp"] \arrow[d, "F(f)"'] & m \arrow[d, "g"] \\ F(y) \arrow[r, "\beta^\sharp"'] & n \end{tikzcd}$$

Since $F(f) \boxslash g$, we can find a morphism $h : F(y) \to m$ rendering the following diagram commutative:

$$\begin{tikzcd} F(x) \arrow[r, "\alpha^\sharp"] \arrow[d, "F(f)"'] & m \arrow[d, "g"] \\ F(y) \arrow[r, "\beta^\sharp"'] \arrow[ur, dashed, "h"] & n \end{tikzcd}$$

Let $h^\flat : y \to G(m)$ be the adjunct of h. We easily see that $h^\flat$ is the desired lifting due to the commutativity of the following diagram:

$$\begin{tikzcd} \hom(F(y), n) \arrow[r, "\cong"] & \hom(y, G(n)) \\ \hom(F(y), m) \arrow[r, "\cong"] \arrow[u, "g_*"] \arrow[d, "F(f)^*"'] & \hom(y, G(m)) \arrow[u, "G(g)_*"'] \arrow[d, "f^*"] \\ \hom(F(x), m) \arrow[r, "\cong"'] & \hom(x, G(m)) \end{tikzcd}$$

□

Definition 1.40. Let $\mathcal{C}$ be a category. An object $x \in \mathcal{C}$ is a **retract** of another object u if there exists a morphism $i : x \to u$ and a morphism $p : u \to x$ such that $p \circ i = \mathrm{Id}_x$.

We say that a morphism $f : x \to y$ is a **retract of a morphism** $g : u \to v$ if it is so in the category of morphisms $\mathrm{Fun}([1], \mathcal{C})$.

A class of morphisms F is **stable under retracts** if any morphism which is a retract of an element of F belongs to F.

Definition 1.41. A class of morphisms F is **stable under pushouts** if, for any push-out square of the form

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y \arrow[r, "b"] & y' \end{tikzcd}$$

if f is in F, then so is f'.

Definition 1.42. Let $\mathcal{C}$ be a cocomplete category, and $I \subset \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. For α an ordinal, which we regard as a category, an α-indexed **transfinite sequence** of elements in I is a cocontinuous functor $X : \alpha \to \mathcal{C}$ such that each successor morphism $\beta \to \beta + 1$ to a morphism in I. The **transfinite composition** is the induced morphism

$$X(0) \to \varinjlim_{i<\alpha} X(i)$$

We say that I is **closed under transfinite composition** if for each α and X the transfinite composition still lies in I.

Definition 1.43. A class of morphisms is **saturated** if it is stable under retracts, under push-outs and under transfinite compositions. If S is a class of morphisms, we use $\overline{S}$ to denote the smallest saturated class of morphisms containing S. $\overline{S}$ is also called the **saturation** of S.

Example 1.44. Consider the category Set. The class of injections is the saturation of $\emptyset \to \{*\}$. Likewise, the class of surjections is the saturation of $\{**\} \to \{*\}$

Lemma 1.45. *$l(F)$ and $r(F)$ are both stable under retracts.*

Proof. We only prove the statement for $r(F)$. Let $g \in r(F)$, so that for each morphism $f \in F$, we have $f \boxslash g$. Let h be a retract of g, exhibited by the diagram

$$\begin{tikzcd}
a \arrow[r, "s"] \arrow[d, "h"'] & x \arrow[r, "t"] \arrow[d, "g"] & a \arrow[d, "h"] \\
b \arrow[r, "i"'] & y \arrow[r, "j"'] & b
\end{tikzcd}$$

We wish to show to that $f \boxslash h$. Given a lifting problem as follows:

$$\begin{tikzcd}
m \arrow[r, "p"] \arrow[d, "f"'] & a \arrow[d, "h"] \\
n \arrow[r, "q"'] & b
\end{tikzcd}$$

Then we may construct a solution of this diagram by considering the following diagram

$$\begin{tikzcd}
m \arrow[r, "p"] \arrow[d, "f"'] & a \arrow[r, "s"] \arrow[d, "h"] & x \arrow[r, "t"] \arrow[d, "g"] & a \arrow[d, "h"] \\
n \arrow[r, "q"'] \arrow[ur, dotted] \arrow[urr, dashed] & b \arrow[r, "i"'] & y \arrow[r, "j"'] & b
\end{tikzcd}$$

$k : n \to x$ is the dashed arrow whose existence is guaranteed by the assumption that $f \boxslash g$, and the dotted arrow is given by $k' := t \circ k$. It is easy to see that k' solves the lifting problem. $\square$

Lemma 1.46. *$l(F)$ is stable under pushout; $r(F)$ is stable under pullback.*

Proof. We only prove the statement for $l(F)$. Let $g \in l(F)$, so that for each $f \in F$, we have $g \boxslash f$. Let h be a pushout of g, witnessed by the following pushout diagram:

$$\begin{tikzcd}
a \arrow[r, "i"] \arrow[d, "g"'] & c \arrow[d, "h"] \\
b \arrow[r, "j"'] & d
\end{tikzcd}$$

and we wish to show that $h \boxslash f$. Consider a lifting problem as follows:

$$\begin{tikzcd}
c \arrow[r, "p"] \arrow[d, "h"'] & x \arrow[d, "f"] \\
d \arrow[r, "q"'] & y
\end{tikzcd}$$

then we can find a solution by considering the following diagram

$$\begin{tikzcd}
a \arrow[r, "i"] \arrow[d, "g"'] & c \arrow[r, "p"] \arrow[d, "h"] & x \arrow[d, "f"] \\
b \arrow[r, "j"'] \arrow[urr, dashed] & d \arrow[r, "q"'] \arrow[ur, dotted] & y
\end{tikzcd}$$

The dashed arrow exists by our assumption that $g \boxslash f$, and the dotted arrow exists by the universal property of pushout, which solves our lifting problem. $\square$

Lemma 1.47. *$l(F)$ is stable under transfinite compositions.*

Proof. Consider the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s"] \arrow[d] \arrow[ddd, bend right=60, "p_0"'] & S \arrow[ddd, "q"] \\
X_1 \arrow[d] \arrow[ur, dashed, "h_1"] \arrow[dd, bend right, "p_1"'] & \\
\cdots \arrow[d] \arrow[uur, dashed, "h_n"] & \\
\varinjlim_n X_n \arrow[r, "t"'] \arrow[uuur, dashed] & T
\end{tikzcd}$$

where $q \in F$ and each connecting map $X_n \to X_{n+1}$ lies in $l(F)$. Hence for each n, there exists a lifting $h_n : X_n \to S$, which induces a morphism $h : \varinjlim_n X_n \to S$. $\square$

Proposition 1.48. *Let $\mathcal{C}$ be a category, together with two classes of morphisms F and F'. We have the following properties:*

1. *$F \subset r(F') \Leftrightarrow F' \subset l(F)$.*
2. *$F \subset F' \Rightarrow l(F') \subset l(F)$, $r(F') \subset r(F)$.*
3. *$r(F) = r(l(r(F)))$; $l(F) = l(r(l(F)))$.*
4. *The class $l(F)$ is saturated and the class $r(F)$ is co-saturated, i.e. is saturated as a class of morphisms of $\mathcal{C}^{\mathrm{op}}$.*

Proposition 1.49 (Retract Lemma)**.** *Assume that a morphism $f : x \to y$ can be factored into $f = p \circ i$. If f has the right (resp. left) lifting property with respect to i (resp. to p), then f is a retract of p (resp. of i).*

Proof. Since $i \boxslash f$, the following lifting problem admits a solution h:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Id}_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r, "p"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

which gives rise to the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{Id}_X"] & T \arrow[r, "h"'] \arrow[d, "p"] & X \arrow[d, "f"] \\
Y \arrow[r, "\mathrm{Id}_Y"'] & Y \arrow[r, "\mathrm{Id}_Y"'] & Y
\end{tikzcd}$$

which exhibits f as a retract of p. $\square$

Definition 1.50. A **weak factorisation system** in a category $\mathcal{C}$ is a couple (A, B) of classes of morphisms satisfying the following properties:

(1) both A and B are stable under retracts;

(2) $A \subset l(B)$ ($\Leftrightarrow B \subset r(A)$);

(3) any morphism $f : x \to y$ of $\mathcal{C}$ admits a factorisation of the form $f = pi$, with $i \in A$ and $p \in B$.

Remark 1.51. It follows from the Retract Lemma that we must have $A = l(B)$ and $B = r(A)$.

The definition of a weak factorization system looks quite complicated and they seem to be quite difficult to construct. Quillen has solved this problem for us: he developed a systematic method to produce weak factorization systems with functorial factorization. As a matter of fact, most weak factorization systems we know are constructed in this way. The following formulation of small object argument is taken from [Cis19].

Proposition 1.52 (Small object argument)**.** *Let $\mathcal{C}$ be a cocomplete category endowed with a small set of morphisms I. Assume that there exists a cardinal κ such that, for any element $i : K \to L$ of I, the functor*

$$\hom_{\mathcal{C}}(K, -) : \mathcal{C} \to \mathrm{Set}$$

commutes with colimits indexed by κ-filtered well-ordered sets. Then the couple $(l(r(I)), r(I))$ is a weak factorisation system with a functorial factorisation. Furthermore, $l(r(I))$ is the smallest saturated class containing I.

Definition 1.53 (Fibrations and anodynes)**.** We will introduce classes of important maps in sSet:

- Let $\Lambda^L = \{\Lambda^n_k \hookrightarrow \Delta^n | 0 \le k < n, n \ge 1\}$. We say that a morphism f is a **left fibration** if $f \in r(\Lambda^L)$. We say that a morphism f is a **left anodyne** if $f \in l(r(\Lambda^L)) = \overline{\Lambda^L}$. $(l(r(\Lambda^L)), r(\Lambda^L))$ form a weak factorisation system.

- Let $\Lambda^R = \{\Lambda^n_k \hookrightarrow \Delta^n | 0 < k \le n, n \ge 1\}$. We say that a morphism f is a **right fibration** if $f \in r(\Lambda^R)$. We say that a morphism f is a **right anodyne** if $f \in l(r(\Lambda^R)) = \overline{\Lambda^R}$. $(l(r(\Lambda^R)), r(\Lambda^R))$ form a weak factorisation system.

- Let $\Lambda^{in} = \{\Lambda^n_k \hookrightarrow \Delta^n | 0 < k < n, n \ge 1\}$. We say that a morphism f is an **inner fibration** if $f \in r(\Lambda^{in})$. We say that a morphism f is an **inner anodyne** if $f \in l(r(\Lambda^{in})) = \overline{\Lambda^{in}}$. $(l(r(\Lambda^{in})), r(\Lambda^{in}))$ form a weak factorisation system.

- Let $\Lambda = \{\Lambda^n_k \hookrightarrow \Delta^n | 0 \le k \le n, n \ge 1\}$. We say that a morphism f is a **Kan fibration** if $f \in r(\Lambda)$. We say that a morphism f is an **anodyne** if $f \in l(r(\Lambda)) = \overline{\Lambda}$. $(l(r(\Lambda)), r(\Lambda))$ form a weak factorisation system.

- Let $\Lambda^{\mathrm{bdy}} = \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \ge 0}$. We say that a morphisms f is a **trivial Kan fibration** if $f \in r(\Lambda^{\mathrm{bdy}})$. We will soon show that a morphism g is injective if and only if $g \in l(r(\Lambda^{\mathrm{bdy}}))$. So $(\mathrm{Mon}, r(\Lambda^{\mathrm{bdy}}))$ form a weak factorisation system.

We immediately get that, the classes of all "fibrations" are stable under pullbacks, and the classes of all "anodynes", together with the class of all monomorphisms, are closed under pushouts.

Here is the proposition we have just promised

Proposition 1.54. $\mathrm{Mon} = l(r(\Lambda^{\mathrm{bdy}}))$.

Proof. First note that the class of injective maps is closed under pushouts, retracts and transfinite compositions, so we see the inclusion $l(r(\Lambda^{\mathrm{bdy}})) = \overline{\Lambda^{\mathrm{bdy}}} \subseteq \mathrm{Mon}$. Conversely, let $f : K \to L$ be injective. Then we show that f can be written as a countable composition of pushouts of coproducts of maps of Λ^{bdy}, there by showing that every injection lies in the saturation $\overline{\Lambda^{\mathrm{bdy}}}$. Define $X_0 = K$. Having defined X_n and $X_n \to L$, let S_n denote the set of n-simplices of L not in the image of

X_n. Each such simplice s is necessarily non-degenerate, and corresponds to a map $\Delta^n \to L$. The restriction of s to $\partial\Delta^n$ factors uniquely through X_n. Define X_{n+1} as the pushout in the diagram

$$\begin{tikzcd} \coprod_{S_n} \partial\Delta^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod_{S_n} \Delta^n \arrow[r] & X_{n+1} \end{tikzcd}$$

The map f is the transfinite composition of the sequences $X_n \to X_{n+1}$.

In a word, the X_n is the subcomplex of L generated by the adjoining of K with all non-degenerate simplex with dimensional smaller than or equal to n in L that is not in K, and $L = \cup_{i=0}^{\infty} X_n$. $\square$

1.4 Leibniz construction

This section is devoted to the so-called Leibniz construction. This section is based on [Rie14], Section 11.1.

Construction 1.55. Consider a two-variable adjunction

$$- \otimes - : \mathcal{M} \times \mathcal{N} \to \mathcal{P}; \quad \{-,-\} : \mathcal{M}^{\mathrm{op}} \times \mathcal{P} \to \mathcal{N}; \quad [-,-] : \mathcal{N}^{\mathrm{op}} \times \mathcal{P} \to \mathcal{M}$$

$$\hom_{\mathcal{P}}(m \otimes n, p) \cong \hom_{\mathcal{N}}(n, \{m,p\}) \cong \hom_{\mathcal{M}}(m, [n,p]).$$

If $\mathcal{P}$ has pushouts and $\mathcal{M}, \mathcal{N}$ have pullbacks, there is an induced two-variable adjunction

$$\begin{aligned} &- \hat{\otimes} - : \mathrm{Fun}([1], \mathcal{M}) \times \mathrm{Fun}([1], \mathcal{N}) \to \mathrm{Fun}([1], \mathcal{P}); \\ &\{-\hat{,}-\} : \mathrm{Fun}([1], \mathcal{M})^{\mathrm{op}} \times \mathrm{Fun}([1], \mathcal{P}) \to \mathrm{Fun}([1], \mathcal{N}); \\ &[-\hat{,}-] : \mathrm{Fun}([1], \mathcal{N})^{\mathrm{op}} \times \mathrm{Fun}([1], \mathcal{P}) \to \mathrm{Fun}([1], \mathcal{M}) \end{aligned} \tag{1}$$

between arrow categories. The left adjoint $\hat{\otimes}$ is the **pushout-product** of $i : m \to m' \in \mathrm{Fun}([1], \mathcal{M})$ and $j : n \to n' \in \mathrm{Fun}([1], \mathcal{N})$ as defined by the induced morphism

$$\begin{tikzcd} m \otimes n \arrow[r] \arrow[d] & m' \otimes n \arrow[d] \arrow[ddr, bend left] & \\ m \otimes n' \arrow[r] \arrow[drr, bend right] & \text{pushout} \arrow[dr, dashed, "i \hat{\otimes} j"] & \\ & & m' \otimes n' \end{tikzcd}$$

For future convenience, we may also refer to the pushout as $m' \otimes n \coprod_{m \otimes n} m \otimes n'$.

The right adjoints $\{-\hat{,}-\}$ and $[-\hat{,}-]$, called **pullback-cotensors** and **pullback-homs**, are defined dually for $f : p \to p'$ by the pullbacks in $\mathcal{N}$ and $\mathcal{M}$ respectively:

$$\begin{tikzcd} \{n',p\} \arrow[dr, dashed, "\exists! \{i \hat{,} f\}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \text{pullback} \arrow[r] \arrow[d] & \{m,p\} \arrow[d] \\ & \{m',p'\} \arrow[r] & \{n,p'\} \end{tikzcd}$$

and similarly:

$$\begin{tikzcd}
{[n',p]} \arrow[dr, dashed, "\exists ! [j\hat{,}f]"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \text{pullback} \arrow[r] \arrow[d] & {[n,p]} \arrow[d] \\
& {[n',p']} \arrow[r] & {[n,p']}
\end{tikzcd}$$

For future convenience we may also refer to the pullback as $[n,p] \times_{[n,p']} [n',p']$.

Theorem 1.56. *The functors defined above in (1) actually form a 2-variable adjunction.*

Proof. We are going to show that, for $f : x \to y$ in $\mathcal{M}$, $g : w \to z$ in $\mathcal{N}$ and $h : m \to n$ in $\mathcal{P}$, there is an isomorphism $\mathrm{Nat}(f \hat{\otimes} g, h) \cong \mathrm{Nat}(f, [g\hat{,}h])$. A natural transformation $\delta : f \hat{\otimes} g \Rightarrow h$ consists of the following data: a morphism $\alpha : x \otimes z \coprod_{x \otimes w} y \otimes w \to m$ and a morphism $\beta : y \otimes z \to n$, such that the following diagram is commutative

$$\begin{tikzcd}
x \otimes z \coprod_{x \otimes w} y \otimes w \arrow[r, "\alpha"] \arrow[d, "f \hat{\otimes} g"'] & m \arrow[d, "h"] \\
y \otimes z \arrow[r, "\beta"'] & n
\end{tikzcd}$$

By the definition of pushout, choosing α is equivalent to choosing two morphisms $\alpha_1 : x \otimes z \to m$, $\alpha_2 : y \otimes w \to m$ such that their restriction to $x \otimes w$ coincide. So the commutativity of the above diagram together with our requirements on α_1 and α_2 translates to the following conditions:

- $\star$ In the following diagrams, α_2 and β are sent to the same element; α_1 and β are sent to the same element; α_1 and α_2 are sent to the same element:

$$\begin{tikzcd}
\beta \arrow[d, mapsto] & \in \hom_{\mathcal{P}}(y \otimes z, n) \arrow[d] \\
\bullet & \in \hom_{\mathcal{P}}(y \otimes w, n) \\
\alpha_2 \arrow[u, mapsto] & \in \hom_{\mathcal{P}}(y \otimes w, m) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_1 \arrow[d, mapsto] & \in \hom_{\mathcal{P}}(x \otimes z, m) \arrow[d] \\
\bullet' & \in \hom_{\mathcal{P}}(x \otimes z, n) \\
\beta \arrow[u, mapsto] & \in \hom_{\mathcal{P}}(y \otimes z, n) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_1 \arrow[d, mapsto] & \in \hom_{\mathcal{P}}(x \otimes z, m) \arrow[d] \\
\bullet'' & \in \hom_{\mathcal{P}}(x \otimes w, m) \\
\alpha_2 \arrow[u, mapsto] & \in \hom_{\mathcal{P}}(y \otimes w, m) \arrow[u]
\end{tikzcd}$$

Similarly, a natural transformation $\rho : f \Rightarrow [g\hat{,}h]$ is equivalent to choosing three morphisms $\theta : x \to [z,m]$, $\gamma_1 : y \to [w,m]$ and $\gamma_2 : y \to [z,n]$ subject to similar restrictions. We conclude by recalling the commutativity of the following diagram, which is part of the definition of adjunction:

$$\begin{tikzcd}
\hom_{\mathcal{P}}(y \otimes z, n) \arrow[r, "\cong"] \arrow[d] & \hom_{\mathcal{M}}(y, [z,n]) \arrow[d] \\
\hom_{\mathcal{P}}(y \otimes w, n) \arrow[r, "\cong"] & \hom_{\mathcal{M}}(y, [w,n]) \\
\hom_{\mathcal{P}}(y \otimes w, m) \arrow[r, "\cong"'] \arrow[u] & \hom_{\mathcal{M}}(y, [w,m]) \arrow[u]
\end{tikzcd}$$

As a result, a choice of the triple $(\beta, \alpha_1, \alpha_2)$ is a equivalent to a choice of the triple $(\theta, \gamma_1, \gamma_2)$. $\square$

Corollary 1.57. *Let R, S, T be three classes of maps in $\mathcal{M}, \mathcal{N}, \mathcal{P}$, respectively. The following lifting properties are equivalent*

$$R\hat{\otimes}S \boxslash T \quad \Leftrightarrow \quad S \boxslash \{R \hat{,} T\} \quad \Leftrightarrow \quad R \boxslash [S \hat{,} T]$$

Proof. We only show $R\hat{\otimes}S \boxslash T \Leftrightarrow R \boxslash [S\hat{,}T]$. The other equivalences follow from a similar argument. We now show that, for a morphism $f : x \to y$ in R, a morphism $g : w \to z$ in S and a morphism $h : m \to n$ in T, we have $f\hat{\otimes}g \boxslash h \Leftrightarrow f \boxslash [g\hat{,}h]$. Suppose we are given a lifting problem (1):

$$\begin{tikzcd} x \otimes z \coprod_{x\otimes w} y \otimes w \arrow[r, "\alpha"] \arrow[d, "f\hat{\otimes}g"'] & m \arrow[d, "h"] \\ y \otimes z \arrow[r, "\beta"'] & n \end{tikzcd}$$

Then according to Theorem 1.56, this lifting problem translates to a lifting problem (2):

$$\begin{tikzcd} x \arrow[r, "\alpha'"] \arrow[d, "f"'] & {[z, m]} \arrow[d, "{[g\hat{,}h]}"] \\ y \arrow[r, "\beta'"'] & {[w, m] \times_{[w,n]} [z, n]} \end{tikzcd}$$

Moreover it is easily seen that a solution of lifting problem (1) translates to a solution lifting problem (2), through the isomorphism $\hom_{\mathcal{P}}(y \otimes z, m) \cong \hom_{\mathcal{M}}(y, [z, m])$. ☐

Example 1.58. The category kTop is Cartesian closed, and thus there is a two-variable adjunction. We use $\mathrm{Map}(A, -)$ to denote the right adjoint of $A \times -$. As introduced in Example 1.38, a map $j : A \to X$ is a Hurewicz cofibration if and only if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Map}(I, Z) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd} \quad \text{or equivalently,} \quad \begin{tikzcd} A \times I \cup_A X \arrow[r] \arrow[d] & Z \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

In particular, take Z to be the mapping cylinder $A \times I \cup_A X$ and the right top arrow to be the identity. Then it is necessary that the map $j \hat{\times} i_0$, which is the canonical inclusion of the mapping cylinder into $X \times I$, has a retraction. Indeed, the universal property of mapping cylinder shows that the existence of such a retraction suffices to characterize Hurewicz cofibrations.

2 Higher category theory

2.1 ∞-categories

Definition 2.1. A simplicial set X is an $(\infty, 1)$**-category/ ∞-category** if the unique morphism $t : X \to \Delta^0$ is an inner fibration. Equivalently, X is an ∞-category if any inner horn $\Lambda^n_k \to X (0 < k < n)$ can be extended to $\Delta^n \to X$. For $\mathcal{C}$ an ∞-category, we will refer to a vertex of $\mathcal{C}$ as an **object** of $\mathcal{C}$, and an edge of $\mathcal{C}$ as a **morphism** of $\mathcal{C}$.

This is the main subject of the note. We will focus on the following questions:

- Why do we call a simplicial set defined as above an ∞-category? How to view it as any kind of category?

- Having seen the categorical nature of an ∞-category, can we generalize various constructions in classical category theory to an ∞-categorical context? For example, how to define functor categories or slice categories? How to define a colimit or how to define a pair of adjunction? Does Yoneda lemma still hold?
- Why do we need an ∞-category? What is the essential advantage of such a concept compared with classical category theory?

Definition 2.2. Let $\mathcal{C}$ be an ∞-category and $(f,g) : \Lambda^2_1 \to \mathcal{C}$ be a pair of composable arrows in $\mathcal{C}$. A **composition of** f **and** g is 2-simplex $\sigma : \Delta \to \mathcal{C}$ such that $d_2(\sigma) = f$ and $d_0(\sigma) = g$. If $h = d_1(\sigma)$ is the third edge of σ, we may say that σ **exhibits** h **as a composition of** f **and** g or simply say that h is a composition of f and g.

Remark 2.3. By the above definition, the composition law of morphisms in an ∞-category is not part of defining data of the category.

The following lemma is direct from the definition of an ∞-category.

Lemma 2.4. *If $X \to \mathcal{C}$ is an inner fibration with $\mathcal{C}$ being an ∞-category, then X is also an ∞-category.*

Proposition 2.5. *Let X be a simplicial set. Then the following conditions are equivalent:*

1. *There exists a category $\mathcal{C}$ and an isomorphism $K \cong N(\mathcal{C})$;*
2. *For each $0 < k < n$ and $\Lambda^n_k \to X$, there exists a* ***unique*** *extension $\Delta^n \to X$.*
3. *For each $\mathrm{Sp}(n) \to X$, there exists a* ***unique*** *extension $\Delta^n \to X$.*

Proof. $1 \Rightarrow 3$ is obvious by the definition of a nerve.

$3 \Rightarrow 1$: Let X be a simplicial set which has unique extension for each spine inclusion. We are able to build a category $\mathcal{C}$ from X as follows: The objects are given by X_0; a morphism from x to y is an edge $f \in X_1$ whose source is x and target is y; identities are given by degenerate edges. We need to explain how to compose morphisms: two composable morphisms determine a map $\mathrm{Sp}(2) = \Lambda^2_1 \to X$ which we can extend over Δ^2 and restrict to the new edge. We omit the work to check that the identities are actually identities and composition of morphisms are associative. Finally, we show that there is a canonical map $\phi : X \to N(\mathcal{C})$ which turns out to be an isomorphism. ϕ is defined as follows: for an n-simplice $\alpha : \Delta^n \to X$ which determines and is uniquely determined by its restriction to the spine $\mathrm{Sp}(n) \to X$, and can be regarded as a spine in $N(\mathcal{C})$, and so can be extended uniquely to $\Delta^n \to N(\mathcal{C})$. ϕ obviously defines a bijection on vertices and edges. Unique extension property can be translated to that $X_n \cong X_1 \times_{X_0} ... \times_{X_0} X_1$, $N(\mathcal{C})_n \cong N(\mathcal{C})_1 \times_{N(\mathcal{C})_0} ... \times_{N(\mathcal{C})_0} N(\mathcal{C})_1$, so ϕ is an isomorphism.

$2 \Rightarrow 3$: We prove this direction by induction. The case $n = 2$ is trivial since $\mathrm{Sp}(2) = \Lambda^2_1$. Now fix n we assume that one can uniquely lift maps $\mathrm{Sp}(k) \to X$ to $\Delta^k \to X$ for $k < n$ consider a map $\mathrm{Sp}(n) \to X$, our purpose is to show that there is a unique extension to $\Delta^n \to X$. We will construct this extension by first extending it uniquely to some $\Lambda^n_k \to X(0 < k < n)$, and then apply condition 2. We first observe that $\mathrm{Sp}(n) \cap \Delta^{n\backslash\{n\}} = \mathrm{Sp}(n-1)$, and similarly $\mathrm{Sp}(n) \cap \Delta^{n\backslash\{0\}} = \mathrm{Sp}(n-1)$. Thus by the inductive hypothesis, there are unique maps $\Delta^{n\backslash\{\epsilon\}} \to X$ extending the map from the spine to X for $\epsilon = 0, n$. Since the intersection of these two faces is given by $\Delta^{n\backslash\{0,n\}}$, which intersect the spine again in a smaller spine, and two extensions agree on this intersection, by the inductive hypothesis. Hence we obtain a map

$$\mathrm{Sp}(n) \cup \Delta^{n\backslash\{0\}} \cup \Delta^{n\backslash\{n\}} = \Delta^{n\backslash\{0\}} \cup \Delta^{n\backslash\{n\}} \to X$$

where the union is in Δ^n. We claim that there exists a unique extension to the union

$$\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}} \cup \Delta^{n\setminus\{1\}}.$$

For this we claim that $\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}}$ contains the spine of $\Delta^{n\setminus\{1\}}$: The edges from $i \to i+1$ for $2 \le i \le n-1$ all lie in $\Delta^{n\setminus\{0\}}$, and edge $0 \to 2$ lies in $\Delta^{n\setminus\{n\}}$ since $n \ge 3$. Hence there is a unique map from $\Delta^{n\setminus\{1\}} \to X$ extending this map on the spine. It remains to be shown that this map agrees with the given one on

$$(\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}}) \cap \Delta^{n\setminus\{1\}} = \Delta^{n\setminus\{0,1\}} \cup \Delta^{n\setminus\{1,n\}}$$

On both of these simplices, the map is determined by its restriction to the spine, which show the claim. Inductively, we find that there exists a unique extension of the map in question to a map $\Lambda^n_{n-1} \to X$. This map can now be uniquely extended to Δ^n by condition 2.

$3 \Rightarrow 2$: We consider an extension problem $\beta : \Lambda^n_i \to X$ for which we want to show that it extends uniquely to $\Delta^n \to X$. We may assume $n \ge 3$ since the case $n = 2$ is trivial. There is an inclusion $\mathrm{Sp}(n) \to \Lambda^n_i$ and we can consider the restricted extension problem $\mathrm{Sp}(n) \to X$. This extension problem can be solved uniquely due to assumption 3, so that we obtain a map $\alpha : \Delta^n \to X$. This map can in turn be restricted to Λ^n_i, and we want to show that this restriction coincide with β. For this purpose, we may restrict the map to the faces of Λ^n_i, i.e., to the union of $\Delta^{n\setminus\{j\}}$ for $j \neq i$. It is easy to see that

$$\alpha_{|\Delta^{n\setminus\{0\}}} = \beta_{|\Delta^{n\setminus\{0\}}}$$

since their restriction on the corresponding spine coincide. Similarly we could show that $\alpha_{|\Delta^{n\setminus\{n\}}} = \beta_{|\Delta^{n\setminus\{n\}}}$. We need to show that

$$\alpha_{|\Delta^{n\setminus\{j\}}} = \beta_{|\Delta^{n\setminus\{j\}}}$$

under the assumption that $j \neq 0, n$. For this we show that $\alpha_{|\Delta^{j-1,j+1}} = \beta_{|\Delta^{j-1,j+1}}$. Since $n \ge 3$, the edge is contained in $\Delta^{n\setminus\{\epsilon\}}$ for ϵ either 0 or 1. Then we can induct that this determines the map from Λ^n_i, which completes the proof. $\square$

Corollary 2.6. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{D}$ be a category and $f : \mathcal{C} \to N(\mathcal{D})$ be a morphism of simplicial sets. Then f is an inner fibration.*

Proof. Consider the following lifting problem ($0 < k < n$)

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \Delta^n \arrow[r] & N(\mathcal{D}) \end{tikzcd}$$

Since $\mathcal{C}$ is an ∞-category, we could find a lift g such that the upper triangle commutes.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "a"] \arrow[d, "\iota"'] & \mathcal{C} \arrow[d, "f"] \\ \Delta^n \arrow[r, "c"'] \arrow[ur, dashed, "g"] & N(\mathcal{D}) \end{tikzcd}$$

The lower triangle commutes since c and $f \circ g$ both extend $f \circ a$ to $\Delta^n \to N(\mathcal{D})$, and they are forced to equal. $\square$

Definition 2.7 (Homotopies between 1-morphisms)**.** Let X be a simplicial set. Let $f, g\colon x \to y$ be parallel edges in X, a **homotopy** from f to g is a 2-cell $\sigma : \Delta^2 \to X$ such that $d_0(\sigma) = s_0(y)$, $d_1(\sigma) = g$ and $d_2(\sigma) = f$. Graphically,

$$\begin{tikzcd}
 & y \arrow[rd, "\mathrm{Id}_y"] & \\
x \arrow[ru, "f"] \arrow[rr, "g"'] & & y
\end{tikzcd}$$

We use the notation $f \simeq g$ if there exists a homotopy from f to g.

Example 2.8. Let $X = N(\mathcal{C})$ and $f, g : x \to y$ be two parallel morphisms in X. Then there exists a homotopy from f to g iff $f = g$.

Proposition 2.9. *If $\mathcal{C}$ is an ∞-category, then $\simeq$ is an equivalence relation. For a morphism $f : x \to y$ in $\mathcal{C}$, we use $[f]$ to denote the corresponding equivalence class modulo $\simeq$. That is, $f \simeq g$ if and only if $[f] = [g]$.*

Proof. It suffices to demonstrate reflectivity and transitivity.

For reflectivity, assume that $f \simeq g$, that is, there is a 2-cell $\sigma : \Delta^2 \to \mathcal{C}$ as follows:

$$\begin{tikzcd}
 & y \arrow[rd, "\mathrm{Id}_y"] & \\
x \arrow[ru, "f"] \arrow[rr, "g"'] & & y
\end{tikzcd}$$

Now consider the inner horn $\kappa : \Lambda^3_1 \to \mathcal{C}$ depicted as

$$\begin{tikzcd}
 & y \arrow[rrd, "="] \arrow[rdd, "="'] & & \\
 & & & y \\
x \arrow[uur, "f"] \arrow[urrr, dashed, "f"] \arrow[rr, "g"'] & & y \arrow[ur, "="'] &
\end{tikzcd}$$

where $\kappa|_{\Delta^{\{0,1,2\}}} = \sigma$, $\kappa|_{\Delta^{\{0,1,3\}}} = s_1(f)$ and $\kappa|_{\Delta^{\{1,2,3\}}} = s_1(s_0(y))$. Since $\mathcal{C}$ is an ∞-category, we may extend κ to a 3-simplex $\tilde{\kappa} : \Delta^3 \to \mathcal{C}$. Now take $\rho := d_1(\tilde{\kappa})$, we see that ρ witnesses a homotopy $g \simeq f$.

For transitivity, suppose that $f \simeq g$, $g \simeq h$, witnessed by the $\varphi, \psi : \Delta^2 \to \mathcal{C}$ respectively:

$$\varphi = \begin{tikzcd}
 & y \arrow[rd, "="] & \\
x \arrow[ru, "f"] \arrow[rr, "g"'] & & y
\end{tikzcd} \quad ; \quad \psi = \begin{tikzcd}
 & y \arrow[rd, "="] & \\
x \arrow[ru, "g"] \arrow[rr, "h"'] & & y
\end{tikzcd}$$

Now consider the inner horn $\theta : \Lambda^3_2 \to \mathcal{C}$ illustrated as

$$\begin{tikzcd}
 & y \arrow[rrd, "="] \arrow[rdd, "="'] & & \\
 & & & y \\
x \arrow[uur, "f"] \arrow[urrr, dashed, "h"] \arrow[rr, "g"'] & & y \arrow[ur, "="'] &
\end{tikzcd}$$

where $\theta|_{\Delta^{\{0,1,2\}}} = \varphi$, $\theta|_{\Delta^{\{0,2,3\}}} = \psi$ and $\theta|_{\Delta^{\{1,2,3\}}} = s_1(s_0(y))$. Since $\mathcal{C}$ is an ∞-category, we can extend θ to a 3-simplex $\tilde{\theta} : \Delta^3 \to \mathcal{C}$. Now take $\delta := d_2(\tilde{\theta})$, we see that δ witnesses a homotopy $f \simeq h$. $\square$

Lemma 2.10. *Let $f : x \to y$ and $g : y \to z$ be two morphisms in an ∞-category $\mathcal{C}$. Let $\sigma_1, \sigma_2 : \Delta^2 \to \mathcal{C}$ be 2-cells illustrated as*

$$\sigma_1 = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h_1"'] & & z \end{tikzcd} \quad ; \quad \sigma_2 = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h_2"'] & & z \end{tikzcd}$$

so that σ_1 witnesses h_1 as a composition of g and f; σ_2 witnesses h_2 as a composition of g and f. Then $h_1 \simeq h_2$.

Proof. Consider the inner horn $\kappa : \Lambda^3_1 \to \mathcal{C}$ illustrated as follows:

$$\begin{tikzcd} & y \arrow[rr, "g"] \arrow[dr, "g"] & & z \\ x \arrow[ur, "f"] \arrow[rr, "h_1"'] \arrow[urrr, dashed, "h_2"] & & z \arrow[ur, "="'] & \end{tikzcd}$$

where $\kappa|_{\Delta^{\{0,1,2\}}} = \sigma_1$, $\kappa|_{\Delta^{\{0,1,3\}}} = \sigma_2$ and $\kappa|_{\Delta^{\{1,2,3\}}} = s_1(g)$. We can extend κ to a simplex $\tilde{\kappa} : \Delta^3 \to \mathcal{C}$. Then $d_1(\tilde{k})$ witness a homotopy $h_1 \simeq h_2$. $\square$

Lemma 2.11. *Let $f, g : x \to y$ be two parallel morphisms in an ∞-category $\mathcal{C}$. Let $h : y \to z$ be a morphism and $k : w \to k$ be a morphism. If $f \simeq g$, then $h \circ f \simeq h \circ g$ and $f \circ k \simeq g \circ k$.*

Proof. We will show that $h \circ f \simeq h \circ g$. Let $\sigma : \Delta^2 \to \mathcal{C}$ be a 2-cell witnessing the homotopy $f \simeq g$. Let $\kappa : \Delta^2 \to \mathcal{C}$ a 2-cell witnessing the composition:

$$\kappa = \begin{tikzcd} & y \arrow[dr, "h"] & \\ x \arrow[ur, "f"] \arrow[rr, "h \circ f"'] & & z \end{tikzcd}$$

Now consider the inner horn $\theta : \Lambda^3_1 \to \mathcal{C}$ illustrated by

$$\begin{tikzcd} & y \arrow[rr, "h"] \arrow[dr, "="] & & z \\ x \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[urrr, dashed, "h \circ f"] & & y \arrow[ur, "h"'] & \end{tikzcd}$$

where $\theta|_{\Delta^{\{0,1,2\}}} = \sigma$, $\theta|_{\Delta^{\{0,1,3\}}} = \kappa$ and $\theta|_{\Delta^{\{1,2,3\}}} = s_0(h)$. Hence we can extend θ to a 3-simplex $\tilde{\theta} : \Delta^3 \to \mathcal{C}$. Take $\rho = d_1(\tilde{\theta})$, we see that ρ exhibits $h \circ f$ as a composition $h \circ g$. The composition is unique up to homotopy, we see that $[h \circ f] = [h \circ g]$. $\square$

Lemma 2.12. *Let $f : x \to y$, $g : y \to z$ and $h : z \to w$ be three morphisms in an ∞-category $\mathcal{C}$. Then $h \circ (g \circ f) \simeq (h \circ g) \circ f$.*

Proof. Let σ be a 2-cell witnessing the composition $g \circ f$, κ be a 2-cell witnessing the composition $h \circ g$ and θ a 2-cell witnessing the composition $h \circ (g \circ f)$. Then σ, κ and θ assembles to an inner horn $\rho : \Lambda^3_2 \to \mathcal{C}$ as follows:

$$\begin{tikzcd}
 & y \arrow[drr, "h\circ g"] \arrow[ddr, "g"] & & \\
 & & & z \\
x \arrow[uur, "f"] \arrow[rr, "g\circ f"'] \arrow[urrr, dashed, "h\circ(g\circ f)"] & & z \arrow[ur, "h"'] &
\end{tikzcd}$$

which extends to a 3-simplex $\tilde{\rho} : \Delta^3 \to \mathcal{C}$. Then $d_2(\tilde{\rho})$ exhibits $h \circ (g \circ f)$ as a composition of f and $h \circ g$, so we have $h \circ (g \circ f) \simeq (h \circ g) \circ f$. $\square$

Definition 2.13. Let $\mathcal{C}$ be an ∞-category. We define a new category $h\mathcal{C}$ as follows:

- The objects of $h\mathcal{C}$ consists of the objects of $\mathcal{C}$, that is, $\mathrm{Ob}(h\mathcal{C}) = \mathcal{C}_0$.
- Morphisms of $h\mathcal{C}$ consists of homotopy equivalent classes of morphisms in $\mathcal{C}$, that is $\mathrm{Mor}(h\mathcal{C}) = \mathcal{C}_1/\simeq$.
- Composition of morphisms is defined by $[g]\circ[f] := [g\circ f]$. The composition law is well defined thanks to Lemma 2.11.
- The composition law is associative thanks to Lemma 2.12.

We will refer to $h\mathcal{C}$ as the **homotopy category** of $\mathcal{C}$.

Proposition 2.14. *Let $\mathcal{C}$ be an ∞-category. Then there is a canonical isomorphism of 1-categories $h\mathcal{C} \cong \pi\mathcal{C}$.*

Definition 2.15. Let $\mathcal{C}$ be an ∞-category. A morphism $f : x \to y$ is an **equivalence** if $[f]$ is an isomorphism in $h\mathcal{C}$.

Definition 2.16. We say that an ∞-category $\mathcal{C}$ is an ∞**-groupoid** if the homotopy category $h\mathcal{C}$ is a groupoid.

Definition 2.17. We say that X is a **Kan complex** if the unique map $X \to \Delta^0$ is a Kan fibration.

Remark 2.18. If $f : X \to Y$ is Kan fibration, then for each $y \in Y_0$ the fiber $f^{-1}(y)$ is either empty or a Kan complex. This is because Kan fibrations are stable under pullbacks:

$$\begin{tikzcd}
f^{-1}(y) = X \times_Y \{y\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^0 \arrow[r, "y"'] & Y
\end{tikzcd}$$

Lemma 2.19. *A Kan complex is an ∞-groupoid.*

Proof. For each morphism $f : x \to y$, consider the horn $(f, \mathrm{Id}_x) : \Lambda^2_0 \to \mathcal{C}$ and $(f, \mathrm{Id}_y) : \Lambda^2_2 \to \mathcal{C}$ depicted by

$$\begin{tikzcd}
 & y \arrow[dr, dashed, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "\mathrm{Id}_x"'] & & x
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[dr, "f"] & \\
y \arrow[ur, dashed, "h"] \arrow[rr, "\mathrm{Id}_y"'] & & y
\end{tikzcd}$$

So that $g \circ f \simeq \mathrm{Id}_x$ and $f \circ h \simeq \mathrm{Id}_y$. This show that $[f]$ admits both a left inverse and a right inverse, so $[f]$ is an isomorphism. Since f is arbitrary, we see that X is an ∞-groupoid. $\square$

We will later show that the converse statement of the above lemma holds true. That is, any ∞-groupoid is a Kan complex. See Proposition 2.73.

2.2 Joins and slices

2.2.1 For ordinary categories

We first introduce the construction of joins and slices for ordinary categories.

Definition 2.20. Let $\mathcal{C}, \mathcal{D}$ be 1-categories. Define a new category $\mathcal{C} \star \mathcal{D}$ as follows

- $\mathrm{Ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \sqcup \mathrm{Ob}(\mathcal{D})$;
- $\hom_{\mathcal{C}\star\mathcal{D}}(x, y) = \begin{cases} \hom_{\mathcal{C}}(x, y) & x, y \in \mathcal{C} \\ \hom_{\mathcal{D}}(x, y) & x, y \in \mathcal{D} \\ \{*\} & x \in \mathcal{C}, y \in \mathcal{D} \\ \emptyset & x \in \mathcal{D}, y \in \mathcal{C}. \end{cases}$

The category $\mathcal{C} \star \mathcal{D}$ is equipped with many special functors. For example, there are canonical embeddings $\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D}$, $\mathcal{D} \hookrightarrow \mathcal{C} \star \mathcal{D}$. Moreover, there is a canonical projection functor $\mathcal{C} \star \mathcal{D} \to [1]$.

Remark 2.21. It is worth mentioning that $- \star -$ is not symmetric.

Example 2.22. There is a canonical isomorphism

$$[m] \star [n] \cong [m + n + 1]$$

Recall that, for a functor $F : \mathcal{C} \to \mathcal{D}$, we can define **slice category** $\mathcal{C}_{/F}$, whose objects are cones over F and whose morphisms are cone transformations. Dually, we have $\mathcal{C}_{F/}$. The following proposition says that the construction of slice categories can be viewed as a right adjoint of the join construction.

Definition 2.23. For a category $\mathcal{E}$, let $\mathrm{Cat}_{\mathcal{E}/}$ be the 2-category of categories under $\mathcal{E}$. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{E} \to \mathcal{D}$ be two objects in $\mathrm{Cat}_{\mathcal{E}/}$, we will write $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ for corresponding hom-category.

Proposition 2.24. *There is a isomorphism of categories*

$$\mathrm{Fun}(\mathcal{E}, \mathcal{D}_{F/}) \cong \mathrm{Fun}_{\mathcal{E}}(\mathcal{C} \star \mathcal{E}, \mathcal{D})$$

Similarly, we have

$$\mathrm{Fun}(\mathcal{E}, \mathcal{D}_{/F}) \cong \mathrm{Fun}_{\mathcal{E}}(\mathcal{E} \star \mathcal{C}, \mathcal{D})$$

Theorem 2.25. *Let $F : \mathcal{C} \to [1]$ be a functor, and let $\mathcal{C}_0 = F^{-1}(0)$ and $\mathcal{C}_1 = F^{-1}(1)$ be the fibers, so that $\mathcal{C}_0$ and $\mathcal{C}_1$ are subcategies of $\mathcal{C}$. Then there is a canonical factorisation*

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{C}_0 \star \mathcal{C}_1 \arrow[dl] \\ & {[1]} & \end{tikzcd}$$

2.2.2 For simplicial sets

For the remaining part, we will generalize the join and slice constructions to the world of simplicial sets.

Definition 2.26. Given a linearly ordered set J, we define a **cuts** of J as a decomposition $J = J_1 \sqcup J_2$ such that $x \in J_1, y \in J_2$ implies that $x < y$. We will write $\mathrm{Cut}(J)$ for the sets of all cuts of J. We will write $\mathrm{Cut}^o(J)$ for the subset consisting of those cuts where both J_1 and J_2 are non-empty.

Remark 2.27. Cut$(-)$ is a covariant functor from linearly ordered sets to sets. Moreover, $\mathrm{Cut}(-) \cong \hom_\Delta(-, [1])$ is representable.

Definition 2.28. Let X and Y be simplicial sets. We define their join $X \star Y$ to be the simplicial set defined as follows. For a finite linearly ordered set J, we set

$$(X \star Y)(J) := \coprod_{(J_1,J_2)\in \mathrm{Cut}(J)} X(J_1) \times X(J_2)$$

where we declare that $X(\emptyset) = \{*\} = Y(\emptyset)$. Alternatively, we my define

$$(X \star Y)(J) := X \coprod Y \coprod (\coprod_{(J_1,J_2)\in \mathrm{Cut}^o(J)} X(J_1) \times Y(J_2))$$

The simplicial structure maps are omitted here.

Example 2.29. For two ordinary categories, there is a canonical map

$$N(\mathcal{C} \star \mathcal{D}) \to N(\mathcal{C}) \star N(\mathcal{D})$$

which is in fact an isomorphism of simplicial sets.

Theorem 2.30. *Let $p : K \to \Delta^1$ be a morphism, and let $K' = p^{-1}(0)$, $K'' = p^{-1}(1)$. Then there is a canonical factorisation*

$$\begin{tikzcd} K \arrow[rr, "p"] \arrow[dr] & & K' \star K'' \arrow[dl] \\ & \Delta^1 & \end{tikzcd}$$

Proof. We need to construct the map $K \to K' \star K''$. For any $n \geq 0$, we have that

$$\hom_{\mathrm{sSet}}(\Delta^n, \Delta^1) = \hom_\Delta([n], [1]) = \mathrm{Cut}([n]).$$

Thus, for every n-simplex $x : \Delta^n \to K$, the composite $px : \Delta^n \to K \to \Delta^1$ determines a cut $([i], [j]) \in \mathrm{Cut}(n)$, so that the map $px : [n] \to [1]$ sends the first i points to 0 and the rest to 1. By definition, this determines a point of $K'([i]) \times K''([j])$ which in turn determines an n-simplex of $K' \star K''$. $\square$

Corollary 2.31. *Given a map $\varphi : K \to X \star Y$ over Δ^1, there is a factorisation into*

$$K \to K' \star K'' \xrightarrow{f \star g} X \star Y$$

Proposition 2.32. *Let $X, Y \in$ sSet. X and Y are ∞-categories if and only if $X \star Y$ is an ∞-category.*

Proof. Consider a map

$$\Lambda^n_j \to X \star Y$$

for $n \geq 2$ and $0 < j < n$. We want to show that this map extends over Δ^n. We can post-compose the map with the canonical map $X \star Y \to \Delta^1$ and obtain a factorisation

$$\Lambda^n_j \to (\Lambda^n_j)' \star (\Lambda^n_j)'' \to X \star Y.$$

There are several possibilities for what the first map is. First recall that any map $\Lambda^n_j \to \Delta^1$ factors uniquely over Δ^n (since Δ^1 is the nerve of a category and Λ^n_j is an inner horn). We need to consider the following three cases:

1 The map $\Lambda^n_j \to \Delta^1$ is constant at 0;

2 The map $\Lambda^n_j \to \Delta^1$ is constant at 1;

3 The map $\Lambda^n_j \to \Delta^1$ is not constant.

In the first case, we find that the map $\Lambda^n_j \to X \star Y$ factors through the embedding $X \hookrightarrow X \star Y$ and can therefore be extended over Δ^n if and only if X is an ∞-category: If an extension of the composite

$$\Lambda^n_j \to X \hookrightarrow X \star Y$$

to Δ^n exists, then the composite $\Delta^n \to X \star Y \to \Delta^1$ is constant at 0, so that the map $\Delta^n \to X \star Y$ factors through the inclusion $X \hookrightarrow X \star Y$. Similarly, in the second case we find that the map $\Lambda^n_j \to X \star Y$ factors through $Y \hookrightarrow X \star Y$, and thus can be extended over Δ^n iff Y is an ∞-category.

It remains to consider the case where the map $\Lambda^n_j \to \Delta^1$ is not constant. First note that this map uniquely factors through a non-constant map $\Delta^n \to \Delta^1$. Such a non-constant map corresponds precisely to a cut of $[n]$ into two non-empty subsets, so there is a $0 \leq k < n$ such that the map $\Delta^n \to \Delta^1$ is isomorphic to the canonical map $\Delta^k \star \Delta^l \to \Delta^1$. It follows that $(\Lambda^n_j)'$ consists of all those m-simplice of Λ^n_j which can be represented a map $f : [m] \to [n]$ subject to the following two conditions:

1. the image of f is contained in $\{0, ..., k\}$, so that it lies in the fiber over 0 and
2. there exists a number $q \neq j$ which is not in the image of f.

Since $k < n$ we can take $q = n$ in condition 2, and so condition 2 is automatically verified if we assume condition 1. In other words, we find that $(\Lambda^n_j)' \cong \Delta^k$, and similarly we have $(\Lambda^n_j)'' \cong \Delta^l$. As a result, the factorisation

$$\Lambda^n_j \to \Delta^k \star \Delta^l \cong \Delta^n \to X \star Y$$

gives the desired extension. It is worth mentioning that, in the last case, the lifting problem can be solved without requiring either X or Y to be an ∞-category. $\square$

The remaining part of this section is devoted to the slice construction for simplicial sets.

Lemma 2.33. *Given a simplicial set X, the join construction determines a functor $X \star - : \mathrm{sSet} \to \mathrm{sSet}_{X/}$. Likewise, it yields a functor $- \star X : \mathrm{sSet} \to \mathrm{sSet}_{X/}$.*

Definition 2.34. Let $p : S \to X$ be a morphism of simplicial sets, the association $n \mapsto \hom_{\mathrm{sSet}_{S/}}(S \star \Delta^n, X)$ determines a simplicial set which we will refer to as $X_{p/}$.

Proposition 2.35. *The functor $(p : S \to X) \mapsto X_{p/}$ is right adjoint to $S \star - : \mathrm{sSet} \to \mathrm{sSet}_{S/}$. Similarly, the functor $(p : S \to X) \mapsto X_{/p}$ is right adjoint to $- \star S : \mathrm{sSet} \to \mathrm{sSet}_{S/}$.*

Proof. We only give a sketchy proof here. Note that the adjunction property for representable simplicial sets

$$\hom_{\mathrm{sSet}_{S/}}(S \star \Delta^n, X) \cong \hom_{\mathrm{sSet}}(\Delta^n, X_{/p})$$

holds automatically by the definition of $X_{/p}$. To extend the adjunction property, it suffices to show that $S \star -$ preserves colimits. But the author is too lazy to write down the details. A nice reference is [Lan21] Section 1.4. $\square$

2.3 Stability properties of anodynes and fibrations

2.3.1 Product-function complex adjunction

Recall that the category of simplicial sets is Cartesian closed. That is, the symmetric monoidal structure given by the product $\times : \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}$ is part of a 2-variable adjunction:

$$\hom_{\mathrm{sSet}}(X \times Y, Z) \cong \hom_{\mathrm{sSet}}(Y, \mathrm{Fun}(X, Z)) \cong \hom_{\mathrm{sSet}}(X, \mathrm{Fun}(Y, Z)).$$

As a result, we may apply the Leibniz construction to this 2-variable adjunction, and get a new 2-variable adjunction, which we denote by

$$\hat{\times} : \mathrm{Fun}([1], \mathrm{sSet}) \times \mathrm{Fun}([1], \mathrm{sSet}) \to \mathrm{Fun}([1], \mathrm{sSet}).$$

and

$$\hat{\mathrm{Fun}}(-,-) : \mathrm{Fun}([1], \mathrm{sSet})^{\mathrm{op}} \times \mathrm{Fun}([1], \mathrm{sSet}) \to \mathrm{Fun}([1], \mathrm{sSet})$$

Note that since the Cartesian product $\times$ is symmetric, the induced map $i \hat{\times} j$ is again symmetric.

Lemma 2.36. *Let S, T be two classes of morphisms in a category $\mathcal{C}$ such that S satisfies the small object argument (Proposition 1.52). Then*

$$S \boxslash T \quad \Leftrightarrow \quad \overline{S} \boxslash T.$$

Proof. The implication $\overline{S} \boxslash T \Rightarrow S \boxslash T$ is obvious. Now we show that $S \boxslash T \Rightarrow \overline{S} \boxslash T$. Since S satisfies the small object argument, we have $\overline{S} = l(r(S))$. We have

$$S \boxslash T \Rightarrow T \subseteq r(S) \Rightarrow l(r(S)) = \overline{S} \subseteq l(T) \Leftrightarrow \overline{S} \boxslash T.$$

□

Lemma 2.37. *Let S, T be two classes of morphisms in* sSet*. Then we have*

$$\overline{\overline{S} \hat{\times} T} = \overline{S \hat{\times} T}; \quad \overline{S \hat{\times} \overline{T}} = \overline{S \hat{\times} T}$$

Proof. The inclusion $\overline{S \hat{\times} T} \subseteq \overline{\overline{S} \hat{\times} T}$ is obvious. To show the reverse inclusion, notice that $\overline{S \hat{\times} T} = l(r(S \hat{\times} T))$. Hence it suffices to show that

$$\overline{\overline{S} \hat{\times} T} \boxslash r(S \hat{\times} T)$$

But according to Lemma 2.36, this is equivalent to

$$\overline{S} \hat{\times} T \boxslash r(S \hat{\times} T)$$

The desired result follows from the following equivalences

$$\begin{aligned} \overline{S} \hat{\times} T \boxslash r(S \hat{\times} T) &\Leftrightarrow \overline{S} \boxslash \hat{\mathrm{Fun}}(T, r(S \hat{\times} T)) \\ \Leftrightarrow S \boxslash \hat{\mathrm{Fun}}(T, r(S \hat{\times} T)) &\Leftrightarrow S \hat{\times} T \boxslash r(S \hat{\times} T) \end{aligned}$$

Note that $S \hat{\times} T \boxslash r(S \hat{\times} T)$ holds true automatically. □

Here is our main theorem of this section.

Theorem 2.38. *Let $i : A \to B$ be a monomorphism. If $j : X \to Y$ is*

(1) a monomorphism, then $i \hat{\times} j$ is again a monomorphism;

(2) a left/right anodyne map, then $i \hat{\times} j$ *is again a left/right anodyne map;*

(3) an anodyne map, then $i \hat{\times} j$ *is again an anodyne map;*

(4) an inner anodyne map, then $i \hat{\times} j$ *is again an inner anodyne map.*

Proof. Let us take statement (4) as an examples. The other statements follow from similar arguments. We need to show that
$$\overline{\overline{\Lambda^{bdy}} \hat{\times} \overline{\Lambda^{in}}} \subseteq \overline{\Lambda^{in}}$$
According to Lemma 2.37, we have
$$\overline{\overline{\Lambda^{bdy}} \hat{\times} \overline{\Lambda^{in}}} = \overline{\Lambda^{bdy} \hat{\times} \Lambda^{in}}.$$
Hence it suffices to show that
$$\Lambda^{in} \hat{\times} \Lambda^{bdy} \subseteq \overline{\Lambda^{in}}$$
We leave this purely combinatoric result to the next lemma. □

Lemma 2.39. *We have*
$$\Lambda^{in} \hat{\times} \Lambda^{bdy} \subseteq \overline{\Lambda^{in}}.$$

Proof. To be added. □

Corollary 2.40. *Let* $i : A \to B$ *be a monomorphism*

1. *If* $f : X \to Y$ *is an inner(resp. left/right/Kan/trivial Kan) fibration, then* $\hat{\mathrm{Fun}}(i, f)$ *is again an inner(resp. left/right/Kan/trivial Kan) fibration;*

2. *If* i *is moreover an inner(resp. left/right) anodyne, and* f *is an inner(resp. left/right) fibration, then* $\hat{\mathrm{Fun}}(i, f)$ *is a trivial Kan fibration.*

Proof. 1. To show that $\hat{\mathrm{Fun}}(i, f)$ is an inner fibration, it suffices to show that for all $j \in l(r(\Lambda^{in}))$ we have $j \boxslash \hat{\mathrm{Fun}}(i, f)$, which is equivalent to $j \hat{\times} i \boxslash f$. This has been proven in the statement 4 of Theorem 2.38.

2. To show that $\hat{\mathrm{Fun}}(i, f)$ is a trivial Kan fibration, it suffices to show that for all monomorphisms $j \in l(r(\Lambda^{bdy}))$, we have $j \boxslash \hat{\mathrm{Fun}}(i, f)$, which is equivalent to $j \hat{\times} i \boxslash f$, this is again the statement 4 of Theorem 2.38. □

Corollary 2.41. *If* $\mathcal{C}$ *is an* ∞*-category, then for each* $X \in \mathrm{sSet}$*, the function complex* $\mathrm{Fun}(X, \mathcal{C})$ *is an* ∞*-category. As a result, for two* ∞*-categories* $\mathcal{C}$ *and* $\mathcal{D}$*, we will refer to* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *as* ***the*** ∞***-category of functors from*** $\mathcal{C}$ ***to*** $\mathcal{D}$*.*

Proof. Let $i : \emptyset \to X$ be the unique map, which is of course a monomorphism. Let $f : \mathcal{C} \to \Delta^1$ be the unique map, which is an inner fibration since $\mathcal{C}$ is an ∞-category. Then $\hat{\mathrm{Fun}}(i, f) : \mathrm{Fun}(X, \mathcal{C}) \to \Delta^0$ is again an inner fibration. □

Corollary 2.42. *If* Y *is a Kan complex, then for any* $X \in \mathrm{sSet}$*, the simplicial set* $\mathrm{Fun}(X, Y)$ *is a Kan complex.*

Corollary 2.43. *For* $\mathcal{C}$ *an* ∞*-category, the morphism* $p : \mathrm{Fun}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_1, \mathcal{C})$ *induced by the canonical inclusion* $i : \Lambda^2_1 \hookrightarrow \Delta^2$ *is a trivial Kan fibration.*

Proof. Let $f : \mathcal{C} \to \Delta^0$ be the unique morphism, which is an inner fibration. Then the induced map p coincide with $\hat{\mathrm{Fun}}(i, f)$, so p is a trivial Kan fibration. □

Definition 2.44. We say that X is a **contractible Kan complex** if the unique map $X \to \Delta^0$ is a trivial Kan fibration.

Definition 2.45. Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets, so there is an induced map $f_* : \mathrm{Fun}(Y, X) \to \mathrm{Fun}(Y, Y)$, which is again a trivial Kan fibration. We will refer to the fiber $f_*^{-1}(\mathrm{Id}_Y)$ as **the space of sections of** f. This Kan complex must be non-empty and contractible.

In particular, any section of $p : \mathrm{Fun}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_1, \mathcal{C})$ can be regarded as a composition law of $\mathcal{C}$, which sends a pair of composable arrows to a 2-cell. As a result that, we could safely say the space of composition laws for an ∞-category is contractible. Now choose a section $q : \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \to \mathrm{Fun}(\Delta^2, \mathcal{C})$, we can write down the corresponding composition law in a more familiar form:

$$\begin{tikzcd}
\mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \arrow[r, "q"] \arrow[d, "\cong"] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d, "(d^1)^*"] \\
\mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r, "c"'] & \mathrm{Fun}(\Delta^1, \mathcal{C})
\end{tikzcd}$$

We will refer to the map

$$c : \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \tag{2}$$

as **the composition law** of $\mathcal{C}$. Note that c is well-defined only up to a contractible space of choices

2.3.2 Join-slice adjunction

Construction 2.46. Given a sequence in sSet as follows

$$A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$$

We could get a commutative diagram as follows:

$$\begin{tikzcd}
X_{\varphi/} \arrow[r] \arrow[d] & X_{\varphi i/} \arrow[d] \\
Y_{f\varphi/} \arrow[r] & Y_{f\varphi i/}
\end{tikzcd}$$

which induces a map

$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}.$$

We will refer to this map as $\langle i, f\rangle_{\varphi/}$.

Likewise we could get a commutative diagram as follows:

$$\begin{tikzcd}
X_{/\varphi} \arrow[r] \arrow[d] & X_{/\varphi i} \arrow[d] \\
Y_{/f\varphi} \arrow[r] & Y_{/f\varphi i}
\end{tikzcd}$$

and again we could get an induced map

$$\langle i, f\rangle_{/\varphi} : X_{/\varphi} \to X_{/\varphi i} \times_{Y_{/f\varphi i}} Y_{/f\varphi}$$

Lemma 2.47. *The slice-join adjunction induces a bijection between solutions of the following two lifting problems:*

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "j"'] & X_{\varphi/} \arrow[d, "\langle i,f\rangle_{\varphi/}"] \\
T \arrow[r] \arrow[ur, dashed] & X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[r] \arrow[rr, bend left, "\varphi"] & A\star T\coprod_{A\star S}B\star S \arrow[r] \arrow[d, "i\hat{\star}j"'] & X \arrow[d, "f"] \\
& B\star T \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Proof. As we have done in Leibniz construction. □

Let us fix $\varphi \in \hom_{\mathrm{sSet}}(B, X)$. If every lifting problem illustrated in the second diagram can be solved, we will use the notation

$$i\hat{\star}j \boxslash_{\varphi/} f$$

Applying this notation, we see that there is an equivalence

$$j \boxslash \langle i, f\rangle_{\varphi/} \quad \Leftrightarrow \quad i\hat{\star}j \boxslash_{\varphi/} f$$

Dually, let us fix $\varphi \in \hom_{\mathrm{sSet}}(T, X)$. Then we can prove that

$$i \boxslash \langle j, f\rangle_{/\varphi} \quad \Leftrightarrow \quad i\hat{\star}j \boxslash_{/\varphi} f$$

It is worth mentioning that we have the following equivalences:

$$i\hat{\star}j \boxslash f \quad \Leftrightarrow \quad i\hat{\star}j \boxslash_{\varphi/} f,\ \forall\varphi \in \hom_{\mathrm{sSet}}(B, X) \quad \Leftrightarrow \quad i\hat{\star}j \boxslash_{/\varphi} f,\ \forall\varphi \in \hom_{\mathrm{sSet}}(T, X).$$

Lemma 2.48. *Let $i : A \to B$, $j : S \to T$ be monomorphisms. Then*

1. *$i\hat{\star}j$ is a monomorphism;*
2. *$i\hat{\star}j$ is inner anodyne if i is right anodyne or j is left anodyne;*
3. *$i\hat{\star}j$ is left(resp. right) anodyne if i(resp. j) is left(resp. right) anodyne.*

Proof. The proof will be similar to what we have done for the product-function complex adjunction: we reduce the problem to generators, and then solve the remaining purely combinatoric problem.

We will prove the first part of statement 2 as an example. Let A_1 be the class of all morphism such that

$$A_1\hat{\star}\mathrm{Mon} \subseteq \overline{\Lambda^{in}}$$

then $A_1 = \overline{A_1}$: a morphism f lies in A_1 if and only if for all monomorphisms $g : S \to T$, and $h : X \to Y$ an inner fibration, we have $f\hat{\star}g \boxslash h$, or equivalently, $f \boxslash \langle g, h\rangle_{/\varphi}, \forall\varphi : T \to X$. As a result, it suffices to show that Λ^R belong to A_1.

Similarly, let A_2 be the class of morphisms such that

$$(\Lambda^R)\hat{\star}A_2 \boxslash r(\Lambda^{in})$$

then A_2 is again saturated. As a result, it suffices to show that $\Lambda^R\hat{\star}\Lambda^{bdy} \boxslash r(\Lambda^{in})$. See the next lemma. □

Lemma 2.49. *We have*

- $\Lambda^n_j \star \Delta^m \cup \Delta^n \star \partial\Delta^m = \Lambda^{n+1+m}_j$;
- $\partial\Delta^m \star \Delta^n \cup \Delta^m \star \Lambda^n_j = \Lambda^{n+1+m}_{m+1+j}$.

Proof. We will determine the following simplicial subsets of Δ^{n+1+m} (Note that these simplicial sets can be canonically identified as simplicial subsets of Δ^{n+m+1} through the isomorphisms $\Delta^{n+m+1} \cong \Delta^n \star \Delta^m \cong \Delta^m \star \Delta^n$):

1. $\partial\Delta^m \star \Delta^n$. By the definition of $- \star -$, a k-simplex of $\partial\Delta^m \star \Delta^n$ is a monotone function $f : [k] \to [n+m+1]$ such that $\mathrm{Im}(f) \cap \{0, ..., m\} \neq \{0, ..., m\}$. Alternatively,
$$(\partial\Delta^m \star \Delta^n)_k = \Delta_k^{m+n+1} - \{f : [k] \to [m+n+1] | \{0, ..., m\} \subseteq \mathrm{Im}(f)\}.$$
2. $\Lambda_j^n \star \Delta^m$. A k-simplex of $\Lambda_j^n \star \Delta^m$ is a monotone function $f : [k] \to [n+m+1]$ such that there exists $0 \leq l \leq n, l \neq j$ such that $l \notin \mathrm{Im}(f) \cap \{0, ..., n\}$. Alternatively,
$$(\Lambda_j^n \star \Delta^m)_k = \Delta_k^{m+n+1} - \{f : [k] \to [m+n+1] | \{0, ..., \hat{j}, ..., n\} \subseteq \mathrm{Im}(f)\}.$$
3. $\Delta^n \star \partial\Delta^m$. A k-simplex of $\Delta^n \star \partial\Delta^m$ is a monotone function $f : [k] \to [m+n+1]$ such that $\mathrm{Im}(f) \cap \{n+1, ..., m+n+1\} \neq \{n+1, ..., m+n+1\}$. Alternatively,
$$(\Delta^n \star \partial\Delta^m)_k = \Delta_k^{m+n+1} - \{f : [k] \to [m+n+1] | \{n+1, ..., m+n+1\} \subseteq \mathrm{Im}(f)\}.$$
4. $\Delta^m \star \Lambda_j^n$. A k-simplex of $\Delta^m \star \Lambda_j^n$ is a monotone function $f : [k] \to [m+n+1]$ such that there exists $m+1 \leq l \leq n+m+1, l \neq j$ such that $l \notin \mathrm{Im}(f) \cap \{m+1, ..., m+n+1\}$. Alternatively,
$$(\Delta^m \star \Lambda_j^n)_k = \Delta_k^{m+n+1} - \{f : [k] \to [m+n+1] | \{m+1, ..., \widehat{j+m+1}, ..., n+m+1\} \subseteq \mathrm{Im}(f)\}.$$

Note that for two simplicial sets S, T we have $(S \cup T)_k = S_k \cup T_k$. By directly compare the set of k-simplices we get the desired result. For example
$$\begin{aligned}
&(\Lambda_j^n \star \Delta^m \cup \Delta^n \star \partial\Delta^m)_k \\
=&\Delta_k^{m+n+1} - \{f | \{0, ..., \hat{j}, ..., n\} \subseteq \mathrm{Im}(f)\} \cap \{f | \{n+1, ..., m+n+1\} \subseteq \mathrm{Im}(f)\} \\
=&\Delta_k^{m+n+1} - \{f | \{0, ..., \hat{j}, ..., n, n+1, ..., n+m+1\} \subseteq \mathrm{Im}(f)\} \\
=&(\Lambda_j^{n+1+m})_k
\end{aligned}$$

□

Corollary 2.50. *Let* $A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$ *be a sequence in* sSet *where* $i \in$ Mon, $f \in r(\Lambda^{in})$. *Then*

1. *The induced map*
$$\langle i, f \rangle_{\varphi/} : X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$
is a left fibration;
2. *If f is a left fibration, then*
$$\langle i, f \rangle_{/\varphi} : X_{/\varphi} \to X_{/\varphi i} \times_{Y_{/f\varphi i}} Y_{/f\varphi}$$
is a left fibration;
3. *If $i : A \to B$ is a right anodyne, then*
$$\langle i, f \rangle_{\varphi/} : X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$
is a trivial Kan fibration;

4. If $f : X \to Y$ is a trivial Kan fibration, then

$$\langle i, f\rangle_{\varphi/} : X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$

is a trivial Kan fibration.

Proof. We only prove statement 1 as an example. To show that $\langle i, f\rangle_{\varphi/}$ is a left fibration, it suffices to show that, for all $j \in l(r(\Lambda^L))$, $j \boxslash \langle i, f\rangle_{\varphi/}$, which is equivalent to $i \hat{\star} j \boxslash_{\varphi/} f$. Since $i \hat{\star} j$ is an inner anodyne, f is an inner fibration, we get the desired result. □

Corollary 2.51. *Let $A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$ be a sequence in* sSet*. Assume that i is a monomorphism, X is an ∞-category and $Y = \Delta^0$ so that f is an inner fibration. Then*

$$\langle i, f\rangle_{\varphi/} : X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/} = X_{\varphi i/}$$

is a left fibration. In particular, taking $A = \emptyset$, we could see that the canonical projection map $X_{\varphi/} \to X$ is a left fibration. Likewise, the morphism

$$\langle i, f\rangle_{/\varphi} : X_{/\varphi} \to X_{/\varphi i}$$

is a right fibration, and the projection $X_{/\varphi} \to X$ is a right fibration.

Corollary 2.52. *Let $\mathcal{C}$ be an ∞-category and $\varphi : K \to \mathcal{C}$ be a morphism of simplicial sets. Then $\mathcal{C}_{/\varphi}$ and $\mathcal{C}_{\varphi/}$ are ∞-categories.*

2.4 Homotopy theory of simplicial sets

Definition 2.53. A **homotopy** between two maps $f, g : X \to Y$ is a map $\phi : X \times \Delta^1 \to Y$ such that $\phi|_{X\times\{0\}} = f$ and $\phi|_{X\times\{1\}} = g$. We will use the notation $\phi : f \Rightarrow g$. We say that f **is homotopic to** g if there exists a homotopy $\phi : f \Rightarrow g$. A morphism $f : X \to Y$ is a **homotopy equivalence** if there exists $g : Y \to X$ and homotopies $\alpha : f \circ g \Rightarrow \mathrm{Id}_Y$ and $\beta : g \circ f \Rightarrow \mathrm{Id}_X$.

Remark 2.54. If we view f and g as two vertexes of the simplicial set $\mathrm{Fun}(X, Y)$, then a homotopy $\phi : f \Rightarrow g$ is nothing but an edge in $\mathrm{Fun}(X, Y)$.

Lemma 2.55. *If Y is a Kan complex, then the relation of homotopy defines an equivalence relation on $\hom_{\mathrm{sSet}}(X, Y)$ for any X. We write $[f]$ for the homotopy class of f.*

Proof. Note that $\mathrm{Fun}(X, Y)$ is a Kan complex, hence an ∞-groupoid. A morphism of an ∞-groupoid admits an inverse, and two morphisms $\phi : f \Rightarrow g$, $\psi : g \Rightarrow h$ can be composed. □

Lemma 2.56. *A homotopy equivalence $f : X \to Y$ of Kan complexes induces a bijection $\pi_0(X) \to \pi_0(Y)$.*

Proof. The key point is to notice that homotopic morphisms will induce the same morphism after applying $\pi_0(-)$. To see this, use the fact that $\pi_0(-)$ preserves products, hence $\pi_0(\Delta^1 \times X) \cong \pi_0(X)$. As a result, let $\phi : \Delta^1 \times X \to Y$ be a homotopy from f to g, then both $\pi_0(f)$ and $\pi_0(g)$ can be identified with $\pi_0(\phi)$.

Suppose $f : X \to Y$ is a homotopy equivalence, then we can choose a homotopy inverse g. Then $\pi_0(f)$ and $\pi_0(g)$ are inverse to each other. □

Proposition 2.57. *Let $f : X \to Y$ be a morphism between Kan complexes. Then TFAE:*

(1) f is a homotopy equivalence.

(2) For all Kan complexes Z, the induced map $f^* : \mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$ is a homotopy equivalence.

(3) For all Kan complexes Z, the induced map $\pi_0(f^*) : \pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$ is a bijection.

Proof. First we show (1) $\Rightarrow$ (2). Since f is a homotopy equivalence, we can choose a homotopy inverse $g : Y \to X$. We claim that $g^* : \mathrm{Fun}(X, Z) \to \mathrm{Fun}(Y, Z)$ is a homotopy inverse of f^*. To see this, let $\phi : X \times \Delta^1 \to X$ be the homotopy $g \circ f \Rightarrow \mathrm{Id}_X$. Then ϕ induces a map $\phi^* : \mathrm{Fun}(X, Z) \to \mathrm{Fun}(X \times \Delta^1, Z) \cong \mathrm{Fun}(\Delta^1, \mathrm{Fun}(X, Z))$. By adjunction we get a map $\Delta^1 \times \mathrm{Fun}(X, Z) \to \mathrm{Fun}(X, Z)$, which gives a homotopy $f^* \circ g^* \Rightarrow \mathrm{Id}_{\mathrm{Fun}(X,Z)}$. Similarly, a homotopy $\psi : f \circ g \Rightarrow \mathrm{Id}_Y$ induces a homotopy from $g^* \circ f^*$ to $\mathrm{Id}_{\mathrm{Fun}(Y,Z)}$.

The implication (2) $\Rightarrow$ (3) follows from Lemma 2.56. It remains to show that (3) implies (1). Take $Z = X$ in condition (3), we find a unique (up to homotopy) map $g : Y \to X$ such that $[g \circ f] = [\mathrm{Id}_X]$. Then take $Z = X$, we see that both $[\mathrm{Id}_Y]$ and $[f \circ g]$ are sent to $[f]$, so that $[\mathrm{Id}_Y] = [f \circ g]$. As a result, g is a homotopy inverse of g, hence f is a homotopy equivalence. $\square$

Lemma 2.58. *Let $f : X \to Y$ be a trivial Kan fibration between Kan complexes. Then f is a homotopy equivalence.*

Proof. Choose a section $s : Y \to X$ of f, so that $f \circ s = \mathrm{Id}_Y$. We wish to show that there is a homotopy between $s \circ f$ and Id_X. To do so, consider the following commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "(s\circ f, \mathrm{Id}_X)"] \arrow[d] & X \arrow[d, "f"] \\ X \times \Delta^1 \arrow[r, "f\circ\pi_X"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

which admits a solution h. Then h gives the desired homotopy. $\square$

We denote the remaining part of this section to the study of weak homotopy equivalences of simplcial sets.

Definition 2.59. We say that a map $f : X \to Y$ of simplicial sets is a **weak homotopy equivalence** if for all Kan complexes K the map of Kan complexes $f^* : \mathrm{Fun}(Y, K) \to \mathrm{Fun}(X, K)$ is a homotopy equivalence.

Example 2.60. Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a weak homotopy equivalence if and only if f is a homotopy equivalence.

Lemma 2.61. *Any anodyne map is a weak homotopy equivalence.*

Proof. Let $f : X \to Y$ be a anodyne map, then for a Kan complex K, the induced map

$$f^* : \mathrm{Fun}(Y, K) \to \mathrm{Fun}(X, K)$$

is a trivial Kan fibration, hence a homotopy equivalence. $\square$

We are going to introduce the following theorems without proof.

Theorem 2.62. *A map $f : X \to Y$ between Kan complexes is a weak homotopy equivalence if and only for all vertexes $x \in X$ and all integers $n \geq 0$, the induced map on homotopy groups*

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism.

2.5 Joyal's horn lifting theorem

Definition 2.63. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We say that F is **conservative** if F reflects equivalences. That is $F(f)$ is an equivalence implies that f is an equivalence.

Proposition 2.64. *Left and right fibrations between ∞-categories are conservative.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left fibration. Let $f \in \mathcal{C}$ be a morphism such that $F(f)$ is an equivalence. Then we have the following lifting problem, which admits a solution σ

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(f,\mathrm{Id}_x)}"] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed, "\sigma"] & \mathcal{D} \end{tikzcd}$$

Let $g = d_0(\sigma)$, and σ witnesses g as a left homotopy inverse of f. Since f is arbitrary, we conclude that every morphism l in $\mathcal{C}$ such that $F(l)$ is an equivalence admits a left homotopy inverse. Since $F(g)$ is an equivalence(it is the homotopy inverse of $F(f)$), we see that g again has a left homotopy inverse, which we denote by $\tilde{f}$. Then

$$\tilde{f} \simeq \tilde{f} \circ g \circ f \simeq f$$

This shows that f and g are homotopy inverse to each other. $\square$

Definition 2.65. An inner fibration $F : \mathcal{C} \to \mathcal{D}$ of ∞-categories is called an **isofibration** if every lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & \mathcal{D} \end{tikzcd}$$

where f is an equivalence in $\mathcal{D}$ has a solution $\tilde{f}$ which is again an equivalence in $\mathcal{C}$.

We omit the proof of the following lemma:

Lemma 2.66. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories. Then F is an isofibration if and only if the induced functor of homotopy categories $hF : h\mathcal{C} \to h\mathcal{D}$ is an isofibration of ordinary categories.*

Corollary 2.67. *$F : \mathcal{C} \to \mathcal{D}$ is an isofibration iff $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an isofibration.*

Proposition 2.68. *Left and right fibrations between ∞-categories are conservative isofibrations.*

Theorem 2.69 (Joyal). *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories and let $\varphi : x \to y$ be a morphism in $\mathcal{C}$ such that $F(\varphi)$ is an equivalence in $\mathcal{D}$. Then TFAE*

(1) φ is an equivalence in $\mathcal{C}$.

(2) $\forall n \geq 2$, any lifting problem illustrated as follows

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "\varphi"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

can be solved.

Proof. We first show the implication (1) $\Rightarrow$ (2). Let $i : \Lambda^1_0 \to \Delta^1$, and $j : \partial\Delta^{n-2} \to \Delta^{n-2}$ be the inclusions. Then $i \hat{\star} j$ is the left horn inclusion $\Lambda^n_0 \to \Delta^n$ by Lemma 2.49. To solve the lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "\varphi"] & \Lambda^n_0 \arrow[r, "u"] \arrow[d] & \mathcal{C} \arrow[d] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

is equivalent to solving the following lifting problem

$$\begin{tikzcd}
0 \arrow[rr] \arrow[d] & & \mathcal{C}_{/\Delta^{n-2}} \arrow[d, "p"] \\
\Delta^1 \arrow[urr, dashed] \arrow[r, "\varphi'"] \arrow[drr, "\varphi"'] & \mathcal{C}_{/\partial\Delta^{n-2}} \times_{\mathcal{D}_{/\partial\Delta^{n-2}}} \mathcal{D}_{/\Delta^{n-2}} \arrow[d, "q"] \\
& & \mathcal{C}
\end{tikzcd}$$

where the morphism $\Delta^{n-2} \to \mathcal{C}$ is given by $u|_{\{\Delta^{2,\ldots,n}\}}$ and the morphism q is obtained as the composition of two right fibrations $\mathcal{C}_{/\partial\Delta^{n-2}} \times_{\mathcal{D}_{/\partial\Delta^{n-2}}} \mathcal{D}_{/\Delta^{n-2}} \to \mathcal{C}_{/\partial\Delta^{n-2}} \to \mathcal{C}$. The two right vertical arrows p and q are both right fibrations, and hence are conservative isofibrations. By assumption φ is an equivalence, so φ' is again an equivalence. Since p is an isofibration, this lifting problem can be solved.

Now we show (2) $\Rightarrow$ (1). Let Let g be an inverse of $F(\varphi)$. Choose a 2-simplex $\sigma : \Delta^2 \to \mathcal{D}$ exhibiting g as the inverse of $F(\varphi)$, so that $\sigma|_{\Delta^{\{0,1\}}} = F(\varphi)$, $\sigma|_{\Delta^{\{0,2\}}} = \mathrm{Id}_{F(x)}$ and $\sigma|_{\Delta^{\{1,2\}}} = g$. Then we have a commutative diagram:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\varphi"] & \Lambda^2_0 \arrow[r, "\rho"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
& \Delta^2 \arrow[r, "\sigma"'] \arrow[ur, dashed, "\delta"] & \mathcal{D}
\end{tikzcd}$$

where $\rho|_{\Delta^{\{0,1\}}} = \varphi$ and $\rho|_{\Delta^{\{0,2\}}} = \mathrm{Id}_x$. By our assumption this lifting problem admits a solution δ, so that $h := \delta|_{\Delta^{\{1,2\}}}$ is a left inverse of φ. Note that $F(h) = g$. We wish to show that h is also a right inverse of φ. Consider $\kappa : \Lambda^3_0 \to \mathcal{C}$ illustrated as follows:

$$\begin{tikzcd}
& y \arrow[rrd, "\mathrm{Id}"] \arrow[rdd, "h"] & & \\
& & & y \\
x \arrow[ruu, "\varphi"] \arrow[rr, "\mathrm{Id}"'] \arrow[rrru, dashed, "\varphi"] & & x \arrow[ru, "\varphi"'] &
\end{tikzcd}$$

where $\kappa|_{\Delta^{\{0,1,2\}}} = \delta$, $\kappa|_{\Delta^{\{0,2,3\}}} = s_0(\varphi)$ and $\kappa|_{\Delta^{\{0,1,3\}}} = s_1(\varphi)$. Note that the first edge of $F \circ \kappa : \Lambda^3_0 \to \mathcal{D}$ is $F(\varphi)$, which is an equivalence. Since we have shown (1) $\Rightarrow$ (2), we can apply it to the inner fibration $\mathcal{D} \to \Delta^0$ and the equivalence $F(\varphi)$. As a result we can extend $F \circ \kappa$ to a 3-simplex

$\theta : \Delta^3 \to \mathcal{D}$. So we get a lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\varphi"] & \Lambda^3_0 \arrow[r, "\kappa"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ & \Delta^3 \arrow[r, "\theta"'] \arrow[ur, dashed, "\tilde{\kappa}"] & \mathcal{D} \end{tikzcd}$$

which admits a solution $\tilde{\kappa}$ by our assumption. Then the 2-simplex $\tilde{\kappa}|_{\Delta^{\{1,2,3\}}}$ exhibits h as a right inverse of φ. □

Remark 2.70. The dual version of Joyal's horn lifting theorem remains true: Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories and let $\varphi : \Delta^1 \to \mathcal{C}$ be an morphism in $\mathcal{C}$ such that $F(\varphi)$ is an equivalence in $\mathcal{D}$. Then φ is an equivalence if and only if $\forall n \geq 2$, any lifting problem illustrated as follows

$$\begin{tikzcd} \Delta^{\{n,n-1\}} \arrow[r, hook] \arrow[rr, bend left, "\varphi"] & \Lambda^n_n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

can be solved.

Corollary 2.71. *An inner fibration $p : \mathcal{C} \to \mathcal{D}$ between ∞-categories is conservative if and only if for every $n \geq 2$ and every lifting problem*

$$\begin{tikzcd} \Delta^{0,1} \arrow[r, hook] \arrow[rr, bend left, "\varphi"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

where $p(\varphi)$ is an equivalence in $\mathcal{D}$ can be solved.

Proof. If p is conservative, then $p(\varphi)$ is an equivalence implies that φ is an equivalence. So the desired lifting problem can be solved by Theorem 2.69.

Conversely, take $n = 2$, every lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\varphi"] & \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ & \Delta^2 \arrow[r] \arrow[ur, dashed, "\sigma"] & \mathcal{D} \end{tikzcd}$$

where $p(\varphi)$ is an equivalence can be solved. Take a morphism f of $\mathcal{C}$ such that $p(f)$ is an equivalence in $\mathcal{D}$, and our purpose is to show that f is itself an equivalence. Take $(f, \mathrm{Id}) : \Lambda^2_0 \to \mathcal{C}$, we see that $g = d_0(\sigma)$ gives a left inverse of f. Now repeat what we have done to g we see that g again has a left inverse, so g and f are homotopy inverse to each other. □

Corollary 2.72. *Conservative maps are stable under pullbacks and co-transfinite compositions.*

Proposition 2.73. *Let $\mathcal{C}$ be an ∞-category. Then TFAE:*

(1) $\mathcal{C}$ *is an* ∞*-groupoid;*

(2) The unique map $\mathcal{C} \to \Delta^0$ *is conservative;*

(3) $\mathcal{C} \to \Delta^0$ *is a left fibration;*

(4) $\mathcal{C} \to \Delta^0$ *is a right fibration;*

(5) $\mathcal{C} \to \Delta^0$ *is a Kan fibration.*

Proof. (1) $\Leftrightarrow$ (2) is obvious; (3), (4) $\Rightarrow$ (2) is Proposition 2.68; (2) $\Rightarrow$ (3), (4) by Corollary 2.71; (3) + (4) $\Leftrightarrow$ (5) by definition. □

2.6 The core of an ∞-category

Let $\mathcal{C}$ be a 1-category. Then we can obtain a goupoid $\mathcal{C}^{\simeq}$ by discarding all the non-invertible morphisms, which is called the **core** of $\mathcal{C}$. In this section we spell a similar construction for ∞-categories.

Definition 2.74. Let $\mathcal{C}$ be an ∞-category. We let $\mathcal{C}^{\simeq}$ be the subcomplex of $\mathcal{C}$ defined by

$$\mathcal{C}^{\simeq}_n := \{x \in \mathcal{C}_n | \text{every edge of } x \text{ is an equivalence in } \mathcal{C}\}.$$

Note that $\mathcal{C}^{\simeq}$ contains all objects of $\mathcal{C}$.

Proposition 2.75. *Let* $\mathcal{C}$ *be an* ∞*-category, let* $h\mathcal{C}$ *be its homotopy category. Then the core* $\mathcal{C}^{\simeq}$ *fits into a pullback diagram as follows:*

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ N((h\mathcal{C})^{\simeq}) \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

Proposition 2.76. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an isofibration of* ∞*-categories. Then the induced map* $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ *is a Kan fibration.*

Proof. We need to show that for all $n > 0$ and all $0 \leq i \leq n$; we wish to show that every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d, "F^{\simeq}"] \\ \Delta^n \arrow[r, "\overline{\sigma}"'] \arrow[ur, dashed, "\sigma"] & \mathcal{D}^{\simeq} \end{tikzcd}$$

admits a solution. When $n = 1$, we get the result directly from the definition of an isofibration and Corollary 2.67. Now assume $n \geq 2$. We claim that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ (not $\mathcal{C}^{\simeq}$!) satisfying $F(\sigma) = \overline{\sigma}$. If $0 < i < n$, this follows from the fact that F is an inner fibration. The extremal cases $i = 0$ and $i = n$ follows from Joyal's special horn lifting theorem. To complete the proof, it will suffice to show that, σ carries each edge of Δ^n to an isomorphism in $\mathcal{C}$. For $n > 2$, this is automatic. In the case $n = 2$ we apply the 2-out-of-3 property for equivalences in $\mathcal{C}$. □

Corollary 2.77. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor of* ∞*-categories, where* $\mathcal{D}$ *is an* ∞*-groupoid. The following conditions are equivalent:*

1. *F is a Kan fibration;*

2. *F is a left fibration;*

3. *F is a right fibration;*

4. *F is conservative isofibration of ∞-categories.*

Proof. It suffice to show the implication $4 \Rightarrow 1$. Since $\mathcal{D}$ is an ∞-groupoid, we have $\mathcal{D} = \mathcal{D}^{\simeq}$. Since F is conservative, we see that every morphism of $\mathcal{C}$ is an equivalence, so that $\mathcal{C} = \mathcal{C}^{\simeq}$. Since F is an isofibration, we see that $F^{\simeq} = F$ is a Kan fibration by Proposition 2.76. □

Corollary 2.78. *Let $\mathcal{C}$ be an ∞-category. Then the core $\mathcal{C}^{\simeq}$ is an ∞-groupoid.*

Proof. Apply Proposition 2.76 to the isofibration $\mathcal{C} \to \Delta^0$. □

By our definition, $\mathcal{C}^{\simeq}$ is the largest Kan complex contained in $\mathcal{C}$, so that any morphism $X \to \mathcal{C}$ whose source is a Kan complex must factor through $\mathcal{C}^{\simeq}$.

Proposition 2.79. *Let $\mathcal{C}$ be an ∞-category and let X be a Kan complex. Then the canonical morphism*

$$\theta : \mathrm{Fun}(X, \mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Fun}(X, \mathcal{C})^{\simeq}$$

is an isomorphism of simplicial sets.

Proof. Let $\sigma : Y \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$ be a morphism of simplicial sets, which we identify with a diagram $F : X \times Y \to \mathcal{C}$. To show that σ factor through the monomorphism θ, it will suffice to show that F factors through the core $\mathcal{C}^{\simeq}$. Equivalently, we wish to show that for every edge $(u, v) : (x, y) \to (x', y')$ in the product simplicial set $X \times Y$, the morphism $F(u, v) : F(x, y) \to F(x', y')$ is an isomorphism in the ∞-category $\mathcal{C}$. Note that $F(u, v)$ can be identified with a composition of morphisms

$$F(x, y) \xrightarrow{F(u, \mathrm{Id}_y)} F(x', y) \xrightarrow{F(\mathrm{Id}_{x'}, v)} F(x', y')$$

Here $F(u, \mathrm{Id}_y)$ is an isomorphism in $\mathcal{C}$ since $F(F(u, \mathrm{Id}_y)) = \sigma(y)(u)$ is the image of an isomorphism in X. $F(\mathrm{Id}_{x'}, v)$ is an isomorphism in $\mathcal{C}$ since by our assumption that σ factors through $\mathrm{Fun}(X, \mathcal{C})^{\simeq}$. □

2.7 Equivalences of ∞-categories

Definition 2.80. We say that a simplicial category $\mathcal{C}$ is **fibrant** is for each pair of objects $x, y \in \mathcal{C}$, the associated simplicial set $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex.

Proposition 2.81. *Let $\mathcal{C}$ a fibrant simplicial category. Then the homotopy coherent nerve $N(\mathcal{C})$ is an ∞-category.*

Construction 2.82 (The ∞-category of spaces)**.** Let $\mathcal{K}\mathrm{an}$ be the full subcategory of the simplicial category sSet spanned by Kan complexes. Then $\mathcal{K}\mathrm{an}$ is a fibrant simplicial category: according to Corollary 2.40, if Y is a Kan complex then so is the function complex $\mathrm{Fun}(X, Y)$. We define a new ∞-category

$$\mathcal{S} := N(\mathcal{K}\mathrm{an}).$$

We will refer to $\mathcal{S}$ as **the ∞-category of spaces**.

Example 2.83. An equivalence in $\mathcal{S}$ is a homotopy equivalence of Kan complexes.

Construction 2.84 (The ∞-category of all ∞-categories)**.** Let QCat^l be the full subcategory of the simplicial category sSet spanned by ∞-categories. Let $\mathrm{QCat} \subseteq \mathrm{QCat}^l$ be the wide subcategory whose mapping spaces are determined by

$$\mathrm{Map}_{\mathrm{QCat}}(x, y) := \mathrm{Map}_{\mathrm{QCat}^l}(x, y)^{\simeq}$$

Then QCat is a fibrant simplicial category. We define a new ∞-category

$$\mathrm{Cat}_\infty := N(\mathrm{QCat}).$$

We will refer to Cat_∞ as **the ∞-category of all ∞-categories**.

The canonical inclusion $\mathcal{K}\mathrm{an} \hookrightarrow \mathrm{QCat}$ induces a monomorphism $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$.

Example 2.85. An object of Cat_∞ is an ∞-category. An edge of Cat_∞ is a functor $F : \mathcal{C} \to \mathcal{D}$. A 2-simplex of Cat_∞ consists of three functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, $H : \mathcal{C} \to \mathcal{E}$ together with a natural isomorphism[2] $\alpha : G \circ F \Rightarrow H$. As a result, two functors $F, F' : \mathcal{C} \to \mathcal{D}$ are homotopic in Cat_∞ if and only if there is a natural isomorphism $\theta : F \Rightarrow F'$, if and only if F and F' lie in the connected component of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$.

Let $h\mathrm{Cat}_\infty$ be the homotopy category of Cat_∞. The objects of $h\mathrm{Cat}_\infty$ are ∞-categories, and the hom sets are described by

$$\hom_{h\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) = \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}).$$

The canonical morphism $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ induces a fully faithful embedding $h\mathcal{S} \hookrightarrow h\mathrm{Cat}_\infty$.

Proposition 2.86. *The canonical embedding $h\mathcal{S} \hookrightarrow h\mathrm{Cat}_\infty$ admits a right adjoint $(-)^{\simeq} : h\mathrm{Cat}_\infty \to h\mathcal{S}$ which sends $\mathcal{C}$ to $\mathcal{C}^{\simeq}$ and F to $F^{\simeq}$.*

Proof. Let X be a Kan complex and $\mathcal{C}$ be an ∞-category. According to Proposition 2.79, composition with $\iota : \mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ induces an isomorphism of Kan complexes $\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \cong \mathrm{Fun}(X, \mathcal{C})^{\simeq}$. As a result we have

$$\hom_{h\mathcal{S}}(X, \mathcal{C}^{\simeq}) = \pi_0(\mathrm{Fun}(X, \mathcal{C}^{\simeq})) \cong \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq}) \cong \hom_{h\mathrm{Cat}_\infty}(X, \mathcal{C}).$$

□

Definition 2.87. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that F is **an equivalence of ∞-categories** if F is an equivalence in Cat_∞.

Remark 2.88. To show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, one needs to find:

- a functor $G : \mathcal{D} \to \mathcal{C}$;
- a natural isomorphism $\alpha : \mathrm{Id}_{\mathcal{C}} \simeq G \circ F$;
- a natural isomorphism $\beta : \mathrm{Id}_{\mathcal{D}} \simeq F \circ G$.

Proposition 2.89. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then TFAE:*

(1) F is an equivalence of ∞-categories.

(2) For any ∞-category $\mathcal{E}$, the induced map

$$[F] \circ - : \pi_0(\mathrm{Fun}(\mathcal{E}, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{E}, \mathcal{D})^{\simeq})$$

is a bijection.

[2] By which we mean an equivalence is the functor ∞-category.

(3) For any ∞-category $\mathcal{E}$, the induced map

$$-\circ[F]:\pi_0(\mathrm{Fun}(\mathcal{D},\mathcal{E})^{\simeq})\to\pi_0(\mathrm{Fun}(\mathcal{C},\mathcal{E})^{\simeq})$$

is a bijection.

(4) For any ∞-category $\mathcal{E}$, the induced map

$$F_*:\mathrm{Fun}(\mathcal{E},\mathcal{C})\to\mathrm{Fun}(\mathcal{E},\mathcal{D})$$

is an equivalence.

(5) For any ∞-category $\mathcal{E}$, the induced map

$$F^*:\mathrm{Fun}(\mathcal{D},\mathcal{E})\to\mathrm{Fun}(\mathcal{C},\mathcal{E})$$

is an equivalence.

Proof. First we demonstrate the implications (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3). By definition $F:\mathcal{C}\to\mathcal{D}$ is an equivalence of ∞-categories if and only if $[F]:\mathcal{C}\to\mathcal{D}$ is an isomorphism in $h\mathrm{Cat}_\infty$. This is equivalent to both (2) and (3) by the 1-categorical Yoneda lemma.

The implications (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3) are easy.

Now we show (1) $\Rightarrow$ (5). Let $G:\mathcal{D}\to\mathcal{C}$ the homotopy inverse of F, and $\alpha:\mathrm{Id}_\mathcal{C}\Rightarrow G\circ F$, $\beta:\mathrm{Id}_\mathcal{D}\Rightarrow F\circ G$ be natural isomorphisms. Then $G^*:\mathrm{Fun}(\mathcal{C},\mathcal{E})\to\mathrm{Fun}(\mathcal{D},\mathcal{C})$ is the homotopy inverse of F^*, witnessed by α^* and β^*. The implication (1) $\Rightarrow$ (4) is similar. $\square$

2.8 Mapping spaces and pointwise criterion for natural isomorphisms

Definition 2.90. Let $\mathcal{C}$ be an ∞-category and $(x,y)\in\mathcal{C}\times\mathcal{C}$. The **mapping space** between x and y is defined by the pullback:

$$\begin{tikzcd}
\mathrm{Map}_\mathcal{C}(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "{(x,y)}"'] & \mathcal{C}\times\mathcal{C}
\end{tikzcd}$$

One of our purposes in this section is to show that $\mathrm{Map}_\mathcal{C}(x,y)$ is Kan complex, thus justifying the name "mapping space".

Lemma 2.91. *Let $K\in\mathrm{sSet}$, and $\mathrm{sk}_0(K)=\coprod_{K_0}\Delta^0$ be the simplicial subset generated by K_0. Then the restriction functor*

$$\mathrm{Fun}(K,\mathcal{C})\to\mathrm{Fun}(\coprod_{K_0}\Delta^0,\mathcal{C})\cong\prod_{K_0}\mathcal{C}$$

is conservative.

Proof. Recall that we have a skeletal filtration on $K=\varinjlim_n\mathrm{sk}_n(K)$. Apply the functor $\mathrm{Fun}(-,\mathcal{C})$ we get

$$\mathrm{Fun}(K,\mathcal{C})\cong\varprojlim_n\mathrm{Fun}(\mathrm{sk}_n(K),\mathcal{C})$$

Note that all connecting maps $\mathrm{Fun}(\mathrm{sk}_n(K),\mathcal{C})\to\mathrm{Fun}(\mathrm{sk}_{n-1},\mathcal{C})$ are inner fibrations.

First recall that conservative maps are stable under co-transfinite compositions, which is proved in Corollary 2.72. As a result, it suffices to show that the connecting maps $\mathrm{Fun}(\mathrm{sk}_n(K),\mathcal{C})\to$

$\mathrm{Fun}(\mathrm{sk}_{n-1}, \mathcal{C})$ are conservative. By applying $\mathrm{Fun}(-, \mathcal{C})$ to the diagram in Proposition 1.13, we obtain a pullback diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{sk}_n(K), \mathcal{C}) \arrow[r] \arrow[d] & \prod \mathrm{Fun}(\Delta^n, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(\mathrm{sk}_{n-1}(K), \mathcal{C}) \arrow[r] & \prod \mathrm{Fun}(\partial\Delta^n, \mathcal{C}) \end{tikzcd}$$

Since conservative maps are stable under pullbacks, it suffices to show that each $\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^n, \mathcal{C})$ is conservative.

First deal with the case $n \geq 2$. Consider the composite

$$\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Sp}(n), \mathcal{C})$$

which is trivial fibration and hence conservative. It follows that $\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^n, \mathcal{C})$ is also conservative. We left the case $n = 1$ to the following proposition. ☐

Proposition 2.92. *Let $\mathcal{C}$ be an ∞-category. Then the restriction functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \cong \mathcal{C} \times \mathcal{C}$ is conservative.*

Proof. It suffices to show that any lifting problem of the form

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(f,\mathrm{Id})}"] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"'] \arrow[ur, dashed] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \end{tikzcd}$$

where f is sent to an equivalence in $\mathrm{Fun}(\partial\Delta^1, \mathcal{C})$, admits a solution. Via adjunction we get an equivalent lifting problem

$$\begin{tikzcd} \Lambda^2_0 \times \Delta^1 \cup \Delta^2 \times \partial\Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^2 \times \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

where $\Delta^{\{0,1\}} \times \Delta^1 \subset \Lambda^2_1 \times \Delta^1$ are sent to equivalences in $\mathcal{C}$. To solve this lifting problem, we construct a filtration

$$\Lambda^2_0 \times \Delta^1 \cup \Delta^2 \times \partial\Delta^1 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 = \Delta^2 \times \Delta^1.$$

where each $F_i \subseteq F_{i+1}$ is obtained by pushing out an inner horn or a left horn whose first edge is an equivalence in $\mathcal{C}$, hence the lifting problem can be solved. Details are described as follows

- We write (i, j) with $i = 0, 1, 2$ and $j = 0, 1$ for a vertex in the simplicial set $\Delta^2 \times \Delta^1$.
- F_1 is obtained from F_0 by glueing a 2-cell along the inner horn specified by $(1, 0) \to (2, 0) \to (2, 1)$.
- F_2 is obtained from F_1 by glueing a 3-cell along the inner horn specified by $(0, 0) \to (1, 0) \to (2, 0) \to (2, 1)$.
- F_3 is obtained from F_2 by glueing a 3-cell along the inner horn specified by $(0, 0) \to (0, 1) \to (1, 1) \to (2, 1)$.
- F_4 is obtained from F_3 by glueing a 3-cell along the left horn specified by $(0, 0) \to (1, 0) \to (1, 1) \to (2, 1)$. Note that first edge $(0, 0) \to (1, 0)$ is an equivalence!

The author doubt that the there is a mistake in the proof of Theorem 2.2.3 in [Lan21], although the author actually steals the proof there. It seems that the order of glueing the cells is wrong there. $\square$

Corollary 2.93. *Let $\mathcal{C}$ be an ∞-category, and $x, y \in \mathcal{C}$. Then the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex.*

Theorem 2.94. *Let $L \to K$ be a map in* sSet *which induces a bijection $L_0 \cong K_0$. Then, for every ∞-category $\mathcal{C}$, the induced functor*

$$\mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(L, \mathcal{C})$$

is conservative.

3 The fundamental theorem of ∞-categories

3.1 The underlying $h\mathcal{S}$-enriched category of an ∞-category

Definition 3.1. Let $\mathcal{C}$ be an ∞-category. We define an $h\mathcal{S}$-enriched category $\mathbf{h}\mathcal{C}$ as follows:

- The objects of $\mathbf{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- The hom-object is defined by $\hom_{\mathbf{h}\mathcal{C}}(x, y) := \mathrm{Map}_{\mathcal{C}}(x, y)$.
- For three objects x, y, z, the composition map $\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, z)$ is defined as follows: recall that we can choose a composition law : $c : \mathrm{Fun}(\Delta^1, \mathcal{C}) \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ and form the following commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "(\pi_1, \pi_3)"'] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

  By considering the induced map of fibres, we obtain a morphism $c_{x,y,z} : \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, z)$.
- The composition is associative, see Lemma 3.2.

We will refer to $\mathbf{h}\mathcal{C}$ as the underlying $h\mathcal{S}$-enriched category of $\mathcal{C}$.

Lemma 3.2. *The following diagram commutes in $h\mathcal{S}$:*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(z, w) \times \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r, "\mathrm{Id} \times c_{x,y,z}"] \arrow[d, "c_{y,z,w} \times \mathrm{Id}"'] & \mathrm{Map}_{\mathcal{C}}(z, w) \times \mathrm{Map}_{\mathcal{C}}(x, z) \arrow[d, "c_{x,z,w}"] \\
\mathrm{Map}_{\mathcal{C}}(y, w) \times \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r, "c_{x,y,w}"'] & \mathrm{Map}_{\mathcal{C}}(x, w)
\end{tikzcd}$$

Proof. Consider the following diagram in sSet:

$$\begin{tikzcd}
\Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} \arrow[d, "\sim"] & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \arrow[l] \arrow[d, "\sim"] \\
\Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] & \Delta^3 & \Delta^{\{0,2,3\}} \arrow[l] \\
\Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[u] & \Delta^{\{0,1,3\}} \arrow[u] & \Delta^{\{0,3\}} \arrow[l] \arrow[u]
\end{tikzcd}$$

where arrows labelled by "$\sim$" are inner anodyne. Now apply the contravariant functor $\mathrm{Fun}(-, \mathcal{C})$, we see that this functor sends all the inner anodyne maps to trivial fibrations. We conclude by taking the induced maps over the fibre $(x, y, z, w) \in \mathcal{C}^{\times 4}$. $\square$

3.2 Joyal equivalences

Definition 3.3. Let $f : X \to Y$ be a morphism of simplicial sets. We say that f is a **Joyal equivalence** if, for every ∞-category $\mathcal{C}$, the induced functor $f^* : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ is an equivalence of ∞-categories.

Example 3.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is a Joyal equivalence if and only if it is an equivalence.

Example 3.5. Let $F : X \to Y$ be a morphism between Kan complexes. Then F is a Joyal equivalence if and only if it is a homotopy equivalence.

Proposition 3.6. *Let $f : X \to Y$ be a morphism of simplicial sets. Then TFAE:*

(1) f is a Joyal equivalence.

(2) For every ∞-category $\mathcal{C}$, precomposition with f induces a homotopy equivalence of Kan complexes $\mathrm{Fun}(Y, \mathcal{C})^{\simeq} \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$.

(3) For every ∞-category $\mathcal{C}$, precomposition with f induces a bijection of connected components:

$$\pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq})$$

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We complete the proof by showing that (3) $\Rightarrow$ (1). Let $f : X \to Y$ be a morphism satisfying condition (3), we wish to show that for each ∞-category $\mathcal{C}$, the induced map $f^* : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ is an equivalence. According to Proposition 2.89, it suffices to show that for every ∞-category $\mathcal{D}$, the induced map

$$\theta : \pi_0(\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(X, \mathcal{C}))^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(Y, \mathcal{C}))^{\simeq})$$

is a bijection. By adjunction, the induced map θ is the same as

$$\pi_0(\mathrm{Fun}(X, \mathrm{Fun}(\mathcal{D}, \mathcal{C}))^{\simeq}) \to \pi_0(\mathrm{Fun}(Y, \mathrm{Fun}(\mathcal{D}, \mathcal{C}))^{\simeq})$$

which is a bijection by our assumption. $\square$

Proposition 3.7. *A trivial Kan fibration $f : X \to Y$ is a Joyal equivalence.*

Proof. Since $f : X \to Y$ is a trivial Kan fibration, the Kan complexes $\mathrm{Fun}_{/Y}(Y, X)$ and $\mathrm{Fun}_{/Y}(X, X)$ are contractible. Choose a 0-simplex $s \in \mathrm{Fun}_{/Y}(Y, X)$, so we have $f_*(s) = \mathrm{Id}_Y$. Then $s \circ f$ determines a 0-simplex in $\mathrm{Fun}_{/Y}(X, X)$, and therefore, there is a 1-simplex connecting $s \circ f$ and Id_X by contractibility. More explicitly, we can find a map

$$\Delta^1 \to \mathrm{Fun}_{/Y}(X, X)$$

whose restriction to 0 is $s \circ f$ and whose restriction to 1 is Id_X. For an arbitrary ∞-category, we can compose this map with the canonical map

$$\mathrm{Fun}_{/Y}(X, X) \to \mathrm{Fun}(X, X) \to \mathrm{Fun}(\mathrm{Fun}(X, \mathcal{C}), \mathrm{Fun}(X, \mathcal{C}))$$

Note that the second map sends $p : X \to X$ to $p^* : \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$. The resulting morphism

$$\Delta^1 \to \mathrm{Fun}_{/Y}(X, X) \to \mathrm{Fun}(\mathrm{Fun}(X, \mathcal{C}), \mathrm{Fun}(X, \mathcal{C}))^{\simeq}$$

determines a natural equivalence between $(s \circ f)^*$ and $\mathrm{Id}_{\mathrm{Fun}(X,\mathcal{C})}$. Since $(f \circ s)^* = \mathrm{Id}_{\mathrm{Fun}(Y,\mathcal{C})}$, we see that s^* and f^* determine equivalences of $\mathrm{Fun}(X, \mathcal{C})$ and $\mathrm{Fun}(Y, \mathcal{C})$, so that f is a Joyal equivalence. $\square$

Corollary 3.8. *An inner-anodyne map is a Joyal equivalence.*

Proof. Let $i : A \to B$ be an inner anodyne map. Then for any ∞-category $\mathcal{C}$, the induced map $i^* : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ is a trivial fibration, hence a Joyal equivalence. □

3.3 Fully faithful and essentially surjective functors

The fundamental theorem of 1-category theory says that, a functor is an equivalence if and only if it is fully faithful and essentially surjective. In this section we will give a similar theorem in the ∞-categorical context. We show that a functor between ∞-categories is a Joyal equivalence if and only if it is fully faithful and essentially surjective.

Definition 3.9. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **fully faithful** if, for each pair of objects $x, y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$ is a homotopy equivalence.

Definition 3.10. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **essentially surjective** in the induced functor $hF : h\mathcal{C} \to h\mathcal{D}$ is essentially surjective. In other words, if for every object $d \in \mathcal{D}$, there exists an object $c \in \mathcal{C}$ such that $F(c) \simeq d$.

Lemma 3.11. *Let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\alpha : F \Rightarrow G$ be a natural transformation. Then the diagram commutes in the homotopy category of Kan complexes $h\mathcal{S}$:*

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(F(x), F(y)) \arrow[d, "{[\alpha_y]\circ -}"] \\ \mathrm{Map}_{\mathcal{D}}(G(x), G(y)) \arrow[r, "{-\circ[\alpha_x]}"'] & \mathrm{Map}_{\mathcal{D}}(F(x), G(y)) \end{tikzcd}$$

Proposition 3.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a Joyal equivalence between ∞-categories. Then f is fully faithful and essentially surjective.*

Proof. Note that if G is an inverse of F, then $hG : h\mathcal{D} \to h\mathcal{C}$ is the inverse of hF, so that hF is an equivalence, hence F is essentially surjective. To show fully faithfulness, fix an inverse G of F. Then, for every pair of objects $x, y \in \mathcal{C}$, we get

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(x), F(y)) \to \mathrm{Map}_{\mathcal{C}}(GF(x), GF(y)) \to \mathrm{Map}_{\mathcal{D}}(FGF(x), FGF(y)).$$

Since the functor $G \circ F$ is naturally equivalent to $\mathrm{Id}_{\mathcal{C}}$ and the functor $F \circ G$ is naturally equivalent to $\mathrm{Id}_{\mathcal{D}}$, we see that both the first two maps and the latter two maps compose into an equivalence. This implies that the middle map is itself an equivalence and thus the first map is also an equivalence. □

Proposition 3.13. *Let $f : X \to Y$ be a map between Kan complexes which induces a bijection on path components. Then f is a homotopy equivalence if and only if, for all points $x \in X$ and all $n \geq 1$, the induced map*

$$f_* : \pi_n(X, x) \to \pi_n(Y, y)$$

is a bijection.

Proposition 3.14. *Suppose we are given a commutative diagram of Kan complexes*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "h"'] & S' \end{tikzcd}$$

where f and f' are Kan fibrations and h is a homotopy equivalence. Then the following conditions are equivalent:

(1) The morphism g is a homotopy equivalence.

(2) For each vertex $s \in S$ with image $s' = h(s)$ in S', the induced maps on fibres $g_s : X_s \to X'_s$ is a homotopy equivalence.

Corollary 3.15. *A fully faithful and essentially surjective functor $f : X \to Y$ between Kan complexes is a homotopy equivalence.*

Theorem 3.16. *A fully faithful and essentially surjective functors $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a Joyal equivalence.*

Theorem 3.17. *A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a Joyal equivalence if and only if it is fully faithful and essentially surjective.*

4 Fat joins, fat slices and mapping spaces (without detail)

This section will not include many details. This section is based on [Lan21].

Definition 4.1. Let $X, Y \in \text{Set}$ be simplicial sets. We define a new simplicial set $X \diamond Y$ to be the pushout

$$\begin{array}{ccc} X \times Y \times \partial\Delta^1 & \xrightarrow{\gamma} & X \coprod Y \\ \downarrow & & \downarrow \\ X \times Y \times \Delta^1 & \longrightarrow & X \diamond Y \end{array}$$

where γ sends the triple $(x, y, 0)$ to x and $(x, y, 1)$ to y. We will refer to $X \diamond Y$ as the **fat join** of X and Y.

Note that there exists a canonical projection map $X \diamond Y \to \Delta^1$.

Lemma 4.2. *Let X and Y be simplicial sets. Then there exists a canonical map $X \diamond Y \to X \star Y$ which commutes with the projection to Δ^1 and the inclusions of X and Y. This map is functorial in X and Y.*

Lemma 4.3. *For fixed Y, the association $X \mapsto X \diamond Y$ extends to a colimit-preserving functor $\text{sSet} \to \text{sSet}_{Y/}$. The same holds for $X \mapsto Y \diamond X$.*

Proposition 4.4. *The functors $Y \diamond - : \text{sSet} \to \text{sSet}$ and $- \diamond Y : \text{sSet} \to \text{sSet}$ both preserves Joyal equivalences.*

Proposition 4.5. *Let X, Y be simplicial sets. Then the canonical map $X \diamond Y \to X \star Y$ is a Joyal equivalence.*

Definition 4.6. Let $p : Y \to W$ be an object of $\text{sSet}_{Y/}$. We define the **fat slice** of p to be the simplicial set $W^{p/}$ by

$$(W^{p/})_n := \hom_{\text{sSet}_{Y/}}(Y \diamond \Delta^n, W),$$

and define the simplicial set $W^{/p}$ to be

$$(W^{/p})_n = \hom_{\text{sSet}_{Y/}}(\Delta^n \diamond Y, W)$$

Lemma 4.7. *The functor $\text{sSet}_{Y/} \to \text{sSet}$ given by sending $p : Y \to W$ to $W^{/p}$ is right adjoint to the functor $- \diamond Y$. Likewise, the functor $p \mapsto W^{p/}$ is right adjoint to the functor $Y \diamond -$.*

Proof. By definition, the adjunction bijection holds for representables, and hence for all simplicial sets, since $- \diamond Y$ and $Y \diamond -$ preserve colimits. $\square$

Lemma 4.8. *Let Y be a simplicial set and $p : Y \to W$ be a map of simplicial sets. Then there are canonical maps $W_{/p} \to W^{/p}$ and $W_{p/} \to W^{p/}$.*

Proof. On n-simplices, we have to provide a map

$$\hom_{\mathrm{sSet}_{Y/}}(Y \star \Delta, W) \to \hom_{\mathrm{sSet}_{Y/}}(Y \diamond \Delta, W)$$

And we recall that there is a canonical map $Y \diamond \Delta^n \to Y \star \Delta^n$ in $\mathrm{sSet}_{Y/}$. □

Lemma 4.9. *Let*

$$S \xrightarrow{i} T \xrightarrow{f} \mathcal{C} \xrightarrow{p} \mathcal{D}$$

be maps of simplicial sets such that i is a monomorphism and p is an isofibration between ∞-categories. Then the induced functor

$$\mathcal{C}^{f/} \to \mathcal{D}^{pf/} \times_{\mathcal{D}^{pfi/}} \mathcal{C}^{fi/}$$

is a left fibration. Similarly, the induced functor

$$\mathcal{C}^{/f} \to \mathcal{D}^{/pf} \times_{\mathcal{D}^{/pfi}} \mathcal{C}^{/fi}$$

is a right fibration.

Proof. Let $j : A \to B$ be a left anodyne map. We wish to solve the following lifting problem:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & \mathcal{C}^{f/} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D}^{pf/} \times_{\mathcal{D}^{pfi/}} \mathcal{C}^{fi/} \end{tikzcd}$$

By adjunction, this is equivalent to the following lifting problem:

$$\begin{tikzcd} T \diamond A \coprod_{S \diamond A} S \diamond B \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ T \diamond B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

where the left vertical map is a monomorphism. We wish to show that the left vertical map is a Joyal equivalence, so that we can use Joyal's model structure (Theorem A.18) to conclude the lemma. For this purpose, we need the following commutative diagram:

$$\begin{tikzcd} T \diamond A \coprod_{S \diamond A} S \diamond B \arrow[r] \arrow[d] & T \star A \coprod_{S \star A} S \star B \arrow[d] \\ T \diamond B \arrow[r] & T \star B \end{tikzcd}$$

The lower horizontal map is a Joyal equivalence by Proposition 4.5. Applying Corollary A.16 to Joyal's model structure, we see that the upper horizontal map is again a Joyal equivalence. The right vertical map is inner anodyne and hence a Joyal equivalence. As a result, the left vertical map is a Joyal equivalence. □

Corollary 4.10. *Let $p : Y \to \mathcal{C}$ be a diagram with $\mathcal{C}$ an ∞-category. Then the functor $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration and the functor $\mathcal{C}^{/p} \to \mathcal{C}$ is a right fibration. In particular, both $\mathcal{C}^{p/}$ and $\mathcal{C}^{/p}$ are ∞-categories.*

Proposition 4.11. *Let $p : Y \to \mathcal{C}$ be a diagram. Then the canonical functor*

$$\mathcal{C}_{p/} \to \mathcal{C}^{p/}$$

is a Joyal equivalence. The same is true for $\mathcal{C}_{/p} \to \mathcal{C}^{/p}$.

Definition 4.12. Let $\mathcal{C}$ be an ∞-category and let x and y be objects of $\mathcal{C}$. We define the **right mapping space** by the pullback

$$\begin{tikzcd} \mathrm{Map}^R_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"'] & \mathcal{C} \end{tikzcd}$$

and the **left mapping space** by the pullback

$$\begin{tikzcd} \mathrm{Map}^L_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"'] & \mathcal{C} \end{tikzcd}$$

Note that there are obvious pullback diagrams

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}^{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}^{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"'] & \mathcal{C} \end{tikzcd}$$

Lemma 4.13. *Consider a diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E}' \arrow[dl, "p'"] \\ & \mathcal{C} & \end{tikzcd}$$

where p and p' are isofibrations and where f is a Joyal equivalence. Then, for all objects x in $\mathcal{C}$, the induced map on fibres $\mathcal{E}'_x \to \mathcal{E}_x$ is again a Joyal equivalence.

Proof. Apply Corollary A.17 to Joyal's model structure. □

Corollary 4.14. *Let $\mathcal{C}$ be an ∞-category and let x and y be objects of $\mathcal{C}$. Then the maps*

$$\mathrm{Map}^R_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,y) \longleftarrow \mathrm{Map}^L_{\mathcal{C}}(x,y)$$

are homotopy equivalences.

5 Grothendieck Constructions

In classical topology, we study fibrations in terms of "classifying spaces" and "classifying maps". For example, for any topological group G, there is a principal G-bundle $EG \to BG$, characterized by the following universal property:

- Let X be a paracompact space. For any principal G-bundle $\xi : E \to X$, there corresponds a unique (up to homotopy) map $f_\xi : X \to BG$, such that ξ is isomorphic to the pullback bundle $X \times_{BG} EG \to X$.

If such a space BG exists we will refer to it as the **classifying space** of G. The map f_ξ will be called **the classifying map** of ξ. In this section we will show that there is a similar pattern for fibrations of (∞-)categories.

5.1 Classical Grothendieck constructions

5.1.1 Categories fibred in sets

Definition 5.1. Let $F : \mathcal{E} \to \mathcal{C}$ be a functor. We say that F is a **left covering functor** if the following condition is verified:

- $*$ For every pair (c, f) such that the outside square commutative, there is a **unique** filler rendering the whole diagram commutative:

$$\begin{tikzcd} {[0]} \arrow[r, "c"] \arrow[d, "d^1"'] & \mathcal{C} \arrow[d, "F"] \\ {[1]} \arrow[r, "f"'] \arrow[ur, dashed, "\exists! \tilde{f}"'] & \mathcal{E} \end{tikzcd}$$

Replacing d^1 by d^0 we get the definition of a **right covering functor**.

Example 5.2. In general, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $d \in \mathcal{D}$ is an object, then the fibre $F^{-1}(d)$ is a subcategory of $\mathcal{D}$. In case F is a left covering map, we easily see that $F^{-1}(d)$ is a discrete category for each $d \in \mathcal{D}$. Hence the title of this subsection.

Example 5.3. Let Set_* be category of pointed sets. Then the forgetful functor $\mathrm{Set}_* \to \mathrm{Set}$ is a left covering functor.

Example 5.4. Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then the canonical projection functor $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$ is a left covering functor. To see this, Let $f : x \to y$ be a morphism in $\mathcal{C}$ and (x, u) an object in $\int^{\mathcal{C}} F$. Then f admits a unique lift $\tilde{f} : (x, u) \to (y, F(f)(u))$. In particular, taking $\mathcal{C} = \mathrm{Set}$ and $F = \mathrm{Id}_{\mathrm{Set}}$ we see that $\mathrm{Set}_* = \int^{\mathrm{Set}} \mathrm{Id}_{\mathrm{Set}}$.

Definition 5.5. For a category $\mathcal{C}$, define the category $\mathrm{LC}(\mathcal{C})$ as the full subcategory of the slice category $\mathrm{Cat}_{/\mathcal{C}}$ spanned by left covering functors. Here we regard Cat as a 1-category.

Theorem 5.6 (Classification of left covering maps)**.** *There is a canonical equivalence of categories*

$$\int^{\mathcal{C}} : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \simeq \mathrm{LC}(\mathcal{C}); \quad F \mapsto (\int^{\mathcal{C}} F, \Pi)$$

Proof. We will construct the quasi-inverse of $\int^{\mathcal{C}}(-)$ directly. Let $F : \mathcal{D} \to \mathcal{C}$ be a left covering functor. Define the map $c \mapsto F^{-1}(c)$. Since F is a left covering map, the category $F^{-1}(c)$ is discrete, and so can be viewed as a set. We need to show that, for a morphism $f : c \to c'$, we can find a map $f_! : F^{-1}(c) \to F^{-1}(c')$, such that $(g \circ f)_! = g_! \circ f_!$.

By the definition of a left covering map, for an object $d \in F^{-1}(c)$ and a morphism $f : c \to c'$, there is a unique lift $\tilde{f} : d \to d'$ with $d' \in F^{-1}(c')$. We define $f_! : F^{-1}(c) \to F^{-1}(c'), d \mapsto d'$. The functoriality condition $(g \circ f)_! = g_! \circ f_!$ easily follows from the uniqueness of the lift. □

Remark 5.7. There is a similar equivalence between the category of right covering functors over $\mathcal{D}$ and the functor category $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Set})$.

Similar to the case of principal G-bundles, any left covering functor can be obtained by pulling back the universal left covering functor:

$$\mathrm{Set}_* \to \mathrm{Set}.$$

Theorem 5.8. *The projection* $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$ *fits into a pullback diagram as follows:*

$$\begin{tikzcd} \int^{\mathcal{C}} F \arrow[r] \arrow[d, "\Pi"'] & \mathrm{Set}_* \arrow[d] \\ \mathcal{C} \arrow[r, "F"] & \mathrm{Set} \end{tikzcd}$$

Proof. The proof of this theorem is just a reformulation of the definition of the category of elements. □

According to Theorem 5.6, the formation of the category of elements does not lose any information of the original functor. As a matter of fact, we can give alternative formulas to compute the limit and colimit of a functor $F : \mathcal{C} \to \mathrm{Set}$. More explicitly:

- Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor, and $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$ the associated category of elements. Let $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \int^{\mathcal{C}} F)$ be the set of sections of the canonical projection Π. Then there is a canonical isomorphism
$$\varprojlim F \cong \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \int^{\mathcal{C}} F).$$
- Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor, and $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$ the associated category of elements. Let $\pi_0(\int^{\mathcal{C}} F)$ be the set of connected components of $\int^{\mathcal{C}} F$. Then there is a canonical isomorphism:
$$\varinjlim F \cong \pi_0(\int^{\mathcal{C}} F).$$

Both of the two isomorphisms are easy to obtain, if the audience is familiar with the explicit construction of limits and colimits in Set. We left the first isomorphism as an exercise. For the second isomorphism, see Lemma 6.41.

5.1.2 Categories fibred in groupoids

Previous discussions focus on categories fibred in sets. Now we turn to study categories fibred in groupoids. Let $\mathcal{G}\mathrm{pd}$ denote the 2-category of all groupoids, functors and natural transformations. All 2-morphisms in $\mathcal{G}\mathrm{pd}$ are invertible.

Definition 5.9. Let $\mathcal{G}\mathrm{pd}_*$ be the 1-category of pointed groupoids, defined as follows:

- objects are pairs $(\mathcal{C}, c)$ where $\mathcal{C}$ is a groupoid and $c \in \mathcal{C}$ is an object;
- a morphism $(\mathcal{C}, c) \to (\mathcal{D}, d)$ is a pair (F, α) where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\alpha : F(c) \to d$ is a morphism $\mathcal{D}$. Note that α is an isomorphism.

Definition 5.10. A **pseudo-functor** is a weak 2-functor from a 1-category (viewed as a 2-category) to a 2-category. For simplicity, we will assume a pseudo-functor to be strictly unital.

Example 5.11. For later convenience, let us unpack the data of a pseudo-functor $F : \mathcal{C} \to \mathrm{Cat}$ where Cat is the 2-category of all categories:

- To each object $x \in \mathcal{C}$ we associate a category $F(x)$;
- To each morphism $f : x \to y$ we associate a functor $F(f) : F(x) \to F(y)$.

- To each composable pair $x \xrightarrow{f} y \xrightarrow{g} z$, we associate an invertible 2-cell $\alpha_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$.

Some extra coherence conditions are imposed on these invertible 2-cells, but we do not write down the details.

Definition 5.12. If we view Cat as a 2-category and $\mathcal{C} \in \mathrm{Cat}$, then we define the slice category $\mathrm{Cat}_{/\mathcal{C}}$ to be the following 2-category:

- An object is a functor $F : \mathcal{E} \to \mathcal{C}$;
- A 1-cell from $F : \mathcal{E} \to \mathcal{C}$ to $G : \mathcal{D} \to \mathcal{C}$ is a functor $H : \mathcal{E} \to \mathcal{D}$ such that $F = G \circ H$ **strictly**;
- If $H_1, H_2 : F \to G$ are 1-cells, 2-cell $\alpha : H_1 \to H_2$ is a natural transformation such that $\forall e \in \mathcal{E}$ we have $G(\alpha_e) = \mathrm{Id}_{F(e)}$.

Definition 5.13. A functor $F : \mathcal{C} \to \mathcal{D}$ is a **left fibration** if

(1) For every pair (c, f) such that the outside square commutative, there is a **not necessarily unique** filler rendering the whole diagram commutative:

$$\begin{tikzcd} {[0]} \arrow[r, "c"] \arrow[d, "d^1"'] & \mathcal{C} \arrow[d, "F"] \\ {[1]} \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{f}"'] & \mathcal{E} \end{tikzcd}$$

(2) For every morphism $f : c \to c_1$ in $\mathcal{C}$ and every object $c_2 \in \mathcal{C}$, the map

$$\hom_{\mathcal{C}}(c_1, c_2) \longrightarrow \hom_{\mathcal{D}}(F(c_1), F(c_2)) \times_{\hom_{\mathcal{D}}(F(c), F(c_2))} \hom_{\mathcal{C}}(c, c_2)$$

is bijective. Equivalently, for each simplicial map $\alpha : \Lambda^2_0 \to N(\mathcal{C})$ such that the square is commutative

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed, "\exists ! g"] & N(\mathcal{D}) \end{tikzcd}$$

Then there exists a **unique** filler $g : \Delta^2 \to N(\mathcal{C})$ rendering the whole diagram commutative.

Proposition 5.14. *In the above definition, the lift in condition (1) is unique up to unique isomorphism in the following sense: if there are two lifts $g : c \to c_1$ and $g : c \to c_2$ of f, then there exists a unique isomorphism $\alpha : c_1 \to c_2$ such that $\alpha \circ g = h$ and $F(\alpha) = \mathrm{Id}_y$, where y is the target of f.*

Proof. Let $f : x \to y$ be a morphism in $\mathcal{D}$ and $c \in \mathcal{C}$ be an object such that $F(c) = x$. Let $g : c \to c_1$ and $h : c \to c_2$ be two lifts of c. Thus we have

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(g,h)"] \arrow[d, hook] & N(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r, "s_1(f)"'] \arrow[ur, dashed] & N(\mathcal{D}) \end{tikzcd}$$

so that there is a unique morphism $\alpha : c_1 \to c_2$ such that $\alpha \circ g = h$. Change the order of g and h we have

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(h,g)"] \arrow[d, hook] & N(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r, "s_1(f)"'] \arrow[ur, dashed] & N(\mathcal{D}) \end{tikzcd}$$

so that there is unique morphism $\beta : c_2 \to c_1$ such that $\beta \circ h = g$. Similarly replace the morphism $\Lambda_0^2 \to N(\mathcal{C})$ by (g, g) and (h, h) respectively, we see that $\alpha \circ \beta = \mathrm{Id}_{c_2}$ and $\beta \circ \alpha = \mathrm{Id}_{c_1}$. □

Corollary 5.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left fibration. Then for each object $d \in \mathcal{D}$, the fiber $F^{-1}(d)$ is a groupoid.*

Proposition 5.16. *A left fibration is conservative.*

Proposition 5.17. *The canonical projection $\mathcal{G}\mathrm{pd}_* \to \mathcal{G}\mathrm{pd}$ is a left fibration if we view $\mathcal{G}\mathrm{pd}$ as a 1-category.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of groupoids, then for each $c \in \mathcal{C}$ we have a lift $(F, \mathrm{Id}_{F(c)}) : (\mathcal{C}, c) \to (\mathcal{D}, F(c))$. Given a diagram $\theta : \Lambda_0^2 \to N(\mathcal{G}\mathrm{pd}_*)$ demonstrated as the following diagram

$$\begin{tikzcd} (\mathcal{C}, c) \arrow[r, "{(F,\alpha)}"] \arrow[dr, "{(G,\beta)}"'] & (\mathcal{D}, d) \\ & (\mathcal{E}, e) \end{tikzcd}$$

where $\alpha : F(c) \cong d$ is an isomorphism, and $\beta : G(c) \cong e$ is an isomorphism. Suppose that there is a functor $H : \mathcal{D} \to \mathcal{E}$ such that $H \circ F = G$ strictly, so that there is a commutative square:

$$\begin{tikzcd} \Lambda_0^2 \arrow[r, "\theta"] \arrow[d, hook] & N(\mathcal{G}\mathrm{pd}_*) \arrow[d] \\ \Delta^2 \arrow[r] & N(\mathcal{G}\mathrm{pd}) \end{tikzcd}$$

Then it is easy to see that there exists a morphism $(H, \gamma) : (\mathcal{D}, d) \to (\mathcal{E}, e)$, where $\gamma : H(d) \cong e$ is defined by the composition

$$\gamma : H(d) \xrightarrow{H(\alpha^{-1})} H(F(c)) = G(c) \xrightarrow{\beta} e$$

such that the equation $(H, \gamma) \circ (F, \alpha) = (G, \beta)$ is easily verified. The uniqueness of γ is again easy to see. Thus we get the unique lift $\rho : \Delta^2 \to N(\mathcal{G}\mathrm{pd}_*)$. □

Definition 5.18. Given a pseudo-functor $F : \mathcal{C} \to \mathcal{G}\mathrm{pd}$, we define $\int^{\mathcal{C}} F$ to be following category:

- Objects of $\int^{\mathcal{C}} F$ are pairs (c, u) where $c \in \mathcal{C}$ is an object and $u \in F(c)$ is an object;
- A morphism $(f, \gamma) : (c, u) \to (d, v)$ consists of a morphism $f : c \to d$ in $\mathcal{C}$ and $\gamma : F(f)(u) \xrightarrow{\cong} v$ an isomorphism in $F(d)$.
- Compositions are defined easily.

Proposition 5.19. *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{G}\mathrm{pd}$ be a pseudo-functor. Then the forgetful functor $\int^{\mathcal{C}} F \to \mathcal{C}$ is a left fibration.*

Proof. The proof is similar to that of Proposition 5.17. □

Definition 5.20. Let $\mathrm{LF}(\mathcal{C}) \subseteq \mathrm{Cat}_{/\mathcal{C}}$ be the full sub-2-category spanned by left fibrations over $\mathcal{C}$. Note that all 2-cells of $\mathrm{LF}(\mathcal{C})$ are invertible.

Proposition 5.21. *Let $\mathcal{C}$ be a category. Then there is a canonical equivalence of 2-categories*

$$\int^{\mathcal{C}}(-) : \mathrm{Psd}(\mathcal{C}, \mathcal{G}\mathrm{pd}) \xrightarrow{\sim} \mathrm{LF}(\mathcal{C}).$$

Here $\mathrm{Psd}(\mathcal{C}, \mathcal{G}\mathrm{pd})$ *is the 2-category of pseudo-functors from* $\mathcal{C}$ *to* $\mathcal{G}\mathrm{pd}$.

Proof. We construct an inverse of $\int^{\mathcal{C}}(-)$ directly. Let $p : \mathcal{D} \to \mathcal{C}$ be a left fibration over $\mathcal{C}$. We define a pseudo-functor $\Phi(p) : \mathcal{C} \to \mathcal{G}\mathrm{pd}$ as follows:

- For each $c \in \mathcal{C}$, $\Phi(p)(c) := p^{-1}(c)$, which is a groupoid due to Corollary 5.15.
- Given a morphism $f : c \to d$ in $\mathcal{C}$, we need to construct a functor $f_! : p^{-1}(c) \to p^{-1}(d)$. For each $\tilde{c} \in p^{-1}(c)$, choose a lift $\tilde{f} : \tilde{c} \to \tilde{d}$, then we define $f_!(\tilde{c}) := \tilde{d}$. The action of $f_!$ on morphisms is define by the following commutative diagram:

$$\begin{tikzcd}
\tilde{c} \arrow[r, "\tilde{f}"] \arrow[d, "k"'] \arrow[dr, "\tilde{f}'\circ k"] & \tilde{d} \arrow[d, dashed, "f_!(k)"] \\
\tilde{c}' \arrow[r, "\tilde{f}'"'] & \tilde{d}'
\end{tikzcd}$$

  Here $\tilde{f}$ and $\tilde{f}' \circ k$ defines a morphism $\Lambda^2_0 \to N(\mathcal{D})$, and $f_!(k)$ is given by the unique filler. The fact that $f_!$ preserves composition of morphisms is illustrated by the following commutative diagram:

$$\begin{tikzcd}
& \tilde{c} \arrow[rr, "\tilde{f}"] \arrow[dl, "k"'] \arrow[dd, dashed] & & \tilde{d} \arrow[dl, "f_!(k)"'] \arrow[dd, "f_!(k'\circ k)"] \\
\tilde{c}' \arrow[rr, "\tilde{f}'"] \arrow[dr, "k'"'] & & \tilde{d}' \arrow[dr, "f_!(k')"'] & \\
& \tilde{c}'' \arrow[rr, "\tilde{f}''"'] & & \tilde{d}''
\end{tikzcd}$$

- Given morphisms $f : c \to d$ and $g : d \to e$ in $\mathcal{C}$, we need to construct the structure morphisms $\mu_{gf} : g_! \circ f_! \cong (g \circ f)_!$ of the pseudo-functor. We will define its component on $\tilde{c} \in p^{-1}(c)$. To do so, choose a lift $\tilde{f} : \tilde{c} \to \tilde{d}$ of f, and then choose a lift $\tilde{g} : \tilde{d} \to \tilde{e}$ of g. Finally we choose a lift $\widetilde{g \circ f} : \tilde{c} \to \tilde{e}'$ of $g \circ f$. Then $\widetilde{g \circ f}$ and $\tilde{g} \circ \tilde{f}$ are both lifts of $g \circ f$, and so there is a unique isomorphism $\gamma : \tilde{e} \to \tilde{e}'$, which serves as the component of μ_{gf}.

Due to the laziness of the author, we omit the proof that $\Phi(-)$ extends to a 2-functor. It is direct to check that $\Phi(-)$ and $\int^{\mathcal{C}}(-)$ are inverse to each other. □

5.1.3 Categories fibred in categories

Definition 5.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $f : c \to c_1$ be a morphism in $\mathcal{C}$. The morphism f is ***F*-coCartesian** if the following property is verified: for all $c_2 \in \mathcal{C}$ the induced map

$$\hom_{\mathcal{C}}(c_1, c_2) \longrightarrow \hom_{\mathcal{D}}(F(c_1), F(c_2)) \times_{\hom_{\mathcal{D}}(F(c),F(c_2))} \hom_{\mathcal{C}}(c, c_2)$$

is a bijection. Or equivalently, the following functor induces a bijection between objects of the two categories:

$$\mathcal{C}_{f/} \to \mathcal{D}_{p(f)/} \times_{\mathcal{D}_{p(c)/}} \mathcal{C}_{c/}.$$

Equivalently, f is F-coCartesian if, for each simplicial map $\alpha : \Lambda^2_0 \to N(\mathcal{C})$ and each $\beta : \Delta \to N(\mathcal{D})$ such that the upper triangle and the lower square is commutative

$$\begin{tikzcd}
\Delta^{0,1} \arrow[d, hook] \arrow[dr, "f"] & \\
\Lambda^2_0 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \arrow[d] \\
\Delta^2 \arrow[r] \arrow[ur, dashed, "\exists ! g"] & N(\mathcal{D})
\end{tikzcd}$$

Then there exists a **unique** filler $g : \Delta^2 \to N(\mathcal{C})$ rendering the whole diagram commutative.

Example 5.23. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $f : d \to d'$ be a morphism in $\mathcal{D}$. Let $g : c \to c'$ and $h : c \to c''$ be two F-coCartesian morphisms such that $F(g) = F(h) = f$ (Note that g and h has the same source!). Then there exists a unique isomorphism $k : c' \to c''$ such that $h = k \circ g$ and that $F(k) = \mathrm{Id}_{d'}$.

Definition 5.24. A functor $F : \mathcal{C} \to \mathcal{D}$ is a **coCartesian fibration** if the following condition is verified:

* For every pair (c, f) such that the outside square commutative, there is a filler rendering the whole diagram commutative:

$$\begin{tikzcd}
{[0]} \arrow[r, "c"] \arrow[d, "d^1"'] & \mathcal{C} \arrow[d, "F"] \\
{[1]} \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{f}"'] & \mathcal{E}
\end{tikzcd}$$

  such that the lift $\tilde{f}$ is F-coCartesian.

There is a dual notion of **Cartesian fibration**. We do not write down the details here.

Lemma 5.25. *The coCartesian lift $\tilde{f}$ in Definition 5.24 is unique up to unique isomorphism.*

Example 5.26. Let $\mathrm{Cat}^{\mathrm{lax}}_*$ be the following category:

- Objects are categories with distinguished objects $(\mathcal{C}, c)$, $c \in \mathcal{C}$.
- A morphism from $(\mathcal{C}, c)$ to $(\mathcal{D}, d)$ consists of a pair (F, u), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $u : F(c) \to d$ is a morphism in $\mathcal{D}$.
- Compositions are defined in an obvious way.

Then there is a canonical projection functor $\mathrm{Cat}^{\mathrm{lax}}_* \to \mathrm{Cat}$, which is a coCartesian fibration.

Definition 5.27. Let $F : \mathcal{C} \to \mathrm{Cat}$ be a pseudo-functor. We define a category $\int^{\mathcal{C}} F$ as follows:

- objects are pairs (c, u) where $c \in \mathcal{C}$ is an object and $u \in F(c)$ is an object.
- A morphism from (c, u) to (c', u') is a pair (f, α) where $f : c \to c'$ is a morphism in $\mathcal{C}$ and $\alpha : F(f)(u) \to u'$ is a morphism in $F(c')$.
- Let $(f, \alpha) : (c, u) \to (c', u')$ and $(g, \beta) : (c', u') \to (c'', u'')$ be a composable pair of morphisms. The composition $(g, \beta) \circ (f, \alpha)$ is defined to be $(g \circ f, \gamma)$ where γ is the morphism given by the composition
$$F(g \circ f)(u) \xrightarrow{\cong} F(g)(F(f)(u)) \xrightarrow{\alpha} F(g)(u') \xrightarrow{\beta} u''.$$

The category $\int^{\mathcal{C}} F$ is equipped with a canonical projection functor $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$.

Proposition 5.28. *Let $F : \mathcal{C} \to \mathrm{Cat}$ be a pseudo-functor and $\Pi : \int^{\mathcal{C}} F \to \mathcal{C}$ the associated category of elements. Then a morphism (f, α) in $\int^{\mathcal{C}} F$ is Π-coCartesian if and only if α is an isomorphism.*

Corollary 5.29. *Let $F : \mathcal{C} \to \mathrm{Cat}$ be a 2-functor. Then the projection functor $\int^{\mathcal{C}} F \to \mathcal{C}$ is a coCartesian fibration.*

Definition 5.30. Let $\mathrm{coCar}(\mathcal{C})$ be the sub-2-category of $\mathrm{Cat}_{/\mathcal{C}}$ consisting of coCartesian fibrations, **coCartesian functors**[3], and all 2-cells.

Theorem 5.31 (Classification of coCartesian fibrations)**.** *There is a canonical 2-equivalence of 2-categories:*

$$\mathrm{Psd}[\mathcal{C}, \mathrm{Cat}] \simeq \mathrm{coCar}(\mathcal{C}); \quad F \mapsto (\int^{\mathcal{C}} F, \Pi)$$

Here $\mathrm{Psd}[\mathcal{C}, \mathrm{Cat}]$ stands for the 2-category of pseudo-functors from $\mathcal{C}$ to Cat.

Proof. The seemingly complicated definition of a coCartesian fibration allows us to construct a pseudo-functor out of each coCartesian fibration $F : \mathcal{E} \to \mathcal{C}$. Mores explicitly, the assignment $c \mapsto F^{-1}(c)$ extends to a pseudo-functor. $\square$

5.2 coCartesian fibrations of ∞-categories

5.2.1 coCartesian arrows in simplicial sets

Definition 5.32. Let $p : X \to S$ be a map of simplicial sets, and $e : \Delta^1 \to X$ be an edge of X. We say that e is **p-coCartesian** if every lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "e"] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

has a solution.

Example 5.33. If $\mathcal{C}$ is an ∞-category and e is an equivalence of $\mathcal{C}$, then for any ∞-category $\mathcal{D}$ and any inner fibration $p : \mathcal{C} \to \mathcal{D}$, e is p-coCartesian according to Theorem 2.69.

Lemma 5.34. *Let $p : X \to Y$ be an inner fibration of simplicial sets. Write $i : \Delta^0 \to \Delta^1$ for the inclusion of 0. Then an edge $f : x \to y$ is p-coCartesian if and only if the morphism*

$$\langle i, p \rangle_{f/} : X_{f/} \to X_{x/} \times_{Y_{p(x)/}} Y_{p(f)/}$$

is a trivial fibration.

Proof. Write $j_n : \partial\Delta^n \hookrightarrow \Delta^n (n \geq 1)$ for the inclusion of boundary of standard n-simplex. $\langle i, p \rangle_{f/}$ is a trivial fibration if and only if, for all $n \geq 0$, $j_n \boxslash \langle i, p \rangle_{f/}$.

Then we have

$$j_n \boxslash \langle i, p \rangle_{f/} \quad \Leftrightarrow \quad i \hat{\star} j_n \boxslash_{f/} p$$

[3] functors preserving coCartesian arrows

The lifting problem on the right hand side reads

$$
\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rrr, bend left, "f"] & \Delta^0 \star \Delta^n \coprod_{\Delta^0 \star \partial\Delta^n} \Delta^1 \star \partial\Delta^n = \Lambda^n_0 \arrow[rr] \arrow[d] & & X \arrow[d] \\
& \Delta^n \arrow[rr] \arrow[urr, dashed] & & S
\end{tikzcd}
$$

which is exactly the lifting problem appearing in the definition of a p-coCartesian edge. ☐

Remark 5.35. Write $j : \Delta^0 \to \Delta^1$ for the inclusion of 1. The morphism $\langle j, p\rangle_{f/} : X_{f/} \to X_{y/} \times_{p(y)/} Y_{p(f)/}$ is a trivial fibration automatically since j is right anodyne.

Lemma 5.36. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration. Let $\sigma : \Delta^2 \to \mathcal{C}$ be a 2-simplex of $\mathcal{C}$ such that $f := d_2(\sigma)$ is p-coCartesian. Let $\rho : \Delta^2 \to \mathcal{C}$ be another 2-simplex such that:*

1. $\sigma|_{\Lambda^2_0} = \rho|_{\Lambda^2_0}$;
2. $p(\sigma) = p(\rho)$.

Then $d_0(\sigma) \simeq d_0(\rho)$.

Lemma 5.37. *Let $p \colon \mathcal{E} \to \mathcal{D}$ and $q \colon \mathcal{D} \to \mathcal{C}$ be inner fibrations between ∞-categories, and let $f : \Delta^1 \to \mathcal{E}$ be a morphism such that $p(f)$ is q-(co)Cartesian. Then f is p-(co)Cartesian if and only if f is $q \circ p$-(co)Cartesian.*

Proof. We only prove for the Cartesian case. Consider the diagram

$$\Delta^0 \xrightarrow{j} \Delta^1 \xrightarrow{f} \mathcal{E} \xrightarrow{p} \mathcal{D}$$

We get the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{E}_{/f} \arrow[r, "\alpha"] \arrow[drr, "\gamma"'] & \mathcal{E}_{/y} \times_{\mathcal{D}_{/p(y)}} \mathcal{D}_{/p(f)} \arrow[r, "\beta"] & \mathcal{E}_{/y} \times_{\mathcal{D}_{/p(y)}} \mathcal{D}_{/p(y)} \times_{\mathcal{C}_{/qp(y)}} \mathcal{C}_{/qp(f)} \arrow[d, "\cong"] \\
& & \mathcal{E}_{/y} \times_{\mathcal{C}_{/qp(y)}} \mathcal{C}_{/qp(f)}
\end{tikzcd}
$$

In the above diagram, α and γ are right fibrations, β is a trivial fibration. Recall that a right fibration is a trivial fibration if and only if it is a Joyal equivalence, and Joyal equivalences satisfy the 2-out-of-3 property. As a result, α is a trivial fibration if and only if γ is a trivial fibration. ☐

Lemma 5.38. *Let $p : \mathcal{C} \to \Delta^0$ be an inner fibration, so that $\mathcal{C}$ is an ∞-category. An edge $f : \Delta^1 \to \mathcal{C}$ is an equivalence if and only if f is p-(co)Cartesian.*

Proof. We only prove the coCartesian case. If f is an equivalence, then f is p-coCartesian according to Theorem 2.69. Conversely, assume that f is p-coCartesian, then the following lifting problem can be solved:

$$
\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^2_0 \arrow[r, "{(f, \mathrm{Id}_x)}"] \arrow[d, hook] & \mathcal{C} \\
& \Delta^2 \arrow[ur, dashed, "\sigma"'] &
\end{tikzcd}
$$

So that $g := d_0(\sigma)$ is a left inverse of f, i.e. $g \circ f \simeq \mathrm{Id}_x$. Now consider $(f, f) : \Lambda^2_0 \to \mathcal{C}$. Then we can find two extensions $\sigma, \rho : \Delta^2 \to \mathcal{C}$ with $d_0(\sigma) = \mathrm{Id}_y$ and $d_0(\rho) = f \circ g$ such that conditions in Lemma 5.36 are verified. So we conclude that $\mathrm{Id}_y \simeq f \circ g$. □

Corollary 5.39. *Let $p : \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories, and let $f : \Delta^1 \to \mathcal{E}$ be a morphism. Then f is an equivalence if and only if it is p-(co)Cartesian and $p(f)$ is an equivalence*

Proof. If f is an equivalence, then f is p-(co)Cartesian and and $p(f)$ is an equivalence. Conversely, consider the composition $\mathcal{E} \to \mathcal{C} \to \Delta^0$, then combine Lemma 5.38 and Lemma 5.37. □

5.2.2 coCartesian arrows and mapping spaces

Recall that in Definition 5.22, we gave several equivalent definitions of coCartesian arrows in 1-categories. In particular there is a characterization in terms of hom-sets:

⋆ Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $f : c \to c_1$ be a morphism in $\mathcal{C}$. The morphism f is **F-coCartesian** if for all $c_2 \in \mathcal{C}$ the following commutative diagram is a pullback

$$\begin{tikzcd} \hom_{\mathcal{C}}(c_1, c_2) \arrow[r, "-\circ f"] \arrow[d] & \hom_{\mathcal{C}}(c, c_2) \arrow[d] \\ \hom_{\mathcal{D}}(F(c_1), F(c_2)) \arrow[r, "-\circ F(f)"'] & \hom_{\mathcal{D}}(F(c), F(c_2)) \end{tikzcd}$$

In this section our purpose is to give a similar characterization of coCartesian arrows in the ∞-categorical context. See Corollary 5.43.

Definition 5.40. A commutative diagram of ∞-categories

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d, "p"] \\ \mathcal{C} \arrow[r] & \mathcal{C}' \end{tikzcd}$$

is called **homotopy Cartesian** if p is an isofibration and the induced map $\mathcal{E} \to \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}'$ is a Joyal equivalence. When $\mathcal{C} = \Delta^0$, we say that the sequence $\mathcal{E} \to \mathcal{E}' \to \mathcal{C}'$ is a **homotopy fibre sequence**.

Lemma 5.41. *Let $p : \mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let x and y be objects of $\mathcal{C}$. Then the induced map*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(p(x), p(y))$$

is a Kan fibration. The same holds true for Map^L and Map^R, even if p is only an inner fibration.

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r, "\beta"] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(p(x), p(y)) \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\ \mathcal{C}^{/y} \arrow[r, "\alpha"] & \mathcal{D}^{/p(y)} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ & \mathcal{D}^{/p(y)} \arrow[r] & \mathcal{D} \end{tikzcd}$$

The upper large square and the right large square are pullbacks. The lower square is a pullback, as a result the square in the top right corner is a pullback. Finally we deduce that the left small square is a pullback. The morphism α is a right fibration, then so is β. Since $\mathrm{Map}_{\mathcal{D}}(p(x), p(y))$ is a Kan complex, we conclude that β is a Kan fibration. □

Lemma 5.42. *Let $p : \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories and let $f : x \to y$ be a p-coCartesian edge in $\mathcal{E}$. Then, for all objects z of $\mathcal{E}$, the induced map*

$$\mathcal{E}^{f/} \times_{\mathcal{E}} \{z\} \to (\mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/}) \times_{\mathcal{E}} \{z\}$$

is a trivial fibration.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{f/} \times_{\mathcal{E}} \{z\} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E}^{f/} \arrow[d, "\gamma"] \\
(\mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/}) \times_{\mathcal{E}} \{z\} \arrow[r] \arrow[d] & \mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/} \arrow[d] \\
\Delta^0 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

Both the outer big square and the lower small square are pullbacks. As a result the upper small square is again a pullback. Since γ is a trivial Kan fibration, then so is α. ☐

Corollary 5.43. *Let $p : \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories, let $f : \Delta^1 \to \mathcal{E}$ be a p-coCartesian morphism of $\mathcal{E}$ from x to y, and let $z \in \mathcal{E}$ be an object. Then the diagram*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(y, z) \arrow[r, "- \circ f"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, y) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(p(y), p(z)) \arrow[r, "- \circ p(f)"] & \mathrm{Map}_{\mathcal{C}}(p(x), p(y))
\end{tikzcd}$$

is a homotopy pullback.

Proof. We consider the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(y, z) \arrow[d] & \mathcal{E}^{f/} \times_{\mathcal{E}} \{z\} \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, z) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(p(y), p(z)) & \mathcal{C}^{p(f)/} \times_{\mathcal{C}} \{p(z)\} \arrow[l, "\simeq"] \arrow[r] & \mathrm{Map}_{\mathcal{C}}(p(x), p(z))
\end{tikzcd}$$

The left horizontal maps are trivial fibrations since $\{1\} \to \Delta$ is right anodyne. It then suffices to show that the right square is homotopy Cartesian. Since f is p-coCartesian, we see that the canonical morphism $\mathcal{E}^{f/} \to \mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/}$ is a trivial fibration. As a result the induced morphism $\mathcal{E}^{f/} \times_{\mathcal{E}} \{z\} \to (\mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/}) \times_{\mathcal{E}} \{z\}$ is again a trivial fibration by Lemma 5.42. We conclude by noticing that $(\mathcal{E}^{x/} \times_{\mathcal{C}^{p(x)/}} \mathcal{C}^{p(f)/}) \times_{\mathcal{E}} \{z\} \cong \mathrm{Map}_{\mathcal{E}}(x, z) \times_{\mathrm{Map}_{\mathcal{C}}(p(x), p(z))} (\mathcal{C}^{p(f)/} \times_{\mathcal{C}} \{p(z)\})$. ☐

Corollary 5.44. *Let $p : \mathcal{E} \to \mathcal{C}$ be an inner fibration and let x and y be objects of $\mathcal{E}$. Let $f : x \to y'$ be a p-coCartesian morphism with $p(y) = p(y')$. Then the diagram*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}_{p(y)}}(y', y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, y) \arrow[d] \\
\Delta^0 \arrow[r, "p(f)"'] & \mathrm{Map}_{\mathcal{C}}(p(x), p(y))
\end{tikzcd}$$

is homotopy-Cartesian.

Proof. According to Corollary 5.43, there is a homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(y', y) \arrow[r, "-\circ f"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, y) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(p(y'), p(y)) \arrow[r, "-\circ p(f)"] & \mathrm{Map}_{\mathcal{C}}(p(x), p(y))
\end{tikzcd}$$

so that the induced map of each vertical fibres is an equivalence. In particular, we consider the fibre over $\mathrm{Id}_{p(y)}$. ☐

5.2.3 coCartesian fibrations

Definition 5.45. Let $p : X \to Y$ be an inner fibration. We say that p is a **coCartesian fibration** if every lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[ur, dashed, "\tilde{f}"] & Y
\end{tikzcd}$$

admits a solution $\tilde{f}$ which is p-coCartesian. Dually, we say that p is a **Cartesian fibration** if every lifting problem

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[ur, dashed, "\tilde{f}"] & Y
\end{tikzcd}$$

admits a solution $\tilde{f}$ which is p-Cartesian.

Example 5.46. Let $p : X \to Y$ be a left fibration. Then every edge in X is p-coCartesian, and p is a coCartesian fibration. Dually, let $p : X \to Y$ be a right fibration. Then every edge in X is p-Cartesian, and p is a Cartesian fibration.

Lemma 5.47. *A coCartesian fibration $p : X \to Y$ is a left fibration if and only if every edge in X is p-coCartesian.*

Definition 5.48. Let $p : X \to Y$ and $p' : X' \to Y$ be (co)Cartesian fibrations. We say that a map $f : X \to X'$ is a **morphism of (co)Cartesian fibrations** if $p'f = p$ and that f **preserves (co)Cartesian arrows**, that is, f sends p-(co)Cartesian arrows to p'-(co)Cartesian arrows.

Theorem 5.49. *Let $f : \mathcal{E} \to \mathcal{E}'$ be a morphism of (co)Cartesian fibration $p : \mathcal{E} \to \mathcal{C}$ and $p' : \mathcal{E}' \to \mathcal{C}$ between ∞-categories. Then f is a Joyal equivalence if and only if, for all objects z of $\mathcal{C}$, the induced map on fibres $\mathcal{E}_z \to \mathcal{E}'_z$ is a Joyal equivalence.*

Proof. Assume that f is a morphism of coCartesian fibration. The Cartesian case will be similar. The "only if" direction holds more generally for morphisms between isofibrations, see Lemma 4.13. Let us assume that all induced maps $\mathcal{E}_x \to \mathcal{E}'_x$ are Joyal equivalences, and our purpose is to show that f is a Joyal equivalence. For this, we will first show that f is fully faithful and essentially surjective, and then conclude the theorem by Theorem 3.17.

That f is essentially surjective is obvious: for an object $y \in \mathcal{E}'$, let $x = p'(y)$. Since $f_x : \mathcal{E}_x \to \mathcal{E}'_x$ is essentially surjective, there exists $z \in \mathcal{E}_x$ such that $f(z) \simeq y$.

In order to show that f is fully faithful, take two objects $x, y \in \mathcal{E}$ and we wish to show that the map

$$\mathrm{Map}_{\mathcal{E}}(x, y) \to \mathrm{Map}_{\mathcal{E}'}(f(x), f(y))$$

is a homotopy equivalence. Consider the following commutative diagram, where the vertical arrows are Kan fibrations by Lemma 5.41.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}'}(f(x),f(y)) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(p(x),p(y)) \arrow[r, "="'] & \mathrm{Map}_{\mathcal{C}}(p(x),p(y))
\end{tikzcd}$$

As a result, it suffices to show that for each vertex $g \in \mathrm{Map}_{\mathcal{C}}(p(x),p(y))$, the induced map on the fibres of g is a homotopy equivalence. For this purpose, we choose a p-coCartesian lift $\alpha : x \to y'$ of g, since f preserves coCartesian morphisms, $f(\alpha) : f(x) \to f(y')$ is a p'-coCartesian lift of g. Now apply Corollary 5.44, the induced morphisms on fibres of g is given by

$$\mathrm{Map}_{\mathcal{E}_{p(y)}}(y',y) \to \mathrm{Map}_{\mathcal{E}'_{p(y)}}(f(y'),f(y))$$

which is a homotopy equivalence by our assumption that f restricts to a Joyal equivalence on each fibre. □

5.3 Straightening-unstraightening equivalence

In the context of ordinary category theory, the theory of coCartesian fibrations can be used to give an alternative description of functors to the category Cat of categories. When dealing with ∞-categories, however, the analogous notion of coCartesian fibrations is far more important: since defining a functor to Cat_∞, the ∞-category of all ∞-categories requires specifying an infinite amount of coherence data, it is in general not feasible to "write down" definitions of functors, as we always do in 1-category theory. As a result, manipulating (co)Cartesian fibrations is often the only reasonable way to define key functors.

Let us first introduce the stability of coCartesian fibrations without proof.

Proposition 5.50 (Stability property for coCartesian fibrations)**.** *Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration and let K be an arbitrary simplicial set, and a natural transformation α is p_*-coCartesian if and only if each component of α is p-coCartesian.*

The constructions of covariant transport and parametrized covariant transport will be useful for our future discussions. The following two theorems are taken from [Lur23].

Proposition 5.51. *Given a coCartesian fibration $p : \mathcal{E} \to \mathcal{C}$ and a morphism $f : \Delta^1 \to \mathcal{C}$ from x to y, there exists a functor $F : \Delta^1 \times \mathcal{E}_x \to \mathcal{E}$ such that the following conditions are verified:*

1. *The following diagram*

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_x \arrow[r, "F"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \arrow[r, "f"'] & \mathcal{C}
\end{tikzcd}$$

   commutes. In particular, this implies that $F(\{1\} \times \mathcal{E}_x) \subseteq \mathcal{E}_y$

2. *The restriction $F|_{\mathcal{E}_x \times \{0\}}$ is the identity map $\mathrm{Id}_{\mathcal{E}_x}$.*

3. *For each $e \in \mathcal{E}_x$, the edge represented by the composite*

$$\Delta^1 \times \{e\} \hookrightarrow \Delta^1 \times \mathcal{E}_x \xrightarrow{F} \mathcal{E}$$

   is p-coCartesian.

In this case, we will denote the map $F|_{\{1\}\times\mathcal{E}_x} : \mathcal{E}_x \to \mathcal{E}_y$ by $f_!$. We will refer to $f_!$ as the ***covariant transport along*** *f.*

Proof. Consider the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r, "g"] \arrow[d] & \mathrm{Fun}(\mathcal{E}_x, \mathcal{E}) \arrow[d, "p_*"] \\ \Delta^1 \arrow[r, "q"'] \arrow[ur, dashed, "k"] & \mathrm{Fun}(\mathcal{E}_x, \mathcal{C}) \end{tikzcd}$$

where g is given by the canonical inclusion $\mathcal{E}_x \hookrightarrow \mathcal{E}$, and q is given by $\mathcal{E}_x \times \Delta_1 \xrightarrow{\pi_2} \Delta^1 \xrightarrow{f} \mathcal{C}$. By Proposition 5.50, p_* is a coCartesian fibration, so there admits an essentially unique coCartesian lift $k : \Delta^1 \to \mathrm{Fun}(\mathcal{E}_x, \mathcal{E})$. By adjunction we get the desired map $\Delta^1 \times \mathcal{E}_x \to \mathcal{E}$, which obviously satisfies the requirements. □

Proposition 5.52 (Parametrized covariant transport)**.** *Let $p : \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration of simplicial sets, and let x and y be vertices of $\mathcal{C}$. Then there exists a map $F : \Delta^1 \times \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \to \mathcal{E}$ such that the following conditions are verified:*

1. *The diagram of simplicial sets*

$$\begin{tikzcd} \Delta^1 \times \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \arrow[r, "F"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \Delta^1 \times \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r] & \mathcal{C} \end{tikzcd}$$

   commutes, where the lower horizontal map is induced by the canonical inclusion $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$.

2. *The restriction $F|_{\{0\}\times\mathrm{Map}_{\mathcal{C}}(x,y)\times\mathcal{E}_x}$ is given by projection onto $\mathcal{E}_x$.*

3. *For every edge $f : x \to y$ and every object $e \in \mathcal{E}_x$, the composite map*

$$\Delta^1 \times \{f\} \times \{e\} \hookrightarrow \Delta^1 \times \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \xrightarrow{F} \mathcal{E}$$

   is a p-coCartesian edge of $\mathcal{E}$.

In this case, we will refer to the restriction $F|_{\{1\}\times\mathrm{Map}_{\mathcal{C}}(x,y)\times\mathcal{E}_x} : \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \to \mathcal{E}_y$ as the ***parametrized covariant transport****.*

Proof. Consider the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathrm{Fun}(\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x, \mathcal{E}) \arrow[d, "p_*"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x, \mathcal{C}) \end{tikzcd}$$

where the upper horizontal arrow is given by $\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \to \mathcal{E}_x \hookrightarrow \mathcal{E}$ and the lower horizontal arrow is given by $\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x \to \mathrm{Map}_{\mathcal{C}}(x, y) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$. So there is a unique lift $h : \Delta^1 \to \mathrm{Fun}(\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathcal{E}_x, \mathcal{E})$ which gives rise to the desired morphism F. □

Now we introduce Lurie's straightening-unstraightening equivalence without details. We write $\mathrm{coCar}(\mathcal{C})$ for the full subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by coCartesian fibrations over $\mathcal{C}$.

Theorem 5.53. *Let $\mathcal{C}$ be an ∞-category, we have a canonical equivalence of ∞-categories:*

$$\mathrm{coCar}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty).$$

Remark 5.54. Lurie's original formulation of the theorem uses the language of model categories. He constructed two model categories and proved the straightening-unstraightening Quillen equivalence between them, see [Lur09] Theorem 3.2.0.1. If we take the underlying ∞-category of these model categories, we obtain the above theorem.

For $\mathcal{C}$ an ∞-category, let $\mathrm{LF}(\mathcal{C})$ be the full category of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by left fibrations over $\mathcal{C}$

Theorem 5.55. *For every ∞-category $\mathcal{C}$, the straightening-unstraightening equivalence restricts to an equivalence*

$$\mathrm{LF}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$$

Remark 5.56. Similarly, Lurie's original formulation of the theorem uses the language of model categories. He constructed two model categories and proved the straightening-unstraightening Quillen equivalence between them, see [Lur09] Theorem 2.2.1.2. If we take the underlying ∞-category of these model categories, we obtain the above theorem.

As a matter of fact we have a commutative diagram

$$\begin{tikzcd}
\mathrm{LF}(\mathcal{C}) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \arrow[d, hook] \\
\mathrm{coCar}(\mathcal{C}) \arrow[r, "\simeq"'] & \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)
\end{tikzcd}$$

The remaining part of this section is devoted to studying the straightening-unstraightening equivalence over an interval Δ^1. First we will introduce the notion of weighted nerves.

Definition 5.57 (The weighted nerve)**.** Let $\mathcal{C}$ be a category equipped with a functor $F : \mathcal{C} \to \mathrm{sSet}$, we going define a simplicial set called $N^F(\mathcal{C})$. For every integer $n \geq 0$, we let $N^F_n(\mathcal{C})$ denote the collection of all pairs (σ, τ) where $\sigma : [n] \to \mathcal{C}$ is an n-simplex of $N(\mathcal{C})$:

$$\sigma = c_0 \to c_1 \to c_2 \to ... \to c_n$$

and τ is a collection of simplices $\{\tau_j : \Delta^j \to F(c_j)\}$, which fit into a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\tau_0"'] & \Delta^1 \arrow[r, hook] \arrow[d, "\tau_1"'] & ... \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\tau_n"] \\
F(c_0) \arrow[r] & F(c_1) \arrow[r] & ... \arrow[r] & F(c_n)
\end{tikzcd}$$

We will refer to $N^F(\mathcal{C})$ as **the F-weighted nerve of** $\mathcal{C}$. Note that there is an evident projection $N^F(\mathcal{C}) \to N(\mathcal{C})$ which sends the pair (σ, τ) to σ.

Example 5.58. Let $F : \mathcal{C} \to \mathrm{sSet}$ be a functor and $N^F(\mathcal{C})$ be its weighted nerve. Then

- A vertex of $N^F(\mathcal{C})$ consists of a pair (c, x), where $c \in \mathcal{C}$ is an object and $x \in F(c)_0$ is a vertex of the simplicial set $F(c)$.
- An edge $(c, x) \to (d, y)$ consists of a morphism $f : c \to d$ and an edge $e : F(f)(x) \to y$ of the simplicial set $F(d)$.

Example 5.59. Let X be a simplicial set, which we may identify with a functor $X : [0] \to \mathrm{sSet}$. Then the weighted nerve of the functor X is the simplicial set X itself.

Let $\mathrm{QC} \subseteq \mathrm{sSet}$ be the full subcategory spanned by ∞-categories.

Proposition 5.60. *Let $\mathcal{C}$ be a category, and $F : \mathcal{C} \to \mathrm{QC}$ be a functor taking values in the full subcategory of ∞-categories. Then*

(1) The canonical projection map $U : N^F(\mathcal{C}) \to N(\mathcal{C})$ is a coCartesian fibration of simplicial sets.

(2) Let $(f, e) : (c, x) \to (d, y)$ be an edge of the simplicial set $N^F(\mathcal{C})$. Then (f, e) is U-coCartesian if and only if $e : F(f)(x) \to y$ is an isomorphism in the ∞-category $F(d)$.

Proof. To be added. □

Definition 5.61. Let $\varphi : X \to Y$ be a morphism of simplicial sets, which we identify with a functor $\varphi : [1] \to \mathrm{sSet}$. We will define the **categorical mapping cone** of $\varphi : X \to Y$ to be the weighted nerve $N^{\varphi}([1])$, and we will denote such a categorical mapping cone as $X \star_Y Y$.

There is a canonical projection $p : X \star_X Y \to \Delta^1$ so that $p^{-1}(0) \cong X$ and $p^{-1}(1) \cong Y$. An n-simplex of the categorical mapping cylinder $X \star_Y Y$ consists of the following data:

- An n-simplex of Δ^1, which can be identified with a cut $([i], [j])$ of $[n]$, where $i + j = n - 1$.
- Three simplexes (σ, ρ, κ) where $\sigma : \Delta^n \to Y$ is an n-simplex of Y, $\rho : \Delta^i \to X$ is an i-simplex of X and $\kappa : \Delta^j \to Y$ is an j-simplex of Y, such that

$$\sigma|_{\Delta^{\{i+1,\ldots,n\}}} = \kappa : \Delta^j \to Y; \quad \sigma|_{\Delta^{\{0,\ldots,i\}}} = \varphi \circ \rho : \Delta^i \to Y.$$

Remark 5.62. The simplicial set $X \star_Y Y$ is equipped with many canonical morphisms. For example:

- There is a canonical morphism $\Delta^1 \times X \to X \star_Y Y$.
- There is a canonical morphism $X \star_Y Y \to \Delta^1 \times Y$.
- There is a canonical morphism $X \star_Y Y \to X \star Y$.

5.4 Application: Symmetric monoidal ∞-categories

5.4.1 The Segal condition

In this section we try to give a definition of symmetric monoidal ∞-categories. We cannot define a symmetric monoidal ∞-category as we have done in the 1-categorical context. We define a symmetric monoidal category by brute force: that is, we specify the product functor and all coherence data, including the unitor, associator, braiding, and the axioms they should verify. However, it is impossible to do so for ∞-categories, since there are infinitely many coherence data. This forces us to look for a cleverer way to organize all the coherence data. Again we start from the 1-categorical case.

Definition 5.63. For any finite set, let I_* denote the set $I \coprod \{*\}$ obtained from I by freely adjoining a new element $*$. For each $n \geq 0$, let $\underline{n}$ denote the set $\{1, 2, ..., n\}$ and $\langle n \rangle = \underline{n}_*$. Let $\mathcal{F}\mathrm{in}_*$ be the following category :

(1) Objects are sets $\langle n \rangle$ for $n \geq 0$;

(2) Given a pair of objects $\langle m \rangle, \langle n \rangle \in \mathcal{F}\mathrm{in}_*$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ is a map $\alpha : \langle m \rangle \to \langle n \rangle$ preserving the distinguished point $*$.

In a word $\mathcal{F}\mathrm{in}_*$ is the skeleton of the category of pointed finite sets.

For every pair of integers $1 \leq i \leq n$, we let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ denote the morphism given by the formula

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise.} \end{cases}$$

Definition 5.64. A morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ is

- **inert** if for each $i \in \underline{n}$, the inverse image $f^{-1}(i)$ contains exactly one element;
- **active** if $f^{-1}(*) = \{*\}$

The morphisms $\{\rho^i\}_{1 \leq i \leq n}$ we have just introduced are important examples of inert morphisms.

Definition 5.65 (Segal)**.** Let $\mathcal{C}$ be a category with finite products. A **commutative monoid in** $\mathcal{C}$ is a functor $A : \mathcal{F}\mathrm{in}_* \to \mathcal{C}$ such that the **Segal condition** is verified:

($\star$) For each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, the canonical morphism $A(\langle n \rangle) \xrightarrow{\prod_{i=1}^n A(\rho^i)} \prod_{i=1}^n A(\langle 1 \rangle)$ induced by $A(\rho^i) : A(\langle n \rangle) \to A(\langle 1 \rangle)$ is an isomorphism.

We write $\mathrm{CM}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ for the full subcategory spanned by commutative monoids.

Let $\mathcal{C}$ be a category with finite products, so that $\mathcal{C}$ is equipped with a canonical symmetric monoidal structure $(\mathcal{C}, \times, \mathbb{1})$, with associators and braidings induced by the universal property of product. Here $\mathbb{1}$ is the final object of $\mathcal{C}$. Recall that there is a notion a **commutative algebra** in a symmetric monoidal category. We write $\mathrm{CAlg}(\mathcal{C})$ for the category of commutative algebras and algebra homomorphisms in $\mathcal{C}$.

Proposition 5.66. *Let $\mathcal{C}$ be a category with finite products. Then there is a canonical equivalence of categories*

$$\Psi : \mathrm{CM}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$$
$$A \mapsto A_1$$

Proof. First I will explain the algebra structure on $A(\langle 1 \rangle)$. For simplicity I will denote $A(\langle n \rangle)$ by A_n. There is a unique active morphism $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ sending $1, 2$ to 1. After applying A we get a morphism $A(\alpha) : A_2 \to A_1$. We define $\mu : A_1 \times A_1 \to A_1$ to be the following composition

$$\mu : A_1 \times A_1 \cong A_2 \xrightarrow{A(\alpha)} A_1$$

where the first isomorphism is the inverse of the Segal map. The associativity of μ is illustrated as follows: we apply A to the following diagram in $\mathcal{F}\mathrm{in}_*$:

$$\begin{tikzcd} \langle 3 \rangle \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & \langle 2 \rangle \arrow[d, "\alpha"] \\ \langle 2 \rangle \arrow[r, "\alpha"] & \langle 1 \rangle \end{tikzcd}$$

where β_1 sends $1, 2$ to 1 and sends 3 to 2, while β_2 sends 1 to 1 and $2, 3$ to 2. Combined with the Segal condition we get the following commutative diagram:

$$\begin{tikzcd} A_1 \times A_1 \times A_1 \arrow[r, "\mu \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mu"'] & A_1 \times A_1 \arrow[d, "\mu"] \\ A_1 \times A_1 \arrow[r, "\mu"'] & A_1 \end{tikzcd}$$

demonstrating the associativity of μ. Similarly by applying A to the unique map $\langle 0\rangle \to \langle 1\rangle$ we get the unit of A, and unitality of μ can be seen in a similar way.

To show that Ψ is an equivalence, we construct its inverse directly. Let

$$\Phi : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CM}(\mathcal{C})$$

be the functor which sends (A, μ, ι) to the following functor $\Phi(A)$:

$$\begin{aligned} \Phi(A) : \mathcal{F}\mathrm{in}_* &\to \mathcal{C} \\ \langle n\rangle &\mapsto A^{\times n} \end{aligned}$$

The action of $\Phi(A)$ on morphisms is easy, noticing that now we have a canonical morphism $A^{\times n} \to A$ for each n. It is direct to see that Ψ and Φ are inverse to each other. $\square$

Now we turn to study symmetric monoidal categories. They can be viewed as commutative monoids in the 2-category of all categories Cat.

Definition 5.67. We say that a pseudo-functor $\mathcal{C} : \mathcal{F}\mathrm{in}_* \to \mathrm{Cat}$ is a **commutative monoid** if the **Segal condition** is verified:

($\star$) For each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, the canonical functor $\mathcal{C}(\langle n\rangle) \xrightarrow{\prod_{i=1}^n \mathcal{C}(\rho^i)} \prod_{i=1}^n \mathcal{C}(\langle 1\rangle)$ induced by $\mathcal{C}(\rho^i) : \mathcal{C}(\langle n\rangle) \to \mathcal{C}(\langle 1\rangle)$ is an equivalence of categories.

We write $\mathrm{CM}(\mathrm{Cat}) \subset \mathrm{Psd}(\mathcal{F}\mathrm{in}_*, \mathrm{Cat})$ for the full sub-2-category spanned by commutative monoids.

Proposition 5.68. *There is a canonical 2-equivalence*

$$\begin{aligned} \Psi : \mathrm{CM}(\mathrm{Cat}) &\simeq \mathrm{SymCat} \\ \mathcal{C} &\mapsto \mathcal{C}_1 := \mathcal{C}(\langle 1\rangle) \end{aligned}$$

where SymCat *is the 2-category of symmetric monoidal categories, braided functors and monoidal natural transformations.*

Proof. I will not give a complete proof. Let me explain the symmetric monoidal structure on $\mathcal{C}_1$. The monoidal structure is offered by

$$\otimes : \mathcal{C}_1 \times \mathcal{C}_1 \simeq \mathcal{C}_2 \xrightarrow{\mathcal{C}(\alpha)} \mathcal{C}_1$$

By applying $\mathcal{C}$ to the commutative diagram (which should be viewed as two trivial 2-cells!)

$$\begin{tikzcd} \langle 3\rangle \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] & \langle 2\rangle \arrow[d, "\alpha"] \\ \langle 2\rangle \arrow[r, "\alpha"] & \langle 1\rangle \end{tikzcd}$$

we get an invertible 2-cell $\alpha : \otimes \circ (\otimes \times \mathrm{Id}) \Rightarrow \otimes \circ (\mathrm{Id} \times \otimes)$. Then by applying $\mathcal{C}$ to the commutative diagram

$$\begin{tikzcd} & \langle 4\rangle \arrow[rr] \arrow[dl] \arrow[dd] & & \langle 3\rangle \arrow[dl] \arrow[dd] \\ \langle 3\rangle \arrow[rr] \arrow[dd] & & \langle 2\rangle \arrow[dd] & \\ & \langle 3\rangle \arrow[rr] \arrow[dl] & & \langle 2\rangle \arrow[dl] \\ \langle 2\rangle \arrow[rr] & & \langle 1\rangle & \end{tikzcd}$$

where all edges are suitable active morphisms, we recover the pentagon axiom for associators. For braiding, consider this commutative diagram in $\mathcal{F}\mathrm{in}_*$:

$$\begin{tikzcd} \langle 2\rangle \arrow[rr, "\tau"] \arrow[dr, "\alpha"'] & & \langle 2\rangle \arrow[dl, "\alpha"] \\ & \langle 1\rangle & \end{tikzcd}$$

where τ is the unique non-trivial automorphism of $\langle 2\rangle$. By applying $\mathcal{C}$ to this diagram we obtain a natural isomorphism $c : \otimes \Rightarrow \otimes \circ T$, where $T : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1 \times \mathcal{C}_1$ is the functor which switch the two factors. By applying $\mathcal{C}$ to certain heptahedrons we recover the hexagon axiom for braidings. One can check that this construction extends to a functor.

It remains to construct the inverse $\Phi : \mathrm{SymCat} \to \mathrm{CM}(\mathrm{Cat})$. Due to the laziness of the author, details will not be given here. It is similar to the construction in Proposition 5.66. □

As a result, we can view each symmetric monoidal category as a pseudo-functor $\mathcal{F}\mathrm{in}_* \to \mathrm{Cat}$ subject to the Segal condition. In view of the classical Grothendieck construction, the data of a symmetric monoidal category is equivalent to a coCartesian fibration $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ such that the Segal condition is verified

(⋆) For each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, the canonical functor $\mathcal{C}^\otimes_{\langle n\rangle} \xrightarrow{\prod_{i=1}^n \rho^i_!} \prod_{i=1}^n \mathcal{C}^\otimes_{\langle 1\rangle}$ induced by $\rho^i_! : \mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$ is an equivalence of categories. Here a $\rho^i_! : \mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$ is the covariant transport determined by ρ^i.

To explain this construction more explicitly, let us take a symmetric monoidal category $\mathcal{C}$, which we view as a functor $\mathcal{C} : \mathcal{F}\mathrm{in}_* \to \mathrm{Cat}$, then the corresponding coCartesian fibration $\pi : \mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ looks like:

- The fibre of π over $\langle n\rangle$, which we denote by $\mathcal{C}^\otimes_{\langle n\rangle}$, can be viewed as the n-fold product $\mathcal{C}^{\times n}$, hence an object of $\mathcal{C}^\otimes$ has the form $(x_1, ..., x_n)$ where each $x_i \in \mathcal{C}$ is an object.
- We write $\underline{x} = (x_1, ..., x_m)$ and $\underline{y} = (y_1, ..., y_n)$. Then the corresponding hom-set is given by

$$\hom_{\mathcal{C}^\otimes}(\underline{x}, \underline{y}) := \coprod_{\alpha : \langle m\rangle \to \langle n\rangle} \prod_{1 \le i \le n} \hom_{\mathcal{C}}(\otimes_{j \in \alpha^{-1}(i)} x_j, y_i).$$

Now we can give the definition of a symmetric monoidal ∞-category.

Definition 5.69. A **symmetric monoidal ∞-category** is a coCarrtesian fibration $\mathcal{C}^\otimes \to N(\mathcal{F}\mathrm{in}_*)$ such that the **Segal condition** is verified:

(⋆) For each $\langle n\rangle \in N(\mathcal{F}\mathrm{in}_*)$, the canonical functor $\mathcal{C}^\otimes_{\langle n\rangle} \xrightarrow{\prod_{i=1}^n \rho^i_!} \prod_{i=1}^n \mathcal{C}^\otimes_{\langle 1\rangle}$ induced by $\rho^i_! : \mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$ is an equivalence of ∞-categories. Here a $\rho^i_! : \mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$ is the covariant transport determined by ρ^i.

5.4.2 Symmetric monoidal functors and lax functors

In this section we study functors between symmetric monoidal categories in the unstraightened world. Again we start from 1-categorical case. We will generalize these notions to operads in Section 8.5.

Definition 5.70. Let $\pi : \mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal category. Then an **inert** morphism in $\mathcal{C}^\otimes$ is a π-coCartesian lift of an inert morphism in $\mathcal{F}\mathrm{in}_*$.

Let $\pi_{\mathcal{C}} : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ and $\pi_{\mathcal{D}} : \mathcal{D}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be symmetric monoidal categories. Then $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ and $\mathcal{D} = \mathcal{D}^{\otimes}_{\langle 1 \rangle}$ are equipped with canonical symmetric monoidal structures. We write $\mathrm{Fun}^{\mathrm{in}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ for the full subcategory of $\mathrm{Fun}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by those functors which preserve inert arrows. We write $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors and monoidal natural transformations.

Proposition 5.71. *There is a canonical equivalence of categories:*

$$\mathrm{Fun}^{\mathrm{in}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \cong \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D})$$

Proof. I do not give a complete proof here, but only illustrate the ideas. Let $F \in \mathrm{Fun}^{\mathrm{in}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ be functor preserving inert morphisms. This condition guarantees that the action of $F_n : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{D}^{\otimes}_{\langle n \rangle}$ on objects is completely determined by $F_1 : \mathcal{C} \to \mathcal{D}$. Recall that an object in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ can be written as $(x_1, ..., x_n)$ where $x_i \in \mathcal{C}$ is an object, then $F_n(x_1, ..., x_n) = (F_1(x_1), ..., F_1(x_n))$. The lax monoidal structure is given by

$$\begin{tikzcd}
(x_1, x_2) \arrow[dr, "\gamma"] & \\
& x_1 \otimes x_2
\end{tikzcd}
\qquad
\begin{tikzcd}
(F_1(x_1), F_1(x_2)) \arrow[r, "\mu"] \arrow[dr, "F(\gamma)"'] & F_1(x_1) \otimes F_1(x_2) \arrow[d, dashed, "\eta_{x_1,x_2}"] \\
& F_1(x_1 \otimes x_2)
\end{tikzcd}$$

$$\begin{tikzcd}
\langle 2 \rangle \arrow[r, "\alpha"] \arrow[dr, "\alpha"] & \langle 1 \rangle \arrow[d, "\mathrm{Id}"] \\
& \langle 1 \rangle
\end{tikzcd}$$

Here $\gamma \in \mathcal{C}^{\otimes}$ is the coCartesian lift of α, and $\mu \in \mathcal{D}^{\otimes}$ is the coCartesian lift of α. Hence γ exhibits $x_1 \otimes x_2$ as the tensor product of x_1 and x_2, while μ exhibits $F_1(x_1) \otimes F_1(x_2)$ as the tensor product of $F_1(x_1)$ and $F_2(x_2)$. Since μ is a coCartesian lift of α, it induces a unique morphism η_{x_1,x_2} which serves as defining data of a lax monoidal functor.

Next let me illustrate that F_1 is braided functor. It suffices to look at the following diagram:

$$\begin{tikzcd}
(x_1, x_2) \arrow[r, "T"] \arrow[d, "\mu"] & (x_2, x_1) \arrow[d, "\mu"] \\
x_1 \otimes x_2 \arrow[r, "c_{x_1,x_2}"'] & x_2 \otimes x_1
\end{tikzcd}$$

$$\begin{tikzcd}
(F_1(x_1), F_1(x_2)) \arrow[rr] \arrow[dd] \arrow[dr] & & (F_1(x_2), F_1(x_1)) \arrow[dd] \arrow[dr] & \\
& F_1(x_1) \otimes F_1(x_2) \arrow[rr] \arrow[dl] & & F_1(x_2) \otimes F_1(x_1) \arrow[dl] \\
F_1(x_1 \otimes x_2) \arrow[rr] & & F(x_2 \otimes x_1) &
\end{tikzcd}$$

$\square$

Definition 5.72. Let $\mathrm{Fun}^{\mathrm{coCar}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \subseteq \mathrm{Fun}^{\mathrm{in}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ be the full subcategory spanned by functors which preserve all coCartesian arrows. Let $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by braided monoidal functors.

Proposition 5.73. *The bijection in Proposition 5.71 restricts to an equivalence*

$$\mathrm{Fun}^{\mathrm{coCar}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \cong \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

Proof. In this case $F(\gamma)$ is also coCartesian lift of α, hence the structure map η_{x_1,x_2} are forces to be isomorphisms.

$$\begin{tikzcd}
(x_1, x_2) \arrow[dr, "\gamma"] & & (F_1(x_1), F_1(x_2)) \arrow[r, "\mu"] \arrow[dr, "F(\gamma)"'] & F_1(x_1) \otimes F_1(x_2) \arrow[d, dashed, "\eta_{x_1,x_2}"] \\
& x_1 \otimes x_2 & & F_1(x_1 \otimes x_2)
\end{tikzcd}$$

$$\begin{tikzcd}
\langle 2 \rangle \arrow[r, "\alpha"] \arrow[dr, "\alpha"] & \langle 1 \rangle \arrow[d, "\mathrm{Id}"] \\
& \langle 1 \rangle
\end{tikzcd}$$

□

Definition 5.74. Let $\pi_{\mathcal{C}} : \mathcal{C}^\otimes \to N(\mathcal{F}\mathrm{in}_*)$ and $\pi_{\mathcal{D}} : \mathcal{D}^\otimes \to N(\mathcal{F}\mathrm{in}_*)$ be symmetric monoidal ∞-categories. Then:

- Let $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}_{/N(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by those functors which preserve coCartesian arrows. An object in $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ is called a **symmetric monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$.
- Let $\mathrm{Fun}^{\otimes,\mathrm{lax}}$ be the full subcategory of $\mathrm{Fun}_{/N(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by those functors which preserve inert arrows. An object in $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ is called a **symmetric lax-monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$.

Remark 5.75. Note that we do not use the words “braided functors” in the ∞-categorical context, but use the word “symmetric functors”. They coincide only in the 1-categorical case.

5.4.3 Commutative algebras as coCartesian sections

Recall that the simplest symmetric monoidal category is the category $[0]$ consisting of only one object $\bullet$, with all associators and braidings being the identity morphism. Then a commutative algebra in $\mathcal{C}$ is the same as a braided lax-monoidal functor $A : [0] \to \mathcal{C}$. Moreover, an algebra homomorphism is the same as a monoidal natural transformation, i.e. there is an equivalence of categories

$$\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lax}}([0], \mathcal{C})$$

Observe that, the coCartesian fibration associated with the trivial symmetric monoidal category $[0]$ is just the identity map $\mathrm{Id} : \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$. Then we have

Proposition 5.76. *Let $\pi : \mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal category, with underlying category $\mathcal{C}$. Then there is a canonical equivalence of categories*

$$\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{in}}_{/\mathcal{F}\mathrm{in}_*}(\mathcal{F}\mathrm{in}_*, \mathcal{C}^\otimes)$$

where an object of the latter category is section $p : \mathcal{F}\mathrm{in}_* \to \mathcal{C}^\otimes$ of π such that p sends inert morphisms to inert maps (i.e. coCartesian lifts of inert maps).

Definition 5.77. Let $\pi : \mathcal{C}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. A **commutative algebra** in $\mathcal{C}$ is a section $p : N(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}^{\otimes}$ which preserves invert morphisms. We write $\mathrm{CAlg}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}_{/N(\mathcal{F}\mathrm{in}_*)}(N(\mathcal{F}\mathrm{in}_*), \mathcal{C}^{\otimes})$ spanned by such sections.

6 Limits, colimits and Kan extensions

6.1 Initial and final objects

Definition 6.1. Let $\mathcal{C}$ be an ∞-category. An object x is said to be **initial** if $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible for all $y \in \mathcal{C}$. Dually, x is said to be **final** if $\mathrm{Map}_{\mathcal{C}}(y, x)$ is contractible for all $y \in \mathcal{C}$.

Lemma 6.2. *Let $\mathcal{C}$ be an ∞-category and x in $\mathcal{C}$ an object. Then TFAE:*

(1) x is final.

(2) The functor $\mathcal{C}_{/x} \to \mathcal{C}$ is a trivial fibration.

(3) For every $n \geq 1$, every lifting problem

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

can be solved.

Proof. To show (1) $\Leftrightarrow$ (2), consider the following diagram, which is a morphism between right fibrations:

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dl, "="] \\ & \mathcal{C} & \end{tikzcd}$$

Our purpose is to show that the horizontal map is a trivial fibration. Since it is a right fibration, it suffices to show that it is a Joyal equivalence if and only if x is final. Apply Theorem 5.49, and note that the induced map on the fibre over y is $\mathrm{Map}^R_{\mathcal{C}}(y, x) \to \Delta^0$.

To show that (2) $\Leftrightarrow$ (3), we consider the lifting problem for $n \geq 1$:

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Note that the canonical inclusion $\partial\Delta^n \to \Delta^n$ is isomorphic to the map

$$\partial\Delta^n \star \Delta^0 \coprod_{\partial\Delta^{n-1}\star\emptyset} \Delta^{n-1} \star \emptyset \to \Delta^{n-1} \star \Delta^0$$

Hence by adjunction the above lifting problem is equivalent to the following one

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & \mathcal{C}_{/x} \arrow[d] \\ \Delta^{n-1} \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

$\square$

Proposition 6.3. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_{\text{final}}$ be the full subcategory spanned by all final objects. Then $\mathcal{C}_{\text{final}}$ is either empty or a contractible Kan complex.*

Proof. Assume that $\mathcal{C}_{\text{final}}$ is not empty, consider the following lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{\text{final}} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

If $n = 0$, a solution exists by the assumption that $\mathcal{C}_{\text{final}}$ is non-empty. For $n \geq 1$, apply Lemma 6.2 and note that each vertex of $\partial\Delta^n$ is mapped to a final object. $\square$

Proposition 6.4. *Let $\mathcal{C}$ be an ∞-category and $f : x \to y$ be a morphism. Then TFAE:*

(1) f is an equivalence.

(2) f is initial as an object of $\mathcal{C}_{x/}$.

(3) f is final as an object of $\mathcal{C}_{/y}$.

(4) f is initial as an object of $\mathcal{C}^{x/}$.

(5) f is final as an object of $\mathcal{C}^{/y}$.

Proof. (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) follow from Proposition 4.11. Now we prove the equivalence (1) $\Leftrightarrow$ (2), and the equivalence (1) $\Leftrightarrow$ (3) follows from a similar argument. f is initial in $\mathcal{C}_{x/}$ if and only if the projection $(\mathcal{C}_{x/})_{f/} \to \mathcal{C}_{x/}$ is a trivial Kan fibration. Note that $(\mathcal{C}_{x/})_{f/} \cong \mathcal{C}_{f/}$, so we have a lifting problem for $n \geq 0$:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{x/} \end{tikzcd}$$

By adjunction, we get an equivalent lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^0_1 \star \Delta^n \cup \Delta^1 \star \partial\Delta^n = \Lambda^{n+2}_0 \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^{n+2} \arrow[ur, dashed] & \end{tikzcd}$$

which can solved if and only if f is an equivalence, according to Joyal's special horn lifting theorem (Theorem 2.69). $\square$

Corollary 6.5. *Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ be an object. Then $\mathrm{Id}_x : x \to x$ is an initial object in $\mathcal{C}_{x/}$ and $\mathcal{C}^{x/}$. Similarly, Id_x is a final object in $\mathcal{C}_{/x}$ and $\mathcal{C}^{/x}$.*

Corollary 6.6. *Let $\mathcal{C}$ be an ∞-category and let $y \in \mathcal{C}$ be an object. Then:*

(1) y is final if and only if the projection map $F : \mathcal{C}_{/y} \to \mathcal{C}$ admits a section G such that $G(y) = \mathrm{Id}_y$.

(2) y is initial if and only if the projection map $F' : \mathcal{C}_{y/} \to \mathcal{C}$ admits a section G' such that $G'(y) = \mathrm{Id}_y$.

Proof. We only prove statement (1). Suppose y is final, then the canonical projection $\mathcal{C}_{/y} \to \mathcal{C}$ is a trivial Kan fibration, hence the following lifting problem admits a solution:

$$\begin{tikzcd} \Delta^0 \arrow[r, "\mathrm{Id}_y"] \arrow[d, "y"'] & \mathcal{C}_{/y} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "="'] \arrow[ur, dashed, "G"] & \mathcal{C} \end{tikzcd}$$

Conversely, suppose that F admits a section $G : \mathcal{C} \to \mathcal{C}_{/y}$ satisfying $G(y) = \mathrm{Id}_y$. Let $x \in \mathcal{C}$ be an object, the functors G and F induces morphisms of Kan complexes

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}_{/y}}(G(x), \mathrm{Id}_y) \xrightarrow{F} \mathrm{Map}_{\mathcal{C}}(x, y),$$

such that the composition is identity. In particular, the Kan complex $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a retract of $\mathrm{Map}_{\mathcal{C}_{/y}}(G(x), \mathrm{Id}_y)$, which is contractible since Id_y is final in $\mathcal{C}_{/y}$. Hence $\mathrm{Map}_{\mathcal{C}}(x, y)$ is again contractible. Allowing x to vary, we see that y is final in $\mathcal{C}$. □

Lemma 6.7. *If i is a monomorphism, then $\Delta^0 \star i$ is left anodyne and $i \star \Delta^0$ is right anodyne.*

Proof. Since any monomorphism can be written as a transfinite composition of pushouts of morphisms in Λ^{bdy}, we see that $\Delta^0 \star i$ lies in the saturated class generated by $\Delta^0 \star \Lambda^{bdy}$. But $\Delta^0 \star (\partial\Delta^n \hookrightarrow \Delta^n)$ is isomorphic to $\Lambda^{n+1}_0 \hookrightarrow \Delta^{n+1}$. □

Corollary 6.8. *Let $\mathcal{C}$ be an ∞-category. Then an object $x \in \mathcal{C}$ is initial if and only if the map $x : \Delta^0 \to \mathcal{C}$ is left anodyne. Dually, x is final if and only if the map $x : \Delta^0 \to \mathcal{C}$ is right anodyne.*

Proof. We will only prove the case for initial objects. Assume that $x \in \mathcal{C}$ is an initial object, then the projection map $F : \mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration which admits a section $G : \mathcal{C} \to \mathcal{C}_{x/}$ such that $G(x) = \mathrm{Id}_x$. By the definition of the slice category $\mathcal{C}_{x/}$, G is equivalent to a morphism $G^\sharp : \mathcal{C}^\triangleleft \to \mathcal{C}$ such that $G^\sharp|_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$, and that $\Delta^0 \star \{x\}$ is carried to Id_x. Now consider the following diagram

$$\begin{tikzcd} \{x\} \arrow[r] \arrow[d] & \{x\}^\triangleleft \arrow[r] \arrow[d] & \{x\} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{C}^\triangleleft \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

where the horizontal compositions are identities, and the mid horizontal morphism is left anodyne, according to Lemma 6.7. Hence the inclusion $\{x\} \to \mathcal{C}$ is left anodyne.

Conversely, suppose that the inclusion $j : \{x\} \to \mathcal{C}$ is left anodyne. We write $i : \emptyset \to \Delta^0$. As a result the pushout join

$$i \hat{\star} j : \{x\}^\triangleleft \cup \mathcal{C} \hookrightarrow \mathcal{C}^\triangleleft$$

is inner anodyne. Consider the following lifting problem where K is determined by the requirements that $K|_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$ and $K(\{x\}^\triangleleft) = \mathrm{Id}_x$:

$$\begin{tikzcd} \{x\}^\triangleleft \cup \mathcal{C} \arrow[r, "K"] \arrow[d, "i \hat{\star} j"'] & \mathcal{C} \\ \mathcal{C}^\triangleleft \arrow[ur, dashed, "F"] & \end{tikzcd}$$

which can be solved since $i \hat{\star} j$ is inner anodyne. We choose a solution $H : \mathcal{C}^\triangleleft \to \mathcal{C}$, then its adjunct $H^\flat : \mathcal{C} \to \mathcal{C}_{x/}$ is a section of the projection map $\mathcal{C}_{x/} \to \mathcal{C}$ such that $H^\flat(x) = \mathrm{Id}_x$. Now apply Corollary 6.6. □

6.2 Yoneda Lemma

6.2.1 Representable fibrations

Let us recall the classical Yoneda lemma. Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then for $x \in \mathcal{C}$, there is a canonical isomorphism of sets

$$\mathrm{Nat}(h_x, F) \cong F(x)$$

where $h_x = \hom_{\mathcal{C}}(x, -)$. Recall that, by Grothendieck construction, we have a canonical equivalence of categories

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \simeq \mathrm{LC}(\mathcal{C}).$$

As a result, we can reformulate Yoneda lemma in the Language of left covering maps:

Lemma 6.9 (Yoneda Lemma)**.** *Let $F : \mathcal{E} \to \mathcal{C}$ be a left covering map of 1-categories, and $x \in \mathcal{C}$ is an object. Then there is a canonical isomorphism of sets:*

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_{x/}, F) \cong F^{-1}(x)$$

The key point is to noticing that: if there is a functor $K : \mathcal{C}_{x/} \to \mathcal{E}$ over $\mathcal{C}$, then K is completely determined by $K(\mathrm{Id}_x) \in F^{-1}(x)$. For example, if we want to determine $K(f)$ for a morphism $f : x \to y$, then we see that

$$K(f) = f_!(K(\mathrm{Id}_x))$$

where $f_! : F^{-1}(x) \to F^{-1}(y)$ is the covariant transport defined by f.

Let $F : \mathcal{E} \to \mathcal{C}$ be a left covering map. Let $e \in \mathcal{E}$ and $c = F(e)$, so that $e \in F^{-1}(c)$. For all object $y \in \mathcal{C}$, e defines a map

$$\rho_y : \hom_{\mathcal{C}}(x, y) \to F^{-1}(y), \quad f \mapsto f_!(e).$$

The proof of the following proposition is easy:

Proposition 6.10. *Let $F : \mathcal{E} \to \mathcal{C}$ be a left covering map. Let $e \in \mathcal{E}$ and $c = F(e)$, so that $e \in F^{-1}(c)$. Let $K : \mathcal{C}_{c/} \to \mathcal{E}$ be the functor determined by e, via Yoneda lemma. Then TFAE:*

(1) e is initial in $\mathcal{E}$.

(2) K is an equivalence of categories.

(3) For all $y \in \mathcal{C}$, the map $\rho_y : \hom_{\mathcal{C}}(x, y) \to F^{-1}(y)$ is an isomorphism of sets.

Definition 6.11. Let $F : \mathcal{E} \to \mathcal{C}$ be a left covering map. We say that F is **representable** if there exists an $e \in \mathcal{E}$ such that either condition of Proposition 6.10 is verified. In this case we say that F **is represented by** $c \in \mathcal{C}$.

Now let us turn to the ∞-categorical context.

Proposition 6.12. *Let $p : \mathcal{D} \to \mathcal{C}$ be a left fibration of ∞-categories. Let $d \in \mathcal{D}$ be an object and $x = p(d)$, so that $d \in p^{-1}(x)$. Then TFAE:*

(1) There exists an equivalence $F : \mathcal{C}_{x/} \to \mathcal{D}$ left fibrations over $\mathcal{C}$ satisfying $F(\mathrm{Id}_x) = d$.

(2) The object $d \in \mathcal{D}$ is initial.

(3) For every left fibration $V : \mathcal{E} \to \mathcal{C}$, evaluation on the object d induces a trivial Kan fibration of Kan complexes $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}) \to V^{-1}(x)$.

(4) For every left fibration $V : \mathcal{E} \to \mathcal{C}$, evaluation on the object d induces a bijection of sets

$$\pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E})) \to \pi_0(V^{-1}(x)).$$

Proof. (1) $\Rightarrow$ (2) is obvious since Id_x is an initial object of $\mathcal{C}_{x/}$. Now we show (2) $\Rightarrow$ (3). According to Corollary 6.8, the morphism $d : \Delta^0 \to \mathcal{D}$ is left anodyne. Hence the pullback-hom

$$\hat{\mathrm{Fun}}(d, V) : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \times_{\mathcal{C}} \mathcal{E}$$

is a trivial Kan fibration. Now consider the following diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \arrow[d] \\
V^{-1}(x) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \arrow[d] \\
\Delta^0 \arrow[r, "p"'] & \mathrm{Fun}(\mathcal{D}, \mathcal{C})
\end{tikzcd}$$

where the outer square and the lower square are both pullbacks, hence the upper square is also a pullback. As a result, the induced map $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}) \to V^{-1}(x)$ is a trivial Kan fibration.

The implication (3) $\Rightarrow$ (4) is obvious. It remains to show that (4) $\Rightarrow$ (1). Note that the left fibration $\mathcal{C}_{x/} \to \mathcal{C}$ satisfies condition (1) and hence satisfies condition (3). It follows that the object $d \in \mathcal{D}$ determines a map $F : \mathcal{C}_{x/} \to \mathcal{D}$ (unique up to homotopy) such that $F(\mathrm{Id}_x) = d$. Now assume that condition (4) is satisfied, we have a commutative diagram of sets

$$\begin{tikzcd}
\pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E})) \arrow[rr, "-\circ[F]"] \arrow[dr] & & \pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_{x/}, \mathcal{E})) \arrow[dl] \\
& \pi_0(V^{-1}(x)) &
\end{tikzcd}$$

where the non-horizontal maps are bijective, hence the horizontal map is again bijective. Take $\mathcal{E} = \mathcal{C}_{x/}$, we can find a map of left fibrations $G : \mathcal{D} \to \mathcal{C}_{x/}$ such that $[G \circ F] = [\mathrm{Id}_{\mathcal{C}_{x/}}]$. Take $\mathcal{E} = \mathcal{D}$, the bijectivity of $- \circ [F]$ forces that $[F \circ G] = [\mathrm{Id}_{\mathcal{D}}]$, so F is an equivalence of left fibrations. □

Proposition 6.13. *Let $p : \mathcal{D} \to \mathcal{C}$ be a left fibration of ∞-categories, and $x \in \mathcal{C}$ is an object. Let $d \in p^{-1}(x)$ be an object in the fibre of x. Then d is an initial object of $\mathcal{D}$ if and only if, for every object $y \in \mathcal{C}$, the composition*

$$\rho_y : \mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\theta} \mathrm{Fun}(p^{-1}(x), p^{-1}(y)) \longrightarrow p^{-1}(y)$$

is a homotopy equivalence. Here the first arrow θ is given by parametrized covariant transport, and the second arrow is given by evaluation on d.

Proof. According to Proposition 6.12, the object $d \in \mathcal{D}$ determines a morphism of left fibrations over $\mathcal{C}$, which we denote by $F : \mathcal{C}^{x/} \to \mathcal{D}$, such that $F(\mathrm{Id}_x) = d$. Moreover F is an equivalence if and only if d is initial. According to Theorem 5.49, F is an equivalence if and only if the induced maps in each fibre

$$F_y : (\mathcal{C}^{x/})_y = \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathcal{D}_y = p^{-1}(y)$$

is a homotopy equivalence. We conclude the proof by noticing that F_y is homotopic to ρ_y. □

Let $\mathcal{C}$ be an ∞-category and $c \in \mathcal{C}$. Let $\mathrm{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathcal{S}$ be the functor classifying the left fibration $\mathcal{C}^{c/} \to \mathcal{C}$.

Corollary 6.14. *Let $\mathcal{C}$ be an ∞-category and $c \in \mathcal{C}$ be an object. Let $F : \mathcal{C} \to CS$ be a functor. Then the canonical morphism*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathrm{Map}_{\mathcal{C}}(c, -), F) \to F(c)$$

given by evaluation at the identity Id_c is a homotopy equivalence.

Proof. Combine straightening-unstraightening equivalence and Proposition 6.12. □

6.2.2 Yoneda embedding

Again we start from 1-categories, and then generalize to ∞-categories.

Definition 6.15. Let $\mathcal{C}$ be a category, then the **twisted arrow category** $\mathrm{Tw}(\mathcal{C})$ of $\mathcal{C}$ is defined as follows:

- An object of $\mathrm{Tw}(\mathcal{C})$ is a morphism of $\mathcal{C}$.
- A morphism from $f : x \to y$ to $f' : x' \to y'$ consists of two morphism $a : x \to x'$, $b : y' \to y$ such that the following diagram is commutative:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "a"'] & y \\ x' \arrow[r, "f'"'] & y' \arrow[u, "b"'] \end{tikzcd}$$

Note that there is a canonical projection $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ sending a morphism to its source and target.

The following proposition is easy to prove:

Proposition 6.16. *The projection $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right covering functor classified by $\mathrm{hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$. Moreover, there are pullback diagrams*

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \times \{x\} \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{C}^{\mathrm{op}})_{/y} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\ \{y\} \times \mathcal{C}^{\mathrm{op}} \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd}$$

Definition 6.17. Let $\epsilon : \Delta \to \Delta$ be the functor $[n] \mapsto [n] \star [n]^{\mathrm{op}} \cong [2n + 1]$. Composition with ϵ induces a functor $\epsilon^* : \mathrm{sSet} \to \mathrm{sSet}$. If $\mathcal{C}$ is an ∞-category we define $\mathrm{Tw}(\mathcal{C}) := \epsilon^*\mathcal{C}$. The natural inclusions $[n], [n]^{\mathrm{op}} \to \epsilon([n])$ induce a natural morphism $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

The proofs of the following two theorems are rather involved, hence we will omit the details.

Theorem 6.18 ([Lur17], Proposition 5.2.1.3)**.** *Let $\mathcal{C}$ be an ∞-category. Then the canonical projection $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right fibration. In particular, $\mathrm{Tw}(\mathcal{C})$ is an ∞-category.*

Theorem 6.19 ([Lur17], Proposition 5.2.1.10)**.** *Let $\mathcal{C}$ be an ∞-category, and $x \in \mathcal{C}$ an object. There is a canonical commutative diagram*

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \times \{x\} \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd}$$

which is homotopy Cartesian.

As a result, the fibre of $\mathrm{Tw}(\mathcal{C})$ at (x, y) is homotopy equivalent to the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$.

Definition 6.20. Let $\mathcal{C}$ be an ∞-category. We will let $\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be the functor classifying the right fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. By adjunction we obtain a functor $\mathcal{Y} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, which we will refer to as the **Yoneda embedding functor**.

Note that we have $\mathcal{Y}(c) = \mathrm{Map}_{\mathcal{C}}(-, c)$. The following proposition justifies the name "Yoneda embedding".

Proposition 6.21. *The Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful.*

Proof. We have the following homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\mathrm{Map}_{\mathcal{C}}(-, x), \mathrm{Map}_{\mathcal{C}}(-, y)) \cong \mathrm{Map}_{\mathcal{C}}(x, y).$$

□

6.3 Limits and colimits

6.3.1 (Co)limit diagrams

Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. For each object $x \in \mathcal{C}$, we will write $\delta_x \in \mathrm{Fun}(K, \mathcal{C})$ for the constant functor with value x. More explicitly, δ_x is defined by

$$\delta_x : K \to \Delta^0 \xrightarrow{x} \mathcal{C}$$

Note that the construction $x \mapsto \delta_x$ determines a functor of ∞-categories $\delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$, carrying each morphism $f : x \to y$ to a natural transformation

$$\delta_f : \delta_x \Rightarrow \delta_y$$

Definition 6.22. Let $\mathcal{C}$ be an ∞-category, and $y \in \mathcal{C}$ an object. Let K be a simplicial set and $F : K \to \mathcal{C}$ be a diagram. We say that a natural transformation $\alpha : \delta_y \Rightarrow u$ **exhibits** y **as a limit of** F if the following condition is satisfied:

- ⋆ For each object $x \in \mathcal{C}$, the composition

  $$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, \delta_y) \xrightarrow{[\alpha]\circ -} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, F)$$

  is an isomorphism in the homotopy category of Kan complexes $h\mathcal{S}$.

Dually, we say that a natural transformation $\beta : F \Rightarrow \delta_y$ **exhibits** y **as a colimit** of F if the following condition is verified

⋆ For each object $y \in \mathcal{C}$, the composition

$$\mathrm{Map}_{\mathcal{C}}(y,x) \to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_y,\delta_x) \xrightarrow{-\circ[\beta]} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(F,\delta_x)$$

is an isomorphism in the homotopy category of Kan complexes $h\mathcal{S}$.

Let us recall that construction of the fat slice category $\mathcal{C}^{/F}$ and $\mathcal{C}^{F/}$. An object of $\mathcal{C}^{F/}$ is a functor $f : K \diamond \Delta^0 \to \mathcal{C}$ such that $f|_K = F$. This is equivalent to choosing an object $x \in \mathcal{C}$ and a natural transformation $\alpha : \delta_x \Rightarrow F$. Similarly, choosing an object of $\mathcal{C}^{F/}$ is equivalent to choosing an object $y \in \mathcal{C}$ and a natural transformation $F \Rightarrow \delta_y$.

Proposition 6.23. *Let $\mathcal{C}$ be an ∞-category and $y \in \mathcal{C}$ is an object. Let $F : K \to \mathcal{C}$ be a diagram, then*

- *A natural transformation $\alpha : \delta_y \Rightarrow F$ exhibits y as a limit of F if and only if it is final when regarded as an object of $\mathcal{C}^{/F}$.*
- *A natural transformation $\alpha : F \Rightarrow \delta_y$ exhibits y as a colimit of F if and only if it is initial when regarded as an object of $\mathcal{C}^{F/}$.*

Proof. We only prove the first assertion; the second follows by a similar argument. Consider the right fibration $\theta : \mathcal{C}^{/F} \to \mathcal{C}$, for an object $x \in \mathcal{C}$, we see that $\theta^{-1}(x) = \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, F)$. By the parametrized covariant transport we have a morphism for all $x \in \mathcal{C}$:

$$\rho_x : \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_y, F) \times \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, u)$$

which factors as a composition

$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_y, F) \times \mathrm{Map}_{\mathcal{C}}(x,y) &\to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_y, F) \times \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, \delta_y)\\ &\to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_x, F).\end{aligned}$$

It follows that a natural transformation $\alpha : \delta_y \Rightarrow F$ exhibits y as a limit of F if and only if for all $x \in \mathcal{C}$, the restriction $\rho_x|_{\{\alpha\}\times \mathrm{Map}_{\mathcal{C}}(x,y)}$ is a homotopy equivalence of Kan complexes, and this is equivalent to the requirement that α is final when regarded as an object of $\mathcal{C}^{/F}$, see Proposition 6.13. □

Corollary 6.24. *Let $\mathcal{C}$ be an ∞-category, and let $F : K \to \mathcal{C}$ be a diagram. Then TFAE:*

(1) The diagram u has a limit in $\mathcal{C}$.

(2) The fat slice ∞-category $\mathcal{C}^{/F}$ has a final object.

(3) The slice ∞-category $\mathcal{C}_{/F}$ has a final object.

Proof. (1) ⇔ (2) follows from Proposition 6.23. (2) ⇔ (3) follows from 4.11. □

The above analysis justifies the following definition.

Definition 6.25. Let $\mathcal{C}$ be an ∞-category, and K be a simplicial set, and let $\overline{F} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets carrying $-\infty$ to an object $y \in \mathcal{C}$. Set $F = \overline{F}|_K$, so that the diagram $\overline{F}$ can be identified with an object of the slice ∞-category $\mathcal{C}_{/F}$. We say that $\overline{F}$ is a **limit diagram** if it is a final object of $\mathcal{C}_{/F}$. If this condition is satisfied, we say that $\overline{F}$ **exhibits y as a limit of F**.

There is a dual notion for colimits, and we do not write down details here.

Proposition 6.26. *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $\overline{F} : K^{\triangleleft} \to \mathcal{C}$ be a morphism and set $F = \overline{F}|_K$. Then TFAE:*

(1) The morphism $\overline{F}$ is a limit diagram.

(2) The map $\mathcal{C}_{/\overline{F}} \to \mathcal{C}_{/F}$ induced by the canonical inclusion $K \to K^{\triangleleft}$ is a trivial Kan fibration.

(3) The map $\mathcal{C}_{/\overline{F}} \to \mathcal{C}_{/F}$ is a Joyal equivalence.

(4) For all $x \in \mathcal{C}$ the induced map on fibres

$$\{x\} \times_{\mathcal{C}} \mathcal{C}_{/\overline{F}} \to \{x\} \times_{\mathcal{C}} \mathcal{C}_{/F}$$

is a homotopy equivalence of Kan complexes.

Proof. Note that $(\mathcal{C}_{/F})_{/\overline{F}} = \mathcal{C}_{/\overline{F}}$, so that (1) $\Leftrightarrow$ (2) follows from Lemma 6.2. (2) $\Leftrightarrow$ (3) follows from the fact that map $\mathcal{C}_{/\overline{F}} \to \mathcal{C}_{/F}$ is a right fibration and hence an isofibration. (3) $\Leftrightarrow$ (4) follows from Theorem 5.49. □

6.3.2 Relative (co)limits

Definition 6.27. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that an object $y \in \mathcal{C}$ is F**-final** if for every object $x \in \mathcal{C}$, F induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(x), F(y)).$$

Dually, y is said to be F-initial if for every object $z \in \mathcal{C}$, the functor F induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{D}}(F(y), F(z)).$$

Example 6.28. If $\mathcal{D} = \Delta^0$ and $F : \mathcal{C} \to \Delta^0$ be the unique projection. Then an object $y \in \mathcal{D}$ is F-final if and only of it is final, and y is F-initial if and only if it is initial.

Example 6.29. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then TFAE:

- The functor F is fully faithful.
- Every object of $\mathcal{C}$ is F-initial.
- Every object of $\mathcal{C}$ is F-final.

Proposition 6.30. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and $c \in \mathcal{C}$ be an object in $\mathcal{C}$. Then TFAE:*

(1) c is F-final.

(2) The canonical morphism morphism G of right fibrations of $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C}^{/c} \arrow[rr, "G"] \arrow[dr] & & \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{/F(c)} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

is an equivalence of ∞-categories.

(3) The canonical morphism morphism H of right fibrations of $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C}_{/c} \arrow[rr, "H"] \arrow[dr] & & \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/F(c)} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

is an equivalence of ∞-categories.

Proof. (2) ⇔ (3) follows from Proposition 4.11. To show that (1) ⇔ (2), apply Theorem 5.49. ☐

Corollary 6.31. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $c \in \mathcal{C}$ be an object. Then TFAE:*

(1) The object c is F-final.

(2) The induced map $\mathcal{C}_{/c} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{C}_{/F(c)}$ is a trivial fibration.

(3) For every $n > 0$, every lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\sigma"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

such that $\sigma(n) = c$, admits a solution.

Proof. (2) ⇔ (3) follows from the observation $(\partial\Delta^{n-1} \hookrightarrow \Delta^{n-1})\hat{\star}(\emptyset \hookrightarrow \Delta^0) = \partial\Delta^n \hookrightarrow \Delta^n$. (1) ⇔ (2) follows from Joyal's model structure, that is a Joyal fibration is a trivial fibration if and only if it is a Joyal equivalence. ☐

Proposition 6.32. *Suppose we are given a commutative diagram of ∞-categories:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\ & \mathcal{E} & \end{tikzcd}$$

where U and V are isofibrations. Let $e \in \mathcal{E}$ be an object and let $F_e : \mathcal{C}_e \to \mathcal{D}_e$ denote the induced morphism of the fibre over e. Then:

(1) If $x \in \mathcal{C}_e$ is F-initial, then x is F-initial.

(2) Assume that U and V are Cartesian fibrations, and F preserves Cartesian edges. Then if x is F_e-initial, then it is F-initial.

Proof. We first prove (1). Assume that x is F-initial. ∀ object $y \in \mathcal{C}_e$, we have a commutative diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Map}_{\mathcal{D}}(F(x),F(y)) \arrow[dl] \\ & \mathrm{Map}_{\mathcal{E}}(e,e) & \end{tikzcd}$$

where the non-horizontal morphisms are Kan fibrations (since U and V are isofibrations, see Lemma 5.41) and ρ is a homotopy equivalence by our assumption. Then apply Proposition 3.14, we conclude that ρ restricts to a homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}_e}(x,y) &\cong \mathrm{Map}_{\mathcal{C}}(x,y) \times_{\mathrm{Map}_{\mathcal{E}}(e,e)} \{\mathrm{Id}_e\} \\ &\to \mathrm{Map}_{\mathcal{C}}(F(x),F(y)) \times_{\mathrm{Map}_{\mathcal{E}}(e,e)} \{\mathrm{Id}_e\} \\ &\cong \mathrm{Map}_{\mathcal{D}_e}(F(x),F(y)). \end{aligned}$$

Since the choice of y is arbitrary, it follows that x is F_e-initial object of $\mathcal{C}$.

Now we prove (2). Assume that U and V are Cartesian fibrations, that F preserves Cartesian edges, and that x is F_e-initial. Fixing an object $z \in \mathcal{C}$, we wish to show that the horizontal morphism in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(x,z) \arrow[rr, "\theta"] \arrow[dr, "U_{x,z}"'] & & \mathrm{Map}_{\mathcal{D}}(F(x),F(z)) \arrow[dl, "V_{x,z}"] \\
& \mathrm{Map}_{\mathcal{E}}(U(x),U(z)) &
\end{tikzcd}$$

is a homotopy equivalence. Since the vertical maps are Kan fibrations, according to Proposition 3.14, it will suffice to show that for each morphism $\tilde{f} \in \mathrm{Map}_{\mathcal{E}}(U(x),U(y))$ the induced map

$$\theta_{\tilde{f}} : \mathrm{Map}_{\mathcal{C}}(x,z) \times_{\mathrm{Map}_{\mathcal{E}}(U(x),U(z))} \{\tilde{f}\} \to \mathrm{Map}_{\mathcal{D}}(F(x),F(z)) \times_{\mathrm{Map}_{\mathcal{E}}(U(x),U(z))} \{\tilde{f}\}$$

is a homotopy equivalence. Let $f : y \to z$ be a U-Cartesian lift of $\tilde{f}$, then $F(f) : F(y) \to F(z)$ is a V-Cartesian lift of f. Applying (the dual version of) Corollary 5.44, we see that there are homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(x,z) \times_{\mathrm{Map}_{\mathcal{E}}(U(x),U(z))} \{\tilde{f}\} \cong \mathrm{Map}_{\mathcal{C}_e}(x,y),$$
$$\mathrm{Map}_{\mathcal{C}}(F(x),F(z)) \times_{\mathrm{Map}_{\mathcal{E}}(U(x),U(z))} \{\tilde{f}\} \cong \mathrm{Map}_{\mathcal{D}_e}(F(x),F(y))$$

so we can replace $\theta_{\tilde{f}}$ by the morphism

$$\mathrm{Map}_{\mathcal{C}_e}(x,y) \to \mathrm{Map}_{\mathcal{D}_e}(F(x),F(y)).$$

$\square$

Now we turn to study relative (co)limits.

Definition 6.33. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a diagram in $\mathcal{C}$, and $f = \overline{f}|_K$ so that $\overline{f}$ can be viewed as an object in $\mathcal{C}_{/f}$. Now we have a functor $U_{/f} : \mathcal{C}_{/f} \to \mathcal{D}_{/U\circ f}$. We will say that $\overline{f}$ is a ***U*-limit diagram** if it is $U_{/f}$-final. Dually, we will say that a diagram $\overline{g} : K^{\triangleright} \to \mathcal{C}$ with restriction $g = \overline{g}|_K$ is a ***U*-colimit diagram** if $\overline{g}$ is $U_{g/}$-initial.

Example 6.34. Let $\mathcal{C}$ be an ∞-category and $F : \mathcal{C} \to \Delta^0$ be the projection map. Then a morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a F-limit if and only if it is a limit diagram.

Example 6.35. Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then

(1) A morphism e of $\mathcal{C}$ is F-Cartesian if and only if it is a F-limit diagram when viewed as a morphism of simplicial sets $(\Delta^0)^{\triangleleft} \to \mathcal{C}$.

(2) A morphism e of $\mathcal{C}$ is F-coCartesian if and only if it is a F-colimit diagram when viewed as a morphism of simplicial sets $(\Delta^0)^{\triangleright} \to \mathcal{C}$.

Proposition 6.36. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a diagram. Then TFAE:*

(1) $\overline{f}$ is a F-limit diagram.

(2) The induced map $\mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f} \times_{\mathcal{D}_{/f}} \mathcal{D}_{/\overline{f}}$ is a trivial Kan fibration.

(3) Every lifting problem

$$\begin{tikzcd}
\partial\Delta^n \star K \arrow[r, "\rho"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^n \star K \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution, provided that $n \geq 1$ and the restriction of ρ to $\{n\} \star K \subseteq \partial\Delta^n \star K$ coincides with $\overline{f}$.

Proof. Combine the definition of a relative limit diagram and Corollary 6.31. $\square$

Corollary 6.37. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a diagram. Then $\overline{f}$ is a F-limit diagram if and only if, for every object $c \in \mathcal{C}$, the diagram of morphism spaces*

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{C})}(\delta_c, \overline{f}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_c|_K, \overline{f}|_K) \arrow[d] \\
\mathrm{Map}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{D})}(F \circ \delta_c, F \circ \overline{f}) \arrow[r] & \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{D})}(F \circ \delta_c|_K, F \circ \overline{f}|_K)
\end{tikzcd}$$

is a homotopy pullback square.

6.4 Final and initial functors

6.4.1 Final/initial functors of ordinary categories

This section is based on [Rie14], Section 8.3.

Let $\mathcal{C}, \mathcal{D}$ be categories. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we can define its colimit to be a universal cone under F, which we denote by $\varinjlim F$. When we need to highlight the source category $\mathcal{C}$, we may also use the notation $\varinjlim_{c \in \mathcal{C}} F(c)$. For $d \in \mathcal{D}$, we use δ_d to denote the constant functor at d.

Now take a new category $\mathcal{E}$ and a functor $G : \mathcal{D} \to \mathcal{E}$, we get a canonical map

$$\varinjlim_{c \in \mathcal{C}} G(F(c)) \to \varinjlim_{d \in \mathcal{D}} G(d)$$

induced by the cone legs $\kappa_c : G(F(c)) \to \varinjlim G$, for all $c \in \mathcal{C}$.

Definition 6.38. A functor $F : \mathcal{C} \to \mathcal{D}$ is **final** if the canonical map defined above is an isomorphism for all categories $\mathcal{E}$ and all functors $G \in \mathrm{Fun}(\mathcal{D}, \mathcal{E})$.

Example 6.39. Let $\mathcal{D}$ be a category with a final object t. Then the functor $T : [0] \to \mathcal{D}$ picking out the final object is a final functor. To see this, we need to check that for all functors $G : \mathcal{D} \to \mathcal{E}$, the canonical morphism

$$G(t) \to \varinjlim_{d \in \mathcal{D}} G(d)$$

is an isomorphism. But this is direct: there is a unique natural transformation $\mathrm{Id}_{\mathcal{D}} \Rightarrow \delta_t$. By applying G we obtain a cone $G \Rightarrow \delta_{G(t)}$, which can be easily checked to be universal.

Here is the main theorem for this section: the "Quilllen's theorem A" for 1-categories.

Theorem 6.40. *A functor $F : \mathcal{C} \to \mathcal{D}$ is final if and only if for each $d \in \mathcal{D}$ the comma category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is non-empty and connected.*

To give a complete proof we need the following lemma:

Lemma 6.41. *Let $\mathcal{C}$ be a category and $X \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$. Let $\int^{\mathcal{C}} X$ be the associated category of elements. Then there is a canonical isomorphism of sets:*

$$\pi_0(\int^{\mathcal{C}} X) \cong \varinjlim X$$

Proof. Let T be a set and $\alpha : X \Rightarrow \delta_T$. Unpacking the definition of a natural transformation we see that α is equivalent to the following data:

- To each $c \in \mathcal{C}$ we associate a map of sets $\alpha_c : X(c) \to T$, that is, to each $(c, u) \in \int^{\mathcal{C}} X$ we assign an element of T;
- Naturality says that if there is a morphism $f : c \to c'$ such that $X(f)(u) = u'$, then $\alpha_c(u) = \alpha_{c'}(u')$, that is, connected elements in $\int^{\mathcal{C}} X$ should be sent to the same element.

So every natural transformation gives rise to a unique map $\pi_0(\int^{\mathcal{C}} X) \to T$. $\square$

Proof of Theorem 6.40. $\Leftarrow$: Let $G : \mathcal{D} \to \mathcal{E}$ be a functor. A cone under G automatically gives a cone under $G \circ F$. Conversely, if the comma category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is non-empty and connected, then every cone under $G \circ F$ gives rise to a cone under G in the following way: Suppose we have an object $e \in \mathcal{E}$ and a natural transformation $\lambda : G \circ F \Rightarrow \delta_e$. Given $d \in \mathcal{D}$, since $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is non-empty, we may choose $(c, \alpha) \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ and we get a leg $\kappa_d : G(d) \overset{G(\alpha)}{\to} G(F(c)) \overset{\lambda_c}{\to} e$. Connectedness of $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ guarantees that the definition of κ_d is independent of the choice of the object $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$. So we get a bijection between the set of cones under G and that under GF, which extends to an isomorphism of categories.

$\Rightarrow$: Arbitrarily choose an object $d \in \mathcal{D}$, we want to show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is non-empty and connected. Consider the following composed functor

$$\mathcal{C} \overset{F}{\to} \mathcal{D} \overset{\hom_{\mathcal{D}}(d,-)}{\longrightarrow} \mathrm{Set}$$

By finality of F, there is a canonical isomorphism:

$$\varinjlim_{x \in \mathcal{C}} \hom_{\mathcal{D}}(d, F(x)) \to \varinjlim_{y \in \mathcal{D}} \hom_{\mathcal{D}}(d, y)$$

By previous discussion the LHS is $\pi_0(\int^{\mathcal{C}} \hom_{\mathcal{D}}(d, F(-))) = \pi_0(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/})$ which is the set we want to compute. The RHS is $\pi_0(\mathcal{D}_{d/}) = \{*\}$ since the category $\mathcal{D}_{d/}$ has an initial object $\mathrm{Id}_d : d \to d$. Note that a category with an initial object must be non-empty and connected. $\square$

Example 6.42. Let $\mathcal{D}$ be a category with a final object t and $T : [0] \to \mathcal{D}$ be the functor picking out the final object. Then $\forall d \in \mathcal{D}$, the comma category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is the one-point category $[0]$. So the functor T is final.

Example 6.43. Any right adjoint functor is final. By the definition of adjunction, a functor $G : \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if the comma category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ has an initial object for all $d \in \mathcal{D}$.

6.4.2 Final/initial functors of ∞-categories

Definition 6.44. Let $p : K \to L$ be a map of simplicial sets. We say that that p is **final** if, for any right fibration $X \to L$, the induced map

$$\mathrm{Fun}_{/L}(L, X) \to \mathrm{Fun}_{/L}(K, X)$$

is a homotopy equivalence of Kan complexes.

Proposition 6.45. *Let $f : K \to L$ and $g : L \to M$ be maps of simplicial sets. If f is final, then $g \circ f$ is final if and only if g is.*

Proof. Arbitrarily choose a right fibration $N \to M$ we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/M}(M, N) \arrow[r] \arrow[dr] & \mathrm{Fun}_{/M}(L, N) \arrow[d] & \mathrm{Fun}_{/L}(L, g^*(N)) \arrow[l, "\cong"'] \arrow[d] \\
& \mathrm{Fun}_{/M}(K, N) & \mathrm{Fun}_{/L}(K, g^*(N)) \arrow[l, "\cong"]
\end{tikzcd}$$

The right horizontal maps are isomorphisms. They are obtained as follows: we apply the limit-preserving functor $\mathrm{Fun}(L,-)$ and $\mathrm{Fun}(K,-)$ to the pullback diagram

$$\begin{tikzcd} g^*(N) \arrow[r] \arrow[d] & N \arrow[d] \\ L \arrow[r, "g"'] & M \end{tikzcd}$$

so there is an induced isomorphism of simplicial sets on each fibre. Since f is final, the right vertical map is a homotopy equivalence, and so is the left vertical map. As a result, g is final if and only if $g \circ f$ is final, due to the 2-out-of-3 property of homotopy equivalences. $\square$

Proposition 6.46. *If f is a monomorphism, then f is final if and only if it is right-anodyne.*

Proof. We claim that if $f : K \to L$ is right anodyne, then for any right fibration $p : T \to L$ the induced map $\mathrm{Fun}_{/L}(L,T) \to \mathrm{Fun}_{/L}(K,T)$ is a trivial Kan fibration. To do so, note that the pullback hom
$$\hat{\mathrm{Fun}}(f,p) : \mathrm{Fun}(L,T) \to \mathrm{Fun}(K,T) \times_{\mathrm{Fun}(K,L)} \mathrm{Fun}(L,L)$$
is a trivial Kan fibration. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}_{/L}(L,T) \arrow[r] \arrow[d] & \mathrm{Fun}(L,T) \arrow[d] \\ \mathrm{Fun}_{/L}(K,T) \arrow[r] \arrow[d] & \mathrm{Fun}(K,T) \times_{\mathrm{Fun}(K,L)} \mathrm{Fun}(L,L) \arrow[d] \\ \Delta^0 \arrow[r, "\mathrm{Id}_L"'] & \mathrm{Fun}(L,L) \end{tikzcd}$$

where the outer square and the lower square are both pullbacks. Hence the upper square is again a pullback.

Conversely, suppose that f is a final monomorphism and let $q : X \to Y$ be a right fibration. To show that f is right anodyne, we need to solve a corresponding lifting problem. By pulling back, we may assume that $Y = L$. Thus we obtain the following lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ L \arrow[r, "="'] \arrow[ur, dashed] & L \end{tikzcd}$$

By assumption, we know that the morphism

$$f^* : \mathrm{Fun}_{/L}(L,X) \to \mathrm{Fun}_{/L}(K,X)$$

is a Joyal equivalence. If we can show that it is in addition an isofibration, then it is a trivial fibration and hence surjective on vertexes, which gives the desired lift. Consider the following diagram

$$\begin{tikzcd} \mathrm{Fun}_{/L}(L,X) \arrow[r] \arrow[d] & \mathrm{Fun}(L,X) \arrow[d] \\ \mathrm{Fun}_{/L}(K,X) \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \times_{\mathrm{Fun}(K,L)} \mathrm{Fun}(L,L) \arrow[d] \\ \Delta^0 \arrow[r, "\mathrm{Id}_L"'] & \mathrm{Fun}(L,L) \end{tikzcd}$$

We see that the upper square is a pullback diagram and hence $f^* : \mathrm{Fun}_{/L}(L, X) \to \mathrm{Fun}_{/L}(K, X)$ is a right fibration. Since $\mathrm{Fun}_{/L}(K, X)$ is a Kan complex, we see that f^* is an isofibration. □

Proposition 6.47. *Let $f : A \to B$ be a morphism of simplicial sets. Then:*

(1) If f is either initial or final, then it is a weak homotopy equivalence.

(2) If f is a weak homotopy equivalence and B is a Kan complex, then f is both final and initial.

Proof. We first prove (1). Let X be a Kan complex, then the projection map $X \times B \to B$ is a Kan fibration, and therefore both a left and a right fibration. Consequently, if f is either initial or final, the induced map

$$\mathrm{Fun}(B, X) \cong \mathrm{Fun}_{/B}(B, X \times B) \to \mathrm{Fun}_{/B}(A, X \times B) \cong \mathrm{Fun}(A, X)$$

is a homotopy equivalence of Kan complexes.

Now we prove (2). Assume that B is a Kan complex and that f is a weak homotopy equivalence, we wish to show that f is final. Let $q : \tilde{B} \to B$ be a right fibration. Since B is a Kan complex, q is a Kan fibration; in particular, $\tilde{B}$ is a Kan complex. We have a commutative diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Fun}(B, \tilde{B}) \arrow[r] \arrow[d] & \mathrm{Fun}(A, \tilde{B}) \arrow[d] \\ \mathrm{Fun}(B, B) \arrow[r] & \mathrm{Fun}(A, B) \end{tikzcd}$$

where the vertical maps are Kan fibrations and the horizontal maps are homotopy equivalences. Applying Proposition 3.14, we see that the induced map $\mathrm{Fun}_{/B}(B, \tilde{B}) \to \mathrm{Fun}_{/B}(A, \tilde{B})$ is a homotopy equivalence. □

Proposition 6.48. *Let $F : A \to B$ be a morphism of simplicial sets. Then TFAE:*

(1) The morphism F is final.

(2) For every ∞-category $\mathcal{C}$ and every diagram $G : B \to \mathcal{C}$, the restriction map $\mathcal{C}_{G/} \to \mathcal{C}_{G\circ F/}$ is an equivalence of ∞-categories.

(3) For every ∞-category $\mathcal{C}$ and every diagram $G : B \to \mathcal{C}$, the restriction map $\mathcal{C}^{G/} \to \mathcal{C}^{G\circ F/}$ is an equivalence of ∞-categories.

(4) For every ∞-category $\mathcal{C}$, every diagram $G : B \to \mathcal{C}$ and every object $x \in \mathcal{C}$, precomposition with F induces a homotopy equivalence of Kan complexes

$$\mathrm{Fun}_{/\mathcal{C}}(B, \mathcal{C}_{/x}) \to \mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{C}_{/x}).$$

(5) For every ∞-category $\mathcal{C}$, every diagram $G : B \to \mathcal{C}$ and every object $x \in \mathcal{C}$, precomposition with F induces a homotopy equivalence of Kan complexes

$$\mathrm{Fun}_{/\mathcal{C}}(B, \mathcal{C}^{/x}) \to \mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{C}^{/x}).$$

Proof. First we show that (2) ⇔ (3) ⇔ (4) ⇔ (5). The equivalences (2) ⇔ (3) and (4) ⇔ (5) follow from Proposition 4.11. Now we show (3) ⇔ (5). According to Theorem 5.49, the morphism of left fibrations $\theta : \mathcal{C}^{G/} \to \mathcal{C}^{G\circ F/}$ is an equivalence if and only if the induced map on each fibre is an equivalence. That is, for all $x \in \mathcal{C}$,

$$\theta_x : (\mathcal{C}^{G/})_x \cong \mathrm{Map}_{\mathrm{Fun}(B,\mathcal{C})}(G, \delta_x) \to \mathrm{Map}_{\mathrm{Fun}(A,\mathcal{C})}(G \circ F, \delta_x) \cong (\mathcal{C}^{G\circ F/})_x$$

is a homotopy equivalence. We obtain the desired equivalence by noticing that there are canonical isomorphisms

$$\mathrm{Map}_{\mathrm{Fun}(B,\mathcal{C})}(G,\delta_x) \cong \mathrm{Fun}_{/\mathcal{C}}(B,\mathcal{C}^{/x}); \quad \mathrm{Map}_{\mathrm{Fun}(A,\mathcal{C})}(G\circ F,\delta_x) \cong \mathrm{Fun}_{/\mathcal{C}}(A,\mathcal{C}^{/x})$$

Next we show that (1) ⇒ (5). To do so, notice that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{C}}(B,\mathcal{C}^{/x}) \arrow[d] & \mathrm{Fun}_{/B}(B,G^*(\mathcal{C}^{/x})) \arrow[l, "\cong"'] \arrow[d] \\ \mathrm{Fun}_{/\mathcal{C}}(A,\mathcal{C}^{/x}) & \mathrm{Fun}_{/B}(A,G^*(\mathcal{C}^{/x})) \arrow[l, "\cong"] \end{tikzcd}$$

where the horizontal arrows are isomorphisms. Since $F : A \to B$ is final, the right vertical map is a homotopy equivalence, hence the left vertical map is a homotopy equivalence.

We now complete the proof by showing that (4) ⇒ (1). Assume that condition (4) is satisfied, and we wish to show that F is final. Let $q : \tilde{B} \to B$ be a right fibration. By the straightening-unstraightening equivalence, $q : \tilde{B} \to B$ fits into a pullback diagram

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] & (\mathcal{S}^{\mathrm{op}})_{/\Delta^0} \arrow[d, "q_{\mathrm{uni}}"] \\ B \arrow[r] & \mathcal{S}^{\mathrm{op}} \end{tikzcd}$$

where q_{uni} is the universal right fibration. Hence it suffices to show that F induces a homotopy equivalence $\mathrm{Fun}_{/\mathcal{S}^{\mathrm{op}}}(B,(\mathcal{S}^{\mathrm{op}})_{/\Delta^0}) \to \mathrm{Fun}_{/\mathcal{S}^{\mathrm{op}}}(A,(\mathcal{S}^{\mathrm{op}})_{/\Delta^0})$, which holds true automatically by our assumption. (Take $\mathcal{C} = \mathcal{S}^{\mathrm{op}}$ and $x = \Delta^0$ in condition (4)). □

6.4.3 Quillen's theorem A for ∞-categories

Definition 6.49. Let $p : Y \to X$ be a map of simplicial sets. We say that p is **smooth**, if for every pullback diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

with i being final, the map j is again final. Dually, it is called **proper** if i is initial implies that j is initial.

Definition 6.50. A map of simplicial sets $p : Y \to X$ is called **universally smooth** if the pullback of p along any morphism is smooth. Dually, p is **universally proper** if the pullback of p along any morphism is proper.

Proposition 6.51. *Consider a pullback diagram*

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

where p is a left fibration and i is right anodyne. Then the map j is again right anodyne.

Proof. To be added. □

Corollary 6.52. *Left fibrations are universally smooth.*

Proof. To be added. $\square$

Theorem 6.53 (Qullen's Theorem A)**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets with $\mathcal{D}$ an ∞-category. Then f is final if and only if the comma category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is weakly contractible for all $d \in \mathcal{D}$.*

Proof. First assume that f is final if and only for all objects $d \in \mathcal{D}$. Consider the following pullback diagram

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r, "f"'] & \mathcal{D} \end{tikzcd}$$

The right vertical map is a left fibration, and thus smooth. It follows that the projection map $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \to \mathcal{D}_{d/}$ is final, and hence a weak homotopy equivalence. Since $\mathcal{D}_{d/}$ is weakly contractible (it has an initial object), $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \to \mathcal{D}_{d/}$ is again contractible.

For the converse direction, we consider a factorisation of f as

$$f : \mathcal{C} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{D}$$

where i is right anodyne and p is a right fibration. We wish to show that p is a trivial fibration. For this purpose, consider the diagram

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \arrow[r, "j"] \arrow[d] & \mathcal{E} \times_{\mathcal{D}} \mathcal{D}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r, "i"'] & \mathcal{E} \arrow[r] & \mathcal{D} \end{tikzcd}$$

where all the squares are pullbacks. Since the very right vertical map is a left fibration, the middle vertical map is again a left fibration. According to Proposition 6.51 j is again right anodyne, and hence a weak equivalence. As a result, $\mathcal{E} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is again weakly contractible. Now we consider the following diagram

$$\begin{tikzcd} \mathcal{E}_d \arrow[r, "k"] \arrow[d] & \mathcal{E} \times_{\mathcal{D}} \mathcal{D}_{d/} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r, "\mathrm{Id}_d"'] & \mathcal{D}_{d/} \arrow[r] & \mathcal{D} \end{tikzcd}$$

where all squares are pullbacks. In this case the rightmost vertical map is a right fibration, hence the middle vertical map is also a right fibration. Furthermore, the map $\mathrm{Id}_d : \Delta^0 \to \mathcal{D}_{d/}$ is left anodyne, hence the map k is left anodyne by Proposition 6.51. As a result, the fibre $\mathcal{E}_d$ is a contractible Kan complex. Therefore p is a trivial Kan fibration. $\square$

6.5 Kan extensions

This section follows from [Har19].

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{E}$ be a third ∞-category. Then we have can associate with φ the restriction functor

$$\varphi^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

In this section we study the reverse process, that is, we want to "extend" a functor $\psi : \mathcal{C} \to \mathcal{E}$ to a functor $\psi' : \mathcal{D} \to \mathcal{E}$.

Definition 6.54. Let $\varphi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{C} \to \mathcal{E}$ be two functors between ∞-categories. **A left extension of ψ along φ** we will mean a pair (ψ', τ) where $\psi' : \mathcal{D} \to \mathcal{E}$ is a functor and $\tau : \psi \Rightarrow \psi' \circ \varphi = \varphi^*(\psi')$ is a natural transformation.

We will organize the collection of all extensions of ψ along φ into an ∞-category. To do so, we introduce:

Definition 6.55. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We define the **left mapping cone** to be the simplicial set obtained by the pushout

$$\begin{tikzcd} \mathcal{C} \times \Delta^{\{1\}} \arrow[r, hook] \arrow[d, "\varphi"'] & \mathcal{C} \times \Delta^1 \arrow[d] \\ \mathcal{D} \times \Delta^{\{1\}} \arrow[r] & \mathrm{Cone}^L_\varphi \end{tikzcd}$$

Note that Cone^L_φ may not be an ∞-category.

According to the definition of Cone^L_φ, a functor $F : \mathrm{Cone}^L_\varphi \to \mathcal{E}$ consists of the following data:

- A functor $\kappa : \mathcal{C} \times \Delta^1 \to \mathcal{E}$.
- A functor $\rho : \mathcal{D} \to \mathcal{E}$, such that $\kappa|_{\mathcal{C}\times\Delta^{\{1\}}} = \varphi^*(\rho)$.

As a result, a functor $F : \mathrm{Cone}^L_\varphi \to \mathcal{E}$ is the same thing as a left extension of $\kappa|_{\mathcal{C}\times\Delta^{\{0\}}}$ along φ.

Restricting along $\mathcal{C} \times \Delta^{\{0\}} \subseteq \mathrm{Cone}^L_\varphi$ then determines a restriction functor

$$\mathrm{Fun}(\mathrm{Cone}^L_\varphi, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

whose fibre over $\psi \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is the desired ∞-category of left extensions of ψ along φ. We denote the corresponding fibre by $\mathrm{Fun}(\mathrm{Cone}^L_\varphi, \mathcal{E})_\psi$.

Definition 6.56. We will say that a left extension (ψ', τ) is a **left Kan extension** if it is initial in $\mathrm{Fun}(\mathrm{Cone}^L_\varphi, \mathcal{E})_\psi$, the ∞-category of left extensions of ψ along φ.

The left mapping cone Cone^L_φ is kind of unsatisfying since it is not an ∞-category. To fix this problem, we exploit the categorical mapping cone $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$. Note that there is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \times \Delta^{\{1\}} \arrow[r] \arrow[d] & \mathcal{C} \times \Delta^1 \arrow[d] \\ \mathcal{D} \times \Delta^{\{1\}} \arrow[r] & \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \end{tikzcd}$$

which induces a canonical morphism $q : \mathrm{Cone}^L_\varphi \to \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$.

Lemma 6.57. *The canonical morphism $q : \mathrm{Cone}^L_\varphi \to \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is Joyal equivalence.*

Proof. To be added. $\square$

According to Lemma 6.57, there is an equivalence $\mathrm{Fun}(\mathcal{C} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}(\mathrm{Cone}^L_\varphi, \mathcal{E})$ for any ∞-category $\mathcal{E}$. Similar to the case of Cone^L_φ, the canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ induces an inner fibration

$$\mathrm{Fun}(\mathcal{C} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

whose fibre over $\psi \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ can be identified with the ∞-category of left extensions of ψ along φ. We say that a functor $\phi : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{E}$ is a **left Kan extension of ψ along φ** if it lies in the fibre $\mathrm{Fun}(\mathcal{C} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E})_\psi$ and is initial in it. To formulate the main theorem of this section we need some preliminaries.

Lemma 6.58. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and $\mathcal{M} = \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is the categorical mapping cylinder. For an object $d \in \mathcal{D}$, there is a canonical isomorphism of simplcial sets*

$$\mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y} \cong \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}.$$

The adjunction of the projection map $\mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y} \to \mathcal{M}_{/y}$ is a map

$$(\mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y})^{\triangleright} \to \mathcal{M}$$

which carries the cone point ∞ to y.

Theorem 6.59. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathcal{M} = \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$. Let $\psi : \mathcal{C} \to \mathcal{E}$ be a functor and suppose that for every $y \in \mathcal{D}$ the composed functor $\mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y} \to \mathcal{C} \xrightarrow{\psi} \mathcal{E}$ admits a colimit in $\mathcal{E}$. Then*

(1) There exists a left Kan extension $\overline{\psi} : \mathcal{M} \to \mathcal{E}$ such that $\overline{\psi}|_{\mathcal{C}} = \psi$.

(2) A functor $\overline{\psi} : \mathcal{M} \to \mathcal{E}$ with $\overline{\psi}|_{\mathcal{C}} = \psi$ is a left Kan extension of ψ along φ if and only if for every $y \in \mathcal{D}$ the composed functor $(\mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y})^{\triangleright} \to \mathcal{M} \to \mathcal{E}$ is a colimit cone.

7 Adjunctions and localizations

8 ∞-operads

8.1 Operator category

We introduce the notion of an operator category in this section. We study this notion in preparation for the introduction of multicategories, since operadic structures in a multicategory are tracked by an operator category. We follow [Bar18], but we will not go very deep since the author can only understand part of this article.

Definition 8.1. Let $\mathcal{C}$ be a 1-category with terminal object 1. For any object $K \in \mathcal{C}$, we write $|K| := \hom_{\mathcal{C}}(1, K)$. An element of $|K|$ is called a **point** of K.

Definition 8.2. An **operator category** Φ is an essentially small category that satisfies the following three conditions:

(1) Φ has a terminal object.

(2) For any morphism $J \to I$ of Φ and for any point $i \in |I|$, the fibre product $J_i := \{i\} \times_I J$ exists:

$$\begin{tikzcd} J_i := \{i\} \times_I J \arrow[r] \arrow[d] & J \arrow[d] \\ 1 \arrow[r, "i"'] & I \end{tikzcd}$$

(3) Φ is locally finite. That is, for any pair of objects $I, J \in \Phi$, the set $\hom_{\Phi}(I, J)$ is finite.

Example 8.3. The following categories are operator categories:

- The trivial category $[0]$.
- For each $n \geq 0$, the partially ordered set $[n]$, viewed as a category, is an operator category.
- The category $\mathbf{O}$ of (possibly empty) linearly ordered finite sets and order-preserving maps.

- The category $\mathbf{F}$ of (possibly empty) finite sets and maps.

Example 8.4. For any operator category Φ and for any integer $n \geq 1$, write $\Phi_{\leq n}$ for the full subcategory of Φ spanned by those objects $I \in \Phi$ such that $\sharp|I| \leq n$. Then the category $\Phi_{\leq n}$ is an operator category.

Example 8.5. Suppose that Φ and Ψ are two operator categories. Then we may define a category $\Psi \wr \Phi$ as follows. An object of $\Psi \wr \Phi$ will be a pair (I, M) consisting of an object $I \in \Phi$ and a collection $M = \{M_i\}_{i \in |I|}$ of objects of Ψ, indexed by the points of I. A morphism $(\eta, \omega) : (J, N) \to (I, M)$ of $\Psi \wr \Phi$ consists of a morphism $\eta : J \to I$ of Φ and a collection

$$\{\omega_j : N_j \to M_{\eta(j)}\}_{j \in |J|}$$

of morphisms of Ψ, indexed by the points of J.

8.2 Φ-multicategories

Almost all contents in this section are stolen from [Bar07].

Definition 8.6. Let Φ be an operator category. A Φ-multicategory $\mathcal{C}$ consists of the following data:

- A set $\mathrm{Ob}(\mathcal{C})$ of objects, or **colors**. If x is an object of $\mathcal{C}$ we will simply write $x \in \mathcal{C}$.
- For each object $I \in \Phi$, and each I-tuple $(x_I) \in \mathrm{Ob}(\mathcal{C})^{\times |I|}$, and each object $y \in \mathcal{C}$, we associate a set of ***I*-multimorphisms** from (x_I) to y. We will refer to this set as $I\mathrm{Mul}_{\mathcal{C}}((x_I), y)$. When $I = 1$ is the terminal object, we will simply denote the set of I-multimorphisms from x to y by $\mathrm{Mul}_{\mathcal{C}}(x, y)$.
- A distinguished element $\mathrm{Id}_x \in \mathrm{Mul}_{\mathcal{C}}(x, x)$, for each $x \in \mathcal{C}$. We will refer to this element as the **identity** of x.
- A composition law defined as follows: for any morphism $J \to I$ in Φ, any J-tuple $(x_J) \in (\mathrm{Ob}\mathcal{C})^{\times |J|}$, any I-tuple $(y_I) \in (\mathrm{Ob}\mathcal{C})^{\times |I|}$ and any object $z \in \mathcal{C}$, we associate a composition map
$$I\mathrm{Mul}_{\mathcal{C}}((y_I), z) \times \prod_{i \in |I|} J_i\mathrm{Mul}_{\mathcal{C}}((x_{J_i}), y_i) \to J\mathrm{Mul}_{\mathcal{C}}((x_J), z).$$

Moreover, the composition maps are required to be associative and unital w.r.t. identities. For details see [Bar07].

Definition 8.7. A **Φ-operad** is a Φ-multicaregory with only one object.

Let's look at a few examples.

Example 8.8. Let's figure out what is a [0]-multicategory. The operator category [0] has a unique object, which is exactly the terminal object. Unpacking the definition, we see that a [0]-multicategory consists of

- a set of objects;
- For two objects x, y, a set of 0-multimorphisms $\mathrm{Mul}_{\mathcal{C}}(x, y)$;
- An identity morphism $\mathrm{Id}_x \in \mathrm{Mul}_{\mathcal{C}}(x, x)$;

- A composition law
$$\mathrm{Mul}_{\mathcal{C}}(y,z)\times \mathrm{Mul}_{\mathcal{C}}(x,y)\to \mathrm{Mul}_{\mathcal{C}}(x,z)$$
induced by the unique morphism $\mathrm{Id}_0 : 0 \to 0$

and the composition law is associative and unital w.r.t. identities. Hence a [0]-category is an ordinary category.

Example 8.9. Let's figure out what is a [1]-multicategory. The category $[1] = 0 \to 1$ has two objects, where 1 is terminal. Also we have $|0| = \emptyset$, $|1| = \{*\}$ is the singleton. Unpacking the definition, we see that a [1]-multicategory consists of the following data:

- A category $\tilde{\mathcal{C}}$, whose composition law is induced by $\mathrm{Id}_1 : 1 \to 1$, and we have $\mathrm{Ob}(\tilde{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$, $\hom_{\tilde{\mathcal{C}}}(x,y) = \mathrm{Mul}_{\mathcal{C}}(x,y)$
- For each object $x \in \mathcal{C}$, we associate a set
$$F(x) := 0\mathrm{Mul}_{\mathcal{C}}(\star, x),$$
where $\star \in \mathrm{Ob}(\mathcal{C})^{\times 0}$ is the unique element.
- The morphism $0 \to 1$ induces a composition law
$$\hom_{\tilde{\mathcal{C}}}(x,y)\times F(x)\to F(y)$$
which extends F to a functor $\tilde{\mathcal{C}} \to \mathrm{Set}$.

As a result, a [1]-multicategory is a pair $(\tilde{\mathcal{C}}, F : \tilde{\mathcal{C}} \to \mathrm{Set})$.

Example 8.10. An **F**-multicategory is a so-called **symmetric multicategory** or a **colored operad** in the sense of [Lur17], Definition 2.1.1.1.

Example 8.11. An **F**-operad is a symmetric operad enriched in Set. To see this let us unpack the definition, so that an **F**-operad $\mathcal{C}$ consists of the following data:

1. For each object $\underline{n} \in \mathbf{F}$, we associate a set $\mathcal{C}_n := \underline{n}\mathrm{Mul}_{\mathcal{C}}((\bullet, ..., \bullet), \bullet)$, where $\bullet \in \mathrm{Ob}(\mathcal{C})$ is the unique object. We refer to $\mathcal{C}_n$ as **the set of n-ary operations** of $\mathcal{C}$.
2. A distinguished object $\mathrm{Id} \in \mathcal{C}_1$, which we refer to as **the identity morphism**.
3. A composition map associated with each morphism in **F**. We identify the group of automorphism of $\underline{n}$ with the symmetric group S_n. Let us take an element $\sigma \in S_n$, then σ induces a composition map
$$(\mathcal{C}_1)^{\times n}\times \mathcal{C}_n \to \mathcal{C}_n$$
Restricting map to $(\mathrm{Id}, ..., \mathrm{Id}) \times \mathcal{C}_n$, we obtain a map $\sigma_* : \mathcal{C}_n \to \mathcal{C}_n$. Combining this with associativity we find that $\mathcal{C}_n$ is equipped with an S_n-action. Moreover, we see that the composition maps are S_n-equivariant.

Example 8.12. An **O**-multicategory is a non-symmetric multicategory in the sense of [GH13], Section 2.1. An **O**-operad is a non-symmetric operad enriched in sets.

To end this section, let us give the definition of a Φ-multifunctor.

Definition 8.13. Let Φ be an operator category, and suppose that $\mathcal{C}$ and $\mathcal{D}$ are Φ-multicategories. A **Φ-multifunctor** $G : \mathcal{C} \to \mathcal{D}$ consists of

- A map $G : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$

- For each object $I \in \Phi$, a map

$$\gamma_I : I\mathrm{Mul}_{\mathcal{C}}((x_I), y) \to I\mathrm{Mul}_{\mathcal{D}}(G(x_I), Gy)$$

for each $|I|$-tuple $(x_I) \in \mathrm{Ob}(\mathcal{C})^{\times |I|}$ and each object $y \in \mathrm{Ob}(\mathcal{C})$.

The such γ_I preserves compositions in the sense that, for any morphism $J \to I \in \Phi$, the following diagram commutes:

$$\begin{tikzcd}
I\mathrm{Mul}_{\mathcal{C}}((y_I), z) \times \prod_{i\in|I|} J_i\mathrm{Mul}_{\mathcal{C}}((x_{J_i}), y_i) \arrow[r] \arrow[d] & I\mathrm{Mul}_{\mathcal{C}}((x_I), y) \arrow[d] \\
I\mathrm{Mul}_{\mathcal{C}}(G(y_I), Gz) \times \prod_{i\in|I|} J_i\mathrm{Mul}_{\mathcal{C}}(G(x_{J_i}), Gy_i) \arrow[r] & I\mathrm{Mul}_{\mathcal{C}}(G(x_I), Gy)
\end{tikzcd}$$

We write MulCat^{Φ} for the 1-category of Φ-multicategories and Φ-multifunctors.

8.3 Perfect operator categories and Leinster categories

Definition 8.14. Let Φ be an operator category. Define the category Φ^{cons} as follows:

- An object of Φ^{cons} is a point in Φ. That is a morphism $i : 1 \to I$ for some object $I \in \Phi$. We denote such an object by (I, i).
- A morphism from (I, i) to (J, j) is a morphism $f : I \to J$ which fits into a pullback diagram

$$\begin{tikzcd}
1 \arrow[r, "i"] \arrow[d, "="'] & I \arrow[d, "f"] \\
1 \arrow[r, "j"'] & J
\end{tikzcd}$$

A **point classifier** for Φ is a terminal object in Φ^{cons}.

Example 8.15. The category $\mathbf{F}$ has a point classifier $(\underline{2} = \{1, 2\}, 2)$.

Example 8.16. The category $\mathbf{O}$ has a point classifier $([2] = \{0 \to 1 \to 2\}, 1)$.

Remark 8.17. In the above two examples, a point classifier also serves as a "subobject classifier" whose detailed meaning will not be discussed here.

Definition 8.18. Let Φ be an operator category, and suppose that (T, t) is a point classifier for Φ. Then for any object (T, i) of Φ^{cons}, we will refer to the unique map $(I, i) \to (T, t)$ as the **classifying map** for i, which will be denoted by χ_i. We shall call the point t the **special point** of Φ. For any morphism $I \to T$, the fibre I_t will be called the **special fibre**. Formation of special fibres defines a functor $\Phi_{/T} \to \Phi$, and we will denote this functor by fib.

Definition 8.19. Let Φ be an operator category with a point classifier (T, t). We say that Φ is **perfect** if the special fibre functor $\mathrm{fib} : \Phi_{/T} \to \Phi$ admits a right adjoint E_{Φ}.

Example 8.20. The category $\mathbf{F}$ is perfect. The functor $E_{\mathbf{F}}$ is defined as

$$E_{\mathbf{F}}(I) := (I \sqcup \{*\}, f)$$

where f sends I to the special point 2 and sends the newly adjoined point $*$ to 1. Then there is an easily established isomorphism

$$\hom_{\mathbf{F}}(\mathrm{fib}(J), I) \cong \hom_{/\underline{2}}(J, E_{\mathbf{F}}(I)).$$

Example 8.21. The category $\mathbf{O}$ is prefect. The functor $E_{\mathbf{O}}$ acts by adding a single point at the beginning and a single point at the end. The morphism $E_{\mathbf{O}}(I) \to [2]$ sends the newly added first point to 0, I to the special point 1 and the newly added last point to 2.

Let Φ is a perfect operator. By abuse of notation, we write $T_\Phi : \Phi \to \Phi$ for the the composition of E_Φ and the forgetful functor from $\Phi_{/T}$ to Φ. A theorem of Barwick ([Bar18] Theorem 5.10) says that the endofunctor T_Φ is equipped with a structure of a monad on Φ. We refer to T_Φ as the **canonical monad** on Φ. When Φ is clear from context, we simply denote T_Φ by T.

Recall that, given a monad $T : \mathcal{C} \to \mathcal{C}$ on a category, the Kleisli category $\mathcal{C}^T$ is defined as follows:

- $\mathrm{Ob}(\mathcal{C}^T) := \mathrm{Ob}(\mathcal{C})$;
- $\hom_{\mathcal{C}^T}(x, y) := \hom_{\mathcal{C}}(x, T(y))$.

We can identify $\mathcal{C}^T$ with the **category of free algebras over** T.

Definition 8.22. Let Φ be a perfect operator category. The **Leinster category** $\Lambda(\Phi)$ of Φ is the Kleisli category of the monad T_Φ.

Example 8.23. For the perfect operator category $\mathbf{F}$, the canonical monad $T : \mathbf{F} \to \mathbf{F}$ acts by freely adjoining a base point. Hence there is an easily established equivalence

$$\begin{aligned} \Lambda(\mathbf{F}) &\simeq \mathcal{F}\mathrm{in}_* \\ I &\mapsto I_* \end{aligned}$$

Example 8.24. For the perfect operator category $\mathbf{O}$, the canonical monad T acts by freely adjoining two points, one at the beginning and one at the end. There is an equivalence

$$\begin{aligned} \Lambda(\mathbf{O}) &\simeq \Delta^{\mathrm{op}} \\ [n] &\mapsto \hom_{\mathbf{O}}([n], [1]) \cong [n+1] \end{aligned}$$

Here note that an element in the set $\hom_{\mathbf{O}}([n], [1])$ is the same as a partition of $[n]$. Hence the set $\hom_{\mathbf{O}}([n], [1])$ is equipped with a canonical total order, which is inherited from that of $[n]$.

8.4 Category of operators and ∞-operads

In this note, we mainly talk about $\mathbf{F}$-multicategories. From now on, by a multicategory, we simply mean a $\mathbf{F}$-multicategory. For the set of $\underline{n}$-multimorphisms $\underline{n}\mathrm{Mul}_{\mathcal{C}}((x_1, ..., x_n), y)$ of a multicategory $\mathcal{C}$, we drop the notation $\underline{n}$ and denote this set by $\mathrm{Mul}_{\mathcal{C}}((x_1, ..., x_n), y)$. By an operad, we mean an $\mathbf{F}$-operad. Similarly, we write MulCat for $\mathrm{MulCat}^{\mathbf{F}}$.

Example 8.25. Here are common examples of operads:

- The operad Ass. Ass has a single element $\bullet$, and the set $\mathrm{Mul}_{\mathrm{Ass}}((\bullet, ..., \bullet)_{i\in I}, \bullet)$ consists of all linear ordering over I, and is equipped with the regular action of S_n.
- The operad Comm. Comm has a single element $\bullet$, and the set $\mathrm{Mul}_{\mathrm{Comm}}((\bullet, ..., \bullet)_{i\in I}, \bullet)$ has a single element whatever I is. S_n acts trivially.
- The operad Poi. Poi has a single element $\bullet$, and the set $\mathrm{Mul}_{\mathrm{Poi}}((\bullet, ..., \bullet)_{i\in I}, \bullet)$ is empty whenever $|I| > 1$ and consists of a single element otherwise. Again, S_n acts trivially.

Example 8.26. A symmetric monoidal category $\mathcal{C}$ gives rise to a multicategory $\mathcal{C}'$ by defining:

- The objects of $\mathcal{C}'$ are those of $\mathcal{C}$;

- $\mathrm{Mul}_{\mathcal{C}'}((x_1, ..., x_n), y) := \hom_{\mathcal{C}}(x_1 \otimes x_2 \otimes ... \otimes x_n, y)$. In particular we define $\mathrm{Mul}_{\mathcal{C}'}(\star, x) := \hom_{\mathcal{C}}(1, x)$.

For convenience we may simply denote this multicategory by $\mathcal{C}$.

Example 8.27. Every multicategory $\mathcal{C}$ has an **underlying category**, which we still denote by $\widetilde{\mathcal{C}}$, defined as follows:

- $\mathrm{Ob}(\widetilde{\mathcal{C}}) := \mathrm{Ob}(\mathcal{C})$;
- $\hom_{\widetilde{\mathcal{C}}}(x, y) := \mathrm{Mul}_{\mathcal{C}}(x, y)$.

For simplicity we still denote this category by $\mathcal{C}$.

Construction 8.28 (Category of operators)**.** Let $\mathcal{C}$ be a multicategory. We define a category $\mathcal{C}^{\otimes}$ as follows:

(1) An object of $\mathcal{C}^{\otimes}$ is an element in $\coprod_{\underline{n}\in\mathbf{F}} \mathrm{Ob}(\mathcal{C})^{\times n}$;

(2) Given two objects

$$(x_1, ..., x_m) \in \mathrm{Ob}(\mathcal{C})^{\times m} \qquad (y_1, ..., y_n) \in \mathrm{Ob}(\mathcal{C})^{\times m}$$

a morphism from $(x_i)_{1\leq i\leq m}$ to $(y_j)_{1\leq j\leq n}$ is given by a map $\alpha : \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$ together with a collection of morphisms

$$\{\phi_j \in \mathrm{Mul}_{\mathcal{O}}((x_i)_{i\in\alpha^{-1}(j)}, y_j)\}_{1\leq j\leq n}$$

(3) Composition of morphisms is defined by combining the composition law on $\mathcal{F}\mathrm{in}_*$ and the composition of the multicategory $\mathcal{C}$.

By our construction, the category $\mathcal{C}^{\otimes}$ comes equipped with a canonical projection $\pi : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ sending $(x_1, ..., x_n)$ to $\langle n\rangle$.

The non-trivial point is that, all information contained in a multicategory is preserved by the formation of category of operator. Hence, this construction provides us with an equivalent way to describe multicategories. Moreover, this description is "unstraightened", hence it can more easily be generalized to the ∞-categorical context.

Definition 8.29. Let CatOpe be the following subcategory of $\mathrm{Cat}_{/\mathcal{F}\mathrm{in}_*}$:

- Objects are functors $p : \mathcal{C} \to \mathcal{F}\mathrm{in}_*$ such that:
  1. any inert morphism in $\mathcal{F}\mathrm{in}_*$ admits a p-coCartesian lift, so that any inert morphism $\alpha : \langle m\rangle \to \langle n\rangle$ induces a functor $\alpha_! : \mathcal{C}_{\langle m\rangle} \to \mathcal{C}_{\langle n\rangle}$;
  2. for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, the morphisms induced by the inert morphisms $(\rho^i)_! : \mathcal{C}_{\langle n\rangle} \to \mathcal{C}_{\langle 1\rangle}$ exhibits $\mathcal{C}_{\langle n\rangle}$ as the product $\mathcal{C}_{\langle n\rangle} \simeq \mathcal{C}_{\langle 1\rangle}^{\times n}$.
  3. for every morphism $f : \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$ and $Y \in \mathcal{C}_{\langle n\rangle}$, composition with coCartesian morphisms $Y \to Y_i$ over the inert morphisms ρ^i gives an isomorphism

$$\hom^f_{\mathcal{C}}(X, Y) \cong \prod_i \hom^{\rho^i\circ f}_{\mathcal{C}}(X, Y_i)$$

     where $\hom^f_{\mathcal{C}}(X, Y)$ means the subset of $\hom_{\mathcal{C}}(X, Y)$ over f.
- Morphisms are those functors preserving coCartesian morphisms over inert arrows.

Theorem 8.30. *The construction of the category of operators defines a functor* $(-)^{\otimes} : \mathrm{MulCat} \to \mathrm{Cat}_{/\mathcal{F}\mathrm{in}_*}$*, which gives rise to an equivalence of categories:*

$$(-)^{\otimes} : \mathrm{MulCat} \simeq \mathrm{CatOpe}$$

Proof. Let's give the inverse functor directly. That is, I will demonstrate how to recover a multicategory from a category of operator. Let $(\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*) \in \mathrm{CatOpe}$. We get a category by taking the fibre $\mathcal{C}^{\otimes}_{\langle 1 \rangle} = \pi^{-1}(\langle 1 \rangle)$, and we will denote it by $\mathcal{C}$. This category serves as the underlying category of our desired multicategory. Then suppose that $n \geq 0$. For $1 \leq i \leq n$, $\rho^i : \langle n \rangle \to \langle 1 \rangle$ induces a functor $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$ and these functors together determine an equivalence of categories $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n$. Hence, an object in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ can be written as an n-tuple $(x_1, ..., x_n)$ where $x_i \in \mathcal{C}$ is an object. Then for each color $y \in \mathcal{C}$, we define the set of multimorphisms as

$$\mathrm{Mul}_{\mathcal{C}}((x_1, ..., x_n), y) := \mathrm{hom}^{\alpha_n}_{\mathcal{C}^{\otimes}}((x_1, ..., x_n), y)$$

Here $\mathrm{hom}^{\alpha_n}_{\mathcal{C}^{\otimes}}((x_1, ..., x_n), y)$ is the set of morphisms over the unique active morphism $\alpha_n : \langle n \rangle \to \langle 1 \rangle$. Hence we defines a multicategory. It is direct to check that these construction is reverse to the construction of category of operators.

□

Example 8.31. Let $\mathcal{C}$ be a symmetric monoidal category, which can be viewed as a multicategory according to Example 8.26. Then the corresponding category of operators coincide with the coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$. Hence we see that category of operators is an "unstraightened" description of multicategory.

Definition 8.32. An ∞-**operad** is a functor $p : \mathcal{O}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$ between ∞-categories satisfying:

1 For every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in $N(\mathcal{F}\mathrm{in}_*)$ and every object $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, there is a p-coCartesian lift $\overline{f} : X \to Y$. In particular, there is an induced functor $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$.

2 Let $Y \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ let $f : \langle m \rangle \to \langle n \rangle$ be a morphism in $\mathcal{F}\mathrm{in}_*$, and let $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(X, Y)$ be the union of those connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(X, Y)$ which lie over f. Choose p-coCartesian morphisms $Y \to Y_i$ lying over the inert morphism $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Then the induced maps

$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(X, Y) \to \prod_{1 \leq i \leq n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(X, Y_i)$$

is a homotopy equivalence.

3 For each $n \geq 0$, the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}\}$ determines an equivalence of ∞-categories $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\times n}$.

We will denote $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ by $\mathcal{O}$, and refer to it as the **underlying** ∞-**category** of $\mathcal{O}$.

Remark 8.33. $\mathcal{O}$ is an ∞-category since the functor $p : \mathcal{O}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$, from an ∞-category to a nerve, must be an inner fibration.

Remark 8.34. As a matter of fact, the most suitable name for what we have defined above is "∞-colored-operad" or an "∞-multicategory". However we will follow Lurie and refer to this stuff as ∞-operad.

Example 8.35. Every symmetric monoidal ∞-category $\pi : \mathcal{C}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$ defines an ∞-operad.

Example 8.36. Let $\mathcal{C}$ be a multicategory, and $\mathcal{C}^{\otimes}$ the associated category of operators. Then the nerve $N(\mathcal{C}^{\otimes})$, together with the morphism $N(\mathcal{C}^{\otimes}) \to N(\mathcal{F}\mathrm{in}_*)$, is an ∞-operad. By preceding examples we get:

- $N(\mathrm{Ass}^{\otimes})$ is an ∞-operad;
- $N(\mathrm{Poi}^{\otimes})$ is an ∞-operad;
- $N(\mathrm{Comm}^{\otimes}) = N(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad.

Example 8.37. By a **simplicial multicategory** we mean a simplicial-enriched version of a multicategory. That is, we require the space of multi-morphisms $\mathrm{Mul}((x_1, ..., x_n), y)$ to be simplicial sets. For a simplicial colored operad $\mathcal{C}$ we may similarly define its **simplicial category of operators** $\mathcal{C}^{\otimes}$ as follows:

- The objects are n-tuples $(x_1, ..., x_n)$;
- The simplicial set of maps from $(x_1, ..., x_n)$ to $(y_1, ..., y_m)$ is given by

$$\coprod_{\alpha:\langle n\rangle \to \langle m\rangle} \prod_{j=1}^{m} \mathrm{Mul}_{\mathcal{C}}((x_i)_{i\in\alpha^{-1}(j)}, y)$$

We say a simplicial multicategory $\mathcal{C}$ is **locally Kan** if $\mathrm{Mul}_{\mathcal{C}}((x_i), y)$ are all Kan complexes. It is shown by Lurie that if $\mathcal{C}$ is locally Kan, then the homotopy coherent nerve $N(\mathcal{C}^{\otimes})$ is an ∞-operad.

Remark 8.38. The audience might wonder why does the category $\mathcal{F}\mathrm{in}_* = \Lambda(\mathbf{F})$ suddenly appears in our definition of the category of operators. As a matter of fact, for each operator category Φ, there is a notion of **Φ-quasioperad**, which is defined as certain inner fibration $\mathcal{P}^{\otimes} \to N(\Lambda(\Phi))$ over the nerve of the Leinster category $\Lambda(\Phi)$, subject to a series of conditions similar to those of an ∞-operad. A Φ-quasioperad is a homotopy-coherent Φ-multicategory, and an ∞-operad is exactly an **F**-quasioperad. In Barwick's paper [Bar18], he developed two different models of homotopy coherent Φ-multicategory, one is Φ-quasioperad, the other is the so-called **complete Segal Φ-operad**. Barwick also established an equivalence between these two models, see [Bar18], Theorem 10.16.

8.5 Algebras over an operad

Definition 8.39. Let $\pi : \mathcal{O}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that a map f is **inert** if f is π-coCartesian and $\pi(f)$ is inert in $\mathcal{F}\mathrm{in}_*$, is **active** if $\pi(f)$ is active.

Remark 8.40. As is indicated by its name, inert maps are in some sense "trivial" and they do not contain any operational information.

Definition 8.41. Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be ∞-operads. A a morphism $f : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ over $N(\mathcal{F}\mathrm{in}_*)$ is a **map of ∞-operads** if it carries inert morphisms to inert morphisms. We will refer to a map of ∞-operads from $\mathcal{O}^{\otimes}$ to $\mathcal{P}^{\otimes}$ as an **$\mathcal{O}$-algebra in** $\mathcal{P}$. We will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ the full subcategory of $\mathrm{Fun}_{/N(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ spanned by maps of ∞-operads.

Remark 8.42. A map of ∞-operads is just like a " symmetric lax monoidal functor", hence preserves algebra objects.

8.6 Operadic model structure

Let us organize the collection of all ∞-operads into an ∞-category, in exactly the same way we construct Cat_∞.

Definition 8.43. Let $\mathrm{Op}_\infty^\Delta$ be the fibrant simplicial category defined as follows:

- An object of $\mathrm{Op}_\infty^\Delta$ is an ∞-operad.
- For every pair of ∞-operads $\mathcal{O}^\otimes, \mathcal{O}'^\otimes \in \mathrm{Op}_\infty^\Delta$, we define $\mathrm{Map}_{\mathrm{Op}_\infty^\Delta}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes) := \mathrm{Alg}_\mathcal{O}(\mathcal{O}')^\simeq$.

We define $\mathrm{Op}_\infty = N(\mathrm{Op}_\infty^\Delta)$ to be the coherent nerve. We will refer to Op_∞ as the ∞-category of all ∞-operads.

Recall that, ∞-categories can be defined as bifibrant objects in a model category where all objects are cofibrant, i.e. the Joyal model structure on sSet. Moreover, the underlying ∞-category of Joyal's model category is exactly Cat_∞. There is a similar phenomenon for ∞-operads, they can be defined as bifibrant objects in a model category where all objects are cofibrant. Moreover, the underlying ∞-category of this model category is Op_∞.

Definition 8.44. A **marked simplcial set** is a pair (X, M) where X is a simplicial set and $M \subseteq X_1$ is a subset such that $s_0(X_0) \subset M$, i.e. M contains all the degenerate edges. An edge in M is said to be **marked**. **A morphism of marked simplicial sets** $f : (X, M) \to (Y, N)$ is a morphism $f : X \to Y$ such that $f_1(M) \subset N$.

Definition 8.45. An **∞-preoperad** is a marked simplicial set (X, M) equipped with a map of simplicial sets $f : X \to N(\mathcal{F}\mathrm{in}_*)$ which sends marked edges to inert ones in $N(\mathcal{F}\mathrm{in}_*)$. A morphism of ∞-preoperads from (X, M, f) to (Y, N, g) is a map of marked simplicial sets $h : (X, M) \to (Y, N)$ such that $g \circ h = f$. In this way we get a category of ∞-preoperads, which we will refer to as $\mathcal{P}\mathrm{Op}_\infty$.

Note that every ∞-operad $\pi : \mathcal{O}^\otimes \to N(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad by marking the inert morphisms. We denote this ∞-preoperad by $\mathcal{O}^{\otimes,\natural}$.

Definition 8.46. Let $h : \overline{X} = (X, M, f) \to \overline{Y} : (Y, N, g)$ be a map of preoperads. We say that f is an **operadic equivalence** if for every ∞-operad $\mathcal{O}^\otimes$, the induced map

$$\mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\overline{Y}, \mathcal{O}^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\overline{X}, \mathcal{O}^{\otimes,\natural})$$

Proposition 8.47 ([Lur17], Proposition 2.1.4.6)**.** *There exists a left proper combinatorial simplicial model structure on $\mathcal{P}\mathrm{Op}_\infty$ such that*

- *All objects are cofibrant.*
- *An ∞-preoperad (X, M, f) is fibrant if and only if the pair (X, f) is an ∞-operad, and M coincide with the set of inert morphisms in X (i.e. f-coCartesian lifts of inert morphisms in $N(\mathcal{F}\mathrm{in}_*)$).*
- *A weak equivalence is an operadic equivalence.*
- *The underlying ∞-category of this model category is Op_∞.*

In practice, the operadic model structure tells us that the formation of ∞-operads is closed under arbitrary homotopy limits and homotopy colimits, allowing us to construct new ∞-operads out of old ones.

Remark 8.48. The operadic model structure is produced by a machinery called **categorical pattern**, see [Lur17], Appendix B. A simplified version is given in [GH13], Section 3.2.

8.7 Little cube operad

Definition 8.49. Let $U, V \subseteq \mathbb{R}^n$ be two open subsets of $\mathbb{R}^n$. We say that a map $f : U \to V$ is a **rectilinear embedding** if it is an open embedding which is given by the a formula of the form

$$f(x_1, ..., x_n) = (a_1x_1 + b_1, ..., a_nx_n + b_n)$$

with $a_i, b_i \in \mathbb{R}$ such that $a_i > 0$. If $\{U_i\}_{i\in I}$ is a collection of open subsets of $\mathbb{R}^n$ indexed by I and $V \subseteq \mathbb{R}^n$ is another open subset then we will say that a map $f : \coprod_i U_i \to V$ is a rectilinear embedding if it is an open embedding and the restriction of f to each U_i is rectilinear. We will denote by $\mathrm{Rect}(\coprod_{i\in I} U_i, V)$ the space of rectilinear embeddings endowed with the topology as a subspace of $(\mathbb{R}^{2n})^I$.

Definition 8.50. Let $\square^n$ the standard n-cube $(-1, 1)^{\times n} \subseteq \mathbb{R}^n$. We consider the following simplicial colored operad $\mathcal{E}_n$:

- It has only one object which we denote by $\bullet$;
- We define the space of multi-morphisms by

$$\mathrm{Mul}_{\mathcal{E}_n}((\bullet, ..., \bullet)_{i\in I}, \bullet) = \mathrm{Sing}(\mathrm{Rect}(\square^k \times I, \square^k))$$

  for every finite set I.

Since $\mathrm{Sing}(\mathrm{Rect}(\square^k \times I, \square^k))$ are Kan complexes, we conclude that $N(\mathcal{E}_n^{\otimes})$ is an ∞-operad, which we denote by $\mathbb{E}_n^{\otimes}$ and refer to it as **the ∞-operad of little n-cubes**.

Example 8.51. Consider the simplicial operad $\mathcal{E}_0$. We have $\square^0 = *$. So the multi-morphism space $\mathrm{Rect}(* \times I, *)$ is empty whenever $|I| > 1$, and consist of single point when $|I| \leq 1$. In particular, $\mathcal{E}_0$ is isomorphic to the operad Poi, thus we have an isomorphism of ∞-operads:

$$\mathbb{E}_0^{\otimes} \cong N(\mathrm{Poi}^{\otimes})$$

Example 8.52. Consider the ∞-operad $\mathbb{E}_1^{\otimes}$. In this case every rectilinear embedding $f : \square^1 \times I \to \square^1$ determines a unique linear ordering on I by $i < i' \Leftrightarrow f(0, i) < f(0, i')$. This association determines a map of simplicial operads

$$\mathcal{E}_1 \to \mathrm{Ass}$$

which in turn gives a map of simplicial categories

$$F : \mathcal{E}_1^{\otimes} \to \mathrm{Ass}^{\otimes}$$

F is bijective on objects and induces weak equivalences on hom-spaces, so it is a Dywer-Kan equivalence. Consequently we have an equivalence of ∞-operads:

$$\mathbb{E}_1^{\otimes} \simeq N(\mathrm{Ass}^{\otimes})$$

Definition 8.53. Let $I \in \mathbf{F}$ be a finite set. Given an open n-manifold M, we write $\mathrm{Conf}(I, M)$ for the space of all injective maps $I \to M$, endowed with topology as a subspace of M^I. We will refer to a point in $\mathrm{Conf}(I, M)$ as an **configuration of $|I|$ points in M**. We will refer to the space $\mathrm{Conf}(I, M)$ as the **configuration space of $|I|$ points in M**.

Lemma 8.54. *Let $I \in \mathbf{F}$ be a finite set. Then the map $\mathrm{ev}_0 : \mathrm{Rect}(\square^n \times I, \square^n) \to \mathrm{Conf}(I, M)$ obtained by restricting along the original points $I \cong \{(0, ..., 0)\} \times I \hookrightarrow \square^n \times I$ is a homotopy equivalence.*

Theorem 8.55. *The space* $\mathrm{Rect}(\square^n \times I, \square^n) \simeq \mathrm{Conf}(I, \square^n)$ *is* $(n-2)$*-connected.*

Note that for every $n \geq 0$ the identification $\square^{n+1} \cong \square^n \times (-1, 1)$. By taking product with the open interval $(-1, 1)$, we get a morphism of simplicial operads $\mathcal{E}_n^{\otimes} \to \mathcal{E}_{n+1}^{\otimes}$. By taking homotopy coherent nerves, we obtain a map of ∞-operads $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$.

Corollary 8.56. *The colimit of the sequence*

$$\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \ldots$$

is equivalent to the commutative ∞*-operad* $\mathrm{Comm}^{\otimes} = N(\mathcal{F}\mathrm{in}_*)$*. As a result, we may sometimes use the notation* $\mathbb{E}_{\infty}^{\otimes} := \mathrm{Comm}^{\otimes}$*.*

9 Presentable ∞-categories

9.1 Basic properties

Definition 9.1. Let κ be a regular cardinal. We say that a simplicial set K is **κ-small** if the collection of non-degenerate simplices in K forms a κ-small set. We say that an ∞-category $\mathcal{C}$ is **κ-filtered** if for any κ-small simplicial set K and any diagram $f : K \to \mathcal{C}$, there is an extension $\overline{f} : K^{\triangleright} \to \mathcal{C}$. We say that $\mathcal{C}$ is **filtered** if it is ω-filtered. We say that simplicial set is κ-filtered if it is categorically equivalent to a κ-filtered ∞-category.

Theorem 9.2. *Let* κ *be a regular cardinal. Then*

- *In the* ∞*-category of spaces* $\mathcal{S}$ *(or more generally, any* ∞*-topos),* κ*-filtered colimits commute with* κ*-small limits.*
- *If an* ∞*-category* $\mathcal{C}$ *admits* κ*-small colimits and* κ*-filtered colimits, then* $\mathcal{C}$ *admits all small colimits.*

Proposition 9.3. *Let* κ *be a regular cardinal, and* $\mathcal{C} \in \mathsf{Cat}_{\infty}$ *be a small* ∞*-category. For a presheaf* $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$*, TFAE:*

1. *The unstraightening of* F*,* $\int_{\mathcal{C}} F$*, is a* κ*-filtered* ∞*-category.*
2. *There exists a* κ*-filtered* ∞*-category* I *and a functor* $J : I \to \mathcal{C}$*, such that*

$$F \simeq \varinjlim(I \xrightarrow{J} \mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})).$$

More explicitly, F *can be written as a filtered colimit of representable presheaves.*

If we further assume that $\mathcal{C}$ *admits finite colimits, then* (1) *and* (2) *are equivalent to*

3. $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ *preserves* κ*-small limits. More explicitly,* F *sends* κ*-small colimits in* $\mathcal{C}$ *to* κ*-small limits in* $\mathcal{S}$*.*

Proof. See [Lur09], Corollary 5.3.5.4. □

Definition 9.4. Let $\mathcal{C}$ be a small ∞-category. We define $\mathsf{Ind}_{\kappa}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ to be the full subcategory spanned by those presheaves satisfying either of the equivalent conditions in Proposition 9.3. We shall refer to $\mathsf{Ind}_{\kappa}(\mathcal{C})$ as the **Ind_{κ}-completion** of $\mathcal{C}$. When $\kappa = \omega$, we simply denote $\mathsf{Ind}_{\omega}(\mathcal{C})$ by $\mathsf{Ind}(\mathcal{C})$. Note that $\mathsf{Ind}_{\kappa}(\mathcal{C})$ is not small in general.

Remark 9.5. Intuitively one may think of $\mathsf{Ind}_{\kappa}(\mathcal{C})$ as the category obtained by formally adjoining κ-filtered colimits to $\mathcal{C}$. When κ gets larger, we are adjoining fewer objects to $\mathcal{C}$: if $\kappa < \lambda$, there is a definitional inclusion $\mathsf{Ind}_{\lambda}(\mathcal{C}) \subseteq \mathsf{Ind}_{\kappa}(\mathcal{C})$.

Remark 9.6. Each representable presheaf over $\mathcal{C}$ manifestly satisfies those conditions in 9.3. Hence the Yoneda embedding $\mathcal{Y} : \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ factors through $\mathsf{Ind}_\kappa(\mathcal{C})$. We abusively denote the full inclusion $\mathcal{C} \hookrightarrow \mathsf{Ind}_\kappa(\mathcal{C})$ by $\mathcal{Y}$, and refer to it as **Yoneda embedding**.

Here we list some properties of $\mathsf{Ind}_\kappa(\mathcal{C})$ construction without proof:

Proposition 9.7. *Let $\mathcal{C}$ be a small ∞-category and κ be a regular cardinal, then*

- $\mathsf{Ind}_\kappa(\mathcal{C})$ *admits κ-filtered colimits,*
- $\mathsf{Ind}_\kappa(\mathcal{C})$ *enjoys the following universal property: Let $\mathcal{D}$ be an ∞-category admitting κ-filtered colimits, then restriction along the Yoneda embedding $\mathcal{Y} : \mathcal{C} \hookrightarrow \mathsf{Ind}_\kappa(\mathcal{C})$ determines an equivalence*
$$\mathrm{Fun}_\kappa(\mathsf{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$
where $\mathrm{Fun}_\kappa(-,-) \subseteq \mathrm{Fun}(-,-)$ denotes the full subcategory spanned by those functors preserving κ-filtered colimts. Moreover, the equivalence preserves fully faithful functors.
- *If $\mathcal{C}$ admits κ-small colimits, then so is $\mathsf{Ind}_\kappa(\mathcal{C})$. In this case $\mathsf{Ind}_\kappa(\mathcal{C})$ admits all small colimits.*

Definition 9.8. We say that $\mathcal{C}$ is **κ-accessible** if there exists a small category $\mathcal{C}_0$ such that $\mathcal{C} \simeq \mathsf{Ind}_\kappa(\mathcal{C}_0)$. We say that $\mathcal{C}$ is **presentable** is $\mathcal{C}$ is κ-accessible for some κ and cocomplete.

Example 9.9. According to Proposition 9.7, if $\mathcal{C}$ is an small ∞-category which admits κ-small colimits, then $\mathsf{Ind}_\kappa(\mathcal{C})$ is presentable. In partcular, if $\mathcal{C}$ admits finite colimits, then $\mathsf{Ind}(\mathcal{C})$ is presentable.

The following theorem demonstrates the power of presentability. In fact this is the main reason why people like presentable categories.

Theorem 9.10 (Adjoint functor theorem)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be presentable ∞-categories. Then*

1. *A functor $F : \mathcal{A} \to \mathcal{B}$ admits a right adjoint if and only if F preserves small colimits.*
2. *A functor $F : \mathcal{A} \to \mathcal{B}$ admits a left adjoint if and only if G preserves small limits and κ-filtered colimits for some κ.*

We write $\widehat{\mathsf{Cat}_\infty}$ for the (very large) ∞-category of large ∞-categories. We define $\mathrm{Pr}^L \subset \widehat{\mathsf{Cat}_\infty}$ to be the non-full subcategory consisting of presentable ∞-categories and left adjoint functors, and we define $\mathrm{Pr}^R \subset \widehat{\mathsf{Cat}_\infty}$ to be the non-full subcategory consisting of presentable ∞-categories and right adjoint functors. The following lemma is immediate.

Notation 9.11. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We write $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by left adjoint functors and $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by right adjoint functors.

Lemma 9.12. *Taking adjoints determines an equivalence*
$$(\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathrm{Pr}^R.$$

Theorem 9.13. *The ∞-category Pr^R admits all limits and colimits. Moreover, the inclusion $\mathrm{Pr}^R \subset \widehat{\mathsf{Cat}_\infty}$ preserves limits, i.e. limits in Pr^R are computed underlying. The ∞-category Pr^L admits all limits and colimits. The inclusion $\mathrm{Pr}^L \subset \widehat{\mathsf{Cat}_\infty}$ preserves limits but not colimits. Let $J : I \to \mathrm{Pr}^L$ be a diagram, then there is an equivalence*
$$\varinjlim(I \to \mathrm{Pr}^L) \simeq \varprojlim(I^{\mathrm{op}} \xrightarrow{J^{\mathrm{op}}} (\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathsf{Pr}^R \hookrightarrow \widehat{\mathsf{Cat}_\infty})$$

We introduce a few stabilibity properties of presentable ∞-categories to conclude this section.

Theorem 9.14. *Let $\mathcal{D}$ be a presentable ∞-category.*

- *Let K be a small simplicial set, then* $\mathrm{Fun}(K, \mathcal{D})$ *is a presentable.*
- *Let $\mathcal{C}$ be a presentable ∞-category, then* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is presentable.*
- *Let $\mathcal{C}$ be a presentable ∞-category, then* $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ *and* $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$ *are presentable.*

9.2 Compact objects and compact generation

Definition 9.15. Let κ be a regular cardinal and $\mathcal{C}$ be an ∞-category which admits all small colimits. We say that an object $x \in \mathcal{C}$ is **κ-compact** if the functor corepresented by x $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}$ commutes with κ-filtered colimits. We write $\mathcal{C}^\kappa \subset \mathcal{C}$ for the full subcategory spanned by κ-compact objects in $\mathcal{C}$.

Remark 9.16. It is immediate that $\mathcal{C}^\kappa \subseteq \mathcal{C}$ is closed under κ-small colimits.

Since a presentable ∞-category admits all small colimits, the inclusion $\mathcal{C}^\kappa \hookrightarrow \mathcal{C}$ uniquely extends to κ-filtered colimits-preserving functor $k : \mathsf{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$, which is again fully faithful by Proposition 9.7.

Definition 9.17. Let $\mathcal{C}$ be an ∞-category which admits all small colimits. We say that $\mathcal{C}$ is **κ-compactly generated** if the functor $k : \mathsf{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$ is an equivalence.

Lemma 9.18. *Let $F : \mathcal{D} \hookrightarrow \mathcal{C}$ be a coflective subcategory, i.e. F is fully faithful and admits a right adjoint G. Then F is an equivalence if and only if G is conservative.*

Proof. If F is an equivalence, then so is G, and hence G is conservative. Conversely, suppose that G is conservative. Since F is fully faithful, the unit of the adjunction $\varepsilon : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ is invertible. We would like to prove that the counit $\eta : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ is also invertible. Now take $d \in \mathcal{D}$ and invoking the zig-zag identity: the following composition

$$G(d) \xrightarrow{\varepsilon_{G(d)}} GFG(d) \xrightarrow{G(\eta_d)} G(d)$$

is homotopic to $\mathrm{Id}_{G(d)}$. Since $\varepsilon_{G(d)}$ is invertible, so $G(\eta_d)$. Since G is conservative, η_d is invertible. $\square$

Proposition 9.19. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. Then TFAE:*

1. *$\mathcal{C}$ is κ-compactly generated.*
2. *Every object of $\mathcal{C}$ can be written as a small colimit of κ-compact objects.*
3. *For every $Z \in \mathcal{C}^\kappa$, the functor* $\mathrm{Map}_{\mathcal{C}}(Z, -) : \mathcal{C} \to \mathcal{S}$ *is conservative.*
4. *The restricted Yoneda embedding* $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S})$ *is fully faithful.*

Proof. We only give a sketchy proof here. For the equivalence (1) $\Leftrightarrow$ (2), one recall that a small colimit can always to written as a κ-filtered colimits of κ-small colimits. Invoking Remark 9.16, we get the the equivalence of (1) and (2).

Now we prove (1) $\Leftrightarrow$ (3). We write j' for the functor $\mathcal{C} \to \mathsf{Ind}_\kappa(\mathcal{C}^\kappa)$ sending X to $\mathrm{Map}_{\mathcal{C}}(-, X)$. We claim that k is left adjoint to j', which amounts the claim that there is a natural homotopy equivalence between $\mathrm{Map}_{\mathsf{Ind}_\kappa(\mathcal{C}^\kappa)}(F, j'X)$ and $\mathrm{Map}_{\mathcal{C}}(kF, X)$. Suppose that F can be written as a formal filtered colimits $F = \text{``}\varinjlim_{i\in I}\text{''} Z_i$, where "$\varinjlim$" means formal colimits and I is a κ-filtered diagram. An object in the former space can be understood as a natural transformation

$$\varinjlim_{i\in I} \mathrm{Map}_{\mathcal{C}}(-, Z_i) \Rightarrow \mathrm{Map}_{\mathcal{C}}(-, X),$$

while an object in the latter space can be understood as a morphism

$$\varinjlim_{i\in I} Z_i \to X.$$

So we can identify the two spaces via Yoneda Lemma. One then note that condition (3) is exactly a reformulation of that

$$j' : \mathcal{C} \to \mathrm{Fun}^{\kappa-sm}((\mathcal{C}^\kappa), \mathcal{S}), \quad X \mapsto \mathrm{Map}_{\mathcal{C}}(-, X).$$

is conservative. Here we have used $\mathrm{Fun}^{\kappa-sm}(-,-)$ to denote the full subcategory spanned by functors preserving κ-small limits.

Now we show that (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3). Note that the restricted Yoneda embedding functor (the composed horizontal arrow in the diagram below) factors through the functor j' defined above. More explicitly we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[dr, "j'"'] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \arrow[r] & \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S}) \\
& \mathsf{Ind}_\kappa \mathcal{C}^\kappa = \mathrm{Fun}^{\kappa-sm}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S}) \arrow[ur, hook] &
\end{tikzcd}$$

If $\mathcal{C}$ is κ-compactly generated, then j' is an equivalence and the restricted Yoneda embedding is a composition of two fully faithful functors. If the restricted Yoneda embedding is fully faithful, then j' is forced to be fully faithful, and hence conservative. $\square$

10 Stable ∞-categories

10.1 An introduction to stable ∞-categories

Definition 10.1. Let $\mathcal{C}$ be an ∞-category. An object $x \in \mathcal{C}$ is a **zero object** if it is both initial and final. $\mathcal{C}$ is **pointed** if it has a zero object. If $\mathcal{C}$ has a zero object, we shall denote it by $0_{\mathcal{C}}$ and simply by 0 when the category $\mathcal{C}$ is clear from the context.

Example 10.2. Let $\mathcal{C}$ be an ∞-category with a terminal object 1. Then the slice category $\mathcal{C}_{1/}$ is pointed, whose zero object is provided by the identity $\mathrm{Id} : 1 \to 1$. This morphism is manifestly initial by Proposition 6.4. To see that it is final, we choose a morphism $f : 1 \to X$ and compute the mapping space $\mathrm{Map}_{\mathcal{C}_{1/}}(f, \mathrm{Id})$, which is definitionally given by the following pullback:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{1/}}(f, \mathrm{Id}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(x, 1) \arrow[d, "f^*"] \\
\Delta^0 \arrow[r, "\mathrm{Id}"] & \mathrm{Map}_{\mathcal{C}}(1, 1).
\end{tikzcd}$$

Since f^* is a map between contractible spaces, it is a homotopy equivalence, and hence f^* has contractible fibres.

Definition 10.3. Let $\mathcal{C}$ be a pointed ∞-category with a zero object 0. A **fibre sequence** is a pullback diagram of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "f"] \\ 0 \arrow[r] & C \end{tikzcd}$$

In this case we say that A is the **fibre** of f and write $A = \mathsf{fib}(f)$. Dually, a **cofibre sequence** is a pushout diagram of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "f"] \\ 0 \arrow[r] & C \end{tikzcd}$$

In this case we say that C is the cofibre of i and write $C = \mathrm{cof}(i)$.

Definition 10.4. We say that a pointed ∞-category $\mathcal{C}$ is **stable** if

- Every morphism in $\mathcal{C}$ admits a fibre and a cofibre.
- A square in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "f"] \\ 0 \arrow[r] & C \end{tikzcd}$$

  is a fibre sequence if and only if it is a cofibre sequence.

Construction 10.5. Let $\mathcal{C}$ be a pointed ∞-category where each morphism admits a cofibre. We are now at the place to construct an improtant endofunctor $\Sigma_{\mathcal{C}}$ of $\mathcal{C}$. For each object $X \in \mathcal{C}$, consider the following pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma_{\mathcal{C}} X \end{tikzcd}$$

It takes some effort to organize the assignments $X \mapsto \Sigma_{\mathcal{C}} X$ into a functor between ∞-categories, and we refer the readers to the lines of [Lur17] before Remark 1.1.2.6 for details.

If $\mathcal{C}$ is pointed and every morphism in $\mathcal{C}$ admits a fibre, then we can define an endofunctor $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ dually. When $\mathcal{C}$ is clear from the context, we simply denote $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$ by Σ and Ω, respectively.

Lemma 10.6. *Let $\mathcal{C}$ be a pointed ∞-category where each morphism admits both a fibre and a cofibre. Then we have an adjunction $\Sigma \dashv \Omega$.*

Proof. Unwinding defintions, we see that ΣX enjoys the following universal property: for each object $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y)$ is homotopy equivalent to the space of diagrams of the following shape:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y, \end{tikzcd}$$

while the latter one is further homotopy equivalent to the mapping space $\mathrm{Map}_{\mathcal{C}}(X, \Omega Y)$, by the universal property of ΩY. Hence we are provided with a canonical homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Omega Y)$. $\square$

Proposition 10.7. *A stable ∞-category has finite limits and finite colimits.*

Proof. We only show that a stable ∞-category admits all pushouts and pullbacks. For the existence of general finite limits/colimits we refer the readers to [Lur17] Corollary 4.4.2.4, which says that any finite limit can be built using pullbacks and the terminal object. Given a diagram in $\mathcal{C}$ of the following shape:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ C & \end{tikzcd}$$

we wish to show that its pushout exsists. To do so, take the fibre of f to get the following diagram:

$$\begin{tikzcd} \mathsf{fib}(f) \arrow[r, "h"] \arrow[d] & A \arrow[r, "g"] \arrow[d, "f"'] & B \\ 0 \arrow[r] & C & \end{tikzcd}$$

Then we take the cofibre of $g \circ h$ to get the following diagram (the outer square is the cofibre sequence), where the dashed arrow (together with suitable higher cells) exists since the left hand square is a pushout (by stability).

$$\begin{tikzcd} \mathsf{fib}(f) \arrow[r, "h"] \arrow[d] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\ 0 \arrow[r] \arrow[rr, bend right] & C \arrow[r, dashed] & \mathsf{cof}(g \circ h). \end{tikzcd}$$

Now since the outer large square and the left hand square are pushout, we conclude that the right hand square is a pushout. The existence of pullbacks can be illustrated dually. $\square$

Theorem 10.8. *Let $\mathcal{C}$ be a pointed ∞-category. Then TFAE:*

1. *$\mathcal{C}$ is stable. That is, every morphism in $\mathcal{C}$ admits a fibre and a cofibre, and a sequence in $\mathcal{C}$ is a fibre sequence if and only if it is a cofibre sequence.*
2. *Every morphism in $\mathcal{C}$ admits a cofibre, and the endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence.*
3. *Every morphism in $\mathcal{C}$ admits a fibre, and the endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence.*
4. *$\mathcal{C}$ has pullbacks and pushouts, and a square in $\mathcal{C}$ is a pushout square if and only if it is a pullback square.*

Proof. The derivations (4) $\Rightarrow$ (1), (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial. Now we prove (1) $\Rightarrow$ (4). Assume that $\mathcal{C}$ is stable, and we are given a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Our goal is to show that the square is also a pushout. To do so, we take the fibre of h, so we get a larger diagram where every square is a pullback:

$$\begin{tikzcd} \mathsf{fib}(h) \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \arrow[r, "k"'] & D \end{tikzcd}$$

By stability, the outside square and the left sqaure are pushout diagrams, so we deduce that the right square is also a pushout. In a similar manner one shows that every pushout square in $\mathcal{C}$ is also a pullback square. It remains to prove (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). To do so we add four extra conditions to our list:

4. Every cofibre sequence in $\mathcal{C}$ is a fibre sequence.

5. Every fibre sequence in $\mathcal{C}$ is a cofibre sequence.

6. Every map in $\mathcal{C}$ has a fibre.

7. Every map in $\mathcal{C}$ has cofibre.

Our plan is to show that (2) $\Rightarrow$ (4), and dually (3) $\Rightarrow$ (5). Then we show that (2) $\Rightarrow$ (6) and dually (3) $\Rightarrow$ (7). The proofs actually reveals that (2) $\Leftrightarrow$ (3). Hence either (2) or (3) gives rise to (1).

Now assume that condition (2) is imposed, and we first show that every cofibre sequence in $\mathcal{C}$ is also a fibre sequence. Suppose we are given a cofibre sequence

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C. \end{tikzcd}$$

We wish to show that the sequence is also a fibre sequence, which amounts to check that A enjoys the universal property of being a fibre. We write $p : \Lambda^2_2 \to \mathcal{C}$ for the functor specifying the diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

and we write $\pi_p : \mathcal{C}_{/p} \to \mathcal{C}$ for the associated right fibration. For an object $X \in \mathcal{C}$, an object in the fibre $\pi_p^{-1}(X)$ can be identified with a square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

For each object $X \in \mathcal{C}$ there is a canonical map $q_X : \mathrm{Map}_{\mathcal{C}}(X, A) \to \pi_p^{-1}(X)$, given by simply composing maps. Then A is the fibre of g if and only if q_X is homotopy equivalence for each X. The key observation is that the homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(X, A) \xrightarrow{\Sigma} \mathrm{Map}_{\mathcal{C}}(\Sigma X, \Sigma A)$ provided by stability actually factor through q_X, i.e. there is a commutative diagram

$$\begin{tikzcd} & \pi_p^{-1}(X) \arrow[dr, "s_X"] & \\ \mathrm{Map}_{\mathcal{C}}(X, A) \arrow[ur, "q_X"] \arrow[rr, "\Sigma"'] & & \mathrm{Map}_{\mathcal{C}}(\Sigma X, \Sigma A) \end{tikzcd}$$

So we see that q_X admits a left inverse s_X. We obtian the desired result by noticing that s_X also admits a left inverse. Let $\Sigma \circ p : \Lambda^2_2 \to \mathcal{C}$ be the composed functor specifying the diagram

$$\begin{tikzcd} & \Sigma B \arrow[d, "\Sigma g"] \\ 0 \arrow[r] & \Sigma C \end{tikzcd}$$

and $\pi_{\Sigma op} : \mathcal{C}_{/\Sigma op} \to \mathcal{C}$ be the associated right fibration. Then there is a canonical factorisation

$$\begin{tikzcd} & \mathrm{Map}_{\mathcal{C}}(\Sigma X, \Sigma A) \arrow[dr] & \\ \pi_p^{-1}(X) \arrow[ur, "s_X"] \arrow[rr, "\Sigma"'] & & \pi_{\Sigma op}^{-1}(\Sigma X) \end{tikzcd}$$

so that s_X also admits a left inverse. We conclude that q_X and s_X are both homotopy equivalences, and we see that (2) $\Rightarrow$ (4). Dually (3) $\Rightarrow$ (5)

Then we show that (2) $\Rightarrow$ (6). Let $p : A \to B$ be a morphism. We can construct the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathsf{cof}(f) \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma f"] \\ & 0 \arrow[r] & \Sigma B \end{tikzcd}$$

where each square is a cofibre sequence, and hence is a fibre sequence. Since Σ is invertible, we apply Σ^{-1} to the lower fibre sequence to get the fibre of f. Dually, one shows that (3) $\Rightarrow$ (7).

Now that the endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$ is well-defined, it has to be the inverse if Σ. In this way we have established the derivation (2) $\Rightarrow$ (3). Dually one shows that (3) $\Rightarrow$ (2). Hence either (2) or (3) gives the stability of $\mathcal{C}$.

$\square$

Proposition 10.9. *A functor between ∞-categories preserves finite limits (resp. colimits) if and only if it preserves pullbacks (resp. pushouts) and terminal (resp. initial objects). As a result, a functor between stable ∞-categories preserves finite limits if and only if it preserves finite colimits.*

Proof. See [Lur17], Corollary 4.4.2.5. $\square$

Definition 10.10. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is **right exact** (resp. **left exact**) if it preserves finite colimits. A functor is **exact** if it is both left and right exact. Note that left/right exact functors between stable ∞-categories are exact. We write $\mathsf{Cat}^{\mathrm{ex}} \subset \mathsf{Cat}_\infty$ for the (non)-full subcategory consisting of (small) stable ∞-categories and exact functors.

10.2 Stabilization

10.2.1 Spanier-Whitehead construction

There are different procedures of stabilization. For example, if we denote by $\mathsf{Cat}_*^{\mathrm{rex}} \subset \mathsf{Cat}_\infty$ the (non-full) subcategory consisting of small pointed ∞-categories which admits finite colimits and right exact functors, then the full inclusion $\mathsf{Cat}^{\mathrm{ex}} \subset \mathsf{Cat}_*^{\mathrm{rex}}$ admits a left adjoint, called **Spanier-Whitehead construction**.

Proposition 10.11. $\mathsf{Cat}_*^{\mathrm{rex}}$ *admits filtered colimits and the inclusion* $\mathsf{Cat}_*^{\mathrm{rex}} \subset \mathsf{Cat}_\infty$ *preserves filtered colimits.*

Definition 10.12. Let $\mathcal{C} \in \mathsf{Cat}_*^{\mathrm{rex}}$ be a pointed small ∞-category with finite colimits. In this case we have an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$. We define $\mathsf{SW}(\mathcal{C})$ to be the following sequential colimit in $\mathsf{Cat}_*^{\mathrm{rex}}$ (hence in Cat_∞):

$$\mathsf{SW}(\mathcal{C}) := \varinjlim(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} ...)$$

An object in $\mathsf{SW}(\mathcal{C})$ can be described by a pair (n, X) where $n \in \mathbb{N}$ is a natural number and $X \in \mathcal{C}$ is an object in $\mathcal{C}$. There are canonical isomorphisms $(n, X) \simeq (n+1, \Sigma X)$ relating different objects. Mapping spaces in $\mathsf{SW}(\mathcal{C})$ can be described as follows:

$$\mathrm{Map}_{\mathsf{SW}(\mathcal{C})}((n, X), (m, Y)) \simeq \varinjlim_{k \in \mathbb{N}} \mathrm{Map}_{\mathcal{C}}(\Sigma^{k-n} X, \Sigma^{k-m} Y). \tag{3}$$

Proposition 10.13. $\mathsf{SW}(\mathcal{C})$ *is a small stable ∞-category. Moreover, there is a canonical functor $i : \mathcal{C} \to \mathsf{SW}(\mathcal{C})$ which induces an equivalence of functor categories:*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathsf{SW}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D})$$

for each stable ∞-category $\mathcal{C}$. This simply says that the Spanier-Whitehead construction defines a functor $\mathsf{SW}(-) : \mathsf{Cat}_*^{\mathrm{rex}} \to \mathsf{Cat}^{\mathrm{ex}}$ left adjoint to the fully faithful inclusion $\mathsf{Cat}^{\mathrm{ex}} \subset \mathsf{Cat}_*^{\mathrm{rex}}$.

Proof. First we note that $\mathsf{SW}(\mathcal{C}) \in \mathsf{Cat}_*^{\mathrm{rex}}$ by definition. In fact $\mathsf{SW}(\mathcal{C})$ has zero objects $(n, 0_{\mathcal{C}})$ where $0_{\mathcal{C}}$ is a zero object of $\mathcal{C}$. According to Theorem 10.8, to demonstrate the stability of $\mathsf{SW}(\mathcal{C})$, it suffices to chech that $\Sigma_{\mathsf{SW}(\mathcal{C})} : \mathsf{SW}(\mathcal{C}) \to \mathsf{SW}(\mathcal{C})$ is an equivalence. So we need to provide an explicit description of $\Sigma_{\mathsf{SW}(\mathcal{C})}$. Since filtered colimits commute with pushouts, $\Sigma_{\mathsf{SW}(\mathcal{C})}$ can be constructed in the following way:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "\Sigma_{\mathcal{C}}"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "\Sigma_{\mathcal{C}}"] & \ldots & \mathsf{SW}(\mathcal{C}) \arrow[d, "\Sigma_{\mathsf{SW}(\mathcal{C})}"] & (n, X) \arrow[d, mapsto] \\
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"'] & \ldots & \mathsf{SW}(\mathcal{C}) & (n, \Sigma X)
\end{tikzcd}$$

One then note that the functor $\Sigma_{\mathsf{SW}(\mathcal{C})}$ sends an object (n, X) to $(n, \Sigma_{\mathcal{C}} X)$. The formula for mapping spaces in the Spanier-Whitehead construction 3 immediately tell us that $\Sigma_{\mathsf{SW}(\mathcal{C})}$ is fully faithful. It is also easy to see that $\Sigma_{\mathsf{SW}(\mathcal{C})}$ is essentially surjective: for each object (n, X), we have $\Sigma_{\mathsf{SW}(\mathcal{C})}(n+1, X) = (n+1, \Sigma X) \simeq (n, X)$.

Definitionally there is a canonical functor $i : \mathcal{C} \to \mathsf{SW}(\mathcal{C})$ sending an object X to $(0, X)$. Since $\mathsf{SW}(\mathcal{C})$ is defined via a sequential colimit, then for another category $\mathcal{D} \in \mathsf{Cat}_*^{\mathrm{rex}}$, a right exact functor $F : \mathsf{SW}(\mathcal{C}) \to \mathcal{D}$ is equivalent to a family of right exact functors $(F_n)_{n \in \mathbb{N}}$ together with specified homotopies $F_n \circ \Sigma_{\mathcal{C}} \simeq F_{n-1}$, as illustrated in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "F_0"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[dl, "F_1"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[dll, "F_2"] & \ldots & \mathsf{SW}(\mathcal{C}) \arrow[d, "F"] \\
\mathcal{D} & & & & \mathcal{D}.
\end{tikzcd}$$

The key observation is that when $\mathcal{D}$ is a stable ∞-category, the family of functors $F = (F_n)_{n \in \mathbb{N}}$ is determined by the component F_0: since each F_n is right exact, $F_n \circ \Sigma_{\mathcal{C}} \simeq F_{n-1}$ implies that $\Sigma_{\mathcal{D}} \circ F_n \simeq F_{n-1}$. By stability of $\mathcal{D}$, we get $F_n \simeq \Omega_{\mathcal{D}} F_{n-1} \simeq \Omega_{\mathcal{D}}^n \circ F_0$. $\square$

10.2.2 Spectrification

In this section we introduce a stabilization procudure which is somtimes referred as **spectrification** in literature. This construction is dual to the Spanier-Whitehead construction in the following sense: let $\mathsf{Cat}_*^{\mathrm{lex}} \subset \mathsf{Cat}_\infty$ be the subcategory consisting of small pointed ∞-categories with finite limits and left exact functors. Spectrification provides us with a right adjpint to the fully faithful inclusion $\mathsf{Cat}^{\mathrm{ex}} \subset \mathsf{Cat}_*^{\mathrm{lex}}$. One easily finds that the spectrification funnctor sends a category $\mathcal{C} \in \mathsf{Cat}_*^{\mathrm{lex}}$ to the inverse limit

$$\varprojlim(\ldots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}).$$

However, we will follow [Lur17] to approach this construction in a different way. As the name spectrification suggests, this is the procedure that sends the category $\mathcal{S}$ to Sp, the ∞-category of spectra.

Definition 10.14. Let $\mathcal{C}$ be pointed ∞-categories admitting finite colimits, and $\mathcal{D}$ be an ∞-category admitting finite limits. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is **excisive** if F sends pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. We say that F is **reduced** if F preserves terminal. We write $\mathsf{Exc}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by excisive ones and write $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by reduced ones. We write $\mathsf{Exc}_*(\mathcal{C}, \mathcal{D}) = \mathsf{Exc}(\mathcal{C}, \mathcal{D}) \cap \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ for their intersection.

Lemma 10.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories such that $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits. Let $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ be a reduced functor. Then TFAE*

1. *F is excisive.*
2. *The canonical natural transformation $F \Rightarrow \Omega_{\mathcal{D}} \circ F \circ \Sigma_{\mathcal{C}}$ is invertible.*

Proof. The proof essentially the same that of Theorem 10.8 and we omit the details. □

Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories such that $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits. Then the category $\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})$ is a pointed ∞-category which admits finite limits (actually it admits all limits that $\mathcal{D}$ admits): the zero object is the constant functor valued at $0_{\mathcal{D}}$, and finite limits of functors are formed pointwise. The endofunctor $\Omega_{\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})}$ sends a functor $F : \mathcal{C} \to \mathcal{D}$ to the composition $\Omega_{\mathcal{D}} \circ F$. The key observation is the following

Theorem 10.16. *The functor $\Omega_{Exc_*(\mathcal{C}, \mathcal{D})}$ is an equivalence. As a result of Theorem 10.8, $\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})$ is a stable ∞-category.*

Proof. We claim that the functor $- \circ \Sigma_{\mathcal{C}} : \mathsf{Exc}_*(\mathcal{C}, \mathcal{D}) \to \mathsf{Exc}_*(\mathcal{C}, \mathcal{D})$ is an inverse of $\Omega_{\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})}$. This amounts to the claim that $F \simeq \Omega_{\mathcal{D}} \circ F \circ \Sigma_{\mathcal{C}}$, which directly follows from reducedness and excisiveness. □

Definition 10.17. Let $\mathcal{S}^{\mathrm{fin}} \subset \mathcal{S}$ be the smallest full subcategory containing $*$ that is closed under isomoprhisms and finite colimits, and let $\mathcal{S}^{\mathrm{fin}}_*$ be its pointed variant. For $\mathcal{C}$ a pointed ∞-category with finite limits, following [Lur17], we write $\mathsf{Sp}(\mathcal{C})$ for $\mathsf{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$, and we refer to an object in $\mathsf{Sp}(\mathcal{C})$ as a **spectrum object** in $\mathcal{C}$.

There is a natural functor $\mathsf{ev}_{S^0} : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ sending a reduced excisive functor to its value at S^0.

Lemma 10.18. *Let $\mathcal{C} \in \mathsf{Cat}^{\mathrm{lex}}_*$. Then $\mathcal{C}$ is stable if and only if $\mathsf{ev}_{s^0} : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Proof. The "if" part follows from Theorem 10.16. For the "only if" part, note that when $\mathcal{C}$ is stable we have

$$\mathsf{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S}) \simeq \mathcal{C}$$

The first equivalence follows from stability, reducedness, excisiveness and Proposition 10.9. The second equivalence is mainifest. The last equivalence follows the universal property of $\mathcal{S}^{\mathrm{fin}}$. □

Proposition 10.19. *For each stable ∞-category $\mathcal{D}$, composition with ev_{S^0} determines an equivalence of functor categories*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C})$$

Proof. We note that there is a chain of canonical equivalences

$$\mathsf{Exc}_*(\mathcal{D}, \mathsf{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})) \simeq \mathsf{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathsf{Exc}_*(\mathcal{D}, \mathcal{C})) \simeq \mathsf{Exc}_*(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}).$$

□

Proposition 10.20. *Let $\mathcal{C}$ be a pointed ∞-category admitting finite limits. Evaluation at S^n determines a family of functor $(F_n = \mathsf{ev}_{S^n} : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C})_{n\in\mathbb{N}}$ together with homotopies $F_{n-1} \simeq \Omega_{\mathcal{C}} \circ F_n$. This family of functors together determines an equivalence*

$$\mathsf{Sp}(\mathcal{C}) \simeq \varprojlim(\ldots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C})$$

Proof. One directly sees that both $\mathsf{Sp}(-)$ and the inverse limit construction provides us with a right adjoint to the fully faithful inclusion $\mathsf{Cat}^{\mathrm{ex}} \subset \mathsf{Cat}_*^{\mathrm{lex}}$.

One can also observe this equivalence by noting that, to specify an object in $\mathsf{Sp}(\mathcal{C})$, it suffices to determine a family of objects $(X_n)_{n\in\mathbb{N}}$ with equivalences $X_{n-1} \simeq \Omega_{\mathcal{C}} X_n$, but this is exactly the amount of data required to specify an object in the inverse limit construction. □

10.2.3 Stabilization of presentable ∞-categories

In this section we study the stabilization process of a presentable ∞-category. It turns out that for presentable ∞-categories the (variant of) Spanier-Whitehead construction coincide with spectrification.

Proposition 10.21 ([Lur17])**.** *Suppose that $\mathcal{C}$ is a small ∞-category which admits finite colimits, and $\mathcal{D}$ be a presentable ∞-category. Then $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, $\mathsf{Exc}(\mathcal{C}, \mathcal{D})$ and $\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})$ are accessible localizations of the presentable ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. As a result, $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, $\mathsf{Exc}(\mathcal{C}, \mathcal{D})$ and $\mathsf{Exc}_*(\mathcal{C}, \mathcal{D})$ are presentable ∞-categories.*

Definition 10.22. For $\mathcal{C} \in \mathrm{Pr}^L$, we use $\mathcal{C}_*$ to denote the slice category $\mathcal{C}_{1/}$, where $1 \in \mathcal{C}$ is the terminal object of $\mathcal{C}$. Since for a small ∞-category $\mathcal{D}$. We simply write $\mathsf{Sp}(\mathcal{C})$ for the $\mathsf{Sp}(\mathcal{C}_*)$. Moreover, we use $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}_*$ to denote the functor ev_{S^0}. Abusively, we also use Ω^∞ to denote the composed functor $\mathsf{Sp}(\mathcal{C}) = \mathsf{Sp}(\mathcal{C}_*) \to \mathcal{C}_* \to \mathcal{C}$.

Remark 10.23. Invoking Proposition 10.21, the construction of the category of spectrum objects defines a functor $\mathsf{Sp}(-) : \mathrm{Pr}^L \to \mathrm{Pr}^L_{\mathrm{st}}$.

As in the case of small ∞-categories, we have a canonical equivalence

$$\mathsf{Sp}(\mathcal{C}) \simeq \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*).$$

where the limit is taken in $\widehat{\mathsf{Cat}_\infty}$, or equivalently in Pr^R. Using the formula for colimits in Pr^L, we see that

$$\mathsf{Sp}(\mathcal{C}) \simeq \varinjlim(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \ldots).$$

Note that the colimit is taken in Pr^L (very important!). The result is not true if we take colimits in $\widehat{\mathsf{Cat}_\infty}$.

Construction 10.24. One note the functor $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}_*$ is a morphism in Pr^R (since the limit is computed in Pr^R). Hence we conclude that it admits a left adjoint Σ^∞, which we refer to as the **based suspension functor**. By composing with the base-point adjoining functor $\mathcal{C} \to \mathcal{C}_*$, we get a functor $\Sigma^\infty_+ : \mathcal{C} \to \mathsf{Sp}(\mathcal{C})$, which we refer to as the **unbased suspension functor**.

Definition 10.25. When $\mathcal{C} = \mathcal{S}$ we write $\mathsf{Sp}(\mathcal{S})$ as Sp, which we refer to as **the ∞-category of spectra**.

We briefly explain the compatibility between stabilization and ind-completion.

Proposition 10.26. *Let $\mathcal{C}$ be a small pointed ∞-category admitting finite colimits. Then there is a canonical equivalence of stable presentable ∞-categories:*

$$\mathsf{Ind}(\mathsf{SW}(\mathcal{C})) \simeq \mathsf{Sp}(\mathsf{Ind}(\mathcal{C}))$$

Proposition 10.27. *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories where $\mathcal{D}$ is stable. Then $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ determines an equivalence of functor categories*

$$\mathrm{Fun}^R(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \to \mathrm{Fun}^R(\mathcal{D}, \mathcal{C}).$$

Proof. By the formula

$$\mathsf{Sp}(\mathcal{C}) \simeq \varprojlim(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*),$$

a functor $F \in \mathrm{Fun}^R(\mathcal{D}, \mathsf{Sp}(\mathcal{C}))$ consists of a family of functors $(F_n)_{n\in\mathbb{N}}$ where each functor admits left adjoints

$$\begin{tikzcd}
& \cdots \arrow[d] \\
& \mathcal{C}_* \arrow[d, "\Omega"] \\
\mathcal{D} \arrow[r, "F_0"'] \arrow[ur, "F_1"'] \arrow[uur] & \mathcal{C}_*
\end{tikzcd}$$

together with homotopies $F_{n-1} \simeq \Omega \circ F_n$. Since $\mathcal{D}$ is stable, each F_n is excisive and reduced, hence the family $(F_n)_{n\in\mathbb{N}}$ is determined by F_0. Thus we get an equivalence

$$\mathrm{Fun}^R(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \simeq \mathrm{Fun}^R(\mathcal{D}, \mathcal{C}_*).$$

We conclude by proving that the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ determines an equivalence $\mathrm{Fun}^R(\mathcal{D}, \mathcal{C}_*) \simeq \mathrm{Fun}^R(\mathcal{D}, \mathcal{C})$. We finish this task by constructing its inverse directly. Given $F : \mathcal{D} \to \mathcal{C}$ a functor in $\mathrm{Fun}^R(\mathcal{D}, \mathcal{C})$, then $F(0_\mathcal{D}) = 1_\mathcal{C}$ where $1_\mathcal{C}$ is the terminal object of $\mathcal{C}$. Then there is an induced functor

$$\tilde{F} : \mathcal{D} \simeq \mathcal{D}_{0_\mathcal{D}/} \to \mathcal{C}_{1_\mathcal{C}/} = \mathcal{C}_*.$$

This construction defines a functor $\mathrm{Fun}^R(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}^R(\mathcal{D}, \mathcal{C}_*)$, which turns out to be the desired inverse. $\square$

Corollary 10.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable. Then composition $\Sigma^\infty_+ : \mathcal{C} \to \mathsf{Sp}(\mathcal{C})$ induces an equivalence*

$$\mathrm{Fun}^L(\mathsf{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}).$$

Proof. Since composition with Ω^∞ determines an equivalence

$$\Omega^\infty \circ - : \mathrm{Fun}^R(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}^R(\mathcal{D}, \mathcal{C}),$$

by taking adjoints, there is an equivalence

$$\mathrm{Fun}^L(\mathsf{Sp}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}).$$

$\square$

10.3 t-structures on stable ∞-categories

Theorem 10.29. *Let $\mathcal{C}$ be a stable ∞-category. Then $h\mathcal{C}$ carries a natural structure of a triangulated category.*

Definition 10.30. Let $\mathcal{C}$ be a triangulated category. A t-structure on $\mathcal{C}$ is defnied to be a pair of full subcategories $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq 0}$ both closed under isomorphisms, such that the following properties are verified:

1. For $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, we have $\hom_{\mathcal{C}}(X, Y[-1]) = 0$.
2. We have inclusions $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$.
3. For any $X \in \mathcal{C}$, there exists a fibre sequence $X' \to X \to X''$, where $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1}$

Notation 10.31. Let $\mathcal{C}$ be a triangulated category equipped with a t-structure. We will write $\mathcal{C}_{\geq n}$ for $\mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq n}$ for $\mathcal{C}_{\leq 0}[n]$.

Notation 10.32. Let $\mathcal{C}$ be a stable ∞-category. We shall use $X[1]$ to denote ΣX and $X[-1]$ to denote ΩX. Similarly we write $X[n]$ for $\Sigma^n X$ and $X[-n]$ for $\Omega^n X$.

Definition 10.33. Let $\mathcal{C}$ be a stable ∞-category. A ***t*-structure** on $\mathcal{C}$ is a t-structure on its homotopy category $h\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure, we write $\mathcal{C}_{\leq n}$ (resp. $\mathcal{C}_{\geq n}$) for the full subcategories of $\mathcal{C}$ spanned by objects belonging to $(h\mathcal{C})_{\leq n}$ (resp. $(h\mathcal{C})_{\geq n}$).

Lemma 10.34. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Then for $X \in \mathcal{C}_{\leq 0}$ and* $\mathrm{Map}_{\mathcal{C}}(Y \in \mathcal{C}_{\leq -1})$*, we have* $\mathrm{Map}_{\mathcal{C}}(X, Y)$ *is constractible.*

Proof. We have $\pi_n(\mathrm{Map}_{\mathcal{C}}(X, Y)) \simeq \pi_0(\mathrm{Map}_{\mathcal{C}}(X, \Omega^n Y)) = \hom_{h\mathcal{C}}(X, Y[-n]) = 0$. □

Proposition 10.35. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. For each $n \in \mathbb{Z}$, the full subcategory $\mathcal{C}_{\leq n}$ is a reflective subcategory of $\mathcal{C}$. Dually, the full subcategory $\mathcal{C}_{\geq n}$ is a coreflective subcategory of $\mathcal{C}$.*

Proof. Without loss of generality we may assume that $n = -1$. We are going to prove that, for each $X \in \mathcal{C}$, there is a localization map $L_X : X \to \overline{X}$ such that

- $\overline{X}$ is a local object, i.e. it lies in the full subcategory $\mathcal{C}_{\leq -1}$.
- L_X induces an equivalence
$$- \circ L_X : \mathrm{Map}_{\mathcal{C}}(\overline{X}, Y) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(X, Y).$$
for each object $Y \in \mathcal{C}_{\leq -1}$.

Invoking condition (3) of Definition 10.30, we can choose L_X to fit it the fibre-cofibre sequence
$$X' \to X \xrightarrow{L_X} \overline{X}. \tag{4}$$
Take an arbitrary object $Y \in \mathcal{C}_{\leq -1}$, and apply the functor $\mathrm{Map}_{\mathcal{C}}(-, Y)$ to the sequence (4). Then we are provided with a fibre sequence of space
$$\mathrm{Map}_{\mathcal{C}}(\overline{X}, Y) \xrightarrow{(L_X)^*} \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X', Y).$$
Since $\mathrm{Map}_{\mathcal{C}}(X', Y)$ is contractible, we conclude that L_X^* is a homotopy equivalence. □

Corollary 10.36. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. For each $n \in \mathbb{Z}$, the full subcategory $\mathcal{C}_{\leq n}$ is stable under all limits which exists in $\mathcal{C}$. Dually, the full subcategory $\mathcal{C}_{\geq n}$ is stable under all limits that exist in $\mathcal{C}$.*

Notation 10.37. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We write $\tau_{\leq n}$ for the left adjoint of the full inclusion $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$, and $\tau_{\geq n}$ for the right adjoint of the full inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$.

11 Application: Factorisation homology and cobordism hypothesis

To be added.

A Model categories

A.1 Basic notions

Definition A.1. A **model category** is a category $\mathcal{M}$ equipped with three classes of maps: W, Cof and Fib, which we will refer to as **weak equivalences**, **cofibrations** and **fibrations** respectively, satisfying the following axioms:

(1) $\mathcal{M}$ is bicomplete.

(2) W satisfies the 2-out-of-3 property.

(3) $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are weak factorisation systems.

We say that an object $c \in \mathcal{M}$ is **cofibrant** if the unique map $\emptyset \to c$ is a cofibration, where $\emptyset$ is the initial object in $\mathcal{M}$. We say that an object $c \in \mathcal{M}$ is **fibrant** if the unique map $c \to *$ is a fibration, where $*$ is the final object in $\mathcal{M}$. We will refer to a morphism in $\mathrm{Cof} \cap W$ as a **trivial cofibration**, and to a morphism in $\mathrm{Fib} \cap W$ as a **trivial fibration**.

Definition A.2. Let $\mathcal{M}$ be a model category. Applying functorial factorisations, we get a **fibrant replacement functor** $R : \mathcal{M} \to \mathcal{M}$ and a **cofibrant replacement functor** $Q : \mathcal{M} \to \mathcal{M}$ equipped with natural weak equivalence[4]:

$$\eta : \mathrm{Id}_{\mathcal{M}} \Rightarrow R \quad \text{and} \quad \epsilon : Q \Rightarrow \mathrm{Id}_{\mathcal{M}}.$$

Model categories are defined to be localized. Among the three classes of morphisms W, Cof and Fib, W is the most important one since this is the class of morphisms we want localize, and the classes Cof and Fib play auxiliary roles.

Definition A.3. Let $(\mathcal{C}, W)$ be a category equipped with a subcategory of weak equivalences W. Define $\mathrm{Ho}(\mathcal{C})$ to be the formal localization $\mathcal{C}[W^{-1}]$, i.e. the category obtained by formally inverting all the morphisms in W. We will refer to $\mathrm{Ho}(\mathcal{C})$ as **the homotopy category** of $\mathcal{C}$.

Definition A.4. A **cylinder object** of an object A is a factorisation of the codiagonal of A into a cofibration followed by a weak equivalence, i.e. a commutative diagram of the form

$$\begin{tikzcd} A \coprod A \arrow[r, "(\partial_0,\partial_1)"'] \arrow[rr, bend left, "(\mathrm{Id}_A,\mathrm{Id}_A)"] & IA \arrow[r, "\sigma"'] & A \end{tikzcd}$$

in which (∂_0, ∂_1) is a cofibration, and σ is s weak equivalence. Dually, a **path object** of an object X is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[rr, bend left, "(\mathrm{Id}_X,\mathrm{Id}_X)"] & X^I \arrow[r, "(d^0,d^1)"'] & X \times X \end{tikzcd}$$

in which s is a weak equivalence, and the map (d^0, d^1) is a fibration.

[4]That is, natural transformations whose components are weak equivalences.

Let us consider two morphisms $f_0, f_1 : A \to X$.

A **left homotopy** (resp. **right homotopy**) from f_0 to f_1 is given by a cylinder object (resp. a path object) of A together with a morphism $h : IA \to X$ (resp. a morphism $k : A \to X^I$), such that, for $i = 0, 1$, we have $h\partial_i = f_i$ (resp. $d^i k = f_i$).

Proposition A.5. *If A is cofibrant and B is fibrant then left and right homotopy define equivalence relations on $\hom_{\mathcal{M}}(A, B)$ and moreover these relations coincide. We will denote this equivalence relation by $\sim$*

Proposition A.6. *Let $f : A \to B$ be a map between bifibrant objects. Then f is a weak equivalence if and only if it has a homotopy inverse.*

Corollary A.7. *Let $\mathcal{M}$ be a model category. Let $\gamma : \mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M}_{cf})$ and $\delta : \mathcal{M}_{cf} \to \mathcal{M}_{cf}/\sim$ be the canonical functors. Then there is a unique isomorphism of categories $\mathcal{M}_{cf}/\sim \to \mathrm{Ho}(\mathcal{M}_{cf})$ such that $j \circ \delta = \gamma$. Furthermore j is identity on objects.*

Theorem A.8. *Let $\mathcal{M}$ be a model category. Let $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ denote the canonical functor. Then:*

(1) The inclusion $\mathcal{M}_{cf} \hookrightarrow \mathcal{M}$ induces an equivalence of categories

$$\mathcal{M}_{cf}/\sim \xrightarrow{\cong} \mathrm{Ho}(\mathcal{M}_{cf}) \to \mathrm{Ho}(\mathcal{M}).$$

(2) There are natural isomorphisms

$$\hom_{\mathcal{M}}(QRX, QRY)/\sim \cong \hom_{\mathrm{Ho}(\mathcal{M})}(\gamma X, \gamma Y) \cong \hom_{\mathcal{M}}(RQX, RQY)/\sim .$$

Lemma A.9 (Ken Brown's lemma)**.** *Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between model categories. Then:*

1. *If F takes trivial cofibrations to weak equivalences, then F takes weak equivalences between cofibrant objects to weak equivalences.*
2. *If F takes trivial fibrations to weak equivalences, then F takes weak equivalences between fibrant objects to weak equivalences.*

Proof. Let $f : x \to y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Consider the following commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & y \arrow[d] \arrow[dr, "b"] \arrow[drr, bend left, "\mathrm{Id}_y"] & & \\
x \arrow[r] \arrow[rr, bend right, "a"'] \arrow[rrr, bend right=50, "f"'] & x \coprod y \arrow[r, "i"] & c \arrow[r, "p"] & Y
\end{tikzcd}$$

The leftmost square is a pushout, and $x \to x \coprod y$ and $y \to x \coprod y$ are both. $p \circ i$ is a factorisation of the canonical map $x \coprod y \to y$, where i is a cofibration and p is a trivial fibration. By the 2-out-of-3 property of weak equivalences, we see that both a and b are trivial cofibrations. Applying F gives the following diagram:

$$\begin{tikzcd}
 & F(y) \arrow[d, "F(b)"'] \arrow[dr, "\mathrm{Id}"] & \\
F(x) \arrow[r, "F(a)"'] & F(c) \arrow[r, "F(p)"'] & F(y)
\end{tikzcd}$$

In the diagram $F(a)$ and $F(b)$ are weak equivalences by the hypothesis, which implies that $F(p)$ is a weak equivalence, and hence $F(f) = F(p) \circ F(a)$ is a weak equivalence. □

Definition A.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. We say that

- a functor $F : \mathcal{C} \to \mathcal{D}$ is a **left Quillen functor** if F is a left adjoint and preserves cofibrations and trivial cofibrations.
- a functor $G : \mathcal{D} \to \mathcal{C}$ is a **right Quillen functor** if G is a right adjoint and preserves fibrations and trivial fibrations.
- a pair of adjunction $F \dashv G$ is a pair of **Quillen adjunction** if F is a left Quillen functor and G is a right Quillen functor.

Remark A.11. Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction between two model categories. If either F is a left Quillen functor or G is a right Quillen functor, then $F \dashv G$ is a Quillen adjunction, see Proposition 1.39.

Remark A.12. According to Ken Brown's Lemma A.9, a left Quillen functor preserves weak equivalences between cofibrant objects and a right Quillen functor preserves weak equivalences between fibrant objects.

A.2 Some constructions

Let $\mathcal{M}$ be a model category, then for a general category $\mathcal{C}$, the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ may not admit a model structure, unless we impose further restrictions on $\mathcal{C}$ or $\mathcal{M}$.

Proposition A.13. *Let $\mathcal{M}$ be a model category and $\mathcal{N} := \mathrm{Fun}([1], \mathcal{M})$. For a morphism $\alpha : X \Rightarrow Y$, we write $\alpha_0 : X(0) \to Y(0)$ and $\alpha_1 : X(1) \to Y(1)$ for its components. Then there exists a model structure on $\mathcal{N}$ such that a morphism $\alpha : X \Rightarrow Y$ is*

- *a weak equivalence if each of its component is a weak equivalence in $\mathcal{M}$.*
- *a cofibration if both α_0 and the map $(\alpha_1, X(01)) : X(1) \coprod_{X(0)} Y(0) \to Y(1)$ are cofibrations in $\mathcal{M}$.*
- *a fibration if each of its component is a fibration in $\mathcal{M}$.*

Proof. It suffices to show that $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, W \cap \mathrm{Fib})$ are weak factorisation systems.

We start with the following observation about lifting problems in $\mathcal{N}$. Given maps $j : A \Rightarrow B$, and $p : X \Rightarrow Y$. Suppose we are given a lifting problem as follows:

$$\begin{tikzcd} A \arrow[r, Rightarrow, "u"] \arrow[d, Rightarrow, "j"'] & X \arrow[d, Rightarrow, "p"] \\ B \arrow[r, Rightarrow, "v"'] \arrow[ur, Rightarrow, dashed, "s"] & Y \end{tikzcd}$$

To solve this lifting problem, we need to find two morphisms $s_0 : B(0) \to X(0)$ and $s_1 : B(1) \to X(1)$ such that the following two diagrams are commutative(we must get s_0 first and then get s_1):

$$\begin{tikzcd} A(0) \arrow[r, "u_0"] \arrow[d, "j_0"'] & X(0) \arrow[d, "p_0"] \\ B(0) \arrow[r, "v_0"'] \arrow[ur, dashed, "s_0"] & Y(0) \end{tikzcd} \qquad \begin{tikzcd} A(1) \coprod_{A(0)} B(0) \arrow[r, "{(u_1, X(01) \circ s_0)}"] \arrow[d, "{(j_1, B(01))}"'] & X(1) \arrow[d, "p_1"] \\ B(1) \arrow[r, "v_1"'] \arrow[ur, dashed, "s_1"] & Y(1) \end{tikzcd}$$

Note that the commutativity of the right-hand-side diagram translates to the commutativity of the following two diagrams:

$$\begin{tikzcd} A(1) \arrow[r, "u_1"] \arrow[d, "j_1"'] & X(1) \arrow[d, "p_1"] \\ B(1) \arrow[r, "v_1"'] \arrow[ur, dashed, "s_1"] & Y(1) \end{tikzcd} \qquad \begin{tikzcd} B(0) \arrow[r, "s_0"] \arrow[d, "B(01)"'] & X(0) \arrow[d, "X(01)"] \\ B(1) \arrow[r, "s_1"] & X(1) \end{tikzcd}$$

After this observation, Cof $\boxslash W \cap$ Fib is easy. Now we show that $W \cap$ Cof $\boxslash$ Fib. To do so, it suffices to show that if $j : A \Rightarrow B$ is a trivial cofibration in $\mathcal{N}$, then the map $(j_1, B(01)) : A(1) \coprod_{A(0)} B(0) \to B(1)$ is a trivial cofibration in $\mathcal{M}$. It is a cofibration by assumption. To show that it is a weak equivalence, consider the composition:

$$\begin{tikzcd} A(1) \arrow[r, "q"] \arrow[dr, "j_1"'] & A(1) \cup_{A(0)} B(0) \arrow[d, "{(j_1, B(01))}"] \\ & B(1) \end{tikzcd} \qquad \begin{tikzcd} A(0) \arrow[r] \arrow[d, "j_0"'] & A(1) \arrow[d, "q"] \\ B(0) \arrow[r] & A(1) \cup_{A(0)} B(0) \end{tikzcd}$$

where j_1 is a weak equivalence by our assumption; q is a weak equivalence since it is a pushout of the trivial cofibration j_0. Applying the 2-out-of-3 property of W we see that $(j_1, B(01))$ is a weak equivalence.

Next, we wish to factorize a map $f : X \Rightarrow Y$ in $\mathcal{N}$ into $f = pj$ with $j : X \Rightarrow U$ is cofibration and $p : U \Rightarrow Y$ a trivial fibration. First we factorize the component f_0 into

$$f_0 : X(0) \xrightarrow{j_0} U(0) \xrightarrow{p_0} Y(0)$$

where j_0 is a cofibration and p_0 is a trivial fibration. Now consider the following diagram:

$$\begin{tikzcd} X(0) \arrow[r, "j_0"] \arrow[d, "X(01)"'] & U(0) \arrow[rr, "p_0"] \arrow[d, "\eta'"] & & Y(0) \arrow[d, "Y(01)"] \\ X(1) \arrow[r, "\eta"'] & X(1) \coprod_{X(0)} U(0) \arrow[r, "g"] \arrow[rr, bend right, "h"'] & U(1) \arrow[r, "p_1"] & Y(1) \end{tikzcd}$$

where the left square is a pushout an (p_1, g) is the factorisation of $h = (f_1, Y(01) \circ p_0) : X(1) \coprod_{X(0)} U(0) \to U(1)$ into a trivial fibration p_1 and a cofibration g. We set $j_1 := g \circ \eta$ and $U(01) := g \circ \eta'$. Factorisation into the composition of a trivial cofibration and a fibration is done in a similar method.

It remains to check that Fib, Fib $\cap\, W$, Cof $\cap\, W$ and Cof are closed under retracts, which we will not write in detail. $\square$

We write Λ_0^2 for the indexing category

$$1 \xleftarrow{01} 0 \xrightarrow{02} 2$$

and Λ_2^2 for the indexing category

$$0 \xrightarrow{02} 2 \xleftarrow{12} 1$$

This should not lead to any confusion.

Proposition A.14. *If $\mathcal{M}$ is a model category, then $\mathcal{K} := \mathrm{Fun}(\Lambda_0^2, \mathcal{M})$ can be equipped with a model category structure where a morphism $\alpha : X \Rightarrow Y$ is*

- *a weak equivalence if each component of α is a weak equivalence in $\mathcal{M}$.*
- *a cofibration if α_0, α_1 and the evident map $X(2)\coprod_{X(0)} Y(0) \to Y(2)$ are cofibrations in $\mathcal{M}$.*
- *a fibration if α_1, α_2 and the evident map $X(0) \to Y(0) \times_{Y(1)} X(1)$ are fibrations in $\mathcal{M}$.*

Proof. The proof will be similar to that of Proposition A.13. Let $j : A \Rightarrow B$ and $p : X \Rightarrow Y$ be morphisms in $\mathcal{K}$. To solving a lifting problem (u, v) in $\mathcal{K}$ as follows:

$$\begin{tikzcd}
A \arrow[r, Rightarrow, "u"] \arrow[d, Rightarrow, "j"'] & X \arrow[d, Rightarrow, "p"] \\
B \arrow[r, Rightarrow, "v"'] \arrow[ur, Rightarrow, dashed, "s"] & Y
\end{tikzcd}$$

is equivalent to finding solutions to the following three lifting problem in $\mathcal{M}$ one-by-one:

$$\begin{tikzcd}
A(1) \arrow[r, "u_1"] \arrow[d, "j_1"'] & X(1) \arrow[d, "p_1"] \\
B(1) \arrow[r, "v_1"'] \arrow[ur, dashed, "s_1"] & Y(1)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A(0) \arrow[r, "u_0"] \arrow[d] & X(0) \arrow[d, "{(p_0, X(01))}"] \\
B(0) \arrow[r, "{(v_0, s_1 \circ B(01))}"'] \arrow[ur, dashed, "s_0"] & Y(0) \times_{Y(1)} X(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
A(2)\coprod_{A(0)} B(0) \arrow[r, "{(u_2, X(01)\circ s_0)}"] \arrow[d, "{(j_2, B(02))}"'] & X(2) \arrow[d, "p_2"] \\
B(2) \arrow[r, "v_2"'] \arrow[ur, dashed, "s_2"] & Y(2)
\end{tikzcd}$$

Note that we must get s_1 first, s_0 second an s_2 last. Similar to the proof of Proposition A.13, apply the 2-out-of-3 property of W we get the desired lifting properties.

It remains to show the required factorisation property. Given $f : X \Rightarrow Y$, we wish to factor f as $f = p \circ j$ where p is a trivial fibration and j is a cofibration. First factorize the component f_1:

$$f_1 : X(1) \xrightarrow{j_1} U(1) \xrightarrow{p_1} Y(1)$$

So that j_1 is a cofibration and p_1 is a trivial fibration. We get a factorisation of s_0 by considering the following diagram

$$\begin{tikzcd}
X(1) \arrow[rr, "j_1"] & & U(1) \arrow[r, "p_1"] & Y(1) \\
X(0) \arrow[u] \arrow[r, "j_0"] & U(0) \arrow[r, "\phi"] \arrow[rr, bend right, "p_0"'] & Y(0) \times_{Y(1)} U(1) \arrow[u, "k'"] \arrow[r, "k"] & Y(0) \arrow[u]
\end{tikzcd}$$

where (j_0, ϕ) is a factorisation of $(f_0, j_1 \circ X(01)) : X(0) \to Y(0) \times_{Y(1)} U(1)$ into a cofibration j_0 and a trivial fibration ϕ. Finally we factorize f_2 as follows

$$\begin{tikzcd}
X(0) \arrow[r, "j_0"] \arrow[d, "X(02)"'] & U(0) \arrow[rr, "p_0"] \arrow[d, "\eta'"] & & Y(0) \arrow[d, "Y(02)"] \\
X(2) \arrow[r, "\eta"'] \arrow[rr, bend right, "j_2"'] & X(2)\coprod_{X(0)} U(0) \arrow[r, "g"] & U(2) \arrow[r, "p_2"] & Y(2)
\end{tikzcd}$$

It remains to check that Fib, $\mathrm{Fib} \cap W$, $\mathrm{Cof} \cap W$ and Cof are closed under retracts, which we will not write in detail. □

Use the model structure in Proposition A.14, we see that a functor X is cofibrant if and only if it satisfies two conditions

- $X(i)$ is cofibrant for each i.
- $X(02) : X(0) \to X(2)$ is a cofibration.

Proposition A.15. *The pair of adjoint functors* $\varinjlim : \mathcal{K} \leftrightarrows \mathcal{M} : \delta$ *is a Quillen pair, where* δ *sends an object* m *to the constant functor* δ_m.

Proof. δ obviously preserves fibrations and trivial fibrations. □

Corollary A.16. *Let* $X, Y \in \mathrm{Fun}(\Lambda^2_0, \mathcal{M})$ *be functors and* $\alpha : X \Rightarrow Y$ *be a natural weak equivalence. Suppose that all objects* $X(i)$ *and* $Y(i)$ *are cofibrant, and the maps* $X(02)$ *and* $Y(02)$ *are cofibrations, then the induced map* $\varinjlim X \to \varinjlim Y$ *is a weak equivalence.*

Proof. Apply Ken Brown's Lemma to the left Quillen functor $\varinjlim : \mathrm{Fun}(\Lambda^2_0, \mathcal{M}) \to \mathcal{M}$. Note that X, Y are cofibrant objects. □

Dually, we have the following proposition for pullbacks:

Corollary A.17. *Let* $X, Y \in \mathrm{Fun}(\Lambda^2_2, \mathcal{M})$ *be functors and* $\alpha : X \Rightarrow Y$ *be a natural weak equivalence. Suppose that all objects* $X(i)$ *and* $Y(i)$ *are fibrant, and the maps* $X(12)$ *and* $Y(12)$ *are fibrations, then the induced map* $\varprojlim X \to \varprojlim Y$ *is a weak equivalence.*

A.3 Examples

Most contents of this section is stolen from [Hau17].

Theorem A.18 (Joyal's model structure)**.** *There is a model structure on* sSet *where:*

- *The cofibrations are the monomorphisms;*
- *The weak equivalences and Joyal equivalences;*
- *The fibrant objects are* ∞*-categories;*
- *Fibrations between fibrant objects are isofibrations.*

We will refer to this model structure as the ***Joyal model structure*** *on* sSet.

The remaining part of this section is devoted to the proof of Theorem A.18.

Definition A.19. A morphism $f : X \to Y$ between simplicial sets is a **categorical fibration** if it has RLP w.r.t. all monomorphism that are Joyal equivalences.

Proposition A.20. *A morphism* $f : X \to Y$ *is both a categorical fibration and a Joyal equivalence if and only if it is a trivial fibration.*

Proof. Clearly a trivial fibration is a categorical fibration. Moreover, we have shown that a trivial fibration is a Joyal equivalence, see Proposition 3.7.

Now suppose that p is a categorical fibration and a Joyal equivalence. We can factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where j is a monomorphism and q is a trivial fibration. Then j is a Joyal equivalence by the 2-out-of-3 property, so there exists a lift f in the square

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

This gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"'] \arrow[d, "p"'] \arrow[rr, bend left, "\mathrm{Id}_X"] & Z \arrow[r, "f"'] \arrow[d, "q"] & X \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{Id}_Y"'] & Y \arrow[r, "\mathrm{Id}_Y"'] & Y \end{tikzcd}$$

which says that p is a retract of q. As a result, p is also a trivial fibration. $\square$

Let Jeq be the class of Joyal equivalences and CFib be the class of categorical fibrations.

Corollary A.21. $(\mathrm{Mon}, \mathrm{Jeq} \cap \mathrm{CFib})$ *is a weak factorisation system.*

It remains to show that $(\mathrm{Mon} \cap \mathrm{Jeq}, \mathrm{CFib})$ is a weak factorisation system. But due to the laziness of the author, I will omit the complete proof.

Theorem A.22. $(\mathrm{Mon} \cap \mathrm{Jeq}, \mathrm{CFib})$ *is a weak factorisation system.*

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