Understanding the Tate construction

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1 Homotopy fixed points and homotopy orbits

Let G be an E_1-group in $\mathcal{S}$, and let BG be its classifying space. Let $\mathcal{C}$ be an ∞-category and $X \in \mathcal{C}$ be an object. **An action of** G **on** $\mathcal{C}$ is a functor $F : BG \to \mathcal{C}$ sending the base point of BG to X. Once X is equipped with a G-action (offered by a functor $F : BG \to \mathcal{C}$), its homotopy fixed points and homotopy orbits are respectively defined by

$$X^{hG} := \varprojlim F; \quad X_{hG} := \varinjlim F.$$

Example 1.1. Let G be a finite group and $X \in \mathrm{Sp}$ be an arbitrary spectrum. Then the spectrum

$$\oplus_{g\in G} X$$

carries a natural G-action by permuting direct summands. Such actions are called **induced action**.

Theorem 1.2 (Homotopy fixed point/homotopy orbits spectral sequence)**.** *Let X be a G-spectrum. There are convergent spectral sequences*

$$E^2_{p,q} = H^{-p}(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X^{hG});$$

$$E^2_{p,q} = H_p(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X_{hG}).$$

Corollary 1.3. *Let G be a finite group and $\oplus_{g\in G}X$ be a spectrum with an induced G-action. Then we have*

$$(\oplus_{g\in G}X)_{hG} \cong X; \quad (\oplus_{g\in G}X)^{hG} \cong X.$$

Proof. Let us compute the E_2-page of the homotopy fixed point spectral sequence. We write $Y = \oplus_{g\in G}X$. The non-zero terms concentrate in the second quadrant, with $E^2_{p,q} = H^{-p}(G, \pi_q(Y))$, $q \geq 0$ and $p \leq 0$. Now $\pi_*(Y) = \oplus_{g\in G}\pi_*(X)$, where $\oplus_{g\in G}\pi_*(X)$ is equipped with the action induced from the trivial group $1 \subset G$. Applying Shapiro's lemma, we have

$$E^2_{p,q} = H^{-p}(G, \pi_q(Y)) \cong H^{-p}(1, \pi_q(X)) = \begin{cases} \pi_q(X) & p = 0, \\ 0 & p < 0. \end{cases}$$

The spectral sequence degenerates at E_2-page, and we have $\pi_*(Y^{hG}) = \pi_*(X)$. Similarly we have $\pi_*(Y_{hG}) = \pi_*(X)$.

□

2 The Tate construction

In this section, we present three different ways to understand the Tate construction:

1. The first way is to understand the Tate construction via norm map. We define the Tate construction to be the cofibre of the norm map.
2. The second way is to approach the Tate construction via the so-called Tate filtration. This is a complete filtration whose graded pieces are easy to describe.
3. The third way is to study the universal property of the Tate construction.

2.1 The norm map

Let G be an E_1-group in $\mathcal{S}$, and let $L \in \mathrm{LMod}_{\Sigma^\infty_+ G}(\mathrm{Sp})$ be a left $\Sigma^\infty_+ G$-module (or equivalently, L is a spectrum equipped with a G-action). Our goal is to construct a morphism

$$\mathsf{Nm}_L : (\Sigma^\infty_+ G)^{hG} \otimes_{\Sigma^\infty_+ G} L \to L^{hG}$$

where $\Sigma^\infty_+ G$ carries a $G \times G$-action induced by the following action of $G \times G$ over G:

$$(G \times G) \times G \to G, \quad ((g, h), k) \mapsto gkh^{-1},$$

and the homotopy quotient $(\Sigma^\infty_+ G)_{hG}$ is taken with respect to the $G \times 1$-action.

Construction 2.1 (Construction of the norm map)**.** Let L be a left $\Sigma^\infty_+ G$-module. The left $\Sigma^\infty_+ G$-module structure over L is encoded by an E_1-ring homomorphism

$$\alpha : \Sigma^\infty_+ G \to \mathsf{map}_{\mathrm{Sp}}(L, L).$$

Since L is a left $\Sigma^\infty_+ G$-module, the mapping spectrum $\mathsf{map}_{\mathrm{Sp}}(L, L)$ carries a natural $\Sigma^\infty_+ G$-$\Sigma^\infty_+ G$-bimodule structure: the right $\Sigma^\infty_+ G$-module structure is induced by the left action on the source L while the left $\Sigma^\infty_+ G$-module structure is induced by the the left action on the target L. By inspection one easily sees that α is also a $\Sigma^\infty_+ G$-$\Sigma^\infty_+ G$-bimodule map. Now taking homotopy fixed points with respect to the left action, we get a right $\Sigma^\infty_+ G$-equivaraint map

$$\alpha_{hG} : (\Sigma^\infty_+ G)^{hG} \to \mathsf{map}_{\mathrm{Sp}}(L, L)^{hG} \simeq \mathsf{map}_{\mathrm{Sp}}(L, L^{hG}),$$

where the last equivalence follows from the fact that $\mathsf{map}_{\mathrm{Sp}}(L, -)$ commutes with limits. By adjunction, we get the desired map

$$\mathsf{Nm}_L : (\Sigma^\infty_+ G)^{hG} \otimes_{\Sigma^\infty_+ G} L \to L^{hG}$$

which we will refer to as the **norm map**. For simplicity we will denote $(\Sigma^\infty_+ G)^{hG}$ by D_G and refer to it as the **dualizing spectrum** of G.

Theorem 2.2 (Klein)**.** *Let G be a Lie group. Then there is an G-equivariant equivalence:*

$$D_G = \Sigma^\infty S^{\mathfrak{g}}.$$

where $\mathfrak{g}$ is the Lie algebra of G, equipped with the adjoint action from G, and $S^{\mathfrak{g}}$ is the one-point compactification of $\mathfrak{g}$.

Theorem 2.3. *Let L be a left $\Sigma^\infty_+ G$-module. We defnine the* ***Tate construction*** *of L, which we denote by L^{tG}, to be the cofibre of the norm map Nm_L:*

$$L^{tG} := \mathsf{cof}(D_G \otimes_{\Sigma^\infty_+ G} L \xrightarrow{\mathsf{Nm}_L} L^{hG}).$$

Example 2.4. Let $G = S^1$, and L be a spectrum with a S^1-action. Then there is an associated norm map

$$\mathsf{Nm}_L : \Sigma\mathbb{S} \otimes_{\Sigma^\infty_+ S^1} L \to L^{hS^1}.$$

Since S^1 is an abelian Lie group, $\Sigma\mathbb{S}$ carries a trivial S^1-action here. Hence the left hand side can be simplified to be ΣL_{hS^1}. The Tate construction is defined via the cofibre sequence

$$\Sigma L_{hS^1} \xrightarrow{\mathsf{Nm}_L} L^{hS^1} \to L^{tS^1}.$$

Example 2.5. Let G be a finite group. Then $\Sigma^\infty_+ G = \oplus_{g\in G}\mathbb{S}$, and $(\oplus_{g\in G}\mathbb{S})^{hG} = \mathbb{S}$. In this case the Tate construction is defined via the cofibre sequence

$$L_{hG} \to L^{hG} \to L^{tG}.$$

Theorem 2.6 (Klein)**.** *Let G be a compact Lie group. Let L be a spectrum, and $\Sigma^\infty_+ G \otimes L$ be the spectrum equipped with induced $\Sigma^\infty_+ G$-action. Then the associated norm map*

$$D_G \otimes_{\Sigma^\infty_+ G} (\Sigma^\infty_+ G \otimes L) = D_G \otimes L \to (\Sigma^\infty_+ G \otimes L)^{hG}$$

is an equivalence. As a result, Tate constructions of induced G-spectrum vanish.

Remark 2.7. As a matter of fact, for an arbitrary G-spectrum L, the tensor product $\Sigma^\infty_+ G \otimes L$ equipped with the diagonal action will have vanishing Tate construction. This is because we can construct a G-equivariant morphism $f : \Sigma^\infty_+ G \otimes L^{triv} \to \Sigma^\infty_+ G \otimes L$ using the fact that $\Sigma^\infty_+ G$ is a "Hopf algebra" in Sp, where L^{triv} means the underlying spectrum of L. The morphism f turns out to be an equivalence, hence any G-spectrum of the form $\Sigma^\infty_+ G \otimes L$ can be identified with an induced G-spectrum $\Sigma^\infty_+ G \otimes L^{triv}$, and thus will have vanishing Tate construction.

2.2 The Tate filtration

Let $\mathbb{C}$ be the complex plane equipped with the natural S^1-action. Let $S(V) = \Sigma^\infty S^{\mathbb{C}}$ be the suspension spectrum of the one-point compactification of $\mathbb{C}$. $S(V)$ naturally carries an S^1-action inherited from $\mathbb{C}$. The inclusion of zero $\{0\} \hookrightarrow \mathbb{C}$ gives us a morphism $e : \mathbb{S} \to S(V)$. Note that e is null homotopic as a map of spectra since on space level it is given by inclusion of south and north poles $S^0 \to S^2$.

The underlying spectrum of $S(V)$ is $\Sigma^2\mathbb{S}$, hence $S(V)$ is an invertible spectrum, whose inverse will be denoted by $S(V)^{-1}$.

Construction 2.8 (Bhartt-Lurie)**.** Let X be an S^1-spectrum,we construct a filtration $\mathrm{Fil}^\bullet_T X$ whose n-stage is given by $\mathrm{Fil}^n_T X = (X \otimes S(V)^{-n})^{hS^1}$, illustrated as follows:

$$... \to (X \otimes S(V)^{-1})^{hS^1} \to X^{hS^1} \to (X \otimes S(V))^{hS^1} \to ... \tag{1}$$

which we will refer to as the **Tate filtration**. In this diagram the connecting morphisms are induced by $e : \mathbb{S} \to S(V)$. We are going to show that the colimit of this sequence coincide with the Tate construction of X, justfying its name. Moreover, we show that the n-th graded piece of the filtration $\mathrm{Fil}^\bullet_T X$ is given by $X[-2n]$.

Lemma 2.9. *There is fibre sequence $\Sigma^\infty_+ S^1 \to \mathbb{S} \xrightarrow{e} S(V)$.*

Proof. We start from the pushout square:

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S^2 = S^{\mathbb{C}}
\end{tikzcd}$$

and we observe that there is an induced pushout diagram

$$\begin{tikzcd} S^1 \coprod * \arrow[r] \arrow[d] & * \coprod * = S^0 \arrow[d] \\ * \arrow[r] & S^2 = S^{\mathbb{C}} \end{tikzcd}$$

Then applying the colimit-preserving functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$, we obtain the desired fibre-cofibre sequence $\Sigma^\infty_+ S^1 \to \mathbb{S} \to S(V)$. $\square$

Lemma 2.10. *The cofibre of the connecting map $e : (X \otimes S(V)^{n-1})^{hS^1} \to (X \otimes S(V)^n)^{hS^1}$ in diagram (1) is given by $X[2n] = \Sigma^{2n} X$.*

Proof. Since the functor $X \otimes S(V)^{n-1} \otimes -$ commutes with colimits, and the functor $(-)^{hG}$ commutes with limits, we have a fibre-cofibre sequence:

$$(X \otimes S(V)^{n-1} \otimes \Sigma^\infty_+ S^1)^{hS^1} \to (X \otimes S(V)^{n-1})^{hS^1} \to (X \otimes S(V)^n)^{hS^1}$$

As is explained in Remark 2.7, the norm map associated with $X \otimes S(V)^{n-1} \otimes \Sigma^\infty_+ S^1$ vanishes, hence we have $(X \otimes S(V)^{n-1} \otimes \Sigma^\infty_+ S^1)^{hS^1} \simeq \Sigma X \otimes S(V)^{n-1}$. So the cofibre of the connecting morphism e is provided by $\Sigma^2 X \otimes S(V)^{n-1}$, whose underlying spectrum is $\Sigma^{2n} X$. $\square$

Lemma 2.11 (Tate periodicity)**.** *Let X be a spectrum with an S^1-action. The morphism $X^{tS^1} \to (X \otimes S)^{tS^1}$ induced by $e : \mathbb{S} \to S(V)$ is an equivalence.*

Proof. Invoking the fibre sequence $\Sigma^\infty_+ S^1 \to \mathbb{S} \to S(V)$, and applying the exact functor $(X \otimes -)^{tS^1}$, we get a fibre sequence

$$(X \otimes \Sigma^\infty_+ S^1)^{tS^1} \to X^{tS^1} \to (X \otimes S(V))^{tS^1}.$$

Recall that $(X \otimes \Sigma^\infty_+ S^1)^{tS^1}$ vanishes, so the induced map $X^{tS^1} \to (X \otimes S(V))^{tS^1}$ is an equivalence. $\square$

Lemma 2.12. *Let X be an S^1-spectrum. Then both the colimit and the limit of the following sequence*

$$\ldots \to X \otimes S(V)^{-1} \to X \to X \otimes S(V) \to \ldots$$

vanish.

Proof. Since $S(V)$ is invertible, the functor $S(V) \otimes - : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence, and hence commutes with all limits and colimits. The morphism $e : \mathbb{S} \to S(V)$ provide us with a morphism

$$\mathbb{S} \otimes \varinjlim_{n \in \mathbb{Z}} (X \otimes S(V)^n) \xrightarrow{e \otimes \mathrm{Id}} S(V) \otimes \varinjlim_{n \in \mathbb{Z}} (X \otimes S(V)^n) \simeq \varinjlim_{n \in \mathbb{Z}} (X \otimes S(V)^{n+1})$$

By inspection one sees that $e \otimes \mathrm{Id}$ is homotopic to the identity of the colimit. Since $e \otimes \mathrm{Id}$ is null-homotopic, we conclude that $\varinjlim_{n \in \mathbb{Z}} (X \otimes S(V)^n) \simeq 0$. In a similar manner one proves that $\varprojlim_{n \in \mathbb{Z}} (X \otimes S(V)^n) \simeq 0$. $\square$

Lemma 2.13. *The limit of the Tate filtration (1) is the trivial spectrum 0. Filtrations satisfying this property are called **complete** ones.*

Proof. Since limits commute with each other, we have

$$\varprojlim_{n\in\mathbb{Z}} (X \otimes S(V)^n)^{hS^1} \simeq (\varprojlim_{n\in\mathbb{Z}} X \otimes S(V)^n)^{hS^1} = 0.$$

□

Proposition 2.14. *The colimit of the Tate filtration (1) is the Tate construction.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d, "\simeq"] \\
\Sigma X_{hS^1} \arrow[r] \arrow[d] & X^{hS^1} \arrow[r] \arrow[d] & X^{tS^1} \arrow[d, "\simeq"] \\
\Sigma(X \otimes S(V))_{hS^1} \arrow[r] \arrow[d] & (X \otimes S(V))^{hS^1} \arrow[r] \arrow[d] & (X \otimes S(V))^{tS^1} \arrow[d] \\
\cdots & \cdots & \cdots
\end{tikzcd}$$

where each horizontal sequence is a cofibre sequence, by the definition of Tate construction. As a result, the induced sequence of filtered colimits is again a cofibre sequence. It suffices to show that the colimit of the left vertical sequence vanishes, which follows directly from Lemma 2.12. □

2.3 The universal property

Via homotopy hypothesis, we can view each space as an ∞-groupoid. Thus to each space X we may associate the functor category $\mathrm{Fun}(X, \mathrm{Sp})$ whose objects are referred to as **local systems over X (with values in Sp)**. We may alternatively write $\mathrm{Fun}(X, \mathrm{Sp})$ as $\mathsf{Loc}(X)$. Recall that the properties of stability and presentability are closed under the formation of functor categories, so $\mathrm{Fun}(X, \mathrm{Sp})$ is again a presentable stable ∞-category. Every functor $f : X \to Y$ induces a chain of adjunctions $f_! \dashv f^* \dashv f_*$:

$$\begin{tikzcd}
\mathrm{Fun}(X, \mathrm{Sp}) \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathrm{Fun}(Y, \mathrm{Sp}) \arrow[l, "f^*"]
\end{tikzcd}$$

The assignments $X \mapsto \mathrm{Fun}(X, \mathrm{Sp})$, $f \mapsto f_!$ extends to a functor $\mathsf{Loc} : \mathcal{S} \to \mathrm{Pr}^L_{\mathrm{st}}$.

Example 2.15. Let $X = BG$ be the classifying space of an E_1-group. Then a local system over X is equivalent to a spectrum equipped with a G-action. Actually we have a canonical equivalences

$$\mathrm{Fun}(BG, \mathrm{Sp}) \simeq \mathrm{RMod}_{\Sigma^\infty_+ G}(\mathrm{Sp}).$$

Let $g : * \to BG$ be the inclusion the base point. Then $g^* : \mathrm{Fun}(BG, \mathrm{Sp}) \to \mathrm{Sp}$ sends a G-spectrum to its underlying spectrum. $g_! : \mathrm{Sp} \to \mathrm{Fun}(BG, \mathrm{Sp})$ will send a spectrum E to the induced G-spectrum $\Sigma^\infty_+ G \otimes E$, while g_* sends E to the coinduced G-spectrum $\mathsf{map}_{\mathrm{Sp}}(\Sigma^\infty_+ G, E)$.

Lemma 2.16. *The presentable ∞-category $\mathrm{Fun}(X, \mathrm{Sp})$ is compactly generated. We view each point $x \in X$ as an inclusion $x : * \to X$, then the functor $x_! : \mathrm{Sp} \to \mathrm{Fun}(X, \mathrm{Sp})$ preserves compact objects. Moreover, the compact local systems $x_!\mathbb{S}$ generate $\mathrm{Fun}(X, \mathrm{Sp})$ when x varies.*

Proof. The evaluation at x functor $x^* : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ preserves filtered colimits, since colimits in a functor category is computed pointwisely. As a result, the left adjoint $x_!$ preserves compact objects. Since $\mathrm{Fun}(X, \mathrm{Sp})$ admits all colimits, the inclusion $\mathrm{Fun}(X, \mathrm{Sp})^\omega \hookrightarrow \mathrm{Fun}(X, \mathrm{Sp})$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathrm{Fun}(X, \mathrm{Sp})^\omega) \to \mathrm{Fun}(X, \mathrm{Sp})$, and F exists a right adjoint G, given by restricted Yoneda embedding. To show that F is an equivalence of ∞-categories, it suffices to show that G is conservative. Given $\alpha : L \to L'$ a morphism of local systems and assume that $G(\alpha)$ is an equivalence. For every compact spectrum E, we have an induced equivalence of mapping spaces:

$$G(\alpha)_* : \mathrm{Map}(x_!E, G(L)) \xrightarrow{\sim} \mathrm{Map}(x_!E, G(L'))$$

Considering the adjunction $F \dashv G$ with F being fully faithful, we can remove G to get the following homotopy equivalence

$$\alpha_* : \mathrm{Map}(x_!E, L) \xrightarrow{\sim} \mathrm{Map}(x_!E, L').$$

This is equivalent to saying that $\alpha_x : L_x \simeq L'_x$ is an equivalence. Hence α is itself an equivalence. One further note that in the above proof we do not need to require that test spectrum E to run over all compact spectra, but only need to require them to run over spectra of the form $\mathbb{S}[i]$. Hence $x_!\mathbb{S}$ are generators. □

Let X be a space and consider the projection $p : X \to *$. Then $p^* : \mathrm{Sp} \to \mathrm{Fun}(X, \mathrm{Sp})$ is the constant local system functors, while $p_!$ (resp. p_*) is the homology (resp. cohomology) functor.

Proposition 2.17. *There is a pair (p_*^T, i), where $p_*^T : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ is a functor vanishing on all compact objects, and i is a natural transformation $p_* \to p_*^T$ such that the pair is initial among all such pairs. The fibre of the natrual transformation $i : p \to p_*^T$ commutes with colimits.*

Proof. In fact there is routine way to do this. We define p_*^T via the formula

$$p_*^T(X) := \varinjlim_{Y \in \mathrm{Fun}(X,\mathrm{Sp})^\omega_{/X}} p_*\mathsf{cof}(Y \to X).$$

Then there is a manifest way to construct a natural transformation by noting that

$$p_*(X) \simeq \varinjlim_{Y \in \mathrm{Fun}(X,\mathrm{Sp})^\omega_{/X}} p_*(X)$$

is the colimit of a constant diagram indexed by the filtered category $\mathrm{Fun}(X, \mathrm{Sp})^\omega_{/X}$. It is direct to check that the pair (p_*^T, i) enjoys the desired universal property. To compute its fibre, note that

$$\mathsf{fib}(p_*(X) \to p_*^T(X)) \simeq \varinjlim_{Y \in \mathrm{Fun}(X,\mathrm{Sp})^\omega_{/X}} p_*Y$$

Assume that $X = \varinjlim_{i \in I} X_i$, then we have

$$\varinjlim_{Y \in \mathrm{Fun}(X,\mathrm{Sp})^\omega_{/X}} p_*Y \simeq \varinjlim_{i \in I} \varinjlim_{Y \in \mathrm{Fun}(X,\mathrm{Sp})^\omega_{/X_i}} p_*(Y) \simeq \varinjlim_{i \in I} \mathsf{fib}(p_*(X_i) \to p_*^T(X_i)).$$

□

Proposition 2.18. *The fibre of the natural transformation $p_* \to p_*^T$ is given by $Y \mapsto p_!(D_X \otimes Y)$ for a unique object $D_X \in \mathrm{Fun}(X, \mathrm{Sp})$. D_X is given by:*

$$D_X : x \mapsto \varprojlim_{y \in X} \Sigma^\infty_+ \mathrm{Map}_X(x, y).$$

Moreover, it is the initial one among those colimit preserving functors over p_*, i.e. it is the assemply map associated to p_*.

Proof. See [NS18]. □

Proposition 2.19. *If p_*^T vanishes on objects of the form $x_!E$ where $x \in X$ is a point and E is a spectra. Then there is a unique lax symmetric monoidal structure on p_*^T which makes the natural transformation $p_* \to p_*^T$ lax symmetric monoidal.*

Proof. See [NS18]. □

We are particularly interested in the case where $X = BG$ is the classying space of an E_1-group. Let us compute the local system D_X associated with X appearing in Proposition 2.18. By the formula, we have

$$D_{BG} : x \mapsto \varprojlim_{y \in X} \Sigma_+^\infty \mathrm{Map}_X(x, y)$$

Since $X = BG$ is a connected ∞-groupoid, the mapping space $\mathrm{Map}_X(x, y) \simeq \mathrm{Map}_{BG}(*, *) \simeq \Omega BG \simeq G$, with $* \in BG$ being the base point. Then D_{BG} sends the base point $*$ to the homotopy invariant of the suspension spectrum $(\Sigma_+^\infty G)^{hG}$. Taking a closer look, one sees that there is an equivalence $D_{BG} \simeq D_G$ as G-spectra, where D_G is the dualizing spectrum of G. As a result, the fibre-cofibre sequence $p_!(D_{BG} \otimes -) \to p_* \to p_*^T$ studied above coincides with the fibre-cofibre sequence defining the Tate construction $(D_G \otimes -)_{hG} = D_G \otimes_{\Sigma_+^\infty G} - \to (-)^{hG} \to (-)^{tG}$. As a result:

1. The norm map $\mathsf{Nm} : (D_G \otimes -)_{hG} \to (-)^{hG}$ is the assembly map associated with $(-)^{hG}$.
2. The Tate construction $(-)^{hG} \to (-)^{tG}$ is the initial functor under $(-)^{hG}$ which vanishes on compact objects in $\mathrm{Fun}(BG, \mathrm{Sp})$. (Actually it is the unique one).
3. Since the Tate construction vanishes on induced G spectra, the functor $(-)^{tG}$ admits a unique lax symmetric monoidal structure making the natural transformation $(-)^{hG} \to (-)^{tG}$ lax symmetric monoidal.

References

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *arXiv preprint*, 2018.