A brief introduction to (co)ends

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Contents

1 Review of category theory

1.1 Basics

We assume the readers are familiar with the basic notions of category theory. We do not take care of size problems in this note. We use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$... to denote categories, and F, G, H... to denote functors. We use $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to denote functor categories and $\mathrm{Nat}(F, G)$ to denote the set of natural transformations between F and G. We use $\varprojlim F$ to denote the limit of F and $\varinjlim F$ to denote the colimit of F. We use $F \dashv G$ to denote a pair of adjunction, with F being the left adjoint of G.

Now we collect some most important examples and theorems in category theory (without proof).

Example 1.1. We introduce the following categories:

- The category of sets and functions, which is denoted by Set;
- the category of topological spaces and continuous functions, which is denoted by Top;
- the category of abelian groups and group homomorphisms, which is denoted by Ab;
- The category vector spaces over some field $\mathbb{k}$ and $\mathbb{k}$-linear maps, which is denoted $\mathrm{Vec}_{\mathbb{k}}$.

Example 1.2. We now define a skeletal category Δ which plays an important role in our seminar: the category of finite ordinals:

- Its objects are finite totally ordered set of the form
$$[n] = \{0 \to 1 \to 2 \to ... \to n\}$$
with $n \geq 0$.
- The morphisms $f : [m] \to [n]$ are (non-strictly) monotone functions from $[m]$ to $[n]$.

We specify are two families of special morphisms:

- For each n, define $d^i : [n-1] \to [n]$ to be the unique injective monotone function whose image does not contain i, for $0 \leq i \leq n$.
- For each n, define $s^i : [n+1] \to [n]$ to be the unique surjective monotone function which sends both i and $i+1$ to i.

It is a easy combinatoric result that s^i and d^i generate all morphisms in Δ. The functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ is called **the category of simplicial sets**, which we denote by sSet.

Note that any partially ordered set can be viewed as a category. In particular, we may either regard the symbol $[n]$ to denote a totally ordered set or a category, depending on the context.

Now we introduce the notion of a comma category:

Definition 1.3. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The **comma category associated with** (F, G) is a category $(F, \downarrow, G)$ such that:

- The objects are pairs (c, d, f) with $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : F(c) \to G(d)$ a morphism;
- A morphism from (c_1, d_1, f_1) to (c_2, d_2, f_2) is a pair of morphisms $g : c_1 \to c_2$ and $h : d_1 \to d_2$ such that $G(h) \circ f_1 = f_2 \circ F(g)$.

Note that there are two canonical projections $\Pi_{\mathcal{C}} : (F, \downarrow, G) \to \mathcal{C}$ and $\Pi_{\mathcal{D}} : (F, \downarrow, G) \to \mathcal{D}$ respectively. In practice, we usually come across the case that one of F or G is a constant functor Δ_e which sends all objects to $e \in \mathcal{E}$ and all morphisms to Id_e. When $F = \Delta_e$, we denote the associated comma category by $(e, \downarrow, G)$; when $G = \Delta_e$, we denote the associated comma category by $(F, \downarrow, e)$.

Theorem 1.4. *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Define the map $F^{\clubsuit}$ by assigning $F^{\clubsuit}(c) := \Pi_{\mathcal{D}}(\chi(c))$ where $\chi(c)$ is the initial object (we assume its existence) in the comma category $(c, \downarrow, F)$. Then $F^{\clubsuit}$ extends to a functor $\mathcal{C} \to \mathcal{D}$ and $F^{\clubsuit} \dashv F$. This theorem has a dual version.*

A direct corollary is:

Corollary 1.5. *Let $I, \mathcal{C}$ be categories. Then $\Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ be the diagonal functor sending c to the constant functor Δ_c. Then there are adjunctions*

$$\varinjlim \dashv \Delta \dashv \varprojlim .$$

Theorem 1.6. *Any limit can be canonically written as a equalizer of a product; dually, any colimit can be canonically written as a coequalizer of a coproduct. As a result, a category $\mathcal{C}$ is complete if and only if it admits products and equalizers; dually. a category $\mathcal{C}$ is cocomplete if and only if it admits coproducts and coequalizers.*

Theorem 1.7. *Left adjoint functors preserve colimits and right adjoint functors preserve limits.*

1.2 Grothendieck construction and density theorem

Let $\mathcal{C}$ be a small category. The **category of presheaves** over $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, which we will denote by $\widehat{\mathcal{C}}$. An object $X \in \widehat{\mathcal{C}}$ is called a presheaf over $\mathcal{C}$. Given an object $c \in \mathcal{C}$, an element in the set $X_c = X(c)$ is called a **section** of X over c. By the Yoneda Lemma there is a fully faithful embedding $\mathcal{Y} : \mathcal{C} \to \widehat{\mathcal{C}}$, $c \mapsto \hom_{\mathcal{C}}(-, c)$. For simplicity we denote the functor $\hom_{\mathcal{C}}(-, c)$ by h_c.

Lemma 1.8. *The category $\widehat{\mathcal{C}}$ is complete and cocomplete.*

Proof. Recall that the category Set is complete and cocomplete. Then the limit of a diagram $J \to \widehat{\mathcal{C}}$ can be defined pointwisely. □

Theorem 1.9. *The Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ perserves limits.*

Proof. Let I be an index category and $\{L \to c_i\}_{i \in I}$ be a universal cone. Since the functor $\hom_{\mathcal{C}}(c, -)$ preserves limits for all $c \in \mathcal{C}$, the new cone $\hom_{\mathcal{C}}(x, L) \to \hom_{\mathcal{C}}(c, c_i)$ is also a universal cone, which means that the diagram $\hom_{\mathcal{C}}(-, L) \Rightarrow \hom_{\mathcal{C}}(-, c_i)$ is universal as well. □

Definition 1.10. Let $X \in \widehat{\mathcal{C}}$. The **category of elements**, or **Grothendieck construction** of X is the category whose objects are pairs (a, s), where $a \in \mathcal{C}$ is an object and s is a section of X over a. A morphism between (a, s) and (b, t) is a morphism $u : a \to b$ in $\mathcal{C}$ such that $X(u)(t) = s$. The category of elements of X is denoted by $\int_{\mathcal{C}} X$.

Remark 1.11. There is a canonical projection functor $\pi_X : \int_{\mathcal{C}} X \to \mathcal{C}$ by sending (a, s) to a.

Remark 1.12. By the Yoneda Lemma a section $s \in X_a$ corresponds to a natural transformation $s \in \mathrm{Nat}(h_a, X)$. Then the requirement that $X(u)(t) = s$ translates to the following diagram:

$$\begin{tikzcd}
h_a \arrow[rr, "u"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\
& X &
\end{tikzcd}$$

So the category of elements $\int_{\mathcal{C}} X$ is isomorphic to the comma category $(\mathcal{Y}, \downarrow, X)$ where $\mathcal{Y}$ is the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$.

Remark 1.13. The category of elements $\int_{\mathcal{C}} X$ is also isomorphic to the comma category $(\{*\}, \downarrow, X)$ where $\{*\}$ denotes the singleton.

The following theorem says that representable presheaves are dense in the category of presheaves.

Theorem 1.14 (Density theorem)**.** *Consider the faithful functor $\varphi_X := \mathcal{Y} \circ \pi_X : \int_{\mathcal{C}} X \to \widehat{\mathcal{C}}$ such that $\varphi_X(a, s) = h_a$, $\varphi_X(u) = u$. There is an obvious cocone from φ_X to X defined by the following collections of maps:*

$$s : h_a \to X, \quad (a, s) \in \int_{\mathcal{C}} X$$

Then these maps exhibits X as a colimit of φ_X.

Proof. Let $\varphi_X \Rightarrow Y$ be another cocone under φ_X. Consider the following diagram

$$\begin{tikzcd}
h_a \arrow[rr, "u"] \arrow[dr, "s"] \arrow[ddr, "p"'] & & h_b \arrow[dl, "t"'] \arrow[ddl, "q"] \\
& X & \\
& Y &
\end{tikzcd}$$

The small triangle is equivalent to the data of two sections $s \in X_a$, $t \in X_b$ such that $X(u)(t) = s$. The large triangle is equivalent to the data of $p \in Y_a$, $q \in Y_b$ such that $Y(u)(q) = p$. Then there is exists a unique natural transformation $\alpha : X \Rightarrow Y$ whose components α_a is defined by $\alpha_a(t) = p$.

$$\begin{tikzcd}
h_a \arrow[rr, "u"] \arrow[dr, "s"] \arrow[ddr, "p"'] & & h_b \arrow[dl, "t"'] \arrow[ddl, "q"] \\
& X \arrow[d, dashed, "\alpha"] & \\
& Y &
\end{tikzcd}$$

□

1.3 Free cocompletion

The following theorem exhibits $\widehat{\mathcal{C}}$ as the free cocompletion of $\mathcal{C}$.

Theorem 1.15 (Free cocompletion)**.** *Let $\mathcal{D}$ be a cocomplete category and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then there is a unique (up to isomorphism) cocontinuous functor $\tilde{F} : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $\tilde{F} \circ \mathcal{Y} \cong F$. Moreover $\tilde{F}$ is left adjoint to the canonical functor $H : \mathcal{D} \to \widehat{\mathcal{C}}$ defined by $H(d) := \hom_{\mathcal{D}}(F(-), d)$.*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{D} \arrow[dl, bend right, "H"'] \\
& \widehat{\mathcal{C}} \arrow[ur, bend right, "\tilde{F}"'] &
\end{tikzcd}$$

Proof. It suffices to show that any such cocontinuous functor gives an adjunction $\tilde{F} \dashv H$. Take $X \in \widehat{\mathcal{C}}$, note that $X \cong \varinjlim_{(a,s) \in \int_{\mathcal{C}} X} h_a$. Since $\tilde{F}$ is cocontinuous, we have

$$\tilde{F}(X) \cong \varinjlim_{(a,s) \in \int_{\mathcal{C}} X} \tilde{F}(h_a) \cong \varinjlim_{(a,s) \in \int_{\mathcal{C}} X} F(a)$$

As a result, we have

$$\begin{aligned}
\hom_{\widehat{\mathcal{C}}}(X, H(d)) &\cong \varinjlim_{(a,s)} \hom_{\widehat{\mathcal{C}}}(h_a, \hom_{\mathcal{D}}(F(-), d)) \\
&\cong \varprojlim_{(a,s)} \hom_{\mathcal{D}}(F(a), d) \\
&\cong \hom_{\mathcal{D}}(\varinjlim_{(a,s)} F(a), d) \\
&\cong \hom_{\mathcal{D}}(\tilde{F}(X), d)
\end{aligned}$$

□

Corollary 1.16. *Let $\mathcal{C}, \mathcal{D}$ be cocomplete categories. If $\mathcal{C}$ is the presheaf category of some small category $\mathcal{A}$, then any cocontinuous functor $\mathcal{C} \to \mathcal{D}$ admits a right adjoint.*

Example 1.17. Let Δ denote the category of finite ordinals whose presheaf category is the category of simplicial sets. There is a functor $|-| : \Delta \to \mathrm{Top}$ sending $[n]$ to the standard topological n-simplice $|\Delta^n|$. Freely extending this functor to sSet by Theorem 1.15 , we get the famous **geometric realization functor**, which we still denote by $|-| : \mathrm{sSet} \to \mathrm{Top}$. Its right adjoint is called **singular functor**, which we denote by $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$.

$$\begin{tikzcd}
\Delta \arrow[rr] \arrow[dr, "\mathcal{Y}"'] & & \mathrm{Top} \arrow[dl, bend right, "\mathrm{Sing}"'] \\
& \mathrm{sSet} \arrow[ur, bend right, "|-|"'] &
\end{tikzcd}$$

Example 1.18. Let Cat denote the category of all categories[1]. Note that this category is cocomplete. Since any partially ordered set can be viewed as a category, there is a functor $i : \Delta \to \mathrm{Cat}$ which sends the totally ordered set $[n]$ to the category $[n]$. Freely extending this functor to sSet by Theorem 1.15 , we get the **fundamental category functor/ categorical realization functor** , which we denote by $h : \mathrm{sSet} \to \mathrm{Top}$. Its right adjoint is the famous **nerve functor** $N : \mathrm{Cat} \to \mathrm{sSet}$.

$$\begin{tikzcd}
\Delta \arrow[rr, "i"] \arrow[dr, "\mathcal{Y}"'] & & \mathrm{Cat} \arrow[dl, bend right, "N"'] \\
& \mathrm{sSet} \arrow[ur, bend right, "h"'] &
\end{tikzcd}$$

[1]Actually the collection of all categories has a structure of a 2-category. Here we view it as a 1-category by discarding all the natural transformations. As a result, it does not make sense to talk about isomorphism of functors.

2 Kan extensions and (co)ends

2.1 Ends and coends

Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$. A **dinatural transformation** $\alpha : F \overset{\bullet\bullet}{\to} G$ consists a family of morphisms $\{\alpha_c : F(c,c) \to G(c,c)\}_{c \in \mathcal{C}}$ such that the following diagram is commutative $\forall c, c' \in \mathcal{C}$ and $f \in \hom(c, c')$:

$$\begin{tikzcd} & F(c,c) \arrow[r, "\alpha_c"] & G(c,c) \arrow[dr, "G(\mathrm{Id}_c, f)"] & \\ F(c',c) \arrow[ur, "F(f,\mathrm{Id}_c)"] \arrow[dr, "F(\mathrm{Id}_{c'}, f)"'] & & & G(c,c') \\ & F(c',c') \arrow[r, "\alpha_{c'}"'] & G(c',c') \arrow[ur, "G(f,\mathrm{Id}_{c'})"'] & \end{tikzcd}$$

Let $\mathrm{Fun}^{\mathrm{din}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$ denote the category whose objects are functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathcal{D}$ and whose morphisms are dinatural transformations. Given $d \in \mathcal{D}$, let $\Delta_d \in \mathrm{Fun}^{\mathrm{din}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$ be the constant functor sending all objects to d and all morphisms to Id_d. Let $\Delta : \mathcal{D} \to \mathrm{Fun}^{\mathrm{din}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$ be the diagonal functor sending $d \in \mathcal{D}$ to the functor Δ_d.

Definition 2.1. Let $F \in \mathrm{Fun}^{\mathrm{din}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$ be a functor. The **end** of a functor F is defined to be the final object in the comma category $(\Delta, \downarrow, F)$, which we denote by $\int_{c \in \mathcal{C}} F(c,c)$. Dually, the **coend** of a functor F is defined to be the initial object in the comma category $(F, \downarrow, \Delta)$, which is denoted by $\int^{c \in \mathcal{C}} F(c,c)$.

To be more explicit, for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, the end of F is a pair $(\int_{c \in \mathcal{C}} F(c,c), \pi)$ where $\int_{c \in \mathcal{C}} F(c,c) \in D$, $\pi : \Delta_{\int_{c \in \mathcal{C}} F(c,c)} \overset{\bullet\bullet}{\to} F$ is a dinatural transformation such that they are universal in the following sense: for each pair $(A, \alpha) \in (\Delta, \downarrow, F)$, there is a unique morphism $\tilde{\alpha} : A \to \int_{c \in \mathcal{C}} F(c,c)$ rendering the whole diagram.

$$\begin{tikzcd} A \arrow[dr, dashed, "\exists ! \tilde{\alpha}"] \arrow[drr, bend left, "\alpha_x"] \arrow[ddr, bend right, "\alpha_y"'] & & \\ & \int_{c \in \mathcal{C}} F(c,c) \arrow[r, "\pi_x"] \arrow[d, "\pi_y"] & F(x,x) \arrow[d, "F(\mathrm{Id}_x, f)"] \\ & F(y,y) \arrow[r, "F(f, \mathrm{Id}_y)"'] & F(x,y) \end{tikzcd}$$

Remark 2.2. The notion of (co)ends is a generalization of the notion of (co)limits. As a matter of fact, for any morphism $F : \mathcal{C} \to \mathcal{D}$, we can extend F trivially to a functor $F^\sharp : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ by sending (x, y) to $F(y)$ and (f, g) to $F(g)$. Then it is direct to see that $\varprojlim F = \int_{c \in \mathcal{C}} F^\sharp(c,c)$.

It is well-known that any limit can be written as an equalizer of a product in a canonical way. So we have:

Lemma 2.3. *The end of F can be written as the following equalizer:*

$$\int_{c\in\mathcal{C}} F(c,c) \longrightarrow \prod_{c\in\mathcal{C}} F(c,c) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{f:c\to c'} F(c,c')$$

Componentwisely, p and q are defined by

$$p_f := F(\mathrm{Id}_c, f)\circ\pi_c, ; q_f := F(f,\mathrm{Id}_{c'})\circ\pi_{c'}$$

for $f : c \to c'$.

Proof. It is just a simple translation of the universal property of $\int_{c\in\mathcal{C}} F(c,c)$. □

Lemma 2.4 (Fubini theorem)**.** *If $F : \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\times\mathcal{E}^{\mathrm{op}}\times\mathcal{E}\to\mathcal{D}$ is a functor, then we have*

$$\int_{(c,e)\in\mathcal{C}\times\mathcal{E}} F(c,c,e,e) \cong \int_{e\in\mathcal{E}}\int_{c\in\mathcal{C}} F(c,c,e,e) \cong \int_{c\in\mathcal{C}}\int_{e\in\mathcal{E}} F(c,c,e,e)$$

As a result, it makes sense to talk about the end

$$\int_{c\in\mathcal{C},e\in\mathcal{E}} F(c,c,e,e).$$

Proof. Limits commute with each other. □

Lemma 2.5. *Let $\mathcal{C},\mathcal{D}$ be categories and $F,G\in\mathrm{Fun}(\mathcal{C},\mathcal{D})$. Then we have*

$$\mathrm{Nat}(F,G) = \int_{c\in\mathcal{C}} \hom_{\mathcal{D}}(F(c),G(c)).$$

Proof. The legs $\pi_c : \mathrm{Nat}(F,G)\to\hom_{\mathcal{D}}(F(c),G(c))$ is given by taking the components at c: $\alpha\mapsto\alpha_c$. The requirement that the family of morphisms $\{\alpha_c\}$ defines a dinatural transformation is simply a reformulation of the naturality condition. Suppose there is another pair: (d,γ) with d a set and $\gamma : \Delta_d \overset{\bullet\bullet}{\to} \hom_{\mathcal{D}}(F(-),G(-))$. Take $\beta\in d$ we obtain a family of morphisms $\beta_c := \gamma_c(\beta)$. These $\{\beta_c\}_{c\in\mathcal{C}}$ define a natural transformation $\overline{\beta}\in\mathrm{Nat}(F,G)$ and so we get a unique morphism $(d,\gamma)\to(\mathrm{Nat}(F,G),\pi)$ in the comma category $(\Delta,\downarrow,\hom_{\mathcal{D}}(F(-),G(-)))$. □

Theorem 2.6 (Yoneda and co-Yoneda Lemma)**.** *For every functor $K : \mathcal{C}\to$ Sets, we have the following isomorphisms:*

$$K(-)\cong\int^{c\in\mathcal{C}} Kc\times\hom_{\mathcal{C}}(c,-); \quad K(-)\cong\int_{c\in\mathcal{C}}\hom_{\mathrm{Set}}(\hom_{\mathcal{C}}(-,c),Kc).$$

If we write down the colimit in detail, we will see that Theorem 1.14 is included as a special case of the above theorem.

Corollary 2.7. *For any simplicial set* $X \in \mathrm{sSet}$*, we have*

$$X \cong \int^{[n]\in\Delta} X_n \times \Delta^n$$

Lemma 2.8. *Let* $\Bbbk$ *be a field and* $\mathrm{Vec}_{\Bbbk}$ *be the category of finite dimensional vector spaces over* $\Bbbk$*. Consider the functor* $H_{W,V}(-,-) = \hom_{\Bbbk}(-,V) \otimes_{\Bbbk} \hom_{\Bbbk}(W,-) : \mathrm{Vec}_{\Bbbk}^{\mathrm{op}} \times \mathrm{Vec}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$*. Then we have*

$$\int^{U\in\mathrm{Vec}_{\Bbbk}} H_{W,V}(U,U) \cong \hom_{\Bbbk}(W,V).$$

Remark 2.9. As a matter of fact, the above lemma can be generalized to any finite tensor category.

Corollary 2.10. *Take* $W = V = \Bbbk$ *we have*

$$\int^{U\in\mathrm{Vec}_{\Bbbk}} U^* \otimes U \cong \Bbbk.$$

2.2 Kan extensions and its computations

Let $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ be functors. Then there is a functor $K^* : \mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$ defined by $G \mapsto G \circ K$, i.e. pulling G puck along K.

Definition 2.11. Given functors $F : \mathcal{C} \to \mathcal{E}, K : \mathcal{C} \to \mathcal{D}$, the **left Kan extension** of F along K is the initial object in the comma category $(F, \downarrow, K^*)$, which we denote by $\mathrm{Lan}_K F$. The **right Kan extension** is the final object in the comma category $(K^*, \downarrow, F)$, which we denote by $\mathrm{Ran}_K F$.

The following lemma is a simple corollary of Theorem 1.4.

Lemma 2.12. *If we view* $\mathrm{Lan}_K(-)$ *and* $\mathrm{Ran}_K(-)$ *as functors from* $\mathrm{Fun}(\mathcal{C},\mathcal{E})$ *to* $\mathrm{Fun}(\mathcal{D},\mathcal{E})$*, then there are adjunctions*

$$\mathrm{Lan}_K(-) \dashv K^* \dashv \mathrm{Ran}_K(-)$$

Remark 2.13. Almost all concepts in basic category theory can be reformulated using the language of Kan extensions. For example, if we take $\mathcal{D}$ to be the final category $[0]$ and K be the unique functor from $\mathcal{C}$ to $[0]$, then $\mathrm{Lan}_K F = \varinjlim F$, $\mathrm{Ran}_K F = \varprojlim F$.

Now our purpose is to find some formulas to compute Kan extensions. Given functors $F \in \mathrm{Fun}(\mathcal{C},\mathcal{E})$ and $K \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$, we define the functor $\varphi_d^{K,F}$ to be the composition $F \circ \Pi_d : (K \downarrow, d) \to \mathcal{E}$, where Π_d is the projection functor from the comma category $(K, \downarrow, d)$ to $\mathcal{C}$. If K and F are clear from the context, we may simply denote $\varphi_d^{K,F}$ by φ_d.

Theorem 2.14. *Given functors $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $K \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. If for every $d \in \mathcal{D}$ the colimit $\varinjlim \varphi_d$ exists then the pointwise assignment $\mathrm{Lan}_K F(d) := \varinjlim \varphi_d \in \mathcal{E}$ defines the left Kan extension of F along K.*

Proof. First we extend the pointwise assignment $\mathrm{Lan}_K F(d) := \varinjlim \varphi_d$ to a functor from $\mathcal{D}$ to $\mathcal{E}$ by using the universal property of $\varinjlim \varphi_d$. Next we construct the universal natural transform $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$. Its components is defined by $\eta_c := \lambda^{Kc}_{\mathrm{Id}_{Kc}: Kc \to Kc}$, where λ^{Kc} is the universal cone over $\mathrm{Lan}_K F(K(c)) := \varinjlim \varphi_{Kc}$. Finally it is direct to check that the pair $(\mathrm{Lan}_K F, \eta)$ is actually initial. $\square$

Example 2.15. Let G be a finite group and $H \leq G$ a subgroup. The inclusion $H \hookrightarrow G$ induces an embedding $i : BH \to BG$, where BG is the delooping of G. This embedding further induces a restriction functor $\mathrm{res}^G_H : \mathrm{Rep}(G) \cong \mathrm{Fun}(BG, \mathrm{Vec}) \to \mathrm{Fun}(BH, \mathrm{Vec}) \cong \mathrm{Rep}(H)$. By Lemma 2.12, to compute the left and right adjoints of res^G_H, it suffices to compute the left and right Kan extension of a representation $X \in \mathrm{Fun}(BH, \mathrm{Vec})$ along the functor i. We denote the left adjoint of res^G_H by ind^G_H and call it the **induced representation**. By Theorem 2.14, $\mathrm{ind}^G_H(X)$ is defined by the colimit of the following diagram

$$\varphi_X : (BH, \downarrow, *_G) \xrightarrow{\Pi} BH \xrightarrow{X} C$$

Objects in the comma category $(BH, \downarrow, *_G)$ are labelled by group elements of G; a morphism $h : g \to g'$ is an element $h \in H$ so that $g'h = g$. The functor φ_X sends each object g to a copy of the H-representation X. $\mathrm{ind}^G_H(X) = \varinjlim \varphi_X$ can be written as a colimit of the form

$$\coprod_{G \times H} X \rightrightarrows \coprod_G X \overset{q}{\twoheadrightarrow} \mathrm{ind}^G_H(X)$$

Thus $\mathrm{ind}^G_H(X)$ is a quotient of $\coprod_G X$ by the following equivalence relation $x \in X_g \sim h_*(x) \in X_{g'}$ where $h : g \to g'$ is a morphism. As a result, the induced representation $\mathrm{ind}^G_H(X)$ is isomorphic to $\coprod_{G/H} X$ as a vector space. $\mathrm{ind}^G_H(X)$ is equipped with a natural G/H-grading. The action of G on $\mathrm{ind}^G_H(X)$ is induced by the G-action over $\coprod_G X$ by permutation of index. From the construction we see that, the induced representation $\mathrm{ind}^G_H(X)$ can be realized by the tensor product $\Bbbk G \otimes_{\Bbbk H} X$.

Dually, the right adjoint functor res^G_H is called **coinduced induction** and is denoted by coind^G_H. $\mathrm{coind}^G_H(X)$ may be presented by the following equalizer

$$\mathrm{coind}^G_H(X) \hookrightarrow \prod_G X \rightrightarrows \prod_{H \times G} X$$

As vector space coind^G_H is equipped with a $H \setminus G$-grading and the G-action is induced by the G-action over $\prod_G X$ by permutation of index. Thus $\mathrm{coind}^G_H(X)$ can be realized by the hom space $\hom_{\Bbbk H}(\Bbbk G, X)$.

As vector space there is a canonical isomorphism $\prod_{H \setminus G} X \cong \coprod_{G/H} X$ since finite products and coproducts coincide to be direct sums in Vec, and this isomorphism actually commutes with G-actions. Thus $\mathrm{coind}^G_H(X) \cong \mathrm{ind}^G_H(X)$.

Now let us return to Theorem 1.15. It turns out that the functor $\tilde{F}$ is exactly the left Kan extension of F along the Yoneda embedding $\mathcal{Y}$. Such a left Kan extension is also called a **Yoneda extension**.

The second formula makes use of the language of (co)ends.

Theorem 2.16. *Let $\mathcal{C}$ be a small category, $\mathcal{D}$ a category and $\mathcal{E}$ a cocomplete category. Let $F : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{C} \to \mathcal{E}$ be functors, then there is a canonical isomorphism of functors:*

$$\mathrm{Lan}_K F \cong \int^{c\in\mathcal{C}} \hom_{\mathcal{D}}(Kc, -) \cdot Fc.$$

First we need to clarify the meaning of the new notation " $\cdot$ ". It is the so-called **tensor** or a **copower**: if S is a set and $e \in \mathcal{E}$ with $\mathcal{E}$ cocomplete, then $S \cdot e$ is the S-indexed coproduct of copies of e. Alternatively, " $\cdot$ " is defined by the adjunction

$$\hom_{\mathcal{E}}(S \cdot e, e') \cong \hom_{\mathrm{Set}}(S, \hom_{\mathcal{E}}(e, e')).$$

Proof of Theorem 2.16. We have

$$\begin{aligned}
\mathrm{Nat}(\int^{c\in\mathcal{C}} \hom_{\mathcal{D}}(Kc, -) \cdot Fc, H) &\cong \int_{c\in\mathcal{C}} \mathrm{Nat}(\hom_{\mathcal{D}}(Kc, -) \cdot Fc, H) \\
&\cong \int_{c\in\mathcal{C}} \mathrm{Nat}(\hom_{\mathcal{D}}(Kc, -), \hom_{\mathcal{E}}(Fc, H(-))) \\
&\cong \int_{c\in\mathcal{C}} \hom_{\mathcal{E}}(Fc, HKc) \\
&\cong \mathrm{Nat}(F(-), HK(-)) \\
&\cong \mathrm{Nat}(\mathrm{Lan}_K F, H)
\end{aligned}$$

Since $H \in \mathrm{Fun}(\mathcal{D}, \mathcal{E})$ is arbitrarily chosen, by Yoneda lemma we conclude that $\mathrm{Lan}_K F \cong \int^{c\in\mathcal{C}} \hom_{\mathcal{D}}(Kc, -) \cdot Fc$. □

Corollary 2.17. *Let $X \in \mathrm{sSet}$ be a simplicial set. Then the geometric realization $|X|$ can be computed as*

$$|X| \cong \int^{[n]\in\Delta} \hom_{\mathrm{sSet}}(\Delta^n, X) \cdot |\Delta^n| \cong \int^{[n]\in\Delta} X_n \times |\Delta^n|.$$

Here $\times$ is the product of topological spaces and we equip X_n with the discrete topology.