Shenzhen-Nagoya workshop on Quantum Science

Center functor and condensation theory

Jiaheng Zhao

Academy of Mathematics and System Sciences, Chinese Academy of Sciences

zhaojiaheng171@mails.ucas.ac.cn

September 11, 2023

Introduction

In this talk I will introduce the notion of a relative center functor, which generalizes Kong and Zheng's work on center functors[Kong-Zheng:2107.03858]. I will develop the calculus of condensable algebras, by which I mean a pair of adjunction between codimension 2 condensations and codimension 1 condensations.

Preliminaries on separable n-categories

This talk relies heavily on the language of separable n-categories developed by Kong and Zheng[Kong-Zheng:2011.02859]. I will first introduce some necessary notions and terminology. Throughout this talk we work over the field $\mathbb{C}$ of complex numbers.

- A **separable n-category** $\mathcal{C}$ a linear n-category such that
  1. $\mathcal{C}$ is Karoubi complete (aka condensation complete[Gaiotto-Johnson-Freyd:1905.09566]);
  2. $\mathcal{C}$ is fully dualizable (in a suitable Morita category).
- The collection of all separable n-categories, linear functors and higher morphisms form a separable $(n+1)$-category, which we denote by $(n+1)\mathrm{Vec}$. We use the convention that $0\mathrm{Vec} = \mathbb{C}$, and $1\mathrm{Vec} = \mathrm{Vec}$, i.e. the category of finite dimensional vector spaces.
- There is a pair of adjunction $\Sigma : \mathrm{Alg}_{\mathbb{E}_{m+1}}(n\mathrm{Vec}) \leftrightarrows \mathrm{Alg}_{\mathbb{E}_m}((n+1)\mathrm{Vec}) : \Omega$, where Σ sends a $\mathbb{E}_{m+1}$-monoidal separable $(n-1)$-category $\mathcal{C}$ to its condensation completion $\Sigma\mathcal{C}$, while Ω sends a $\mathbb{E}_m$-monoidal separable n-category $\mathcal{D}$ to $\hom_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$.

Preliminaries on separable n-categories

- We have $\Sigma(n\mathrm{Vec}) = (n+1)\mathrm{Vec}$ and $\Omega((n+1)\mathrm{Vec}) = n\mathrm{Vec}$. In particular, there is a simple but astonishing formula $n\mathrm{Vec} = \Sigma^n\mathbb{C}$, which says that **everything can be obtained by iteratedly condensing $\mathbb{C}$, the field of complex numbers**.
- A $\mathbb{E}_1$-monoidal separable n-category is called a **multi-fusion n-category**; a $\mathbb{E}_2$-monoidal separable n-category is called a **braided multi-fusion n-category**. We say that a (braided) multi-fusion n-category is **(braided) fusion** if its tensor unit is simple.
- Let $\mathcal{A}, \mathcal{B}$ be indecomposable multi-fusion n-categories. We say that a separable $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$ is **closed** if the canonical linear monoidal functor $\mathcal{A} \boxtimes \mathcal{B}^{\mathrm{rev}} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})$ is invertible. We say that $\mathcal{A}$ is **non-degenerate** if the $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}$ is closed.

Preliminaries on separable n-categories

- Let $\mathcal{A}, \mathcal{B}$ be braided fusion n-categories. A **multi-fusion $\mathcal{A}$-$\mathcal{B}$-bimodule** is a multi-fusion n-category $\mathcal{X}$ equipped with a linear braided monoidal functor $\psi_{\mathcal{X}} : \mathcal{A} \boxtimes \overline{\mathcal{B}} \to \mathfrak{Z}_1(\mathcal{X})$. We say that $\mathcal{X}$ is **closed** if $\psi_{\mathcal{X}}$ is invertible. We say that $\mathcal{A}$ is **non-degenerate** if the multi-fusion $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}$ is closed.
- A multi-fusion $\mathcal{A}$-module is equivalent to a separable algebra in $\Sigma\mathcal{A}$.
- An $(n+1)$D (spacetime dimension) anomaly-free topological order $\mathcal{A}$ can be described in terms of the following equivalent ways:
  1 . A braided fusion $(n-1)$-category $\mathcal{C}$, whose objects correspond to codimension 2 defects of $\mathcal{A}$.
  2 . A fusion n-category $\Sigma\mathcal{C}$, whose objects correspond to codimension 1 defects of $\mathcal{A}$.

Let us fix a non-degenerate multi-fusion n-category $\mathcal{C}$. Let $\mathcal{M}, \mathcal{N}$ be separable left $\mathcal{C}$-modules, we use $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ to denote the separable n-category of $\mathcal{C}$-module functors. When $\mathcal{M} = \mathcal{N}$, the functor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is multi-fusion.

Notation

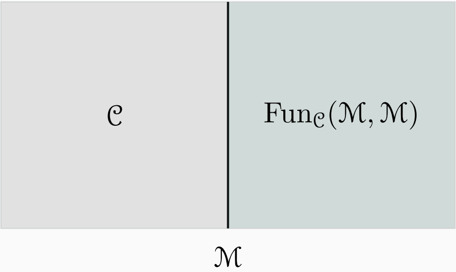
We use $\mathrm{LMod}_{\mathcal{C}}^{\mathrm{ind}}((n+1)\mathrm{Vec})$ to denote the 1-category where an object is an indecomposable separable left $\mathcal{C}$-module, and a morphism is an equivalence class of $\mathcal{C}$-module functors.

Notation

We use $\mathcal{F}us_n^{\mathrm{cl}}$ to denote the symmetric monoidal 1-category where an object is a non-degenerate fusion n-category $\mathcal{A}$ and a morphism $\mathcal{A} \to \mathcal{B}$ is an equivalence class of pairs $(\mathcal{M}, X)$ where $\mathcal{M}$ is a closed $\mathcal{B}$-$\mathcal{A}$-bimodule, and X is a non-zero object of $\mathcal{M}$.

Relative E_0-center functor

For a left $\mathcal{C}$-module $\mathcal{M}$, the left $\mathcal{C}$-module structure is equivalent to a monoidal functor $F : \mathcal{C} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M}) = \mathfrak{Z}_0(\mathcal{M})$. $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is nothing but the E_1-centralizer $\mathfrak{Z}_1(\mathcal{C}, \mathrm{Fun}(\mathcal{M}, \mathcal{M}))$. When $\mathcal{M}$ is separable, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a non-degenerate multi-fusion category. We can view $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ as "the E_0-center of $\mathcal{M}$ relative to $\mathcal{C}$". Geometrically, the relationship between $\mathcal{C}$, $\mathcal{M}$ and $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is demonstrated in the following diagram:



Theorem

Let $\mathcal{C}$ be a non-degenerate fusion n-category. We may define a relative center functor

$$\mathfrak{Z}_0^{rel} : \mathrm{LMod}_{\mathcal{C}}^{\mathrm{ind}}((n+1)\mathrm{Vec}) \to \mathcal{F}us_n^{\mathrm{cl}}$$

which sends a indecomposable separable left $\mathcal{C}$-module $\mathcal{M}$ to the non-degenerate fusion category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ and sends a $\mathcal{C}$-module functor $F:\mathcal{M}\to\mathcal{N}$ to the pair $(\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N}),F)$

Proof.

It suffices to check functoriality. We have

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{N},\mathcal{P}) \boxtimes_{\mathrm{Fun}_{\mathcal{C}}(\mathcal{N},\mathcal{N})} \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N}) \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{P}).$$

□

Relative E_1-center functor

Notation

Let $\mathcal{BF}us_n^{\mathrm{cl}}$ denote the symmetric monoidal 1-category where an object is an non-degenerate braided multi-fusion n-category and a morphism $\mathcal{A} \to \mathcal{B}$ is an equivalence class of closed multi-fusion $\mathcal{B}$-$\mathcal{A}$-bimodules.

Notation

Let $\mathcal{C}$ be a non-degenerate braided monoidal fusion $(n-1)$-category, so that $\Sigma\mathcal{C}$ is a non-degenerate fusion n-category. Let $\mathcal{MF}us_{\mathcal{C}}^{\mathrm{ind}}$ be the 1-category whose objects are indecomposable multi-fusion $\mathcal{C}$-modules. A morphism $\mathcal{M} \to \mathcal{N}$ is an equivalence class of separable $\mathcal{C}$-central $\mathcal{M}$-$\mathcal{N}$-bimodule $\mathcal{L}$.

Theorem

The assignment $\mathcal{M} \mapsto \mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))$ *together with the assignment* ${}_{\mathcal{M}}\mathcal{L}_{\mathcal{N}} \mapsto \mathrm{Fun}^{\mathcal{C}}_{\mathcal{M}|\mathcal{N}}(\mathcal{L}, \mathcal{L}) := \Omega(\mathrm{Fun}_{\Sigma\mathcal{C}}(\Sigma\mathcal{M}, \Sigma\mathcal{N}), - \otimes_{\mathcal{M}} \mathcal{L})$ *define a functor*

$$\mathfrak{Z}_1^{rel} : \mathcal{MF}us^{\mathrm{ind}}_{\mathcal{C}} \to \mathcal{BF}us^{\mathrm{cl}}_{n-1}$$

Proof.

Our assignments coincide with the composition

$$\mathcal{MF}us^{\mathrm{ind}}_{\mathcal{C}} \xrightarrow{\Sigma} \mathrm{LMod}^{\mathrm{ind}}_{\Sigma\mathcal{C}}((n+1)\mathrm{Vec}) \xrightarrow{\mathfrak{Z}_0^{rel}} \mathcal{F}us^{\mathrm{cl}}_n \xrightarrow{\Omega} \mathcal{BF}us^{\mathrm{cl}}_{n-1}$$

□

Calculus of condensable algebras

Recall that, for a braided fusion n-category $\mathcal{C}$, a multi-fusion $\mathcal{C}$-module is the same as a separable algebra object in the fusion $(n+1)$-category $\Sigma\mathcal{C}$.

Notation

We use $\mathrm{Alg}^{\mathrm{sep}}_{\mathbb{E}_1}(\Sigma\mathcal{C})$ to denote the category where an object is a separable algebra in the fusion $(n+1)$-category $\Sigma\mathcal{C}$, and a morphism is an equivalence class of algebra homomorphisms.

Given $\mathcal{M}, \mathcal{N} \in \mathrm{Alg}^{\mathrm{sep}}_{\mathbb{E}_1}(\Sigma\mathcal{C})$, our theory of relative center functor allows us to give a geometric construction of an algebra homomorphism $\mathcal{M} \to \mathcal{N}$ in $\Sigma\mathcal{C}$. This construction is inspired by Kong and Zheng's work[Kong-Zheng:2107.03858].

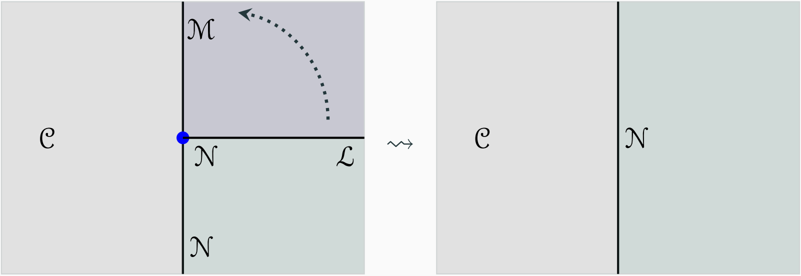
Proposition

Let $\mathcal{M}, \mathcal{N} \in \mathrm{Alg}^{\mathrm{sep}}_{\mathbb{E}_1}(\Sigma\mathcal{C})$, *then*

- *Let* $F : \mathcal{M} \to \mathcal{N}$ *be an algebra homomorphism in* $\Sigma\mathcal{C}$. *Then we have an equivalence of algebras in* $\Sigma\mathcal{C}$*:*
$$\mathrm{Fun}^{\mathcal{C}}_{\mathcal{M}|\mathcal{N}}(\mathcal{N}, \mathcal{N}) \boxtimes_{\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))} \mathcal{M} \simeq \mathcal{N}, \quad f \boxtimes_{\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))} X \mapsto fF(X).$$
- *Conversely, suppose we are given a multi-fusion* $\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))$*-module* $\mathcal{E}$ *and an equivalence* $\phi : \mathcal{E} \boxtimes_{\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))} \mathcal{M} \simeq \mathcal{N}$ *of algebras in* $\Sigma\mathcal{C}$, *then the composition* $\mathcal{M} \xrightarrow{\mathbb{1}_{\mathcal{E}} \boxtimes_{\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))} \mathrm{Id}_{\mathcal{M}}} \mathcal{E} \boxtimes_{\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))} \mathcal{M} \overset{\phi}{\simeq} \mathcal{N}$ *defines a algebra homomorphism from* $\mathcal{M}$ *to* $\mathcal{N}$.

Moreover, these constructions are inverse to each other.

According the proposition, an algebra homomorphism $F : \mathcal{M} \to \mathcal{N}$ is equivalent to the following data: a multi-fusion $\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{N}))$-$\mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M}))$ bimodule $\mathcal{L}$ such that $\mathcal{L} \boxtimes_{\mathfrak{Z}_2(\mathcal{C},\mathfrak{Z}_1(\mathcal{M}))} \mathcal{M} \simeq \mathcal{N}$ as multi-fusion $\mathcal{C}$-modules. Physically, this means a gapped domain wall $\mathcal{L}$ such that the dimensional reduction of $(\mathcal{L}, \mathfrak{Z}_2(\mathcal{C}, \mathfrak{Z}_1(\mathcal{M})), \mathcal{M})$ is $\mathcal{N}$:



Calculus of condensable algebras

Let us return to the 1-categorical case. Fix a non-degenerate braided fusion category $\mathcal{C}$, recall that a ($\mathbb{E}_2$-) **condensable algebra** in $\mathcal{C}$ is a connected commutative separable algebra. We use $\mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_2}(\mathcal{C})$ to denote the category of condensable algebras and algebra homomorphisms. We use $\mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_1}(\Sigma\mathcal{C})$ to denote the category of condensable algebras and (equivalence classes of) algebra homomorphisms.

Define a functor

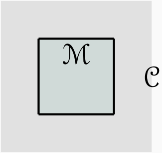
$$\begin{aligned} \mathrm{RMod}_{-}(\mathcal{C}) : \mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_2}(\mathcal{C}) &\to \mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_1}(\Sigma\mathcal{C}) \\ A &\mapsto \mathrm{RMod}_A(\mathcal{C}) \\ (f : A \to B) &\mapsto - \otimes_A {}_f B_B \end{aligned}$$

Here the left A-module structure on B is induced by the algebra homomorphism f.

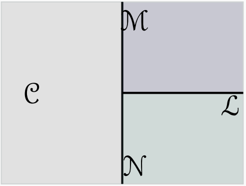
We will define a functor in the reverse direction. Let $\mathfrak{Z}$ be the following functor

$$\mathfrak{Z} : \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{con}}(\Sigma\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{con}}(\mathcal{C})$$
$$(F : \mathcal{C} \to \mathcal{M}) \mapsto [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}]_{\mathcal{C}}$$

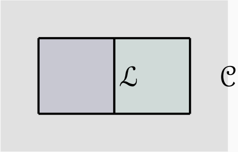
It is well known that the internal hom $[\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}]_{\mathcal{C}}$ is equipped with a canonical structure of condensable algebra in $\mathcal{C}$[Davydov-Müger-Nikshych-Ostrik:1009.2117v2]. This algebra has an interesting geometric construction: we do dimensional reduction to the "bubble" in $\mathcal{C}$, as is depicted in the following picture



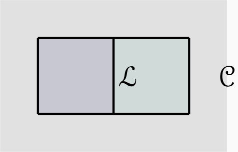
I will now explain the action of $\mathfrak{Z}$ on morphisms. Given an algebra homomorphism $F : \mathcal{M} \to \mathcal{N}$, recall that this can be given by a gapped domain wall $\mathcal{L}$:



We define $\mathfrak{Z}(F)$ to be the dimensional reduction of the following object in $\mathcal{C}$:



What does this have to do with an algebra homomorphism from $A = [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}]_{\mathcal{C}}$ to $B = [\mathbb{1}_{\mathcal{N}}, \mathbb{1}_{\mathcal{N}}]_{\mathcal{C}}$? Let us observe this "bubble"



This object has the following properties:

- The dimension reduction of this "bubble" is B, hence it acquires an algebra structure;
- This object is equipped a canonical left A-action, and this left action is compatible with the algebra structure, in an obvious sense.

Then, we could see that such a left A-module structure on B is equivalent to an algebra homomorphism $f : A \to B$! This completes our definition of the functor $\mathfrak{Z}$.

Now I will claim the main result, without writing down more details.

Theorem

Let $\mathcal{C}$ be a non-degenerate braided fusion category. There is a pair of adjunction

$$\mathrm{RMod}_{-}(\mathcal{C}) : \mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_2}(\mathcal{C}) \leftrightarrows \mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_1}(\Sigma\mathcal{C}) : \mathfrak{Z}$$

Such that $\mathrm{RMod}_{-}(\mathcal{C})$ *is fully faithful, and* $\mathfrak{Z} \circ \mathrm{RMod}_{-}(\mathcal{C}) = \mathrm{Id}_{\mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_2}(\mathcal{C})}$. *Using some mathematical terminology, we say that* $\mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_2}(\mathcal{C})$ *is a* **coreflective localization** *of* $\mathrm{Alg}^{\mathrm{con}}_{\mathbb{E}_1}(\Sigma\mathcal{C})$.

Thank You For Listening!