Equivalent definitions for the non-degeneracy of a braided fusion category II

Jiaheng Zhao

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Last time we gave a brief proof of the following theorem:

Theorem

Let $\mathcal{C}$ be a braided fusion category. TFAE:

(1) *$\mathcal{C}$ is non-degenerate;*

(2) *$\mathcal{C}$ is factorisable;*

(3) *The Müger center of $\mathcal{C}$ is trivial;*

(4) *$\mathcal{C}$ is weakly-factorisable.*

This time, we further show that any of the above conditions is verified if and only if $\mathcal{C}$ is an invertible object in certain Morita category. The main reference is [Brochier-Jordan-Safronov-Snyder: Invertible braided tensor categories].

Conventions

- All categories we mention in this presentation are $\mathbf{k}$-linear categories, where $\mathbf{k}$ is an algebraically closed field of characteristic 0. All functors we use are $\mathbf{k}$-linear.
- We use $(\mathcal{S}, \boxtimes)$ to denote a closed symmetric monoidal $(\infty, 2)$-category which admits geometric realizations. An E_n-algebra in $\mathcal{S}$ is a symmetric monoidal functor $\mathcal{A} : \mathrm{Disk}_n^{fr} \to \mathcal{S}$. The factorisation homology with coefficients in $\mathcal{A}$ is the left Kan extension of $\mathcal{A}$ along the embedding $\mathrm{Disk}_n^{fr} \hookrightarrow \mathrm{Mfld}_n^{fr}$, and is denoted by $\int_{\_} \mathcal{A}$. We use $\mathrm{Alg}_n(\mathcal{S})$ to denote the collection E_n-algebras in $\mathcal{S}$.
- We use $\mathbf{Pr}$ to denote the symmetric monoidal 2-category of locally presentable categories[1], cocontinuous functors and natural transformations. The symmetric monoidal structure is given by the Deligne tensor product.

[1] Finite categories are not locally presentable, but their ind-completions are. We can study finite categories thorough their ind-completions.

Conventions

- Let $\mathcal{C}$ be an E_1-algebra. We use $\mathcal{C}^{\otimes\mathrm{op}}$ to denote the opposite E_1-algebra whose multiplication direction is reverse to $\mathcal{C}$. Let $\mathcal{A}$ be an E_2-algebra, we can either reverse the multiplication in the x-direction or in the y-direction. The two E_2-algebras obtained in the two ways can be canonically identified and we denote it as $\mathcal{A}^{\sigma\mathrm{op}}$.

Enveloping algebra

Definition

Let $\mathcal{A}$ be an E_n-algebra. Its **enveloping algebra** is the E_1-algebra

$$U^n_{\mathcal{A}} = \int_{S^{n-1}\times\mathbb{R}} \mathcal{A}.$$

The enveloping algebra has a natural left action on $\mathcal{A} = \int_{\mathbb{R}^n} \mathcal{A}$.

If we take $\mathcal{S} = \mathbf{Pr}$, then:

- $n = 1$: $U^1_{\mathcal{A}} = \mathcal{A} \boxtimes \mathcal{A}^{\otimes\mathrm{op}}$. We also denote this algebra by $\mathcal{A}^e$;
- $n = 2$: $U^2_{\mathcal{A}} = \mathcal{A} \boxtimes_{\mathcal{A}\boxtimes\mathcal{A}^{\sigma\mathrm{op}}} \mathcal{A}^{\otimes\mathrm{op}}$. We also denote this algebra by $\mathrm{HC}(\mathcal{A})$.

E_n-center

Definition

Let $\mathcal{A}$ be an E_n-algebra. Its E_n-**center** is the object

$$\mathcal{Z}_n(\mathcal{A}) = \mathrm{End}_{U^n_{\mathcal{A}}}(\mathcal{A}).$$

If we take $\mathcal{S} = \mathbf{Pr}$, then

- $n = 1$: $\mathcal{Z}_1(\mathcal{A}) = \mathrm{Fun}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$ and is canonically equivalent to the Drinfeld center of $\mathcal{A}$.
- $n = 2$: $\mathcal{Z}_2(\mathcal{A}) = \mathrm{Fun}_{\mathrm{HC}(\mathcal{A})}(\mathcal{A}, \mathcal{A})$, which is canonically equivalent to the Müger center of $\mathcal{A}$.

Morita category

The collection of E_n-algebras in a fixed $\mathcal{S}$, $\mathrm{Alg_n}(\mathcal{S})$, carries a structure of a symmetric monoidal $(\infty, n+2)$ category, called the Morita category of $\mathcal{S}$.

Definition (Sketchy)

The $(\infty, n+2)$-category $\mathrm{Alg}_n(\mathcal{S})$ consists of

- Objects are E_n algebras, which we denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}$...
- 1-morphisms are E_{n-1}-algebra objects in $(\mathcal{A}, \mathcal{B})$-bimodules ;
- ...
- $n+1$-morphisms are bimodule 1-morphisms;
- $n+2$-morphisms are bimodule 2-morphisms.

Review: Dualizability of bimodules

Let $\mathcal{C}$ and $\mathcal{D}$ be E_1-algebras and $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule. We say that $\mathcal{M}$ is **right dualizable** if there is a $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{N}$ together with a evaluation bimodule map $v : \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N} \to \mathcal{C}$ and a coevaluation bimodule map $u : \mathcal{D} \to \mathcal{N} \otimes_{\mathcal{C}} \mathcal{M}$ such that the compositions

$$\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{D}} \mathcal{D} \xrightarrow{\mathrm{id}_{\mathcal{M}} \otimes_{\mathcal{D}} u} \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N} \otimes_{\mathcal{C}} \mathcal{M} \xrightarrow{v \otimes_{\mathcal{C}} \mathrm{id}_{\mathcal{M}}} \mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{M},$$

$$\mathcal{N} \simeq \mathcal{D} \otimes_{\mathcal{D}} \mathcal{N} \xrightarrow{u \otimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}} \mathcal{N} \otimes_{\mathcal{C}} \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\mathrm{id}_{\mathcal{N}} \otimes_{\mathcal{C}} v} \mathcal{N} \otimes_{\mathcal{C}} \mathcal{C} \simeq \mathcal{N}.$$

are isomorphic to the identity maps. In this case we say that $\mathcal{N}$ is the **right dual** of $\mathcal{M}$, or equivalently, $\mathcal{M}$ is the **left dual** of $\mathcal{N}$.

Duality theory of bimodules can be regarded as a generalization of the usual duality theory in a monoidal category. Recall that in a monoidal category $\mathcal{T}$, the tensor unit $\mathbb{1}$ is equipped with a canonical algebra structure, and the forgetful functor $U : \mathrm{BMod}_{\mathbb{1}|\mathbb{1}}(\mathcal{T}) \to \mathcal{T}$ is obviously a monoidal equivalence. We can see that an object T is left (resp. right) dualizable as a $\mathbb{1}$-$\mathbb{1}$-bimodule if and only if it is left (resp. right) dualizable as an object in $\mathcal{C}$.

Through the obvious equivalences $\mathrm{BMod}_{\mathbb{1}|\mathcal{A}}(\mathcal{T}) \simeq \mathrm{RMod}_{\mathcal{A}}(\mathcal{T})$ and $\mathrm{BMod}_{\mathcal{A}|\mathbb{1}}(\mathcal{T}) \simeq \mathrm{LMod}_{\mathcal{A}}(\mathcal{T})$ one can also talk about the dualizability of a left/right module over some algebra $\mathcal{A}$. For example, we say a left $\mathcal{A}$-module is dualizable if it is right dualizable as a $\mathcal{A}$-$\mathbb{1}$-bimodule.

Remark

Our convention of left/right duality follows [Kong-Zheng: The center functor is fully faithful] and maybe different from the conventions used by other authors.

Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be E_1-algebras and $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule. A morphism of E_1-algebras $f : \mathcal{B} \to \mathcal{D}$ gives rise to a pull-back functor $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}} \to \mathrm{BMod}_{\mathcal{C}|\mathcal{B}}$. It turns out that dualizability of a bimodule is preserved by the pull-back functor:

Lemma

Let $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}$ be right dualizable with right dual $\mathcal{N} \in \mathrm{BMod}_{\mathcal{D}|\mathcal{C}}$. Then $\mathcal{M}'$, the image of $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{B}}$ is also right dualizable with right dual $\mathcal{N}'$, the image of $\mathcal{N}$ in $\mathrm{BMod}_{\mathcal{B}|\mathcal{C}}$.

Remark

Note that $\mathcal{M} = \mathcal{M}'$, $\mathcal{N} = \mathcal{N}'$ as objects in $\mathcal{T}$. Only the module structure changes when we change scalars.

In fact, the converse of the previous theorem is also true:

Theorem (J.Lurie)

Let $\mathcal{C}, \mathcal{D}$ be E_1-algebras in some suitable monoidal ∞-category $\mathcal{T}$ and $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule. TFAE:

(1) *$\mathcal{M}$ is right dualizable as a $\mathcal{C}$-$\mathcal{D}$-bimodule;*

(2) *For every algebra homomorphism $f : \mathcal{B} \to \mathcal{D}$, the image of $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{B}}(\mathcal{T})$ is right dualizable;*

(3) *There exists an algebra homomorphism $f : \mathcal{B} \to \mathcal{D}$ such that the image of $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{B}}$ is right dualizable.*

(4) *Let $\mathbb{1}$ be the tensor unit of $\mathcal{T}$, then the image of $\mathcal{M}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathcal{T}) \simeq \mathrm{BMod}_{\mathcal{C}|\mathbb{1}}(\mathcal{T})$ is right dualizable.*

For a complete proof we refer to [Lurie: Higher algebra, Theorem 4.6.2.13]

Remark

The preceding theorem says that left (right) dualizability of a bimodule $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathcal{T})$ depends only on the right (left) action. Now let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule. If $\mathcal{M}$ is right dualizable, its right dual must have the form $\mathcal{M}^R = \hom_{\mathcal{C}}(\mathcal{M}, \mathcal{C})$. As an object of $\mathcal{T}$, the dual $\mathcal{M}^R$ does not depend on the action on the right side.

Theorem

Let $\mathcal{M}$ be a left $\mathcal{C}$-module. Then $\mathcal{M}$ is right dualizable if and only if the canonical map

$$\hom_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \boxtimes_{\mathcal{C}} \mathcal{N} \to \hom_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \quad F \boxtimes_{\mathcal{C}} n \mapsto F(-) \odot n$$

is an equivalence for all left $\mathcal{C}$-module $\mathcal{N}$.

Proof.

Let us view $\mathcal{C}$ as a $\mathcal{C}$-$\mathbb{1}$-bimodule. Then $\hom_{\mathcal{C}}(\mathcal{M},\mathcal{C})$ is a right dual of $\mathcal{C}$ if and only if $\hom_{\mathcal{C}}(\mathcal{M},\mathcal{C}) \boxtimes_{\mathcal{C}} -$ is a right adjoint of $\mathcal{M} \boxtimes -$, i.e.

$$\hom_{\mathcal{C}}(\mathcal{M} \boxtimes \mathbb{1}, \mathcal{N}) \simeq \hom_{\mathcal{T}}(\mathbb{1}, \hom_{\mathcal{C}}(\mathcal{M},\mathcal{C}) \boxtimes_{\mathcal{C}} \mathcal{N})$$

$\square$

It is a well-known fact that any E_n algebra is n-dualizable in the Morita category[Gwilliam-Scheibauer: Duals and adjoints in higher Morita categories].

Example

Any E_1-algebra $\mathcal{C} \in \mathrm{Alg}_1(\mathcal{S})$ is 1-dualizable with dual $\mathcal{C}^\vee = \mathcal{C}^{\otimes\mathrm{op}}$, with evaluation and co-evaluation map given by:

- ev is $\mathcal{C}$ as a $(\mathcal{C}^{\otimes\mathrm{op}} \boxtimes \mathcal{C}, \mathbf{1}_{\mathcal{S}})$-bimodule.
- coev is $\mathcal{C}$ as a $(\mathbf{1}_{\mathcal{S}}, \mathcal{C} \boxtimes \mathcal{C}^{\otimes\mathrm{op}})$-bimodule.

Example

Every E_2-algebra $\mathcal{A} \in \mathrm{Alg}_2(\mathcal{S})$ is 2-dualizable with dual $\mathcal{A}^\vee = \mathcal{A}^{\sigma\mathrm{op}}$, and with evalution and coevaluation given by the regular central algebra:

- ev is $\mathcal{A}$ as a $(\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}, \mathbf{1}_{\mathcal{S}})$;
- coev is $\mathcal{A}$ as a $(\mathbf{1}_{\mathcal{S}}, \mathcal{A} \boxtimes \mathcal{A}^{\sigma\mathrm{op}})$-algebra.

The right adjoints to evaluation and coevaluation are given by

- ev^R is $\mathcal{A}^{\otimes\mathrm{op}}$ as a $(\mathbf{1}_{\mathcal{S}}, \mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A})$-central algebra,
- coev^R is $\mathcal{A}^{\otimes\mathrm{op}}$ as a $(\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}, \mathbf{1}_{\mathcal{S}})$-central algebra.

Higher unit and counit morphisms are omitted here.

It is natural to ask for higher dualizability an E_n-algebra. When is an E_n-algebra (n+1)-dualizable? A conjecture is proposed in [Brochier-Jordan-Safronov-Snyder: Invertible braided tensor categories] to answer this question:

Conjecture

An E_n-algebra $\mathcal{A}$ is $(n+1)$-dualizable if and only if it is dualizable over the factorization homologies $\int_{S^{k-1}\times\mathbb{R}^{n-k+1}}\mathcal{A}$ for all $k=0,...,n$.

The case $n=1$ has been proved in [Lurie: On the classification of topological field theories] and the case $n=2$ has been proved in [Brochier-Jordan-Safronov-Snyder: Invertible braided tensor categories].

Question

How to view the action of $\int_{S^{k-1}\times\mathbb{R}^{n-k+1}}\mathcal{A}$ on $\mathcal{A}=\int_{\mathbb{R}^n}\mathcal{A}$ topologically?

Example

An E_1-algebra $\mathcal{C}$ is 2-dualizable if and only if it is dualizable as an object of $\mathcal{S}$, and as a left $\mathcal{C}^e$-module.

To see this recall that the evaluation and coevaluation maps witnessing the 1-dualizability of $\mathcal{C}$ are given by:

- ev is $\mathcal{C}$ as a $(\mathcal{C}^{\otimes\mathrm{op}} \boxtimes \mathcal{C}, \mathbf{1}_{\mathcal{S}})$-bimodule.
- coev is $\mathcal{C}$ as a $(\mathbf{1}_{\mathcal{S}}, \mathcal{C} \boxtimes \mathcal{C}^{\otimes\mathrm{op}})$-bimodule.

A theorem of Lurie [Lurie: On the classification of topological field theories, Proposition 4.2.3] says that, a 1-dualizable object $\mathcal{C}$ of a symmetric monoidal 2-category is 2-dualizable if and only if the evaluation and coevaluation maps each admit a right adjoint. By our previous discussions on base changes, it suffices to require that $\mathcal{C}$ is dualizable as an object of $\mathcal{S}$, and as a $\mathcal{C}^e$-module.

Invertibility of E_n-algebras

We know that, an equivalence between two categories can always be promoted to part of an adjoint equivalence. Conversely, a pair of adjunction $F \dashv G$ are mutually inverse to each other if and only if the evaluation and coevaluation maps are isomorphisms. These patterns still work for higher categories. We may conclude that

- Requiring a dualizable object to be invertible is equivalent to requiring the corresponding evaluation and coevaluation maps to be invertible;
- Requiring the evaluation and coevaluation maps to be invertible is equivalent to requiring these maps are dualizable and the corresponding higher (co)evaluations are isomorphisms;
- ...
- Invertibility is stronger than dualizability.

This implies us to study invertibility of E_n-algebras inductively.

Let's start from E_1-algebras:

Theorem

Let $\mathcal{B} \in \mathrm{Alg}_2(\mathcal{S})$ be an E_2-algebra and $\mathcal{C}$ an $\mathcal{B}$-central algebra viewed as a 1-morphism $\mathcal{B} \to \mathbf{1}_{\mathcal{S}}$. Then $\mathcal{C}$ is invertible if and only if $\mathcal{C}$ is 2-dualizable as an E_1-algebra and the following maps are equivalences:

1. *The evaluation map $\mathcal{C} \boxtimes_{\mathcal{B}} \hom_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}^e) \to \mathcal{C}^e$.*
2. *The map $\mathcal{B} \to \mathfrak{Z}_1(\mathcal{C})$ given by the $\mathcal{B}$-central structure on $\mathcal{C}$.*
3. *The evaluation map $\hom_{\mathcal{S}}(\mathcal{C}, \mathbf{1}_{\mathcal{S}}) \boxtimes_{\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{C}^{\otimes \mathrm{op}}} \mathcal{C} \to \mathbf{1}_{\mathcal{S}}$.*
4. *The map $\mathcal{C}^{\otimes \mathrm{op}} \boxtimes_{\mathcal{B}} \mathcal{C} \to \hom_{\mathcal{S}}(\mathcal{C}, \mathcal{C})$ given by the left and right action of $\mathcal{C}$ on itself.*

As a matter of fact, the four maps are exactly the characteristic maps witnessing the 2-dualizability of $\mathcal{C}$.

We only look at the first two maps in detail. $\mathcal{C}$ is 1-dualizable with right dual $\mathcal{C}^{\otimes\mathrm{op}}$, which should be viewed as $\mathbf{1}_{\mathcal{S}}$-$\mathcal{B}$-bimodule. The counit is given by $\eta = \mathcal{C}$ viewed as a $\mathcal{C} \boxtimes \mathcal{C}^{\otimes\mathrm{op}}$-$\mathcal{B}$-bimodule, the counit is given by $\epsilon = \mathcal{C}$ viewed as a $\mathbf{1}_{\mathcal{S}}$-$\mathcal{C}^{\otimes\mathrm{op}} \boxtimes_{\mathcal{B}} \mathcal{C}$-bimodule.

Since $\mathcal{C}$ is 2-dualizable, η and ϵ are both right dualizable, with right duals given by $\hom_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}^e)$ and $\hom(\mathcal{C}, \mathbf{1}_{\mathcal{S}})$ respectively. The corresponding (co)evaluation maps are:

1. $\mathcal{C} \boxtimes_{\mathcal{B}} \hom_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}^e) \to \mathcal{C}^e$;
2. $\mathcal{B} \to \hom_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}^e) \boxtimes_{\mathcal{C}^e} \mathcal{C} \simeq \hom_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}) \simeq \mathfrak{Z}_1(\mathcal{C})$;
3. $\mathcal{C} \boxtimes_{\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{C}^{\otimes\mathrm{op}}} \hom_{\mathcal{S}}(\mathcal{C}, \mathbf{1}_{\mathcal{S}}) \to \mathbf{1}_{\mathcal{S}}$.
4. $\mathcal{C}^{\otimes\mathrm{op}} \boxtimes_{\mathcal{B}} \mathcal{C} \to \hom(\mathcal{C}, \mathbf{1}_{\mathcal{S}}) \boxtimes \mathcal{C} \simeq \hom(\mathcal{C}, \mathcal{C})$.

Theorem

An E_2-algebra $\mathcal{A} \in \mathrm{Alg}_2(\mathcal{S})$ is invertible if, and only if, it is 3-dualizable and the following maps are isomorphisms:

$1'$. $\mathrm{HC}(\mathcal{A}) \to \hom_{\mathcal{S}}(\mathcal{A}, \mathcal{A})$.

$2'$. $\mathcal{A} \boxtimes \mathcal{A}^{\sigma\mathrm{op}} \to \mathfrak{Z}_1(\mathcal{A})$.

$3'$. *The inclusion of the unit* $\mathbf{1}_{\mathcal{S}} \to \mathfrak{Z}_2(\mathcal{A})$.

Fact

Taking $\mathcal{S} = \mathrm{Pr}$, the conditions $1'$, $2'$ and $3'$ are equivalent.

Proof.

If $\mathcal{A}$ is invertible, then it is 3-dualizable. Moreover, the evaluation map $\mathcal{A} : \mathcal{A} \boxtimes \mathcal{A}^{\sigma\mathrm{op}} \to \mathbf{1}_{\mathcal{S}}$ is an invertible 1-morphism. Then by the preceding theorem, the conditions 1,2,3,4 hold with $\mathcal{B} = \mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}$. The condition 2 is exactly condition $2'$, condition 4 is exactly condition $1'$ and condition 3 translates into condition $3'$ after applying the functor $\hom(-, \mathbf{1}_{\mathcal{S}})$.

Now assume that $\mathcal{A}$ is 3-dualizable and conditions $1'$, $2'$ and $3'$ hold. Then conditions 2 and 4 trivially hold. Condition 3 holds because $\hom(-, \mathbf{1}_{\mathcal{S}})$ is an equivalence due to the dualizability of $\mathcal{A}$. Finally, condition 1 is equivalent to condition 2 after applying the functor $\hom_{\mathcal{A}^e}(-, \mathcal{A}^e)$, which is also an equivalence. $\square$

1. $\mathcal{A} \boxtimes_{\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}} \hom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) \to \mathcal{A}^e$;
2. $\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A} \to \hom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) \boxtimes_{\mathcal{A}^e} \mathcal{A} \simeq \hom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) \simeq \mathfrak{Z}_1(\mathcal{A})$;
3. $\mathcal{A} \boxtimes_{\mathcal{A} \boxtimes_{\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}} \mathcal{A}^{\otimes\mathrm{op}}} \hom_{\mathcal{S}}(\mathcal{A}, \mathbf{1}_{\mathcal{S}}) \to \mathbf{1}_{\mathcal{S}}$.
4. $\mathcal{A}^{\otimes\mathrm{op}} \boxtimes_{\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}} \mathcal{A} \to \hom(\mathcal{A}, \mathbf{1}_{\mathcal{S}}) \boxtimes \mathcal{A} \simeq \hom(\mathcal{A}, \mathcal{A})$.

$1'$. $\mathrm{HC}(\mathcal{A}) \to \hom_{\mathcal{S}}(\mathcal{A}, \mathcal{A})$.

$2'$. $\mathcal{A} \boxtimes \mathcal{A}^{\sigma\mathrm{op}} \to \mathfrak{Z}_1(\mathcal{A})$.

$3'$. The inclusion of the unit $\mathbf{1}_{\mathcal{S}} \to \mathfrak{Z}_2(\mathcal{A})$.

Apply the functor $\hom_{\mathcal{S}}(-, \mathbf{1}_{\mathcal{S}})$ to condition 3 we get

$$\hom(\mathbf{1}_{\mathcal{S}}, \mathbf{1}_{\mathcal{S}}) \simeq \mathbf{1}_{\mathcal{S}} \to \hom(\mathcal{A} \boxtimes_{\mathcal{A} \boxtimes_{\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}} \mathcal{A}^{\otimes\mathrm{op}}} \hom_{\mathcal{S}}(\mathcal{A}, \mathbf{1}_{\mathcal{S}}), \mathbf{1}_{\mathcal{S}}) \simeq \hom_{\mathrm{HC}(\mathcal{A})}(\mathcal{A}, \mathcal{A}).$$

Apply the functor $\hom_{\mathcal{A}^e}(-, \mathcal{A}^e)$ to condition 1 we get

$$\hom_{\mathcal{A}^e}(\mathcal{A}^e, \mathcal{A}^e) \simeq \mathcal{A}^e \to \hom_{\mathcal{A}^e}(\mathcal{A} \boxtimes_{\mathcal{A}^{\sigma\mathrm{op}} \boxtimes \mathcal{A}} \hom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e), \mathcal{A}^e) \simeq \mathfrak{Z}_1(\mathcal{A}).$$