On final functors

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The first part of these slides contain a detailed introduction to final/initial functors. Final/initial functors are elementary and useful in category theory, but they were missed from my previous studies. The theory of final functors and initial functors are dual to each other, so I will only talk about final functors.

In the second part of these slides I will give a brief introduction to sifted colimits and its relation with final functors. Few details will be given there.

In the last part I will briefly introduce the generalization of final functors in homotopy theory. Even some definitions will not explained there.

Let $\mathcal{C}, \mathcal{D}$ be categories. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we can define its colimit to be a universal cone under F, which we denote by $\varinjlim F$. When we need to highlight the source category $\mathcal{C}$, we may also use the notation $\varinjlim_{c\in\mathcal{C}} F(c)$. For $d \in \mathcal{D}$, we use δ_d to denote the constant functor at d.

Now take a new category $\mathcal{E}$ and a functor $G : \mathcal{D} \to \mathcal{E}$, we get a canonical map

$$\varinjlim_{c\in\mathcal{C}} G(F(c)) \to \varinjlim_{d\in\mathcal{D}} G(d)$$

induced by the cone legs $\kappa_c : G(F(c)) \to \varinjlim G$, for all $c \in \mathcal{C}$.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is **final** if the canonical map defined above is an isomorphism for all categories $\mathcal{E}$ and all functors $G \in \mathrm{Fun}(\mathcal{D}, \mathcal{E})$.

Example

Let $\mathcal{D}$ be a category with a terminal object t. Then the functor $T : [0] \to \mathcal{D}$ picking out the terminal object is a final functor. Here $[0]$ is the category with only one morphism.

To see this, we need to check that for all functors $G : \mathcal{D} \to \mathcal{E}$, the canonical morphism

$$G(t) \to \varinjlim_{d \in \mathcal{D}} G(d)$$

is an isomorphism. But this is direct: there is a unique natural transformation $\mathrm{Id}_{\mathcal{D}} \Rightarrow \delta_t$. By applying G we obtain a cone $G \Rightarrow \delta_{G(t)}$, which can be easily checked to be universal.

The main theorem today says that finality of functors has a "geometric" characterization:

Theorem

A functor $F : \mathcal{C} \to \mathcal{D}$ is final if and only if for each $d \in \mathcal{D}$ the comma category $(d, \downarrow, F)$ is non-empty and connected.

First let us recall the definition of the comma category $(d, \downarrow, F)$: its objects are pairs (c, α) where $c \in \mathcal{C}$ is an object and $\alpha : d \to F(c)$ is a morphism; a morphism $f : (c, \alpha) \to (c', \beta)$ is a morphism $f : c \to c'$ rendering an obvious diagram commutative.

To give a complete proof we need some preparations. Recall the following definition:

Definition

Let $X : \mathcal{C} \to \mathrm{Set}$ be a functor. The **category of elements of X $\mathbf{el}(X)$** is the following category:

- An object of $\mathbf{el}(X)$ is a pair (c, u) where $c \in \mathcal{C}$ is an object and $u \in X(c)$ is an element;
- A morphism $f : (c, u) \to (c', u')$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $X(f)(u) = u'$.

Example

Let $c \in \mathcal{C}$ be an object, then $\mathbf{el}(\hom_{\mathcal{C}}(c, -)) = \mathcal{C}_{c/}$ is the slice category of $\mathcal{C}$ under c.

Example

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $\hom_{\mathcal{D}}(d, -) : \mathcal{D} \to \mathrm{Set}$ be a representable functor. Consider the composed functor $\hom_{\mathcal{D}}(d, F(-)) : \mathcal{C} \to \mathrm{Set}$. Then its category of elements $\mathbf{el}(\hom_{\mathcal{D}}(d, F(-)))$ is nothing but the comma category $(d, \downarrow, F)$ which we have just introduced.

Lemma

There is a canonical isomorphism of sets:

$$\pi_0(\mathbf{el}(X)) \cong \varinjlim X$$

Proof.

Let T be a set and $\alpha : X \Rightarrow \delta_T$. Unpacking the definition of a natural transformation we see that α is equivalent to the following data:

- To each $c \in \mathcal{C}$ we associate a map of sets $\alpha_c : X(c) \to T$, that is, to each $(c, u) \in \mathbf{el}(X)$ we assign an element of T;
- Naturality says that if there is a morphism $f : c \to c'$ such that $X(f)(u) = u'$, then $\alpha_c(u) = \alpha_{c'}(u')$, that is, connected elements in $\mathbf{el}(X)$ should sent to the same elements

So every natural transformation gives rise to a unique map $\pi_0(\mathbf{el}(X)) \to T$. $\square$

Theorem

A functor $F : \mathcal{C} \to \mathcal{D}$ is final if and only if for each $d \in \mathcal{D}$ the comma category $(d, \downarrow, F)$ is non-empty and connected.

Proof of the main theorem

$\Leftarrow$: Let $G : \mathcal{D} \to \mathcal{E}$ be a functor. A cone under G automatically gives a cone under $G \circ F$. Conversely, if the comma category $(d, \downarrow, F)$ is non-empty and connected, then every cone under $G \circ F$ gives rise to a cone under G in the following way: Suppose we have an object $e \in \mathcal{E}$ and a natural transformation $\lambda : G \circ F \Rightarrow \delta_e$. Given $d \in \mathcal{D}$, since $(d, \downarrow, F)$ is non-empty, we may choose $(c, \alpha) \in (d, \downarrow, F)$ and we get a leg $\kappa_d : G(d) \overset{G(\alpha)}{\to} G(F(c)) \overset{\lambda_c}{\to} e$. Connectedness of $(d, \downarrow, F)$ guarantees that the definition of κ_d is independent of the choice of the object (c, α). So we get a bijection between the set of cones under G and that under GF, which extends to an isomorphism of categories.

Proof(Continue).

$\Rightarrow$: Arbitrarily choose an object $d \in \mathcal{D}$, we want to show that $(d, \downarrow, F)$ is non-empty and connected. Consider the following composed functor

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\hom_{\mathcal{D}}(d,-)} \mathrm{Set}$$

By finality of F, there is a canonical isomorphism:

$$\varinjlim_{x \in \mathcal{C}} \hom_{\mathcal{D}}(d, F(x)) \to \varinjlim_{y \in \mathcal{D}} \hom_{\mathcal{D}}(d, y)$$

By previous discussion the LHS is $\pi_0(\mathbf{el}(\hom_{\mathcal{D}}(d, F(-)))) = \pi_0((d, \downarrow, F))$ which is the set we want to compute. The RHS is $\pi_0(\mathcal{D}_{d/}) = \{*\}$ since the category $\mathcal{D}_{d/}$ has an initial object $\mathrm{Id}_d : d \to d$. Note that a category with an initial object must be non-empty and connected. $\square$

Example

Let $\mathcal{D}$ be a category with a terminal object t and $T : [0] \to \mathcal{D}$ be the functor picking out the terminal object. Then $\forall d \in \mathcal{D}$, the comma category $(d, \downarrow, T)$ is the one-point category $[0]$. So the functor T is final.

Example

Any right adjoint functor is final. By the definition of adjunction, a functor $G : \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if the comma category $(d, \downarrow, G)$ has an initial object for all $d \in D$.

Example

Let Δ be the category of finite ordinals. Let $[0] \rightrightarrows [1]$ be the subcategory of Δ consisting of $[0]$ and $[1]$ and inclusions of two endpoints. Note that this category is nothing but the indexing category for (co)equalizers. We claim that the inclusion $\iota : ([0] \rightrightarrows [1])^{\mathrm{op}} \hookrightarrow \Delta^{op}$ is final. Then it suffices to show that the comma category $([n], \downarrow, \iota)$ is empty and connected. This fact become obvious after the following observation:

- Objects of $([n], \downarrow, \iota)$ are point inclusions $[0] \to [n]$ and interval inclusions $[1] \to [n]$.
- Each interval inclusion $[1] \to [n]$ is connected with its endpoints inclusions $[0] \to [n]$;
- Any two point inclusions $f, g : [0] \to [n]$ are in the same connected components, since there is an internal $[1] \to [n]$ whose endpoints are f, g.

Final functors simplify the computation of colimits greatly.

Proposition

Let $\mathcal{C}$ be a category and $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. Then the colimit of the simplicial object $\varinjlim X$ is the coequalizer

$$\varinjlim X \cong \mathrm{coeq}(X_1 \rightrightarrows X_0)$$

In particular, the colimit of a simplicial set X is the set of its connected components $\pi_0(X)$.

Example

Let $(P, \leq)$ be a directed poset. That is, $(P, \leq)$ is poset with an additional property that $\forall a, b \in P$ there exists $c \in P$ such that $a \leq c$ and $b \leq c$. Let $Q \subset P$ be a subset which inherits a preorder from P, and Q satisfies the following condition:

$$(*) \quad \forall p \in P, \quad \exists q \in Q, \quad s.t.\ p \leq q.$$

Let $\iota : Q \hookrightarrow P$ be the inclusion. Then the comma category $(p, \downarrow, \iota)$ is non-empty and connected for all $p \in P$. (The above $*$-condition guarantees that the comma category is non-empty, and it is connected since P is a poset)

Here is a result which everyone uses every day. But when people use it, maybe they are not aware of the general theory behind it.

Proposition

Let $J : (P, \leq) \to \mathcal{C}$ *be a diagram indexed by a directed poset. Then for any sub-poset* $Q \subseteq P$ *subject to the* $*$*-condition, there is a canonical isomorphism*

$$\varinjlim_{q \in Q} J(q) \cong \varinjlim_{p \in P} J(p)$$

Application: Sifted categories

Definition

We say that a category $\mathcal{D}$ is **sifted** if any $\mathcal{D}$-shaped colimits in Set commute with finite products. More explicitly, let $F, G \in \mathrm{Fun}(\mathcal{D}, \mathrm{Set})$ be functors, then the canonical morphism

$$\varinjlim_{d \in \mathcal{D}} (F(d) \times G(d)) \longrightarrow (\varinjlim_{d \in \mathcal{D}} F(d)) \times (\varinjlim_{d \in \mathcal{D}} G(d))$$

is an isomorphism.

To understand this definition, lets recall some facts about filtered categories:

Definition

A category $\mathcal{C}$ is **filtered** if any finite diagram in $\mathcal{C}$ can be extended to a cone **under** it. More explicitly the condition translates into:

- for each pair of objects c, c' there is some c'' together with maps $c \to c''$ and $c \to c''$;
- for each pair of morphisms $c \rightrightarrows c'$ there is some $c' \to c''$ such that the two composites coincide.

An important and well-known fact about filtered categories is that, any colimit of filtered diagrams in Set commute with finite limits. A less well known fact is that, the converse of the statement is also true. That is, a category $\mathcal{C}$ is filtered if and only if any $\mathcal{C}$-colimit in Set commute with finite limits. As a result, sifted categories can be viewed as a kind of generalization of filtered categories.

Example

Every filtered category is sifted.

Example

The category Δ^{op} is a sifted category. To see this, it suffices to show that for two simplicial sets $X, Y \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$, there is a canonical isomorphism

$$\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y),$$

which is clear.

Theorem (Gabriel-Ulmer)

A category $\mathcal{D}$ is sifted if and only if the diagonal functor $D : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is final.

Example

Let $(\Delta_{\leq 1})^{\mathrm{op}}$ be the full subcategory of Δ^{op} spanned by $[0]$ and $[1]$. This category looks like

$$\begin{tikzcd} {[1]} \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & {[0]} \arrow[l, "s_0"] \end{tikzcd}$$

such that $d_0 \circ s_0 = d_1 \circ s_0 = \mathrm{Id}_{[0]}$. A colimit over $(\Delta_{\leq 1})^{\mathrm{op}}$ is called a **reflective coequalizer**. The category $(\Delta_{\leq 1})^{\mathrm{op}}$ is sifted: we need to show that $(([a], [b]) \downarrow, D)$ is non-empty and connected, and this can be done case by case since there are only four choices for the pair $([a], [b])$. As a result, reflective coequalizers commute with finite products in Set.

Generalization: Homotopy final functors

In the remaining part of the slides, I would like give a brief proof of Quillen's theorem A. To formulate the theorem, we need some preparations:

Definition

The **classifying space** of a category $\mathcal{C}$, which we denote by $B\mathcal{C}$, is the geometric realization of its nerve,

$$B\mathcal{C} := |N(\mathcal{C})|$$

Note that a category is connected if and only if its classifying space is connected.

Example

If $\mathcal{D}$ is a category with an initial object t, then $B\mathcal{D}$ is contractible. To see this let $\delta_t : \mathcal{D} \to \mathcal{D}$ be the constant functor at t, then there is a unique natural transformation $\delta_t \Rightarrow \mathrm{Id}_{\mathcal{D}}$. The data of such a natural transformation is equivalent to that of a functor $H : \mathcal{D} \times [1] \to \mathcal{D}$, whose restriction to $\mathcal{D} \times \{0\}$ is Δ_t and to $\mathcal{D} \times \{1\}$ is $\mathrm{Id}_{\mathcal{D}}$. By taking nerve we get $H : N(\mathcal{D}) \times \Delta^1 \to N(\mathcal{D})$. Then by taking geometric realization we get $B\mathcal{D} \times I \to B\mathcal{D}$ whose restriction to $B\mathcal{D} \times \{0\}$ is the constant map at the basepoint t and to $B\mathcal{D} \times \{1\}$ is the identity map. As a result, the unique map $B\mathcal{D} \to \{*\}$ is the homotopy inverse of the basepoint inclusion $t : \{*\} \to B\mathcal{D}$. Note that both nerve functor and geometric realization functor preserve finite products.

Definition

A functor $K : \mathcal{C} \to \mathcal{D}$ is **homotopy final** if the classifying space of the comma category $(d, \downarrow, K)$ is contractible, for all $d \in \mathcal{D}$.

The definition is justified by the following theorem:

Theorem

Let $F : \mathcal{D} \to \mathcal{M}$ be a functor taking value in a simplicial model category. If $K : \mathcal{C} \to \mathcal{D}$ is homotopy final, then the natural map $\varinjlim^h FK \to \varinjlim^h F$ is a weak equivalence. Here by $\varinjlim^h F$ I mean the homotopy colimit of F which I will define soon.

Theorem (Quillen's Theorem A)

If $K : \mathcal{C} \to \mathcal{D}$ is homotopy final, then the induced map $BK : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

Admitting the previous theorem, it takes just a few words to prove Quillen's theorem A. Before doing so, we need to say something about homotopy colimits.

Definition

Let $\mathcal{C}$ be a cocomplete category enriched, tensored, and cotensored over sSet. We use $-\otimes - : \mathrm{sSet} \times \mathcal{C} \to \mathcal{C}$ to denote the action functor. Let $F : I \to \mathcal{C}$ be a functor. Define the **simplicial replacement** $S(F)$ to be the following simplicial object in $\mathcal{C}$:

$$S(F)_n := \coprod_{i_0 \to \ldots \to i_n} F(i_0)$$

that is, $S(F)_n$ is a coproduct indexed by n-simplices of $N(I)$. The **homotopy colimit** of F is the geometric realization of $S(F)_\bullet$:

$$\varinjlim{}^h F := \int^{[n] \in \Delta^{\mathrm{op}}} \Delta^n \otimes S(F)_n$$

This is the famous **Bousfield-Kan formula** for homotopy colimits.

Remark

Note that the colimit of the simplicial object $S(F)_\bullet$ is nothing but the usual colimit of F. This can be shown as follows: by our previous discussions, the colimit of a simplicial object X can be computed by the coequalizer $\mathrm{coeq}(X_1 \rightrightarrows X_0)$, and

$$\mathrm{coeq}(S(F)_1 \rightrightarrows S(F)_0) = \mathrm{coeq}(\coprod_{f:i_0\to i_1} F(i_0) \rightrightarrows \coprod_{i\in I} F(i))$$

which is exactly the definition of colimit. As a result, there is a comparison map $\varinjlim^h(F) \to \varinjlim F$ defined by

$$\varinjlim{}^h(F) = \int^{[n]\in\Delta^{\mathrm{op}}} \Delta^n \otimes S(F)_n \to \int^{[n]\in\Delta^{\mathrm{op}}} * \otimes S(F)_n \cong \varinjlim F$$

Lemma

Let Top *be the simplicial model category of topological spaces. Let* $t : \mathcal{C} \to \mathrm{Top}$ *be the constant functor at the final object* t*. Then* $\varinjlim^h t \simeq B\mathcal{C}$

Proof.

We see that the simplicial replacement of t is nothing but the nerve of $\mathcal{C}$:

$$S(t)_n := \coprod_{i_0 \to \ldots \to i_n} t(i_0) = \coprod_{i_0 \to \ldots \to i_n} * \cong N(\mathcal{C})_n$$

$\square$

Proof of Quillen's theorem A.

Consider the diagram

$$\mathcal{C} \xrightarrow{K} \mathcal{D} \xrightarrow{t} \mathrm{Top}$$

Then the map $BK : B\mathcal{C} \to B\mathcal{D}$ is exactly the canonical map

$$B\mathcal{C} \simeq \varinjlim{}^h (t \circ K) \to \varinjlim{}^h t \simeq B\mathcal{D}$$

which is a weak equivalence since K is homotopy final. Note that a weak equivalence between CW-complexes is a homotopy equivalence. $\square$

Example

By our previous discussions, the category Δ^{op} is sifted. By Gabriel-Ulmer theorem, the diagonal functor $D^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is final. Here let us prove a stronger result: the diagonal functor D^{op} is homotopy final. We'll show that $D : \Delta \to \Delta \times \Delta$ is homotopy initial, i.e. $(D, \downarrow, ([p], [q]))$ is contractible for arbitrary choices of p, q. Define the following two endofunctors functors of $(D, \downarrow, ([p], [q]))$

- $S : (\alpha : [k] \to [p], \beta : [k] \to [q]) \mapsto (S\alpha : [k+1] \to [p], S\beta : [k+1] \to [q])$ by setting $S\alpha(0) = 0$ and $S\alpha(i) = \alpha(i-1)$ for $i > 1$.
- E is the constant functor at $([0] \xrightarrow{0} [p], [0] \xrightarrow{0} [q])$.

Then there is a natural $\theta : E \Rightarrow S$ given by inclusion of 0. Moreover, there is a natural transformation $d^0 : \mathrm{Id} \Rightarrow S$ given by $d^0 : [k] \to [k+1]$. After taking classifying space, we get a homotopy between the constant map E and the identity map.