Homework 3

Discrete Random Sequences

Name: Jinhan Zhang

Class: ECE 251A Digital Signal Processing I

Date: 01/30/2017

Discrete Random Sequences

• Objective

Use Matlab to analyze discrete-time random signals, generate and plot the autocorrelation sequence estimate and analyze the cross-correlation sequence estimate between the original signal and the one after low-pass filter.

• Background

In many real-world situations, there are processes which generate signal that is too complex to make precise description of the signal desirably and intuitively. In such cases, modeling the signal as a random process can be analytically useful. Thus we need to know how to analyze the discrete random sequences, use the result we compute from assumed probability laws to deduct the property of the signal.

• Approach

I. Random Numbers

We firstly generate two 1024-point sequences containing independent variables, one with uniform distribution in [0,1], the other zero-mean Gaussian distribution with variance to be 1. Then we plot the histogram corresponding to the sequences. Finally, we use N=256 to generate autocorrelation sequence estimate $c_{xx}(m)$ of the sequences above respectively in cases of $\mathrm{m} = 0,1,\ldots,15$, and compare the result with our theoretical calculation result.

II. Filtering

Firstly, we generate a low-pass filter which has impulse response like:

$$\mathrm{h(n)} = \begin{cases} 1, & \mathrm{n} = 0,1,\ldots,7 \\ 0, & otherwise \end{cases}$$

and plot its dB magnitude response to see its low-pass property. Then we pass our Gaussian sequence generated above through this filter and observe the input and output impulse response. Finally, we calculate the autocorrelation of the output and the cross-correlation between the input and the output, and compare them with our calculation result.

• Results

I. Random Numbers

Fig.1 shows the 1024-point uniform sequence; Fig.2 shows the 1024-point Gaussian sequence we generate. Fig.3 and Fig.4 show the histograms corresponding to uniform and Gaussian sequences respectively. We can see from them that these two histograms perform distributions of uniform and Gauss clearly. Fig.5 shows the autocorrelation sequence estimate $c_{xx}(m)$ of uniform sequence for $\mathrm{m} = 0,1,\ldots,15$. We can see from the plot that when

$\mathrm{m} = 0$ $c_{xx}(m)$ is around 0.33 while when $0 < \mathrm{m} \le 15$, $c_{xx}(m)$ is around 0.25. Fig.6 shows the autocorrelation sequence estimate $c_{xx}(m)$ of Gaussian sequence for $\mathrm{m} = 0,1,\ldots,15$. We can see from the plot that when $\mathrm{m} = 0$, $c_{xx}(m)$ is around 1 while when $0 < \mathrm{m} \le 15$, $c_{xx}(m)$ is around 0. Theoretical calculation is shown as below:

For uniform sequence:

1) m=0:

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-1} E[x^2(n)]$$

Because $E[X]$=$(E[X])^2 + Var[X]$, so we have

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-1}\{(E[X])^2 + Var[X]\}$$

For uniform sequence distributed in [0,1], we have $E[X] = \frac{1}{2}$ and $Var[X] = \frac{1}{12}$, so

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{3}$$

2) $\mathrm{m} > 0$:

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-m-1} E[x(n)x(n+m)]$$

Because $\mathrm{x(n)}$ and $x(n+m)$ are independent when $\mathrm{m} \ne 0$, so

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-m-1} E[x(n)]E[x(n+m)]$$

For uniform sequence distributed in [0,1], we have $E[X] = \frac{1}{2}$, ignore the bias resulting from 1/N then we have:

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{4}$$

We can see that the result in Fig.5 matches the result above.

For Gaussian sequence:

1) m=0:

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-1} E[x^2(n)]$$

Similarly, we have

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-1} \{(E[X])^2 + Var[X]\}$$

For Gaussian sequence with zero-mean and one-variance, we have $E[X] = 0$ and $Var[X] = 1$, so

$$\mathrm{E}[c_{xx}(\mathrm{m})] = 1$$

2) $\mathrm{m} > 0$:

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-m-1} E[x(n)x(n+m)]$$

Because $\mathrm{x(n)}$ and $x(n+m)$ are independent when $\mathrm{m} \neq 0$, so

$$\mathrm{E}[c_{xx}(\mathrm{m})] = \frac{1}{N}\sum_{n=0}^{N-m-1} E[x(n)]E[x(n+m)]$$

For Gaussian sequence with zero-mean and one-variance, we have

$E[X] = 0$, so

$$\mathrm{E}[c_{xx}(\mathrm{m})] = 0$$

We can see that the result in Fig.6 also matches this result.

II. Filtering

Fig.7 and Fig.8 show the zeros/poles and dB magnitude response of the low-pass filter respectively. We can see that the low-pass filter we create has 7 zeros and 7 poles, and also a big gain at low frequencies. Fig.9 and Fig.10 are a piece of Gaussian sequence and its corresponding sequence after passing through the low-pass filter. We can see clearly that the output has been smoothed due to the low-pass filter. Fig.11 shows the autocorrelation sequence estimate $c_{yy}(m)$ of output sequence for $\mathrm{m} = 0,1,\dots,15$. We can see from the plot that $c_{yy}(m)$ decreases step by step for m=0 to 7, then stay around 0 when $\mathrm{m} \geq 8$. Fig.12 shows the cross-correlation sequence estimate $c_{xy}(m)$ between input and output for $\mathrm{m} = -15,\dots 0,\dots,15$. We can see from the plot that $c_{xy}(m)$ is around 1 for m=0 to 7, and around 0 otherwise. Theoretical calculation for the expected values of $c_{xx}(m)$ is shown as Part I. And calculation for the expected values of $c_{yy}(m)$ and $c_{xy}(m)$ is shown below.

For real sequence,

$$\emptyset_{yy}(m) = E[y(n)]E[y(n+m)]$$

$$\emptyset_{yy}(m) = E\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(k)h(l)x(n-k)x(n-l+m)\right]$$

$$\emptyset_{yy}(m) = \sum_{k=-\infty}^{\infty} h(k) \sum_{l=-\infty}^{\infty} h(l)\, E[X(n-k)X(n+m-l)]$$

Because x(n) is assumed to be stationary,

$$\emptyset_{yy}(m) = \sum_{k=-\infty}^{\infty} h(k) \sum_{l=-\infty}^{\infty} h(l)\, \emptyset_{xx}(m+k-l)$$

Assume $r = l - k$,

$$\emptyset_{yy}(m) = \sum_{r=-\infty}^{\infty} \emptyset_{xx}(m-r) \sum_{k=-\infty}^{\infty} h(k)h(r+k)$$

For Gaussian sequence, $\emptyset_{xx}(m-r)$ is 1 only when r=m, and 0 otherwise. So

$$\emptyset_{yy}(m) = 1 \times \sum_{k=-\infty}^{\infty} h(k)h(m+k)$$

Thus the expected values of $c_{yy}(m)$ (ignore the bias resulting from 1/N in the expression) is $8 - m$ for $m = 0,1,\dots,7$ and 0 for $m = 8,9,\dots,15$, which matches the result in Fig. 11.

For real sequence,

$$\emptyset_{xy}(m) = E[x(n)]E[y(n+m)]$$

$$\emptyset_{xy}(m) = E\left[x(n) \sum_{k=-\infty}^{\infty} h(k)x(n+m-k)\right]$$

$$\emptyset_{xy}(m) = \sum_{k=-\infty}^{\infty} h(k)\, \emptyset_{xx}(m-k)$$

For Gaussian sequence, $\emptyset_{xx}(m-k)$ is 1 only when k=m, and 0 otherwise. So

$$\emptyset_{xy}(m) = h(m)$$

Thus the expected values of $c_{xy}(m)$ (ignore the bias resulting from 1/N in the expression) is 1 for $\mathrm{m} = 0,1,\dots,7$ and 0 for $\mathrm{m} = -15,\dots,-1$ and $8,\dots,15$ which matches the result in Fig. 12.

- **Summary**

In this assignment, we have generated discrete random sequences and analyze their properties by calculating histogram, autocorrelation and cross-correlation between original signal and signal after low-pass filter. We see how low-pass filter work on discrete random sequence and computation of the expected values of $c_{xx}(m)$, $c_{yy}(m)$ and $c_{xy}(m)$.

- **Plots**

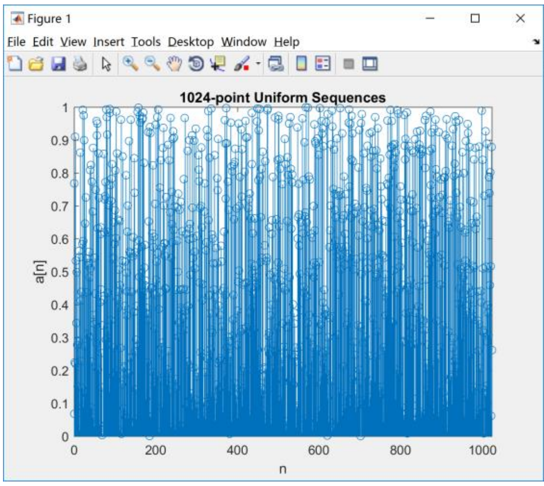

Fig. 1

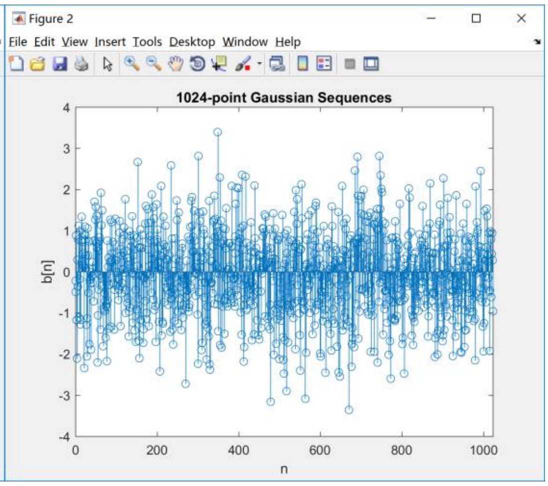

Fig. 2

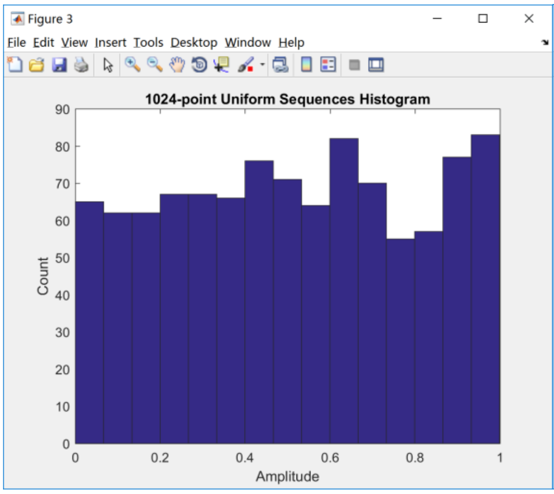

Fig. 3

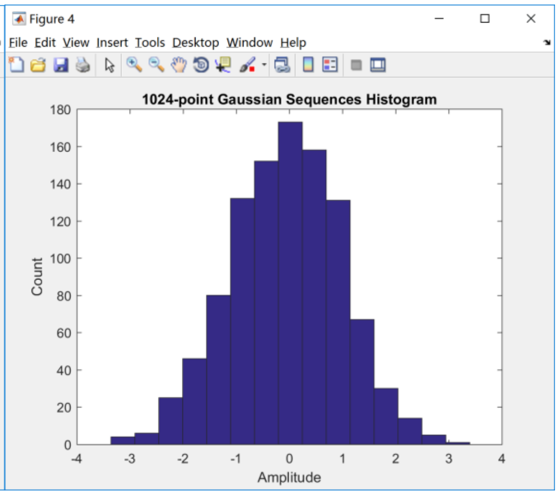

Fig. 4

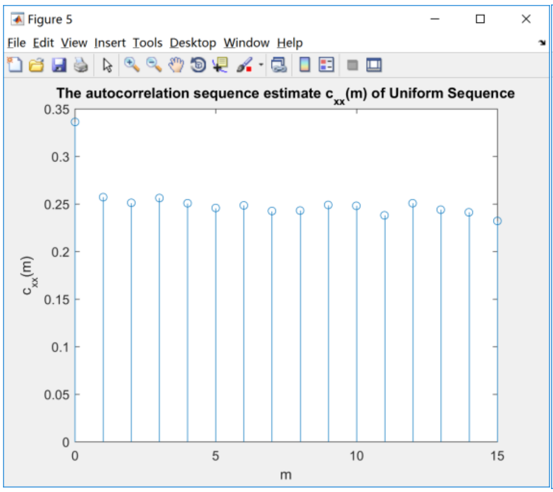

Fig. 5

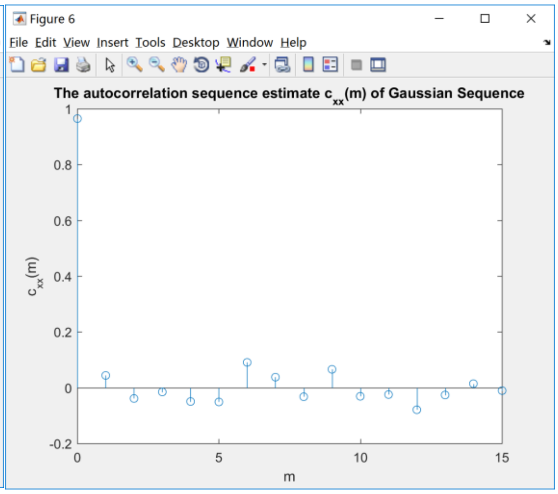

Fig. 6

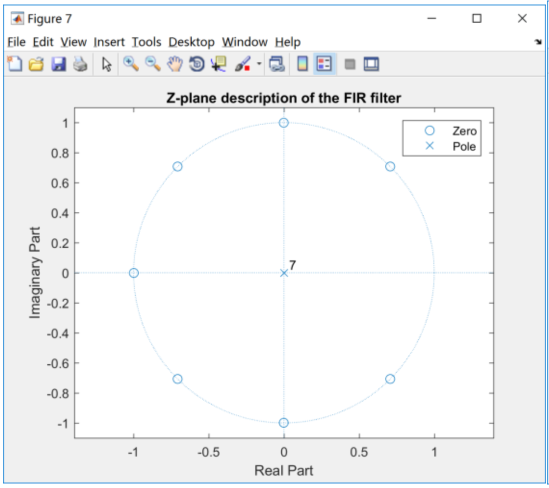

Fig. 7

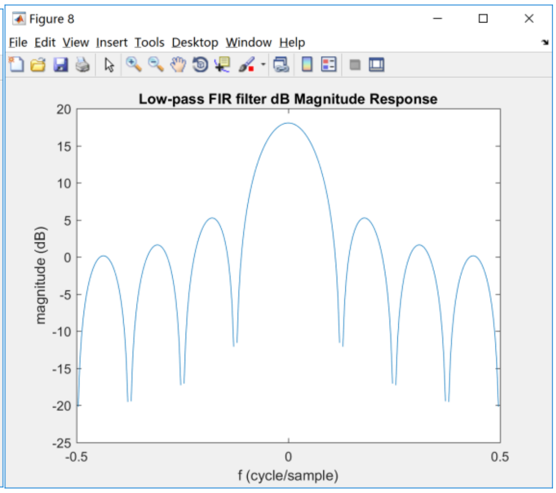

Fig. 8

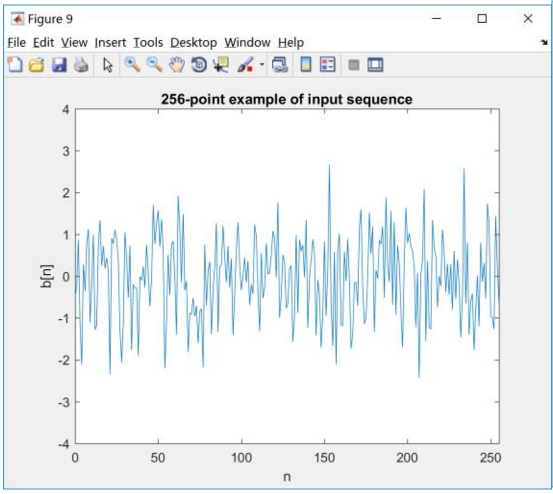

Fig. 9

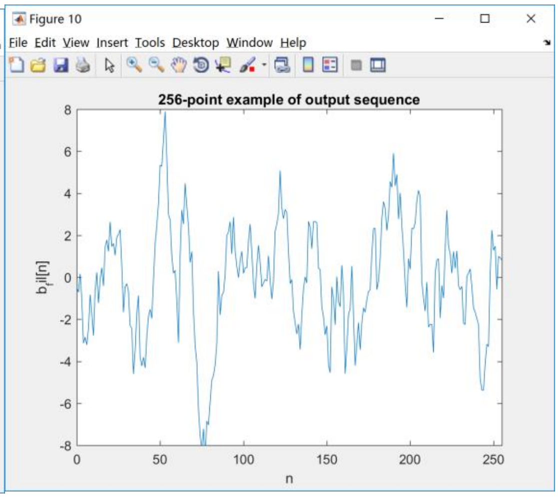

Fig. 10

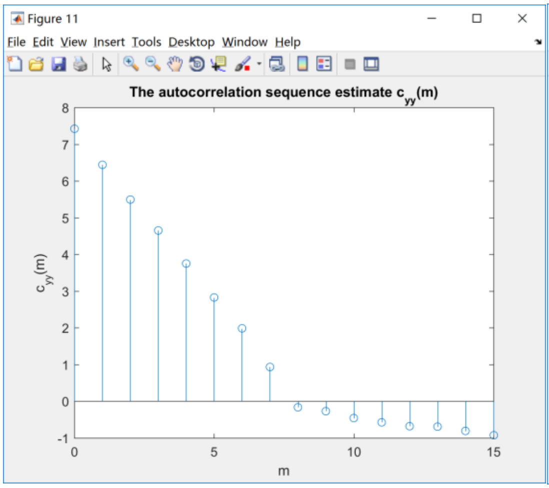

Fig. 11

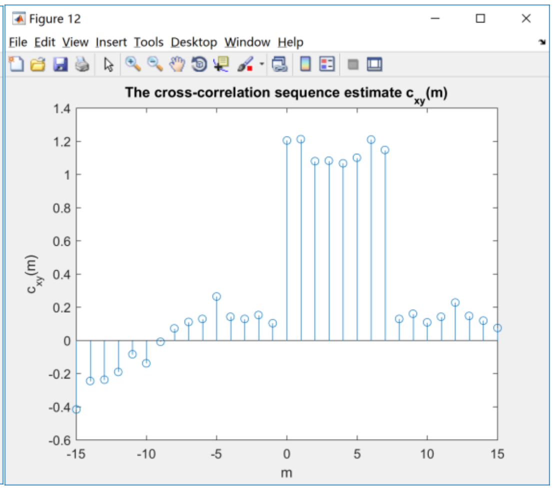

Fig. 12

- Appendix

Script:

```
close all;clc;clear;
%% IA
N=1024;
a=rand(1,N);
b=randn(1,N);
figure(1);
stem(0:N-1,a);
title('1024-point Uniform Sequences');
xlabel('n');ylabel('a[n]');
axis([0 1023 0 1]);
figure(2);
stem(0:N-1,b);
title('1024-point Gaussian Sequences');
xlabel('n');ylabel('b[n]');
axis([0 1023 -4 4]);
%% IB
figure(3);
hist(a,15);
title('1024-point Uniform Sequences Histogram');
xlabel('Amplitude');ylabel('Count');
figure(4);
hist(b,15);
title('1024-point Gaussian Sequences Histogram');
xlabel('Amplitude');ylabel('Count');
%% IC
m=15;
figure(5);
a_auto=xcorr(a(1:256),m,'biased');
a_auto=a_auto(m+1:2*m+1);
stem(0:m,a_auto);
title('The autocorrelation sequence estimate c_{xx}(m)
of Uniform Sequence');
xlabel('m');ylabel('c_{xx}(m)');

figure(6);
b_auto=xcorr(b(1:256),m,'biased');
b_auto=b_auto(m+1:2*m+1);
stem(0:m,b_auto);
title('The autocorrelation sequence estimate c_{xx}(m)
of Gaussian Sequence');
```

```
xlabel('m');ylabel('c_{xx}(m)');

%% IIA
lpf=ones(1,8);
figure(7);
zplane(lpf);
legend('Zero','Pole'); title('Z-plane description of the
FIR filter');

figure(8);
NN=256;
LPF=fftshift(fft(lpf,NN));
plot([-0.5:1/NN:0.5-1/NN],20*log10(abs(LPF)));
title('Low-pass FIR filter dB Magnitude Response');
xlabel('f (cycle/sample)');ylabel('magnitude (dB)');
%% IIB
b_fil=conv(b,lpf);
figure(9);
plot(0:NN-1,b(1:NN));
title('256-point example of input sequence');
xlabel('n');ylabel('b[n]');
axis([0 255 -4 4]);

figure(10);
plot(0:NN-1,b_fil(1:NN));
title('256-point example of output sequence');
xlabel('n');ylabel('b_fil[n]');
axis([0 255 -8 8]);

figure(11);
bfil_auto=xcorr(b_fil(1:NN),m,'biased');
bfil_auto=bfil_auto(m+1:2*m+1);
stem(0:m,bfil_auto);
title('The autocorrelation sequence estimate
c_{yy}(m)');
xlabel('m');ylabel('c_{yy}(m)');

figure(12);
bxy=xcorr(b_fil(1:NN),b(1:NN),m,'biased');
stem(-m:m,bxy);
title('The cross-correlation sequence estimate
c_{xy}(m)');
xlabel('m');ylabel('c_{xy}(m)');
```