Distorting filter: $h[n] = \delta[n] - \frac{1}{2}\delta[n-n_0]$

(a) The Z transform of $h[n]$

$$
\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\
H(z) &= 1 - \frac{1}{2}z^{-n_0}
\end{aligned}
$$

The N-pt DFT of $h[n]$: $(N = 4n_0)$

$$
\begin{aligned}
H[k] &= \sum_{n=0}^{4n_0-1} h[n]W_{4n}^{kn_0}, \quad 0 \le k \le (4n_0 - 1) \\
&= 1 - \frac{1}{2}W_{4n_0}^{kn_0} \\
H[k] &= 1 - \frac{1}{2}e^{-j(\pi/2)k}
\end{aligned}
$$

(b)

$$
\begin{aligned}
H_i(z) &= \frac{1}{1-(1/2)z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{(1/n_0)} \text{ for causality} \\
h_i[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta[n - kn_0]
\end{aligned}
$$

The filter is IIR.

(c)

$$
G[k] = \frac{1}{H[k]} = \frac{1}{1-\frac{1}{2}e^{-j(\pi/2)k}}, \quad 0 \le k \le (4n_0 - 1)
$$

The impulse response, $g[n]$, is just $h_i[n]$ time-aliased by $4n_0$ points:

$$
\begin{aligned}
g[n] &= \left(1 + \frac{1}{16} + \frac{1}{256} + \cdots\right)\delta[n] + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \cdots\right)\delta[n-n_0] \\
&+ \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \cdots\right)\delta[n-2n_0] + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \cdots\right)\delta[n-3n_0] \\
g[n] &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n-n_0] + \frac{4}{15}\delta[n-2n_0] + \frac{2}{15}\delta[n-3n_0]
\end{aligned}
$$

(d) Indeed,

$$
G[k]H[k] = 1, \quad 0 \le k \le (4n_0 - 1)
$$

However, this relationship is only true at $4n_0$ distinct frequencies. This fact does not imply that for all ω:

$$
G(e^{j\omega})H(e^{j\omega}) = 1
$$

(e)

$$
\begin{aligned}
y[n] &= g[n] * h[n] \\
&= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n-n_0] + \frac{4}{15}\delta[n-2n_0] + \frac{2}{15}\delta[n-3n_0] - \frac{8}{15}\delta[n-n_0] \\
&\quad - \frac{4}{15}\delta[n-2n_0] - \frac{2}{15}\delta[n-3n_0] - \frac{1}{15}\delta[n-4n_0] \\
y[n] &= \frac{16}{15}\delta[n] - \frac{1}{15}\delta[n-4n_0]
\end{aligned}
$$