Hilbert Transforms

Hilbert Transform Relations for Complex Sequences

- consider complex sequences when the real and imaginary parts can be related through a Hilbert transform

  "Causality" will mean the periodic frequency domain will be zero in the second half of each period

- useful in the representation of bandpass signals as complex signals

  correspond to "analytic signal" in analog signal theory

  complex sequence: $s(n) = s_r(n) + j\, s_i(n) \longrightarrow S(e^{j\omega})$

$$s_r(n) \longrightarrow S_r(e^{j\omega}) = \frac{1}{2}\left[S(e^{j\omega}) + S^*(e^{-j\omega})\right]$$

conjugate even

$$j\, s_i(n) \longrightarrow j\, S_i(e^{j\omega}) = \frac{1}{2}\left[S(e^{j\omega}) - S^*(e^{-j\omega})\right]$$

conjugate odd

complex quantities

Hilbert Transforms

Assume: $S(e^{j\omega}) = 0 \quad -\pi \le \omega < 0$ ("causality")

Thus: $$S(e^{j\omega}) = \begin{cases} 2\,S_r(e^{j\omega}) & 0 \le \omega < \pi \\ 0 & -\pi \le \omega < 0 \end{cases}$$

and $$S(e^{j\omega}) = \begin{cases} 2j\,S_i(e^{j\omega}) & 0 \le \omega < \pi \\ 0 & -\pi \le \omega < 0 \end{cases}$$

Also $$S_i(e^{j\omega}) = \begin{cases} -j\,S_r(e^{j\omega}) & 0 \le \omega < \pi \\ j\,S_r(e^{j\omega}) & -\pi \le \omega < 0 \end{cases}$$

$e^{-j\pi/2} S_r(e^{j\omega})$

$S_r(n)$ and $S_i(n)$ are real sequences $\longrightarrow$ Fourier transforms have complex conjugate symmetry

or $S_i(e^{j\omega}) = H(e^{j\omega})\, S_r(e^{j\omega})$

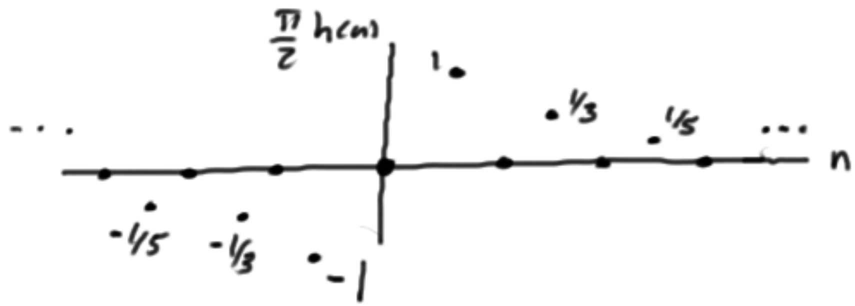
where: $$H(e^{j\omega}) = \begin{cases} -j & 0 \le \omega < \pi \\ j & -\pi \le \omega < 0 \end{cases}$$

$90°$ phase shifter or Hilbert transformer

Hilbert Transforms

Impulse response of $H(e^{j\omega})$

$$h(n) = \begin{cases} \dfrac{2}{\pi} \dfrac{\sin^2\left(\frac{\pi n}{2}\right)}{n} & n \neq 0 \\ 0 & n = 0 \end{cases}$$



For practical use, design a finite length Hilbert transformer using a FIR filter design routine

e.g. "firpm"

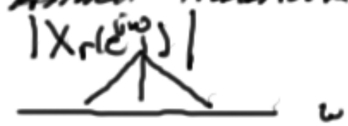
Hilbert Transforms

Representation of Bandpass Signals in terms of Complex Lowpass Signals

Complex lowpass signal

$$x(n) = x_r(n) + j\,x_i(n)$$ where $x_i(n)$ is the Hilbert transform of $x_r(n)$

$$= A(n)\, e^{j\phi(n)}$$ complex envelope



thus $$X(e^{j\omega}) = 0 \qquad -\pi \le \omega < 0$$

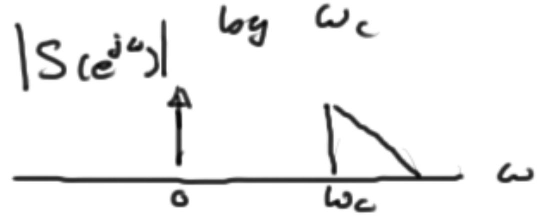
$$s(n) = x(n)\, e^{j\omega_c n} = s_r(n) + j\,s_i(n)$$ } mult by a^n property, $a = e^{j\omega_c}$

$$= A(n)\, e^{j(\omega_c n + \phi(n))}$$

Thus $$S(e^{j\omega}) = X\left(e^{j(\omega - \omega_c)}\right)$$ rotation of z-plane counter clockwise by ω_c

$x(n)$ is complex

$\longrightarrow$



$s(n)$ is complex

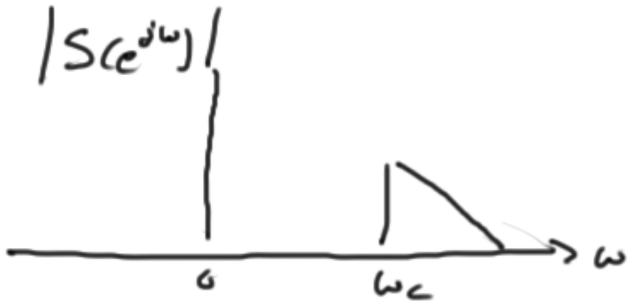
Hilbert Transforms

$$s_r(n) = \mathrm{Re}\{s(n)\} = x_r(n)\cos\omega_c n \underset{+}{-} x_i(n)\sin\omega_c n$$

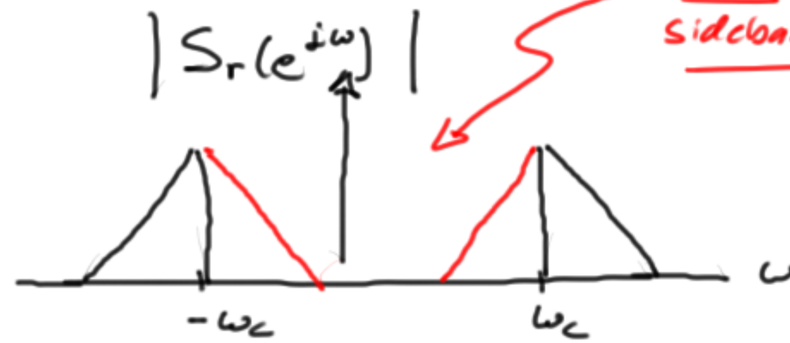
upper sideband

lower sideband

$$= A(n)\cos(\omega_c n + \phi(n))$$

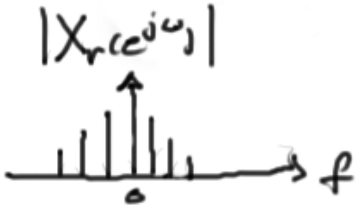


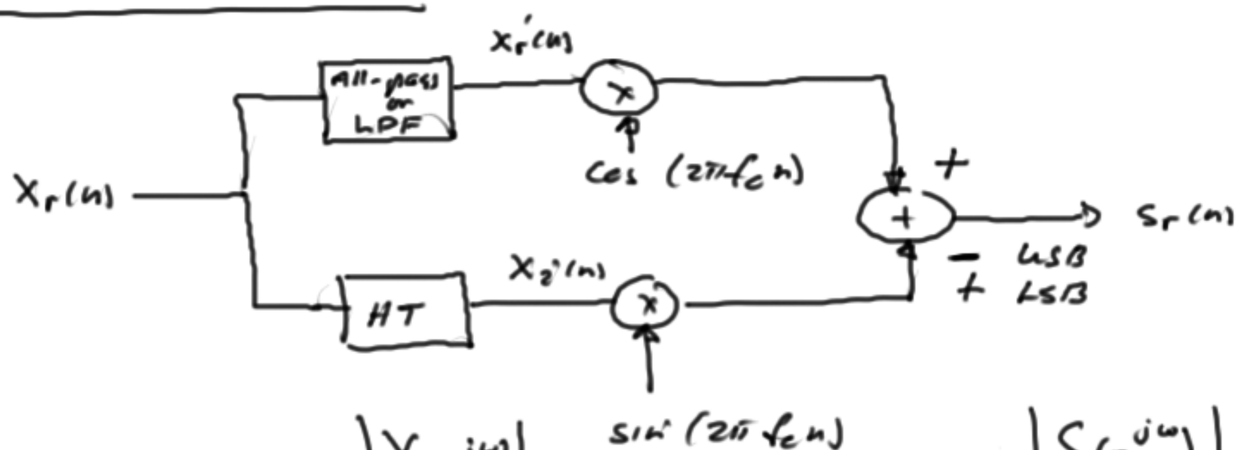
$\Longrightarrow$

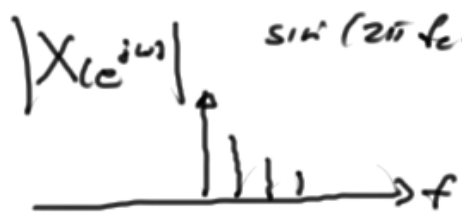


Hilbert Transforms

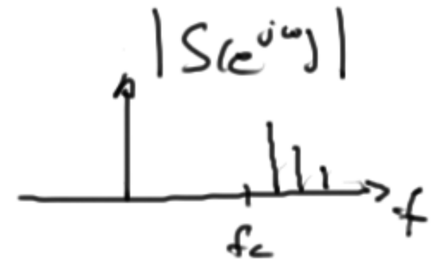
HW #2 - Single Sideband Generation





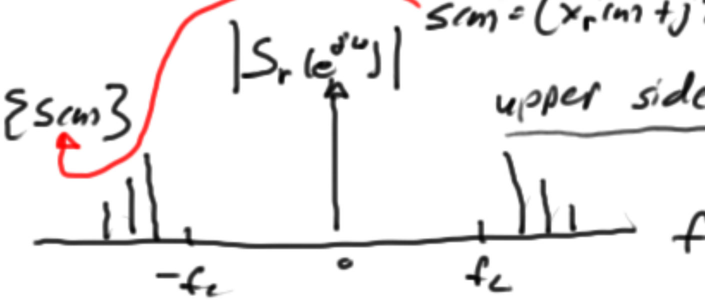


$x(n) = x_r'(n) + j\,x_i'(n)$



$s(n) = (x_r'(n) + j\,x_i'(n))\,e^{j2\pi f_c n}$

$s_r(n) = \mathrm{Re}\{s(n)\}$



upper side band