Coherence and Transfer Function Estimation

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Coherence and Transfer Function Estimation

System Example

I. Functions of interest

A. Transfer Function - The relationship between x(n) and y(n)

B. Coherence Function - The degree of causality between x(n) and y(n)

Coherence and Transfer Function Estimation

II. Components and Their Definitions

A. Auto-Power Spectra

conventional power spectral estimation

$$G_{xx}(\omega) \rightarrow \overline{\hat{G}_{xx}(k)} = \overline{X(k)\, X^*(k)}$$

where $X(k) =$ DFT $\{x(n)\}$

$$G_{yy}(\omega) \rightarrow \overline{\hat{G}_{yy}(k)} = \overline{Y(k)\, Y^*(k)}$$

where $Y(k) =$ DFT $\{y(n)\}$

B. Cross-Power Spectra

$$G_{yx}(\omega) \rightarrow \overline{\hat{G}_{yx}(k)} = \overline{Y(k)\, X^*(k)}$$

$$G_{yx}(\omega) = H(\omega)\, G_{xx}(\omega) + G_{nx}(\omega)$$

$$= H(\omega)\, G_{xx}(\omega) \quad \text{assume } G_{nx}(\omega) = 0$$

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$$\overline{\hat{G}_{yx}(\omega)} = \overline{Y(k)\,X^*(k)}$$

$$= \overline{[H(k)\,X(k) + N(k)]\,X^*(k)}$$

$$= H(k)\,\overline{X(k)\,X^*(k)} + \overline{N(k)\,X^*(k)}$$

$$= H(k)\,\overline{\hat{G}_{xx}(k)} + \overline{N(k)\,X^*(k)}$$

Not zero on a single estimate basis

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Cross-Correlation / Normalized Cross-Correlation

$$R_{yx}(n) = h(n) * R_{xx}(n) + R_{nx}(n)$$

$$= h(n) * R_{xx}(n) \quad \text{when } R_{nx}(n) = 0$$

$$\hat{R}_{yx}(n) = h(n) * \hat{R}_{xx}(n) + \hat{R}_{nx}(n)$$

not zero when sample size is small

$$\rho_{yx}(n) = \frac{R_{yx}(n)}{\left(R_{yy}(0)\, R_{xx}(0)\right)^{1/2}}$$

Note $|\rho_{yx}(n)| \leq 1$

$$\hat{\rho}_{yx}(n) = \frac{\hat{R}_{yx}(n)}{\left(\hat{R}_{yy}(0)\, \hat{R}_{xx}(0)\right)^{1/2}}$$

Note significance of $\hat{\rho}_{yx}(n)$ not equal to 0 or ± 1 is not clear

Coherence and Transfer Function Estimation

Transfer Function Estimation

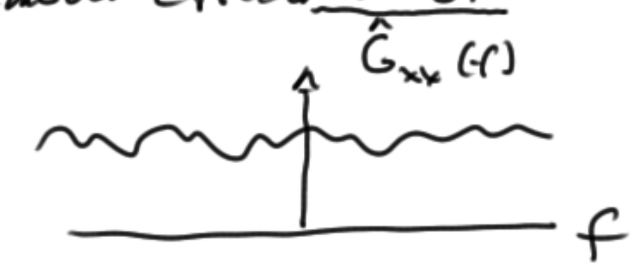
$$H(\omega) = \frac{G_{yx}(\omega)}{G_{xx}(\omega)}$$

$$\hat{H}(\omega) = \frac{\overline{\hat{G}_{yx}(\omega)}}{\overline{\hat{G}_{xx}(\omega)}} = \frac{H(k)\,\overline{\hat{G}_{xx}(k)} + \overline{N(k)X^*(k)}}{\overline{\hat{G}_{xx}(k)}}$$

Note: (1) removes $N(k)X^*(k)$ component by averaging

(2) removes effects of non-white $G_{xx}(\omega)$ (determinative)

(3) removes effects of statistical fluctuation in $\overline{\hat{G}_{xx}(k)}$



Coherence and Transfer Function Estimation

Coherence Function Estimation

$\overline{\hat{G}_{yx}(k)}$ will not answer question of causality between $x(n)$ and $y(n)$ since $\overline{N(k)X^*(k)} \simeq 0$

magnitude squared coherence

$$\gamma^2(\omega) = \frac{|G_{yx}(\omega)|^2}{G_{xx}(\omega)\,G_{yy}(\omega)} = \frac{|H(\omega)|^2 G_{xx}(\omega)}{|H(\omega)|^2 G_{xx}(\omega) + G_{nn}(\omega)}$$

(assuming $G_{nx}(\omega)=0$)

<u>Note</u> $0 \le \gamma^2(\omega) \le 1$

$$\hat{\gamma}^2(\omega) = \frac{\left|\overline{\hat{G}_{yx}(k)}\right|^2}{\overline{\hat{G}_{xx}(k)}\;\overline{\hat{G}_{yy}(k)}}$$

Coherence and Transfer Function Estimation

Power Spectrum at Output

Due to system input (i.e. $x(n)$) $\qquad \gamma^2(\omega)\, G_{yy}(\omega)$

"coherent output power"

Due to additive noise (i.e. $n(n)$) $\qquad (1-\gamma^2(\omega))\, G_{yy}(\omega)$