Conventional Power Spectral Estimation

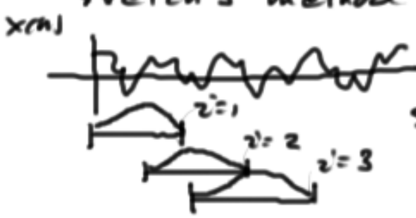
Normalization Factors for Broadband Power Spectra

$$P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m)\, e^{-j\omega m}$$

Refs: Oppenheim & Schafer (2010)
Ch. 10.2 Window functions
Ch. 10.5 Periodogram

$$\sigma^2 = \phi_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\, d\omega = \int_{-0.5}^{0.5} P_{xx}(f)\, df$$

Welch's method of windowed and overlapped FFTs



50% overlapped segments

$$\hat{P}_{xx}(f) = \frac{1}{f_s M U}\, \overline{|X(k)|^2}$$

Power/Hz

f_s = sample rate

$$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2(n)$$

$$X^{(i)}(k) = \sum_{n=0}^{M-1} w(n)\, x^{(i)}(n)\, e^{-j\left(\frac{2\pi}{M}\right)nk}$$

M = FFT length

$$\omega_k = 2\pi f_k = \left(\frac{2\pi}{M}\right)k$$

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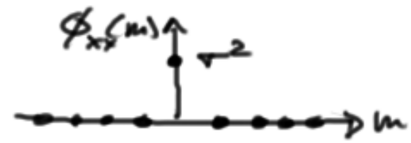
Periodogram

$$I_N(\omega) = \frac{1}{N}\left|X(e^{j\omega})\right|^2 = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)\, e^{-j\omega m}$$

$$\phi_{xx}(m) = \begin{cases} \sigma^2, & m=0 \\ 0, & m\neq 0 \end{cases}$$

Consider uncorrelated Gaussian noise $x(n) \sim N(0,\sigma^2)$ $\longrightarrow$

$$E[I_N(\omega)] = E\left[\frac{1}{N}\left|X(e^{j\omega})\right|^2\right] = \sum_{m=-(N-1)}^{N-1} E[c_{xy}(m)]\, e^{-j\omega m} = \sigma^2$$



$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$\left|X(e^{j\omega})\right|^2 = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x(n)\, x(m)\, e^{-j\omega n}\, e^{j\omega m}$$

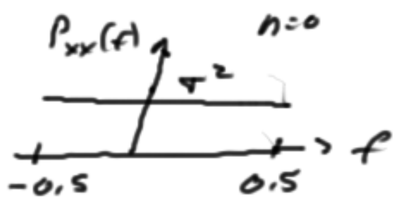
$$E\left[\left|X(e^{j\omega})\right|^2\right] = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E[x(n)\, x(m)]\, e^{-j\omega n}\, e^{j\omega m}$$

$$= \sum_{n=0}^{N-1} \sigma^2 = N\sigma^2$$

If a window w(n) had been used, this factor would be $\sum_{n=0}^{N-1} w^2(n)$

$$E[x^2(n)] = \sigma^2 \quad n=m$$

$$E[x(n)]\,E[x(m)] = 0 \quad n\neq m$$



Thus $\int_{-0.5}^{0.5} P_{xx}(f)\,df = \int_{-0.5}^{0.5} \sigma^2\, df = \sigma^2$

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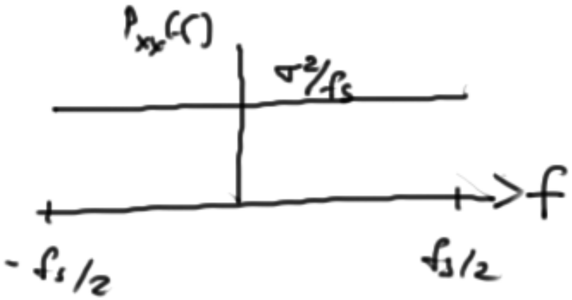

thus $\int_{-f_s/2}^{f_s/2} P_{xx}(f)\,df = \int_{-f_s/2}^{f_s/2} \frac{\sigma^2}{f_s}\,df = \sigma^2$

$$\hat{P}_{xx}(f) = \frac{1}{f_s M U} \overline{|X(k)|^2} \qquad \omega_k = 2\pi f_k \left(\frac{2\pi}{M}\right) k$$

"Power"/Hz

↑

volts^2

(A/D counts)^2

(physical units)^2

U is the normalization corrects for the distortion (tapering) of the segment time series by the window function

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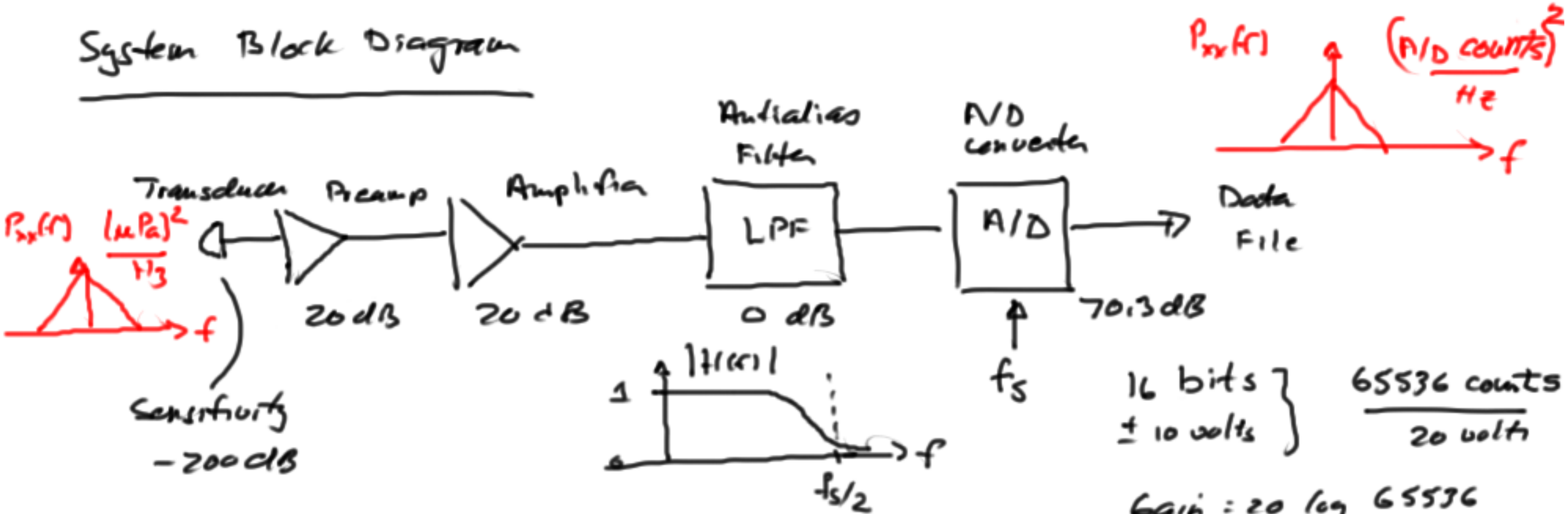
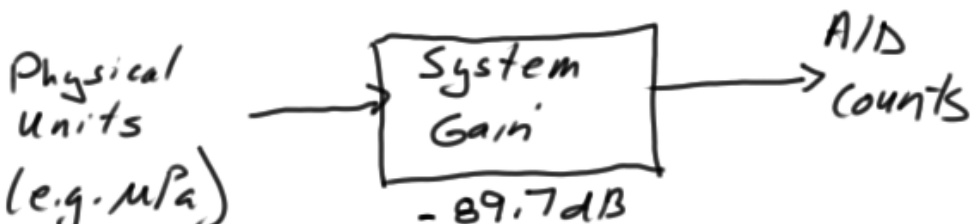
System Block Diagram



System Gain $= -200\,dB + 20\,dB + 20\,dB + 70.3\,dB$
$= -89.7\,dB$



Thus, to go from $P_{xx}(f)$ in $(A/D\ counts)^2/Hz$ to $P_{xx}(f)$ in $(\mu Pa)^2/Hz$ must add 89.7 dB