Homework 8

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Problem 1

Solution. **(i)** Use coordinates $u = (u^1, \dots, u^n)$ on U and the parametrization

$$X(u) = (u^1, \dots, u^n, f(u)).$$

Then the coordinate tangent vectors are

$$X_i := \frac{\partial X}{\partial u^i} = e_i + f_i e_{n+1}, \qquad f_i := \frac{\partial f}{\partial u^i}.$$

The induced metric is

$$g_{ij} = \langle X_i, X_j \rangle = \delta_{ij} + f_i f_j.$$

A normal vector is $(-f_1, \dots, -f_n, 1)$, whose length is

$$W := \sqrt{1 + \sum_{k=1}^{n} f_k^2}.$$

So the upward unit normal is

$$N = \frac{1}{W}(-f_1, \dots, -f_n, 1).$$

The second fundamental form is, by Lee, Prop. 8.23

$$h(X_i, X_j) = \left\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, N \right\rangle.$$

We have

$$\frac{\partial^2 X}{\partial u^i \partial u^j} = f_{ij} e_{n+1}, \qquad f_{ij} := \frac{\partial^2 f}{\partial u^i \partial u^j},$$

so

$$h_{ij} := h(X_i, X_j) = \langle f_{ij} e_{n+1}, N \rangle = f_{ij} N_{n+1} = \frac{f_{ij}}{W}.$$

By def of the shape operator s,

$$\langle sX_i, X_j \rangle = h(X_i, X_j),$$

so if $s(X_j) = s^k{}_j X_k$, then

$$h_{ij} = g_{ik} s^k{}_j.$$

Thus, the components of the shape operator in graph coordinates are

$$s^k{}_j = g^{ki} h_{ij} = \frac{1}{W}\, g^{ki} f_{ij},$$

where (g^{ij}) is the inverse matrix of $(g_{ij}) = (\delta_{ij} + f_i f_j)$.

(ii) Now let

$$f(x) = |x|^2 = \sum_{i=1}^{n} (x^i)^2.$$

Then

$$f_i = \frac{\partial f}{\partial x^i} = 2x^i, \qquad f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} = 2\delta_{ij}.$$

Hence

$$|\nabla f|^2 = \sum_i f_i^2 = 4|x|^2 = 4r^2, \quad r := |x|, \qquad W = \sqrt{1+4r^2}.$$

From (i),

$$h_{ij} = \frac{f_{ij}}{W} = \frac{2}{W}\, \delta_{ij}.$$

The metric in these coordinates is

$$g_{ij} = \delta_{ij} + f_i f_j = \delta_{ij} + 4x^i x^j,$$

which is of the matrix

$$G = I + 4xx^T.$$

To find principal curvatures, we should find the eigenvalues of s, which is of the matrix

$$S = (s^i{}_j) = G^{-1} H, \quad H = (h_{ij}) = \frac{2}{W} I.$$

So $S = \frac{2}{W} G^{-1}$, and its eigenvalues are $\frac{2}{W}$ times the eigenvalues of G^{-1}.

Choose an orthonormal basis in $\mathbb{R}^n$ so that

$$e_1 = \frac{x}{r}, \qquad e_2, \dots, e_n \perp x.$$

Then

$$Ge_1 = (1+4r^2)e_1, \qquad Ge_\alpha = e_\alpha \quad (\alpha = 2, \dots, n).$$

So the eigenvalues of G are $1+4r^2$ and 1. Thus the eigenvalues of G^{-1} are

$$\frac{1}{1+4r^2}, \qquad 1.$$

Therefore the principal curvatures at a point with $|x| = r > 0$ are

$$\kappa_{\mathrm{rad}}(r) = \frac{2}{W} \cdot \frac{1}{1+4r^2} = \frac{2}{(1+4r^2)^{3/2}},$$

in the direction of ∇f, and

$$\kappa_{\tan}(r) = \frac{2}{W} = \frac{2}{\sqrt{1+4r^2}}$$

in each of the $n-1$ directions orthogonal to ∇f.

At the vertex $x = 0$, we have $r = 0$, so $G(0) = I$, $h_{ij}(0) = 2\delta_{ij}$, hence

$$S(0) = 2I,$$

and all principal curvatures at $x = 0$ are

$$\kappa_1(0) = \cdots = \kappa_n(0) = 2.$$

$\square$

Problem 2

Proof. **(i)** Compute first derivatives:

$$X_t = (a' \cos\theta,\ a' \sin\theta,\ b'), \qquad X_\theta = (-a\sin\theta,\ a\cos\theta,\ 0).$$

$$\begin{aligned} E &= \langle X_t, X_t \rangle = (a')^2 + (b')^2 = 1, \\ F &= \langle X_t, X_\theta \rangle = 0, \\ G &= \langle X_\theta, X_\theta \rangle = a^2. \end{aligned}$$

So in the basis $\{X_t, X_\theta\}$,

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

and the coordinate lines $t = \text{const}$ (meridians) and $\theta = \text{const}$ (parallels) are orthogonal.

Compute

$$X_t \times X_\theta = (-ab' \cos\theta,\ -ab' \sin\theta,\ aa'),$$

whose length is $|X_t \times X_\theta| = a$. Thus a unit normal is

$$N = \frac{1}{a}(X_t \times X_\theta) = (-b' \cos\theta,\ -b' \sin\theta,\ a').$$

Second derivatives:

$$X_{tt} = (a'' \cos\theta,\ a'' \sin\theta,\ b''), \qquad X_{t\theta} = (-a' \sin\theta,\ a' \cos\theta,\ 0),$$

$$X_{\theta\theta} = (-a\cos\theta,\ -a\sin\theta,\ 0).$$

Then

$$\begin{aligned} e &:= h(X_t, X_t) = \langle X_{tt}, N \rangle = -a''b' + a'b'', \\ f &:= h(X_t, X_\theta) = \langle X_{t\theta}, N \rangle = 0, \\ g &:= h(X_\theta, X_\theta) = \langle X_{\theta\theta}, N \rangle = ab'. \end{aligned}$$

So, the matrices of the first and second fundamental forms in $\{X_t, X_\theta\}$ are

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}, \qquad \mathrm{II} = \begin{pmatrix} -a''b' + a'b'' & 0 \\ 0 & ab' \end{pmatrix}.$$

The shape operator S satisfies $\mathrm{II}(v,w)=\langle Sv,w\rangle$. In matrix form, $S=\mathrm{I}^{-1}\mathrm{II}$, so

$$S=\begin{pmatrix}1 & 0\\ 0 & a^{-2}\end{pmatrix}\begin{pmatrix}-a''b'+a'b'' & 0\\ 0 & ab'\end{pmatrix}=\begin{pmatrix}-a''b'+a'b'' & 0\\ 0 & \dfrac{b'}{a}\end{pmatrix}.$$

Thus,

$$S(X_t)=k_1X_t,\quad k_1(t)=-a''(t)b'(t)+a'(t)b''(t),$$
$$S(X_\theta)=k_2X_\theta,\quad k_2(t)=\frac{b'(t)}{a(t)}.$$

So, X_t and X_θ are eigenvectors of S. Therefore, the direction tangent to the meridian is a principal direction; the direction tangent to the latitude circle is a principal direction.

(ii) For a surface in $\mathbb{R}^3$, the Gaussian curvature is

$$K=\det S=k_1k_2.$$

From above,

$$k_1k_2=\big(-a''b'+a'b''\big)\cdot\frac{b'}{a}=\frac{b'}{a}\big(a'b''-b'a''\big).$$

Now use the unit-speed condition, we get

$$(a')^2+(b')^2=1.$$

Differentiate it, we have

$$2a'a''+2b'b''=0\implies a'a''+b'b''=0\implies b'b''=-a'a''.$$

Then

$$(a'b''-b'a'')\,b'=a'b''b'-b'^2a''=a'(-a'a'')-b'^2a''=-a'^2a''-b'^2a''=-(a'^2+b'^2)a''=-a''.$$

So

$$K=k_1k_2=\frac{(a'b''-b'a'')b'}{a}=\frac{-a''}{a}.$$

Thus, the Gaussian curvature at $X(t,\theta)$ is

$$K(t,\theta)=-\frac{a''(t)}{a(t)}.$$

$\square$

Problem 3

Proof. **(i)** Let a unit–speed curve likes in problem 2 such that

$$\gamma(t)=(a(t),b(t)),\qquad (a')^2+(b')^2=1,\ a(t)>0,$$

revolved around the z–axis gives a surface of revolution

$$X(t,\theta)=(a(t)\cos\theta,\ a(t)\sin\theta,\ b(t)).$$

From Problem 2 we know

$$k_1(t) = -\frac{a''(t)}{b'(t)}, \qquad k_2(t) = \frac{b'(t)}{a(t)}, \qquad K(t,\theta) = k_1(t)k_2(t) = -\frac{a''(t)}{a(t)}.$$

To get $K \equiv 1$, we need

$$-\frac{a''(t)}{a(t)} = 1 \quad \Longleftrightarrow \quad a''(t) + a(t) = 0.$$

For example, let

$$a(t) = 2\cos t,$$

and restrict to the interval $I = (-\pi/6, \pi/6)$ so that $a(t) > 0$. Define b so that γ is unit speed as

$$(a')^2 + (b')^2 = 1 \implies 4\sin^2 t + (b')^2 = 1 \implies b'(t) = \sqrt{1 - 4\sin^2 t},$$

which is smooth and positive on I. Let

$$b(t) = \int_0^t \sqrt{1 - 4\sin^2 s}\, ds.$$

Then $\gamma(t) = (a(t), b(t))$ is unit speed, $a > 0$, and the corresponding surface S is a surface of revolution. For this a we have $a''(t) = -2\cos t = -a(t)$, so

$$K(t,\theta) = -\frac{a''(t)}{a(t)} = 1$$

for all $(t,\theta) \in I \times (0, 2\pi)$. Now look at one principal curvature

$$k_2(t) = \frac{b'(t)}{a(t)} = \frac{\sqrt{1 - 4\sin^2 t}}{2\cos t}.$$

At $t = 0$ we get

$$k_2(0) = \frac{1}{2}.$$

As $t \to \pi/6$, we have $\sin^2 t \to 1/4$, so $b'(t) \to 0$ while $\cos(\pi/6) = \sqrt{3}/2 \neq 0$, hence

$$\lim_{t \to \pi/6} k_2(t) = 0.$$

Thus, k_2 is not constant, so the principal curvatures of S are not constant, even though $K \equiv 1$.

(ii) For $w > 0$, consider

$$\gamma(t) = (a(t), b(t)) = \big(w\cosh(t/w),\, t\big), \qquad t \in \mathbb{R},$$

and revolve it around the z–axis

$$X(t,\theta) = \big(a(t)\cos\theta,\ a(t)\sin\theta,\ b(t)\big).$$

Here, $a(t) = w\cosh(t/w)$, $b(t) = t$, so

$$a'(t) = \sinh(t/w), \qquad a''(t) = \frac{1}{w}\cosh(t/w), \qquad b'(t) = 1, \quad b''(t) = 0.$$

Computing first derivatives

$$X_t = (a' \cos\theta,\ a' \sin\theta,\ b'), \quad X_\theta = (-a\sin\theta,\ a\cos\theta,\ 0).$$

So, the first fundamental form is

$$E = \langle X_t, X_t \rangle = a'^2 + b'^2 = \sinh^2(t/w) + 1 = \cosh^2(t/w),$$

$$F = \langle X_t, X_\theta \rangle = 0, \qquad G = \langle X_\theta, X_\theta \rangle = a^2 = w^2 \cosh^2(t/w).$$

Then we have

$$X_t \times X_\theta = (-ab' \cos\theta,\ -ab' \sin\theta,\ aa') = (-a\cos\theta,\ -a\sin\theta,\ aa'),$$

so

$$|X_t \times X_\theta| = a\sqrt{E},$$

and a unit normal is

$$N = \frac{1}{a\sqrt{E}}(X_t \times X_\theta) = \frac{1}{\sqrt{E}}(-b'\cos\theta,\ -b'\sin\theta,\ a') = \frac{1}{\cosh(t/w)}(-\cos\theta,\ -\sin\theta,\ \sinh(t/w)).$$

Computing second derivatives

$$X_{tt} = (a'' \cos\theta,\ a'' \sin\theta,\ b''), \quad X_{t\theta} = (-a'\sin\theta,\ a'\cos\theta,\ 0), \quad X_{\theta\theta} = (-a\cos\theta,\ -a\sin\theta,\ 0).$$

Then the second fundamental form are

$$e = \langle X_{tt}, N \rangle = \frac{-a''b' + a'b''}{\sqrt{E}} = \frac{-a''}{\sqrt{E}},$$

$$f = \langle X_{t\theta}, N \rangle = 0,$$

$$g = \langle X_{\theta\theta}, N \rangle = \frac{ab'}{\sqrt{E}} = \frac{a}{\sqrt{E}}.$$

Plug for a, a'', E:

$$E = \cosh^2(t/w), \quad \sqrt{E} = \cosh(t/w), \quad a = w\cosh(t/w), \quad a'' = \frac{1}{w}\cosh(t/w).$$

Hence,

$$e = -\frac{a''}{\sqrt{E}} = -\frac{\frac{1}{w}\cosh(t/w)}{\cosh(t/w)} = -\frac{1}{w},$$

$$g = \frac{a}{\sqrt{E}} = \frac{w\cosh(t/w)}{\cosh(t/w)} = w.$$

So e and g are constants. Since $F = f = 0$, the shape operator matrix in the basis $\{X_t, X_\theta\}$ is

$$S = \mathrm{I}^{-1}\mathrm{II} = \begin{pmatrix} E^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} \dfrac{e}{E} & 0 \\ 0 & \dfrac{g}{G} \end{pmatrix}.$$

Thus, the principal curvatures are

$$k_1 = \frac{e}{E} = \frac{-1/w}{\cosh^2(t/w)} = -\frac{1}{w\cosh^2(t/w)},$$

$$k_2 = \frac{g}{G} = \frac{w}{w^2 \cosh^2(t/w)} = \frac{1}{w \cosh^2(t/w)}.$$

Therefore,

$$k_1 + k_2 = 0,$$

so the mean curvature

$$H = \frac{k_1 + k_2}{2} = 0.$$

This implies M_w is a minimal surface for every $w > 0$. $\square$

Problem 4

Proof. **(i)** Fix a point $p \in M$ and write $V = T_pM$ with inner product induced by g. For $w, x, y, z \in V$, define

$$B(w, x, y, z) := \langle w, y\rangle\langle x, z\rangle - \langle w, z\rangle\langle x, y\rangle.$$

For fixed y, z, this is alternating in (w, x), and for fixed w, x it is alternating in (y, z). By property of the exterior product in Lee's review, there exists a unique bilinear map

$$\langle\cdot,\cdot\rangle_{\Lambda^2} : \Lambda^2 V \times \Lambda^2 V \to \mathbb{R}$$

such that

$$\langle w \wedge x,\ y \wedge z\rangle_{\Lambda^2} = B(w, x, y, z) = \langle w, y\rangle\langle x, z\rangle - \langle w, z\rangle\langle x, y\rangle$$

for all $w, x, y, z \in V$.

We first check that $\langle\cdot,\cdot\rangle_{\Lambda^2}$ is an inner product. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Then

$$\langle e_i \wedge e_j,\ e_k \wedge e_\ell\rangle_{\Lambda^2} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}.$$

In particular, for $i < j$ and $k < \ell$, this is 1 if $(i, j) = (k, \ell)$ and 0 otherwise. Thus, the family

$$\{e_i \wedge e_j : 1 \le i < j \le n\}$$

is an orthonormal basis of $\Lambda^2 V$, so $\langle\cdot,\cdot\rangle_{\Lambda^2}$ is symmetric and positive definite. Hence, it is an inner product on $\Lambda^2 V$. Then we compute the associated norm on decomposable 2-vectors

$$|w \wedge x|^2 = \langle w \wedge x,\ w \wedge x\rangle_{\Lambda^2} = \langle w, w\rangle\langle x, x\rangle - \langle w, x\rangle^2 = |w|^2|x|^2 - \langle w, x\rangle^2.$$

This shows the existence.

For uniqueness, suppose $(\cdot,\cdot)'$ is any other fiber metric on $\Lambda^2 V$ whose associated norm satisfies

$$|w \wedge x|'^2 = |w|^2|x|^2 - \langle w, x\rangle^2 \quad \text{for all } w, x \in V.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. For $i < j$, we have

$$|e_i \wedge e_j|'^2 = |e_i|^2|e_j|^2 - \langle e_i, e_j\rangle^2 = 1,$$

so each $e_i \wedge e_j$ has norm 1 with respect to $(\cdot,\cdot)'$. To see that they are orthogonal, fix distinct i, k and some j. Take $w = e_i + e_k$ and $x = e_j$. Then

$$w \wedge x = (e_i + e_k) \wedge e_j = e_i \wedge e_j + e_k \wedge e_j.$$

On one hand, by the given norm formula and orthonormality of e_i, e_j, e_k,

$$|w \wedge x|'^2 = |w|^2|x|^2 - \langle w, x\rangle^2 = 2 \cdot 1 - 0 = 2.$$

On the other hand, expanding with the (unknown) inner product,

$$|w \wedge x|'^2 = |e_i \wedge e_j + e_k \wedge e_j|'^2 = |e_i \wedge e_j|'^2 + |e_k \wedge e_j|'^2 + 2(e_i \wedge e_j,\, e_k \wedge e_j)'.$$

Since each norm is 1, this is

$$2 = 1 + 1 + 2(e_i \wedge e_j,\, e_k \wedge e_j)' \quad \Longrightarrow \quad (e_i \wedge e_j,\, e_k \wedge e_j)' = 0.$$

Similar choices of w, x show that $(e_i \wedge e_j,\, e_k \wedge e_\ell)' = 0$ whenever $(i, j) \neq (k, \ell)$ with $i < j$, $k < \ell$. Thus, $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis for $(\cdot, \cdot)'$ as well. But on a finite-dimensional vector space, an inner product is uniquely determined by declaring a basis to be orthonormal. Hence, $(\cdot, \cdot)'$ coincides with $\langle \cdot, \cdot \rangle_{\Lambda^2}$. Therefore, the fiber metric constructed above is the unique one whose associated norm satisfies

$$|w \wedge x|^2 = |w|^2|x|^2 - \langle w, x\rangle^2.$$

(ii) Fix $p \in M$ and write $V = T_pM$. Denote by Rm_p the curvature tensor at p:

$$\mathrm{Rm}_p(w, x, y, z) = g_p(R(w, x)y, z).$$

Then it is multilinear and alternating in (w, x) and in (y, z). Define first a bilinear form

$$B : \Lambda^2 V \times \Lambda^2 V \to \mathbb{R}$$

by

$$B(w \wedge x,\ y \wedge z) := -\mathrm{Rm}_p(w, x, y, z),$$

and extend by bilinearity. This is well-defined by the universal property of $\Lambda^2 V$, because Rm_p is alternating in both (w, x) and (y, z).

Now $(\Lambda^2 V, \langle \cdot, \cdot \rangle_{\Lambda^2})$ is a finite-dimensional inner product space. By linear algebra, any bilinear form B on such a space is represented by a unique linear operator $R_p : \Lambda^2 V \to \Lambda^2 V$ via

$$B(\alpha, \beta) = \langle R_p(\alpha), \beta \rangle_{\Lambda^2} \qquad \text{for all } \alpha, \beta \in \Lambda^2 V.$$

Thus, there exists a unique linear map

$$R_p : \Lambda^2 T_pM \to \Lambda^2 T_pM$$

such that

$$\langle R_p(w \wedge x),\, y \wedge z \rangle_{\Lambda^2} = -\mathrm{Rm}_p(w, x, y, z)$$

for all $w, x, y, z \in T_pM$.

These maps R_p vary smoothly with p. In a local orthonormal frame $\{E_i\}$ of TM, the frame $\{E_i \wedge E_j\}$ of $\Lambda^2(TM)$ is orthonormal, and we can write

$$R_p(E_i \wedge E_j) = \sum_{k<\ell} \big(-R_{ijk\ell}(p)\big)\, E_k \wedge E_\ell,$$

where $R_{ijk\ell}$ are the components of the curvature tensor; these coefficients are smooth functions. Thus, R is a smooth bundle endomorphism

$$\mathcal{R} : \Lambda^2(TM) \to \Lambda^2(TM),$$

called the curvature operator of g, and it satisfies

$$\langle \mathcal{R}(w \wedge x),\, y \wedge z \rangle = -\mathrm{Rm}(w, x, y, z)$$

for all tangent vectors w, x, y, z. Also, the uniqueness of $\mathcal{R}$ follows from the fiberwise uniqueness of R_p. $\square$

Problem 5

Proof. Write
$$v(t) := \gamma'(t), \qquad a(t) := D_t\gamma'(t)$$
for its velocity and covariant acceleration. Set
$$T(t) := \frac{v(t)}{|v(t)|},$$
the unit tangent field along γ. Choose an arc-length parameter s for γ, so that
$$\sigma(s) := \gamma(t(s)), \qquad |\sigma'(s)| \equiv 1.$$
By def, the geodesic curvature of γ at t is
$$k(t) := \big|D_sT\big|_{s=s(t)},$$
where $T = \sigma'$ regarded as a unit tangent field along the unit-speed reparametrization. We express D_sT in terms of t. Since
$$\frac{ds}{dt} = |v(t)|, \qquad \frac{d}{ds} = \frac{1}{|v|}\,D_t,$$
we obtain
$$D_sT = \frac{1}{|v|}\,D_tT.$$
Now compute D_tT. Using $T = v/|v|$ and the product rule,
$$D_tT = D_t\Big(\frac{v}{|v|}\Big) = \frac{1}{|v|}a - \frac{1}{|v|^3}\,\langle a, v\rangle\, v.$$
Define the component of a orthogonal to v by
$$a_\perp := a - \frac{\langle a, v\rangle}{|v|^2}v.$$
Then
$$D_tT = \frac{1}{|v|}a_\perp, \qquad D_sT = \frac{1}{|v|^2}a_\perp.$$
Hence
$$k(t) = |D_sT| = \frac{|a_\perp|}{|v|^2}.$$
By def of $a_\perp$,
$$|a_\perp|^2 = |a|^2 - \frac{\langle a, v\rangle^2}{|v|^2}.$$
On the other hand, by Problem 4,
$$|v \wedge a|^2 = |v|^2|a|^2 - \langle v, a\rangle^2.$$
Combining these,
$$|a_\perp|^2 = \frac{|v \wedge a|^2}{|v|^2}.$$

Therefore,

$$k(t)^2 = \frac{|a_\perp|^2}{|v|^4} = \frac{|v \wedge a|^2}{|v|^6},$$

and taking square roots gives

$$k(t) = \frac{|\,\gamma'(t) \wedge D_t\gamma'(t)\,|}{|\gamma'(t)|^3}.$$

Here the norm in the numerator is the one induced on $\Lambda^2(TM)$ as in Problem 4. Now assume $M = \mathbb{R}^3$ with the Euclidean metric. There is a natural identification between 2–vectors and vectors via the cross product. For $u, v \in \mathbb{R}^3$, $u \times v$ corresponds to $u \wedge v \in \Lambda^2\mathbb{R}^3$, and this identification is an isometry, so

$$|u \times v| = |u \wedge v|.$$

Thus, for $\gamma : I \to \mathbb{R}^3$,

$$k(t) = \frac{|\,\gamma'(t) \wedge \gamma''(t)\,|}{|\gamma'(t)|^3} = \frac{|\,\gamma'(t) \times \gamma''(t)\,|}{|\gamma'(t)|^3}.$$

□

Problem 6

Proof. Fix $p \in M$ and write $V = T_pM \subset \mathbb{R}^{n+1}$. For any $X \in V$, consider N as a vector field along M in $\mathbb{R}^{n+1}$. Then

$$d\nu_p(X) = D_X N \in T_{\nu(p)}S^n \subset \mathbb{R}^{n+1},$$

where D is the ordinary derivative in $\mathbb{R}^{n+1}$.

Because $|N|^2 \equiv 1$, we have

$$0 = X\big(\langle N, N\rangle\big) = 2\langle D_X N, N\rangle,$$

so $D_X N \perp N$. Since both T_pM and $T_{\nu(p)}S^n$ are the hyperplane $\{v \in \mathbb{R}^{n+1} : \langle v, N_p\rangle = 0\}$, we have $D_X N \in T_pM$ and $D_X N \in T_{\nu(p)}S^n$ simultaneously.

By definition of the shape operator S of M (Lee, §8.2),

$$S_p(X) = -D_X N \in T_pM.$$

Thus, viewing T_pM and $T_{\nu(p)}S^n$ as the same subspace of $\mathbb{R}^{n+1}$,

$$d\nu_p = -S_p : T_pM \to T_{\nu(p)}S^n.$$

Both T_pM and $T_{\nu(p)}S^n$ are n–dimensional oriented inner product spaces, with volume forms $(dV_g)_p$ and $(dV_{g_0})_{\nu(p)}$. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of T_pM. Since N_p is the outward unit normal for both M and S^n, we may regard $e_1, \ldots, e_n$ also as an oriented orthonormal basis of $T_{\nu(p)}S^n$ (the condition that $(N_p, e_1, \ldots, e_n)$ is an oriented orthonormal basis of $\mathbb{R}^{n+1}$ is the same in both cases).

By definition of the pullback of a top-degree form,

$$(\nu^* dV_{g_0})_p(e_1, \ldots, e_n) = (dV_{g_0})_{\nu(p)}\big(d\nu_p(e_1), \ldots, d\nu_p(e_n)\big).$$

In an oriented orthonormal basis, a volume form is just the determinant:

$$(dV_{g_0})_{\nu(p)}\big(d\nu_p(e_1), \ldots, d\nu_p(e_n)\big) = \det(d\nu_p)\,(dV_{g_0})_{\nu(p)}(e_1, \ldots, e_n).$$

But $(dV_{g_0})_{\nu(p)}(e_1, \dots, e_n) = 1$ and $(dV_g)_p(e_1, \dots, e_n) = 1$, so

$$(\nu^* dV_{g_0})_p(e_1, \dots, e_n) = \det(d\nu_p) = \det(-S_p).$$

If $\kappa_1, \dots, \kappa_n$ are the principal curvatures of M at p, then S_p is diagonalizable in an orthonormal basis with eigenvalues $\kappa_1, \dots, \kappa_n$, so

$$\det(S_p) = \kappa_1 \cdots \kappa_n =: K(p)$$

is the Gaussian curvature of M at p (product of principal curvatures). Hence

$$\det(d\nu_p) = \det(-S_p) = (-1)^n \det(S_p) = (-1)^n K(p).$$

Therefore,

$$(\nu^* dV_{g_0})_p(e_1, \dots, e_n) = (-1)^n K(p) = \big((-1)^n K \, dV_g\big)_p(e_1, \dots, e_n).$$

Since this holds for all oriented orthonormal bases of T_pM, we conclude

$$\nu^* dV_{g_0} = (-1)^n K \, dV_g.$$

□