

Math 5223 Homework 5

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Problem 1

Proof. (a) Let $r(\theta, \varphi) = (R \sin \varphi \cos \theta, R \sin \varphi \sin \theta, R \cos \varphi)$, $-\pi < \theta < \pi, 0 < \varphi < \pi$. Note that

$$r_\theta = \frac{\partial r}{\partial \theta} = (-R \sin \varphi \sin \theta, R \sin \varphi \cos \theta, 0),$$

$$r_\varphi = \frac{\partial r}{\partial \varphi} = (R \cos \varphi \cos \theta, R \cos \varphi \sin \theta, -R \sin \varphi).$$

Compute,

$$\langle \mathbf{r}_\theta, \mathbf{r}_\theta \rangle = R^2 \sin^2 \varphi (\sin^2 \theta + \cos^2 \theta) = R^2 \sin^2 \varphi,$$

$$\langle \mathbf{r}_\varphi, \mathbf{r}_\varphi \rangle = R^2 (\cos^2 \varphi + \sin^2 \varphi) = R^2,$$

$$\langle \mathbf{r}_\theta, \mathbf{r}_\varphi \rangle = 0.$$

Hence, in the coordinates (θ, φ) , the round metric is

$$g_R = \langle \mathbf{r}_\varphi, \mathbf{r}_\varphi \rangle d\varphi^2 + 2\langle \mathbf{r}_\theta, \mathbf{r}_\varphi \rangle d\theta d\varphi + \langle \mathbf{r}_\theta, \mathbf{r}_\theta \rangle d\theta^2 = R^2 d\varphi^2 + R^2 \sin^2 \varphi d\theta^2,$$

as desired.

(b) In the coordinates (θ, φ) , the only nonzero components of the metric are

$$g_{\varphi\varphi} = R^2, \quad g_{\theta\theta} = R^2 \sin^2 \varphi,$$

so the inverse metric satisfies

$$g^{\varphi\varphi} = \frac{1}{R^2}, \quad g^{\theta\theta} = \frac{1}{R^2 \sin^2 \varphi}.$$

Since $g_{\theta\theta}$ depends only on φ and $g_{\varphi\varphi}$ is constant,

$$\partial_\varphi g_{\theta\theta} = 2R^2 \sin \varphi \cos \varphi, \quad \partial_\theta g_{\theta\theta} = 0, \quad \partial_\varphi g_{\varphi\varphi} = 0, \quad \partial_\theta g_{\varphi\varphi} = 0$$

Hence the only nonzero Christoffel symbols are

$$\Gamma^\varphi_{\theta\theta} = \frac{1}{2} g^{\varphi\varphi} (0 + 0 - \partial_\varphi g_{\theta\theta}) = -\frac{1}{2} \cdot \frac{1}{R^2} \cdot 2R^2 \sin \varphi \cos \varphi = -\sin \varphi \cos \varphi,$$

$$\Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \frac{1}{2} g^{\theta\theta} \partial_\varphi g_{\theta\theta} = \frac{1}{2} \cdot \frac{1}{R^2 \sin^2 \varphi} \cdot 2R^2 \sin \varphi \cos \varphi = \cot \varphi.$$

(c) With the metric $g_R = R^2 d\varphi^2 + R^2 \sin^2 \varphi d\theta^2$ the only nonzero Christoffel symbols are

$$\Gamma^\varphi_{\theta\theta} = -\sin \varphi \cos \varphi, \quad \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \cot \varphi.$$

Hence the geodesic equations in coordinates $(\theta(t), \varphi(t))$ are

$$\begin{aligned} \theta'' + 2\Gamma^\theta_{\theta\varphi} \theta' \varphi' &= \theta'' + 2 \cot \varphi \theta' \varphi' = 0, \\ \varphi'' + \Gamma^\varphi_{\theta\theta} (\theta')^2 &= \varphi'' - \sin \varphi \cos \varphi (\theta')^2 = 0. \end{aligned}$$

Consider a meridian: $\theta(t) \equiv \theta_0$ and $\varphi(t) = t$. Then $\theta' = \theta'' = 0$, so the first equation holds trivially, and the second becomes $\varphi'' = 0$, which is satisfied by $\varphi(t) = at + b$. Therefore $(\theta(t), \varphi(t)) = (\theta_0, t)$ is a geodesic. \square

Problem 2

Proof. On the unit sphere with spherical coordinates (θ, φ) , the round metric is $g = d\varphi^2 + \sin^2 \varphi d\theta^2$. From Problem 1(b), the only nonzero Christoffel symbols are

$$\Gamma^\varphi_{\theta\theta} = -\sin \varphi \cos \varphi, \quad \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \cot \varphi.$$

Let $V = \frac{\partial}{\partial \varphi}$. Using $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$ for the coordinate frame, we have

$$\nabla_{\frac{\partial}{\partial \theta}} V = \nabla_{\partial_\theta} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\theta\varphi} \partial_\varphi = \cot \varphi \frac{\partial}{\partial \theta},$$

since $\Gamma^\varphi_{\theta\varphi} = 0$. Also,

$$\nabla_{\frac{\partial}{\partial \varphi}} V = \nabla_{\partial_\varphi} \partial_\varphi = \Gamma^\theta_{\varphi\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = 0,$$

because $\Gamma^\theta_{\varphi\varphi} = \Gamma^\varphi_{\varphi\varphi} = 0$.

Along the equator $\varphi = \pi/2$, the tangent is ∂_θ and

$$\nabla_{\partial_\theta} V = \cot(\pi/2) \partial_\theta = 0,$$

so V is parallel along the equator. Along any meridian $\theta = \theta_0$, the tangent is ∂_φ and $\nabla_{\partial_\varphi} V = 0$, so V is parallel along each meridian. \square

Problem 3

Proof. Fix $p \in M$. Using the standard identification $T_{(p,0)}(TM) \cong T_pM \oplus T_pM$ (horizontal \oplus vertical), write a tangent vector as (X, W) . Consider the map $G : TM \rightarrow M$, $G(q, V) = \exp_q V$. Then

$$dG_{(p,0)}(X, W) = X + W.$$

Indeed, if $V \equiv 0$ and $q(s)$ is a curve with $q(0) = p$, $q'(0) = X$, then $G(q(s), 0) = q(s)$, hence $\frac{d}{ds}\big|_0 G = X$. If $q \equiv p$ and $V(s)$ is a curve in T_pM with $V(0) = 0$, $V'(0) = W$, then $\frac{d}{ds}\big|_0 \exp_p V(s) = d(\exp_p)_0(W) = W$ since $d(\exp_p)_0 = \text{id}_{T_pM}$. By linearity the general variation gives $X + W$.

Thus, for $F(q, V) = (q, G(q, V))$,

$$dF_{(p,0)}(X, W) = (X, dG_{(p,0)}(X, W)) = (X, X + W) \in T_pM \oplus T_pM \cong T_{(p,p)}(M \times M).$$

With respect to the decompositions $T_{(p,0)}(TM) \cong T_pM \oplus T_pM$ and $T_{(p,p)}(M \times M) \cong T_pM \oplus T_pM$, this is the block matrix

$$dF_{(p,0)} = \begin{pmatrix} \text{id} & 0 \\ \text{id} & \text{id} \end{pmatrix}, \quad \text{whose inverse is } \begin{pmatrix} \text{id} & 0 \\ -\text{id} & \text{id} \end{pmatrix}.$$

Hence $dF_{(p,0)}$ is an isomorphism.

By the inverse function theorem, there exist neighborhoods $\mathcal{U}_p \subset TM$ of $(p, 0)$ and $\mathcal{V}_p \subset M \times M$ of (p, p) such that $F : \mathcal{U}_p \rightarrow \mathcal{V}_p$ is a diffeomorphism. Since p was arbitrary, shrinking and taking the union over p shows that F is a local diffeomorphism from a neighborhood of the zero section in TM onto a neighborhood of the diagonal $\Delta \subset M \times M$.

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