Math 5223 Homework 5

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Problem 1

Proof. (**a**) Let $r(\theta, \varphi) = \big(R\sin\varphi\cos\theta, ; R\sin\varphi\sin\theta, ; R\cos\varphi\big), -\pi < \theta < \pi, 0 < \varphi < \pi$. Note that

$$r_\theta = \frac{\partial r}{\partial \theta} = (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0),$$

$$r_\phi = \frac{\partial r}{\partial \phi} = (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\theta).$$

Compute,

$$\begin{aligned}
\langle \mathbf{r}_\theta, \mathbf{r}_\theta \rangle &= R^2\sin^2\varphi(\sin^2\theta + \cos^2\theta) = R^2\sin^2\varphi, \\
\langle \mathbf{r}_\varphi, \mathbf{r}_\varphi \rangle &= R^2(\cos^2\varphi + \sin^2\varphi) = R^2, \\
\langle \mathbf{r}_\theta, \mathbf{r}_\varphi \rangle &= 0.
\end{aligned}$$

Hence, in the coordinates (θ, φ), the round metric is

$$g_R = \langle \mathbf{r}_\varphi, \mathbf{r}_\varphi \rangle \, d\varphi^2 + 2\langle \mathbf{r}_\theta, \mathbf{r}_\varphi \rangle \, d\theta \, d\varphi + \langle \mathbf{r}_\theta, \mathbf{r}_\theta \rangle \, d\theta^2 = R^2 \, d\varphi^2 + R^2\sin^2\varphi \, d\theta^2,$$

as desired.

(**b**) In the coordinates (θ, φ), the only nonzero components of the metric are

$$g_{\varphi\varphi} = R^2, \qquad g_{\theta\theta} = R^2\sin^2\varphi,$$

so the inverse metric satisfies

$$g^{\varphi\varphi} = \frac{1}{R^2}, \qquad g^{\theta\theta} = \frac{1}{R^2\sin^2\varphi}.$$

Since $g_{\theta\theta}$ depends only on φ and $g_{\varphi\varphi}$ is constant,

$$\partial_\varphi g_{\theta\theta} = 2R^2 \sin\varphi\cos\varphi, \qquad \partial_\theta g_{\theta\theta} = 0, \qquad \partial_\varphi g_{\varphi\varphi} = 0, \qquad \partial_\theta g_{\varphi\varphi} = 0$$

Hence the only nonzero Christoffel symbols are

$$\Gamma^\varphi{}_{\theta\theta} = \frac{1}{2} g^{\varphi\varphi}(0 + 0 - \partial_\varphi g_{\theta\theta}) = -\frac{1}{2} \cdot \frac{1}{R^2} \cdot 2R^2 \sin\varphi\cos\varphi = -\sin\varphi\cos\varphi,$$

$$\Gamma^\theta{}_{\theta\varphi} = \Gamma^\theta{}_{\varphi\theta} = \frac{1}{2} g^{\theta\theta}\, \partial_\varphi g_{\theta\theta} = \frac{1}{2} \cdot \frac{1}{R^2 \sin^2\varphi} \cdot 2R^2 \sin\varphi\cos\varphi = \cot\varphi.$$

(**c**) With the metric $g_R = R^2\, d\varphi^2 + R^2 \sin^2\varphi\, d\theta^2$ the only nonzero Christoffel symbols are

$$\Gamma^\varphi{}_{\theta\theta} = -\sin\varphi\cos\varphi, \qquad \Gamma^\theta{}_{\theta\varphi} = \Gamma^\theta{}_{\varphi\theta} = \cot\varphi.$$

Hence the geodesic equations in coordinates $(\theta(t), \varphi(t))$ are

$$\theta'' + 2\,\Gamma^\theta{}_{\theta\varphi}\, \theta'\varphi' = \theta'' + 2\cot\varphi\, \theta'\varphi' = 0,$$
$$\varphi'' + \Gamma^\varphi{}_{\theta\theta}\, (\theta')^2 = \varphi'' - \sin\varphi\cos\varphi\, (\theta')^2 = 0.$$

Consider a meridian: $\theta(t) \equiv \theta_0$ and $\varphi(t) = t$. Then $\theta' = \theta'' = 0$, so the first equation holds trivially, and the second becomes $\varphi'' = 0$, which is satisfied by $\varphi(t) = at + b$. Therefore $(\theta(t), \varphi(t)) = (\theta_0, t)$ is a geodesic. $\square$

Problem 2

Proof. On the unit sphere with spherical coordinates (θ, φ), the round metric is $g = d\varphi^2 + \sin^2\varphi\, d\theta^2$. From Problem 1(b), the only nonzero Christoffel symbols are

$$\Gamma^\varphi{}_{\theta\theta} = -\sin\varphi\cos\varphi, \qquad \Gamma^\theta{}_{\theta\varphi} = \Gamma^\theta{}_{\varphi\theta} = \cot\varphi.$$

Let $V = \frac{\partial}{\partial\varphi}$. Using $\nabla_{\partial_i}\partial_j = \Gamma^k{}_{ij}\, \partial_k$ for the coordinate frame, we have

$$\nabla_{\frac{\partial}{\partial\theta}} V = \nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta{}_{\theta\varphi}\, \partial_\theta + \Gamma^\varphi{}_{\theta\varphi}\, \partial_\varphi = \cot\varphi\, \frac{\partial}{\partial\theta},$$

since $\Gamma^\varphi{}_{\theta\varphi} = 0$. Also,

$$\nabla_{\frac{\partial}{\partial\varphi}} V = \nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta{}_{\varphi\varphi}\, \partial_\theta + \Gamma^\varphi{}_{\varphi\varphi}\, \partial_\varphi = 0,$$

because $\Gamma^\theta{}_{\varphi\varphi} = \Gamma^\varphi{}_{\varphi\varphi} = 0$.

Along the equator $\varphi = \pi/2$, the tangent is ∂_θ and

$$\nabla_{\partial_\theta} V = \cot(\pi/2)\, \partial_\theta = 0,$$

so V is parallel along the equator. Along any meridian $\theta = \theta_0$, the tangent is ∂_φ and $\nabla_{\partial_\varphi} V = 0$, so V is parallel along each meridian. $\square$

Problem 3

Proof. Fix $p \in M$. Using the standard identification $T_{(p,0)}(TM) \cong T_pM \oplus T_pM$ (horizontal $\oplus$ vertical), write a tangent vector as (X, W). Consider the map $G : TM \to M$, $G(q, V) = \exp_q V$. Then

$$dG_{(p,0)}(X, W) = X + W.$$

Indeed, if $V \equiv 0$ and $q(s)$ is a curve with $q(0) = p,\ q'(0) = X$, then $G(q(s), 0) = q(s)$, hence $\frac{d}{ds}\big|_0 G = X$. If $q \equiv p$ and $V(s)$ is a curve in T_pM with $V(0) = 0,\ V'(0) = W$, then $\frac{d}{ds}\big|_0 \exp_p V(s) = d(\exp_p)_0(W) = W$ since $d(\exp_p)_0 = \mathrm{id}_{T_pM}$. By linearity the general variation gives $X + W$.

Thus, for $F(q, V) = (q, G(q, V))$,

$$dF_{(p,0)}(X, W) = \big(X,\ dG_{(p,0)}(X, W)\big) = \big(X,\ X+W\big) \in T_pM \oplus T_pM \cong T_{(p,p)}(M \times M).$$

With respect to the decompositions $T_{(p,0)}(TM) \cong T_pM \oplus T_pM$ and $T_{(p,p)}(M \times M) \cong T_pM \oplus T_pM$, this is the block matrix

$$dF_{(p,0)} = \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} & \mathrm{id} \end{pmatrix}, \qquad \text{whose inverse is } \begin{pmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{pmatrix}.$$

Hence $dF_{(p,0)}$ is an isomorphism.

By the inverse function theorem, there exist neighborhoods $\mathcal{U}_p \subset TM$ of $(p, 0)$ and $\mathcal{V}_p \subset M \times M$ of (p, p) such that $F : \mathcal{U}_p \to \mathcal{V}_p$ is a diffeomorphism. Since p was arbitrary, shrinking and taking the union over p shows that F is a local diffeomorphism from a neighborhood of the zero section in TM onto a neighborhood of the diagonal $\Delta \subset M \times M$.

□

Problem 4

Proof. (**a**) Let $T(s, t) = \partial_t \Gamma(s, t)$. Note that

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) &= \frac{d}{ds}\bigg|_{s=0} \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |T(s,t)|, dt \\ &= \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \frac{\langle \nabla_{\partial_s} T, T \rangle}{|T|}\bigg|_{s=0} dt \\ &\overset{|T|=1 \text{ at } s=0}{=} \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \langle \nabla_{\partial_s} T, \dot\gamma \rangle, dt. \end{aligned}$$

Since the Levi-Civita connection is torsion free, we have $\nabla_{\partial_s} T = \nabla_{\partial_t} \partial_s \Gamma$. Thus, with $V(t) = \partial_s \Gamma(0,t)$, we have

$$\frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) = \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \langle D_t V, \dot\gamma \rangle, dt.$$

By integration by part,

$$\int_{a_i}^{a_{i+1}} \langle D_t V, \dot\gamma \rangle, dt = \big[\langle V, \dot\gamma \rangle\big]_{t=a_i^+}^{t=a_{i+1}^-} - \int_{a_i}^{a_{i+1}} \langle V, D_t \dot\gamma \rangle, dt.$$

Take the sum over all i, we have

$$\sum_{i=1}^{k-1} \big[\langle V, \dot\gamma \rangle\big]_{a_i^+}^{a_{i+1}^-} = \langle V(b), \dot\gamma(b) \rangle - \langle V(a), \dot\gamma(a) \rangle - \sum_{i=2}^{k-1} \langle V(a_i), , \Delta_i \dot\gamma \rangle,$$

since the interior endpoint contributions telescope and leave the jumps of $\dot\gamma$ at the break points a_i. Therefore,

$$\frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) = -\int_a^b \langle V, D_t \dot\gamma \rangle, dt + \langle V(b), \dot\gamma(b) \rangle - \langle V(a), \dot\gamma(a) \rangle - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i \dot\gamma \rangle.$$

(**b**) Fix any $w \in T_q N$. Take a smooth curve $\eta : (-\delta, \delta) \to N$ with $\eta(0) = q$ and $\eta'(0) = w$. For $|s|$ small, join p to $\eta(s)$ by a unit–speed minimizing geodesic γ_s, and define a smooth variation

$$\Gamma : (-\varepsilon, \varepsilon) \times [0,1] \to M, \qquad \Gamma(s,t) = \gamma_s(t).$$

Let $V(t) := \partial_s \Gamma(0,t)$ be its variation field along γ. Then $V(0) = 0$ and $V(1) = \eta'(0) = w \in T_q N$.

Since q realizes the distance from p to N, the length function $L(\Gamma_s)$ has a minimum at $s = 0$, hence

$$\frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) = 0.$$

Applying the first variation formula from part (a),

$$\frac{d}{ds}\bigg|_{s=0} L(\Gamma_s) = -\int_0^1 \langle V, D_t \dot\gamma \rangle \, dt + \langle V(1), \dot\gamma(1) \rangle - \langle V(0), \dot\gamma(0) \rangle.$$

Because γ is a geodesic, $D_t \dot\gamma = 0$, and $V(0) = 0$, we obtain

$$0 = \langle V(1), \dot\gamma(1) \rangle = \langle w, \dot\gamma(1) \rangle.$$

This holds for every $w \in T_q N$, so $\dot\gamma(1) \perp T_q N$. Therefore any minimizing geodesic from p to a nearest point $q \in N$ meets N orthogonally. $\square$

Problem 5

Proof. Let (M, g) and (M', g') be Riemannian manifolds and suppose $\varphi, \psi : M \to M'$ are local isometries with

$$\varphi(p) = \psi(p) =: q \qquad \text{and} \qquad \varphi_*|_p = \psi_*|_p.$$

Because φ and ψ are local isometries, there are open neighborhoods $U \ni p$ and $U' \ni q$ such that $\varphi|_U : U \to U'$ and $\psi|_U : U \to U'$ are Riemannian isometries (diffeomorphisms onto U'). Define

$$h := (\psi|_U)^{-1} \circ (\varphi|_U) : U \longrightarrow U.$$

Then h is a local isometry of (U, g), with

$$h(p) = p \qquad \text{and} \qquad h_*|_p = (\psi|_U)_*^{-1}|_q \circ \varphi_*|_p = \mathrm{id}_{T_pM}.$$

Claim. h is the identity on some neighborhood of p.

Let $v \in T_pM$ be small and let $\gamma_v(t) = \exp_p(tv)$ be the unit-speed geodesic with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Since h is a local isometry, it preserves the Levi–Civita connection and therefore maps geodesics to geodesics. Hence $h \circ \gamma_v$ is a geodesic with

$$(h \circ \gamma_v)(0) = h(p) = p, \qquad \frac{d}{dt}\Big|_{t=0}(h \circ \gamma_v) = h_*|_p(v) = v.$$

By uniqueness of geodesics with given initial data, $h \circ \gamma_v = \gamma_v$ for all t sufficiently small. Since points near p are of the form $\exp_p(tv)$, we obtain $h = \mathrm{id}$ on a neighborhood $W \subset U$ of p.

Next, we propagate the identity along broken geodesics. Let $x \in M$. Because M is connected, there exists a piecewise smooth geodesic

$$p = x_0 \xrightarrow{\gamma_0} x_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{m-1}} x_m = x$$

such that each segment γ_i lies in a normal neighborhood of x_i. Assume inductively that $h(x_i) = x_i$ and $h_*|_{x_i} = \mathrm{id}$. Then $h \circ \gamma_i$ is a geodesic with the same initial point x_i and the same initial velocity as γ_i, hence (by uniqueness) $h \circ \gamma_i = \gamma_i$ on its whole segment. In particular $h(x_{i+1}) = x_{i+1}$ and $h_*|_{x_{i+1}} = \mathrm{id}$. Starting from $x_0 = p$ (where this holds by construction), we conclude $h(x) = x$ for all $x \in M$. Therefore $h = \mathrm{id}_M$.

Finally, on U we have

$$\varphi = \psi \circ h = \psi.$$

Since the set $\{x \in M : \varphi(x) = \psi(x)\}$ is closed and contains a nonempty open set (the neighborhood W), by connectedness of M it must be all of M. Hence $\varphi \equiv \psi$. $\square$