

# ***Left-Leaning Red-Black Trees***

***Robert Sedgewick***  
***Princeton University***

Original version: Data structures seminar at Dagstuhl (Feb 2008)

- red-black trees made simpler (!)
- full delete() implementation

This version: Analysis of Algorithms meeting at Maresias (Apr 2008)

- back to balanced 4-nodes
- back to 2-3 trees (!)
- scientific analysis

Addendum: observations developed after talk at Maresias

Java code at [www.cs.princeton.edu/~rs/talks/LLRB/Java](http://www.cs.princeton.edu/~rs/talks/LLRB/Java)

Movies at [www.cs.princeton.edu/~rs/talks/LLRB/movies](http://www.cs.princeton.edu/~rs/talks/LLRB/movies)

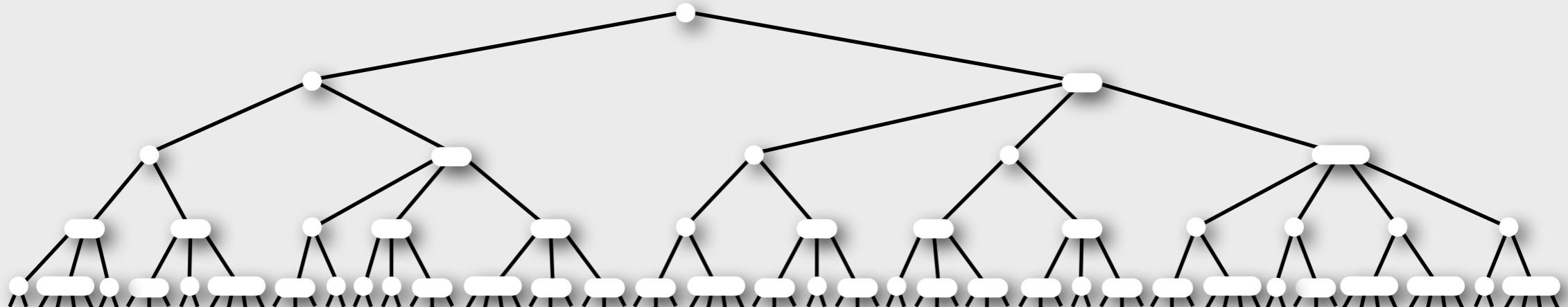
# ***Introduction***

## 2-3-4 Trees

# *Red-Black Trees*

# *Left-Leaning RB Trees*

# *Deletion*

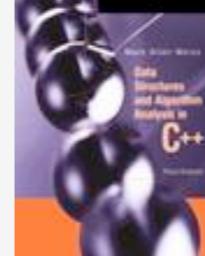
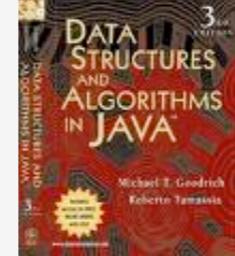
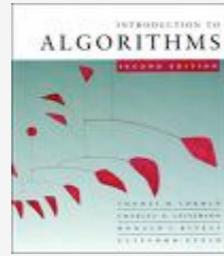
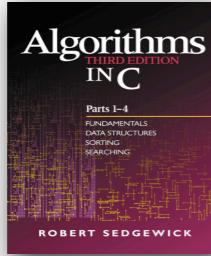


# Red-black trees

are now found throughout our computational infrastructure

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

Textbooks on algorithms



...

Library [search function](#) in many programming environments



...

[Popular culture \(stay tuned\)](#)

[Worth revisiting?](#)

# Red-black trees

are now found throughout our computational infrastructure

*Introduction  
2-3-4 Trees  
LLRB Trees  
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*Typical:*

> ya thanks,  
> i got the idea  
> but is there some other place on the web where only the algorithms  
> used by STL is  
> explained. (that is the underlying data structures etc. ) without  
> explicit reference to the code (as it is pretty confusing) if I try to  
> read through).  
>  
> thanks[/color]

The standard does not specify which algorithms the STL must use.  
Implementers are free to choose which ever algorithm or data structure that  
fulfils the functional and efficiency requirements of the standard.

There are some common choices however. For instance every implementation of  
map, multimap, set and multiset that I have ever seen uses a structure  
called a red black tree. Typing 'red black tree algorithm' in google  
produces a number of likely looking links.

john

## Digression:

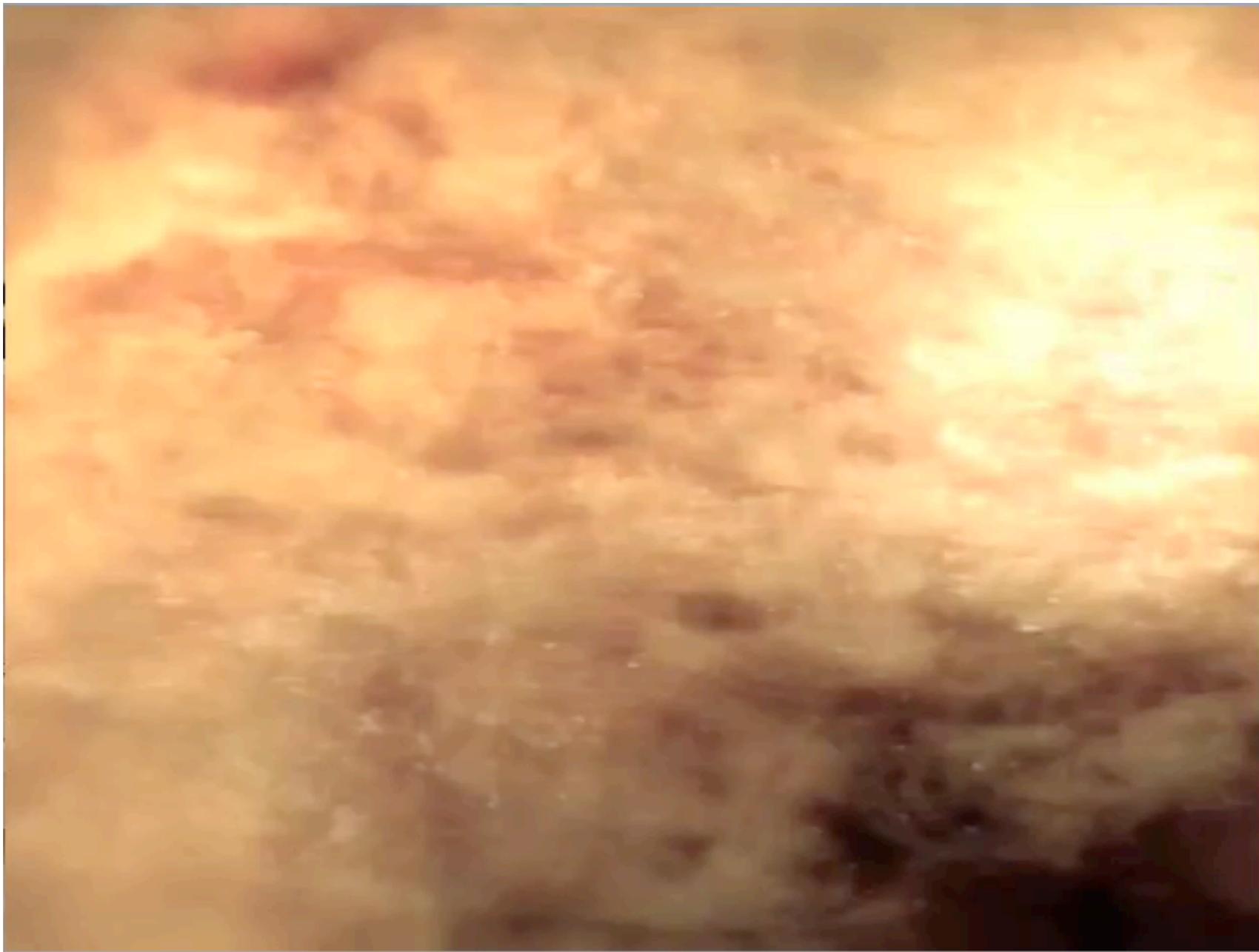
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Red-black trees are found in popular culture??

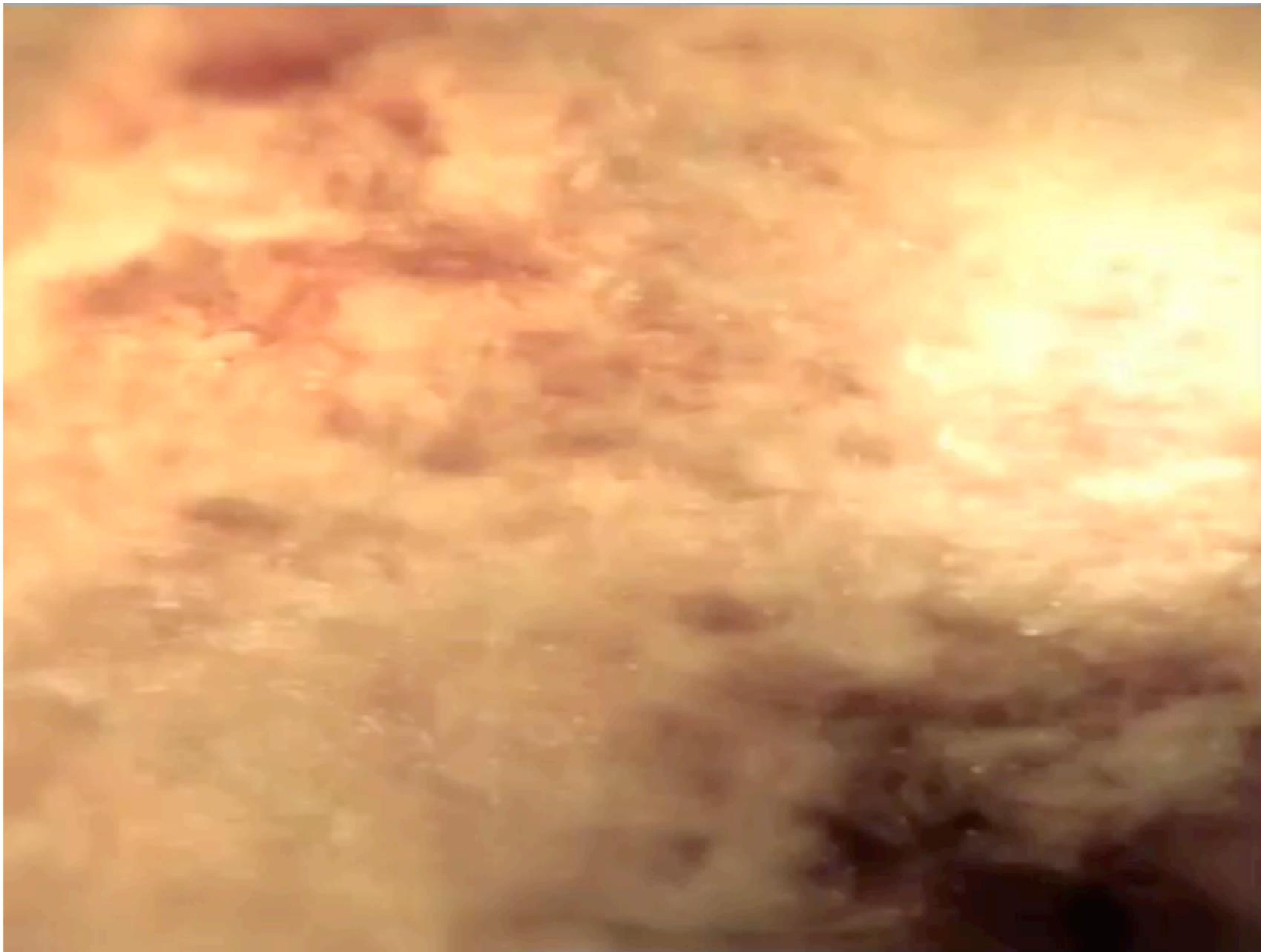
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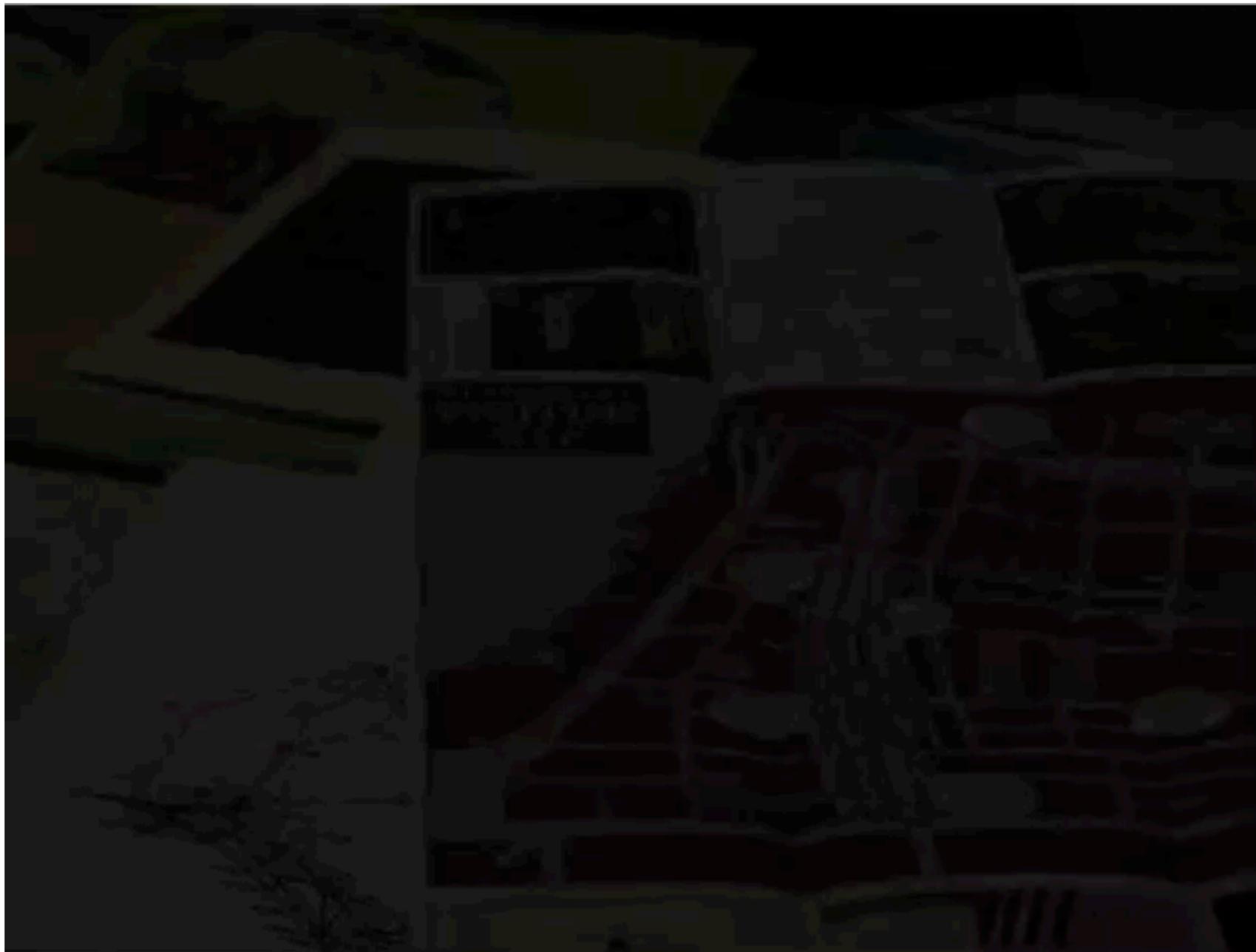
Mystery: black door?



Mystery: red door?



An explanation ?



## Red-black trees (Guibas-Sedgewick, 1978)

- reduce code complexity
- minimize or eliminate space overhead
- unify balanced tree algorithms
- single top-down pass (for concurrent algorithms)
- find version amenable to average-case analysis

## Current implementations

- maintenance
- migration
- space not so important (??)
- guaranteed performance
- support full suite of operations

Worth revisiting ?

Red-black trees (Guibas-Sedgewick, 1978)

- reduce code complexity
- minimize or eliminate space overhead
- unify balanced tree algorithms
- single top-down pass (for concurrent algorithms)
- find version amenable to average-case analysis

Current implementations

- maintenance
- migration
- space not so important (??)
- guaranteed performance
- support full suite of operations

Worth revisiting ? YES. Code complexity is out of hand.

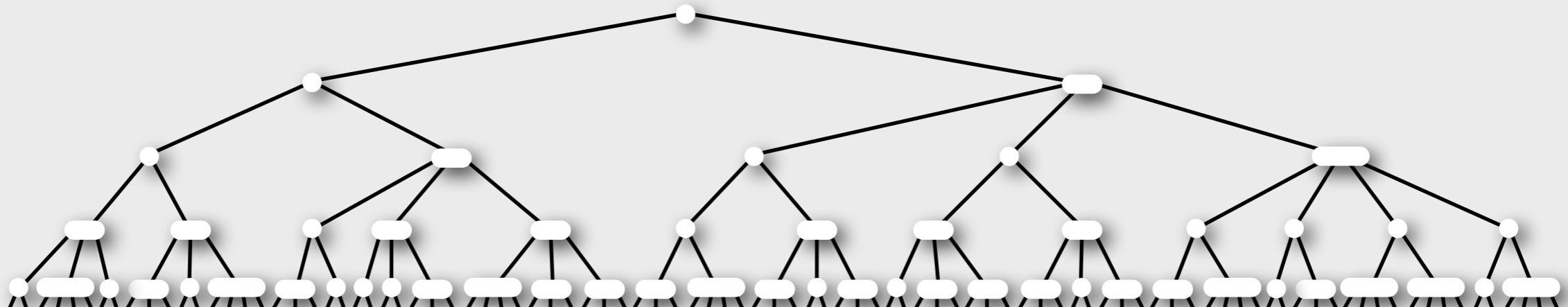
*Introduction*

## **2-3-4 Trees**

*LLRB Trees*

*Deletion*

*Analysis*



# 2-3-4 Tree

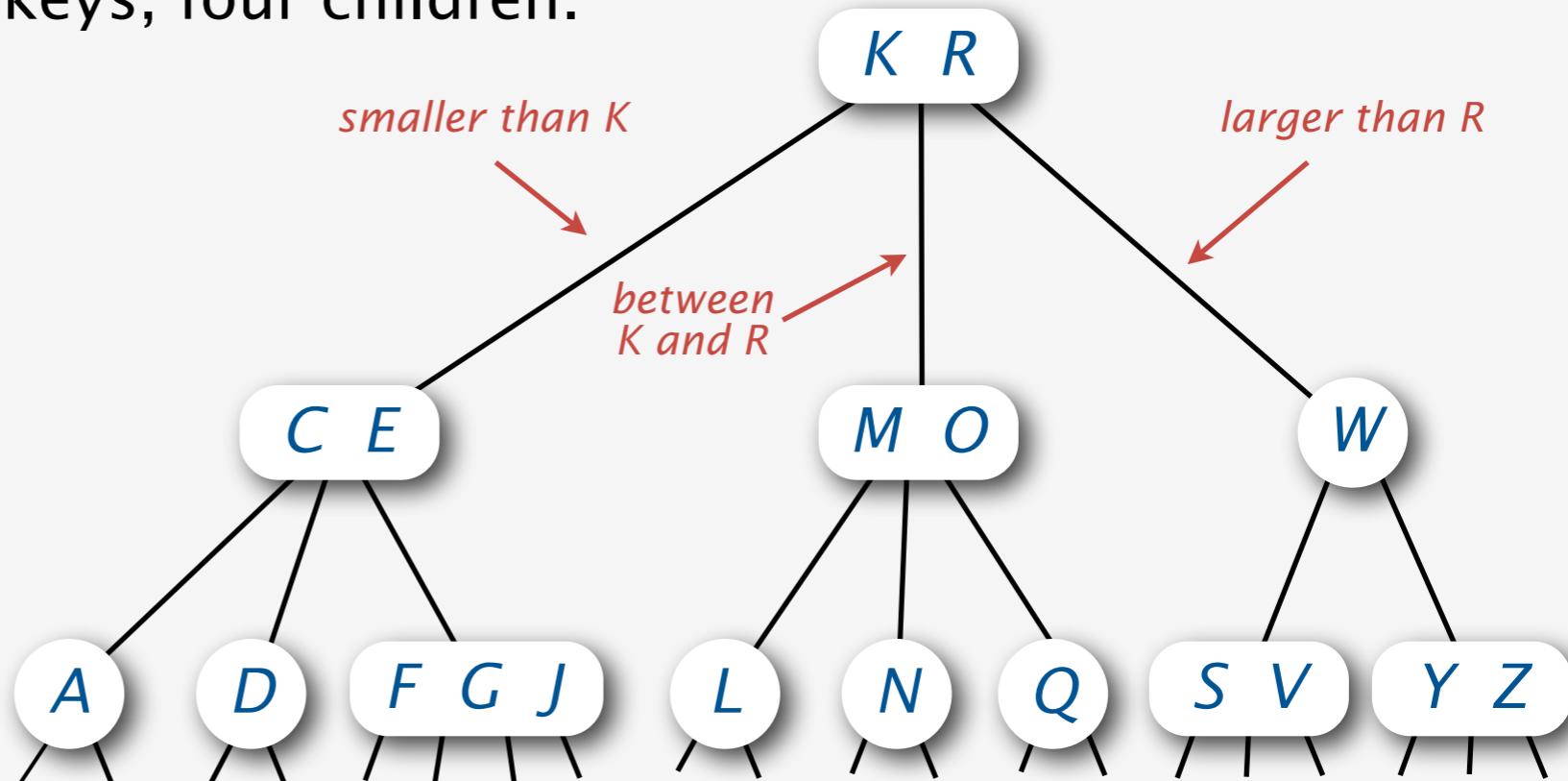
Introduction  
2-3-4 Trees  
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Analysis

Generalize BST node to allow multiple keys.  
Keep tree in perfect balance.

**Perfect balance.** Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.
- 4-node: three keys, four children.



# Search in a 2-3-4 Tree

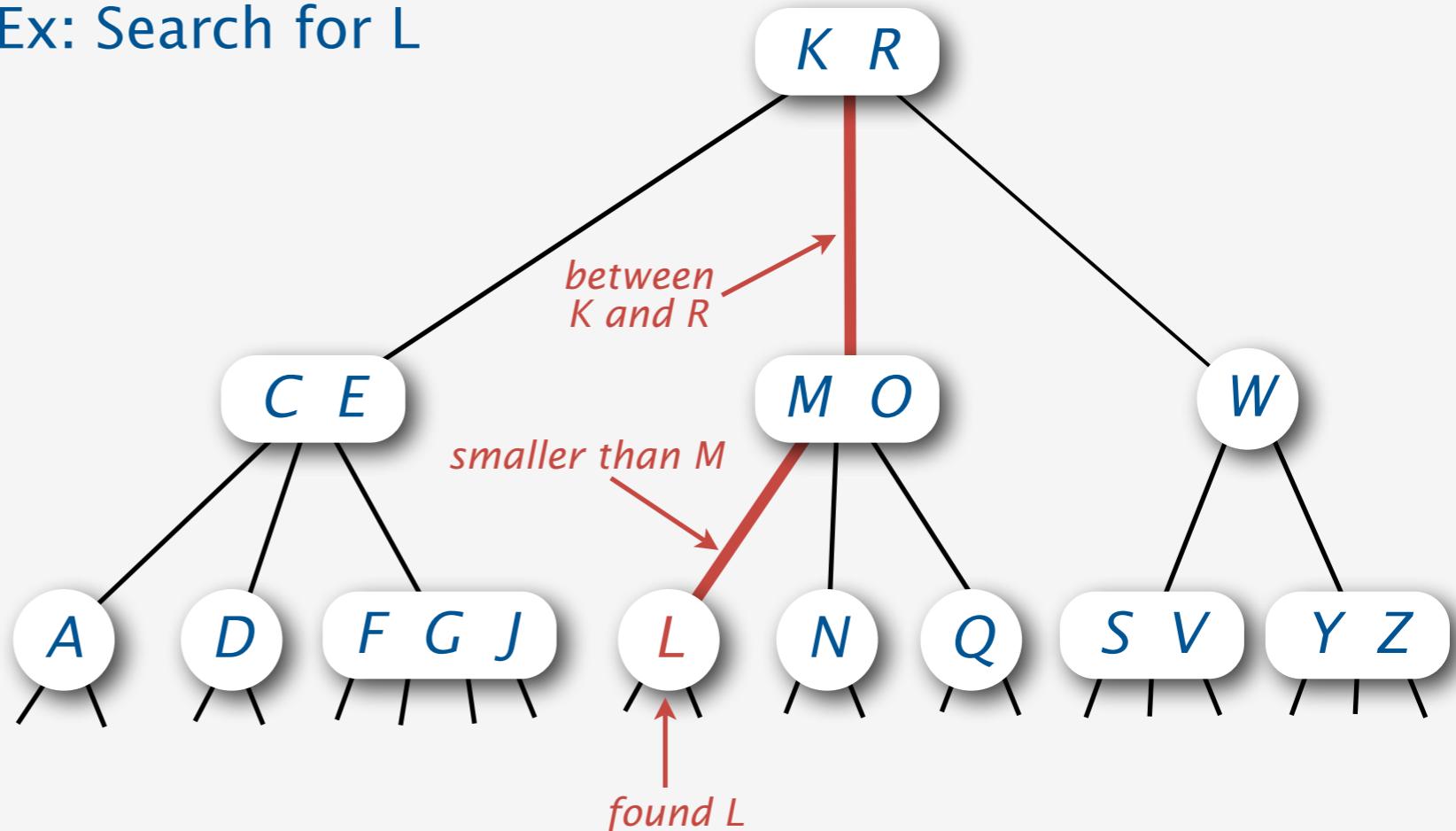
Compare node keys against search key to guide search.

Introduction  
2-3-4 Trees  
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Analysis

## Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

Ex: Search for L



# Insertion in a 2-3-4 Tree

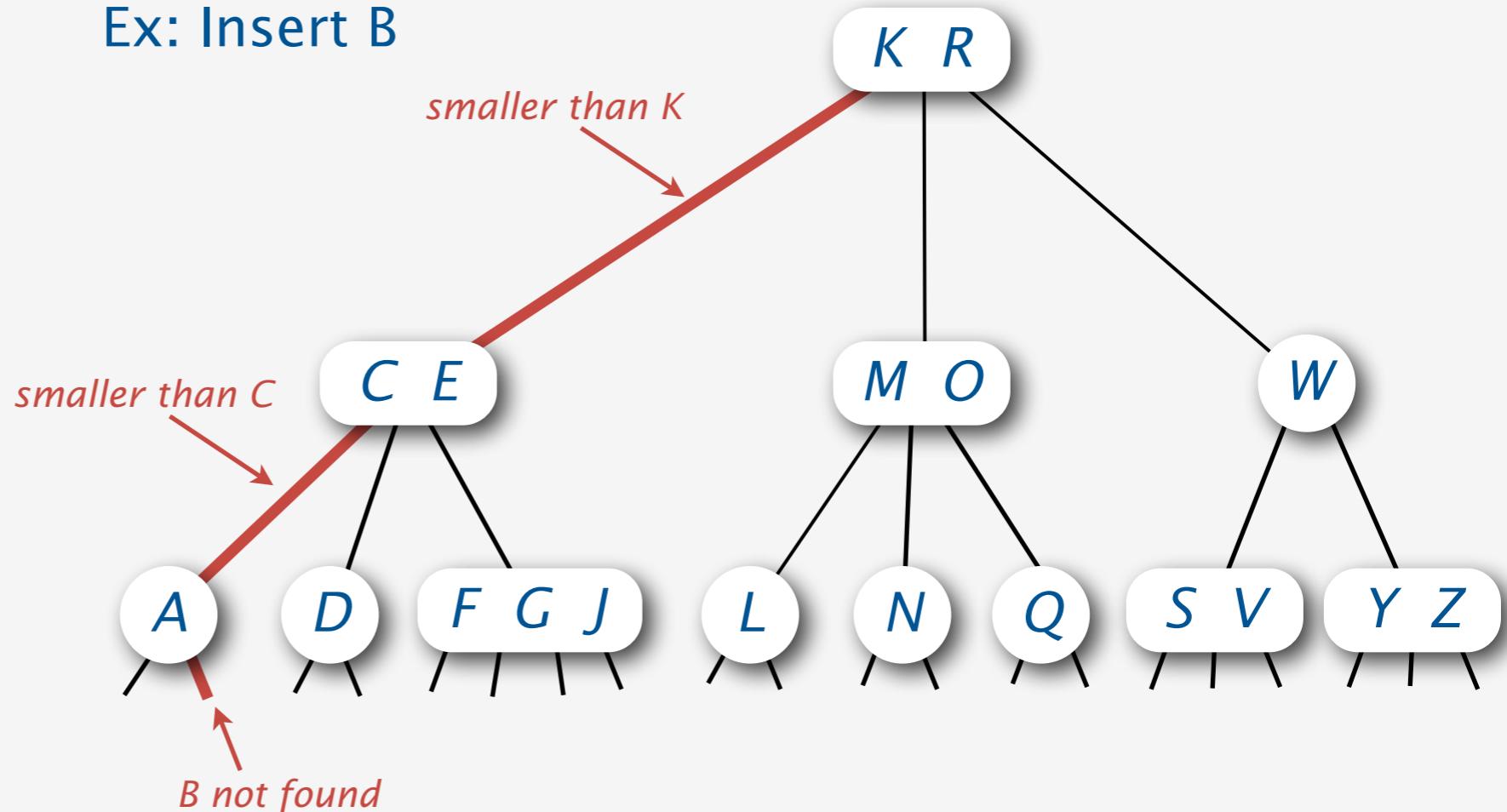
Add new keys at the bottom of the tree.

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## Insert.

- Search to bottom for key.

Ex: Insert B



# Insertion in a 2-3-4 Tree

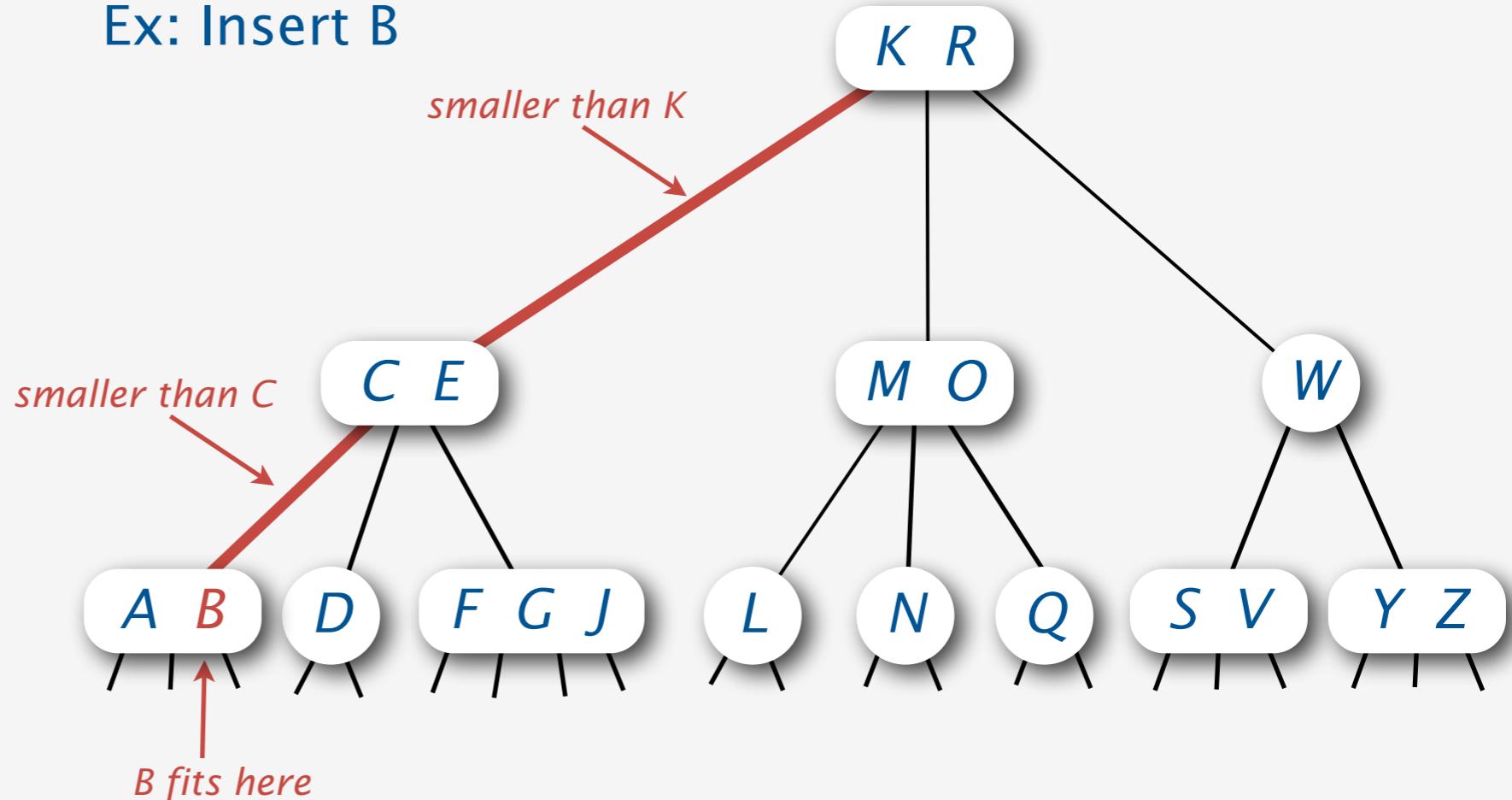
Add new keys at the bottom of the tree.

Introduction  
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Analysis

## Insert.

- Search to bottom for key.
- 2-node at bottom: convert to a 3-node.

Ex: Insert B



# Insertion in a 2-3-4 Tree

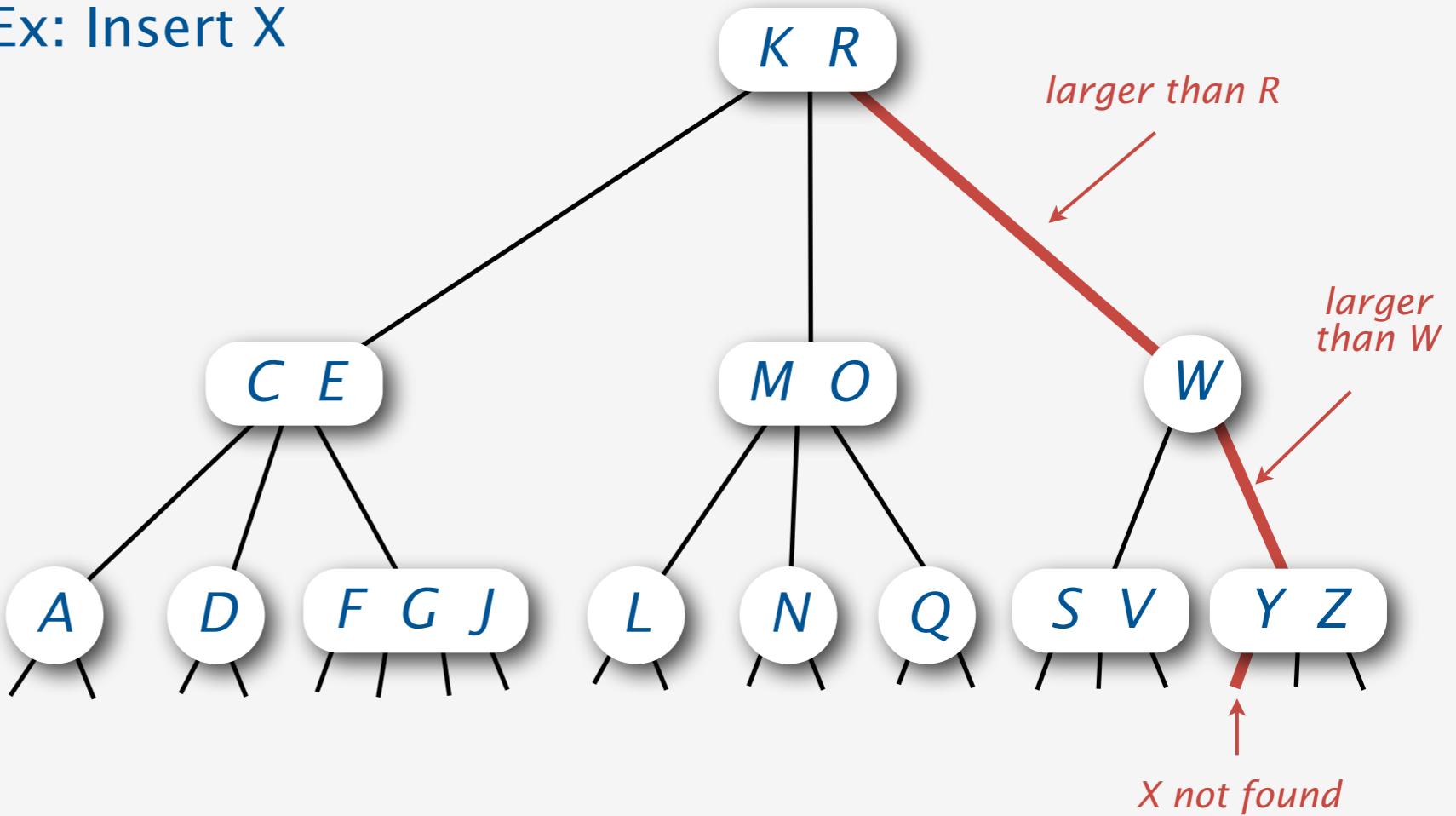
Add new keys at the bottom of the tree.

Introduction  
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Analysis

## Insert.

- Search to bottom for key.

Ex: Insert X



## Insertion in a 2-3-4 Tree

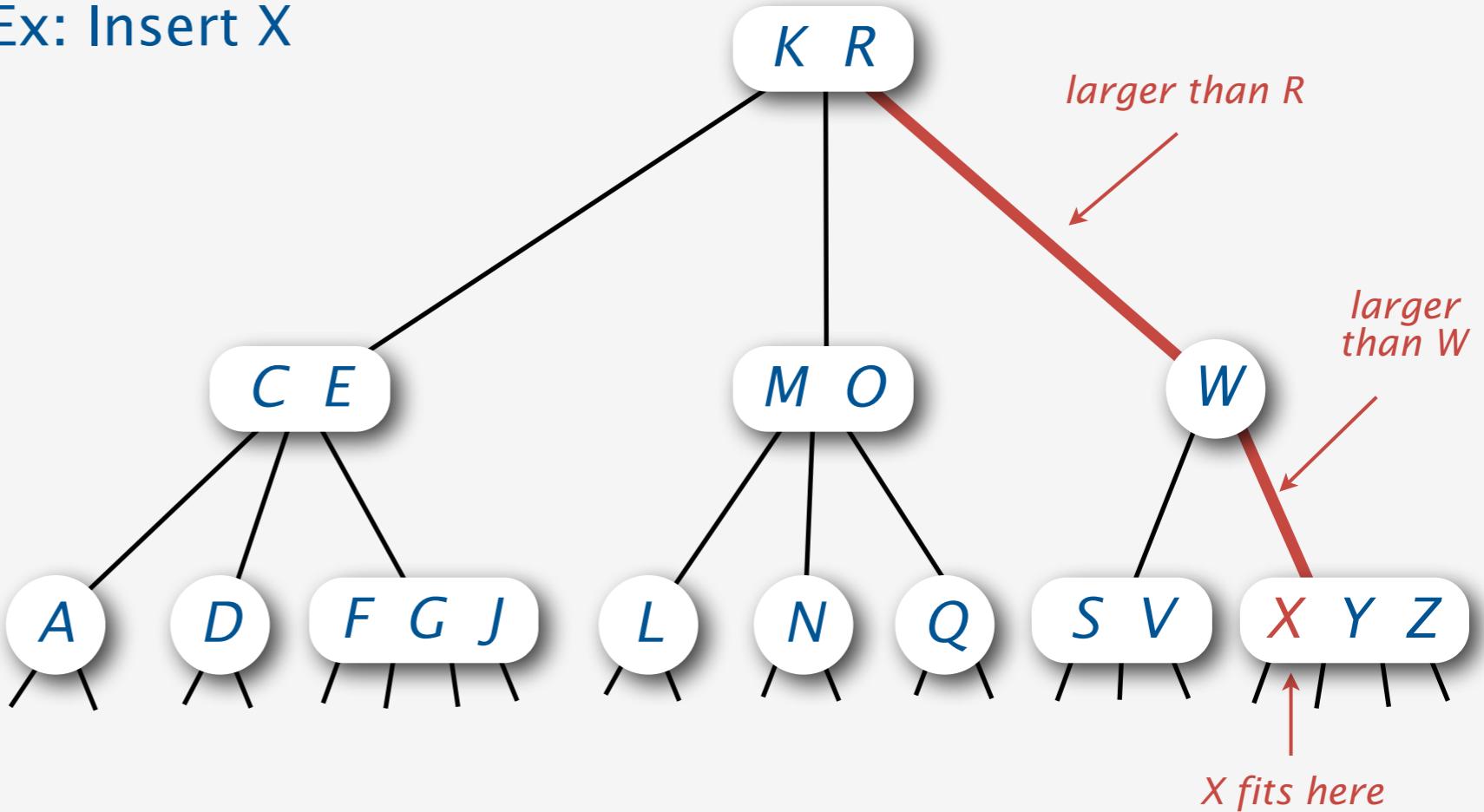
# *Introduction 2-3-4 Trees LLRB Trees Deletion Analysis*

Add new keys at the bottom of the tree.

# Insert.

- Search to bottom for key.
  - 3-node at bottom: convert to a 4-node.

## Ex: Insert X



# Insertion in a 2-3-4 Tree

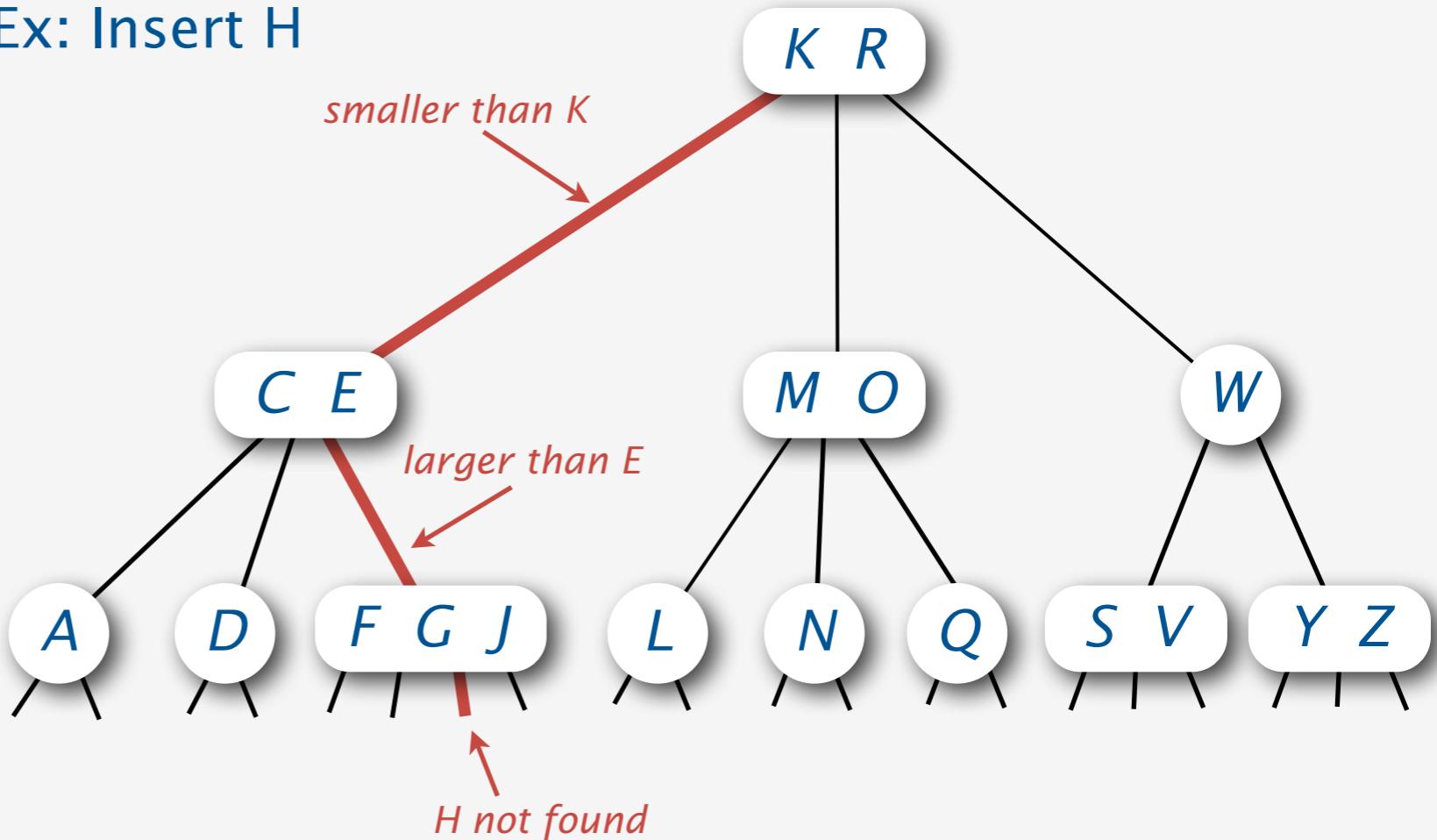
Add new keys at the bottom of the tree.

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Deletion  
Analysis

## Insert.

- Search to bottom for key.

Ex: Insert H



# Insertion in a 2-3-4 Tree

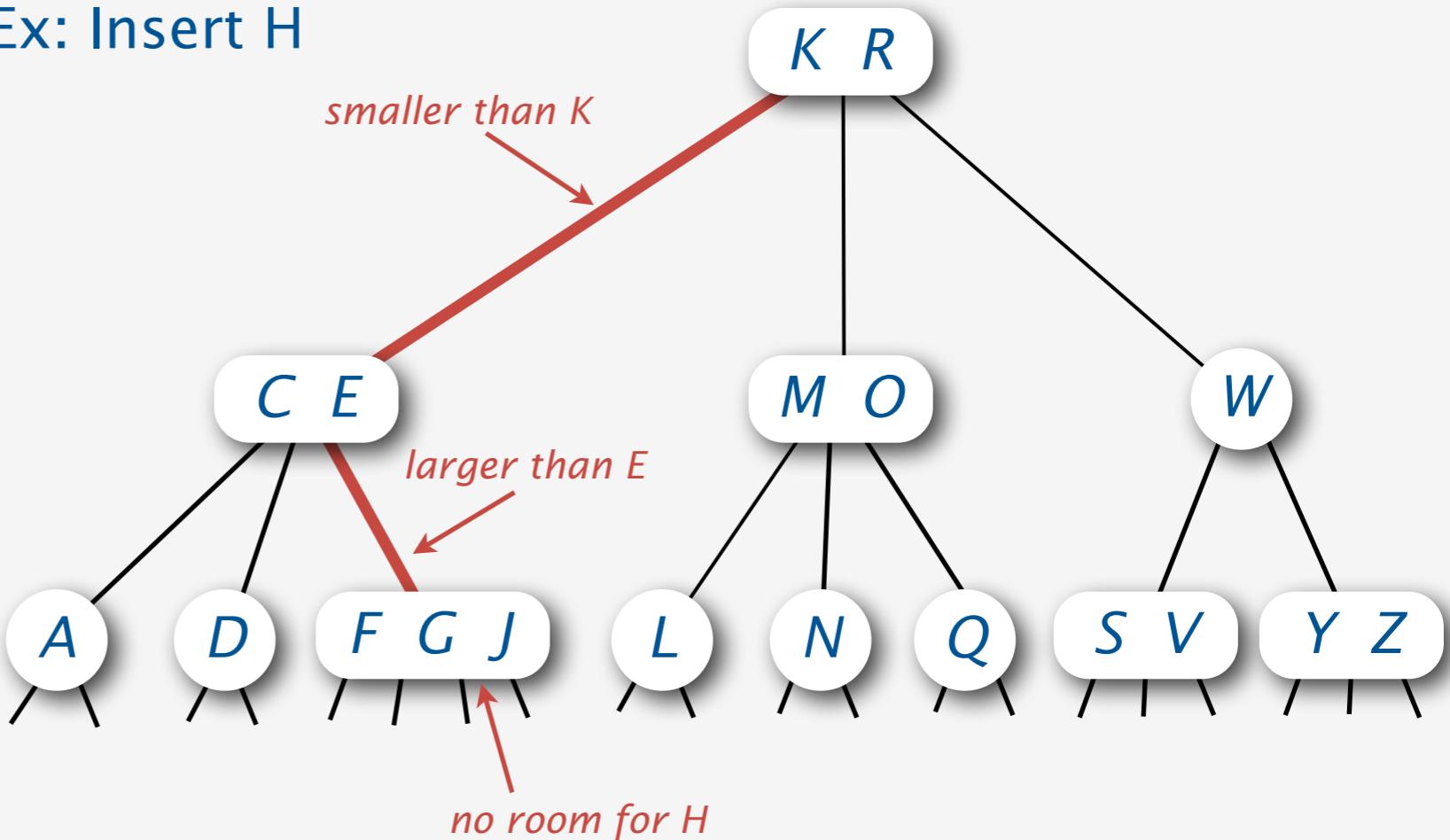
Add new keys at the bottom of the tree.

Introduction  
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Analysis

## Insert.

- Search to bottom for key.
  - 2-node at bottom: convert to a 3-node.
  - 3-node at bottom: convert to a 4-node.
- 4-node at bottom: no room for new key.

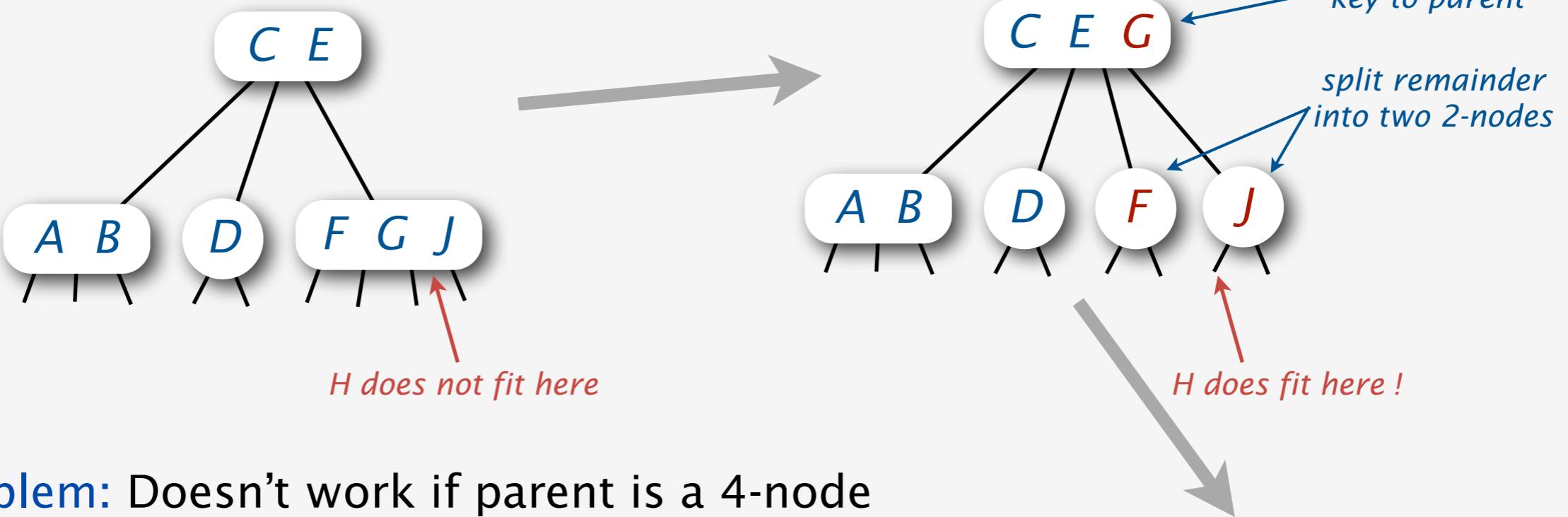
Ex: Insert H



# Splitting 4-nodes in a 2-3-4 tree

is an effective way to make room for insertions

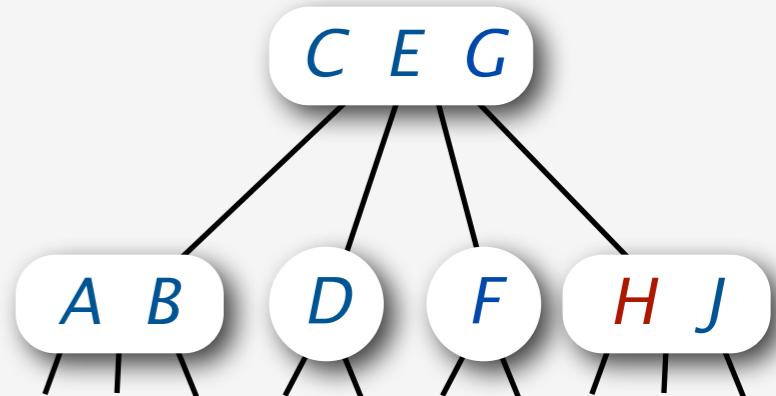
Introduction  
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LLRB Trees  
Deletion  
Analysis  
*move middle key to parent*



Problem: Doesn't work if parent is a 4-node

Bottom-up solution (Bayer, 1972)

- Use same method to split parent
- Continue up the tree while necessary



Top-down solution (Guibas-Sedgewick, 1978)

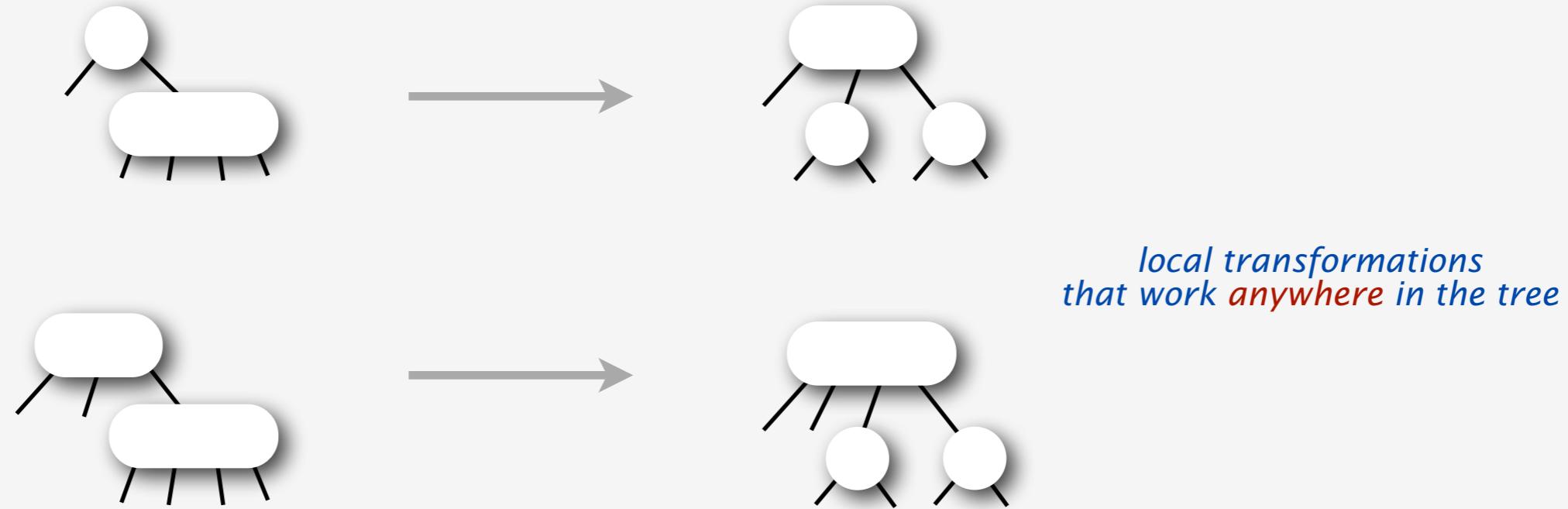
- Split 4-nodes on the way **down**
- Insert at bottom

# Splitting 4-nodes on the way down

ensures that the “current” node is not a 4-node

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Transformations to split 4-nodes:



Invariant: “Current” node is not a 4-node

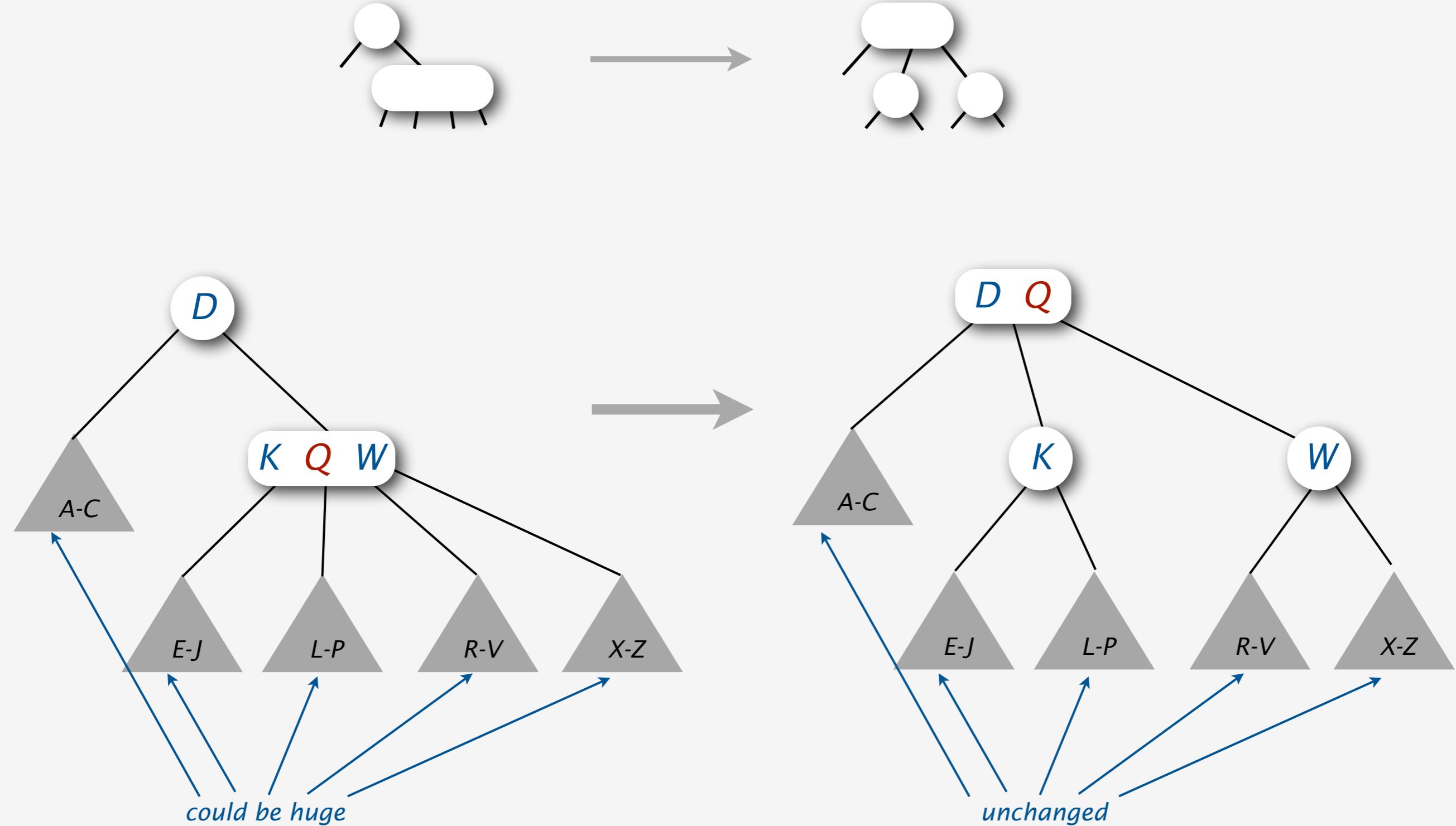
Consequences:

- 4-node below a 4-node case never happens
- Bottom node reached is always a 2-node or a 3-node

# Splitting a 4-node below a 2-node

is a **local** transformation that works anywhere in the tree

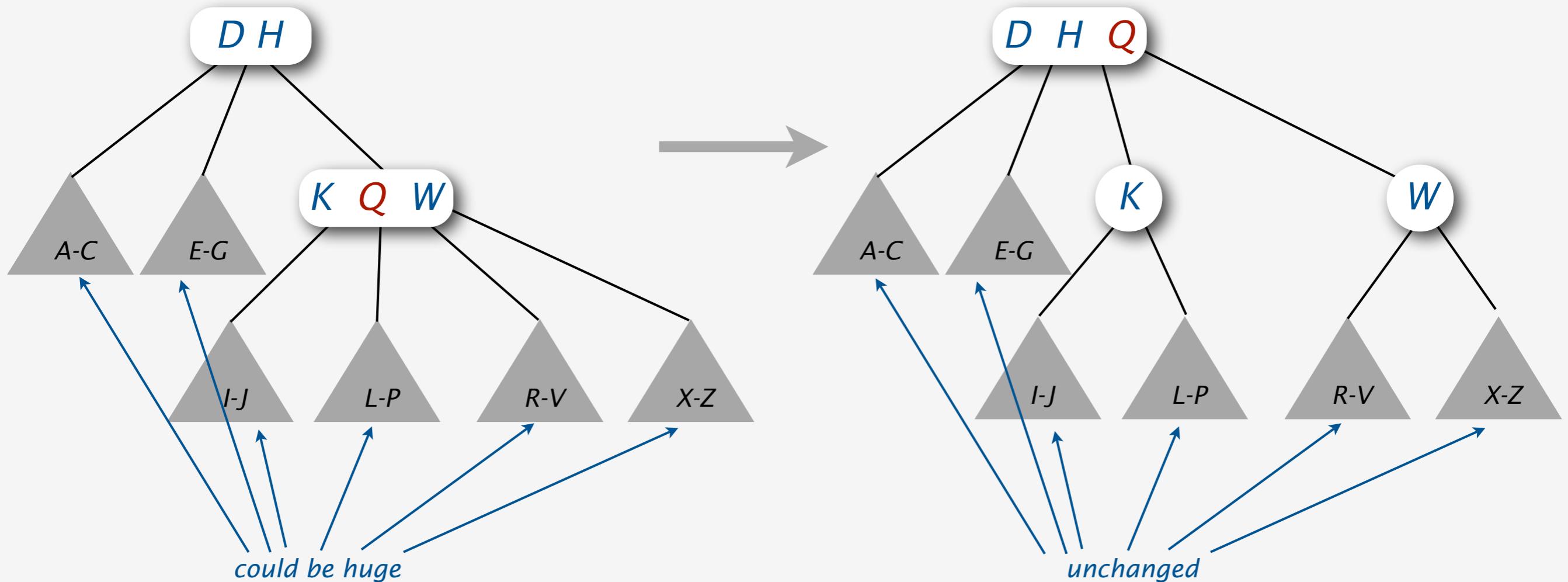
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# Splitting a 4-node below a 3-node

is a **local** transformation that works anywhere in the tree

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Analysis



# Growth of a 2-3-4 tree

happens upwards from the bottom

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*insert A*



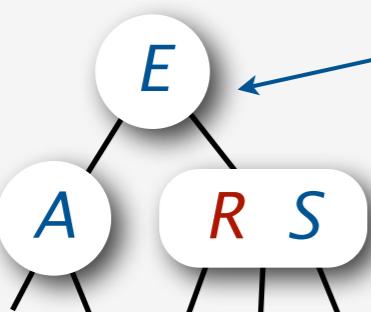
*insert S*



*insert E*

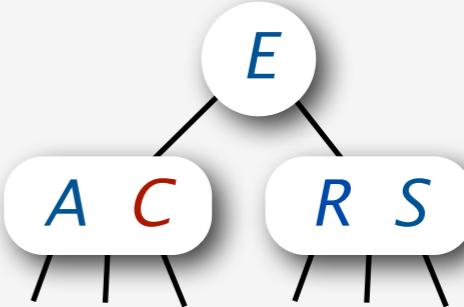


*insert R*

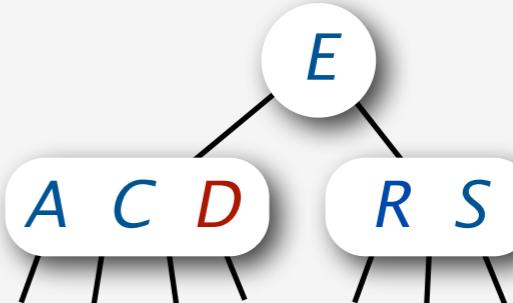


split 4-node to  
and then insert

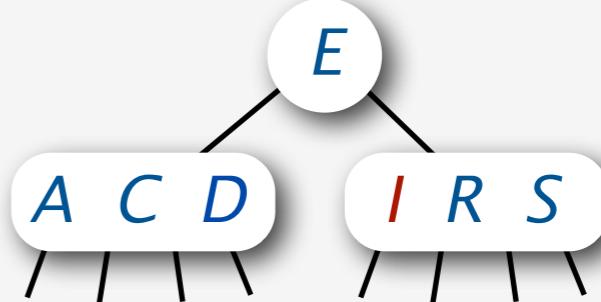
*insert C*



*insert D*



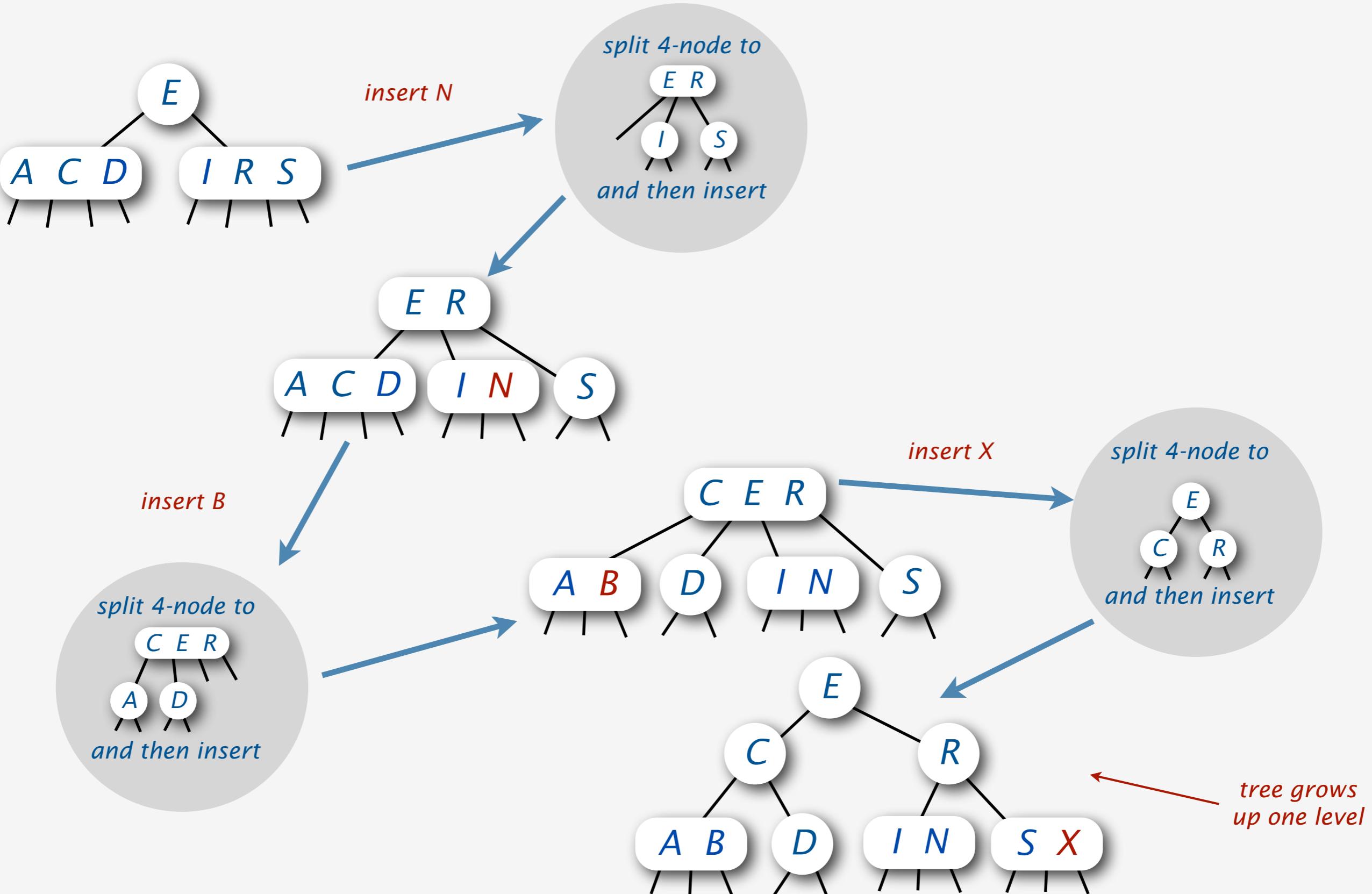
*insert I*



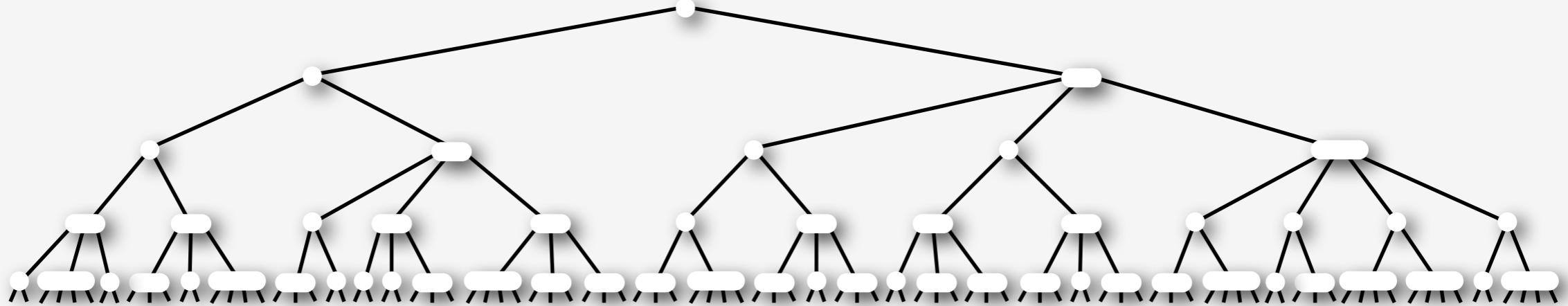
# Growth of a 2-3-4 tree (continued)

happens upwards from the bottom

Introduction  
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**Key property:** All paths from root to leaf are the same length



## Tree height.

- Worst case:  $\lg N$  [all 2-nodes]
- Best case:  $\log_4 N = \frac{1}{2} \lg N$  [all 4-nodes]
- Between 10 and 20 for 1 million nodes.
- Between 15 and 30 for 1 billion nodes.

Guaranteed logarithmic performance for both search and insert.

# Direct implementation of 2-3-4 trees

is complicated because of code complexity.

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Maintaining multiple node types is cumbersome.

- Representation?
- Need multiple compares to move down in tree.
- Large number of cases for splitting.
- Need to convert 2-node to 3-node and 3-node to 4-node.

```
private void insert(Key key, Val val)      fantasy  
{                                         code  
    Node x = root;  
    while (x.getTheCorrectChild(key) != null)  
    {  
        x = x.getTheCorrectChild(key);  
        if (x.is4Node()) x.split();  
    }  
    if (x.is2Node()) x.make3Node(key, val);  
    else if (x.is3Node()) x.make4Node(key, val);  
    return x;  
}
```

Bottom line: Could do it, but stay tuned for an easier way.

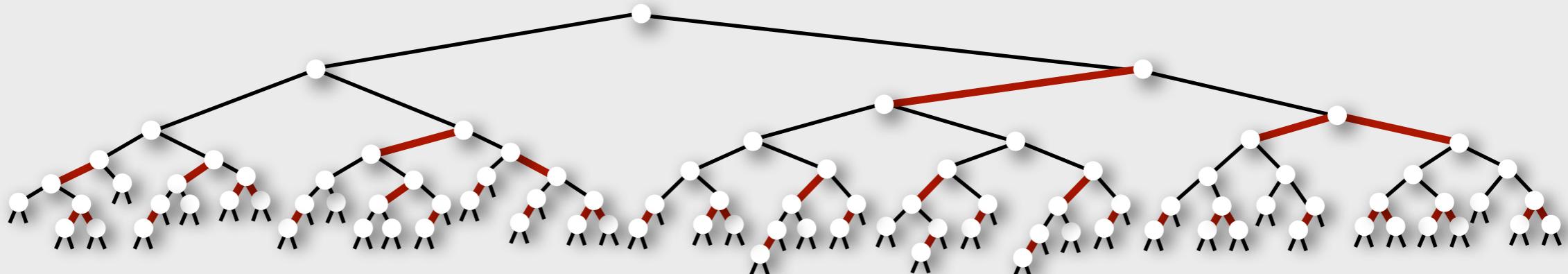
# *Introduction*

## *2-3-4 Trees*

## **LLRB Trees**

### *Deletion*

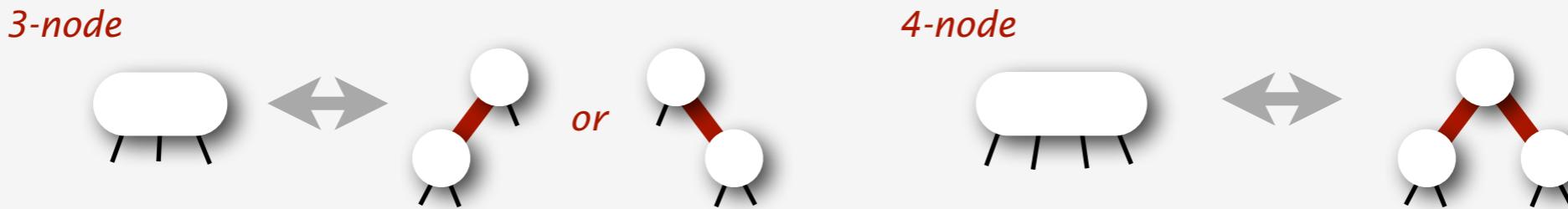
### *Analysis*



# Red-black trees (Guibas-Sedgewick, 1978)

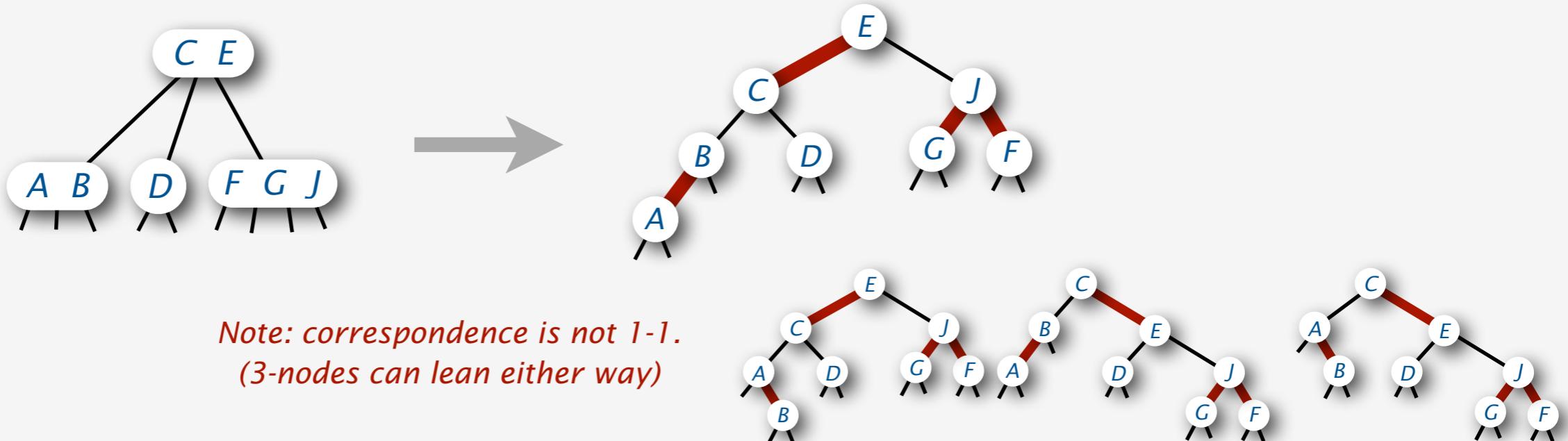
Introduction  
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Analysis

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.



## Key Properties

- elementary BST search works
- easy to maintain a correspondence with 2-3-4 trees  
(and several other types of balanced trees)



Many variants studied ( details omitted. )

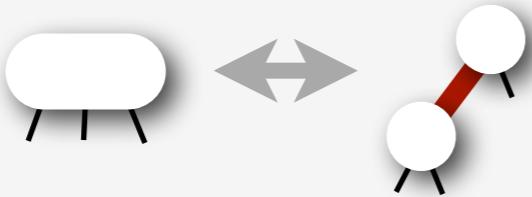
NEW VARIANT (this talk): Left-leaning red-black trees

# Left-leaning red-black trees

Introduction  
2-3-4 Trees  
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Deletion  
Analysis

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.
3. Require that 3-nodes be left-leaning.

3-node

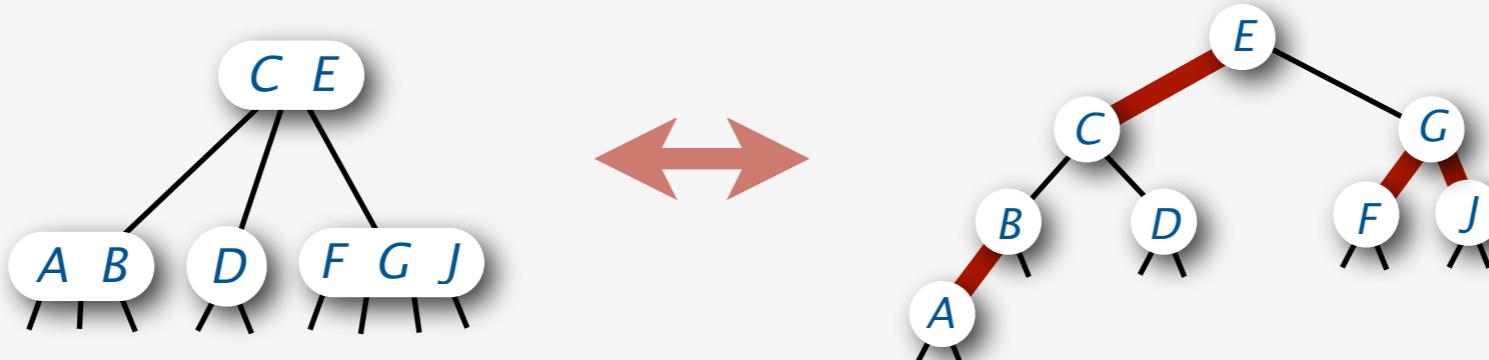


4-node



## Key Properties

- elementary BST search works
- easy-to-maintain **1-1** correspondence with 2-3-4 trees
- trees therefore have perfect black-link balance

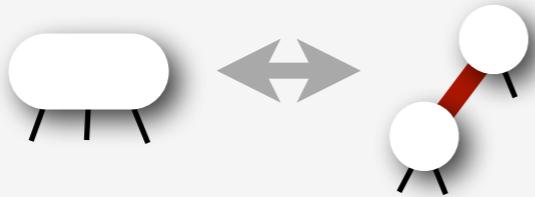


# Left-leaning red-black trees

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Deletion  
Analysis

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.
3. Require that 3-nodes be left-leaning.

*3-node*

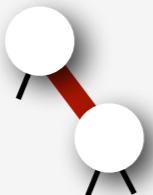


*4-node*



## Disallowed

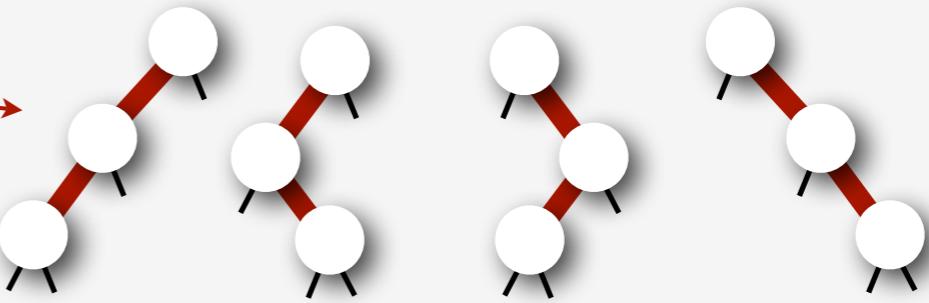
- right-leaning 3-node representation



*standard red-black trees  
allow this one*

- two reds in a row

*original version of left-leaning trees  
used this 4-node representation*



*single-rotation trees  
allow all of these*

# Java data structure for red-black trees

adds one bit for color to elementary BST data structure

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```
public class BST<Key extends Comparable<Key>, Value>
{
```

```
    private static final boolean RED = true; ← constants
    private static final boolean BLACK = false; ←
```

```
    private Node root;
```

```
    private class Node
    {
```

```
        Key key;
```

```
        Value val;
```

```
        Node left, right;
```

```
        boolean color; ←
```

```
        Node(Key key, Value val, boolean color)
```

```
        {
```

```
            this.key = key;
```

```
            this.val = val;
```

```
            this.color = color;
```

```
        }
```

```
    }
```

```
    public Value get(Key key)
```

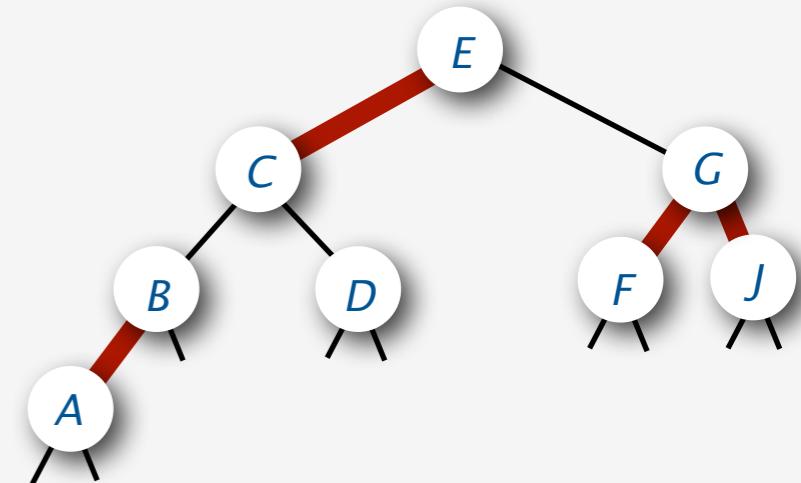
```
    // Search method.
```

```
    public void put(Key key, Value val)
```

```
    // Insert method.
```

```
}
```

*color of incoming link*



*helper method to test node color*

```
private boolean isRed(Node x)
{
    if (x == null) return false;
    return (x.color == RED);
}
```

# Search implementation for red-black trees

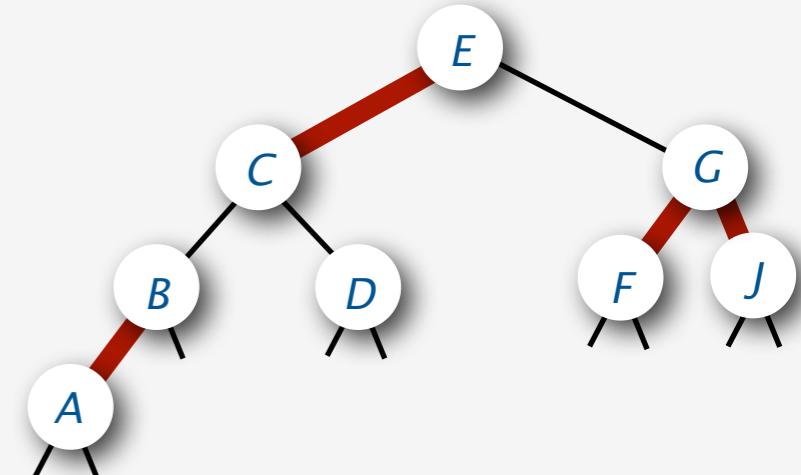
is the same as for elementary BSTs

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( but typically runs faster because of better balance in the tree).

*BST (and LLRB tree) search implementation*

```
public Value get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if (cmp == 0)      return x.val;
        else if (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
    }
    return null;
}
```



Important note: Other BST methods also work

- order statistics
- iteration

*Ex: Find the minimum key*

```
public Key min()
{
    Node x = root;
    while (x != null) x = x.left;
    if (x == null) return null;
    else           return x.key;
}
```

# Insert implementation for LLRB trees

is best expressed in a **recursive** implementation

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*Recursive insert() implementation for elementary BSTs*

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val);

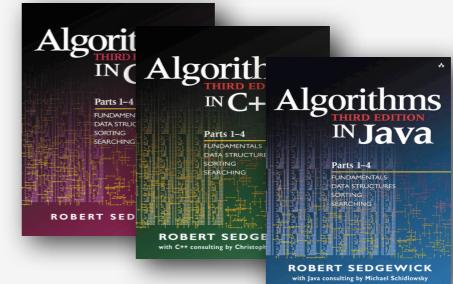
    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val; ← associative model
    if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

    return h;
}
```

*Nonrecursive*



*Recursive*



**Note:** effectively travels down the tree and then up the tree.

- simplifies correctness proof
- simplifies code for balanced BST implementations
- could remove recursion to get stack-based single-pass algorithm

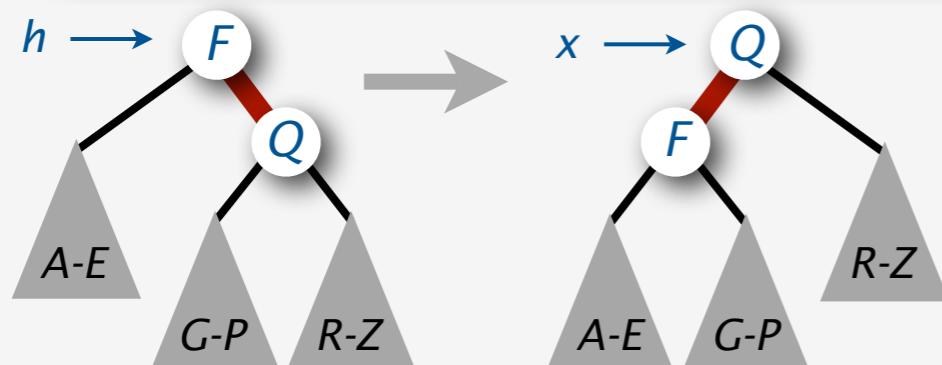
# Balanced tree code

is based on local transformations known as **rotations**

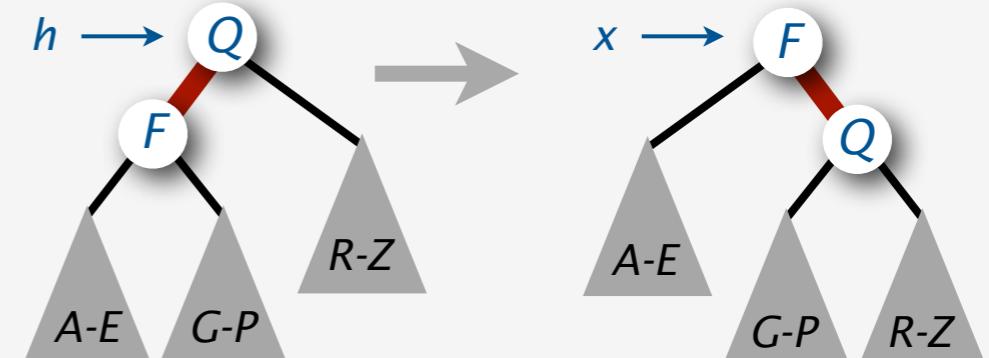
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In red-black trees, we only rotate red links  
(to maintain perfect black-link balance)

```
private Node rotateLeft(Node h)
{
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = x.left.color;
    x.left.color = RED;
    return x;
}
```



```
private Node rotateRight(Node h)
{
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = x.right.color;
    x.right.color = RED;
    return x;
}
```

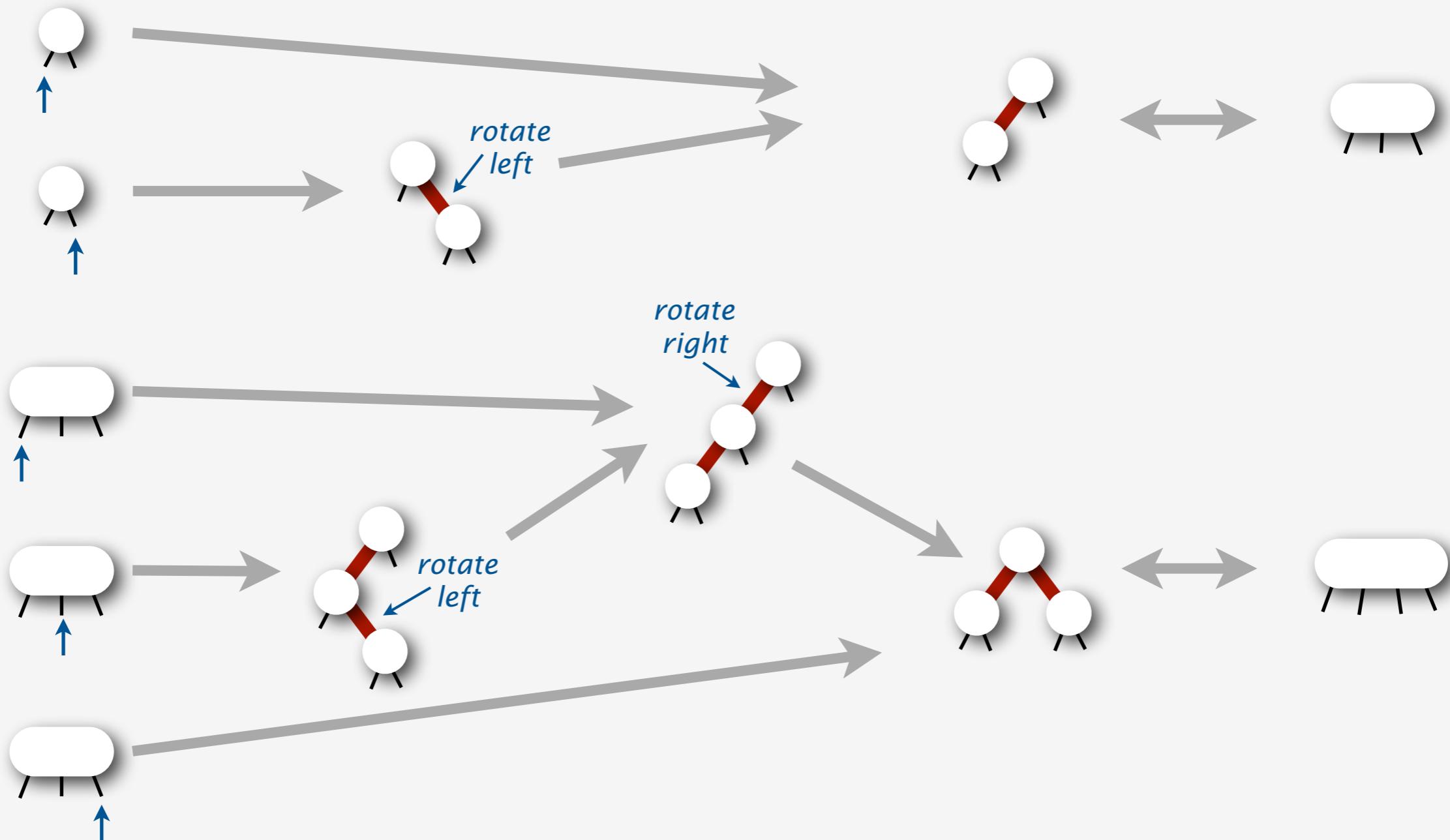


# Insert a new node at the bottom in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

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1. Add new node as usual, with red link to glue it to node above
2. **Rotate if necessary** to get correct 3-node or 4-node representation



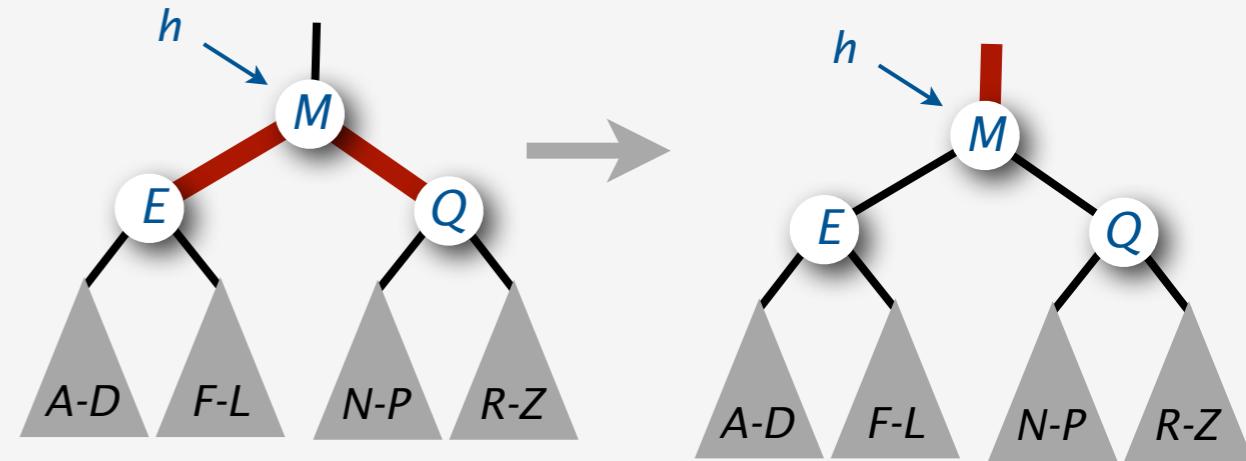
# Splitting a 4-node

is accomplished with a **color flip**

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Flip the colors of the three nodes

```
private Node colorFlip(Node h)
{
    x.color      = !x.color;
    x.left.color = !x.left.color;
    x.right.color = !x.right.color;
    return x;
}
```



Key points:

- preserves perfect black-bin balance
- passes a **RED** link up the tree
- reduces problem to inserting (that link) into parent

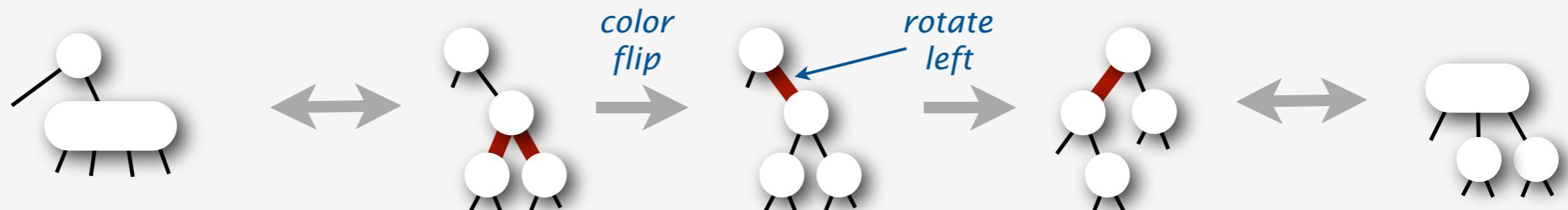
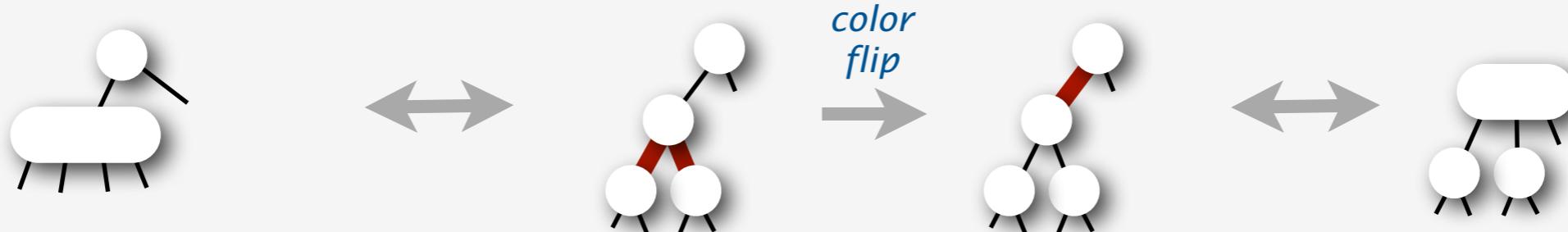
# Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. Flip colors, which passes red link up one level
2. Rotate if necessary to get correct representation in parent  
(using precisely the same transformations as for insert at bottom)

*Parent is a 2-node: two cases*



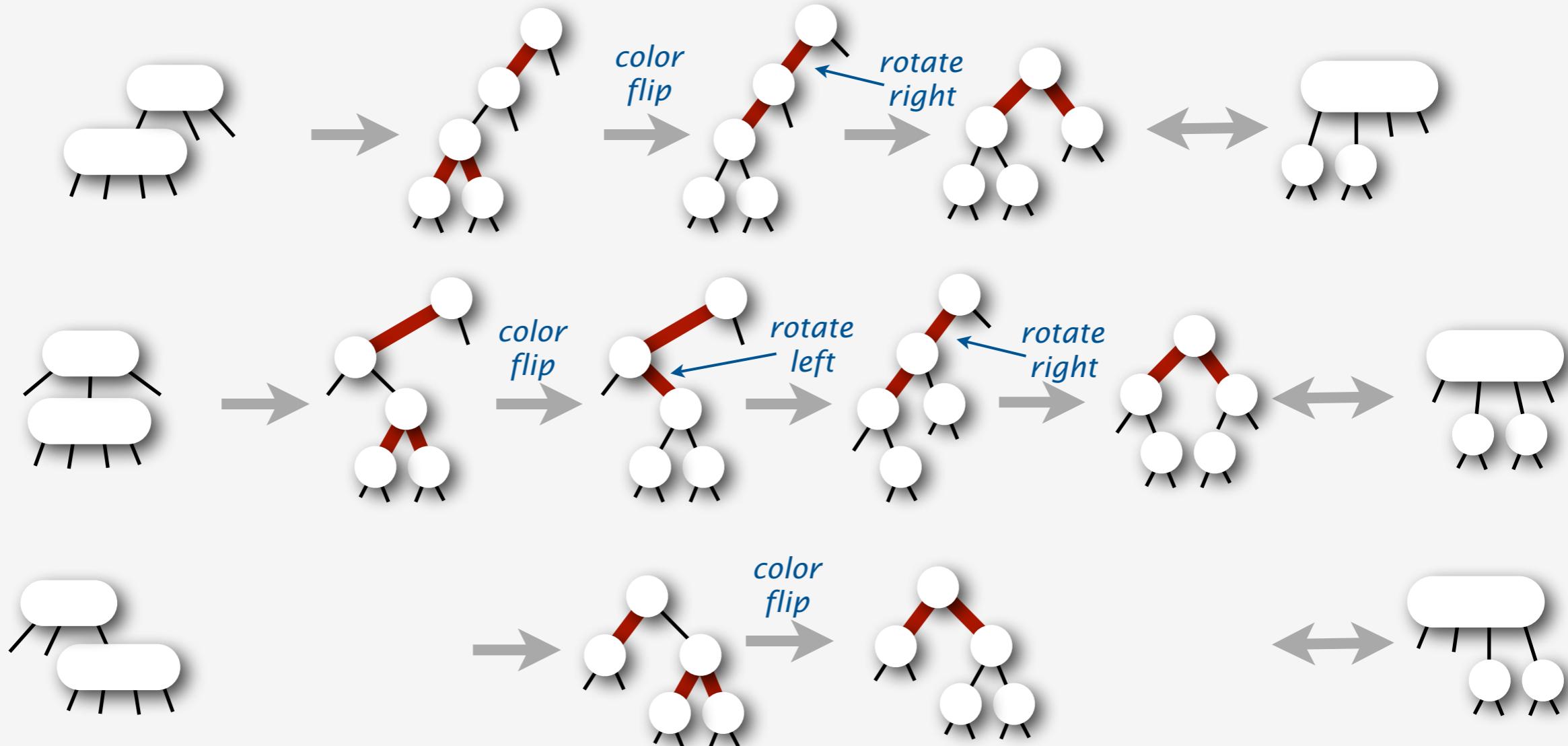
# Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. Flip colors, which passes red link up one level
2. Rotate if necessary to get correct representation in parent  
(using precisely the same transformations as for insert at bottom)

*Parent is a 3-node: three cases*



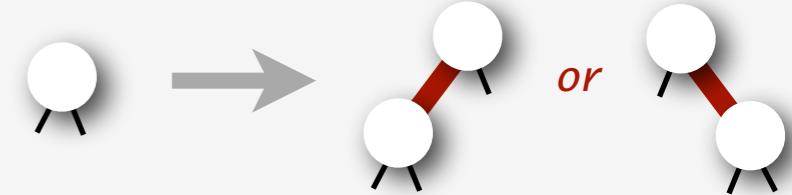
# Inserting and splitting nodes in LLRB trees

are easier when rotates are done on the way **up** the tree.

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

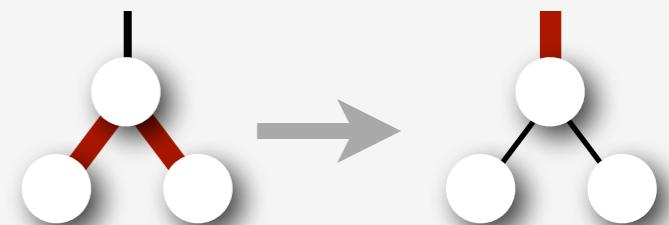
Search as usual

- if key found reset value, as usual
- if key not found insert new red node at the bottom
- might leave right-leaning red or two reds in a row higher up in the tree



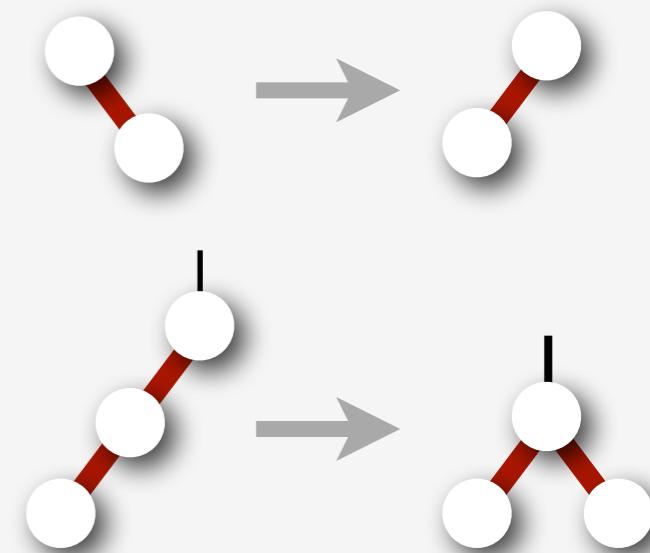
Split 4-nodes on the way down the tree.

- flip color
- might leave right-leaning red or two reds in a row higher up in the tree



**NEW TRICK:** Do rotates on the way **UP** the tree.

- left-rotate any right-leaning link on search path
- right-rotate top link if two reds in a row found
- trivial with recursion (do it after recursive calls)
- no corrections needed elsewhere



# Insert code for LLRB trees

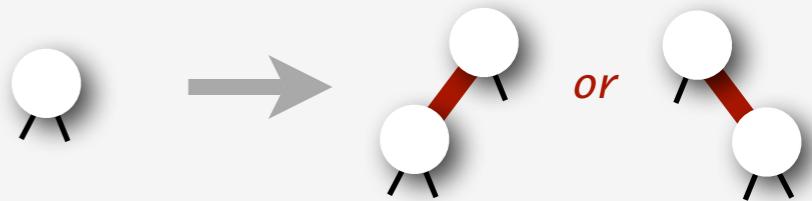
Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

is based on four simple operations.

## 1. Insert a new node at the bottom.

```
if (h == null)  
    return new Node(key, value, RED);
```

could be  
right or left



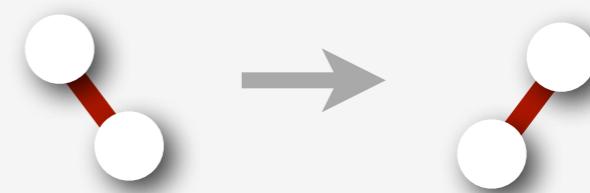
## 2. Split a 4-node.

```
if (isRed(h.left) && isRed(h.right))  
    colorFlip(h);
```



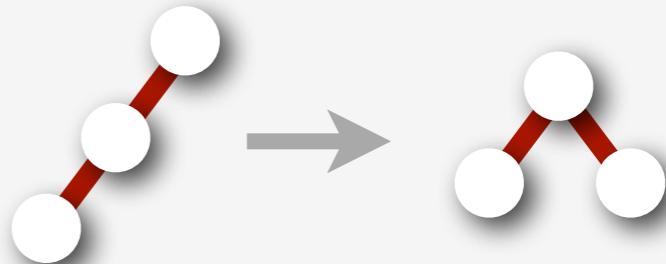
## 3. Enforce left-leaning condition.

```
if (isRed(h.right))  
    h = rotateLeft(h);
```



## 4. Balance a 4-node.

```
if (isRed(h.left) && isRed(h.left.left))  
    h = rotateRight(h);
```



# Insert implementation for LLRB trees

is a few lines of code added to elementary BST insert

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);           ← insert at the bottom

    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);                         ← split 4-nodes on the way down

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);      ← standard BST insert code
    else
        h.right = insert(h.right, key, val);

    if (isRed(h.right))
        h = rotateLeft(h);                   ← fix right-leaning reds on the way up

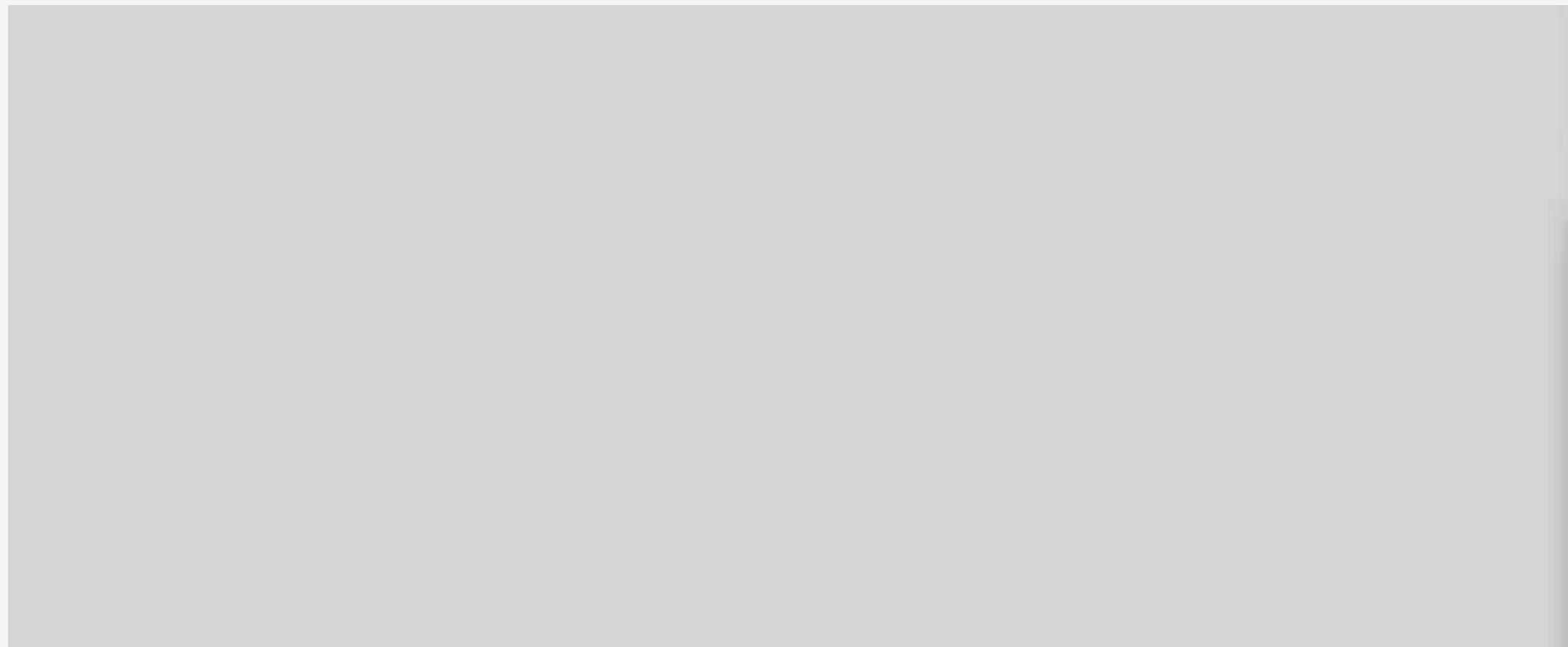
    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);                  ← fix two reds in a row on the way up

    return h;
}
```

# LLRB (top-down 2-3-4) insert movie

---

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*



# A surprise

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

Q. What happens if we move color flip to the end?

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);

    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

    if (isRed(h.right))
        h = rotateLeft(h);

    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);

    return h;
}
```

# A surprise

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

Q. What happens if we move color flip to the end?

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

    if (isRed(h.right))
        h = rotateLeft(h);

    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);

    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);

    return h;
}
```

# A surprise

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

Q. What happens if we move color flip to the end?

A. It becomes an implementation of 2-3 trees (!)

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

    if (isRed(h.right))
        h = rotateLeft(h);

    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);

    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);

    return h;
}
```

Insert in 2-3 tree:  
*attach new node  
with red link*  
*2-node → 3-node*  
*3-node → 4-node*  
*split 4-node*  
*pass red link up to  
parent and repeat*  
*no 4-nodes left!*

# Insert implementation for 2-3 trees (!)

is a few lines of code added to elementary BST insert

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);           ← insert at the bottom

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);      ← standard BST insert code

    if (isRed(h.right))
        h = rotateLeft(h);                     ← fix right-leaning reds on the way up

    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);                   ← fix two reds in a row on the way up

    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);                      ← split 4-nodes on the way up

    return h;
}
```

# LLRB (bottom-up 2-3) insert movie

---

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

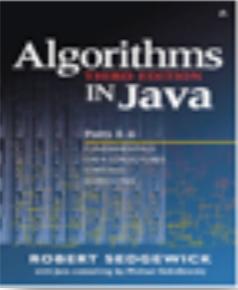


# Why revisit red-black trees?

Which do you prefer?

```
private Node insert(Node x, Key key, Value val, boolean sw)
{
    if (x == null)
        return new Node(key, value, RED);
    int cmp = key.compareTo(x.key);

    if (isRed(x.left) && isRed(x.right))
    {
        x.color = RED;
        x.left.color = BLACK;
        x.right.color = BLACK;
    }
    if (cmp == 0) x.val = val;
    else if (cmp < 0)
    {
        x.left = insert(x.left, key, val, false);
        if (isRed(x) && isRed(x.left) && sw)
            x = rotR(x);
        if (isRed(x.left) && isRed(x.left.left))
        {
            x = rotR(x);
            x.color = BLACK; x.right.color = RED;
        }
    }
    else // if (cmp > 0)
    {
        x.right = insert(x.right, key, val, true);
        if (isRed(h) && isRed(x.right) && !sw)
            x = rotL(x);
        if (isRed(h.right) && isRed(h.right.right))
        {
            x = rotL(x);
            x.color = BLACK; x.left.color = RED;
        }
    }
    return x;
}
```



```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

    if (isRed(h.right))
        h = rotateLeft(h);
    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);
    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);

    return h;
}
```

**Left-Leaning  
Red-Black Trees**  
Robert Sedgewick  
Princeton University

↑  
*straightforward*

*very  
tricky*

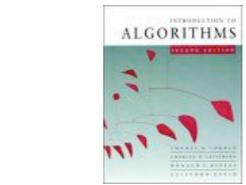
# Why revisit red-black trees?

Take your pick:

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

TreeMap.java

Adapted from  
CLR by  
experienced  
professional  
programmers  
(2004)



150

wrong scale!

Why revisit red-black trees?

Which do you prefer?

```
private Node insert(Node x, Key key, Value val, boolean sw)
{
    if (x == null)
        return new Node(key, value, RED);
    int cmp = key.compareTo(x.key);

    if (isRed(x.left) && isRed(x.right))
    {
        x.color = RED;
        x.left.color = BLACK;
        x.right.color = BLACK;
    }
    if (cmp == 0) x.val = val;
    else if (cmp < 0)
    {
        x.left = insert(x.left, key, val, false);
        if (isRed(x) && isRed(x.left) && sw)
            x = rotR(x);
        if (isRed(x.left) && isRed(x.left.left))
        {
            x = rotR(x);
            x.color = BLACK; x.right.color = RED;
        }
    } else // if (cmp > 0)
    {
        x.right = insert(x.right, key, val, true);
        if (isRed(h) && isRed(x.right) && !sw)
            x = rotL(x);
        if (isRed(h.right) && isRed(h.right.right))
        {
            x = rotL(x);
            x.color = BLACK; x.left.color = RED;
        }
    }
    return x;
}
```



```
private Node insert(Node h, Key key, Value val)
{
    if (h == null)
        return new Node(key, val, RED);

    int cmp = key.compareTo(h.key);
    if (cmp == 0) h.val = val;
    else if (cmp < 0)
        h.left = insert(h.left, key, val);
    else
        h.right = insert(h.right, key, val);

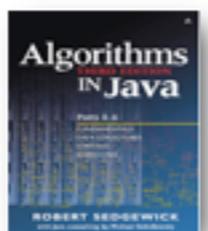
    if (isRed(h.right))
        h = rotateLeft(h);
    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);
    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);

    return h;
}
```

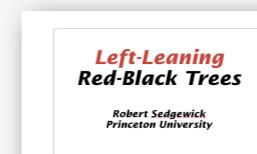


straightforward

very  
tricky



46



33

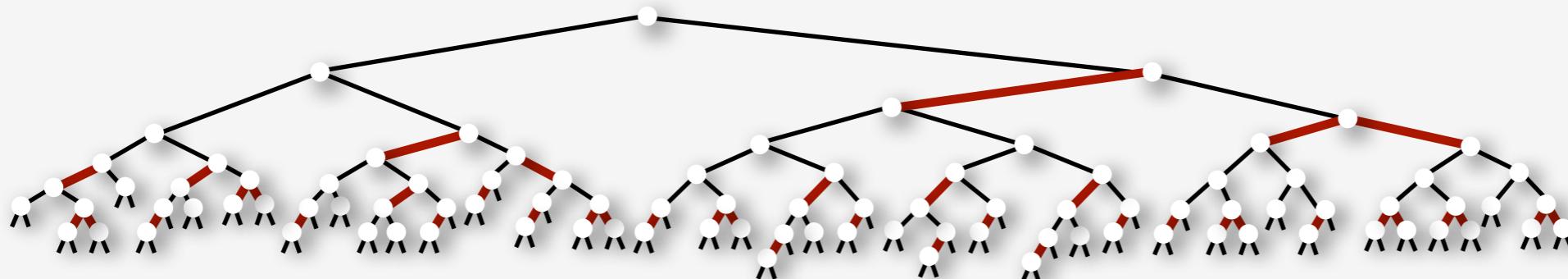
← lines of code for insert  
(lower is better!)

# Why revisit red-black trees?

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

LLRB implementation is **far simpler** than previous attempts.

- left-leaning restriction reduces number of cases
- recursion gives two (easy) chances to fix each node
- take your pick: **top-down 2-3-4 or bottom-up 2-3**



2008  
1978

1972

Improves widely used implementations

- AVL, 2-3, and 2-3-4 trees
- red-black trees

Same ideas simplify implementation of other operations

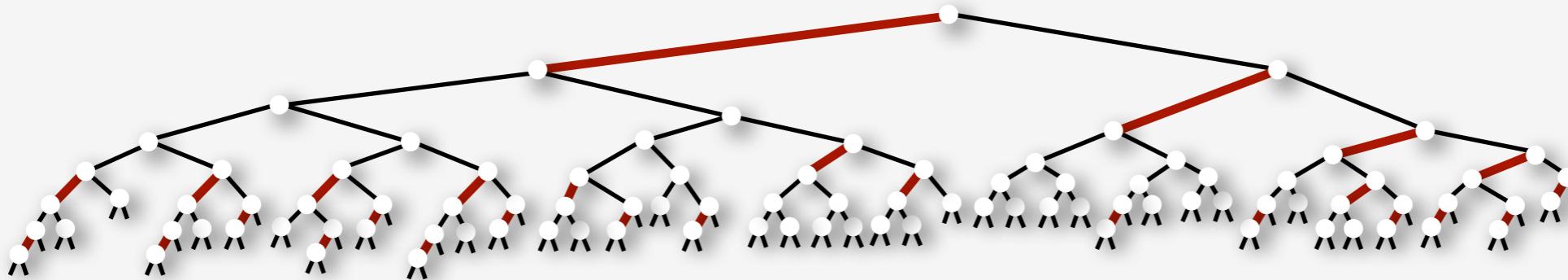
- delete min, max
- arbitrary delete

# Why revisit red-black trees?

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

LLRB implementation is **far simpler** than previous attempts.

- left-leaning restriction reduces number of cases
- recursion gives two (easy) chances to fix each node
- take your pick: top-down 2-3-4 or **bottom-up 2-3**



2008  
1978

1972

Improves widely used implementations

- AVL, 2-3, and 2-3-4 trees
- red-black trees

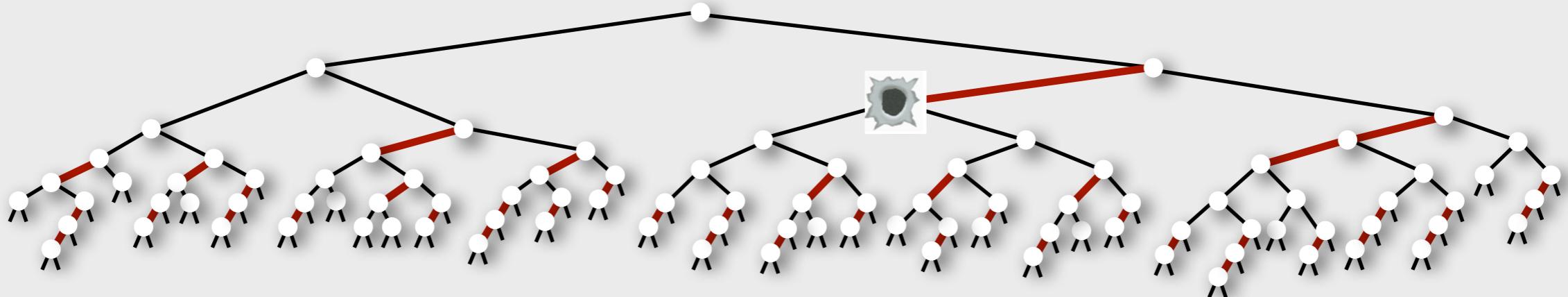
Same ideas simplify implementation of other operations

- delete min, max
- arbitrary delete

*Introduction*  
*2-3-4 Trees*  
*LLRB Trees*

***Deletion***

*Analysis*



# Lessons learned from insert() implementation

also simplify delete() implementations

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. Color flips and rotations preserve perfect black-link balance.
2. Fix right-leaning reds and eliminate 4-nodes on the way up.

```
private Node fixUp(Node h)
{
    if (isRed(h.right))
        h = rotateLeft(h);
    if (isRed(h.left) && isRed(h.left.left))
        h = rotateRight(h);
    if (isRed(h.left) && isRed(h.right))
        colorFlip(h);
    return h;
}
```

← *rotate-left right-leaning reds*  
← *rotate-right red-red pairs*  
← *split 4-nodes*

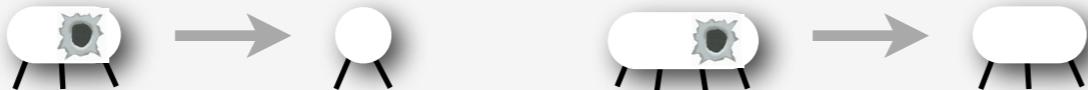
## Delete strategy (works for 2-3 and 2-3-4 trees)

- invariant: **current node is not a 2-node**
- introduce 4-nodes if necessary
- remove key from bottom
- eliminate 4-nodes on the way up

# Warmup 1: delete the maximum

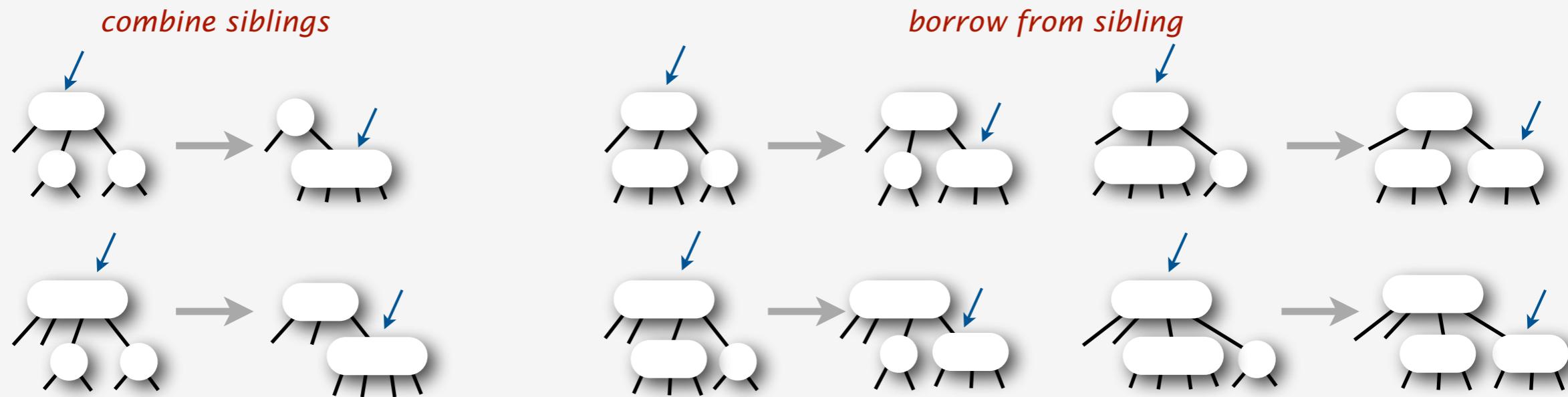
Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. Search down the right spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.



3. Removing a 2-node would destroy balance

- transform tree on the way down the search path
- Invariant: **current node is not a 2-node**



Note: LLRB representation reduces number of cases (as for insert)

# Warmup 1: delete the maximum

by carrying a red link **down** the right spine of the tree.

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

Invariant: either **h** or **h.right** is **RED**

Implication: deletion easy at bottom



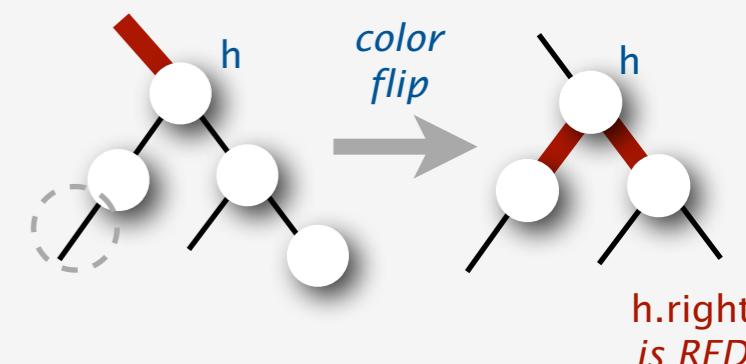
1. Rotate red links to the right



*Easy case: h.left.left is BLACK*

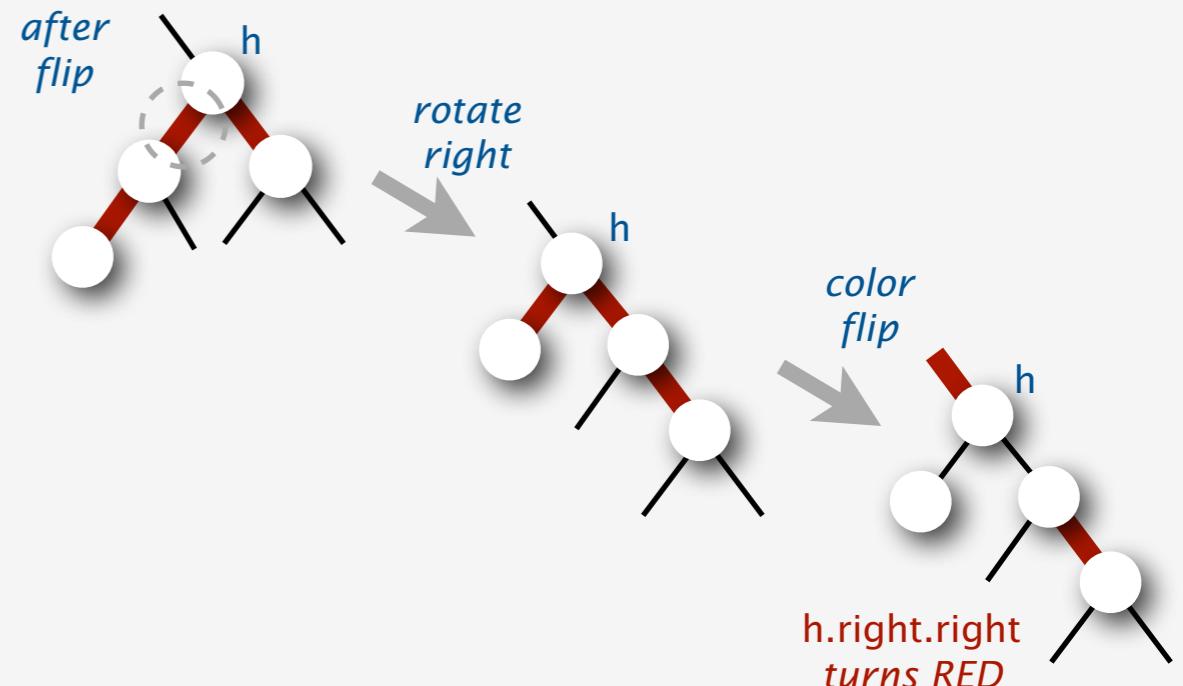
2. Borrow from sibling if necessary

- when **h.right** and **h.right.left** are both **BLACK**
- Two cases, depending on color of **h.left.left**



```
private Node moveRedRight(Node h)
{
    colorFlip(h);
    if (isRed(h.left.left))
    {
        h = rotateRight(h);
        colorFlip(h);
    }
    return h;
}
```

*Harder case: h.left.left is RED*



# deleteMax() implementation for LLRB trees

is otherwise a few lines of code

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

```
public void deleteMax()
{
    root = deleteMax(root);
    root.color = BLACK;
}

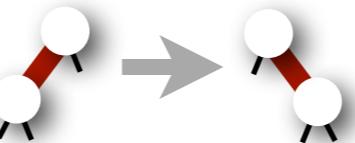
private Node deleteMax(Node h)
{
    if (isRed(h.left))
        h = rotateRight(h);

    if (h.right == null)
        return null;

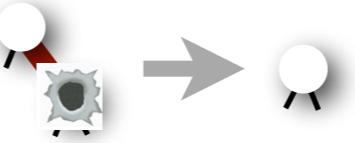
    if (!isRed(h.right) && !isRed(h.right.left))
        h = moveRedRight(h);

    h.left = deleteMax(h.left);

    return fixUp(h);
}
```



*lean 3-nodes to the right*



*remove node on bottom level  
(h must be RED by invariant)*

*borrow from sibling if necessary*

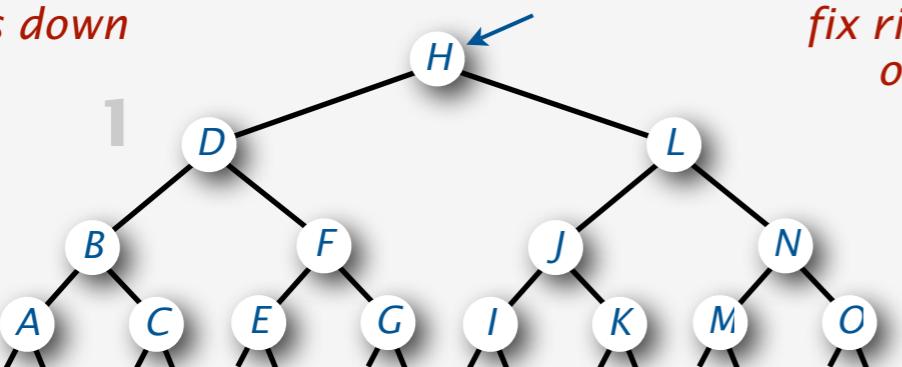
*move down one level*

*fix right-leaning red links  
and eliminate 4-nodes  
on the way up*

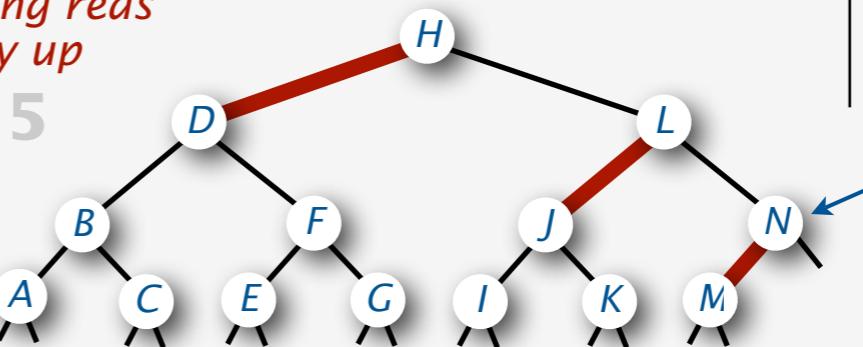
# deleteMax() example 1

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

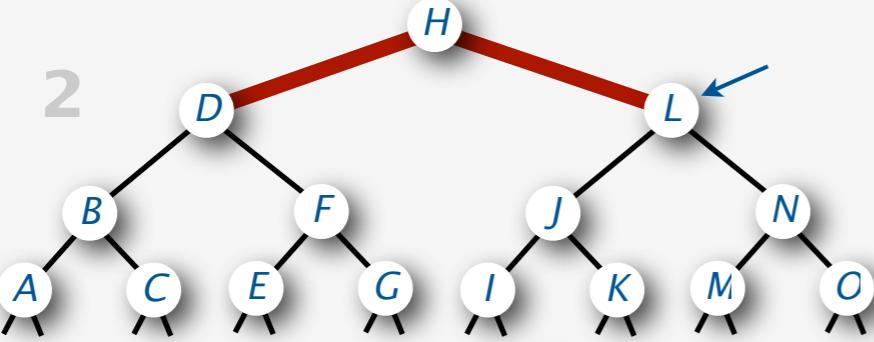
*push reds down*



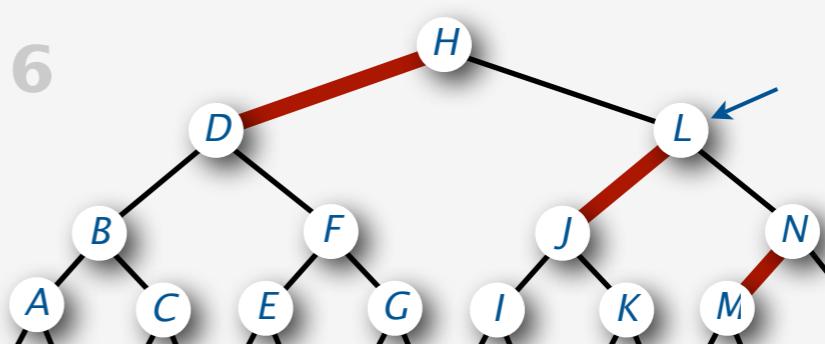
*fix right-leaning reds  
on the way up*



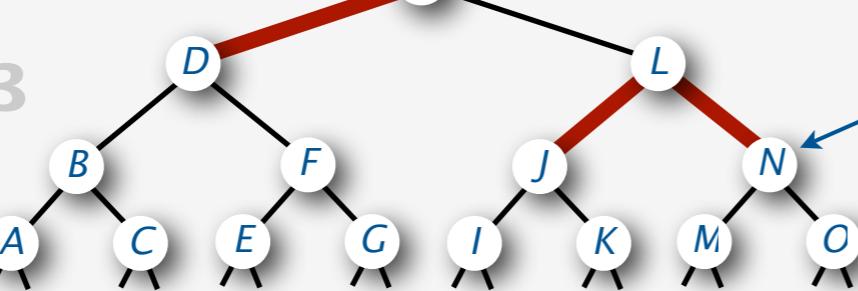
2



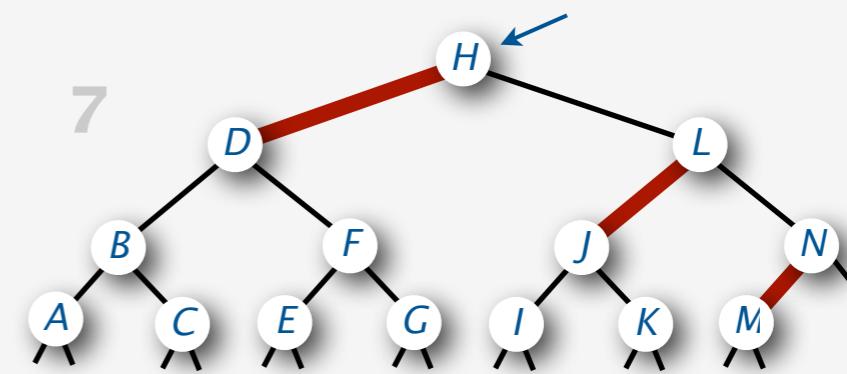
6



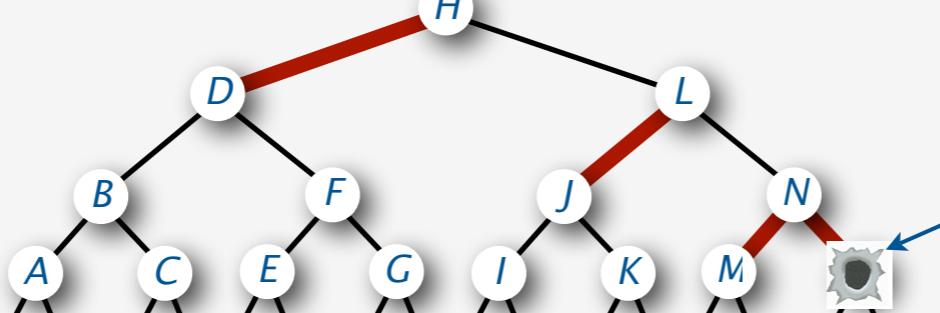
3



7



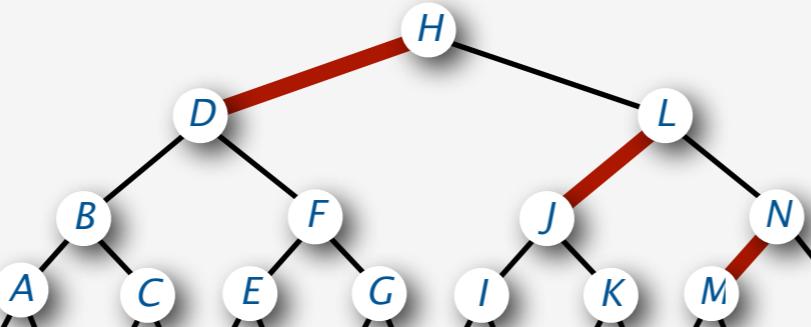
4



*(nothing to fix!)*

*remove maximum*

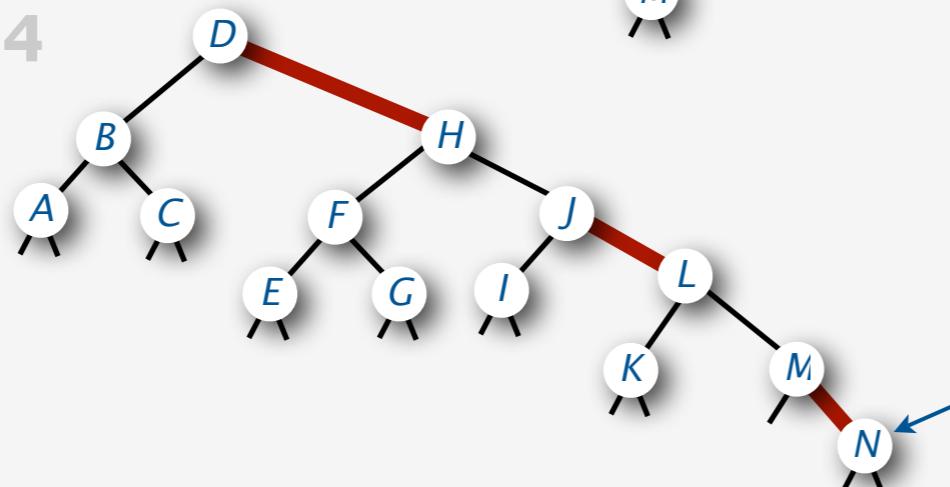
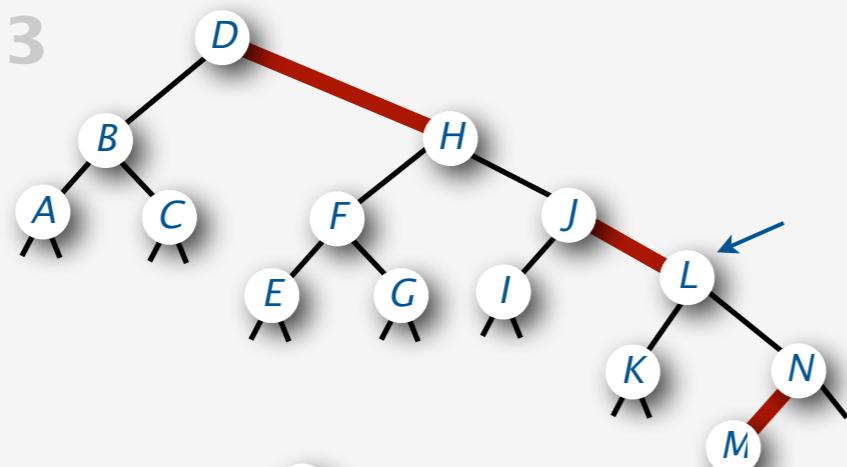
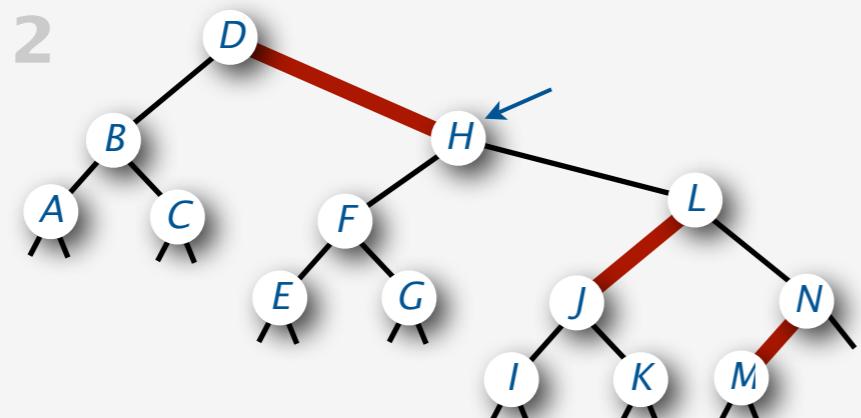
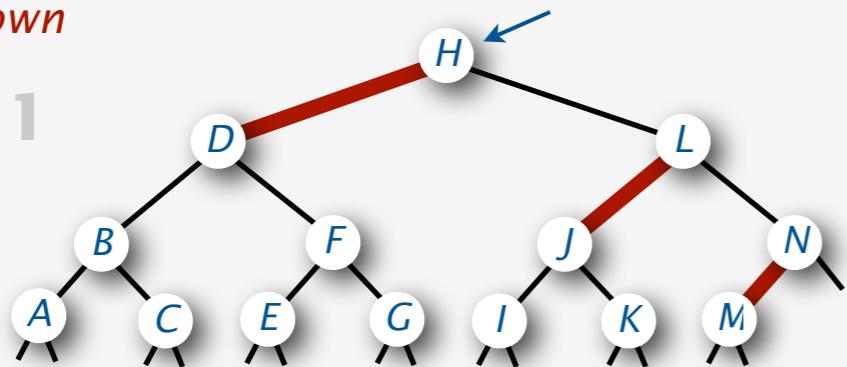
5



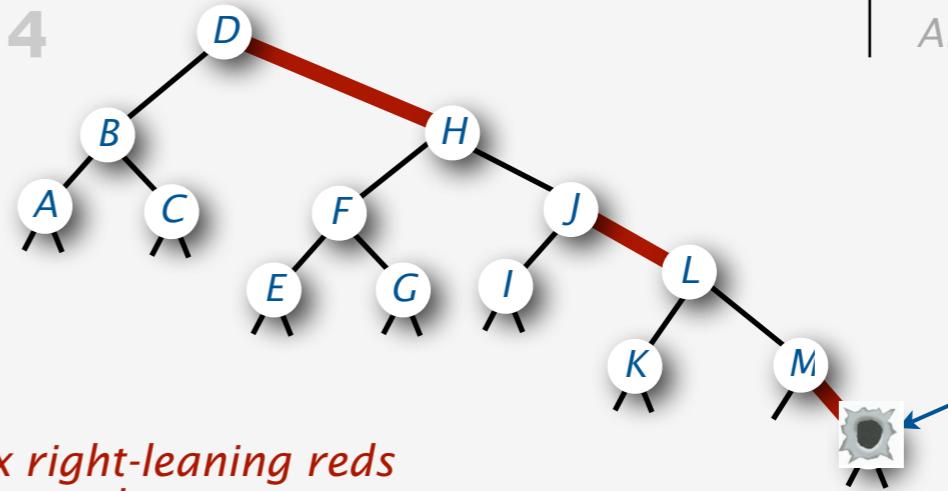
# deleteMax() example 2

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

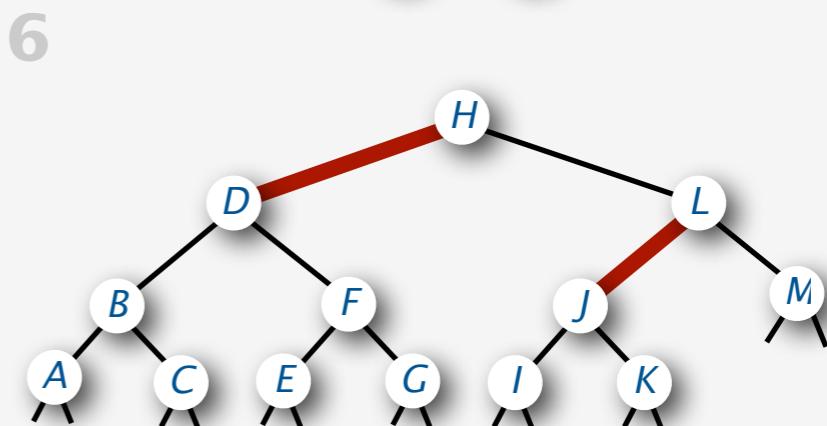
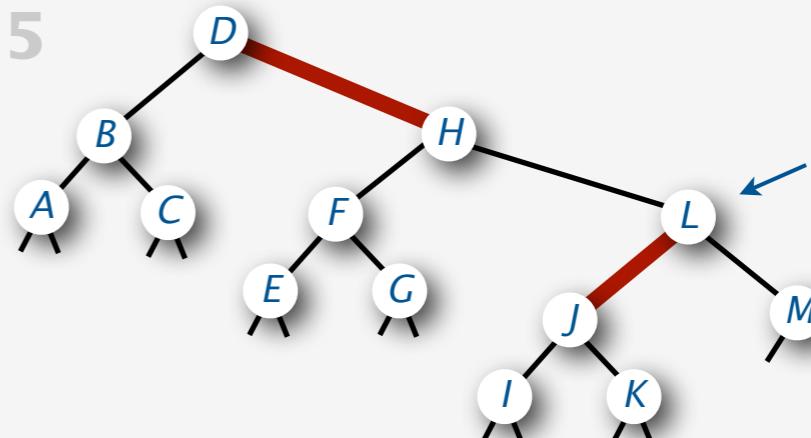
*push reds down*



*remove maximum*



*fix right-leaning reds  
on the way up*



# LLRB deleteMax() movie

---

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*



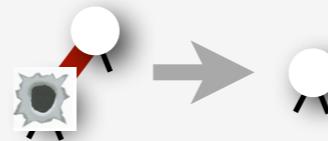
# Warmup 2: delete the minimum

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

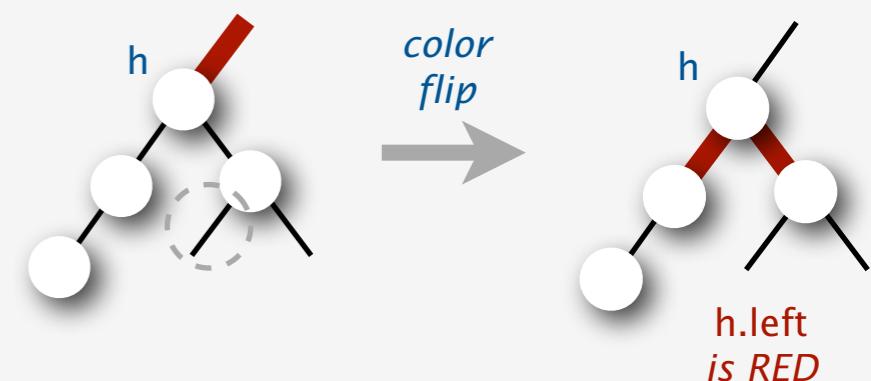
is similar but slightly different (since trees lean left).

Invariant: either **h** or **h.left** is **RED**

Implication: deletion easy at bottom



Easy case: **h.right.left** is **BLACK**

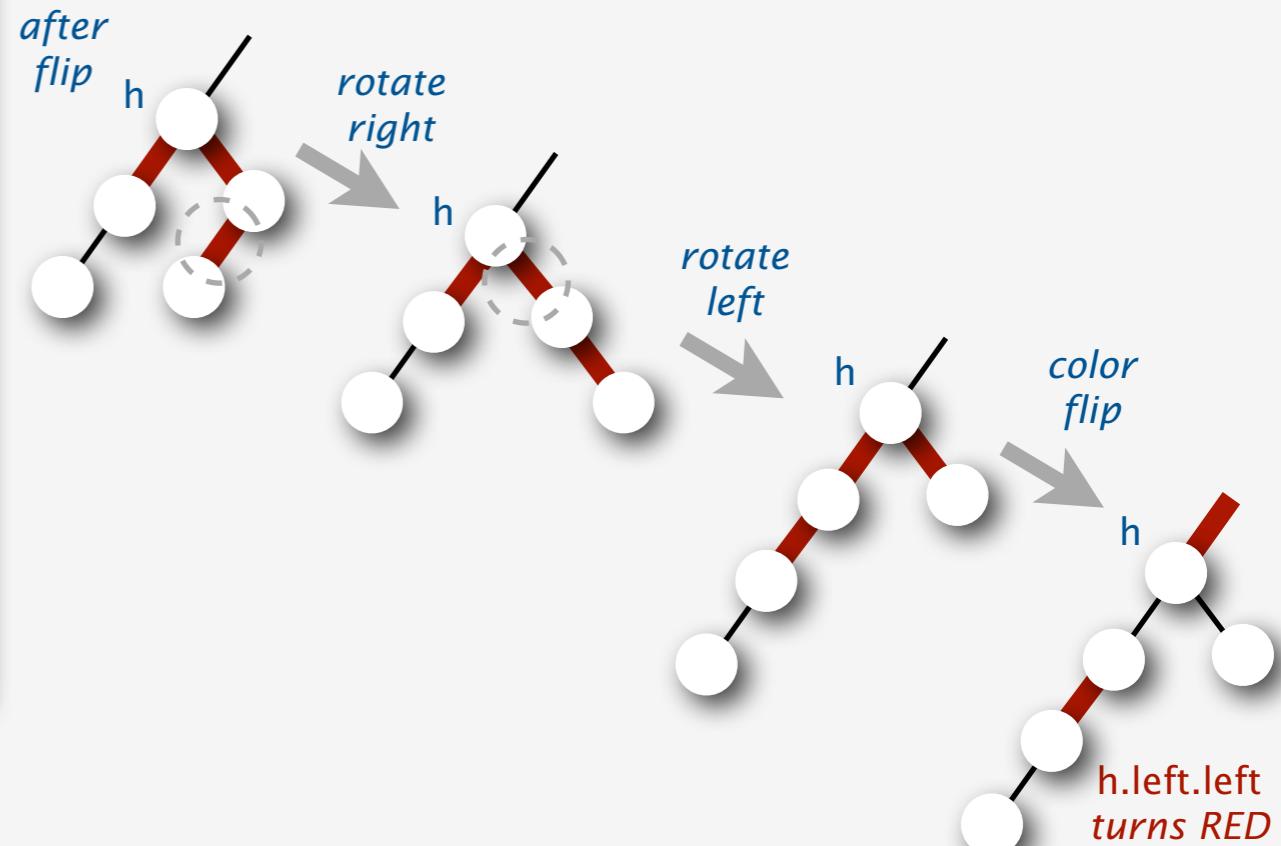


Borrow from sibling

- if **h.left** and **h.left.left** are both **BLACK**
- two cases, depending on color of **h.right.left**

```
private Node moveRedLeft(Node h)
{
    colorFlip(h);
    if (isRed(h.right.left))
    {
        h.right = rotateRight(h.right);
        h = rotateLeft(h);
        colorFlip(h);
    }
    return h;
}
```

Harder case: **h.right.left** is **RED**

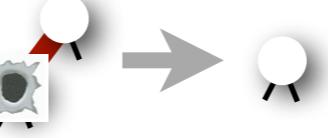


# deleteMin() implementation for LLRB trees

is a few lines of code

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

```
public void deleteMin()
{
    root = deleteMin(root);
    root.color = BLACK;
}

private Node deleteMin(Node h)
{
    if (h.left == null)
        return null; 
```

*remove node on bottom level  
(h must be RED by invariant)*

*push red link down if necessary*

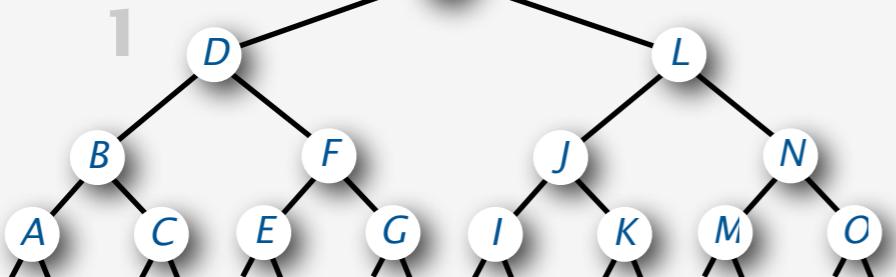
*move down one level*

*fix right-leaning red links  
and eliminate 4-nodes  
on the way up*

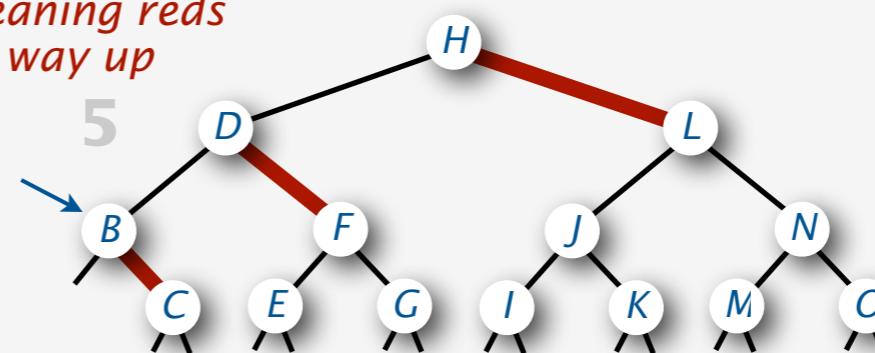
# deleteMin() example

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

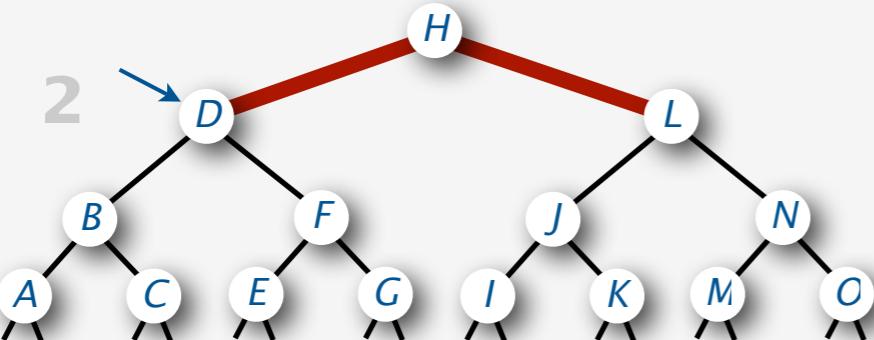
*push reds down*



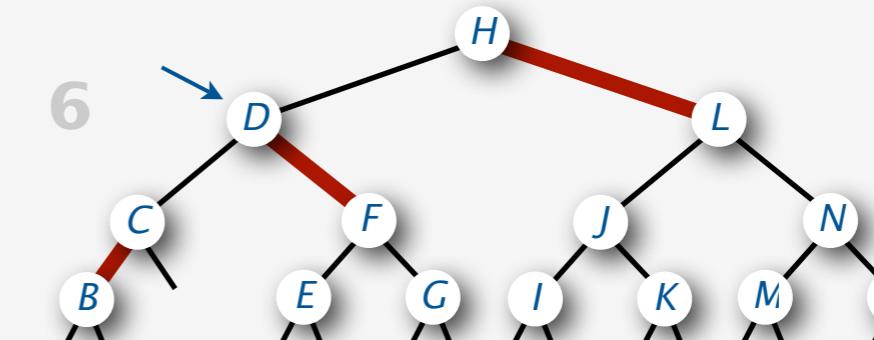
*fix right-leaning reds  
on the way up*



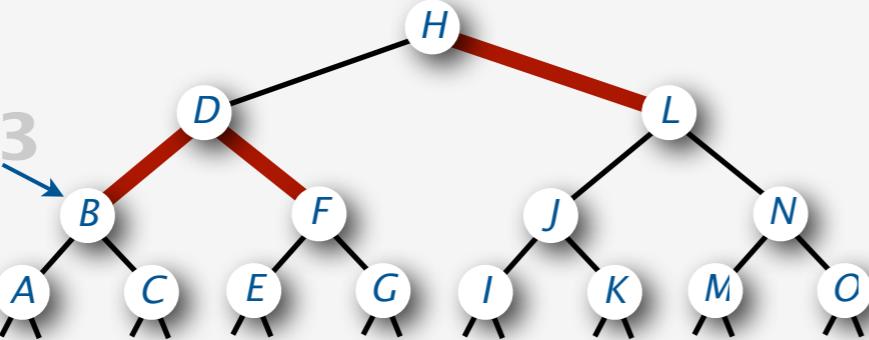
2



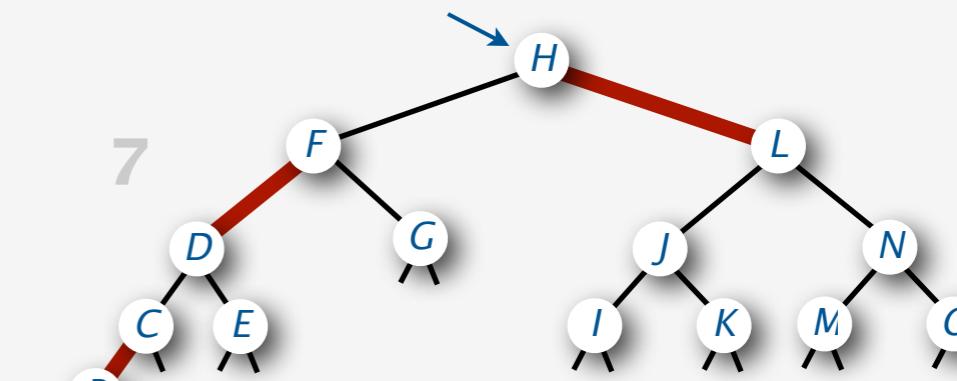
6



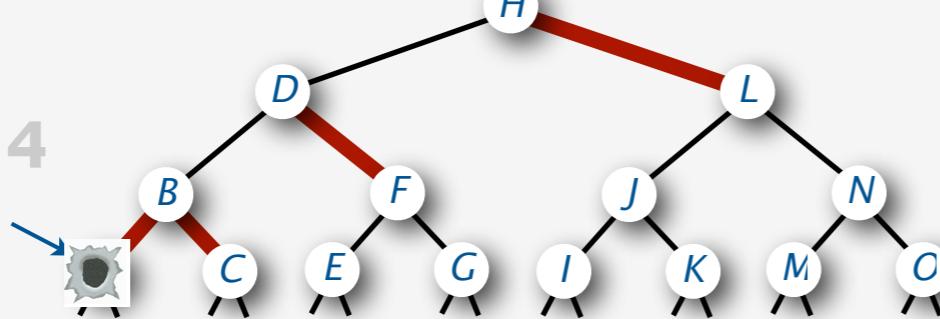
3



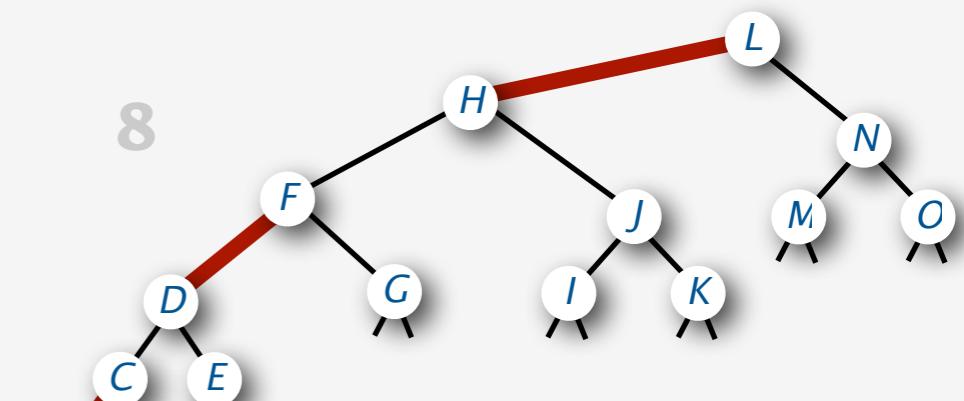
7



4

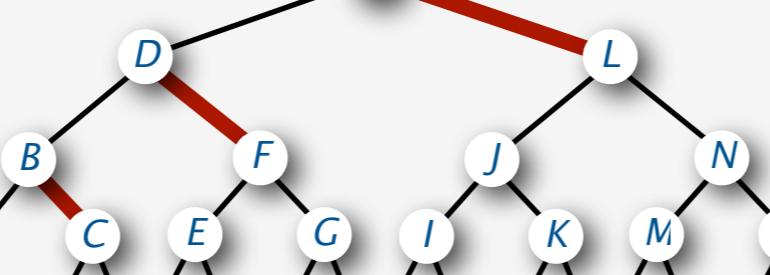


8



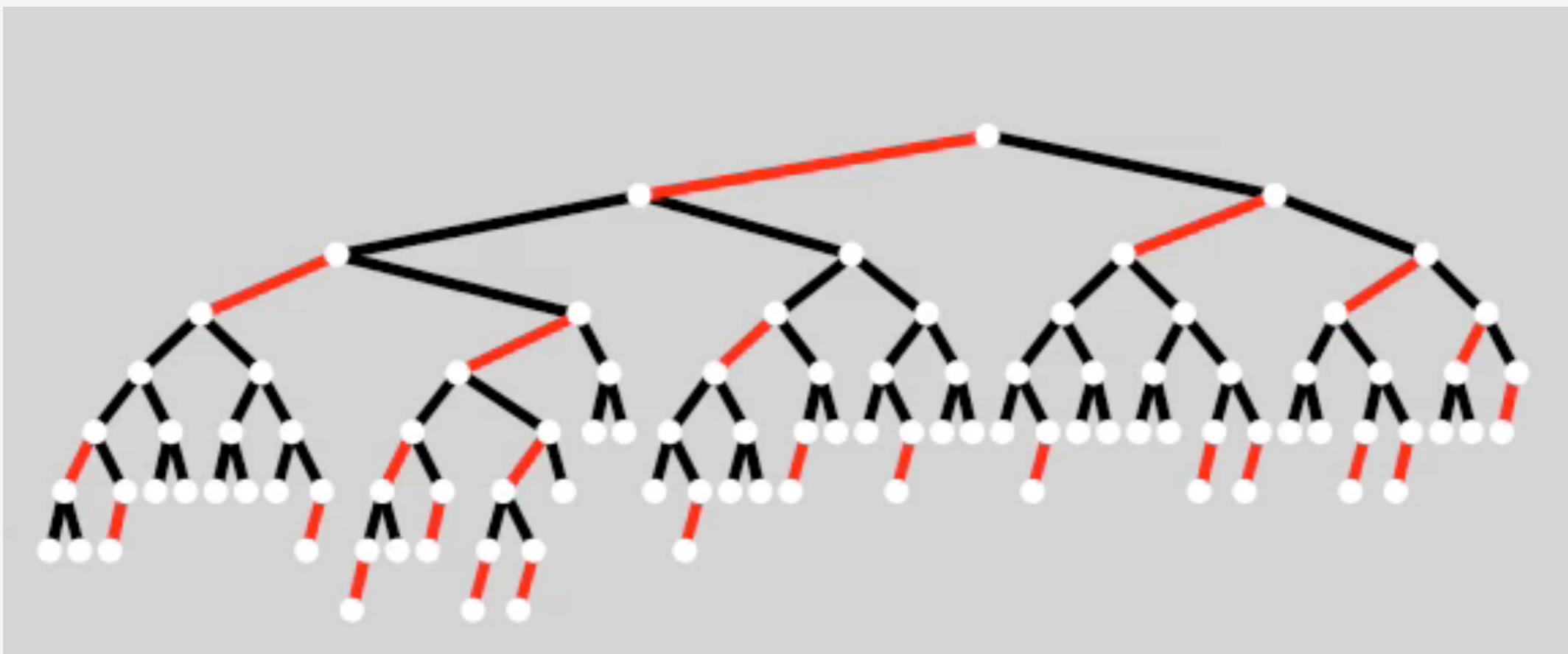
*remove minimum*

5



# LLRB deleteMin() movie

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

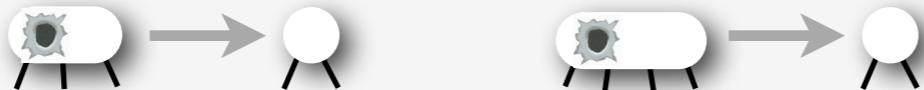


# Deleting an arbitrary node

involves the same general strategy.

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. Search down the left spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.



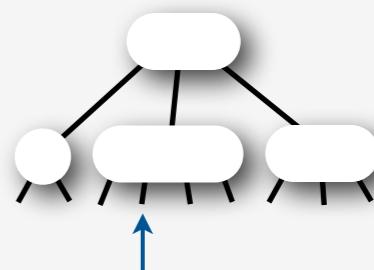
3. Removing a 2-node would destroy balance
  - transform tree on the way down the search path
  - Invariant: current node is not a 2-node

## Difficulty:

- Far too many cases!
- LLRB representation **dramatically** reduces the number of cases.

**Q:** How many possible search paths in **two** levels ?

**A:**  $9 * 6 + 27 * 9 + 81 * 12 = 1269 (! !)$

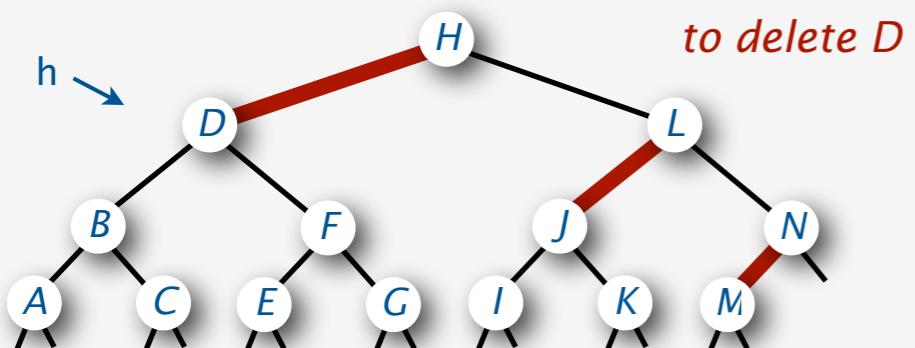


# Deleting an arbitrary node

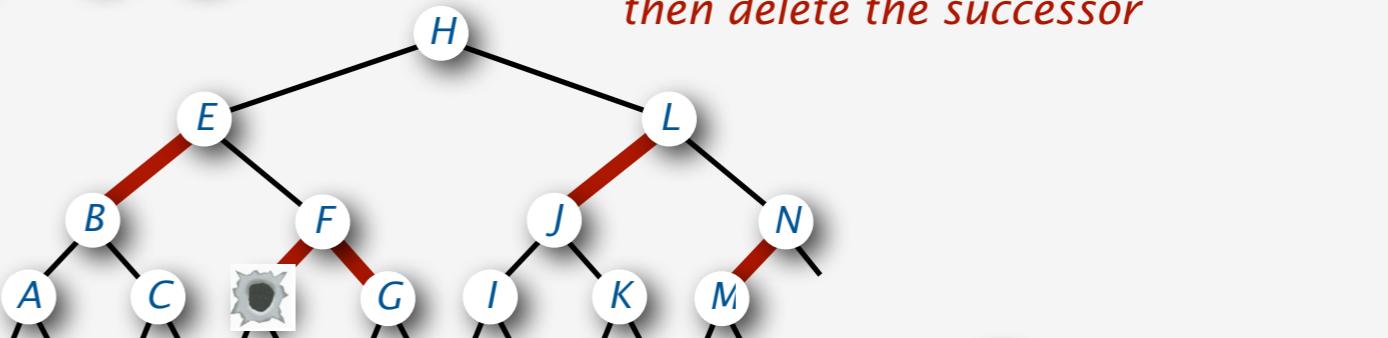
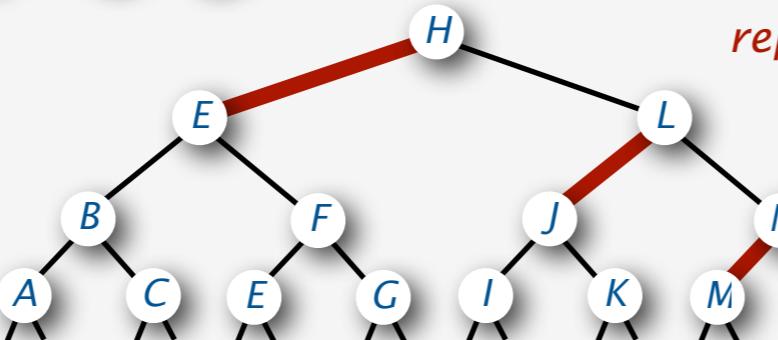
reduces to `deleteMin()`

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

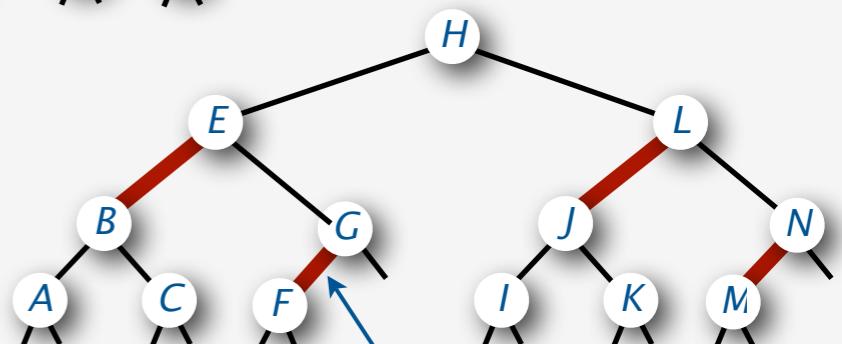
A standard trick:



```
h.key    = min(h.right);  
h.value = get(h.right, h.key);  
h.right = deleteMin(h.right);
```



fix right-leaning red link



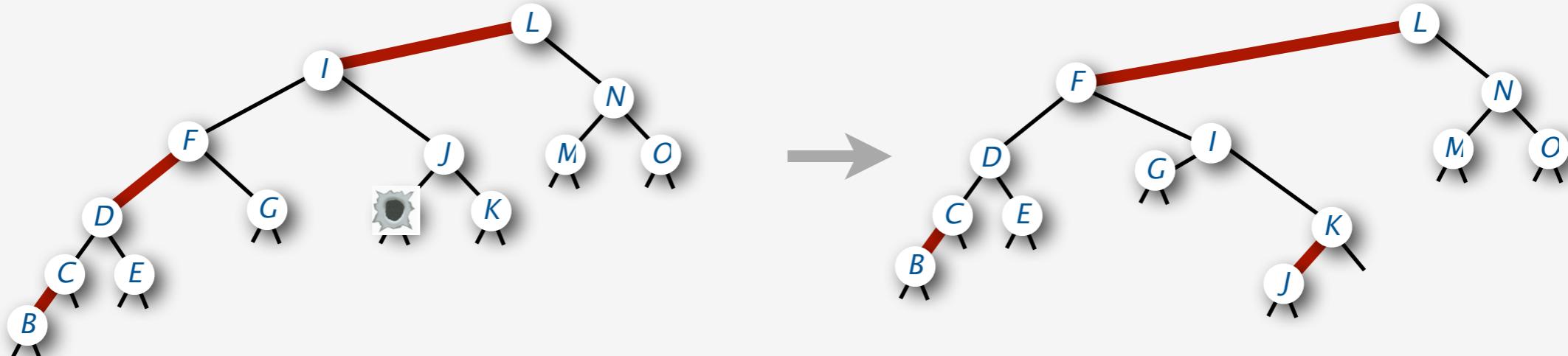
# Deleting an arbitrary node at the bottom

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

can be implemented with the **same** helper methods used for deleteMin() and deleteMax().

Invariant: **h** or one of its children is **RED**

- search path goes left: use moveRedLeft().
- search path goes right: use moveRedRight().
- delete node at bottom
- fix right-leaning reds on the way up



# delete() implementation for LLRB trees

```
private Node delete(Node h, Key key)
{
    int cmp = key.compareTo(h.key);
    if (cmp < 0)
    {
        if (!isRed(h.left) && !isRed(h.left.left))
            h = moveRedLeft(h);
        h.left = delete(h.left, key);
    }
    else
    {
        if (isRed(h.left)) h = leanRight(h);

        if (cmp == 0 && (h.right == null))
            return null;

        if (!isRed(h.right) && !isRed(h.right.left))
            h = moveRedRight(h);

        if (cmp == 0)
        {
            h.key = min(h.right);
            h.value = get(h.right, h.key);
            h.right = deleteMin(h.right);
        }
        else h.right = delete(h.right, key);
    }

    return fixUp(h);
}
```

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

LEFT

*push red right if necessary  
move down (left)*

RIGHT or EQUAL

*rotate to push red right  
EQUAL (at bottom)  
delete node*

*push red right if necessary*

*EQUAL (not at bottom)*

*replace current node with  
successor key, value*

*delete successor*

*move down (right)*

*fix right-leaning red links  
and eliminate 4-nodes  
on the way up*

# LLRB delete() movie

---

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*



Red-black-tree implementations in widespread use:

- are based on pseudocode with “case bloat”
- use parent pointers (!)
- 400+ lines of code for core algorithms

## Left-leaning red-black trees

- you just saw all the code
- single pass (remove recursion if concurrency matters)
- <80 lines of code for core algorithms
- less code implies faster insert, delete
- less code implies easier maintenance and migration

2008  
1978

1972



insert

delete

helper

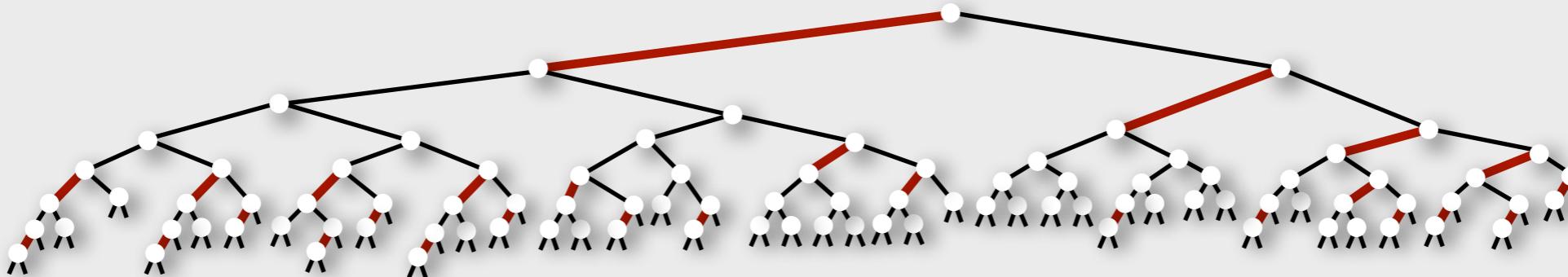
insert

delete

helper

← accomplishes the same result with less than 1/5 the code

*Introduction*  
*2-3-4 Trees*  
*LLRB Trees*  
*Deletion*  
***Analysis***



# Worst-case analysis

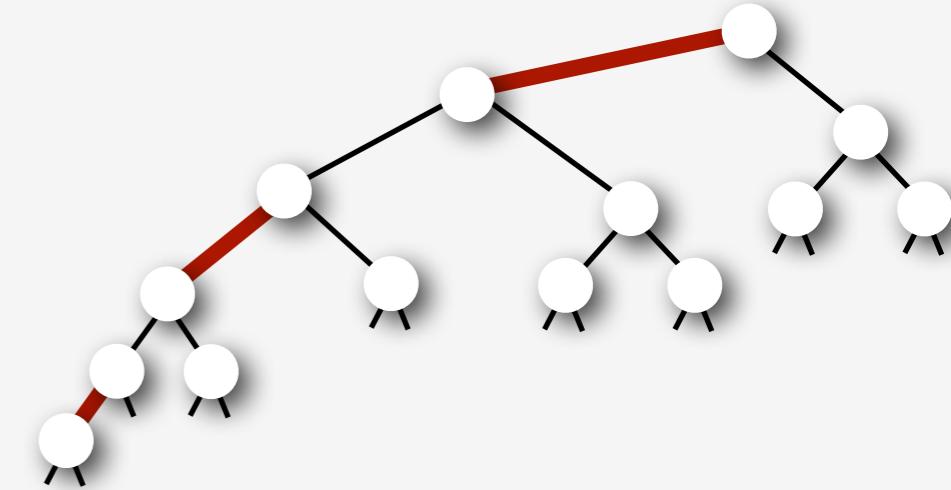
follows immediately from 2-3-4 tree correspondence

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

1. All trees have perfect black balance.
2. No two red links in a row on any path.

Shortest path:  $\lg N$  (all black)

Longest path:  $2 \lg N$  (alternating red-black)



**Theorem:** *With red-black BSTs as the underlying data structure, we can implement an ordered symbol-table API that supports insert, delete, delete the minimum, delete the maximum, find the minimum, find the maximum, rank, select the kth largest, and range count in guaranteed logarithmic time.*

Red-black trees are the method of choice for many applications.

# One remaining question

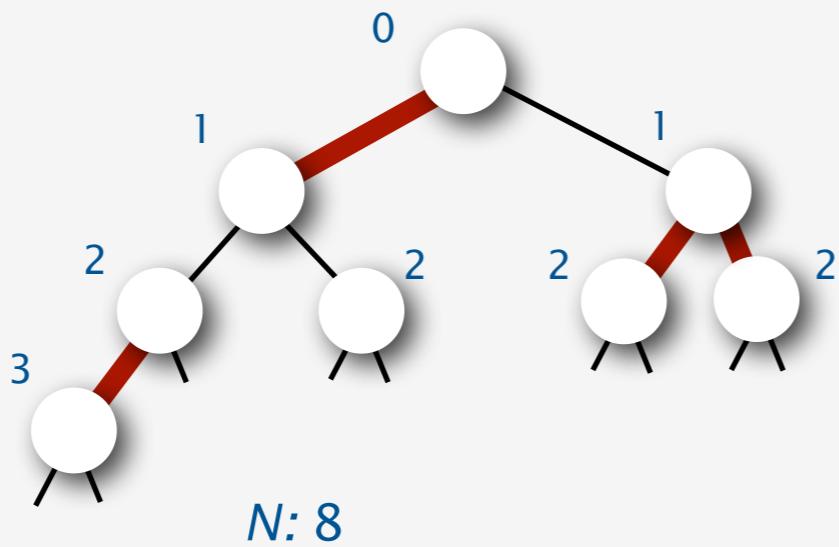
that is of interest in typical applications

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

The number of **searches** far exceeds the number of inserts.

Q. What is the cost of a typical search?

A. If each tree node is equally likely to be sought, compute the internal path length of the tree and divide by N.



*internal path length:  $0 + 1 + 1 + 2 + 2 + 2 + 2 + 3 = 13$*

*average search cost:  $13/8 = 1.625$*

Q. What is the **expected internal path length** of a tree built with randomly ordered keys (average cost of a search)?

# Average-case analysis of balanced trees

---

deserves another look!

Main questions:

Is average path length in tree built from random keys  $\sim c \lg N$  ?  
If so, is  $c = 1$  ?

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

# Average-case analysis of balanced trees

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

deserves another look!

Main questions:

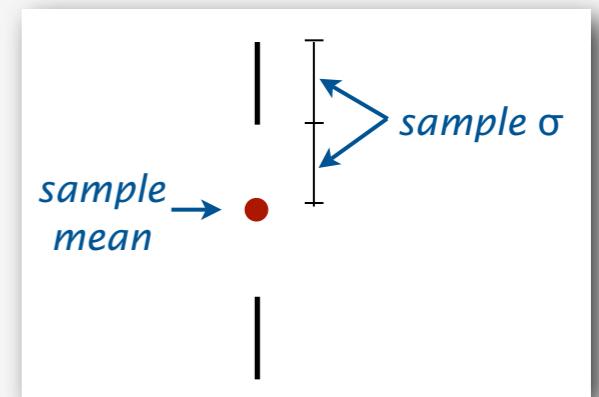
Is average path length in tree built from random keys  $\sim c \lg N$ ?  
If so, is  $c = 1$ ?

Experimental evidence

Ex: Tufte plot of average path length in 2-3 trees

- $N = 100, 200, \dots, 50,000$
- 100 trees each size

Tufte plot



# Average-case analysis of balanced trees

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis

deserves another look!

Main questions:

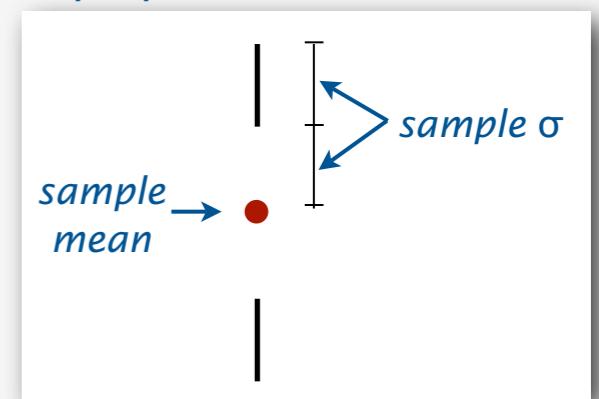
Is average path length in tree built from random keys  $\sim c \lg N$ ?  
If so, is  $c = 1$ ?

Experimental evidence strongly suggests YES!

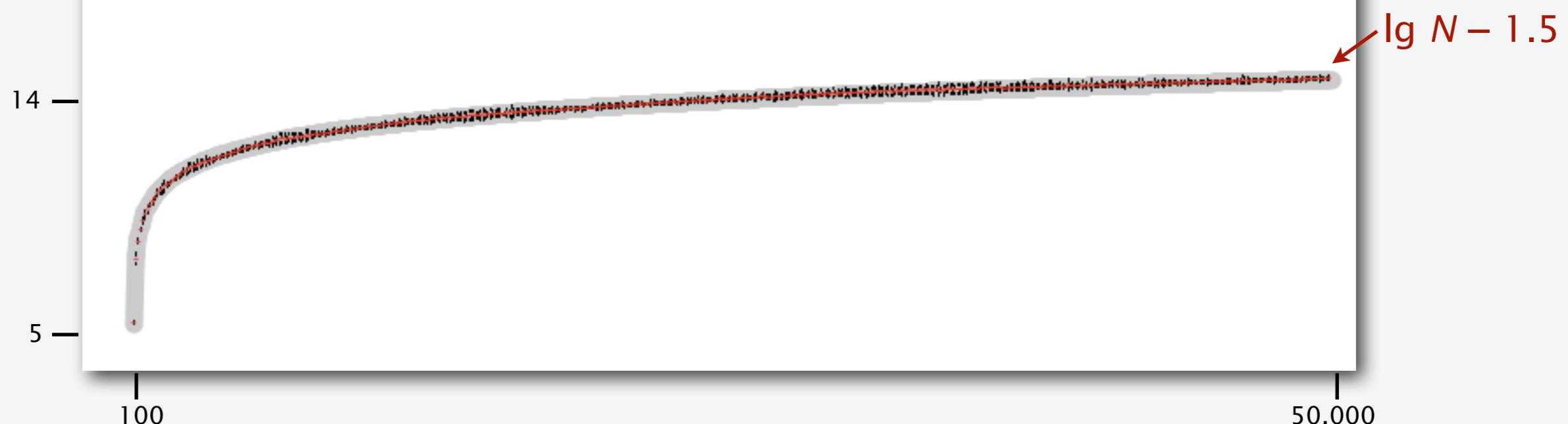
Ex: Tufte plot of average path length in 2-3 trees

- $N = 100, 200, \dots, 50,000$
- 100 trees each size

Tufte plot



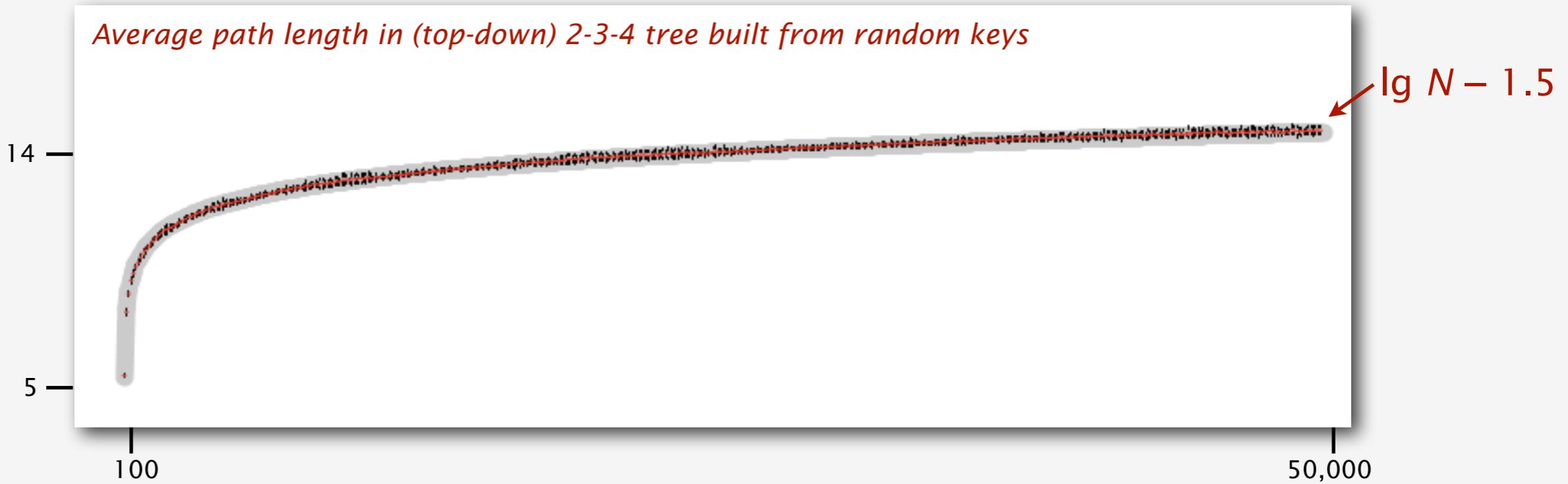
Average path length in 2-3 tree built from random keys



# Experimental evidence

can suggest and confirm hypotheses

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



deserves another look!

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

Main questions:

Is average path length in tree built from random keys  $\sim c \lg N$ ?  
If so, is  $c = 1$ ?

Some known facts:

- worst case gives easy  $2 \lg N$  upper bound
- fringe analysis gives upper bound of  $c_k \lg N$  with  $c_k > 1$
- analytic combinatorics gives path length in random trees

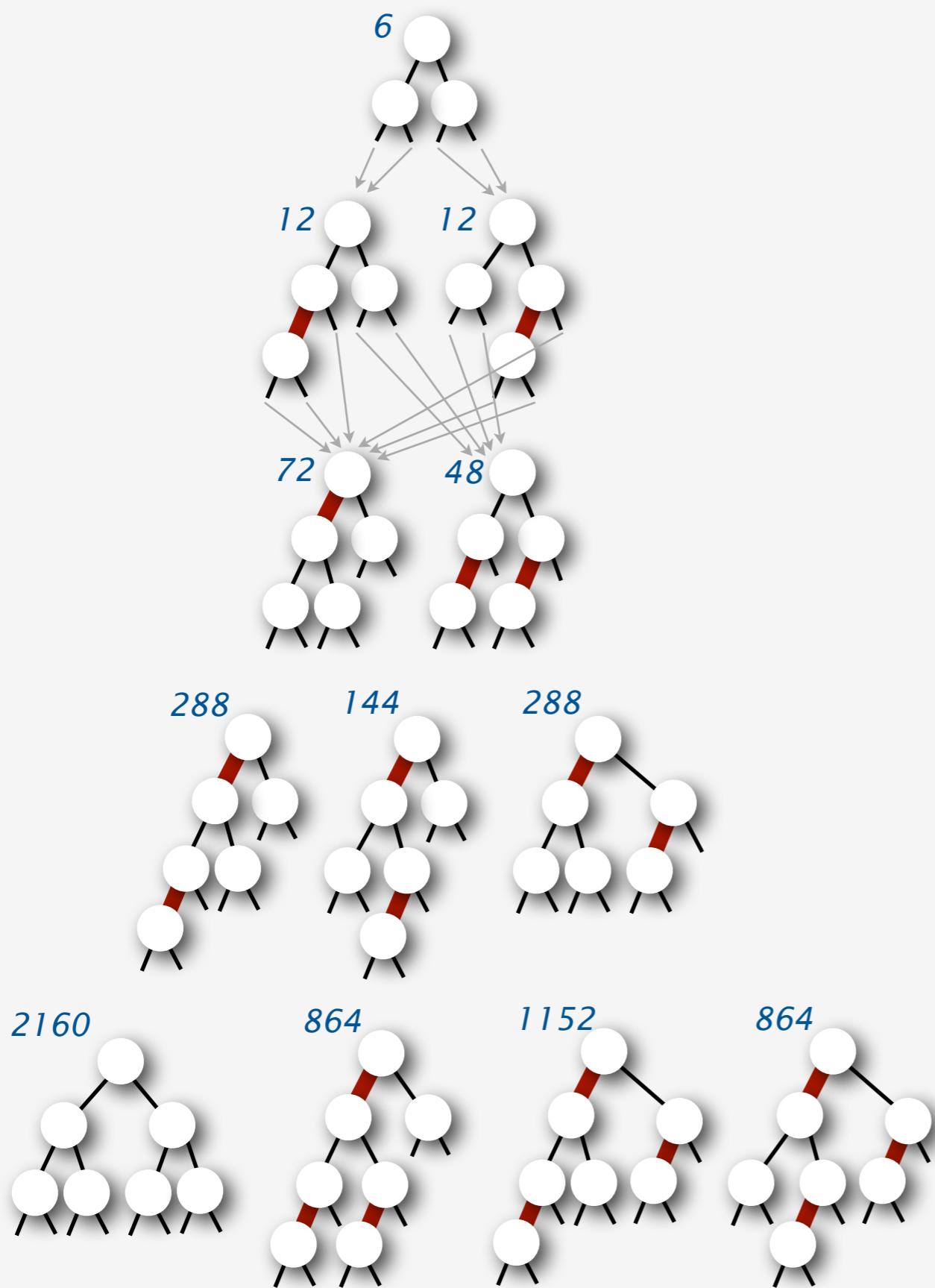
Are simpler implementations simpler to analyze?

Is the better experimental evidence that is now available helpful?

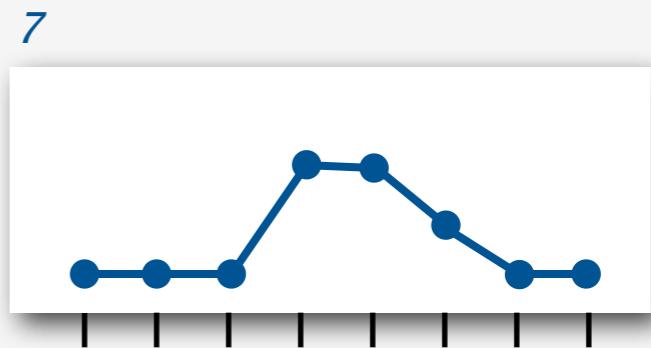
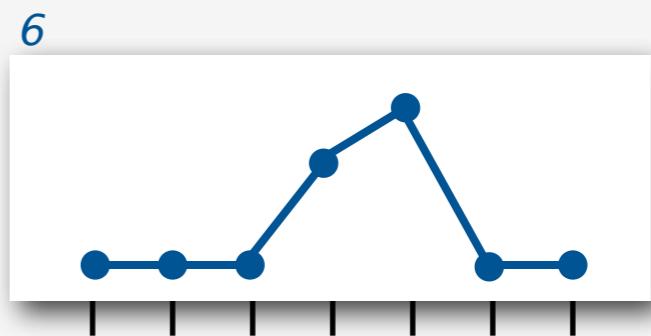
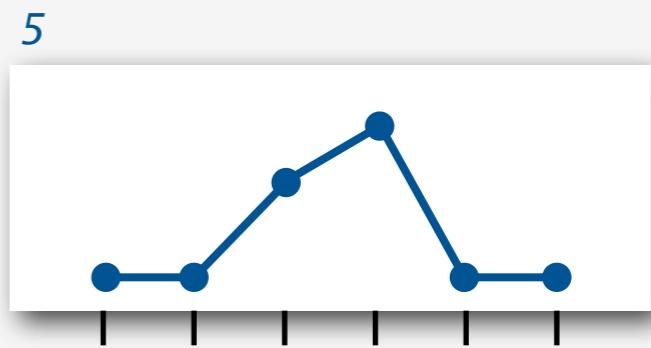
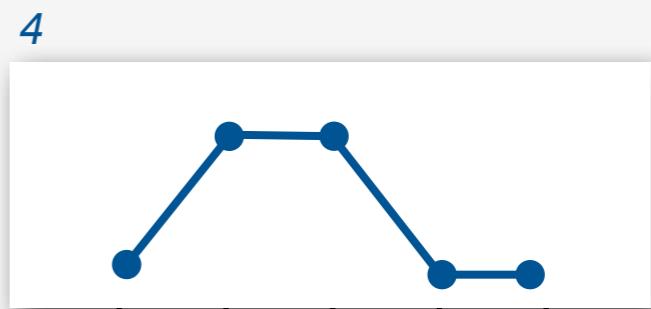
A starting point: study balance at the root (left subtree size)

# Left subtree size in left-leaning 2-3 trees

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



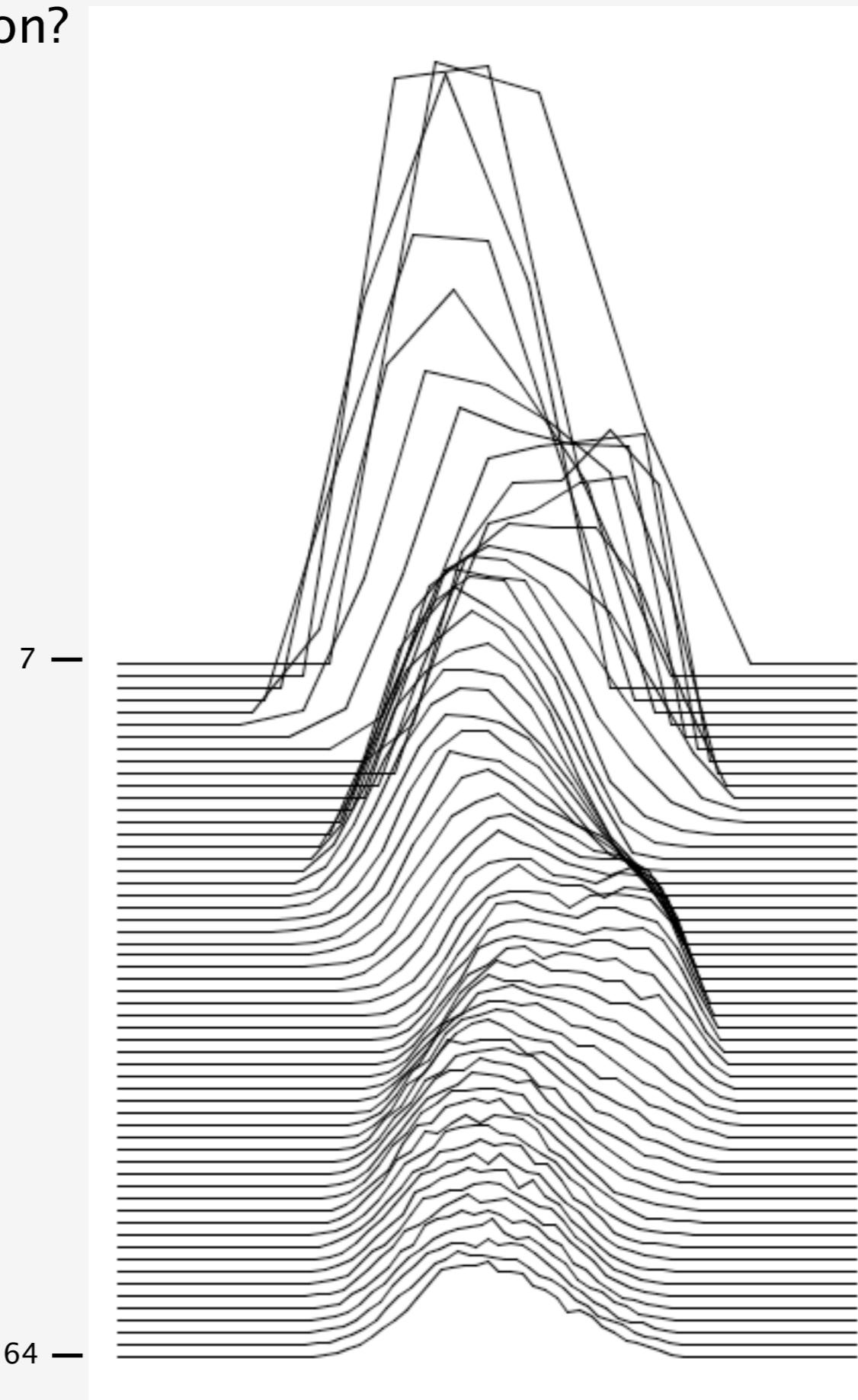
*Exact distributions*



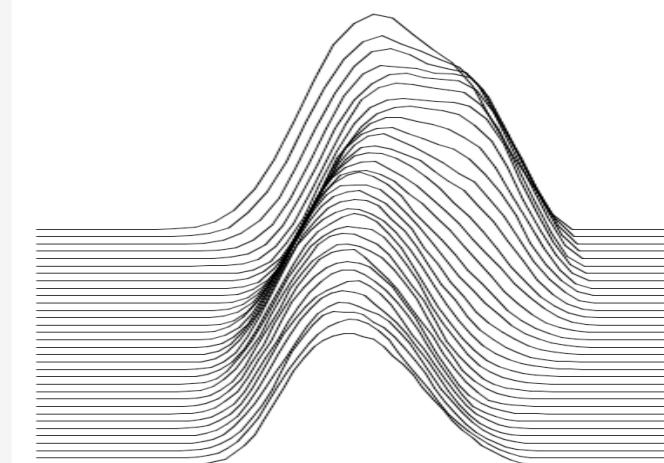
# Left subtree size in left-leaning 2-3 trees

Limiting distribution?

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



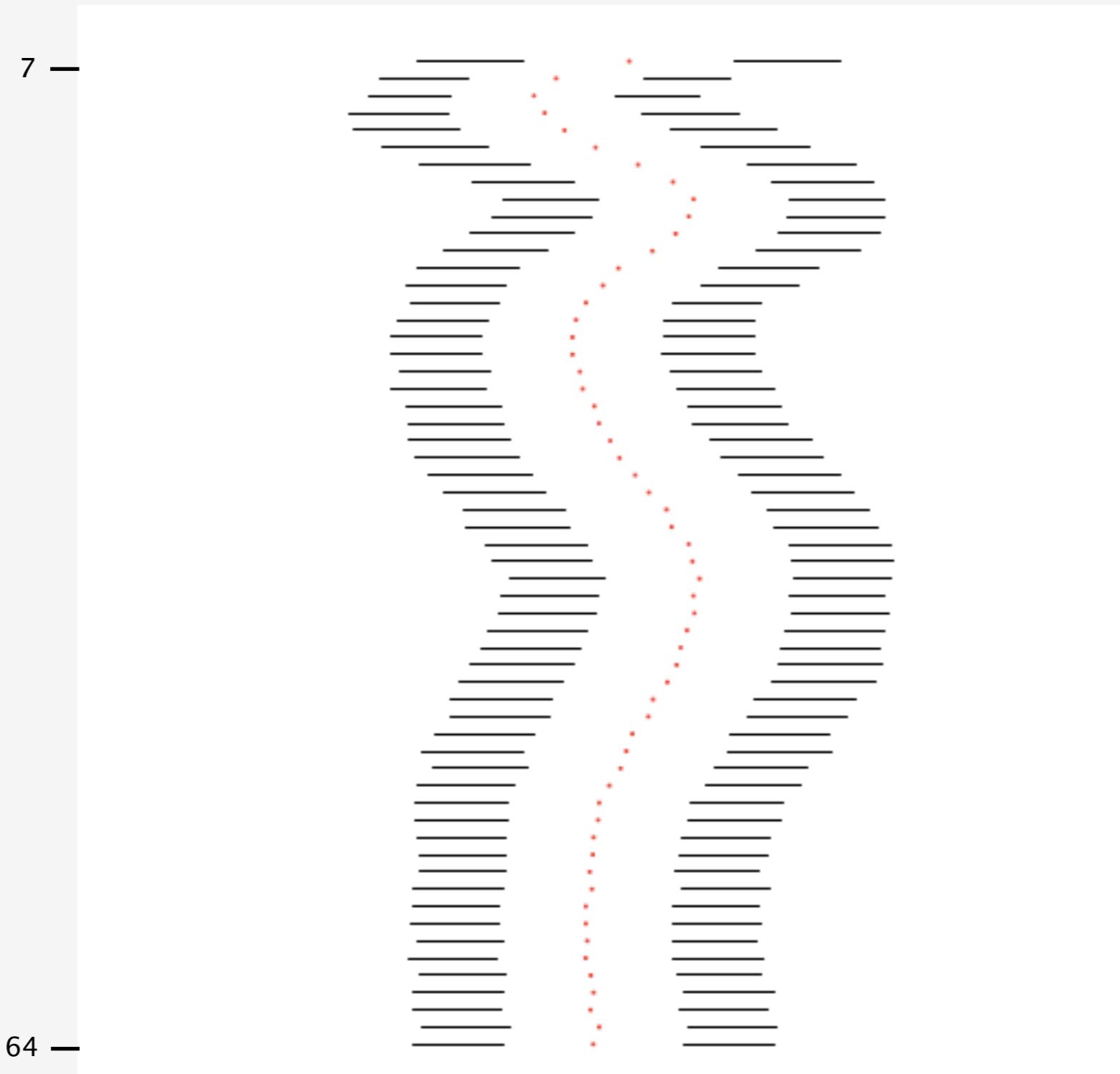
*smoothed version (32-64)*



# Left subtree size in left-leaning 2-3 trees

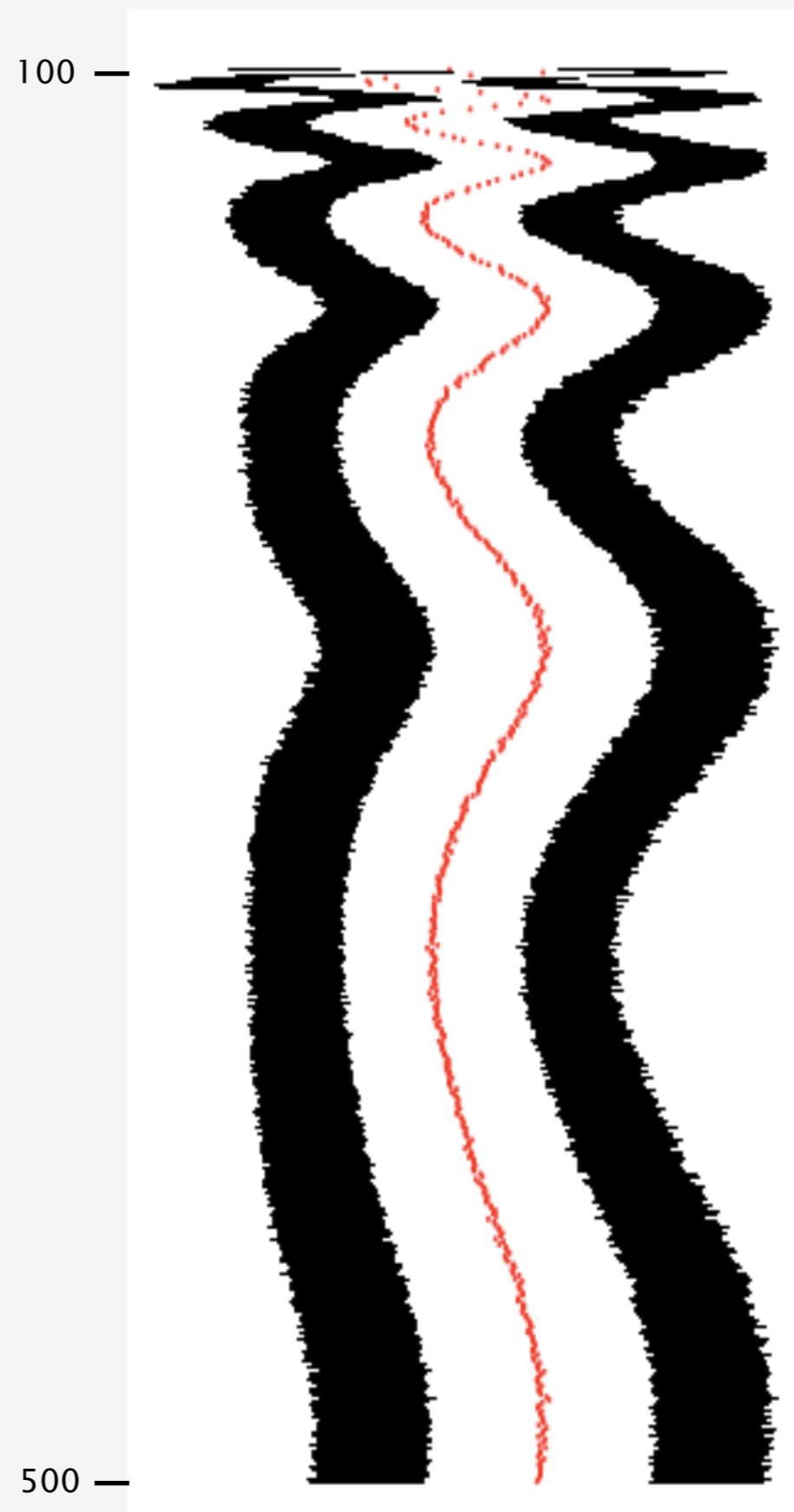
## Tufte plot

*Introduction*  
*2-3-4 Trees*  
*LLRB Trees*  
*Deletion*  
*Analysis*



# Left subtree size in left-leaning 2-3 trees

Tufte plot



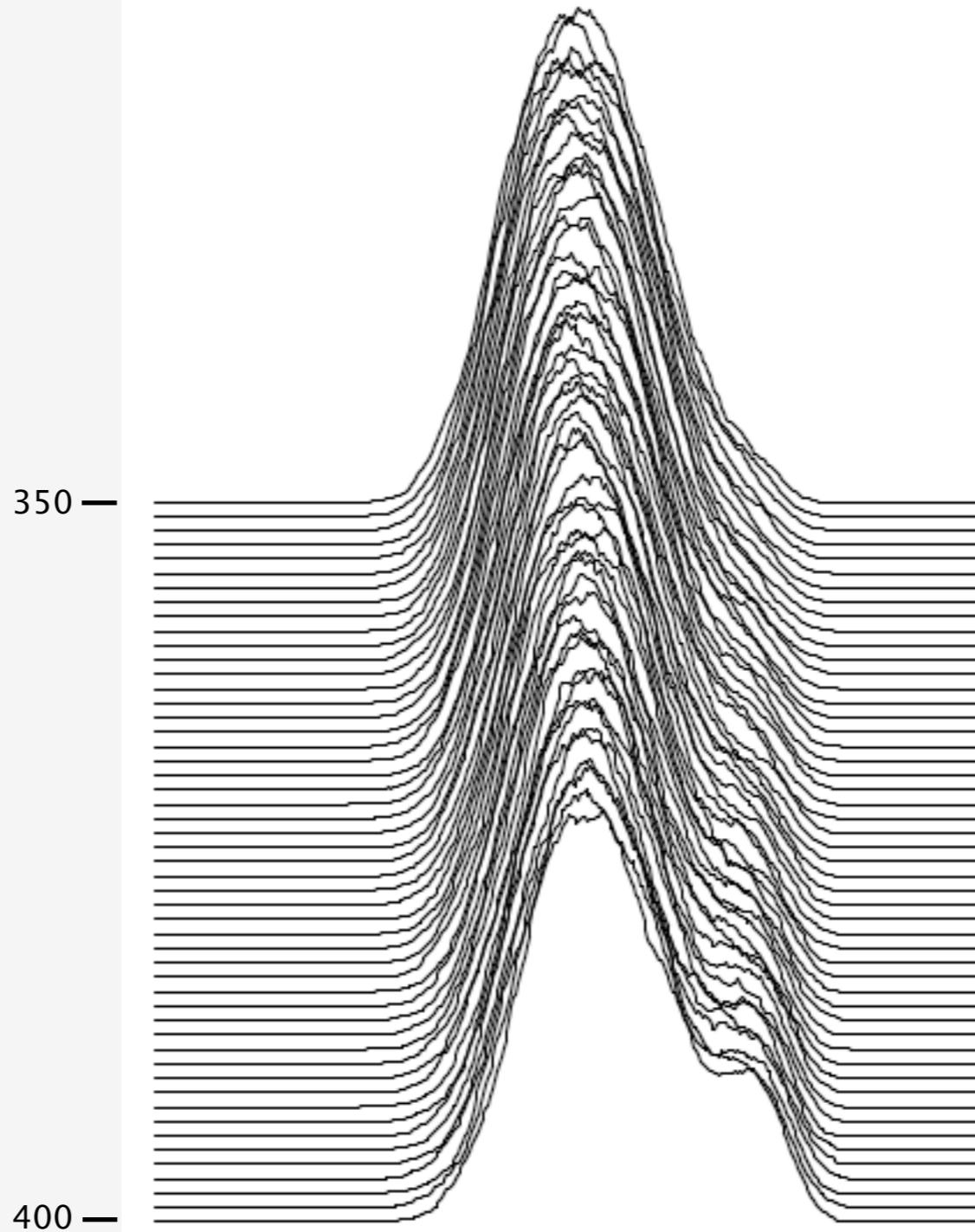
*view of highway for bus driver who  
has had one Caipirinha too many ?*

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

# Left subtree size in left-leaning 2-3 trees

Limiting distribution?

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

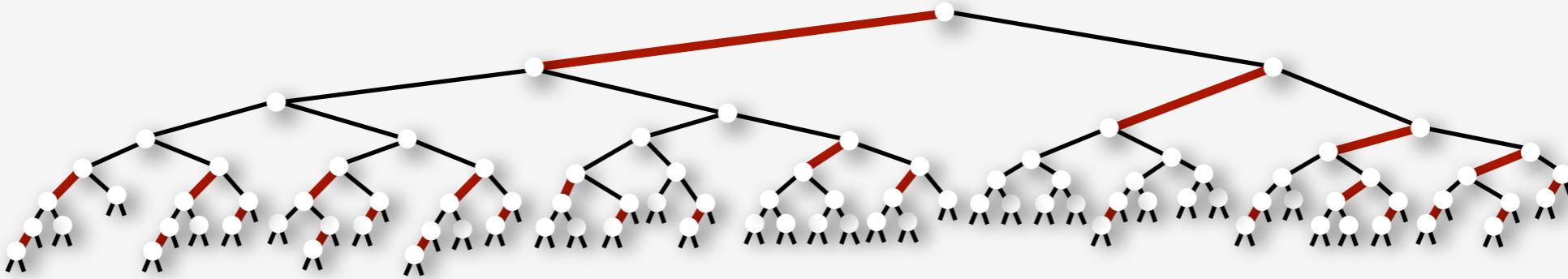


*10,000 trees for each size  
smooth factor 10*

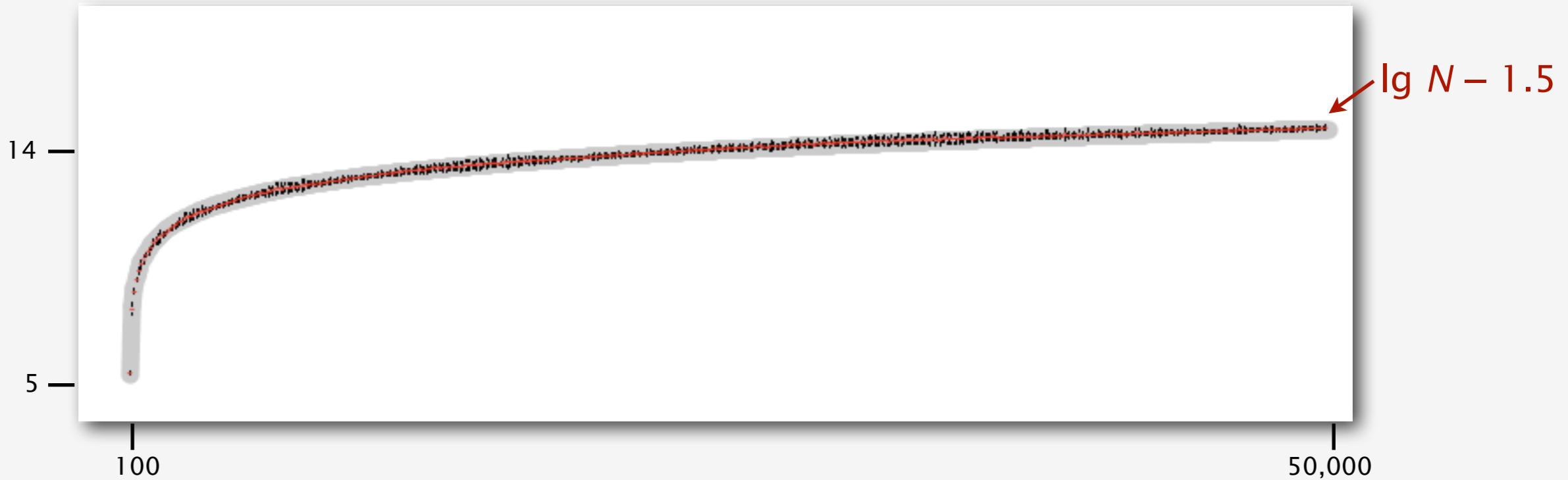
# An exercise in the analysis of algorithms

Find a proof!

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



*Average path length in 2-3 tree built from random keys*

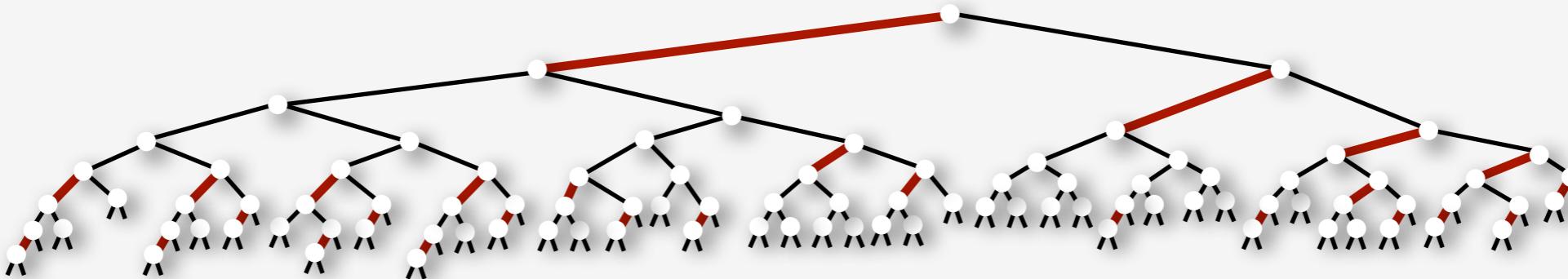


***Addendum:***  
***Observations***

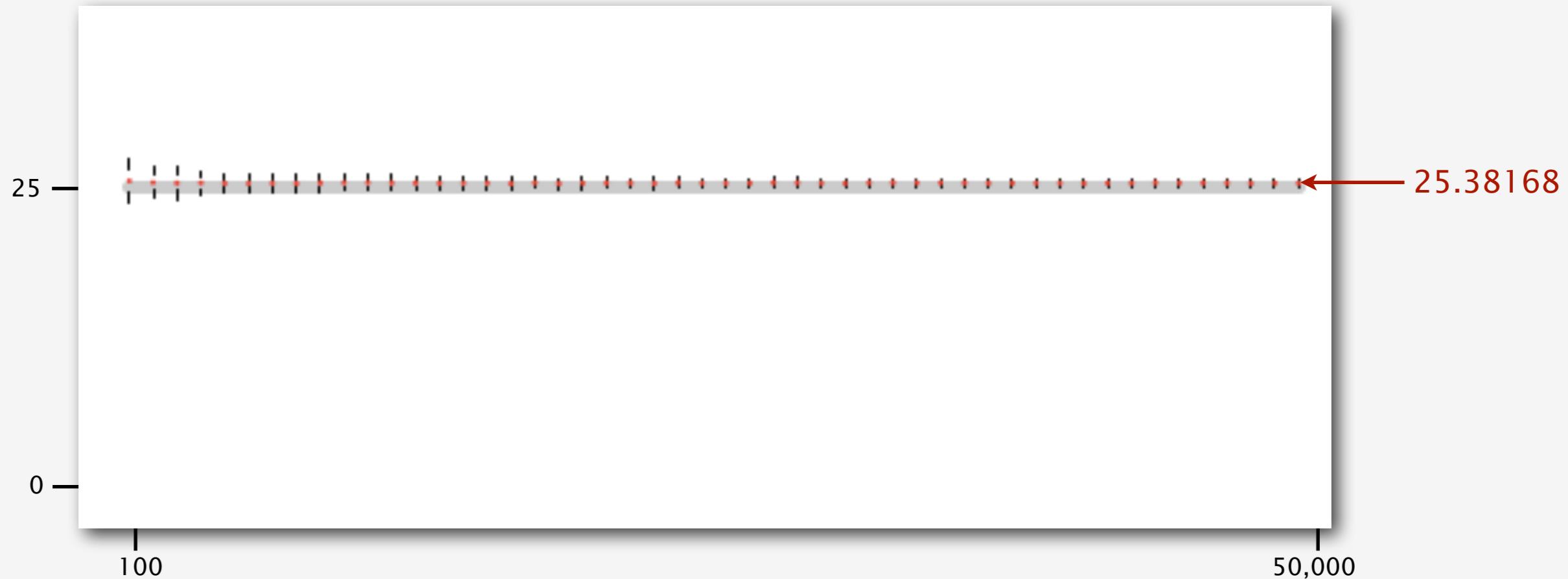
# Observation 1

The percentage of red nodes in a 2-3 tree is between 25 and 25.5%

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



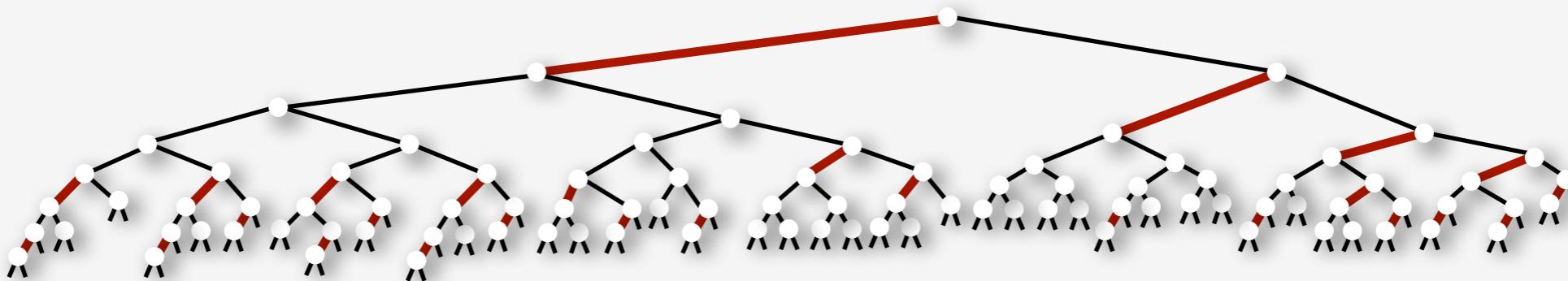
*Percentage of red nodes in 2-3 tree built from random keys*



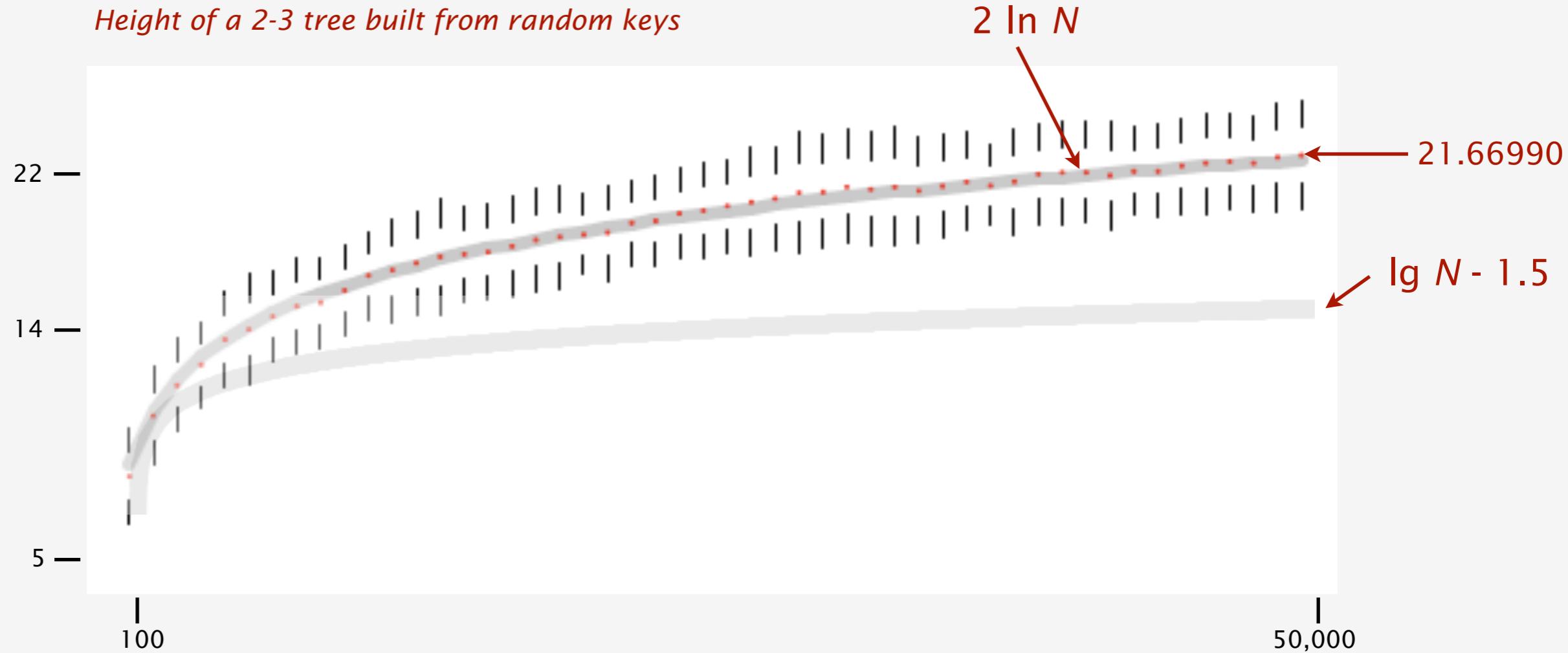
## Observation 2

The **height** of a 2-3 tree is  $\sim 2 \ln N$  (!!!)

Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis



*Height of a 2-3 tree built from random keys*

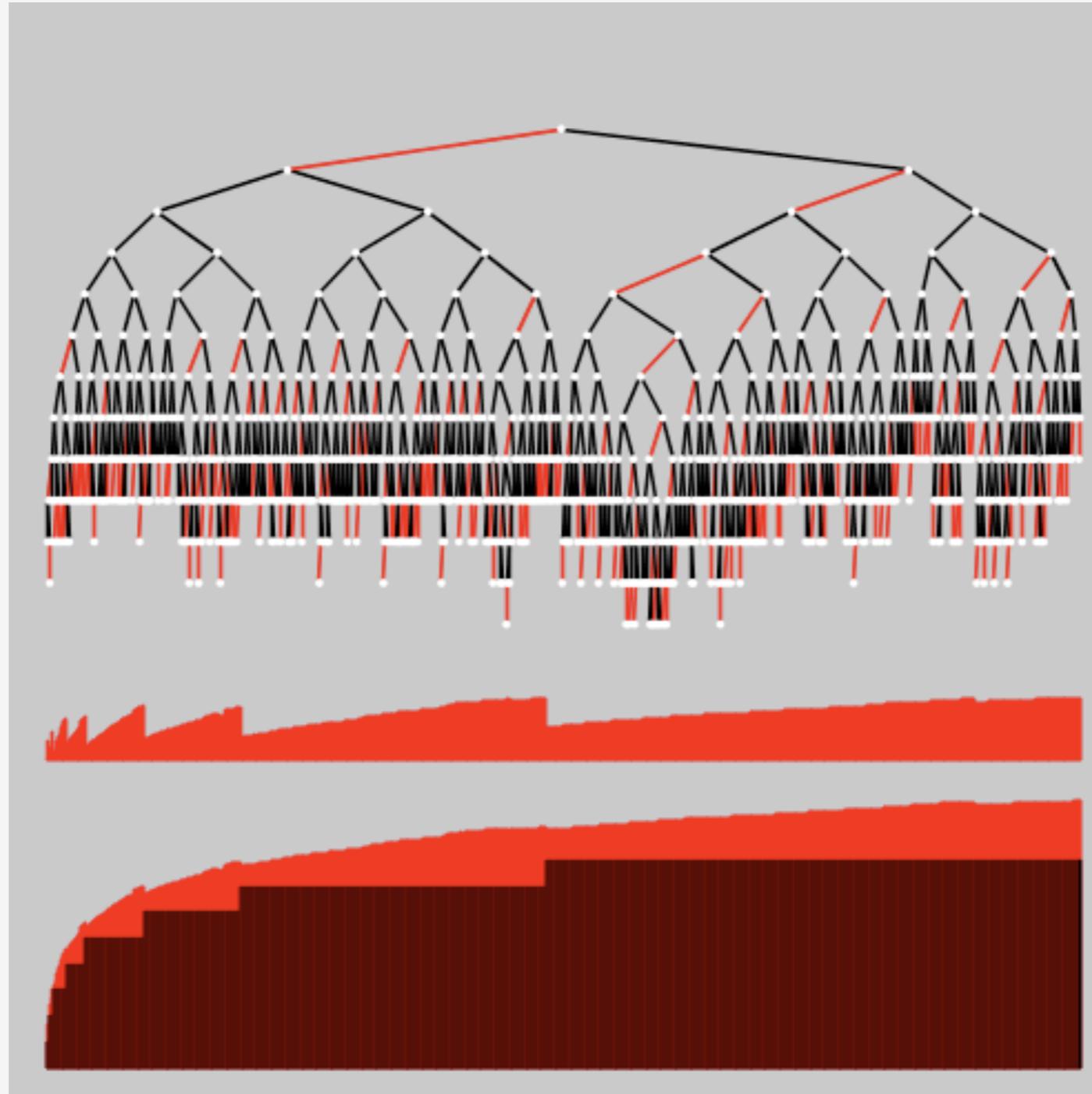


*Very surprising because the average path length in an elementary BST is also  $\sim 2 \ln N \approx 1.386 \lg N$*

## Observation 3

The percentage of red nodes on each **path** in a 2-3 tree rises to about 25%, then drops by 2 when the root splits

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*



## Observation 4

In aggregate, the observed number of red links per path log-alternates between periods of steady growth and not-so-steady decrease (because root-split times vary widely)

*Introduction  
2-3-4 Trees  
LLRB Trees  
Deletion  
Analysis*

