Generalized Fiducial Inference for Accelerated Life Tests With Weibull Distribution and Progressively Type-II Censoring

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***Abstract*—In addition to conventional censoring schemes such as Type-I or Type-II censoring, progressively censoring is a useful method to reduce cost and obtain additional reliability information in accelerated life testing. Statistical inference for accelerated life testing (ALT) with Weibull distribution and progressively censoring is found to be difficult. This paper develops generalized fiducial inference techniques for the constant-stress ALT model with Weibull distribution and progressively Type-II censoring. Simulation studies reveal the good performance of the proposed inference methods. An example is given for illustration.**

***Index Terms*—Accelerated life test, fiducial inference, Gibbs sampling, progressively censoring, Weibull distribution.**

Acronyms

ALT	Accelerated life testing.
CSALT	Constant-stress ALT.
SSALT	Step-stress ALT.
ML	Maximum likelihood.
MLEs	Maximum likelihood estimates.
PDF	Probability density function.
SF	Survival function.
CDF	Cumulative distribution function.
RMSE	Root mean squared errors.
FIs	Fiducial intervals.

Notation

k	Number of levels in constant-stress ALT.
S_i	Stress level for the ith stress environment.
n_i	Number of test units placed at stress level S_i.
R_{ij}	jth progressive censoring scheme at stress level S_i.
t_{ij}	jth failure time at stress level S_i.
β	Shape parameter of Weibull distribution.
θ_i	Scale parameter at stress level S_i.

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I. Introduction

NOWADAYS, manufacturers face strong pressure to develop products which can operate without failures for years, decades or even longer. The requirements for high reliability have increased the difficulty to obtain failure times at normal use conditions. Under such circumstances, an accelerated life testing (ALT) is widely used to obtain timely information of the product lifetime. In the ALT, the level of stress such as temperature or voltage is increased. Based on whether the stress level changes during the testing, there are two important methods of ALT, i.e., constant-stress ALT (CSALT) and step-stress ALT (SSALT). Both CSALT and SSALT have received much attention in the literature (see [1]–[4] for example).

In general, shorter lifetimes are expected with higher stress levels. However, it is not uncommon that some products have long lifetimes even under acceleration. In order to reduce test time and expense, the test is usually stopped before all products fail [5]. Most test plans proposed in the literature are based on the ideas of Type-I or Type-II censoring (e.g., [6] and [7]). In Type-I censoring, the test is terminated after a prefixed time while in Type-II censoring the test is terminated after a prefixed number of failures observed. It is true that these two conventional censoring schemes are the most commonly used. However, as stated in [8], periodical removal of products at time points other than the final termination point is sometimes desirable for various reasons in life testing. On one hand, the allowance of removal saves the test time and expense. On the other hand, the removed units can be used for other tests (e.g., degradation related) and the corresponding information can be obtained. Therefore, progressively censoring is attracting more and more attention since it allows removal of prefixed number of units in each failure during the test. For a detailed introduction of progressively censoring, one may refer to [9], or [10] for a book length treatment.

In most of the research works on ALTs under progressively censoring, the exponential distribution for lifetime is assumed. For example, by assuming the exponential model, the optimal SSALT plan with progressive Type-I censoring and progressive Type-II censoring is studied in [11] and [12], respectively. Wang [13] proposed interval estimation methods for SSALT with progressive Type-II censoring under the exponential assumption. One reason for using exponential distribution assumption is that statistical inference is relatively easy, as the maximum likelihood (ML) estimators of parameters have closed forms. However, the exponential distribution is rarely suitable in practice [5], [14]. A much more suitable distribution is the Weibull distribution because it is found to provide a good fit to many types of lifetime data [15, p. 18]. A lot of studies have focused on the ALT with Weibull distribution and Type-II censoring (e.g., [7] and [16]), a special case of the progressively Type-II censoring. ML estimates (MLEs) of the parameters can be obtained numerically and Fisher information matrix is often used for interval estimation [16]. However, according to Wang *et al.* [4], the MLEs and corresponding confidence intervals based on the normal approximation may not work well since the sample sizes are often small in ALTs. This argument is also supported by their simulation study as well as our simulation in Section IV. On the other hand, there is still a lack of research directly focusing on the ALT with Weibull distribution and progressively Type-II censoring.

The main objective of this study is to make statistical inference for CSALT with Weibull distribution and progressively Type-II censoring. Once the inference methods are proposed, they should also be applicable for CSALT with Weibull distribution and conventional Type-II censoring. Wang *et al.* [4] proposed new point and interval estimation methods for the (functions of) parameters. On one hand, the Gauss–Markov theorem is used to obtain the unbiased estimators of the parameters. On the other hand, the generalized confidence interval method is used for interval estimation. According to their simulation results, their proposed methods are quite effective in terms of biases and coverage probabilities. However, one shortcoming of their methods is that they are not easy to implement. In their study, it is necessary to solve an equation for the point estimation and the procedure for the interval estimation is not straightforward to follow. In this study, we propose new inference methods for CSALT with Weibull distribution and progressively Type-II censoring based on generalized fiducial methods [17]–[19]. Performance of the proposed methods will be investigated both theoretically and by simulation.

The remainder of the paper is organized as follows. Section II states the setting of the problem, i.e., CSALT with Weibull distribution and progressively Type-II censoring. Section III introduces the generalized fiducial inference for the inference of the problem. Two generalized fiducial inference methods including both the point and interval estimation are proposed based on the methods in [4] and [19], respectively. In Section IV, a simulation study is carried out to examine the proposed inference methods. An illustrative example is provided in Section V. Section VI concludes the paper.

II. Problem Statement

Consider a k level CSALT under a progressive Type-II censoring scheme. At each stress level S_i, a number of n_i identical units are tested, $i = 1, \ldots, k$. Prior to the experiment, we need to specify the failure number r_i and the progressive censoring scheme $R_i = (R_{i1}, \ldots, R_{ir_i})$ for $i = 1, \ldots, k$, where $R_{ij} \geq 0$ and $\sum_{j=1}^{r_i} R_{ij} + r_i = n_i$. Let $t_{i1} < \ldots < t_{ir_i}$ be the observed failure times at stress level S_i, $i = 1, \ldots, k$. At the jth failure time $t_{ij}, j = 1, \ldots, r_i$, R_{ij} units together with the failure one are removed. Let $n = n_1 + \cdots + n_k$, $r = r_1 + \cdots + r_k$, and S_0 the use condition stress level. Denote $\boldsymbol{t}_i = (t_{i1}, \ldots, t_{ir_i})$ and $\boldsymbol{t} = (\boldsymbol{t}_i, \ldots, \boldsymbol{t}_k)$.

At each stress level S_i, the lifetime X_i of an item is assumed to follow a Weibull distribution $\mathrm{wb}(\theta_i, \beta_i)$ with a scale parameter θ_i and a shape parameter β_i, $i = 0, 1, \ldots, k$. The probability density function (PDF) and survival function are given by

$$f_i(t) = \frac{\beta_i t^{\beta_i - 1}}{\theta_i^{\beta_i}} \exp\left\{-\left(\frac{t}{\theta_i}\right)^{\beta_i}\right\}$$

and

$$R_i(t) = \exp\left\{-\left(\frac{t}{\theta_i}\right)^{\beta_i}\right\}$$

respectively. We also assume $\beta_0 = \beta_1 = \cdots = \beta_k = \beta$. This is a reasonable assumption when the failure mechanisms are the same under different stress levels, as β is empirically found to be dependent on the failure mechanism.

In addition to a lifetime distribution, the other important component of ALT models is the relationship between one (or more) of the distribution parameters and the stress. Under the Weibull assumption, the scale parameter θ_i of Weibull distribution is often assumed to have a log-linear relationship with the function of the stress [4], [6], [20]

$$\log \theta_i = a + b\varphi(S_i), i = 0, 1, \ldots, k$$

where a and $b > 0$ are unknown parameters and $\varphi(S)$ is a given decreasing function of stress level S. This Weibull log-linear model inherently includes many commonly used ALT models with Weibull distribution [6]. In particular, when the temperature is used as the stress for acceleration, it offers the Arrhenius model with $\varphi(S) = 1/S$. On the other hand, when voltage is the stress, it turns to be the inverse power law model with $\varphi(S) = -\log(S)$. For simplicity, we standardize the stress levels as $x_i = (\varphi(S_i) - \varphi(S_0))/(\varphi(S_k) - \varphi(S_0))$, so that $x_i \in [0, 1]$. Then,

$$\log \theta_i = \alpha_0 + \alpha_1 x_i \tag{1}$$

where $\alpha_0 = a + b\varphi(S_0)$ and $\alpha_1 = b(\varphi(S_k) - \varphi(S_0))$. Note that α_1 is negative, indicating that failure probability increases with x_i.

III. Generalized Fiducial Inference

In this section, we will propose new inference methods for the CSALT with Weibull distribution and progressively Type-II censoring based on the generalized fiducial method. This method is found easy to implement and quite effective in a variety of scenarios [18]. The basic idea of fiducial inference is to switch the role of the parameters and the observed data. Suppose that there is a data generation process

$$Y = H(\xi, U) \tag{2}$$

where ξ is a vector of parameters and U is a random vector with completely known distribution independent of any parameters. Equation (2) is also called a structural equation, which can be seen as a detailed description of the noise process U that combines with the signal ξ to yield observed data Y. Thus, for any fixed value of the parameter ξ, the distribution of U and the structural equation imply the distribution of the data Y. After observing the data Y, we can switch the role of data and the parameters ξ by solving the structural equation (2), that is,

$$\xi = W(Y, U). \tag{3}$$

Thus, ξ is treated as random because of U. The distribution associated with ξ is called the fiducial distribution of ξ and it can be obtained based on the observed data Y and U. If the fiducial distribution is hard to derive, then one can get a random realization from the fiducial distribution of ξ by first generating U and then plugging it into (3). In the following, two generalized fiducial inference procedures are proposed based on the methods given in [4] and [19], respectively.

A. Generalized Fiducial Inference Method I

For each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r_i$, let

$$Z_{ij}(\beta) = \sum_{l=1}^{j}(R_{il}+1)t_{il}^{\beta} + \left[n_i - \sum_{l=1}^{j}(R_{il}+1)\right]t_{ij}^{\beta}.$$

From [4], we know that

$$G_i(\beta) = 2\sum_{j=1}^{r_i-1}\log\left(\frac{Z_{ir_i}(\beta)}{Z_{ij}(\beta)}\right) \sim \chi^2(2r_i - 2), \quad i = 1, 2, \ldots, k$$

and $G_i(\beta)$ is a strictly monotone increasing function of β. Note that $G_1(\beta), \ldots, G_k(\beta)$ are independent and they are governed by a common parameter β. Define

$$G(\beta) = \sum_{i=1}^{k} G_i(\beta) = U_0$$

where $U_0 \sim \chi^2\left(2\sum_{i=1}^{k} r_i - 2k\right)$. Thus, we have

$$\beta = G^{-1}(U_0). \tag{4}$$

Let $Z_i(\beta) = \sum_{j=1}^{r_i}(R_{ij}+1)t_{ij}^{\beta}$, $i = 1, 2, \ldots, k$. Then $2Z_i(\beta)/\theta_i^{\beta}$ follows a χ^2 distribution with $2r_i$ degrees of freedom. The results can be written as

$$2Z_i(\beta) = U_i\theta_i^{\beta}, \quad i = 1, 2, \ldots, k \tag{5}$$

where $U_i \sim \chi^2(2r_i)$. Taking logarithm with respect to (5) leads to

$$[\log(2Z_i(\beta)) - \log(U_i)]/\beta = \alpha_0 + \alpha_1 x_i. \tag{6}$$

Note that the χ^2 distribution is a special case of the gamma distribution. According to the property of the log-Gamma distribution, the variance of $\log(U_i)$ is $\mathrm{Var}(\log(U_i)) = \psi(r_i)$, where $\psi(\cdot)$ is the trigamma function. Let $V_i = \log[2Z_i(\beta)] - \log(U_i)$. Using weighted least square methods, the solution of (α_0, α_1) in (5) is

$$\alpha_0 = \frac{A_1A_4 - A_2A_5}{\beta(A_1A_3 - A_2^2)}, \ \alpha_1 = \frac{A_1A_5 - A_2A_4}{\beta(A_1A_3 - A_2^2)} \tag{7}$$

where $A_1 = \sum_{i=1}^{k}\psi^{-1}(r_i)$, $A_2 = \sum_{i=1}^{k}\psi^{-1}(r_i)x_i$, $A_3 = \sum_{i=1}^{k}\psi^{-1}(r_i)x_i^2$, $A_4 = \sum_{i=1}^{k}\psi^{-1}(r_i)V_i$, and $A_5 = \sum_{i=1}^{k}\psi^{-1}(r_i)x_iV_i$. It is difficult to derive the joint fiducial distribution of $(\beta, \alpha_0, \alpha_1)$ based on (4) and (7). However, the estimates of $(\beta, \alpha_0, \alpha_1)$ can be obtained based on the following Monte Carlo simulation procedure:

1) For a given dataset (n, m, t, R), generate $U_0^{(1)}, \ldots, U_0^{(M)}$ from $\chi^2\left(2\sum_{i=1}^{k} r_i - 2k\right)$, and $U_i^{(1)}, \ldots, U_i^{(M)}$ from $\chi^2(2r_i)$, $i = 1, \ldots, k$.
2) From (4), we have $\beta^{(1)} = G^{-1}\left(U_0^{(1)}\right), \ldots, \beta^{(M)} = G^{-1}\left(U_0^{(M)}\right)$. These $\beta^{(j)}$s are the sample of β. Then the point estimate of β is $\frac{1}{M}\sum_{j=1}^{M}\beta^{(j)}$. A $(1-\gamma)$ fiducial interval (FI) for β is
$$\left[G^{-1}\left\{\chi^2_{1-\gamma/2}\left(2\sum_{i=1}^{k} r_i - 2k\right)\right\},\right.$$
$$\left.G^{-1}\left\{\chi^2_{\gamma/2}\left(2\sum_{i=1}^{k} r_i - 2k\right)\right\}\right] \tag{8}$$
where $\chi^2_{\gamma}(\nu)$ is the upper γ percentile of the χ^2 distribution with ν degrees of freedom.
3) In terms of (7), compute $\alpha_l^{(j)}$, $l = 0, 1$, where β and V_i are replaced with $\beta^{(j)}$ and $\log\left[2Z_i\left(\beta^{(j)}\right)\right] - \log\left(U_i^{(j)}\right)$, respectively. α_l can be estimated by $\frac{1}{M}\sum_{j=1}^{M}\alpha_l^{(j)}$, $l = 0, 1$. Let α_l^{γ} be the γ percentile of $\alpha_l^{(j)}$s. Then, a $(1-\gamma)$ FI for α_l is approximated by
$$\left[\alpha_l^{\gamma/2}, \alpha_l^{1-\gamma/2}\right], \quad l = 0, 1.$$
4) For any function of $(\beta, \alpha_0, \alpha_1)$, say $g(\beta, \alpha_0, \alpha_1)$, substitute $\left(\beta^{(j)}, \alpha_0^{(j)}, \alpha_1^{(j)}\right)$ for $(\beta, \alpha_0, \alpha_1)$, and we obtain the samples of $g(\beta, \alpha_0, \alpha_1)$ as follows:
$$g\left(\beta^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}\right), \ldots, g\left(\beta^{(M)}, \alpha_0^{(M)}, \alpha_1^{(M)}\right).$$
Similar to the cases of α_0 and α_1, the point estimate and $(1-\gamma)$ FI of $g(\beta, \alpha_0, \alpha_1)$ can be easily computed.

The following theorem shows that the fiducial inference for β is exact in terms of coverage probability. The proof is placed in Appendix A.

Theorem 1: Consider a CSALT under Weibull distribution and a progressive Type-II censoring scheme as stated in Section II. For a given dataset (n, m, t, R), the $1-\gamma$ FI for the shape parameter β based on expression (8) is exact.

Remark: In [4], the point estimate of β is obtained by solving an equation and then it is used to get the estimates of α_0 and α_1. Generalized confidence intervals of the parameters are then constructed. In our proposed method, the point and interval estimates can be obtained simultaneously by the above Monte Carlo Simulation. Note that the FIs in this section coincide with the generalized confidence intervals in their paper. This is not surprising because generalized confidence intervals can be obtained by the fiducial method when the structure equation (2) can be inverted [17]. However, our proposed fiducial procedure for interval estimation is more straightforward to follow and implement compared with the method given in [4]. In this regard, the fiducial method in this section can be treated as an improved version of the method given in [4].

B. Generalized Fiducial Inference Method II

Usually, the inverse of the structural equation (2) does not exist. This may happen for two cases: for some value of y_0 and u (y_0 is an observation of Y, and u is a realization of U), either there is more than one ξ or there is no ξ satisfying $y_0 = H(\xi, u)$. In [19], Hannig gave solutions to such cases. In our problem, though there is no such nonexistence problem, we can still borrow the inference method in [19]. Simulation study in Section IV shows that this generalized fiducial inference method is preferred.

Assume that the observed data are $y = (y_1, \ldots, y_n)$, and that $U = (U_1, \cdots, U_n)$ are independent identically distributed (i.i.d.) samples from Uniform (0, 1). Denote the structural equation as $H = (H_1, \cdots, H_n)$, so that $y_i = H_i(\xi, U_i)$ for $i = 1, \ldots, n$. Assume that the parameter $\xi \in \Xi \subseteq \mathbb{R}^p$ is p-dimensional. Given some differentiability assumptions, Hannig [19] showed that the generalized fiducial distribution is absolutely continuous with density

$$\tau(\xi) = \frac{J(y,\xi)L(y,\xi)}{\int_\Xi J(y,\xi')L(y,\xi')\,d\xi'} \tag{9}$$

where $L(y, \xi)$ is the joint likelihood function of the observed data and

$$J(y,\xi) = \sum_{\substack{i=(i_1,\ldots,i_p)\\ 1\le i_1<\cdots<i_p\le n}} \left| \det\left(\left(\frac{d}{dy}H^{-1}(y,\xi)\right)^{-1} \frac{d}{d\xi}H^{-1}(y,\xi)\right)_i \right| \tag{10}$$

where the sum goes over all p-tuples of indexes $i = (1 \le i_1 < \cdots < i_p \le n) \subset \{1, \ldots, n\}$. $dH^{-1}(y, \xi)/d\xi$ and $dH^{-1}(y, \xi)/dy$ are the $n \times p$ and $n \times n$ Jacobian matrices, respectively. For any $n \times p$ matrix D, $(D)_i$ is the $p \times p$ matrix comprising of the rows $i_1, \ldots, i_p$ of D. In [19], Hannig showed that if the sample y is i.i.d. from an absolutely continuous distribution having a cumulative distribution function (CDF) $F_\xi(y)$ and let $H^{-1} = (F_\xi(y_1), \ldots, F_\xi(y_n))$, (10) will be simplified to

$$J(y,\xi) = \sum_{\substack{i=(i_1,\ldots,i_p)\\ 1\le i_1<\cdots<i_p\le n.}} \left| \frac{\det\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\left(F_\xi(y_{i_1}),\ldots,F_\xi(y_{i_p})\right)\right)}{f_\xi(y_{i_1})\cdots f_\xi(y_{i_p})} \right| \tag{11}$$

Based on this method, we can derive the joint fiducial distribution of $(\beta, \alpha_0, \alpha_1)$ as shown in the following theorem. The proof is placed in Appendix B.

TABLE I
SIMULATION DESIGN SCENARIOS

	$x_1, \ldots, x_k$	$n_1, \ldots, n_k$	$r_1, \ldots, r_k$	$R_1, \ldots, R_k$
1	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (0, \ldots, 0, 8)$ $R_2 = (0, \ldots, 0, 4)$
2	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (8, 0, \ldots, 0)$ $R_2 = (4, 0, \ldots, 0)$
3	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (4, 0, \ldots, 0, 4)$ $R_2 = (2, 0, \ldots, 0, 2)$
4	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (0, \ldots, 0, 8)$ $R_2 = (0, \ldots, 0, 6)$ $R_3 = (0, \ldots, 0, 4)$
5	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (8, 0, \ldots, 0)$ $R_2 = (6, 0, \ldots, 0)$ $R_3 = (4, 0, \ldots, 0)$
6	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (4, 0, \ldots, 0, 4)$ $R_2 = (3, 0, \ldots, 0, 3)$ $R_3 = (2, 0, \ldots, 0, 2)$
7	(0.5, 0.75, 0.9, 1)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (0, \ldots, 0, 12)$ $R_2 = (0, \ldots, 0, 8)$ $R_3 = (0, \ldots, 0, 6)$ $R_4 = (0, \ldots, 0, 4)$
8	(0.5, 0.75, 0.9, 1)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (12, 0, \ldots, 0)$ $R_2 = (8, 0, \ldots, 0)$ $R_3 = (6, 0, \ldots, 0)$ $R_4 = (4, 0, \ldots, 0)$
9	(0.5, 0.75, 0.9, 1)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (6, 0, \ldots, 0, 6)$ $R_2 = (4, 0, \ldots, 0, 4)$ $R_3 = (3, 0, \ldots, 0, 3)$ $R_4 = (2, 0, \ldots, 0, 2)$

Theorem 2: Consider a CSALT under Weibull distribution and a progressive Type-II censoring scheme as stated in Section II. For a given dataset (n, m, t, R), the joint fiducial distribution of $(\beta, \alpha_0, \alpha_1)$ is proportional to

$$\frac{1}{\beta}\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{\beta t_{ij}^{\beta-1}}{\exp[\beta\alpha_0+\beta\alpha_1 x_i]} \times \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{ij}+1)t_{ij}^{\beta}\exp[-\beta\alpha_0-\beta\alpha_1 x_i]\right\}.$$

Having the joint fiducial distribution, we can generate random realizations of $(\beta, \alpha_0, \alpha_1)$ based on the following Gibbs sampling procedure:

1) Given (β, α_1), the conditional fiducial distribution of $\exp(-\alpha_0\beta)$ is

$$\mathrm{Gam}\left(\sum_{i=1}^{k} r_i, \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{ij}+1)t_{ij}^{\beta}\exp[-\beta\alpha_1 x_i]\right)$$

where $\mathrm{Gam}(c, d)$ is the gamma distribution with PDF proportional to $x^{c-1}\exp(-dx)$.

TABLE II
RELATIVE BIAS AND RMSE OF THE FIDUCIAL ESTIMATES BASED ON 10 000 REPLICATIONS

	Relative bias											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	0.1691	−0.0031	0.0480	−0.0328	0.0550	0.0024	−0.0012	−0.0758	0.0628	0.0145	0.0972	−0.0279
2	0.1051	0.0009	0.0465	−0.0233	0.0315	−0.0027	−0.0151	−0.0827	0.0373	0.0102	0.0848	−0.0313
3	0.1683	0.0004	−0.1208	−0.0089	0.0474	0.0008	0.0013	−0.0749	0.0546	0.0130	0.0979	−0.0264
4	0.1023	−0.0036	0.0180	−0.0461	0.0367	0.0006	−0.0063	−0.0819	0.0393	0.0118	0.0800	−0.0360
5	0.0642	0.0005	0.0270	−0.0318	0.0216	−0.0014	−0.0054	−0.0800	0.0238	0.0098	0.0768	−0.0341
6	0.1150	0.0006	−0.1339	−0.0178	0.0292	0.0012	0.0020	−0.0746	0.0313	0.0123	0.0853	−0.0249
7	0.0571	−0.0028	0.0070	−0.0357	0.0190	0.0014	0.0033	−0.0561	0.0192	0.0061	0.0393	−0.0357
8	0.0373	−0.0015	0.0046	−0.0319	0.0152	−0.0006	−0.0020	−0.0588	0.0157	0.0041	0.0323	−0.0382
9	0.0852	−0.0090	−0.1782	−0.0528	0.0171	−0.0001	−0.0032	−0.0592	0.0176	0.0047	0.0326	−0.0385
	RMSE											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	0.6581	0.3656	0.5203	0.2203	0.2536	0.0731	0.5184	0.2892	0.2531	0.0622	0.4251	0.2498
2	0.4638	0.3717	0.5305	0.2147	0.1943	0.0738	0.5220	0.2919	0.1928	0.0609	0.4235	0.2460
3	0.5901	0.3500	0.5085	0.2097	0.2315	0.0724	0.5185	0.2878	0.2303	0.0609	0.4257	0.2457
4	0.4587	0.3662	0.5081	0.2150	0.2032	0.0734	0.5084	0.2894	0.1952	0.0620	0.4191	0.2458
5	0.3368	0.3664	0.5089	0.2089	0.1564	0.0721	0.5026	0.2851	0.1514	0.0601	0.4136	0.2384
6	0.4232	0.3489	0.4991	0.2049	0.1825	0.0727	0.5068	0.2873	0.1760	0.0608	0.4157	0.2410
7	0.3148	0.2937	0.3943	0.1748	0.1461	0.0586	0.4059	0.2463	0.1405	0.0529	0.3606	0.2206
8	0.2442	0.2982	0.4004	0.1748	0.1199	0.0592	0.4098	0.2494	0.1156	0.0530	0.3644	0.2223
9	0.3158	0.2873	0.4198	0.1783	0.1364	0.0579	0.4012	0.2445	0.1313	0.0524	0.3581	0.2190

2) Given (β, α_0), the conditional fiducial density function of α_1 is proportional to

$$\exp\left\{-\beta\alpha_1\sum_{i=1}^{k} r_i x_i - \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{ij}+1)t_{ij}^{\beta} \times \exp[-\beta\alpha_0 - \beta\alpha_1 x_i]\right\}$$

which is a log-concave. Thus, an adaptive rejection sampling [21] can be used for sampling.

3) Given (α_0, α_1), the conditional fiducial density function of β is

$$\beta^{\sum_{i=1}^{k} r_i - 1} \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\Big(\beta(\alpha_0 + \alpha_1 x_i - \log(t_{ij})) + (R_{ij}+1)t_{ij}^{\beta}\exp[-\beta\alpha_0 - \beta\alpha_1 x_i]\Big)\right\}$$

which is also a log-concave and thus an adaptive rejection sampling is again invoked.

After running the above procedure N times and discarding the initial L burn-in samples, we have the remaining $N - L$ samples. Then, both the point estimates and confidence intervals of these parameters or functions of them can be obtained in a similar vein to that in Section III-A.

IV. SIMULATION

In this section, simulation studies are conducted to evaluate the performance of the two proposed fiducial methods. For comparison purpose, we also consider the ML estimation. In the ML estimation, the point estimates of the parameters are obtained by setting the first partial derivatives of the likelihood function with respect to each parameter to 0 and solving the equations jointly, and the confidence intervals are obtained based on the covariance matrices, which can be calculated by inverting the Fisher information matrix. We take the number of stress levels $k = 2, 3, 4$, and three progressive censoring schemes are considered for each k. Thus, totally nine scenarios are considered in the simulation studies (see Table I for more details). For each scenario, 10 000 replicates of progressively Type-II censored samples are generated with the model parameters $(\beta, \alpha_0, \alpha_1) = (2, 5, -1)$. When applying the generalized fiducial inference method II, the number of Gibbs sampling iteration is 10 000, and the number of burn-in samples L for Gibbs sampling procedure is set as 5000. We calculate the relative biases, root mean squared errors (RMSE), 90% (95%) coverage probabilities, and their average interval lengths of the estimates based on 10 000 replicates. The results are listed in Tables II–IV, where $R(100)$ is the reliability function at time 100 h under the normal stress level. In these tables, "ML" refers to the ML estimation. "Fiducial I" and "Fiducial II" refer to generalized fiducial inference method I and II, respectively.

As we can see from Table II, the relative biases and RMSEs by the ML method are generally the largest among all the three methods. Between Fiducial I and Fiducial II, the relative biases

TABLE III
COVERAGE PROBABILITY AND AVERAGE INTERVAL LENGTH FOR 90% CONFIDENCE INTERVAL BASED ON 10 000 REPLICATIONS

	Coverage probability											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	0.8819	0.8341	0.8278	0.7669	0.9023	0.8987	0.8995	0.9001	0.9028	0.9018	0.8980	0.8989
2	0.8889	0.8497	0.8431	0.7758	0.9004	0.9007	0.8992	0.9002	0.9018	0.8994	0.9009	0.9000
3	0.8881	0.8475	0.8392	0.7772	0.8951	0.9012	0.9003	0.8999	0.8994	0.9073	0.8997	0.9010
4	0.8915	0.8576	0.8593	0.7904	0.8976	0.8943	0.8965	0.8968	0.9000	0.9027	0.9058	0.9058
5	0.8929	0.8674	0.8678	0.7948	0.8985	0.9047	0.9003	0.9027	0.8999	0.8994	0.9000	0.9005
6	0.8989	0.8650	0.8547	0.7939	0.8988	0.8970	0.8959	0.8960	0.9007	0.8969	0.8991	0.9070
7	0.8968	0.8753	0.8739	0.8295	0.9044	0.8967	0.8943	0.8933	0.9052	0.9062	0.8951	0.9061
8	0.8943	0.8795	0.8851	0.8281	0.8967	0.8957	0.8929	0.8946	0.8933	0.9038	0.9005	0.9028
9	0.8838	0.8725	0.8428	0.8370	0.8993	0.9029	0.9022	0.9005	0.9015	0.9053	0.8994	0.8984
	Average interval length											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	1.6519	1.0554	1.4856	0.6014	1.5664	1.2842	1.8184	0.5874	1.5623	1.1378	1.5333	0.5307
2	1.2632	1.1008	1.5494	0.6184	1.2305	1.2612	1.7897	0.5999	1.2196	1.1025	1.5133	0.5390
3	1.5409	1.0389	1.4688	0.6016	1.4334	1.2669	1.8033	0.5919	1.4308	1.1154	1.5251	0.5331
4	1.2551	1.1110	1.5353	0.6199	1.2583	1.2527	1.7380	0.5859	1.2158	1.1108	1.4953	0.5313
5	0.9801	1.1401	1.5766	0.6266	0.9962	1.2376	1.7190	0.5956	0.9631	1.0924	1.4874	0.5376
6	1.1842	1.0811	1.4982	0.6105	1.1521	1.2428	1.7297	0.5868	1.1153	1.1024	1.4974	0.5319
7	0.9298	0.9242	1.2474	0.5258	0.9474	0.9849	1.3613	0.4997	0.9114	0.9211	1.2532	0.4691
8	0.7400	0.9374	1.2661	0.5324	0.7614	0.9760	1.3483	0.5079	0.7328	0.9072	1.2401	0.4748
9	0.8906	0.8869	1.2003	0.5324	0.8746	0.9777	1.3551	0.5043	0.8431	0.9117	1.2474	0.4720

TABLE IV
COVERAGE PROBABILITY AND AVERAGE INTERVAL LENGTH FOR 95% CONFIDENCE INTERVAL BASED ON 10 000 REPLICATIONS

	Coverage probability											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	0.9448	0.8932	0.8906	0.8153	0.9495	0.9486	0.9481	0.9474	0.9481	0.9495	0.9509	0.9515
2	0.9496	0.9070	0.9012	0.8228	0.9501	0.9513	0.9488	0.9502	0.9529	0.9476	0.9520	0.9541
3	0.9530	0.9040	0.8994	0.8229	0.9460	0.9486	0.9496	0.9507	0.9476	0.9472	0.9501	0.9506
4	0.9505	0.9155	0.9137	0.8367	0.9466	0.9460	0.9466	0.9480	0.9470	0.9480	0.9495	0.9484
5	0.9485	0.9251	0.9226	0.8420	0.9476	0.9495	0.9505	0.9528	0.9492	0.9530	0.9533	0.9542
6	0.9552	0.9240	0.9163	0.8394	0.9467	0.9484	0.9475	0.9497	0.9495	0.9531	0.9424	0.9528
7	0.9511	0.9299	0.9325	0.8793	0.9513	0.9482	0.9496	0.9490	0.9526	0.9535	0.9451	0.9537
8	0.9473	0.9356	0.9365	0.8782	0.9446	0.9458	0.9468	0.9452	0.9440	0.9514	0.9513	0.9519
9	0.9497	0.9303	0.9082	0.8844	0.9489	0.9490	0.9512	0.9511	0.9489	0.9463	0.9463	0.9465
	Average interval length											
	ML				Fiducial I				Fiducial II			
	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$	β	α_0	α_1	$R(100)$
1	1.9684	1.2576	1.7701	0.7166	1.8659	1.5791	2.2372	0.6759	1.8630	1.3702	1.8181	0.6096
2	1.5052	1.3117	1.8462	0.7369	1.4649	1.5340	2.1802	0.6883	1.4546	1.3165	1.7827	0.6176
3	1.8361	1.2379	1.7502	0.7169	1.7084	1.5515	2.2088	0.6808	1.7063	1.3385	1.8037	0.6119
4	1.4955	1.3238	1.8294	0.7387	1.4985	1.5257	2.1182	0.6735	1.4488	1.3248	1.7653	0.6091
5	1.1679	1.3586	1.8786	0.7467	1.1861	1.4968	2.0802	0.6837	1.1478	1.2953	1.7477	0.6149
6	1.4111	1.2882	1.7852	0.7275	1.3715	1.5084	2.1017	0.6749	1.3293	1.3112	1.7642	0.6093
7	1.1080	1.1013	1.4863	0.6265	1.1279	1.1880	1.6427	0.5814	1.0857	1.0976	1.4854	0.5431
8	0.8818	1.1169	1.5086	0.6344	0.9058	1.1732	1.6217	0.5906	0.8731	1.0767	1.4656	0.5488
9	1.0612	1.0568	1.4302	0.6344	1.0410	1.1776	1.6323	0.5867	1.0043	1.0841	1.4762	0.5459

TABLE V
TIME TO BREAKDOWN DATA OF A CERTAIN INSULTING FLUID FROM [4]

Voltage (S_i)	n_i	R_i	Breakdown times
30 kV	11	(2, 0,. . ., 0, 2)	7.74, 17.05, 21.02, 43.40, 47.30, 139.07, 144.12
36 kV	15	(4, 0,. . ., 0, 1)	0.35, 0.96, 1.69, 1.97, 2.58, 2.71, 3.67, 3.99, 5.35, 13.77

TABLE VI
MLES, 90%, AND 95% FIS OF THE PARAMETERS FOR THE DATA FROM [4]

	Fiducial I			Fiducial II		
	MLEs	90% FIs	95% FIs	MLEs	90% FIs	95% FIs
β	0.975	(0.683, 1.288)	(0.632, 1.364)	0.959	(0.679, 1.269)	(0.620, 1.346)
α_0	19.83	(15.04, 25.15)	(13.93, 26.35)	19.50	(14.79, 24.51)	(13.71, 25.37)
α_1	−0.503	(−0.657, −0.359)	(−0.695, −0.325)	−0.498	(−0.643, −0.371)	(−0.689, −0.312)
μ	16727.7	(2383.9, 156836.7)	(1500.1, 255310.9)	16931.4	(2484.8, 153741.3)	(1899.2, 241315.0)
$R(2000)$	0.873	(0.439, 0.983)	(0.298, 0.991)	0.876	(0.450, 0.985)	(0.337, 0.991)

of β, α_0, and α_1 by Fiducial I are smaller while the relative biases of $R(100)$ by Fiducial I are larger. On the other hand, the RMSEs based on Fiducial II are smaller than those from Fiducial I for parameters as well as reliability function. Tables III and IV show that the coverage probabilities based on the ML method significantly deviates the nominal values in some scenarios, while the coverage probabilities based on the two proposed methods are very close to the nominal levels in all the settings. As far as the average interval lengths are concerned, Fiducial II seems to perform better than Fiducial I. It is worth noting that since Fiducial I can be treated as an improved version of the method discussed in [4], our proposed methods should also work better than the method discussed in [4].

V. DATA ANALYSIS

In [22], Nelson studied the time to breakdown data of a certain insulting fluid. Each test was run at a constant voltage stress and the authors tried to estimate the distribution of time to breakdown at 20 kV. Based on this dataset, Wang *et al.* [4] randomly generated two Type-II progressively censored data with parameter settings $(n_1, S_1) = (11, 30)$ and $(n_2, S_2) = (15, 36)$ as shown in Table V, which is the same with Table VI in their paper. In this study, we use the proposed inference methods for this dataset. We set $\varphi(S_i) = S_i$ in the log linear model. MLEs and FIs of the parameters, as well as the mean time to failure and reliability function with $t = 2000$ h, are shown in Table VI. As can be seen, PEs based on the two methods are close to each other, while the lengths of the interval estimates based on the generalized fiducial inference method II are shorter, which tally with the simulation studies.

VI. CONCLUSION AND DISCUSSION

When the Weibull lifetime distribution is assumed, statistical inference for the combination of ALT and progressively censoring is difficult. In this study, we have proposed two inference methods for data from CSALT with a Weibull distribution under progressively Type-II censoring. The proposed inference methods are based on the idea of generalized fiducial inferences. The point estimates and the interval estimates of the parameters as well as the related functions can be obtained simultaneously rather than separately as in [4]. Compared with their method, another advantage of our proposed methods is that they are more straightforward to follow and implement. The simulation results have shown the excellent performance of these methods. It is worth mentioning that since the CSALT with a Weibull distribution under conventional Type-II censoring is a special case of our problem, the proposed fiducial inference methods should also work well for this case. Most existing interval estimation methods for ALT with Weibull distribution and Type-II censoring are based on the Fisher information matrix, which do not work well when the sample size is limited. Therefore, our proposed methods actually fill part of this gap.

APPENDIX A
PROOF OF THEOREM 1

Because $U_0 \sim \chi^2\left(2\sum_{i=1}^k r_i - 2k\right)$, then we have

$$P\left(\chi^2_{1-\gamma/2}\left[2\sum_{i=1}^k r_i - 2k\right] \le U_0 \le \chi^2_{\gamma/2}\left[2\sum_{i=1}^k r_i - 2k\right] \mid n, m, t, R\right) = 1 - \gamma.$$

The left side of the above equation is equal to

$$P\left(G^{-1}\left\{\chi^2_{1-\gamma/2}\left(2\sum_{i=1}^k r_i - 2k\right)\right\} \le \beta \le G^{-1}\left\{\chi^2_{\gamma/2}\left(2\sum_{i=1}^k r_i - 2k\right)\right\} \mid n, m, t, R\right)$$

because $G(\beta) = U_0$ and $G(\cdot)$ is a strictly monotone increasing function.

APPENDIX B
PROOF OF THEOREM 2

Based on the dataset (n, m, t, R), the likelihood function is

$$L(n, m, t, R|\beta, \alpha_0, \alpha_1) = \prod_{i=1}^{k}\prod_{j=1}^{r_i} \frac{\beta t_{ij}^{\beta-1}}{\theta_i^{\beta}} \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{ij}+1)\left(t_{ij}/\theta_i\right)^{\beta}\right\}$$

where $\log(\theta_i) = \alpha_0 + \alpha_1 x_i$.

Since the CDF of Weibull distribution is absolutely continues, (11) is used to derive $J(t, \beta, \alpha_0, \alpha_1)$. For any failure times t_1, t_2, and t_3, after some algebraic calculations, we have that

$$\left|\frac{\det\left(\frac{\mathrm{d}}{\mathrm{d}(\beta,\alpha_0,\alpha_1)}\left(F(t_1), F(t_2), F(t_3)\right)\right)}{f(t_1)f(t_2)f(t_3)}\right| = \begin{cases} \frac{t_1 t_2 t_3\left[x_i \log(t_3/t_2) + x_j \log(t_1/t_3) + x_k \log(t_2/t_1)\right]}{\beta}, & \text{case1} \\ \frac{t_1 t_2 t_3 |\log(t_1/t_2)(x_i - x_j)|}{\beta}, & \text{case2} \\ \frac{t_1 t_2 t_3 |\log(t_1/t_3)(x_i - x_j)|}{\beta}, & \text{case3} \\ \frac{t_1 t_2 t_3 |\log(t_2/t_3)(x_i - x_j)|}{\beta}, & \text{case4} \\ 0, & \text{others} \end{cases}$$

where case1 means that three failure are from different stress levels x_i, x_j, and x_k, respectively; case2–case4 mean that two of t_1, t_2, and t_3 are from the same stress level x_i (i.e., t_1 and t_2 for case2) and the rest one is from stress level x_j. Thus, the result follows.

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