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RESEARCH ARTICLE

Yield-based process capability indices for nonnormal continuous data

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ABSTRACT
Process capability indices (PCIs) are widely used to assess whether an in-control process meets manufacturing specifications. In most applications of classical PCIs, the process characteristic is assumed normally distributed. However, the normal distribution has been found inappropriate in various applications. In the literature, the percentile-based PCIs are widely used to deal with the nonnormal process. One problem associated with the percentile-based PCIs is that they do not provide a quantitative interpretation to the process capability. In this study, new PCIs that have a consistent quantification to the process capability for both normal and nonnormal processes are proposed. The proposed PCIs are generalizations of the classical normal PCIs in the sense that they are the same as the classical PCIs when the process characteristic follows a normal distribution, and they offer the same interpretation to the process capability as the classical PCIs when the process characteristic is nonnormal. We then discuss nonparametric and parametric estimation of the proposed PCIs. The nonparametric estimator is based on the kernel density estimation and confidence limits are obtained by the nonparametric bootstrap, while the parametric estimator is based on the maximum likelihood estimation and confidence limits are constructed by the method of generalized pivots. The proposed methodologies are demonstrated using a real example from a manufacturing factory.

KEYWORDS
confidence limits; coverage probability; kernel estimation; nonparametric bootstrap

1. Introduction

Process capability index (PCI) is widely used as an indicator of process capability in terms of the ability to produce output within specification limits (Polansky 2014). Numerous PCIs have been proposed in manufacturing industries. Some commonly recognized PCIs include C_p, C_{pk}, C_{pm}, and C_{pmk} (Ryan 2011, chap. 7). When the process characteristic X has a normal distribution $N(\mu, \sigma^2)$, these four PCIs are given by

$$C_p = \frac{USL-LSL}{6\sigma}, C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\},$$
$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}, C_{pmk} = \frac{\min\{USL-\mu, \mu-LSL\}}{3\sqrt{\sigma^2+(\mu-T)^2}}, \quad [1]$$

where USL and LSL are respectively the upper and lower specification limits of the process characteristic, and T is the target value. The values of USL, LSL, and T are often determined from engineering tolerances and/or from customers' needs (Ryan 2011, chap. 7). To see how the process capability can be measured based on these PCIs, assume the value of C_{pk} is known. The process yield, which is defined as the percentage of the process characteristic falling within the specification limits, is then bounded by $2\Phi(3C_{pk})-1$ and $\Phi(3C_{pk})$ (Chen and Ye 2018), where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution. If both C_{pk} and C_p are known, the process yield is exactly $1-\Phi(3C_{pk}-6C_p)-\Phi(-3C_{pk})$ (Perakis and Xekalaki 2016).

The establishments of the aforementioned classical PCIs are based on the (approximate) normality assumption on the process characteristic X (Ryan 2011, chap. 7). However, the process characteristic is likely to be nonnormal in various applications. For example, the distribution of cycle time data is often skewed in the procurement process of some companies (Aldowaisan et al. 2015). Similar observations are found in some chemical processes, such as zinc plating (Wang et al. 2016). In addition, many nonnegative quality characteristics in manufacturing processes, such as diameter and roundness, are often nonnormal (Ryan 2011, chap. 7). The normality assumption is

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also questionable in many service and transaction systems (Aldowaisan et al. 2015).

When the process characteristic X is nonnormal, a serious consequence is that the process capability measured based on C_p, C_{pk}, C_{pm}, and C_{pmk} in Eq. [1] can be misleading (Wang et al. 2016). One natural remedy is to first identify an appropriate distribution for the process characteristic data and then use PCIs tailored for this distribution (Clements 1989; Rodriguez 1992). For a nonnormal process, the percentile-based PCIs are probably the most popular ones (Kotz and Lovelace 1998). The percentile-based PCIs were proposed by Clements (1989) and are given by

$$C_{p(q)} = \frac{USL - LSL}{X_{0.99865} - X_{0.00135}} \quad \text{and}$$
$$C_{pk(q)} = \min\left\{\frac{USL - X_{0.5}}{X_{0.99865} - X_{0.5}}, \frac{X_{0.5} - LSL}{X_{0.5} - X_{0.00135}}\right\}, \qquad [2]$$

where X_γ is the γ percentile of X. It is obvious that if X follows the normal distribution, then $C_{p(q)} = C_p$ and $C_{pk(q)} = C_{pk}$. The other two PCIs $C_{pm(q)}$ and $C_{pmk(q)}$ could be similarly obtained (Pearn and Kotz 1994). Unlike the classical PCIs, the percentile-based PCIs do not bear a quantitative relationship with the process capability. In other words, one cannot infer the process yield based merely on the percentile-based PCIs. Moreover, processes with different distributions but the same percentile-based PCIs are very likely to have different capabilities. Therefore, percentile-based PCIs for different distributions cannot be compared directly. This is undesirable, as the PCIs are commonly used as indices for tracking performance and comparing several processes (Chen and Chen 2004). For example, the process capability is an important criterion for companies in selecting suppliers (e.g., Linn et al. 2006). If the process distributions of the suppliers are different, it would be misleading to compare the process capabilities of these suppliers based on the percentile-based PCIs.

In brief, the classical PCIs can only be used for a normal process and the process capability cannot be quantified using the percentile-based PCIs. This study aims to propose new PCIs that are capable of quantifying the process capability for both normal and nonnormal processes. The underlying idea of the proposed PCIs is to transform the process characteristic X to normality. In the literature, transforming the process characteristic to normality and then using the classical PCIs is an alternative method for the nonnormal process. The primary impediment to the transformation method is that the closeness of the transformed data to normality cannot be guaranteed. Although several transformation methods such as the Box–Cox transformation and the Johnson transformation have been proposed (Tang and Than 1999), none of these methods is exact and thus the transformation method is often not tempting for practitioners (Ryan 2011, chap. 7). On the contrary, given a known continuous CDF of X, our method to transform X to normality is exact. In addition, the proposed PCIs are generalizations of the classical PCIs as they degenerate to the classical PCIs when the process is normal, and they offer the same interpretation to the process capability as the classical PCIs when the process is nonnormal. As a result, the proposed PCIs can be safely used in various practical applications, such as the aforementioned supplier-selection problem.

Both nonparametric and parametric inference procedures for the proposed PCIs are developed. The nonparametric inference is based on the kernel density estimation, which is a popular approach to smooth nonparametric estimation of a density (Wand and Jones 1994). In addition, nonparametric confidence limits for the proposed PCIs are constructed by the nonparametric bootstrap. On the other hand, the parametric estimator is based on the maximum likelihood (ML) estimation, and confidence limits are obtained by the generalized pivotal quantity (GPQ) method. Simulation studies are used to assess the proposed inference methods.

The remainder of the article is organized as follows. The second section introduces the new PCIs and investigates their properties. The third section discusses the nonparametric inference of the proposed PCIs, and the fourth section discusses the parametric inference. The proposed PCIs and the developed methodologies are illustrated by a real example from a manufacturing factory in the fifth section. The sixth section concludes the article.

2. The proposed PCIs

As argued, the percentile-based PCIs for a nonnormal process fail to quantify the process yield, and they cannot be compared across different distribution families. This section proposes new PCIs that have a consistent quantification to the process yield. Consider a process characteristic X with CDF $F(x)$. By the fundamental theorem of simulation, the random variable $F(X)$ follows the standard uniform distribution $U(0, 1)$ when $F(\cdot)$ is continuous (Casella and Berger 2002, Theorem 2.1.10). Therefore, the random variable $\Phi^{-1}(F(X))$ follows the standard normal distribution $N(0, 1)$, where $\Phi^{-1}(\cdot)$ is the standard normal quantile.

This means that given X with a continuous CDF, the transformation $Y = \Phi^{-1}(F(X))$ always ensures $Y \sim N(0,1)$. This relationship motivates the definition of our new PCIs.

Assume that the process characteristic X has a continuous CDF. Let USL and LSL be the upper and lower specification limits of X. After the transformation $Y = \Phi^{-1}(F(X))$, the upper and lower specification limits of Y are then $\Phi^{-1}(F(USL))$ and $\Phi^{-1}(F(LSL))$, respectively. Because Y is standard normal, its PCIs can be determined using Eq. [1]. Since the process characteristic X is a transformation of Y, it is reasonable to require that the PCIs for X are the same as the classical PCIs for Y, that is,

$$C_{p(Q)} = \frac{\Phi^{-1}(F(USL)) - \Phi^{-1}(F(LSL))}{6} \quad [3]$$

and

$$C_{pk(Q)} = \frac{\min\left\{\Phi^{-1}(F(USL)), -\Phi^{-1}(F(LSL))\right\}}{3}. \quad [4]$$

The PCIs with respect to C_{pm} and C_{pmk} can be similarly constructed as

$$C_{pm(Q)} = \frac{\Phi^{-1}(F(USL)) - \Phi^{-1}(F(LSL))}{6\sqrt{1 + [\Phi^{-1}(F(T))]^2}} \quad [5]$$

and

$$C_{pmk(Q)} = \frac{\min\{\Phi^{-1}(F(USL)), -\Phi^{-1}(F(LSL))\}}{3\sqrt{1 + [\Phi^{-1}(F(T))]^2}}, \quad [6]$$

where T is the target value for X.

Because the proposed PCIs for X are essentially the same as the classical PCIs for Y, it is not surprising to find that the proposed PCIs inherit most of the good properties from the classical PCIs. Some of the key properties are summarized below, and the proofs are given in the Appendix.

Property 1. When X is normally distributed, the proposed PCIs are the same as the classical PCIs in Eq. [1] defined for normally distributed characteristics.

Property 2. If the proposed PCI $C_{pk(Q)}$ in Eq. [4] is known, the process yield $P(LSL \leq X \leq USL)$ is bounded as

$$2\Phi\left(3C_{pk(Q)}\right) - 1 \leq \text{Yield} \leq \Phi\left(3C_{pk(Q)}\right).$$

Property 3. If the proposed PCIs $C_{p(Q)}$ *and* $C_{pk(Q)}$ in Eq. [3] and Eq. [4] are known, the process yield $P(LSL \leq X \leq USL)$ is exactly

$$\text{Yield} = 1 - \Phi\left(3C_{pk(Q)} - 6C_{p(Q)}\right) - \Phi\left(-3C_{pk(Q)}\right).$$

The preceding properties reveal the same properties between the proposed PCIs and the classical PCIs in the case of two-sided specification limits. In fact, the case of one-sided specification limit is also common in industry. Usually, a smaller-the-better process only has an upper specification limit USL, and a larger-the-better process only has a lower specification limit LSL (e.g., Hubele et al. 2005; Wang and Tamirat 2016). In such cases, Property 4 shows that the process yield has an exact relationship with the proposed $C_{pk(Q)}$.

Property 4. Consider a process characteristic X with either an upper specification limit USL or a lower specification limit LSL. If the proposed PCI $C_{pk(Q)}$ in Eq. [4] is known, then the process yield $P(X \leq USL)$ (or $P(X \geq LSL)$) is exactly

$$\text{Yield} = \Phi\left(3C_{pk(Q)}\right).$$

The intimate relations with the yield are a major advantage of the proposed PCIs over the percentile-based PCIs when X is nonnormal, as the percentile-based PCIs do not have a quantitative relationship with the process yield. In addition, because the proposed PCIs quantify the process yield regardless of the distribution families, they can be compared directly even if the underlying distributions of the processes are different. As an example, consider two process characteristics X_1 and X_2 with the same one-sided specification limit. If the distributions of X_1 and X_2 are different, the process yields of the two processes cannot be compared based on the percentile-based PCIs. On the contrary, the proposed $C_{pk(Q)}$ can be safely used for capability comparison. Next, we show that the proposed PCIs have the following invariance property.

Property 5. If $g(\cdot)$ is a strictly increasing function on the support of X, the proposed PCIs are invariant under the transformation $X \to g(X)$.

The invariance property is useful in various cases. For example, consider the lifetime X as the process characteristic. The distribution of the lifetime is often nonnormal, and the logarithm transformation of the lifetime is often seen in many practical applications (Lawless 2003). In such cases, the percentile-based PCIs based on the original data and the log-transformed data would be different, which may confuse the practitioners. On the contrary, the proposed PCIs are consistent as the logarithm transformation is monotonic.

3. Nonparametric estimation

This section develops nonparametric inference procedures for the proposed PCIs. Nonparametric estimation of the proposed PCIs in Eqs. [3]–[6] requires a nonparametric estimator for the CDF $F(\cdot)$. Let $X_1, \ldots, X_n$ be n independent and identically distributed (iid) copies of X. A natural nonparametric estimator for the distribution function is the empirical distribution function $F_n(x)$ defined as the sample proportion of the X_i's that are not larger than x. However, $F_n(\cdot)$ is a step function and thus it is not continuous. Moreover, $F_n(USL) = 1$ as long as the upper specification limit USL is larger than the maximum observation. As a result, estimates of the proposed PCIs in Eqs. [3]–[6] are not well defined because $\Phi^{-1}(1) = \infty$. To overcome this problem, the kernel density estimation is used to obtain a smooth estimator of $F(x)$.

3.1. The proposed method

Let $f(x)$ be the probability density function (PDF) of X. A well-known smooth nonparametric estimator of the density function is the kernel estimator, defined as

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i),$$

where $K_h(u) = K(u/h)/h$ with K the kernel function and h the bandwidth parameter. Using the relationship between $F(x)$ and $f(x)$, it is easy to construct a kernel estimator for the distribution function as

$$\hat{F}_h(x) = \int_{-\infty}^{x} \hat{f}_h(t)dt,$$

which can also be written in a similar form as the kernel estimator of the density, by

$$\hat{F}_h(x) = \frac{1}{n}\sum_{i=1}^{n} H\left(\frac{x - X_i}{h}\right), \quad [7]$$

where $H(x) = \int_{-\infty}^{x} K(t)dt$. Substituting $\hat{F}_h(x)$ in Eq. [7] gives the kernel estimators of the proposed PCIs.

As seen from Eq. [7], we need to determine the kernel function K and the bandwidth h in estimating the distribution function. The choice of K is an issue of less importance as many kernels such as the Guassian kernel, the Epanechnikov kernel, and the triweight kernel generally work well (Wand and Jones 1994, sec.2.7). In practice, the selection of h based on the data is a more complicated problem. If h is small, the kernel estimator may be undersmoothed and has high variability. On the other hand, if h is large, the kernel estimator may be oversmoothed and much of the underlying structure is obscured. In the context of distribution estimation, there are by and large two methods in bandwidth selection, namely, plug-in and cross-validation (Polansky and Baker 2000). Since the cross-validation method tends to choose smaller bandwidths more frequently than predicted and it generally requires a large sample size to ensure good estimation of the distribution function (Altman and Leger 1995), we suggest the plug-in method in estimating the proposed PCIs, and the bandwidth has the form $\hat{h} = Cn^{-1/3}$, where C can be estimated based on the observed data and different estimation methods can be found in Altman and Leger (1995) and Polansky and Baker (2000). According to our extensive simulation (not shown), the triweight kernel and the plug-in method in Polansky and Baker (2000) achieve the highest accuracy in estimating the proposed PCIs. Therefore, we would recommend such a combination of K and h in real applications of the proposed PCIs.

In practice, a lower confidence limit of PCIs is much more useful than a single point estimator (Chang 2009; Weber et al. 2016). For example, many companies have established threshold values for PCIs, and many purchasers would specify a minimum value of the PCI in the purchasing contract (Wu et al. 2012). Because of the equivalence between hypothesis testing and the confidence intervals (Casella and Berger 2002, sec. 9.2.1), the lower confidence limit can be used to construct the acceptance region in a demonstration test where a supplier demonstrates the process capability to the purchaser. In the parametric setting for PCIs, many methods have been proposed in constructing the lower confidence limits of the classical PCIs (e.g., Pearn and Shu 2003; Chang 2009). However, few studies have focused on the nonparametric setting. In this section, the nonparametric bootstrap-t method is used for interval estimation. Algorithm 1 summarizes the detailed procedure for constructing a nonparametric lower confidence limit of $C_{p(Q)}$. Nonparametric confidence limits for the other proposed PCIs can be constructed in a similar vein.

Algorithm 1. Constructing a nonparametric lower confidence limit for $C_{p(Q)}$.

Step 1. Compute the kernel estimate $\hat{F}$ and $\hat{C}_{p(Q)}$ based on the original data $x_1, \ldots, x_n$.
Step 2. Generate M bootstrap samples each with size n from $\hat{F}$ and then compute M kernel estimates of $C_{p(Q)}$. Denote their standard deviation by $\hat{se}$.
Step 3. Generate B bootstrap samples each with size n by re-sampling from the original data $x_1, \ldots, x_n$ (with replacement).

Step 4. For each bootstrap sample in Step 3, compute the kernel estimate $\hat{F}_b$ and $\hat{C}^{(b)}_{p(Q)}, b = 1, ..., B$. In addition, compute the standard deviation $\hat{se}_b$ by Step 2 with $\hat{F}$ replaced by $\hat{F}_b$ and let $t_b = (\hat{C}^{(b)}_{p(Q)} - \hat{C}_{p(Q)})/\hat{se}_b, b = 1, ..., B$.
Step 5. The $100(1-\alpha)$ percent nonparametric lower confidence limit of $C_{p(Q)}$ is computed as $\hat{C}_{p(Q)} - t_{1-\alpha}\hat{se}$, where t_α is the α percentile of t_b's.

3.2. Simulation study

A simulation is conducted to assess the performance of Algorithm 1. Specifically, 95 percent nonparametric lower confidence limits of $C_{p(Q)}, C_{pk(Q)}$ and $C_{pm(Q)}$ are constructed by generating data from $WB(k, 1), LN(\mu, 1)$, and $GA(\vartheta, 1)$, respectively. Here, $WB(k, \lambda)$ denotes a Weibull distribution with shape k and scale λ, $LN(\mu, \sigma^2)$ denotes a log-normal distribution with location μ and scale σ, and $GA(\vartheta, \beta)$ denotes a gamma distribution with shape ϑ and rate β. For each considered PCI, k, μ, and ϑ are set as $0.5, 1, ..., 4.5, 5$. The *USL* and *LSL* are respectively set as the 0.999 and 0.001 percentiles of the true distribution, which ensures that the process yield is as high as 99.8 percent. The target value T, if required, is set as the mean of the true distribution. The kernel estimators of the proposed PCIs are then obtained by using the triweight kernel and the plug-in method in Polansky and Baker (2000). In addition, B is set as 10,000 and M is set as 1,000 in Algorithm 1, and sample sizes $n = 50$, 100, 200 are considered. Based on 10,000 Monte Carlo replications, the estimated coverage probabilities are shown in Figure 1. As expected, the performance of Algorithm 1 improves with the sample size n. When n is large enough (e.g., $n = 200$), the coverage probability is reasonably close to the nominal value.

4. Parametric estimation

As shown in the third section, it is possible to use the proposed PCIs in a nonparametric setting. When the sample sizes of the process data are not large enough, or when there is sufficient prior information on the distribution of the process characteristic, it is useful to assume a parametric model for the process characteristic X (Ryan 2011, chap. 7). In this section, both point and interval estimations of the proposed PCIs are discussed in the parametric setting.

4.1. The proposed method

Assume that the CDF of X is $F(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the model parameters. Commonly used models include distributions in the location-scale family (e.g., the log-normal distribution and the Weibull distribution) and the gamma distribution, which are recognized as useful models for the process data (Ryan 2011, chap. 3). Point estimation of the proposed PCIs is generally easy as ML estimators of the parameters can be utilized. ML estimation for most of the popular parametric distributions can be found in Lawless (2003, chaps. 4 and 5).

As argued, the lower confidence limit is of most interest in practice. In the literature, a multitude of studies focus on constructing the lower confidence limits of the classical PCIs (e.g., Pearn and Shu 2003; Chang 2009). Among all the possible methods for the classical PCIs, the GPQ method seems to have the best performance in terms of coverage probabilities (Mathew et al. 2007). In this section, we show that the GPQ method can also be used for interval estimation of the proposed PCIs.

The GPQ method proposed by Weerahandi (1993) is a powerful tool for interval estimation of parameters in the presence of nuisance parameters. When

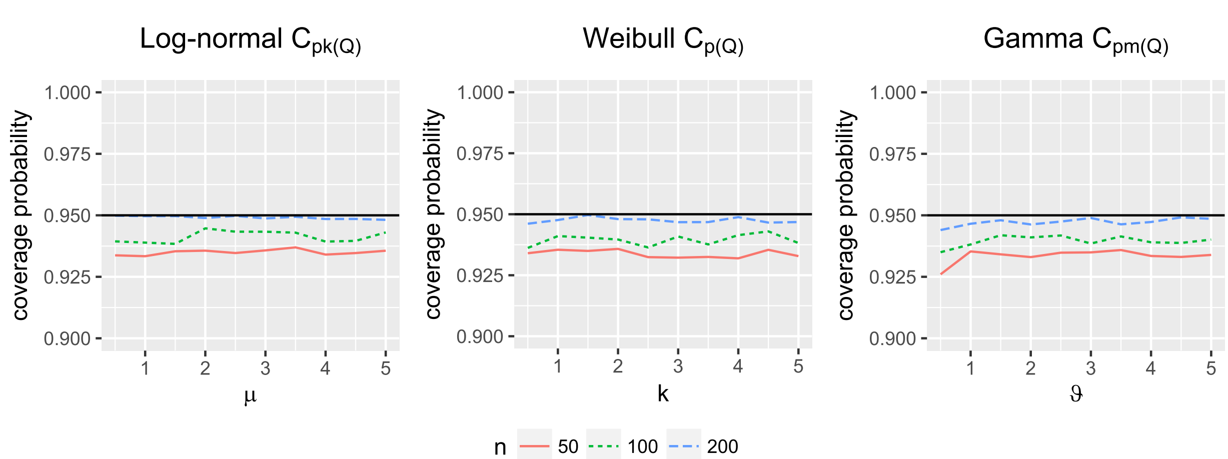

Figure 1. Coverage probabilities for the 95 percent nonparametric lower confidence limit when $n = 50$, 100, 200.

GPQs for the model parameters are available, a GPQ for a function of the model parameters can be obtained by simply plugging in the parameter GPQs, and then the confidence interval for the function can be constructed. Under mild assumptions, Hannig et al. (2006) showed that the GPQ method has asymptotically correct frequentist coverage. Successful applications of the GPQ methods have been reported in Hannig et al. (2006, 2016), among others.

The first step in applying the GPQ method to the porposed PCIs is to find GPQs for the parameters $\boldsymbol{\theta}$. GPQs for parameters of the location-scale parametric distributions can be found in Krishnamoorthy and Mathew (2009, sec. 1.4.2), and GPQs for the gamma distribution are given in Appendix B. Once the GPQs of $\boldsymbol{\theta}$, denoted by $\mathcal{G}_{\boldsymbol{\theta}}$, are available, they could be plugged into Eqs. [3]–[6] to obtain the GPQs of the proposed PCIs. For instance, the GPQ of $C_{p(Q)}$ can be constructed as

$$\mathcal{G}_{C_{p(Q)}} = \frac{\Phi^{-1}(F(USL;\mathcal{G}_{\boldsymbol{\theta}}))-\Phi^{-1}(F(LSL;\mathcal{G}_{\boldsymbol{\theta}}))}{6}, \qquad [8]$$

and the α percentile of this can then be used for constructing a $100(1-\alpha)$ percent lower confidence limit of $C_{p(Q)}$. Although the exact distribution of Eq. [8] may not be easy to obtain, it can be estimated through the Monte Carlo simulation. Algorithm 2 summarizes the procedure for constructing the lower confidence limit of $C_{p(Q)}$. Similar procedures can be applied to construct lower confidence limits of the other proposed PCIs.

Algorithm 2. Constructing a parametric lower confidence limit for $C_{p(Q)}$.

Step 1. Generate B realizations of $\boldsymbol{\theta}$ from the distributions of their GPQs $\mathcal{G}_{\boldsymbol{\theta}}$, and denote them as $(\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(B)})$.

Step 2. For each $\boldsymbol{\theta}^{(b)}, b=1,\ldots,B$, compute the value of $C_{p(Q)}$ defined in Eq. [4] and denote it as $C^{(b)}_{p(Q)}$.

Step 3. Use the α percentile of $\{C^{(1)}_{p(Q)}\ldots C^{(B)}_{p(Q)}\}$ as the $100(1-\alpha)$ percent lower confidence limit of $C_{p(Q)}$.

4.2. Simulation study

Simulation is conducted to assess the performance of the GPQ method in constructing the lower confidence limits of the proposed PCIs. The general setting is almost identical to that in the third section, second subsection. In brief, 95 percent lower confidence limits of $C_{p(Q)}, C_{pk(Q)}$, and $C_{pm(Q)}$ are constructed by assuming that X follows $WB(k,1), LN(\mu,1)$, and $GA(\vartheta,1)$, respectively. The parameters k,μ, and ϑ are set as $0.5, 1, \ldots, 4.5, 5$. The respective ULS and LSL are set as the 0.999 and 0.001 percentiles of the true distribution, and the target value T is set as the mean of the true distribution. In addition, B is set as 10,000 and sample sizes $n=10$, 20, 50 are considered. The estimated coverage probabilities shown in Figure 2 are obtained based on 10,000 replications. As can be seen, the coverage probabilities are very close to the nominal values regardless of the sample size n.

5. Illustrative example

An example from a manufacturing factory is used to illustrate the proposed methods. In the production process of the factory, cutting machines are used. The drill is one of the important components in the cutting machine, and drills of different sizes are needed in the production process. In this study, we focus on drills of size 1.88 mm. Currently, the factory purchases the 1.88-mm drills from two different suppliers. To

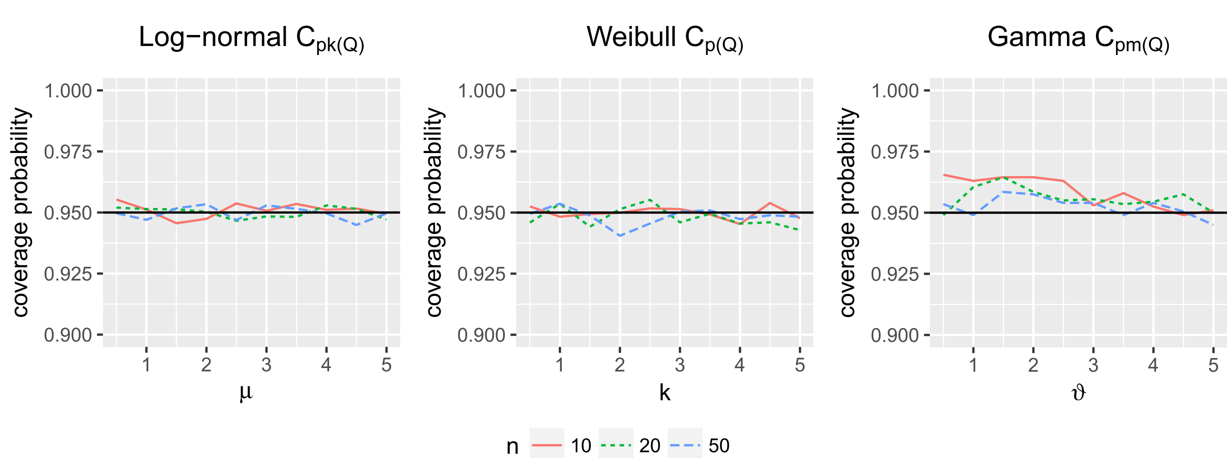

Figure 2. Coverage probabilities for the log-normal $C_{p(Q)}$, the Weibull $C_{pk(Q)}$, and the gamma $C_{pm(Q)}$ when $n=10$, 20, 50 and the nominal value is 0.95.

make a subsequent purchase decision, the factory is interested in knowing which supplier is more reliable. The lifetime of the 1.88-mm drill is treated as the quality characteristic and the comparison is based on the PCIs. Lifetime data for the drills are collected during the production process, as shown in Table 1. According to the process record, all the drill lifetime data are collected from an in-control process.

Based on the histograms in Figure 3, the drill lifetimes of both suppliers seem to have skewed distributions. The log-normal distribution, the Weibull distribution, and the gamma distribution are then used to fit the lifetime data. Based on the values of Akaike information criterion (AIC) in Table 2, the gamma distribution seems to provide the best fit.

Because the lifetime is larger-the-better, the factory sets $LSL = 80$ and puts no restriction on USL, that is, $ULS = \infty$. Given the one-sided specification limit, Property 4 in the second section shows that the process yield is exactly $\Phi(3C_{pk(Q)})$. Therefore, $C_{pk(Q)}$ defined in Eq. [4] is used for comparison. On one hand, the kernel estimator of $C_{pk(Q)}$ is obtained with the kernel function K set as the triweight function and the bandwidth h is selected by the plug-in method in Polansky and Baker (2000). Algorithm 1 is then used to construct the nonparametric lower confidence limits of $C_{pk(Q)}$. On the other hand, the gamma distribution is used to fit the lifetime data, and Algorithm 2 is used for interval estimation. All these estimation results are shown in Table 3. As seen, the difference between the nonparametric confidence limit and the nonparametric point estimate is smaller than that of the parametric estimation. This is reasonable as the nonparametric bootstrap generally requires a large sample size to ensure a satisfactory nonparametric confidence limit (see the third section, second subsection), while the parametric lower confidence limit based on the GPQ method can be well estimated regardless of the sample size (see the fourth section, second subsection). Therefore, we suggest using parametric models when the sample size is not very large (e.g., $n \leq 100$). At last, all the inference results indicate that the first supplier is more reliable, and this information may help the factory to select a better supplier.

Table 1. Lifetimes (in minutes) of 1.88-mm drill from two suppliers.

X_1	135	98	114	137	138	144	99	93	115	106	132	122	94	98	127
	122	102	133	114	120	93	126	119	104	119	114	125	107	98	117
	111	106	108	127	126	135	112	94	127	99	120	120	121	122	96
	109	123	105												
X_2	105	105	95	87	112	80	95	97	77	103	78	87	107	96	79
	91	108	97	80	76	92	85	76	96	77	80	100	94	82	104
	91	95	93	99	99	94	84	99	91	85	86	79	89	89	100

6. Conclusion

This study is the first attempt at developing yield-based PCIs for nonnormal processes. In the literature, the use of classical PCIs such as C_p and C_{pk} is based on the normality assumption of the process characteristic X. If X is nonnormal, the percentile-based PCIs cannot quantify the process yield, which limits their usefulness in various applications such as the supplier-selection problem. On the contrary, our proposed PCIs degenerate to the classical PCIs when X is normally distributed, and they have the same

Table 2. AIC values based on the log-normal, Weibull, and gamma distributions.

	Log-normal	Weibull	Gamma
X_1	390.08	391.96	389.87
X_2	335.38	337.80	335.27

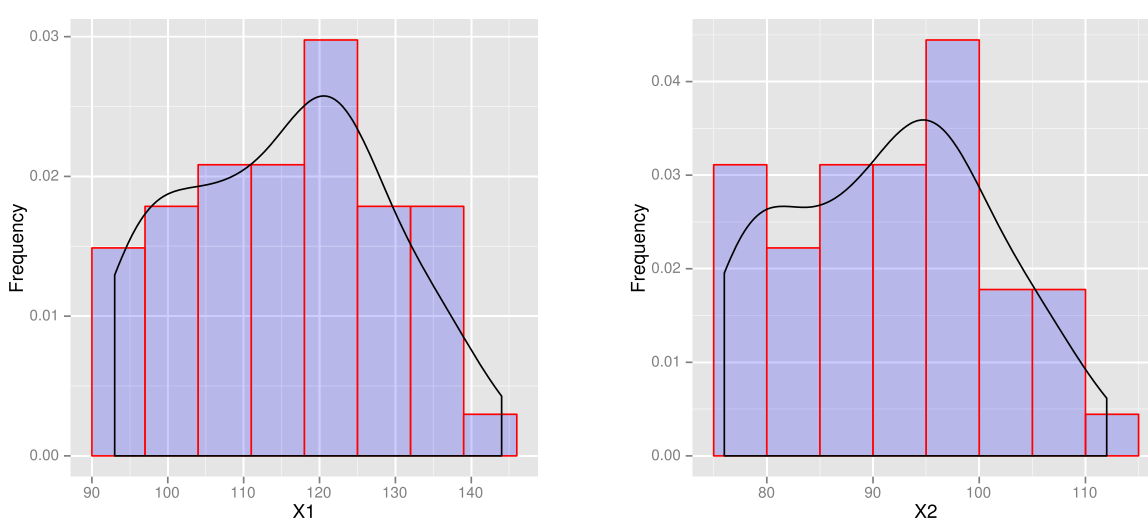

Figure 3. Histograms of the 1.88-mm drill lifetimes from two suppliers.

Table 3. Point estimates and 95 percent lower confidence limits (in parentheses) of $C_{pk(Q)}$ and process yield by the kernel estimation and by assuming the gamma distribution.

		Kernel	Gamma
X_1	$C_{pk(Q)}$	1.022 (0.938)	0.953 (0.768)
	Yield (%)	99.89 (99.75)	99.78 (98.93)
X_2	$C_{pk(Q)}$	0.322 (0.237)	0.397 (0.287)
	Yield (%)	83.30 (76.17)	88.32 (80.54)

quantitative interpretation to the process capability as the classical PCIs when X is non-normal. Because the proposed PCIs have a consistent quantitative relationship with the process yield regardless of the distributions, they cover a much wider class of applications than the classical PCIs and the percentile-based PCIs. In addition to the appealing capability-related properties, the proposed PCIs are also invariant under monotonic transformation of the process data, which further expands their range of applications. Both nonparametric and parametric inference procedures for the proposed PCIs were developed, and their satisfactory performance was revealed by simulation. Finally, the proposed PCIs and the inference procedures have been successfully applied to a practical example.

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References

Aldowaisan, T., M. Nourelfath, and J. Hassan. 2015. Six Sigma performance for non-normal processes. *European Journal of Operational Research* 247 (3):968–77.

Altman, N., and C. Leger. 1995. Bandwidth selection for kernel distribution function estimation. *Journal of Statistical Planning and Inference* 46 (2):195–214.

Casella, G., and R. L. Berger. 2002. *Statistical inference*. Duxbury Pacific Grove, CA: Brooks/Cole Publishing Company.

Chang, Y. 2009. Interval estimation of capability index *cpmk* for manufacturing processes with asymmetric tolerances. *Computers & Industrial Engineering* 56 (1):312–22.

Chen, J.-P., and K. Chen. 2004. Comparing the capability of two processes using C_{pm}. *Journal of Quality Technology* 36 (3):329–35.

Chen, P., and Z.-S. Ye. 2017. Estimation of field reliability based on aggregate lifetime data. *Technometrics* 59 (1): 115–25.

Chen, P., and Z.-S. Ye. 2018. A systematic look at the gamma process capability indices. *European Journal of Operational Research* 265 (2):589–97.

Clements, J. A. 1989. Process capability calculations for non-normal distributions. *Quality Progress* 22 (9):95–7.

Hannig, J., H. Iyer, R. C. Lai, and T. C. Lee. 2016. Generalized fiducial inference: A review and new results. *Journal of the American Statistical Association* 111 (515): 1346–61.

Hannig, J., H. Iyer, and P. Patterson. 2006. Fiducial generalized confidence intervals. *Journal of the American Statistical Association* 101 (473):254–69.

Hubele, N. F., A. Berrado, and E. S. Gel. 2005. A Wald test for comparing multiple capability indices. *Journal of Quality Technology* 37 (4):304–7.

Iliopoulos, G. 2016. Exact confidence intervals for the shape parameter of the gamma distribution. *Journal of Statistical Computation and Simulation* 86 (8):1635–42.

Kotz, S., and C. Lovelace. 1998. *Process capability indices in theory and practice*. London, UK: Arnold.

Krishnamoorthy, K., and T. Mathew. 2009. *Statistical tolerance regions: Theory, applications, and computation*. New York, NY: John Wiley & Sons.

Krishnamoorthy, K.,. T. Mathew, and S. Mukherjee. 2008. Normal-based methods for a gamma distribution: Prediction and tolerance intervals and stress-strength reliability. *Technometrics* 50 (1):69–78.

Krishnamoorthy, K., and X. Wang. 2016. Fiducial confidence limits and prediction limits for a gamma

distribution: Censored and uncensored cases. *Environmetrics* 27 (8):479–93.
Lawless, J. F. 2003. *Statistical models and methods for lifetime data*. New York, NY: John Wiley & Sons.
Linn, R. J., F. Tsung, and L. W. C. Ellis. 2006. Supplier selection based on process capability and price analysis. *Quality Engineering* 18 (2):123–9.
Mathew, T., G. Sebastian, and K. Kurian. 2007. Generalized confidence intervals for process capability indices. *Quality and Reliability Engineering International* 23 (4):471–81.
Pearn, W., and S. Kotz. 1994. Application of Clements'method for calculating second-and third-generation process capability indices for non-normal Pearsonian populations. *Quality Engineering* 7 (1):139–45.
Pearn, W., and M.-H. Shu. 2003. Lower confidence bounds with sample size information for C_{pm} applied to production yield assurance. *International Journal of Production Research* 41 (15):3581–99.
Perakis, M., and E. Xekalaki. 2016. On the relationship between process capability indices and the proportion of conformance. *Quality Technology & Quantitative Management* 13 (2):207–20.
Polansky, A. M. 2014. Assessing the capability of a manufacturing process using nonparametric Bayesian density estimation. *Journal of Quality Technology* 46 (2):150–70.
Polansky, A. M., and E. R. Baker. 2000. Multistage plug-in bandwidth selection for kernel distribution function estimates. *Journal of Statistical Computation and Simulation* 65 (1–4):63–80.
Rodriguez, R. N. 1992. Recent developments in process capability analysis. *Journal of Quality Technology* 24 (4):176–87.
Ryan, T. P. 2011. *Statistical methods for quality improvement*. New York, NY: John Wiley & Sons.
Shao, J., and D. Tu. 2012. *The jackknife and bootstrap*. New York, NY: Springer Science & Business Media.
Tang, L. C., and S. E. Than. 1999. Computing process capability indices for nonnormal data: A review and comparative study. *Quality and Reliability Engineering International* 15 (5):339–53.
Wand, M. P., and M. C. Jones. 1994. *Kernel smoothing*. Boca Raton, FL: CRC Press.
Wang, B. X., and F. Wu. 2017. Inference on the gamma distribution. *Technometrics* 60 (2):235–244.
Wang, F.-K., and Y. Tamirat. 2016. Multiple comparisons with the best for process selection for linear profiles with one-sided specifications. *Quality and Reliability Engineering International* 32 (2):697–704.
Wang, H., J. Yang, and S. Hao. 2016. Two inverse normalizing transformation methods for the process capability analysis of non-normal process data. *Computers & Industrial Engineering* 102:88–98.
Weber, S., T. Ressurreição, and C. Duarte. 2016. Yield prediction with a new generalized process capability index applicable to non-normal data. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* 35 (6):931–42.
Weerahandi, S. 1993. Generalized confidence intervals. *Journal of the American Statistical Association* 88 (423):899–905.
Wu, C.-W., M. Aslam, and C.-H. Jun. 2012. Variables sampling inspection scheme for resubmitted lots based on the process capability index C_{pk}. *European Journal of Operational Research* 217 (3):560–6.

Appendix A: Proofs of the properties

We first prove Property 5, which includes Property 1 as its special case. We only need to show that $C_{p(Q)}$ is invariant under the transformation $X \to g(X)$, as the invariance property of the other proposed PCIs can be proved in a similar way. Let $Z = g(X)$ and the CDF of Z be $F_Z(\cdot)$. Then,

$$\Phi^{-1}\left(F_Z\left(g(USL)\right)\right) = \Phi^{-1}\left(P\left(Z \le g(USL)\right)\right) = \Phi^{-1}(P(X \le USL)) = \Phi^{-1}(F(USL)).$$

Similarly, we can show that $\Phi^{-1}(F_Z(g(LSL))) = \Phi^{-1}(F(LSL))$. Therefore, $C_{p(Q)}$ for Z can be expressed as

$$\frac{\Phi^{-1}\left(F_Z\left(g(USL)\right)\right) - \Phi^{-1}\left(F_Z\left(g(LSL)\right)\right)}{6} = \frac{\Phi^{-1}(F(USL)) - \Phi^{-1}(F(LSL))}{6},$$

which completes the proof.

To prove Property 2 and Property 3, we need to show that the process yields of X and $Y = \Phi^{-1}(F(X))$ are the same, that is,

$$P(LSL \le X \le USL) = P\left(\Phi^{-1}(F(LSL)) \le Y \le \Phi^{-1}(F(USL))\right).$$

Based on Theorem 2.1.10 in Casella and Berger (2002), we have $P(LSL \le X \le USL) = P(F(LSL) \le F(X) \le F(USL))$. On the other hand, since $\Phi^{-1}(\cdot)$ is strictly increasing, the preceding equality obviously holds. Because the proposed PCIs for X are the same as the classical PCIs for Y, and the classical PCIs have the same relationship with the process yield as that in Property 2 and Property 3, the proof is complete. Given Property 3, Property 4 is obvious as $C_{p(Q)} = \infty$ in the case of one-sided specification limit.

Appendix B: Gamma distribution

Consider a random variable X following a gamma distribution $GA(\vartheta, \beta)$ with shape parameter ϑ and rate parameter β. The GPQ of the shape parameter ϑ can be constructed based on the Cornish–Fisher expansion to $W = \log(\tilde{X}/\bar{X})$ (Wang and Wu 2017), where $\bar{X} = \sum_i X_i/n, \tilde{X} = \prod_i X_i^{1/n}$. The Cornish–Fisher expansion is a useful tool to approximate the quantiles of a distribution by using its cumulants (Shao and Tu 2012). Since W is independent of the rate parameter β, its pth quantile W_p can be approximated by $c_1(\vartheta) + [c_2(\vartheta)]^{1/2} Z(\vartheta, p)$, where c_i is the ith cumulant of W and Z is a function of the c_i's. For example, if the first five cumulants of W are used for the approximation, then

$$Z(\vartheta, p) = z_p + \frac{1}{6}\tilde{c}_3\left(z_p^2 - 1\right) + \frac{1}{24}\tilde{c}_4\left(z_p^3 - 3z_p\right) - \frac{1}{36}(\tilde{c}_3)^2\left(2z_p^3 - 5z_p\right) + \frac{1}{120}\tilde{c}_5\left(z_p^4 - 6z_p^2 + 3\right) - \frac{1}{24}\tilde{c}_3\tilde{c}_4\left(z_p^4 - 5z_p^2 + 2\right) + \frac{1}{324}(\tilde{c}_3)^3\left(12z_p^4 - 53z_p^2 + 17\right),$$

where $\tilde{c}_i = c_i/(c_2)^{i/2}$ and z_p is the pth quantile of a standard normal distribution. After some manipulation, it can be shown that $c_1 = \psi(\vartheta) - \psi(n\vartheta) + \log(n)$ and $c_i = \psi^{(i-1)}(\vartheta)/n^{i-1} - \psi^{(i-1)}(n\vartheta), i = 2, 3, \ldots$ for W, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\psi^{(m)}(x) = d^m\psi(x)/dx^i$.

Since W is a monotone function of ϑ (Iliopoulos 2016) based on the observed data, a GPQ $\mathcal{G}_\vartheta$ for ϑ can be the solution to $\log(\tilde{X}/\bar{X}) = c_1(\mathcal{G}_\vartheta) + [c_2(\mathcal{G}_\vartheta)]^{1/2} Z[\mathcal{G}_\vartheta, F_W(W^*)]$, where W^* is a random copy of W. Because $F_W(W^*)$ follows the standard uniform distribution, $\mathcal{G}_\vartheta$ is actually the solution to

$$\log\left(\tilde{X}/\bar{X}\right) = c_1(\mathcal{G}_\vartheta) + [c_2(\mathcal{G}_\vartheta)]^{1/2} Z(\mathcal{G}_\vartheta, U_\vartheta),$$

where U_ϑ follows the standard uniform distribution.

As far as the rate parameter β is considered, we have $U_\beta = 2n\beta\bar{X} \sim \chi^2(2n\vartheta)$ (Chen and Ye 2017). Conditioning on U_ϑ, we have $U_\beta | U_\vartheta \sim \chi^2(2n\mathcal{G}_\vartheta)$, where $U_\vartheta \sim U(0,1)$. Given the observed data $x_1, \ldots, x_n$, the unconditional distribution of U_β can be obtained by integrating U_ϑ out, and is free of the unknown parameters. A GPQ for β can then be constructed as

$$\mathcal{G}_\beta = \frac{U_\beta}{2n\bar{x}}.$$

An alternative way to construct the GPQs of the gamma parameters is to use the normal-based method proposed in Krishnamoorthy et al. (2008). The underlying idea of the normal-based method is that if $X \sim GA(\vartheta, \beta)$, then $\sqrt[3]{X}$ approximately follows $N(\mu, \sigma^2)$, where the parameters have the following relationship:

$$\mu = \left(\frac{\vartheta}{\beta}\right)^{1/3}\left(1 - \frac{1}{9\vartheta}\right) \quad \text{and} \quad \sigma^2 = \frac{1}{9\vartheta^{1/3}\beta^{2/3}}. \qquad [A.1]$$

Once the data are transformed to normality, we can construct GPQs of the normal parameters based on the transformed data. By using Eq. [A.1], we can then obtain the GPQs of the gamma parameters (Krishnamoorthy and Wang 2016). According to our simulation results (not shown) in constructing the lower confidence limits of the proposed PCIs, the performance of the GPQs based on the normal-based method is quite similar to that based on the the Cornish–Fisher expansion as long as the shape parameter is not too small. Because we aim to provide a general framework of parametric estimation of the proposed PCIs, we do not show the simulation results based on the normal-based method. However, the normal-based method is definitely a popular alternative when the gamma distribution is selected for the process data and the estimated shape parameter is not too small.