Random Effects Models for Aggregate Lifetime Data

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***Abstract*—Field data provide important information about product quality and reliability. Many large organizations have developed ambitious reliability databases to trace field failure data of a variety of components on the systems they operate and maintain. Due to the exponential distribution assumption for the component lifetimes, the data in these databases are often aggregated. Specifically, individual lifetimes of the components are not available. Instead, each recorded data point is the cumulative operating time of one component position from system installation to the last component replacement, and the number of replacements in between. In the literature, the gamma distribution and the inverse Gaussian (IG) distribution have been used to fit the aggregate data, while the operating environment of different systems is often assumed the same. In order to capture possible heterogeneities among the systems, this study proposes the gamma random effects model and the IG random effects model. The expectation–maximization algorithm is used for point estimation of the parameters and an algorithm based on the generalized fiducial inference method is proposed for interval estimation. Simulation studies are conducted to assess the performance of the proposed inference methods. A real aggregate dataset is used for illustration.**

***Index Terms*—Confidence interval, expectation–maximization (EM) algorithm, gamma distribution, generalized fiducial inference, inverse Gaussian (IG) distribution.**

I. Introduction

FIELD failure data contain valuable information about quality and reliability of a product in the field. Because they inherently capture the actual usage environment, the field data are highly desirable in the defense, automotive, railroad, and power utility industries [1]. These organizations have developed dedicated field-data collection programs on a diversity of components, and used these data to form a large reliability database. In many of the databases (e.g., see [2]–[4]), the component lifetime is assumed to follow an exponential distribution. With this assumption, lifetimes of the components are often aggregated. As we can see from Fig. 1, for each component position in different systems, the only recorded data are the cumulative operating time from system installation to the last component replacement, and the number of replacements in between. However, the precise individual failure times are not available. This is because the failure rate of the exponentially distributed component lifetime can be estimated with just the cumulative operating time and the failure quantities.

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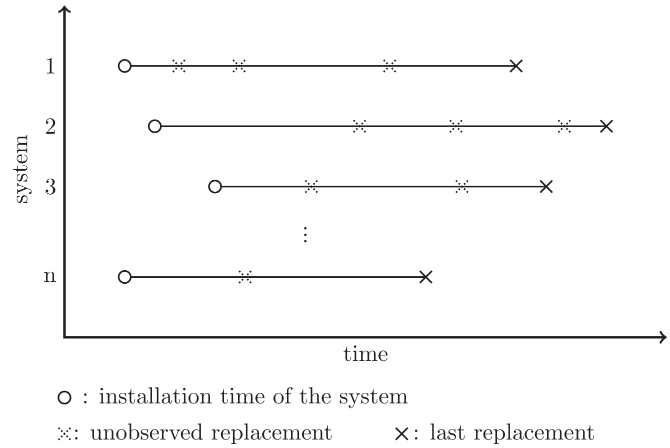

Fig. 1. Diagram of the aggregate data.

Because of its constant hazard function, however, the exponential distribution is not appropriate for components whose failure mechanism is attributed to fracture, fatigue, corrosion, or wearout. In fact, the exponential distribution seems to be a poor choice for most components recorded in the databases. Hypothesis testing in [1] also supports this argument. The incorrect assumption of the exponential distribution may lead to a serious consequence, as the component reliabilities estimated from the databases are often used for estimating the reliability of a new system. Therefore, a more appropriate lifetime distribution is desired. Nevertheless, this is not an easy task as the convolutions of most distributions are mathematically intractable, and hence, the likelihood function based on the aggregate data does not have a closed form. As a result, the parameters cannot be easily estimated. For example, it is difficult to fit the aggregate data by the Weibull distribution and the lognormal distribution. When the observed number of failures is large in some systems, numerically optimizing the likelihood function involving high-dimensional integral is computationally demanding and the maximum likelihood (ML) estimators so obtained usually have large biases. Among all the common lifetime distributions, the gamma distribution and the IG distribution are the two exceptions, as the sum of gamma (IG) distributed lifetimes are again gamma (IG) distributed. In addition to their mathematical convenience, there are several reasons to use the gamma distribution and the IG distribution for the aggregate data. On one hand, these two distributions adequately fit a variety of lifetime data. The recent applications of the gamma distribution as a lifetime model can be found in [5]–[7]; for the IG distribution, see [8]–[10] for a book-length account. On the other hand, simulation study in [11] shows that these two distributions are quite robust to the Weibull or lognormal distribution in terms of small quantiles even under model misspecification.

To the best of our knowledge, Coit and Jin [12] first assumed a gamma distribution for the component lifetime. The likelihood function based on the aggregate data was derived and the ML estimators were obtained by the quasi-Newton search. Later, the authors in [11] successfully fit the aggregate data by the IG distribution and obtained the ML estimators. In addition, they proposed interval estimation methods for both the gamma and IG models. When applying these two models in the literature, a basic assumption is that all systems are under the same environment, and hence, the component lifetimes in different systems are independent and identically distributed (i.i.d.). However, failure data in a reliability database are often complied from many different data sources. For example, the Reliability Information Analysis Center collected data from reports and papers, government sponsored studies, military systems and many other sources to form the large database [3]. It also utilized data from generic sources like commercial warranty repair systems and industrial maintenance databases to form the database [4]. The Offshore Reliability Data Handbook [2] collected failure data from different companies in the oil and gas industry, and has established a comprehensive databank with reliability and maintenance data for exploration and production equipment from a wide variety of geographic areas, equipment types, and operating conditions. In these databases, components reported from different sources may be installed in totally different systems, and the operating environments of these systems may dramatically differ. This means the failure behavior of components reported in the same database may be quite heterogeneous, even when the components are made by the same manufacturer.

In order to capture these heterogeneities, random effects models for the aggregate data are proposed in this study. Random effects models or frailty models are effective in capturing these unobserved factors, and thus, are widely applied in lifetime data analysis. For example, Gorfine *et al.* [13] used a gamma frailty model to represent a family-to-family heterogeneity in genetic linkage analysis. Hanagal and Dabade [14] applied an IG frailty model to bivariate survival data. In the literature, most random effects models assume that heterogeneities exist in all components. In our study, however, random effects exist between systems while not within systems. Components installed successively in one system share the same working environment, and hence, their lifetimes can be regarded as homogeneous, while components installed in different systems may have totally different operating environment [15] and their lifetimes are heterogeneous. Therefore, random effects models for our problem will be different from traditional random effects models available in the literature.

After the random effects models are developed, another important task is to estimate the parameters in the models. In this study, the EM algorithm is used to find the ML estimators of the unknown parameters, as the random effects lead to a complicated likelihood function whose maximization is not an easy task. In addition to the point estimation, an algorithm based on the generalized fiducial inference method [16] is proposed for constructing confidence intervals of the parameters. According to our simulation, conventional interval estimation methods such as the large-sample approximation method and the bootstrap method do not work well for the random effects models when the sample sizes are small. The proposed method, on the other hand, generally guarantees intervals with shorter length and conservative coverage probabilities.

The remainder of this paper is organized as follows. Section II gives a mathematical description of the aggregate data. The gamma and IG models without random effects are also given. In Sections III and IV, the gamma and IG random effects models are proposed, respectively. Procedures of the EM algorithm to these two models are also presented. Section V provides an algorithm for constructing confidence intervals of the unknown parameters in the random effects models. In Section VI, simulation is conducted to show the good performance of the proposed inference methods. Section VII provides an illustrative example. Section VIII concludes this paper.

II. Problem Statement

Consider a component position in a system. Suppose a component with random lifetime X is installed in this position. Once the component fails, it will be replaced by a new one immediately. The replacement time is assumed negligible. For components installed in this position, we assume that individual failure times of the components are unavailable. Instead, the failure times are aggregated, which is the case in most reliability databases. Let T be the cumulative operating time of the component position from system installation to the last component replacement, and let M be the number of replacements. When aggregate data from n systems containing such as component position are collected, the data are denoted as $\mathbf{D} = \{\mathbf{t} = (t_1, \ldots, t_n), \mathbf{m} = (m_1, \ldots, m_n)\}$.

Previous studies assume that all the systems have the same operating environment, and hence, the component lifetimes are i.i.d. among all the systems. Under this assumption, the gamma distribution and the IG distribution have been used to fit the aggregate data. A good property of these two distributions is that the sum of gamma (IG) random variables is again gamma (IG) distributed [17]. Suppose the component lifetime X follows a gamma distribution Gamma(α, β) with probability density function (pdf)

$$f(x, \boldsymbol{\theta}) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \;\; x > 0$$

where α is the shape parameter and β is the rate parameter. Then, the ith cumulative operating time t_i follows a gamma distribution with the shape parameter $m_i\alpha$ and rate parameter β [12].

The likelihood function can be derived based on $\mathbf{D}$ and the ML estimators can be obtained numerically [12]. In addition, interval estimation of the parameters of the gamma distribution is discussed in [11]. On one hand, a χ^2 approximation to a modified ratio of the geometric mean to the arithmetic mean is used to construct the confidence interval of the shape parameter. On the other hand, the generalized pivotal quantity method is used to construct the confidence interval of the rate parameter. In view of the unclear physical meaning of the gamma distribution, the IG distribution is then assumed for the component lifetime, as the IG distribution is the first-passage-time distribution of the Wiener degradation process to a fixed threshold. The pdf of an IG distribution $\mathcal{IG}(\mu, \lambda)$ is

$$f(x, \boldsymbol{\theta}) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right], \ x > 0$$

where μ and λ are the mean and shape parameters, respectively. Similarly, if the component lifetime $X \sim \mathcal{IG}(\mu, \lambda)$, then the ith aggregate lifetime t_i follows $\mathcal{IG}(m_i\mu, m_i^2\lambda)$. Both the point and interval estimation methods for the parameters in the IG model can be found in [11].

As introduced before, field data are often collected from systems operated at different locations and used by different users. There might be some unobservable factors such as different use rates and different ambient environments. In order to capture these heterogeneities, the random effects models are proposed in this study. In our problem, random effects exist on the system level rather than the component level, which is different with the common random effects models seen in the literature. Take the gamma distribution for the component lifetime as an example. The gamma parameters α and β can be regarded as the same for components installed in the same system. However, components in different systems will have different realizations of the two parameters. Therefore, we need a system-level random effects model for the aggregate data. In the next two sections, the gamma random effects model and IG random effects model are proposed, respectively. Point estimation of the parameters is also discussed.

III. Gamma Random Effects Model

A common way to introduce random effects into the gamma distribution is to let the rate parameter be a random variable (e.g., see [18], [19], and [20, Sec. 5.2.2]). We can borrow the idea in our system-level model. Assume the component lifetime $X \sim \text{Gamma}(\alpha, \beta_i), i = 1, \ldots, n$, where α and β_i are shape and rate parameters, respectively. Then, the observed aggregate lifetime in the ith system $t_i \sim \text{Gamma}(m_i\alpha, \beta_i)$. Here, β_i is used to represent the system-to-system heterogeneity and assume $\beta_i \sim \text{Gamma}(w, \delta), i = 1, \ldots, n$. The log-likelihood function of $\boldsymbol{\theta} = (\alpha, w, \delta)$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}; \mathbf{D}) = &\sum_{i=1}^{n} [\log\Gamma(\alpha m_i + w) - \log\Gamma(w) + (\alpha m_i - 1)\log t_i \\ &+ w\log\delta - (\alpha m_i + w)\log(\delta + t_i) - \log\Gamma(\alpha m_i)]. \end{aligned}$$

The ML estimators of the unknown parameters $\boldsymbol{\theta}$ can be obtained by directly maximizing the log-likelihood function. However, because the likelihood function is quite flat in some directions around the ML estimators (e.g., when δ is large), the direct maximization may not be stable. Alternatively, the EM algorithm is found to be more efficient in finding the ML estimators. This is because the maximization problem in the EM algorithm is much easier to solve, as shown in the following detailed procedures.

Recall that $\mathbf{D}$ is the observed data. Let the complete data $\mathbf{D}_c = \mathbf{x} \cup \boldsymbol{\beta}$, where $\mathbf{x} = \{x_{ij}; i = 1, \ldots, n, j = 1, \ldots, m_i\}$ are the individual failure times and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$. If the complete data $\mathbf{D}_c$ are available, the log-likelihood function can be specified as

$$\begin{aligned} l(\boldsymbol{\theta}; \mathbf{D}_c) = &\sum_i \sum_j [\alpha\log\beta_i + (\alpha - 1)\log x_{ij} \\ &- \beta_i x_{ij} - \log\Gamma(\alpha)] \\ &+ \sum_i [w\log\delta + (w-1)\log\beta_i - \delta\beta_i - \log\Gamma(w)]. \end{aligned}$$

In the expectation step, $E(\log x_{ij}|\mathbf{D})$, $E(\log\beta_i|\mathbf{D})$, and $E(\beta_i|\mathbf{D})$ need to be calculated. Given $\boldsymbol{\theta}$ from the last maximization step as well as $\mathbf{D}$, we have

$$\begin{aligned} f(\beta_i|\mathbf{D}, \boldsymbol{\theta}) = f(\beta_i|t_i, \boldsymbol{\theta}) &= \frac{f(t_i; \boldsymbol{\theta}|\beta_i) f(\beta_i)}{f(t_i; \boldsymbol{\theta})} \\ &= \frac{\frac{\beta_i^{\alpha m_i}}{\Gamma(\alpha m_i)} t_i^{\alpha m_i - 1} e^{-\beta_i t_i} \frac{\delta^w}{\Gamma(w)} \beta_i^{w-1} e^{-\delta\beta_i}}{\frac{\Gamma(\alpha m_i + w)}{\Gamma(\alpha m_i)\Gamma(w)} \delta^w (\delta + t_i)^{-(\alpha m_i + w)} t_i^{\alpha m_i - 1}} \\ &= \frac{(t_i + \delta)^{\alpha m_i + w}}{\Gamma(\alpha m_i + w)} \beta_i^{\alpha m_i \alpha + w - 1} e^{-(t_i + \delta)\beta_i}. \end{aligned}$$

Therefore

$$(\beta_i|\mathbf{D}, \boldsymbol{\theta}) \sim \text{Gamma}(\alpha m_i + w, t_i + \delta).$$

Then, $E(\log\beta_i|\mathbf{D})$ and $E(\beta_i|\mathbf{D})$ can be obtained as

$$\begin{aligned} p_i &= E(\log\beta_i|\mathbf{D}) = \psi(\alpha m_i + w) - \log(t_i + \delta) \\ q_i &= E(\beta_i|\mathbf{D}) = (\alpha m_i + w)/(t_i + \delta) \end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In addition, $\sum_j E(\log x_{ij}|\mathbf{D})$ can be expressed as

$$\begin{aligned} u_i &= \sum_j E(\log x_{ij}|\mathbf{D}) \\ &= m_i \left[\log(t_i) + \psi(\alpha) - \psi(\alpha m_i)\right]. \end{aligned}$$

In the maximization step, based on the values of p_i, q_i, and u_i from the last expectation step, $\hat{\alpha}$ can be obtained by

$$\hat{\alpha} = \psi^{-1}\left[\frac{\sum_i (m_i p_i + u_i)}{\sum_i m_i}\right].$$

Next, $\hat{w}$ can be obtained by solving

$$\log\hat{w} - \psi(\hat{w}) = \log\bar{q} - \bar{p}$$

where $\bar{p}$ and $\bar{q}$ are equal to $\sum p_i/n$ and $\sum q_i/n$, respectively. At last, $\hat{\delta}$ can be obtained by substituting $\hat{w}$ into $w/\bar{q}$.

IV. IG Random Effects Model

Since the mean of the IG random variable $\mathcal{IG}(\mu, \lambda)$ is μ, many researchers have proposed random μ for component lifetimes (e.g., see [21]–[23]). This idea is extended to our system-level random effects model. Assume the component lifetime $X \sim \mathcal{IG}(\mu_i, \lambda), i = 1, \ldots, n$, where μ_i and λ are the mean parameter and shape parameter. Then, the ith aggregate lifetime $t_i \sim \mathcal{IG}(m_i\mu_i, m_i^2\lambda)$. Assume $z_i = \mu_i^{-1}$ follows a normal distribution $N(\gamma, \sigma^2)$, where γ and σ^2 are the mean and variance, respectively. Similar to the gamma random effects model, we can obtain the log-likelihood function of $\boldsymbol{\theta} = (\gamma, \sigma, \lambda)$, up to a constant, as

$$l(\boldsymbol{\theta}; \mathbf{D}) = \sum_i \left[\log \bar{\sigma}_i - \log \sigma + \frac{\log \lambda}{2} + \frac{1}{2}\left(\frac{\bar{\gamma}_i^2}{\bar{\sigma}_i^2} - \frac{\gamma^2}{\sigma^2}\right) - \frac{\lambda m_i^2}{2t_i} \right]$$

where

$$\bar{\sigma}_i = \frac{1}{\sqrt{1/\sigma^2 + \lambda t_i}} \quad \text{and} \quad \bar{\gamma}_i = \left(\frac{\gamma}{\sigma^2} + \lambda m_i \right) \bar{\sigma}_i^2. \tag{1}$$

The EM algorithm can again be invoked to obtain the ML estimators. Let the complete data $\mathbf{D}_c = \mathbf{x} \cup \mathbf{z}$, where $\mathbf{x} = \{x_{ij}; i = 1, \ldots, n, j = 1, \ldots, m_i\}$ are the individual failure times and $\mathbf{z} = (z_1, \ldots, z_n)$. The log-likelihood function with the complete data can be expressed as

$$l(\boldsymbol{\theta}; \mathbf{D}_c) = \sum_i \sum_j \left[\frac{1}{2} \log \lambda - \frac{\lambda (x_{ij} z_i - 1)^2}{2 x_{ij}} \right] - \sum_i \left[\frac{(z_i - \gamma)^2}{2\sigma^2} + \log \sigma \right].$$

In the expectation step, $E(z_i|\mathbf{D})$, $E(z_i^2|\mathbf{D})$ and $E(x_{ij}^{-1}|\mathbf{D})$ need to be calculated. Based on $\boldsymbol{\theta}$ and $\mathbf{D}$, we have

$$f(z_i|\mathbf{D}, \boldsymbol{\theta}) = f(z_i|t_i, \boldsymbol{\theta}) = \frac{f(t_i; \boldsymbol{\theta}|z_i) f(z_i)}{f(t_i; \boldsymbol{\theta})}$$

$$\propto \frac{\sqrt{\frac{m_i^2 \lambda}{t_i^3}} \exp\left[\frac{-\lambda (t_i z_i - m_i)^2}{2t_i} \right] \frac{1}{\sigma} \exp\left[\frac{-(z_i - \gamma)^2}{2\sigma^2} \right]}{\frac{\bar{\sigma}_i}{\sigma} \sqrt{\lambda} \exp\left[\frac{1}{2} \left(\frac{\bar{\gamma}_i^2}{\bar{\sigma}_i^2} - \frac{\gamma^2}{\sigma^2} - \frac{\lambda m_i^2}{t_i} \right) \right]}$$

$$\propto \frac{1}{\bar{\sigma}_i} \exp\left[\frac{-(z_i - \bar{\gamma}_i)^2}{2\bar{\sigma}_i^2} \right]$$

which shows

$$(z_i|\mathbf{D}, \boldsymbol{\theta}) \sim N(\bar{\gamma}_i, \bar{\sigma}_i^2)$$

where $\bar{\gamma}_i$ and $\bar{\sigma}_i$ are defined in (1). On the other hand, based on [24, Th. 4], we have

$$E(x_{ij}^{-1}|\mathbf{D}) = \frac{m_i}{t_i} + \frac{m_i - 1}{m_i \lambda}.$$

With these results, we can complete the E-step as

$$p_i = E(z_i|\mathbf{D}) = \bar{\gamma}_i, \quad q_i = E(z_i^2|\mathbf{D}) = \bar{\gamma}_i^2 + \bar{\sigma}_i^2$$

$$u_i = \sum_j E(x_{ij}^{-1}|\mathbf{D}) = m_i^2/t_i + (m_i - 1)/\lambda.$$

The maximization step can then be completed as

$$\hat{\lambda} = \frac{\sum_i m_i}{\sum_i (t_i q_i - 2 m_i p_i + u_i)}, \quad \hat{\gamma} = \bar{p}, \quad \hat{\sigma} = \sqrt{\bar{q} - \bar{p}^2}$$

where $\bar{p}$ and $\bar{q}$ are equal to $\sum p_i/n$ and $\sum q_i/n$, respectively.

V. Confidence Interval

In order to capture uncertainties in the point estimation, a more important task is to construct confidence intervals of the parameters. Classical methods such as large-sample approximation and the bootstrap may not work well in small samples. In addition, according to our simulation in Section VI, their performance is quite poor in some parameter settings for the random effects models. The inadequacy of the classical approaches calls for a uniformly well-performed method. In this section, an algorithm based on the generalized fiducial inference is proposed for constructing the confidence intervals. As we will see in Section VI, it often outperforms the classical methods by a large margin.

The basic idea of the generalized fiducial method expresses a functional relationship between the data $\mathbf{X}$ and the model parameters $\boldsymbol{\theta}$ as

$$X_i = g_i(\boldsymbol{\theta}, \mathbf{U}), \quad i = 1, \ldots, n \tag{2}$$

where $\mathbf{G} = (g_1, \ldots, g_n)$ is called the structural equation and $\mathbf{U}$ is a random vector with a completely known distribution independent of any parameters. Assume that $\mathbf{U} = (U_1, \ldots, U_n)$ are i.i.d. samples from the standard uniform distribution and $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^p$ is p-dimensional. Under some differentiability assumptions on the structural equation $\mathbf{G} = (g_1, \ldots, g_n)$, Hannig [16] showed that the generalized fiducial distribution of $\boldsymbol{\theta}$ is absolutely continuous with density

$$\tau(\boldsymbol{\theta}) = \frac{J(\mathbf{x}, \boldsymbol{\theta}) L(\mathbf{x}, \boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} J(\mathbf{x}, \boldsymbol{\theta}') L(\mathbf{x}, \boldsymbol{\theta}')\, d\boldsymbol{\theta}'} \tag{3}$$

where $L(\mathbf{x}, \boldsymbol{\theta})$ is the likelihood function of the observed data $\mathbf{x}$ and

$$J(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\substack{\mathbf{i} = (i_1, \ldots, i_p) \\ 1 \le i_1 < \cdots < i_p \le n}} \left| \det \left(\left(\frac{d}{d\mathbf{x}} \mathbf{G}^{-1}(\mathbf{x}, \boldsymbol{\theta}) \right)^{-1} \frac{d}{d\boldsymbol{\theta}} \mathbf{G}^{-1}(\mathbf{x}, \boldsymbol{\theta}) \right)_{\mathbf{i}} \right| \tag{4}$$

where the sum goes over all p-tuples of indexes $\mathbf{i} = (1 \le i_1 < \cdots < i_p \le n) \subset \{1, \ldots, n\}$, $d\mathbf{G}^{-1}(\mathbf{x}, \boldsymbol{\theta})/d\boldsymbol{\theta}$ and $d\mathbf{G}^{-1}(\mathbf{x}, \boldsymbol{\theta})/d\mathbf{x}$ are the $n \times p$ and $n \times n$ Jacobian matrices, respectively, and for any $n \times p$ matrix H, $(H)_{\mathbf{i}}$ is the $p \times p$ matrix comprising of the rows $i_1, \ldots, i_p$ of H. A natural choice of the structural equation is to let $\mathbf{G}^{-1} = (F_1(x_1, \boldsymbol{\Theta}), \ldots, F_n(x_n, \boldsymbol{\theta}))$, where $F_i(\cdot, \boldsymbol{\theta})$ is the cumulative distribution function (cdf) of X_i. Then, (4) is simplified t

$$J(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\substack{\mathbf{i} = (i_1, \ldots, i_p) \\ 1 \le i_1 < \cdots < i_p \le n}} \left| \frac{\det \left(\frac{d}{d\boldsymbol{\theta}} \left(F_{i_1}(x_{i_1}, \boldsymbol{\theta}), \ldots, F_{i_p}(x_{i_p}, \boldsymbol{\theta}) \right) \right)}{f_{i_1}(x_{i_1}, \boldsymbol{\theta})) \ldots f_{i_p}(x_{i_p}, \boldsymbol{\theta})} \right| \tag{5}$$

where $f_i(\cdot, \boldsymbol{\theta})$ is the pdf of X_i. After obtaining the generalized fiducial distribution of $\boldsymbol{\theta}$, one can generate realizations of the parameters, and then, use them to construct confidence intervals. Extensive studies have shown the success of the generalized fiducial inference (e.g., [25]–[28]). In fact, Hannig [16] proved that under mild conditions, the coverage is guaranteed by the generalized fiducial inference.

Because the cdfs and pdfs of the aggregated lifetimes T_i's are numerically available, the generalized fiducial inference method is applicable to our problem by replacing X in the aforementioned procedure with T. Unfortunately, the Jacobian in (5) for our problem does not have a closed form, as the cdf $F_i(\cdot, \boldsymbol{\theta})$ for both gamma and IG random effects models involves an integration. It is also difficult to evaluate the Jacobian numerically, as a large number of determinants are required in (5) and another integration is involved in (3). In view of this, we propose an algorithm to avoid computing any determinants for the random effects models. Our algorithm is different from that proposed by [29] in the sense that T_i's are not i.i.d. in our problem while an i.i.d sample is a primary assumption in [29].

In order to make use of the generalized fiducial distribution for $\boldsymbol{\theta}$, we need to approximate the integral in the denominator of (3). Instead of dealing with the whole parameter space $\boldsymbol{\Theta}$ directly, the key idea in [29] is to transform the integral from $\boldsymbol{\Theta}$ to $(0,1)^p$, where p is the number of parameters and $p = 3$ in our problem. Assume for each $\mathbf{i} = (i_1, i_2, i_3)$, there is an unique solution of $\boldsymbol{\theta}$ to the three equations $F_i(t_i, \boldsymbol{\theta}) = u_i, i \in \mathbf{i}$ for any $\mathbf{u} = (u_1, u_2, u_3) \in \Omega_{\mathbf{i}} \subset (0,1)^3$ and no solution for $\mathbf{u} \notin \Omega_{\mathbf{i}}$. Let the solution be $Q_{\mathbf{i}}(\mathbf{u})$. Then

$$\begin{aligned}
&\int_{\boldsymbol{\Theta}} J(\mathbf{t}, \boldsymbol{\theta}) L(\mathbf{t}, \boldsymbol{\theta})\, d\boldsymbol{\theta} \\
&= \sum_{\mathbf{i}} \int_{\boldsymbol{\Theta}} \left| \det\left(\frac{d}{d\boldsymbol{\theta}} \left(F_{i_1}(t_{i_1}, \boldsymbol{\theta}), F_{i_2}(t_{i_2}, \boldsymbol{\theta}), F_{i_3}(t_{i_3}, \boldsymbol{\theta})\right)\right)\right| \\
&\quad \times \prod_{j \notin \mathbf{i}} f_j(t_j, \boldsymbol{\theta}) d\boldsymbol{\theta} \\
&= \sum_{\mathbf{i}} \int_{\Omega_{\mathbf{i}}} \prod_{j \notin \mathbf{i}} f_j(t_j, Q_{\mathbf{i}}(\mathbf{u})) d\mathbf{u}.
\end{aligned}$$

By partitioning $(0,1)^3$ into some cubes, we are able to approximate $\int_{\Omega_{\mathbf{i}}} \prod_{j \notin \mathbf{i}} f_j(t_j, Q_{\mathbf{i}}(\mathbf{u})) d\mathbf{u}$ for each $\mathbf{i}$ based on the Rieman rule. However, an accurate approximation to the integral usually requires a huge number of small enough cubes, which may become computationally burdensome. Fortunately, the approximation is still satisfactory only if cubes near $Q_{\mathbf{i}}^{-1}(\hat{\boldsymbol{\theta}}) = (F_{i_1}(t_{i_1}, \hat{\boldsymbol{\theta}}), F_{i_2}(t_{i_2}, \hat{\boldsymbol{\theta}}), F_{i_3}(t_{i_3}, \hat{\boldsymbol{\theta}}))$ are made smaller [29]. On the other hand, in order not to compute the integral for each $\mathbf{i}$, it is better to deal with the generalized fiducial distribution $\int \tau(\boldsymbol{\theta}) d\boldsymbol{\theta}$ instead of the density $\tau(\boldsymbol{\theta})$ directly. In such cases, a random sampling scheme can be used to reduce the number of computing the integrals. The following algorithm summarizes the procedure of constructing the confidence intervals for the unknown parameters in both the gamma and IG random effects models. To ease notation, denote the unknown parameters as $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ for both models.

Remark: In Step 4, the Newton's method can be used to solve the equations and we may restrict our attention to those $E_{k,l}$'s where the solution exists [29]. Generally speaking, the larger K and L, the more accurate approximation we have. However, due to limited computing resource, we would suggest using $K = 100$ and $L = 1000$ and the performance is satisfactory according to our simulation in the next section.

Algorithm: Algorithm for constructing confidence intervals of $\boldsymbol{\theta}$ in the random effects models.

Step 1: Sample K sets $\mathbf{i}_k = (1 \le i_{k,1} < i_{k,2} < i_{k,3} \le n)$, $k = 1, \ldots, K$.
Step 2: For each k, compute $Q_{\mathbf{i}_k}^{-1}(\hat{\boldsymbol{\theta}}) = (F_{i_{k,1}}(t_{i_{k,1}}, \hat{\boldsymbol{\theta}}), F_{i_{k,2}}(t_{i_{k,2}}, \hat{\boldsymbol{\theta}}), F_{i_{k,3}}(t_{i_{k,3}}, \hat{\boldsymbol{\theta}}))$.
Step 3: For each k, partition $(0,1)^3$ into L cubes $E_{k,l}$, $1 \le l \le L$. Cubes near $Q_{\mathbf{i}_k}^{-1}(\hat{\boldsymbol{\theta}})$ could be made small than cubes further away. Let $\tilde{\mathbf{u}}_{k,l}$ be the center of $E_{k,l}$.
Step 4: For each k and l, compute $\tilde{\boldsymbol{\theta}}_{k,l} = Q_{\mathbf{i}_k}(\tilde{\mathbf{u}}_{k,l})$, i.e., $\tilde{\boldsymbol{\theta}}_{k,l}$ satisfies

$$(F_{i_{k,1}}(t_{i_{k,1}}, \tilde{\boldsymbol{\theta}}_{k,l}), F_{i_{k,2}}(t_{i_{k,2}}, \tilde{\boldsymbol{\theta}}_{k,l}), F_{i_{k,3}}(t_{i_{k,3}}, \tilde{\boldsymbol{\theta}}_{k,l})) = \tilde{\mathbf{u}}_{k,l}.$$

Step 5: For each k and l, calculate $J_{k,l} = \text{Volume}(E_{k,l}) \prod_{j \notin \mathbf{i}_k} f_j(t_j, \tilde{\boldsymbol{\theta}}_{k,l})$, where $\text{Volume}(E_{k,l})$ is the volume of the cube $E_{k,l}$. The generalized fiducial probability $\int_C \tau(\boldsymbol{\theta}) d\boldsymbol{\theta}$ for any set C is approximated by

$$\int_C \tau(\boldsymbol{\theta}) d\boldsymbol{\theta} = \frac{\sum_{k=1}^{K} \sum_{l=1}^{L} J_{k,l} \mathbf{1}_C(\tilde{\boldsymbol{\theta}}_{k,l})}{\sum_{k=1}^{K} \sum_{l=1}^{L} J_{k,l}}$$

where $\mathbf{1}(\cdot)$ is the indicator function.
Step 6: To construct confidence interval of certain parameter, say θ_1, first generate B random variables from the standard uniform distribution, denoted as $u_1, \ldots, u_B$. For each $b \in \{1, 2, \ldots, B\}$, a realization of θ_1, denoted as $\theta_{1,b}$, is obtained by solving $\int_C \tau(\boldsymbol{\theta}) d\boldsymbol{\theta} = u_b$, where $C = (0, \theta_{1,b}) \times (0, \infty)^2$. Sort the values $\theta_{1,1}, \ldots, \theta_{1,B}$ in ascending order and use the percentile to obtain confidence interval of θ_1 with any confidence level.

VI. Simulation

The purpose of the simulation in this section is twofold: To show that the EM algorithm is effective in finding the ML estimators and that the proposed algorithm performs uniformly well in constructing the confidence intervals. General settings of the simulation are as follows. The number of component replacements m_i in any system i is randomly chosen from $\{1, \ldots, 10\}$. For the gamma random effects models, we consider two parameter settings $(\alpha, w, \delta) = (1, 5, 2)$ and $(2, 20, 1)$. For the IG random effects models, the parameters are set as $(\gamma, \sigma, \lambda) = (0.1, 0.01, 5)$ and $(0.2, 0.01, 2)$.

We first examine the performance of the EM algorithm. The system number is set as $n = 10$, 20, 50. The estimated biases and root mean square errors (RMSEs) are computed based on 10 000 replications. The results for the gamma and IG random effects models are presented in Tables I and II, respectively. As we can see, generally the parameters can be quite accurately estimated by the EM algorithm, and the accuracy improves with the sample size n. In fact, according to our simulation (not reported), the performance of the EM algorithm also improves with the number of observed failures m_i's. This is understandable as aggregate data with larger m_i's inherently contain more information about the component lifetime.

TABLE I
BIASES AND RMSES (IN PARENTHESIS) FOR ML ESTIMATORS OF THE PARAMETERS IN THE GAMMA RANDOM EFFECTS MODEL BY USING THE EM ALGORITHM

n	$\alpha = 1$	$w = 5$	$\delta = 2$	$\alpha = 2$	$w = 20$	$\delta = 1$
10	0.579	1.731	– 0.276	1.914	1.373	– 0.481
	(0.652)	(3.113)	(1.974)	(1.914)	(14.04)	(0.977)
20	0.162	0.540	– 0.072	0.664	– 0.503	– 0.259
	(0.436)	(2.458)	(1.931)	(0.976)	(12.84)	(0.955)
50	0.037	0.193	– 0.007	0.221	– 0.204	– 0.127
	(0.285)	(1.578)	(1.223)	(0.577)	(9.550)	(0.721)

TABLE II
BIASES AND RMSES (IN PARENTHESIS) FOR ML ESTIMATORS OF THE PARAMETERS IN THE IG RANDOM EFFECTS MODEL BY USING THE EM ALGORITHM

n	$\gamma = 0.1$	$\sigma = 0.01$	$\lambda = 5$	$\gamma = 0.2$	$\sigma = 0.01$	$\lambda = 2$
10	0.006	– 0.001	3.956	0.017	0.010	1.843
	(0.023)	(0.017)	(10.45)	(0.053)	(0.038)	(6.117)
20	0.005	0.000	2.270	0.012	0.008	0.881
	(0.016)	(0.016)	(7.159)	(0.037)	(0.035)	(3.355)
50	0.002	– 0.001	0.642	0.006	0.007	0.280
	(0.010)	(0.013)	(2.229)	(0.022)	(0.027)	(0.782)

TABLE III
COVERAGE PROBABILITIES AND AVERAGE LENGTHS (IN PARENTHESIS) FOR THE GAMMA RANDOM EFFECTS MODEL WITH TRUE PARAMETERS $(\alpha, w, \delta) = (1, 5, 2)$ BASED ON THE PROPOSED ALGORITHM (FIDUCIAL), LARGE-SAMPLE APPROXIMATION (NORM) AND PERCENTILE BOOTSTRAP (BTR)

		Fiducial	Norm	BTR
$n = 5$	α	99.9 (27.3)	88.6 (24.9)	82.6 (37.1)
	w	99.1 (64.2)	91.1 (61.5)	92.0 (77.7)
	δ	100 (46.8)	100 (52.7)	100 (54.9)
$n = 10$	α	99.2 (11.9)	92.9 (9.1)	85.7 (23.9)
	w	98.8 (26.4)	94.1 (21.5)	94.7 (41.7)
	δ	100 (20.8)	100 (22.7)	100 (21.3)
$n = 20$	α	96.8 (7.9)	94.5 (7.0)	91.9 (18.3)
	w	97.1 (14.1)	95.1 (12.8)	96.2 (23.5)
	δ	96.3 (13.5)	100 (14.6)	99.0 (15.0)
$n = 50$	α	94.7 (2.9)	93.6 (3.1)	95.7 (4.9)
	w	95.5 (7.3)	95.5 (7.2)	93.5 (14.3)
	δ	94.7 (5.2)	100 (6.0)	94.7 (6.3)

TABLE IV
COVERAGE PROBABILITIES AND AVERAGE LENGTHS (IN PARENTHESIS) FOR THE GAMMA RANDOM EFFECTS MODEL WITH TRUE PARAMETERS $(\alpha, w, \delta) = (2, 20, 1)$ BASED ON THE PROPOSED ALGORITHM (FIDUCIAL), LARGE-SAMPLE APPROXIMATION (NORM) AND PERCENTILE BOOTSTRAP (BTR)

		Fiducial	Norm	BTR
$n = 5$	α	100 (22.7)	94.6 (14.9)	68.6 (37.1)
	w	98.1 (73.2)	85.0 (71.5)	92.0 (77.7)
	δ	98.3 (10.8)	100 (12.7)	100 (14.9)
$n = 10$	α	100 (16.1)	96.5 (6.9)	81.1 (31.3)
	w	96.8 (46.7)	88.9 (46.3)	98.4 (60.5)
	δ	96.8 (4.3)	100 (6.1)	100 (5.9)
$n = 20$	α	98.5 (11.4)	96.4 (5.7)	87.7 (23.3)
	w	96.3 (37.6)	92.0 (28.1)	98.9 (53.8)
	δ	96.1 (1.9)	100 (3.6)	100 (3.4)
$n = 50$	α	97.3 (8.2)	94.5 (3.6)	91.8 (10.4)
	w	95.7 (29.5)	94.1 (25.5)	97.1 (33.0)
	δ	95.9 (1.8)	97.4 (3.3)	96.5 (3.1)

TABLE V
COVERAGE PROBABILITIES AND AVERAGE LENGTHS (IN PARENTHESIS) FOR THE IG RANDOM EFFECTS MODEL WITH TRUE PARAMETERS $(\gamma, \sigma, \lambda) = (0.1, 0.01, 5)$ BASED ON THE PROPOSED ALGORITHM (FIDUCIAL), LARGE-SAMPLE APPROXIMATION (NORM) AND PERCENTILE BOOTSTRAP (BTR)

		Fiducial	Norm	BTR
$n = 5$	γ	92.1 (0.241)	74.7 (0.168)	75.9 (0.174)
	σ	100 (0.136)	58.3 (0.071)	100 (0.521)
	λ	98.3 (27.7)	91.0 (18.3)	44.2 (10.1)
$n = 10$	γ	92.5 (0.153)	81.3 (0.136)	83.8 (0.162)
	σ	100 (0.071)	71.0 (0.052)	100 (0.317)
	λ	97.5 (19.3)	94.3 (15.3)	61.4 (8.9)
$n = 20$	γ	92.8 (0.108)	88.1 (0.087)	86.1 (0.083)
	σ	100 (0.054)	81.1 (0.034)	95.9 (0.145)
	λ	96.4 (11.3)	95.4 (9.4)	77.0 (7.6)
$n = 50$	γ	96.2 (0.037)	91.5 (0.025)	90.5 (0.021)
	σ	99.2 (0.035)	84.4 (0.024)	98.9 (0.054)
	λ	94.9 (5.3)	94.6 (5.3)	86.6 (4.7)

The second simulation demonstrates the good performance of the proposed algorithm in constructing the confidence intervals of the parameters. We compare the proposed algorithm with two well-known methods, i.e., the large-sample approximation and the bootstrap. The large-sample approximation is based on a normal approximation to the ML estimators. The inverse of the observed information matrix is used as the covariance matrix of the normal approximation. For the bootstrap, the percentile bootstrap is used and the number of bootstrap samples is set as 10 000. As an example of the percentile bootstrap, assume a parameter θ is of interest. We generate 10 000 bootstrap samples $\hat{\theta}_b, b = 1, 2, \ldots, 10\,000$ for an ML estimator $\hat{\theta}$ and use the respective sample quantiles for the lower and upper limits of the confidence interval. Because all the parameters considered in this paper are positive, the confidence intervals by the large sample approximation and bootstrap are constructed based on the log-transformation of the parameters. For the proposed algorithm, we set $K = 100$ and $L = 1000$. We are interested in the 95% equal-tailed confidence intervals, and the system number is set as $n = 5, 10, 20,$ and 50. The coverage probability and the average interval obtained from 10 000 replications are used for comparison, and they are reported in Tables III–VI. These tables show that the proposed algorithm has a stable performance in all the settings. Most of the coverage probabilities are no less than 95% and the average intervals are comparable to the other two methods. It is not surprising to find that the other two methods perform poorly in small samples, though their performance improves with the sample size.

VII. EXAMPLE

In this section, airplane light data from the large reliability database [3] are used for illustration. The aggregate data are presented in Table VII. As we can see, there are totally six airplanes (systems) and in all the airplanes, indicator lights (components) made by the same manufacturer are installed. The indicator light was replaced immediately once it failed. Because

TABLE VI
COVERAGE PROBABILITIES AND AVERAGE LENGTHS (IN PARENTHESIS) FOR THE IG RANDOM EFFECTS MODEL WITH TRUE PARAMETERS $(\gamma, \sigma, \lambda) = (0.2, 0.01, 2)$ BASED ON THE PROPOSED ALGORITHM (FIDUCIAL), LARGE-SAMPLE APPROXIMATION (NORM) AND PERCENTILE BOOTSTRAP (BTR)

		Fiducial	Norm	BTR
$n = 5$	γ	98.2 (0.723)	77.0(0.579)	79.4 (0.601)
	σ	100 (0.591)	64.2 (0.392)	100 (1.761)
	λ	94.7 (13.3)	89.9 (11.5)	52.8 (8.9)
$n = 10$	γ	96.2 (0.321)	88.3 (0.273)	88.0 (0.261)
	σ	100 (0.169)	69.6 (0.131)	100 (0.932)
	λ	94.2 (7.3)	94.6 (7.1)	65.2 (5.9)
$n = 20$	γ	94.4 (0.141)	88.2 (0.103)	86.6 (0.107)
	σ	99.5 (0.106)	74.3 (0.091)	99.7 (0.591)
	λ	94.4 (3.7)	95.9 (3.5)	77.0 (2.3)
$n = 50$	γ	93.6 (0.082)	94.0 (0.075)	92.9 (0.071)
	σ	99.5 (0.066)	74.4 (0.053)	99.9 (0.091)
	λ	93.1 (1.8)	94.5 (2.1)	86.3 (1.3)

TABLE VII
AIRPLANE INDICATOR LIGHT DATA

System Index i	m_i	t_i (in 1000 h)
1	2	51.0
2	9	194.9
3	8	45.3
4	8	112.4
5	6	104.0
6	5	44.8

TABLE VIII
ML ESTIMATORS (MLES) AND 95% LOWER CONFIDENCE LIMITS (LCLS) OF THE PARAMETERS IN THE GAMMA RANDOM EFFECTS MODEL (LEFT PANEL) AND IG RANDOM EFFECTS MODEL (RIGHT PANEL)

	α	w	δ	λ	γ	σ
MLE	0.846	25.63	452.9	7.834	0.069	4.9×10^{-7}
LCL	0.793	0.434	0.596	2.015	0.051	2.2×10^{-8}

of the exponential distribution assumption and a lack of elapsed time meters, the individual failure times of the indicator lights are not available in the database. Instead, the only recorded data are the cumulative operating time t_i from commencement of an airplane to the last replacement of the light, and the number of replacements m_i in between.

To deal with this aggregate dataset, Coit and Jin [12] first assumed a gamma distribution for the indicator light lifetime and numerically obtained the ML estimators of the parameters. Further, Chen and Ye [11] constructed confidence intervals of the gamma parameters. In view of the appealing physical meaning of the IG distribution, they then assumed the IG distribution for the indicator light lifetime, and obtained both ML estimators and confidence intervals of the parameters. In this section, the gamma and IG random effects models are used to fit this dataset. Table VIII shows the ML estimators by the EM algorithm and the 95% lower confidence limits by the proposed algorithm. As we can see, in the gamma random effects model, the estimated δ is too large that the estimated variance of the random effects, which is ω/δ^2, is extremely small. Similar results are found in the estimated variance σ^2 of the random effects in the IG

TABLE IX
AICS VALUE OF GAMMA AND IG MODELS WITHOUT RANDOM EFFECTS (GAMMA SIMPLE AND IG SIMPLE) AND WITH RANDOM EFFECTS (GAMMA RE AND IG RE)

Gamma Simple	Gamma RE	IG Simple	IG RE
65.63	67.71	66.33	68.33

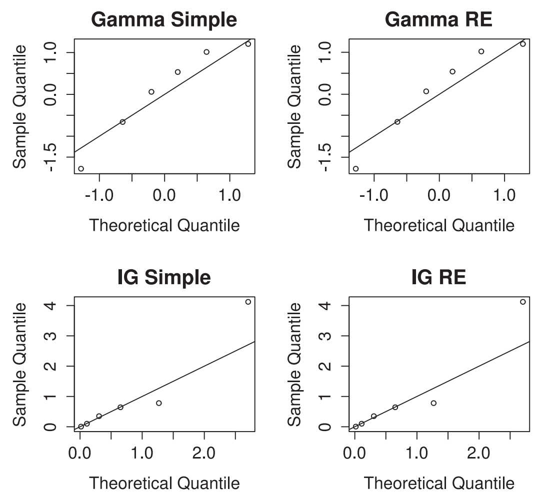

Fig. 2. Q-Q plots of gamma and IG models without random effects (Gamma Simple and IG Simple) and with random effects (Gamma RE and IG RE).

distribution model. The small variances indicate no random effects in the data. The Akaike information criterion (AIC) values of models without and with random effects, as shown in Table IX, also support our argument. This can be explained by the source of the data. The data were collected from six airplanes and the operating environments in different airplanes are almost the same. Usually, all the airplanes will provide pleasant environment for passengers, including moderate temperature and humidity. Therefore, it is convincing that lights in different systems are operated in almost the same environments. According to the AIC values in Table IX, the gamma distribution without random effects tends to fit the data better than the IG distribution.

Next, the goodness of fit of these models is assessed graphically. For the gamma model without random effects, if $X \sim \text{Gamma}(\alpha, \beta)$, where α and β are shape and rate parameters, then $3\sqrt{X\hat{\beta}/\hat{\alpha}}$ is approximately $N(1 - 1/9\hat{\alpha}, 1/9\hat{\alpha})$ distributed [30]. If we incorporate random effects into the gamma distribution, we can simply let $\hat{\beta}_i = E(\beta_i|\mathbf{D}, \boldsymbol{\theta}), i = 1, \ldots, n$. For the IG model without random effects, if $X \sim IG(\mu, \lambda)$, then $\lambda(X - \mu)^2/(\mu^2 X) \sim \chi^2(1)$ [31]. The ML estimators $\hat{\mu}$ and $\hat{\lambda}$ can be substituted into the equation to assess the adequacy of the IG distribution. Similar to the gamma random effects model, $\hat{\mu}_i = E(\mu_i|\mathbf{D}, \boldsymbol{\theta}), i = 1, \ldots, n$ when random effects are incorporated. The quantile–quantile (Q-Q) plot for these four models are shown in Fig. 2. As can be seen, both gamma models with and without random effects seem to provide good fits to the aggregate data. In addition, these Q-Q plots also suggest no random effects for the data, as the plots for the models with and without random effects are almost the same.

VIII. Conclusion

Aggregate lifetime data are common in many large databases. In the literature, the gamma distribution and the IG distribution have been used to analyze the aggregate data. In the implementation of these two models, it is often assumed that all the systems are operated under the same condition, and hence, the component lifetimes in different systems are i.i.d. However, it is not uncommon that the operating environments of different systems are different. In view of the possible heterogeneities among the systems, two random effects models, i.e., the gamma random effects model and the IG random effects model, were proposed in this paper. The EM algorithm was successfully used to obtain the ML estimators.

Another major contribution of this paper comes from the proposed interval estimation algorithm, which performs well and outperforms classical methods. Because the proposed algorithm is essentially based on the generalized fiducial inference, the confidence intervals so obtained have correct coverage under some mild conditions [16]. Of course, in order to improve the accuracy, one need to choose larger values of K and L in the algorithm, which may make the algorithm more computationally demanding than other methods. According to our simulation, however, the extra expense of computation is worthwhile for a better performance. From a practical point of view, we would suggest using $K = 100$ and $L = 1000$, and the performance is satisfactory. At last, the proposed algorithm should also work well for random effects models for the individual lifetime data, which is a special case of the aggregate lifetime data. Existing methods include the large-sample approximation and the bootstrap, which may fail in small samples. Our proposed algorithm, on the other hand, seems to fill this gap.

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