IISE Transactions

Taylor & Francis
Taylor & Francis Group

ISSN: 2472-5854 (Print) 2472-5862 (Online) Journal homepage: https://www.tandfonline.com/loi/uiie21

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To cite this article: Piao Chen, Zhi-Sheng Ye & Qingqing Zhai (2020) Parametric analysis of time-censored aggregate lifetime data, IISE Transactions, 52:5, 516-527, DOI: 10.1080/24725854.2019.1628374

To link to this article: https://doi.org/10.1080/24725854.2019.1628374

Accepted author version posted online: 10 Jun 2019.
Published online: 25 Jul 2019.

IISE TRANSACTIONS
2020, VOL. 52, NO. 5, 516–527
https://doi.org/10.1080/24725854.2019.1628374

Parametric analysis of time-censored aggregate lifetime data

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ABSTRACT

Many large organizations have developed ambitious programs to build reliability databases by collecting field failure data from a large variety of components. To make the database concise, only the number of component replacements in a component position during an operation time interval is reported in these databases. This leads to time-censoring in the aggregate failure data. Statistical inference for the time-censored aggregate data is challenging, because the likelihood function based on some common lifetime distributions can be intractable. In this study, we propose a general parametric estimation framework for the aggregate data. We first use the gamma distribution and the Inverse Gaussian (IG) distribution to model the aggregate data. Bayesian inference for the two models is discussed. Unlike the gamma/IG distribution, other lifetime distributions involve multiple integrals in the likelihood function, making the standard Bayesian inference difficult. To address the estimation problem, an approximate Bayesian computation algorithm that does not require evaluating the likelihood function is proposed, and its performance is assessed by simulation. As there are several candidate distributions, we further propose a model selection procedure to identify an appropriate distribution for the time-censored aggregate data. A real aggregate dataset extracted from a reliability database is used for illustration.

ARTICLE HISTORY
Received 1 August 2018
Accepted 24 May 2019

KEYWORDS
Weibull distribution; gamma distribution; log-normal distribution; inverse Gaussian distribution; approximate Bayesian computation; Monte Carlo sampling

1. Introduction

Field failure data contain valuable reliability information for a product working in actual use conditions. Therefore, many large organizations, such as the US Air Force and US Navy have developed comprehensive programs to collect field failure data on a variety of components. The data have been collected from various sources, and have been combined into reliability databases. Real examples can be found in OREDA (2009), Mahar *et al.* (2011), and Denson *et al.* (2014). As there are numerous components, the component failure data in the databases are often incomplete. For example, Coit and Jin (2000) reported that individual failure times of 90% of components in the nonelectronic parts reliability databases collected by the Reliability Analysis Center in the US are missing. Instead, the failure information is aggregated to make the database concise.

A class of aggregate data was reported in Coit and Jin (2000), as illustrated in Figure 1. Consider a component position in a system. Instead of observing individual component failure times, the observed data are the cumulative operating time and the number of replacements from system installation to the last component replacement. As the observation is censored by a failure event, the resulting data may be called *failure-censored* aggregate data. One reason to record the failure-censored aggregate data is that the exponential distribution is often assumed by industry for a component's lifetime. Based on the exponential distribution assumption, the cumulative operating time together with the total number of replacements are sufficient to summarize all the information contained in individual failure times. However, due to its constant hazard function, the exponential distribution often fails to provide an adequate fit to components whose failure mechanism is attributed to wear-out, fatigue or corrosion (Coit and Dey, 1999). To rectify the deficiency of the exponential distribution, Coit and Jin (2000) proposed the use of the gamma distribution for the failure-censored aggregate data. Interval estimation procedures for the model were developed in Chen and Ye (2017).

The failure-censored aggregate data in Figure 1 are reasonable when the data are recorded in response to an event such as a failure. In many other applications, however, data may be recorded based on a scheduled inspection or a planned data reporting interval. In such cases, an observation is censored at some fixed time, and the resulting aggregate data are *time-censored*. A schematic of the time-censored aggregate data is shown in Figure 2. As can be seen, the end of the observation does not correspond to a failure event, and again the individual failure times of the components are missing. Statistical analysis of the time-censored aggregate data tends to be more difficult than that of the failure-censored aggregate data, as the likelihood function based on the time-censored aggregate data does not have an analytic form under most common lifetime

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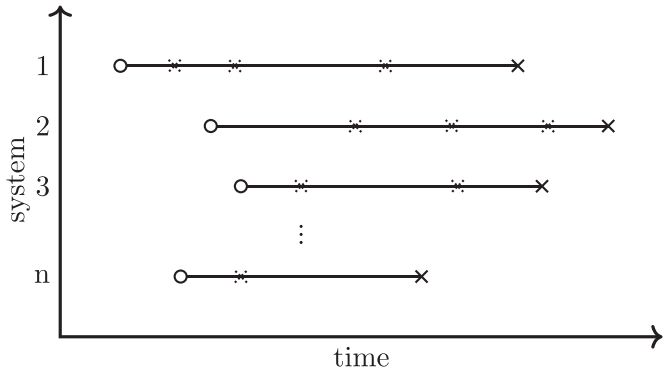

Figure 1. A schematic of the failure-censored aggregate data: the circle denotes the commission date, the cross at the right denotes the recorded last replacement, and the dashed crosses in between denote the unobserved replacements.

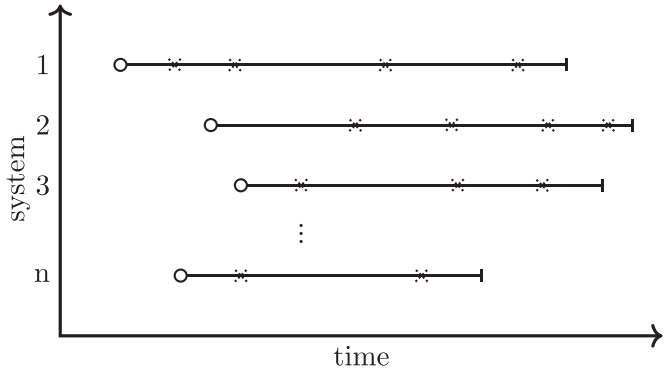

Figure 2. A schematic of the time-censored aggregate data: the circle denotes the commission date, the vertical line at the right denotes the end of observation, and the dashed crosses in between denote the unobserved replacements.

distributions. As a result of this handicap, statistical inference methods for the time-censored aggregate data are underdeveloped.

The objective of this study is to rise to the inference challenge set by the time-censored aggregate data. Hereafter, we refer to the time-censored aggregate data as aggregate data unless otherwise specified. In accordance with the work of Coit and Jin (2000) and Chen and Ye (2016, 2017), we first use the gamma distribution and the Inverse Gaussian (IG) to fit the aggregate data. The gamma/IG distribution is a flexible lifetime model, and recent applications of these two distributions in lifetime data analysis can be found in Balakrishnan and Mitra (2013), Liao and Tian (2013), and Zhai *et al.* (2018), to name a few. One appealing property of the gamma/IG distribution when used to model the aggregate data is its infinite divisibility. That is, the sum of independent identically distributed (iid) gamma/IG random variables is again gamma/IG distributed. As a result, under the gamma/IG distribution assumption, the likelihood function based on the aggregate data only involves a one-dimensional integral, and it can be easily evaluated. Therefore, standard Bayesian methods can be used to estimate the parameters, as well as other reliability characteristics.

In reliability analysis, the Weibull distribution and the log-normal distribution are the most widely used lifetime models (Lawless, 2002, sec 1.3). See Zhang and Yang (2015), Zhao and Xie (2017), and Zhao *et al.* (2019) for a few examples. However, analysis of the aggregate data under these distributions is much more challenging than that under the gamma/IG distribution. This is due to the distribution of the sum of iid Weibull/log-normal random variables being intractable; and hence, the likelihood function based on these two lifetime distributions involves multiple integrals, or even high-dimensional integrals. Generally, the difficulty of computing multiple internals increases dramatically with the dimension of the integrals (Bakhvalov, 2015). As the number of component failures in one system can be very large (e.g., 200) in the reliability databases, it is almost impossible to evaluate the likelihood function based on the aggregate data. To overcome the computational difficulty, this study proposes an Approximate Bayesian Computation (ABC) algorithm that does not require the specification of the likelihood function. The basic idea of the ABC algorithm is to compare simulated data to the observed data. If their difference is small, then the parameter values that generate the simulated data are selected as a sample from the posterior distribution. Successful applications of the ABC algorithm for intractable likelihoods can be found in various areas, including engineering (e.g., Mason (2016)), biology (e.g., Liepe *et al.* (2014)), environment (e.g., Sadegh and Vrugt (2014)), archaeology (e.g., Crema *et al.* (2014)), and many other research fields. As the number of failures is usually large in the aggregate data, the rejection rate of the basic ABC rejection algorithm can be extremely high. Therefore, a Population Monte Carlo (PMC) sampling scheme is proposed to improve the efficiency of the ABC algorithm, and its performance will be assessed by simulation.

Since there are several popular lifetime distributions, we further consider model selection based on the aggregate data. In the literature, the Bayesian evidence, which is defined as the probability that the observed data are generated under the model, is widely used for Bayesian model selection (Everitt *et al.*, 2017; Ong *et al.*, 2018). Under the gamma and IG model, the Bayesian evidence is easy to compute. However, under other lifetime models with intractable likelihoods, computation of the Bayesian evidence is generally difficult. To tackle this problem, a method building on the idea of synthetic likelihood (Wood, 2010) is proposed. The proposed method proceeds by assuming a normal distribution to some summary statistics and using its likelihood, which is called the synthetic likelihood, to approximate the true likelihood. Successful applications of synthetic-likelihood-related methods can be found in Everitt *et al.* (2017), Ong *et al.* (2018), and Price *et al.* (2018), to name a few. In this study, simulation will be conducted to assess the accuracy of the proposed method in approximating the Bayesian evidence.

The remainder of this article is organized as follows. Section 2 presents a general statistical model for the aggregate data. Section 3 introduces the gamma distribution and the IG distribution for the aggregate data, and discusses the Bayesian inference. Section 4 considers other lifetime models, and develops an ABC PMC algorithm to estimate the model parameters as well as other reliability characteristics. Section 5 proposes a model selection procedure by

separately computing the Bayesian evidence for each model. A real aggregate dataset is used for illustration in Section 6. Conclusions are drawn in Section 7.

2. Aggregate data and the likelihood

Consider a component position in a system. Suppose a component with random lifetime X is installed in this position. Once the component fails, it is immediately replaced by a new one with replacement times assumed negligible. We assume that the component's lifetimes are iid with common Probability Density Function (PDF) $f(x)$ and Cumulative Distribution Function (CDF) $F(x)$, $x \geq 0$. For components installed in the position, the individual failure time of each component is unavailable. Instead, only the number of replacements M during an observation window of length T is recorded. The observation window is from the system installation date to the scheduled data report date. The randomness in the duration T results from the randomness in the installation date. Therefore, it is reasonable to assume that T does not contain statistical information on the component reliability. Suppose aggregate data from n systems containing the same component position are collected, and the data are denoted as $\mathbf{D} = \{(t_1, m_1), ..., (t_n, m_n)\}$.

In order to derive the likelihood function, let $f^{(m)}(s), s \geq 0$ be the PDF of $S_m = \sum_{j=1}^{m} X_j$, the sum of m iid component lifetimes. Then $f^{(m)}(s)$ can be expressed as

$$f^{(m)}(s) = \int_{\sum_{j=1}^{m-1} x_j \leq s} f(x_1) f(x_2) ... f\left(s - \sum_{j=1}^{m-1} x_j\right) dx_1 \cdots dx_{m-1}. \tag{1}$$

Conditional on T, the probability to observe $M = m$ replacements within the interval can be obtained as

$$P(M = m|T) = \int_0^T f^{(m)}(s)[1 - F(T - s)]ds. \tag{2}$$

Using the above display, the likelihood function based on the aggregate data $\mathbf{D}$ can be readily obtained. Conditional on T, the event $\{M = m\}$ is equivalent to $\{S_m \leq T, S_{m+1} > T\}$. On the other hand, the event $\{S_m \leq T\}$ can be expressed as the union of two mutually exclusive events $\{S_m \leq T, S_{m+1} \leq T\} \cup \{S_m \leq T, S_{m+1} > T\}$, where the first event is equal to $\{S_{m+1} \leq T\}$. The above argument implies that $\{M = m\}$ is the difference of $\{S_m \leq T\}$ and $\{S_{m+1} \leq T\}$, with the later being included in the former. Therefore, if we define $F^{(m)}(\cdot)$ to be the CDF of $\sum_{j=1}^{m} X_j$, then Equation (2) is equal to

$$P(M = m|T) = F^{(m)}(T) - F^{(m+1)}(T). \tag{3}$$

When $F^{(m)}(\cdot)$ has a simple expression, which is the case when the gamma and the IG distributions are assumed, Equation (3) greatly simplifies Equation (2).

3. Gamma and IG component lifetimes

In this section, we first consider the gamma distribution for the component lifetime. A gamma distribution, denoted as Gamma (α, β), has PDF:

$$f_{\text{gam}}(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, x > 0, \tag{4}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. A good property of the gamma distribution is that the sum of identical gamma random variables is again gamma distributed. Specifically, the distribution of $S_m = \sum_{j=1}^{m} X_j$ is Gamma $(m\alpha, \beta)$. Based on this appealing property, Coit and Jin (2000) successfully used the gamma distribution to fit the failure-censored aggregate data. Here, we further use the gamma distribution to fit the time-censored aggregate data. Since the distribution of S_m is known, Equation (3) can be used to write the likelihood function of $\boldsymbol{\theta} = (\alpha, \beta)$, which is given by

$$L(\boldsymbol{\theta}|\mathbf{D}) = \prod_i \left[F_{\text{gam}}(t_i; m_i\alpha, \beta) - F_{\text{gam}}(t_i; (m_i + 1)\alpha, \beta)\right], \tag{5}$$

where $F_{\text{gam}}(\cdot; \alpha, \beta)$ is the CDF of Gamma (α, β).

Another popular lifetime model that has the infinite divisibility property is the IG distribution, and it has also been used in the literature to fit the failure-censored aggregate data (Chen and Ye, 2017). An IG distribution $\mathcal{IG}(\mu, \theta)$ has the following PDF:

$$f_{\text{ig}}(x) = \sqrt{\frac{\theta}{2\pi x^3}} \exp\left[\frac{-\theta(x-\mu)^2}{2\mu^2 x}\right], x > 0, \tag{6}$$

where $\mu > 0$ and $\theta > 0$ are the mean and shape parameters, respectively. Similar to the case for the gamma distribution, the sum of identical IG random variables is again IG distributed, i.e., $S_m = \sum_{j=1}^{m} X_j \sim \mathcal{IG}(m\mu, m^2\theta)$. Based on Equation (3), the likelihood function of the IG parameters $\boldsymbol{\theta} = (\mu, \theta)$ can be specified as

$$L(\boldsymbol{\theta}|\mathbf{D}) = \prod_i \left[F_{\text{ig}}\left(t_i; m_i\mu, m_i^2\theta\right) - F_{\text{ig}}\left(t_i; (m_i + 1)\mu, (m_i + 1)^2\theta\right)\right], \tag{7}$$

where $F_{\text{ig}}(\cdot; \mu, \theta)$ is the CDF of $\mathcal{IG}(\mu, \theta)$.

The likelihoods (5) and (7) involve functions of the gamma CDF and the IG CDF, respectively. Their values can be easily evaluated at any given value of $\boldsymbol{\theta}$. As a result, the standard Bayesian methods can be used for estimation. Specifically, we first assign a prior $\pi(\boldsymbol{\theta})$ to the parameters $\boldsymbol{\theta}$. Some common priors for the gamma/IG parameters are the uniform distribution, the gamma distribution and the Jeffrey's non-informative prior (e.g., Meshkani *et al.* (2016) and Louzada and Ramos (2018)). Then, the random-walk Metropolis algorithm can be used for sampling from the posterior distribution, which is proportional to $\pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\mathbf{D})$. The algorithm can be implemented by using the `metrop` function in the R package `mcmc`, where the multivariate normal distribution is set as the default proposal distribution as well as the default distribution of the random increment. A detailed treatment of the random-walk Metropolis algorithm may be found in Givens and Hoeting (2012, sec 7.1). Using samples from the posterior distribution, point estimates and credible intervals of the parameters, as well as

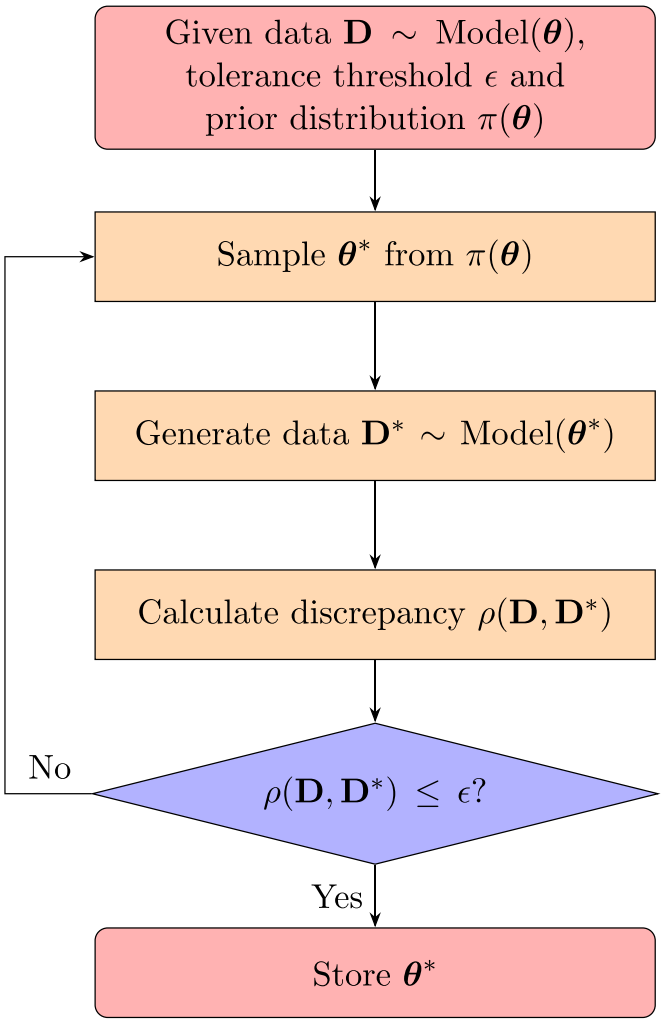

Figure 3. The ABC procedure to sample from the posterior distribution.

other reliability characteristics such as the lifetime CDF and the lifetime quantile, can be readily obtained.

4. Other component lifetime models

Other than the gamma/IG distribution, convolution of a general distribution is most likely intractable. For instance, the distribution of $S_m = \sum_{j=1}^{m} X_j$ does not have a closed form when X is Weibull or log-normal. As a result, the likelihood function (3) involves multiple integrals, which is difficult to evaluate when m is large, say $m \geq 10$. To circumvent this difficulty, an ABC algorithm is developed in this section.

ABC is a Bayesian approach that does not require the specification of a likelihood function. Given a prior $\pi(\cdot)$ for the parameters $\boldsymbol{\theta}$, the ABC algorithm proceeds in the following way. First, we sample a candidate parameter $\boldsymbol{\theta}^*$ from $\pi(\boldsymbol{\theta})$. With this candidate $\boldsymbol{\theta}^*$, a dataset $\mathbf{D}^*$ is generated from the assumed model that has the same generating mechanism as the observed dataset $\mathbf{D}$, so that the distributional properties of the simulated data $\mathbf{D}^*$ can match those of the observed data $\mathbf{D}$. We then compute the distance $\rho(\mathbf{D}^*, \mathbf{D})$ between the simulated data $\mathbf{D}^*$ and the observed data $\mathbf{D}$, where the distance function ρ is usually the difference between some summary statistics. If $\rho(\mathbf{D}^*, \mathbf{D})$ is not larger than a prefixed ϵ, then the simulated data $\mathbf{D}^*$ is "close enough" to the observed data $\mathbf{D}$, and thus we keep $\boldsymbol{\theta}^*$ as a sample from the posterior. Otherwise, $\boldsymbol{\theta}^*$ is rejected. A flowchart of the ABC sampling procedure is shown in Figure 3.

In order to develop an ABC algorithm suitable for our aggregate data problem, we need to determine the prior distribution $\pi(\boldsymbol{\theta})$, the distance function $\rho(\cdot)$ and the tolerance threshold ϵ. These issues are discussed in Section 4.1 and Section 4.2. In addition, to improve the efficiency of the ABC algorithm, a sequential Monte Calo sampling scheme is provided in Section 4.3. Its performance will be assessed by simulation in Section 4.4.

4.1. Prior selection and data generation

An advantage of the Bayesian inference is that the prior π can be determined based on past information, if any. Since no past information is available for our aggregate data problem, non-informative priors are adopted for the parameters. In our problem, the inverse gamma distribution is used as priors for positive parameters, such as the Weibull shape parameter. A random variable $X \sim \text{InvGam}(\alpha, \beta)$ if $1/X \sim \text{Gamma}(\alpha, \beta)$. The inverse gamma distribution is commonly used as a prior in Bayes analysis; see Huang *et al.* (2013) and Rhodes *et al.* (2015) for a few examples. In practice, we may set both α and β small so that the prior is flat and the parameter can explore the whole positive real line. On the other hand, when the parameter can be any real number, we let the normal distribution $N(0, \sigma^2)$ be the prior. When σ is large, the distribution is quite flat and disperses over a wide range of the real line. Applications of the normal distribution as a prior in ABC can be found in Liepe *et al.* (2014) and Jennings and Madigan (2017), to name a few.

After the prior π is fixed, we generate $\boldsymbol{\theta}^*$ from π, based on which a pseudo aggregate dataset $\mathbf{D}^*$ is generated. The observed aggregate data are $\mathbf{D} = \{(t_i, m_i), i = 1, \ldots, n\}$. The mechanism that generates the pseudo data should mimic that for the observed data. Therefore, the pseudo data should contain n component positions. For the ith component position, we assume that the observation window t_i is exogenous and it does not contain statistical information on $\boldsymbol{\theta}$. Based on this consideration, we use a conditional argument and treat t_i as fixed when generating the pseudo number of failures m_i^* for the ith position. Consequently, a pseudo aggregate dataset based on $\boldsymbol{\theta}^*$ is $\mathbf{D}^* = \{(t_i, m_i^*), i = 1, \ldots, n\}$.

4.2. Choice of $\rho(\mathbf{D}^*, \mathbf{D})$ and ϵ

Since the observations window t_i is the same for both $\mathbf{D}$ and $\mathbf{D}^*$, we only need to consider the number of replacements m_i in the distance function $\rho(\cdot)$. A simulated aggregate dataset that has a good match with the observed aggregate dataset should have m_i^*s that are close to m_is for each system i. Therefore, we set $\rho(\mathbf{D}^*, \mathbf{D}) = \sum_i |m_i - m_i^*|$. When $\rho(\mathbf{D}^*, \mathbf{D}) = 0$, the simulated aggregate data have exactly the same data structure as the observed data. In fact, this distance function simultaneously considers n summary

statistics, i.e., the number of replacements $\mathbf{M} = \{M_1, ..., M_n\}$ for the n systems. Generally, considering more than one summary statistic in the distance function will increase the amount of information available to the ABC algorithm, and hence improve the estimation accuracy (Marin *et al.*, 2012). In case that t_is are the same for all the systems, due to the iid assumption of the component lifetimes in different systems, we could compare the ordered m_is and m_i^*s in the distance function, which helps to reduce the computational efforts.

As shown above, the ideal choice of ϵ would be $\epsilon = 0$. However, such a setting would increase the computational burden. On the other hand, when ϵ is small enough, reducing it further would not change the approximate posterior distribution (Beaumont *et al.*, 2009). Generally, there is no uniform guideline for selecting ϵ, and a good value of ϵ often varies case by case (Marin *et al.*, 2012). In terms of the aggregate data, according to our numerical studies in Section 4.4 and Section 6, a good value of ϵ should depend on the number of systems n and the inspection times t_is. When n or t_i is large, a moderate ϵ indicates a close agreement between $\mathbf{D}$ and $\mathbf{D}^*$. When n and t_is are small, however, a very small ϵ should be considered.

4.3. Population Monte Carlo sampling

The ABC algorithm shown in Figure 3 is the simplest rejection algorithm. It discards the candidate value $\boldsymbol{\theta}^*$ when $\rho(\mathbf{D}^*, \mathbf{D}) > \epsilon$. When there is a large discrepancy between the prior and the posterior, which is usually true if a non-informative prior is adopted, the distance between the observed data and the pseudo data is likely to be large. In our aggregate data case, ϵ is set small, and hence, the rejection rate is high based on the basic ABC rejection sampling algorithm. To improve the efficiency of the ABC algorithm, a PMC sampling scheme is proposed in this subsection.

The PMC belongs to the category of sequential Monte Carlo sampling, and it is a convenient and attractive approach to obtain the posterior distribution (Beaumont *et al.*, 2009). Instead of drawing one candidate $\boldsymbol{\theta}^*$ at a time, the PMC algorithm simultaneously generates a large pool of candidates, which are also called particles, at each iteration. Associated with each particle is a weight representing the probability of being chosen in the next iteration. The weight is often computed based on the current particle, the weights of all particles from the previous iteration, and a Gaussian kernel (Beaumont *et al.*, 2009). In particular, the weights for particles at the first iteration are the same. When ABC is applied to the PMC sampling, a sample is rejected when the distance between the generated pseudo data and the observed data is larger than ϵ, while ϵ is allowed to change over the PMC iterations. The ABC PMC uses a decreasing sequence of tolerances $\boldsymbol{\epsilon}$. A large ϵ at the outset leads to a small rejection rate, whereas a small ϵ at the end ensures that the distribution of the final particles are close to the posterior. A number of studies have shown that the PMC strategy makes the ABC much more efficient (e.g., Beaumont *et al.*, (2009) and Liepe *et al.* (2014)).

Now let us formally state the ABC PMC algorithm for estimating the unknown parameters $\boldsymbol{\theta}$ based on the aggregate data. Let N be the number of particles generated at each iteration. At the first iteration $v = 1$, we generate N particles $\boldsymbol{\theta}_{1:N,1} = \{\boldsymbol{\theta}_{1,1}^*, ..., \boldsymbol{\theta}_{N,1}^*\}$ by the ABC rejection algorithm (see Figure 3) with ϵ_1. Note that ϵ_1 is set large, and hence, the first iteration takes a relatively short computation time. In addition, the weights for these particles are set as $w_{p,1} = 1/N, p = 1, ..., N$, and the covariance matrix $\boldsymbol{\Sigma}_1$ for the Gaussian kernel is set as twice of the sample covariance of $\boldsymbol{\theta}_{1:N,1}$, i.e., $\boldsymbol{\Sigma}_1 = 2\text{Cov}(\boldsymbol{\theta}_{1:N,1})$. This choice makes the sequential sampling procedure efficient, and its theoretical support can be found in Beaumont *et al.* (2009).

At iteration $v \geq 2$, one particle is sampled as follows. We first sample $\boldsymbol{\theta}^\dagger$ from the particles $\boldsymbol{\theta}_{1:N,v-1}$ at the previous iteration with probabilities $w_{1:N,v-1}$. Then we perturb $\boldsymbol{\theta}^\dagger$ by sampling $\boldsymbol{\theta}^*$ from the multivariate normal distribution $N(\boldsymbol{\theta}^\dagger, \boldsymbol{\Sigma}_{v-1})$. Afterwards, a pseudo dataset $\mathbf{D}^*$ is generated based on $\boldsymbol{\theta}^*$, and $\boldsymbol{\theta}^*$ is accepted only when $\rho(\mathbf{D}^*, \mathbf{D}) \leq \epsilon_v$. In the literature, there is no general guideline to select ϵ_v (Marin *et al.*, 2012). In this study, we determine ϵ_v based on the distances at the last iteration. For example, ϵ_v can be set as the 10th percentile of the N distances at iteration v – 1. Such a procedure is intuitive, and it balances the number of iterations and the rejection rate within each iteration. The above procedure is repeated until N particles are accepted, after which their weights $w_{p,v}, p = 1, ..., N$ are updated as

$$w_{p,v} = \frac{\pi(\boldsymbol{\theta}_{p,v})}{\sum_{q=1}^{N} w_{q,v-1}\phi(\boldsymbol{\theta}_{p,v}|\boldsymbol{\theta}_{q,v-1}, \boldsymbol{\Sigma}_{v-1})},$$

where $\phi(\cdot|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the PDF of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. In addition, $\boldsymbol{\Sigma}_v$ is updated as $\boldsymbol{\Sigma}_v = 2\text{Cov}(\boldsymbol{\theta}_{1:N,v})$. The above PMC algorithm is terminated when ϵ_v is less than the prefixed tolerance ϵ.

With an appropriate $\rho(\cdot)$ and ϵ, the ABC PMC algorithm approximately generates samples from the posterior distribution (Beaumont *et al.*, 2009). Therefore, once N particles at the last iteration are generated. Their mean and percentiles can be used for point and interval estimations of $\boldsymbol{\theta}$. For better illustration, the ABC PMC algorithm for estimating $\boldsymbol{\theta}$ is summarized in Algorithm 1.

Algorithm 1. The ABC PMC algorithm for estimating the unknown parameters $\boldsymbol{\theta}$.

Input: the observed data $\mathbf{D}$, the threshold ϵ_1 and ϵ, and prior distribution $\pi(\boldsymbol{\theta})$.

procedure First Iteration
  for $1 \leq p \leq N$ **do**
    repeat
      Generate a $\boldsymbol{\theta}^* \sim \pi(\boldsymbol{\theta})$
      Generate an aggregate data set $\mathbf{D}^*$ based on $\boldsymbol{\theta}^*$
    until $\rho(\mathbf{D}^*, \mathbf{D}) \leq \epsilon_1$
    $\boldsymbol{\theta}_{p,1} \leftarrow \boldsymbol{\theta}^*$
    $w_{p,1} \leftarrow 1/N$
  end for
  $\boldsymbol{\Sigma}_1 \leftarrow 2\text{Cov}(\boldsymbol{\theta}_{1:N,1})$
  $v \leftarrow 1$
end procedure

procedure NEXT ITERATIONS
  repeat
    $v \leftarrow v+1$
    $\epsilon_v \leftarrow$ the 10th percentile of the N distances at iteration $v-1$
    for $1 \le p \le N$ **do**
      repeat
        Generate a $\boldsymbol{\theta}^{\dagger} \sim \boldsymbol{\theta}^{*}_{1:N,v-1}$ with probabilities $w_{1:N,v-1}$
        Generate a $\boldsymbol{\theta}^{*} \sim N(\boldsymbol{\theta}^{\dagger}, \boldsymbol{\Sigma}_{v-1})$
        Generate an aggregate data set $\mathbf{D}^{*}$ based on $\boldsymbol{\theta}^{*}$
      until $\rho(\mathbf{D}^{*}, \mathbf{D}) \le \epsilon_v$
      $\boldsymbol{\theta}_{p,v} \leftarrow \boldsymbol{\theta}^{*}$
      $w_{p,v} = \frac{\pi(\boldsymbol{\theta}_{p,v})}{\sum_{q=1}^{N} w_{q,v-1}\phi(\boldsymbol{\theta}_{p,v}|\boldsymbol{\theta}_{q,v-1},\boldsymbol{\Sigma}_{v-1})}$
    end for
    $\boldsymbol{\Sigma}_v \leftarrow 2\mathrm{Cov}(\boldsymbol{\theta}_{1:N,v})$
  until $\epsilon_v \le \epsilon$
end procedure

Output: use the mean of $\boldsymbol{\theta}_{1:N,v}$ as the point estimate of $\boldsymbol{\theta}$, and percentiles of $\boldsymbol{\theta}_{1:N,v}$ as the credible intervals of $\boldsymbol{\theta}$.

4.4. Simulation study

Simulation is used to assess the performance of Algorithm 1. Two popular lifetime distributions, i.e., the Weibull distribution and the log-normal distribution, are considered. The Weibull distribution $WB(k,\lambda)$ has a PDF:

$$f_{\mathrm{wb}}(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^{k}\right], x \ge 0, \tag{8}$$

where $k>0$ is the shape parameter and $\lambda>0$ is the scale parameter. On the other hand, the PDF of a log-normal distribution $LN(\mu,\sigma)$ with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma>0$ is

$$f_{\ln}(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log x-\mu)^2}{2\sigma^2}\right], x>0. \tag{9}$$

Throughout this section, the prior distribution for the parameter μ is $N(0,100^2)$ and the prior distribution for parameters k,λ, and σ is $\mathrm{InvGam}(0.1,0.1)$. The number of systems n is set as 5, 10, and 20, and the inspection time is set as 20 for all the systems. In addition, a total of $N=2000$ parameter samples are generated in each iteration. Based on the selection criterion for ϵ in Section 4.2, we set $\epsilon = 1,3,6$ corresponding to each value of $n=5$, 10, 20. Another input ϵ_1 in Algorithm 1 is set as $10n$, which ensures quick computation at the first iteration.

We first consider the Weibull distribution for the component lifetime. We set $\boldsymbol{\theta} = (k,\lambda) = (0.5,1)$, (2, 1), and (5, 1). Figures 4 and 5, respectively, show the simulation results for k and λ, where the absolute bias and the Root Mean Square Error (RMSE) are estimated based on 2000 replications. As seen, the parameters can be accurately estimated and the accuracy improves with n. Based on our other simulation trials (not reported), the estimation accuracy also improves with the inspection duration T if the system number n is fixed. This is reasonable, as longer inspection durations lead to larger numbers of replacements in each system, which increases the amount of information available to the proposed algorithm. In addition to the parameters, we further consider estimating the lifetime quantiles, which are often of more interest in practice. The parameter samples from Algorithm 1 are first used to get samples of the lifetime quantiles, whose percentiles are then used to construct the 95% credible intervals of the $0.1, 0.2, 0.3, 0.4, 0.5$ lifetime quantiles. Figure 6 shows the coverage probabilities based on 2000 replications. As seen, the proposed algorithm generally performs well and when $n=20$, the coverage probability is very close to the nominal value.

In the second simulation study, we assume that the Weibull shape parameter k is known and fixed at one. Under this setting, the Weibull distribution degenerates to the exponential distribution, which is also a special case of the gamma distribution. As a result, the Metropolis algorithm in Section 3 can be used for posterior sampling. On the other hand, if the proposed ABC PMC algorithm could approximate the posterior samples well, it should perform similarly as the Metropolis algorithm. To verify this, we consider different values of the scale parameters, e.g., $\lambda = 0.5, 1$, and 2. A total of 2000 posterior samples are generated from both the Metropolis algorithm and Algorithm 1. Based on 2000 replications, the RMSEs for λ are shown in Figure 7. As seen, the two algorithms have similar performance, indicating that the proposed algorithm is adequate for posterior sampling.

Finally, we consider the log-normal distribution for the component lifetime. We set $\boldsymbol{\theta} = (\mu,\sigma) = (-2,1)$, (0, 1), and (2, 1). Based on 2000 replications, the biases and RMSEs of the parameters are shown in Figures 8 and 9, and the

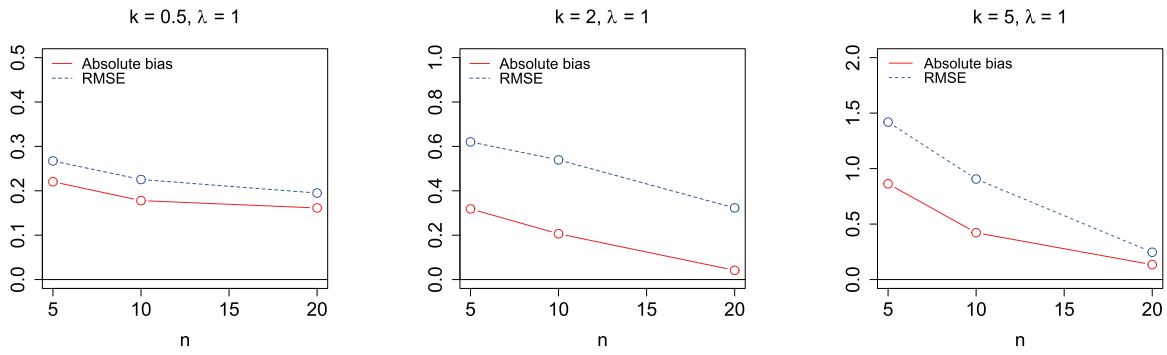

Figure 4. Absolute bias and RMSE for the Weibull shape parameter k based on 2000 replications.

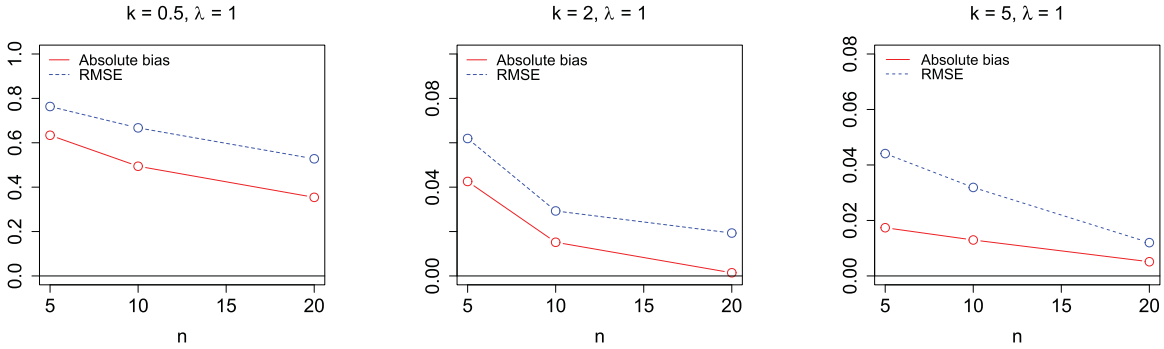

Figure 5. Absolute bias and RMSE for the Weibull scale parameter λ based on 2000 replications.

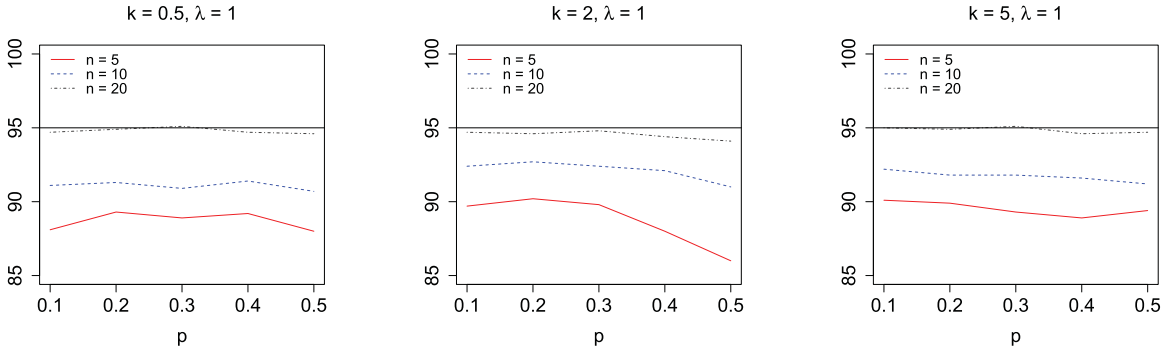

Figure 6. Coverage probabilities for the lifetime quantiles based on 2000 replications by assuming the Weibull distribution.

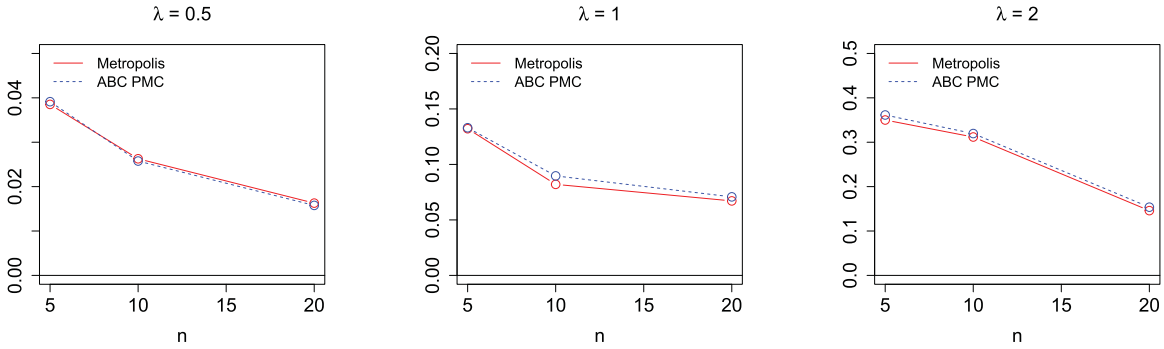

Figure 7. RMSE for the Weibull scale parameter λ based on the Metropolis algorithm and the ABC PMC algorithm by fixing $k = 1$.

coverage probabilities of the lifetime quantiles are shown in Figure 10. As expected, both the parameters and the lifetime quantiles can be well estimated, and the performance improves with the sample size n.

5. Model selection

As argued, it is important to select an appropriate model for the aggregate data. Suppose there are Q candidate models $\mathcal{M}_1, \ldots, \mathcal{M}_Q$ and consider the model index $\mathcal{M}$, which has the prior distribution $\pi(\mathcal{M})$. In the Bayesian framework, model selection is often made based on the the posterior probability $P(\mathcal{M}|\mathbf{D})$, and the model with the highest posterior probability is selected (e.g., Friel and Pettitt, (2008) and Barnes *et al.* (2012)).

In the literature, the ABC algorithm brings a straightforward solution to approximate $P(\mathcal{M}|\mathbf{D})$ (e.g., Toni *et al.* (2009) and Barnes *et al.* (2012)). Briefly, a model is generated from $\pi(\mathcal{M})$ and the model is stored only if the difference between the simulated data under the model and the observed data is less than a prefixed threshold. The posterior probability $P(\mathcal{M}|\mathbf{D})$ is then approximated by the acceptance frequency of $\mathcal{M}$. This procedure enforces the use of the ABC algorithm in posterior sampling for each model. In terms of our aggregate data problem, however, such a procedure is computationally expensive when the gamma/IG distribution is one of the candidate models, as posterior sampling is generally easy for these two distributions (see Section 3). Therefore, we deal with the candidate models separately in this study.

Note that $P(\mathcal{M}|\mathbf{D}) \propto P(\mathbf{D}|\mathcal{M})\pi(\mathcal{M})$. Since no prior information is available for the preference of the models, assume that $\pi(\mathcal{M})$ follows a uniform distribution. As a result, comparing $P(\mathcal{M}|\mathbf{D})$ is the same as comparing $P(\mathbf{D}|\mathcal{M})$, and the model with the largest value of $P(\mathbf{D}|\mathcal{M})$ should be selected. In the literature, $P(\mathbf{D}|\mathcal{M})$ is known as the Bayesian evidence or the marginal likelihood under model $\mathcal{M}$, and it has also been widely used for Bayesian model selection (e.g., Everitt *et al.* (2017) and Ong *et al.*

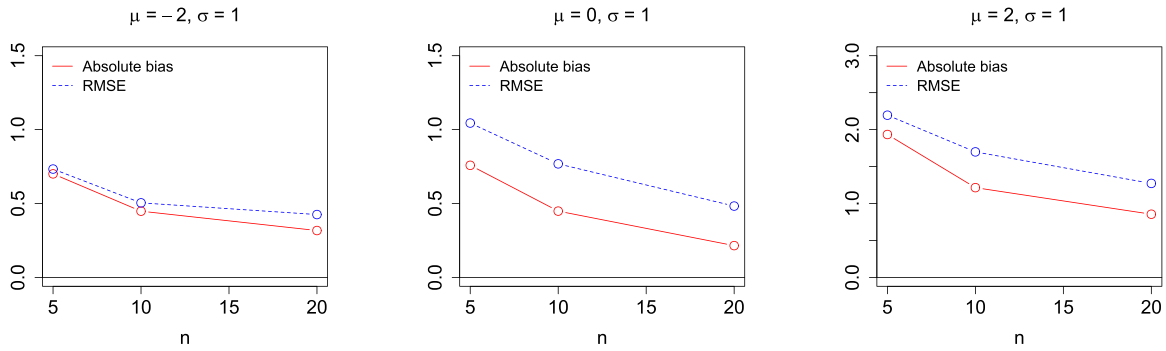

Figure 8. Absolute bias and RMSE for the log-normal location parameter μ based on 2000 replications.

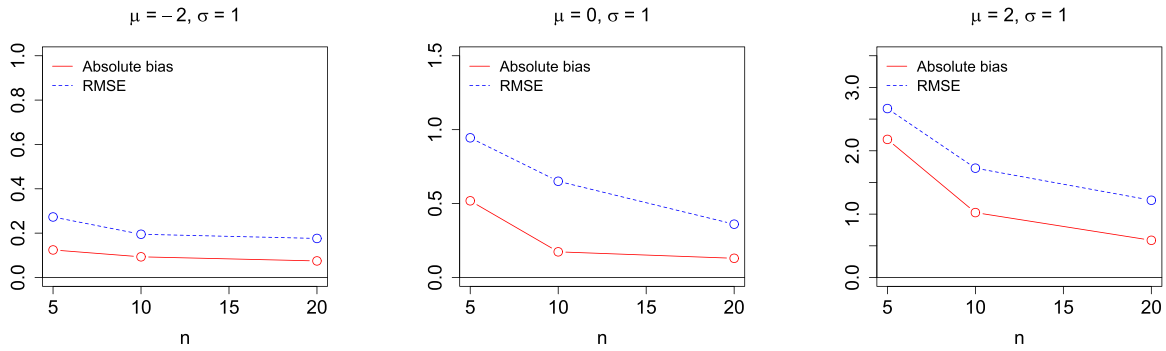

Figure 9. Absolute bias and RMSE for the log-normal scale parameter σ based on 2000 replications.

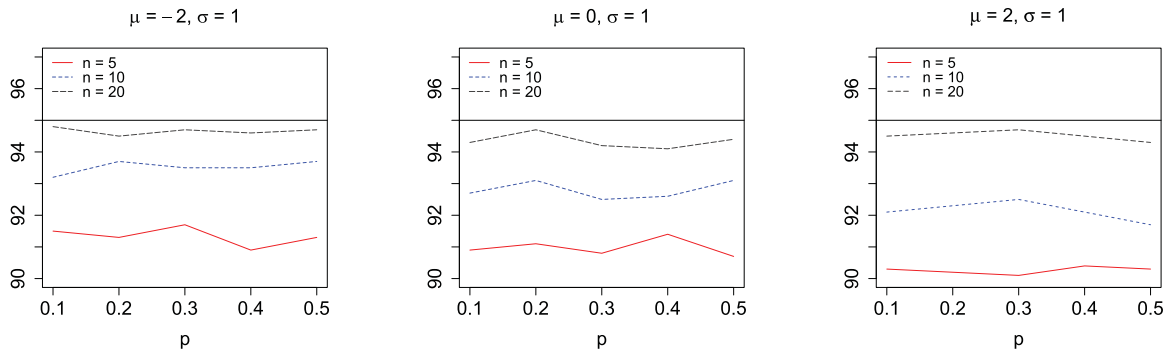

Figure 10. Coverage probabilities for the lifetime quantiles based on 2000 replications by assuming the log-normal distribution.

(2018)). In the following, we compute the Bayesian evidence $P(\mathbf{D}|\mathcal{M})$ for each model separately.

5.1. Bayesian evidence under gamma and IG models

As stated in Section 3, the likelihood function based on the aggregate data is easy to evaluate under the gamma and IG models. See Equations (5) and (7). In such cases, the Bayesian evidence can be easily computed. Note that $P(\mathbf{D}|\mathcal{M})$ can be expressed as

$$P(\mathbf{D}|\mathcal{M}) = \int P(\mathbf{D}|\boldsymbol{\theta}, \mathcal{M})\pi(\boldsymbol{\theta}|\mathcal{M})d\boldsymbol{\theta}. \tag{10}$$

Since we focus on one model at a time, we may drop the notational dependence on $\mathcal{M}$. Then, Equation (10) becomes:

$$P(\mathbf{D}) = \int P(\mathbf{D}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta} = \int L(\boldsymbol{\theta}|\mathbf{D})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{11}$$

where $L(\boldsymbol{\theta}|\mathbf{D})$ is the likelihood function. The simple Monte Carlo integration estimator of $P(\mathbf{D})$ is to sample N realizations from $\pi(\theta)$ and average over $P(\mathbf{D}|\boldsymbol{\theta})$. Since the prior may be far away from the posterior, $P(\mathbf{D}|\boldsymbol{\theta})$ can be very small, and hence, a huge sample size may be needed to achieve a good approximation (Kass and Raftery, 1995; Zeng *et al.*, 2016). In this study, we use a more stable estimator of $P(\mathbf{D})$ based on a normal approximation to the posterior distribution (Tierney and Kadane, 1986). Let $\tilde{l}(\boldsymbol{\theta}) = \log[P(\mathbf{D}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})]$. According to Kass and Raftery (1995), a Taylor expansion of $\tilde{l}(\boldsymbol{\theta})$ about its mode $\tilde{\boldsymbol{\theta}}$ yields a normal distribution approximation to the posterior distribution with mean $\tilde{\boldsymbol{\theta}}$ and covariance matrix $\tilde{\boldsymbol{\Sigma}} = [-\partial^2\tilde{l}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}]^{-1}$. Integrating this approximation out in Equation (11) yields:

$$\hat{P}(\mathbf{D}) = (2\pi)^{d/2}|\tilde{\boldsymbol{\Sigma}}|^{1/2}P(\mathbf{D}|\tilde{\boldsymbol{\theta}})\pi(\tilde{\boldsymbol{\theta}}), \tag{12}$$

where d is the dimension of $\boldsymbol{\theta}$. In practice, the posterior mode $\tilde{\boldsymbol{\theta}}$ and the covariance matrix $\tilde{\boldsymbol{\Sigma}}$ can be estimated based on the sample $\{\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_N\}$ from the posterior distribution: $\tilde{\boldsymbol{\theta}}$ can be estimated as the $\boldsymbol{\theta}$ that maximizes $P(\mathbf{D}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})$ and $\tilde{\boldsymbol{\Sigma}}$ can be estimated as the sample covariance Cov $(\boldsymbol{\theta}_{1:N})$ (Kass

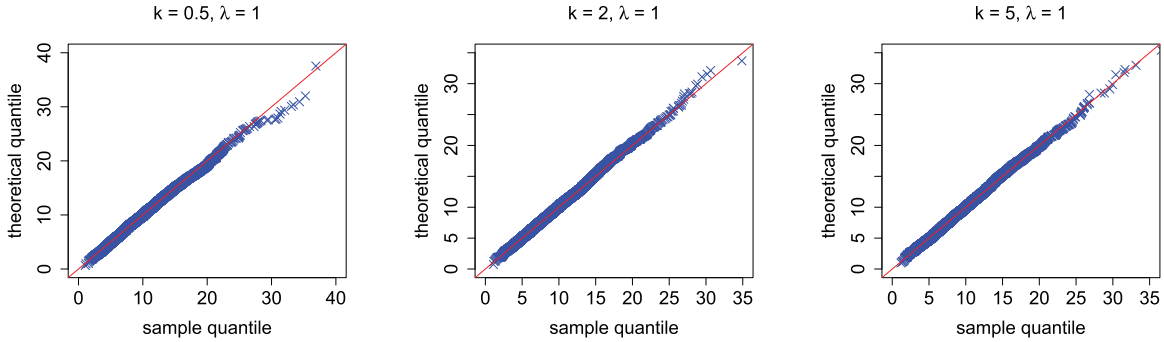

Figure 11. Q-Q plots by assuming the Weibull distribution.

and Raftery, 1995). The approximation in Equation (12) is known as the Laplace approximation of integrals, and a multitude of studies have revealed its good performance under moderate sample size (e.g., Wood, (2011) and Drton *et al.* (2017)).

5.2. Bayesian evidence under other lifetime models

The difficulty in computing Bayesian evidence for other lifetime models comes from their intractable likelihood. In this section, the synthetic likelihood approach proposed by Wood (2010) is used to approximate $P(\mathbf{D}|\boldsymbol{\theta})$. The basic idea of the synthetic likelihood approach is to first transform the original data to some summary statistics and then assume a normal distribution for the summary statistics. The original likelihood is then approximated by the likelihood for the summary statistics. Once $P(\mathbf{D}|\boldsymbol{\theta})$ is approximated, it can be substituted into Equation (12) to obtain an estimator of the Bayesian evidence $P(\mathbf{D})$.

In the proposed ABC framework for the aggregate data, the number of replacements $\mathbf{M} = \{M_1, ..., M_n\}$ are treated as the summary statistics. Following the procedure of the synthetic likelihood approach, $\mathbf{M}$ is approximated by a normal distribution with mean $\boldsymbol{\mu}_{\boldsymbol{\theta}}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$. Then, $P(\mathbf{D}|\boldsymbol{\theta})$ is estimated as

$$\hat{P}(\mathbf{D}|\boldsymbol{\theta}) \approx P(\mathbf{m}|\boldsymbol{\theta}) = \phi\left(\mathbf{m}|\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right), \tag{13}$$

where $\phi(\cdot|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the PDF of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{m} = \{m_1, ..., m_n\}$ are the observed number of replacements. In practice, $\boldsymbol{\mu}_{\boldsymbol{\theta}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ can be estimated by simulation. Based on the model parameters $\boldsymbol{\theta}$, K pseudo sets $\mathbf{m}_1, ..., \mathbf{m}_K$ are first generated, where $\mathbf{m}_k = \{m_{k,1}, ..., m_{k,n}\}, k = 1, ..., K$. Let $\mathbf{m}^{(i)} = \{m_{1,i}, ..., m_{K,i}\}, i = 1, ..., n$. The estimators of $\boldsymbol{\mu}_{\boldsymbol{\theta}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ are then given by

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}} = \frac{1}{K}\sum_k \mathbf{m}_k \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}} = \mathrm{Cov}\left(\mathbf{m}^{(1)}, ..., \mathbf{m}^{(n)}\right). \tag{14}$$

By substituting Equation (13) into Equation (12), the Bayesian evidence for models with intractable likelihoods can be readily computed.

The accuracy of the synthetic likelihood approach depends on two factors. The first factor is about the accuracy of the normal approximation to the summary statistics $\mathbf{M}$. Theoretically, we could expand the joint density of $\mathbf{M}$ as a quadratic about its mode, which yields a normal approximation to $\mathbf{M}$ (Wood, 2010). However, a good approximation requires the closeness between the observed $\mathbf{m}$ and the mode, and hence, a normality check is necessary in practice (Wood, 2010). On the other hand, the second factor is about how accurate $P(\mathbf{D}|\boldsymbol{\theta})$ can be approximated by $P(\mathbf{m}|\boldsymbol{\theta})$. A body of literature has shown the good performance of the approximation by using the summary statistics (e.g., Everitt *et al.* (2017) and Ong *et al.* (2018)), however, numerical evaluation in terms of the aggregate data is still needed. Based on these considerations, simulation studies are conducted in the next subsection.

5.3. Simulation study

Our first simulation is to assess the normality assumption on $\mathbf{M}$. Note that if $\mathbf{M} \sim N(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})$, then $(\mathbf{M}-\boldsymbol{\mu}_{\boldsymbol{\theta}})'\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{M}-\boldsymbol{\mu}_{\boldsymbol{\theta}}) \sim \chi_n^2$, where χ_n^2 denotes a chi-squared distribution with degrees of freedom n. Therefore, by generating K pseudo sets $\mathbf{m}_1, ..., \mathbf{m}_K$ based on the model parameters $\boldsymbol{\theta}$, we can plot the K ordered values of $(\mathbf{m}_k-\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}})'\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}^{-1}(\mathbf{m}_k-\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}})$ against the quantiles of a χ_n^2 distribution to check the normality of $\mathbf{M}$. The expressions of $\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$ are given in Equation (14). We consider generating aggregate data from the Weibull distribution and the log-normal distribution. We fix $n = 10$ and the inspection time $t_i = 40$, $i = 1, ..., n$. We consider $\boldsymbol{\theta} = (0.5, 1), (2, 1), (5, 1)$ for the Weibull distribution and $\boldsymbol{\theta} = (-2, 1), (0, 1), (2, 1)$ for the log-normal distribution. In each scenario, $K = 5000$ is considered. The quantile-quantile (Q-Q) plots for the Weibull model and the log-normal model are shown in Figure 11 and 12, respectively. As seen, the points are close to the red reference line $y = x$, indicating that the normality assumption of $\mathbf{M}$ is reasonable.

Our second simulation is to assess the accuracy of the synthetic likelihood $P(\mathbf{m}|\boldsymbol{\theta})$ in approximating the likelihood $P(\mathbf{D}|\boldsymbol{\theta})$. Since the likelihood is substituted into Equation (12) to obtain the Bayesian evidence $P(\mathbf{D})$, we directly compare Bayesian evidence in this simulation. As the exact likelihood based on the Weibull/log-normal model is intractable, we consider a gamma distribution with $\boldsymbol{\theta} = (0.5, 1), (2, 1), (5, 1)$. Under each parameter setting, 100 aggregate datasets are generated. Based on each dataset, 10 000 posterior samples of the gamma parameters are

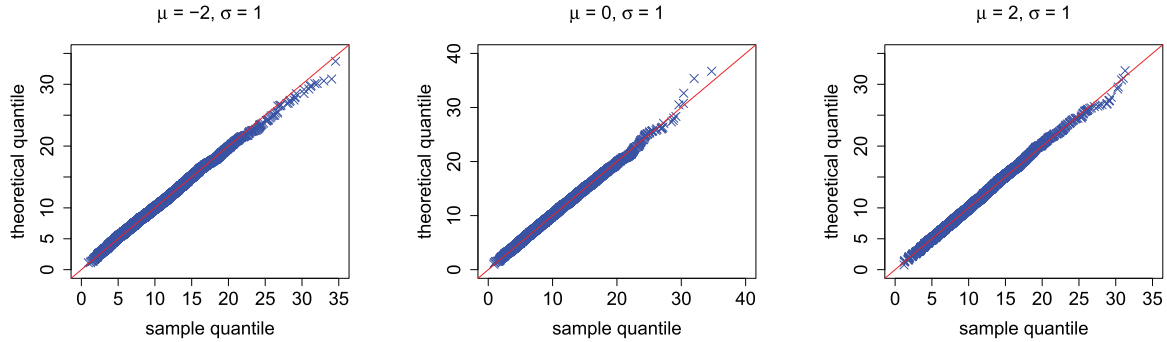

Figure 12. Q-Q plots by assuming the log-normal distribution.

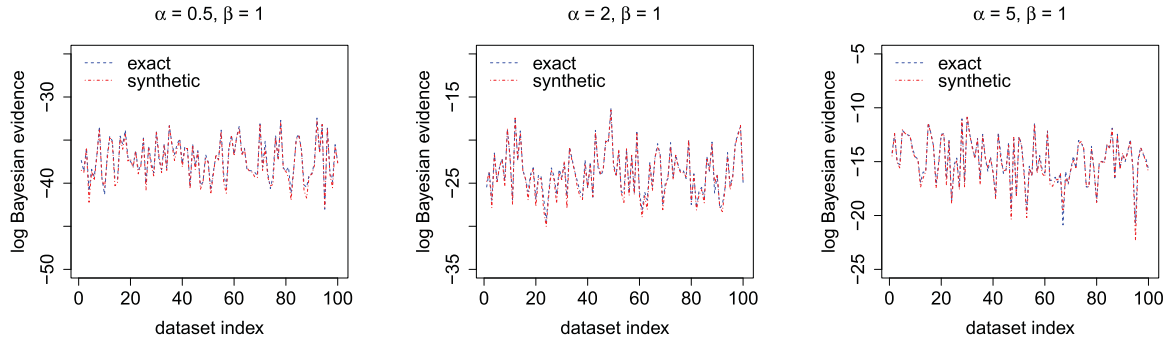

Figure 13. Logarithm of Bayesian evidence based on the exact likelihood and the synthetic likelihood by assuming the gamma distribution.

Table 1. Aggregate data of electromagnetic relay.

m_i	3	1	3	1	1	9	2	1	1	5	1	1
t_i (10^3 hours)	25.5	25.5	25.5	25.5	25.5	25.5	25.5	25.5	25.5	25.5	25.5	25.5

Table 2. Logarithm of the Bayesian evidence of each model.

Log-normal	*Weibull*	*Gamma*	*IG*
−32.64	−34.40	−27.88	−33.90

generated following the procedure in Section 3, and the Bayesian evidence is computed by using the exact likelihood and the synthetic likelihood. Figure 13 shows the logarithm of Bayesian evidences under different settings. As seen, the synthetic likelihood generally approximates the exact likelihood well in computing the Bayesian evidence.

6. Illustrative example

In this section, a real aggregate dataset from the reliability database of Mahar *et al.* (2011) is used for illustration. This reliability database consists of reliability data of various components from fielded systems, and it is maintained by the Reliability Information Analysis Center (RIAC), which is a center of the U.S. department of Defence's Center for reliability/maintainability and quality. Due to a variety of reasons, aggregate data are common for most components in the database.

Table 1 shows the recorded data for the electromagnetic relay component in 12 different systems (Army aircraft). As seen, a component position for the electromagnetic relay exists in each of the 12 aircraft. For each component position, the electromagnetic relay was replaced once it failed. However, individual failure times of the electromagnetic relays are missing. In the ith aircraft, the lifetime data are aggregated and recorded as the total number of component replacement m_i during a scheduled inspection duration t_i. As the operation environments of the 12 aircraft are almost identical, we assume that the lifetimes of the electromagnetic relay in different systems are iid.

Four popular lifetime distributions, i.e., the log-normal distribution, the Weibull distribution, the gamma distribution and the IG distribution, are used to fit this aggregate dataset. We first consider model selection based on the Bayesian evidence. $N = 5000$ posterior samples for each model are first generated based on the methods proposed in Section 3 and Section 4. The prior distribution for the log-normal location parameter μ is set as $N(0, 100^2)$ and the prior distribution for other parameters is set as $\text{InvGam}(0.1, 0.1)$. In addition, $\epsilon_1 = 10$ and $\epsilon = 1$ in Algorithm 1. Based on the posterior samples, the Bayesian evidence for the gamma distribution and the IG distribution are directly obtained by Equation (12). As for the Weibull distribution and the log-normal distribution, their Bayesian evidences are obtained by substituting the synthetic likelihood (13) into Equation (12). In computing the synthetic likelihood, we set $K = 2000$ in Equation (14). Table 2 shows the Bayesian evidence of each model. As seen, the gamma distribution has the largest model evidence, and hence, it is selected for this aggregate dataset.

Estimation of the gamma model is simultaneously done in the above model-selection procedure. The point estimates of α and β are 1.566 and 0.166, respectively. The 95% credible intervals for α and β are $(0.335, 11.93)$ and $(0.026, 1.356)$, respectively. In addition to the parameters, estimation of the lifetime quantiles and CDF are also

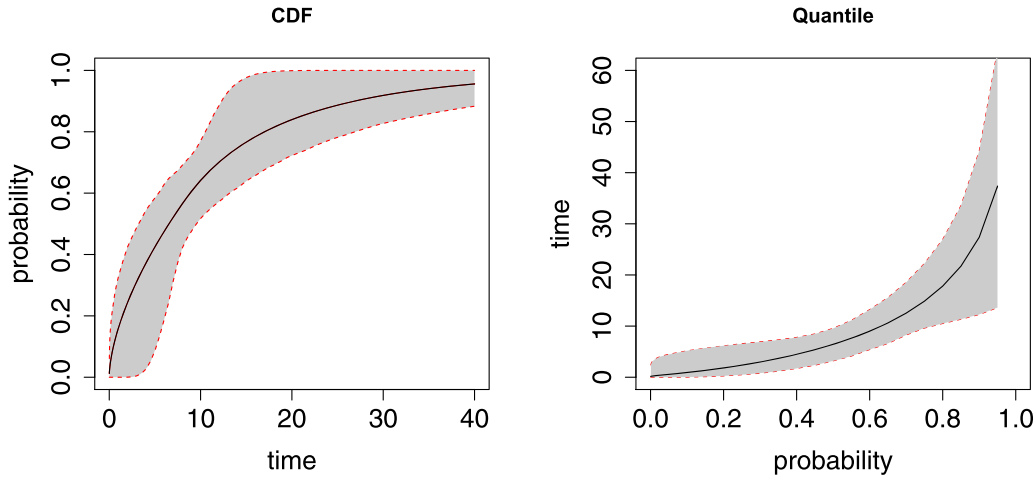

Figure 14. The 95% point-wise credible bands of the lifetime quantiles (left panel) and the lifetime CDFs (right panel) for the aggregate data based on the gamma distribution.

considered. These two reliability characteristics are often of more interest to practitioners. Figure 14 shows the point estimates and 95% credible intervals of the lifetime quantiles and CDFs by assuming the gamma distribution.

7. Conclusion

Time-censored aggregation is widely used when recording field failure data in major reliability databases. This study is the first attempt to develop statistical inference procedures for the time-censored aggregate data. Under the gamma/IG distribution, the likelihood function is a function of gamma/IG CDFs, and it can be easily evaluated by conventional Bayesian methods. On the other hand, it is numerically difficult to evaluate the likelihood function under other lifetime distributions such as the Weibull distribution and the log-normal distribution. An ABC algorithm equipped with the PMC sampling scheme was proposed for these distributions. Since there are several candidate lifetime models, we further proposed a model selection procedure by evaluating the Bayesian evidence for each individual model. When the likelihood function is easy to evaluate, the Laplace method was used to approximate the Bayesian evidence. When the likelihood function is intractable, the synthetic likelihood was first used to approximate the likelihood. Finally, the proposed methods were successfully applied to a practical aggregate dataset.

Based on the simulation results of the proposed ABC procedures, we observe that there is still room for improvement in the estimation accuracy, especially when the number of systems is small (e.g., $n \leq 10$). In practice, such a dataset is not uncommon in the reliability databases (e.g., Coit and Jin (2000)), and hence, it is of interest to develop efficient inference procedures for the aggregate data with a small size. In addition, nonparametric inference for the aggregate data could be another topic for future research. This should not be easy as the likelihood function (3) is intractable under most nonparametric models and further substantial efforts are needed to adapt the ABC algorithm to a nonparametric setting.

Acknowledgments

We are grateful to the editor, the associate editor and two referees for their insightful comments that have lead to a substantial improvement to an earlier version of the paper.

Funding

This work was supported by National Natural Science Foundation of China under grants 71601138, 71771221, and 71871191.

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