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Uncertainty quantification for monotone stochastic degradation models

Piao Chen[a] and Zhi-Sheng Ye[b]

[a]Institute of High Performance Computing, Singapore; [b]Department of Industrial Systems Engineering and Management, National University of Singapore, Singapore

ABSTRACT
Degradation data are an important source of product reliability information. Two popular stochastic models for degradation data are the Gamma process and the inverse Gaussian (IG) process, both of which possess monotone degradation paths. Although these two models have been used in numerous applications, the existing interval estimation methods are either inaccurate given a moderate sample size of the degradation data or require a significant computation time when the size of the degradation data is large. To bridge this gap, this article develops a general framework of interval estimation for the Gamma and IG processes based on the method of generalized pivotal quantities. Extensive simulations are conducted to compare the proposed methods with existing methods under moderate and large sample sizes. Degradation data from capacitors are used to illustrate the proposed methods.

KEYWORDS
coverage probability; Gamma process; generalized pivotal quantity; inverse Gaussian process

1. Introduction

As products become highly reliable, it is challenging to obtain product lifetimes through traditional life tests within an affordable amount of time. Although accelerated life tests have been employed in the industry to overcome this challenge, the acceleration usually provides limited assistance because few failures are likely to occur during the tests. In such cases, degradation experiments are widely adopted. In a degradation experiment, degradation for each individual unit is regularly monitored and the degradation measurements are recorded. Generally, far fewer testing samples are required in a degradation test to achieve the same estimation accuracy as in the life test (Meeker and Escobar 1998). Application areas of degradation analysis include inertial navigation systems (Wang et al. 2014), nanoscale transistors (Ko et al. 2010), vacuum fluorescent displays (e.g., Bae and Kvam 2004), lithium batteries (e.g., Hu et al. 2015), rolling bearings (e.g., Liao and Tian 2013), hard disk drives (Ye et al. 2013), and gear boxes (e.g., Moghaddass and Zuo 2014; Wang et al. 2015), to name a few.

In view of the fact that degradation is usually continuous, continuous-state stochastic processes are popular models for degradation modeling. In the literature, three classes of degradation processes (i.e., the Wiener process, the Gamma process, and the inverse Gaussian (IG) process) have been well exploited. Various variants—such as time-scale transformation, random effects, covariates, and measurement errors—of these three processes have been proposed. The Wiener process has long been a popular method of modeling the degradation data. Recent applications of the Wiener process in degradation modeling can be found in Wang (2010), Zhai and Ye (2017a), and Zhai and Ye (2017b). A distinct feature of the Wiener process is that its sample path is not necessarily monotone, which might not be meaningful in many degradation applications. When the degradation is monotone or when there is a requirement on the monotonicity (e.g., wear), the Gamma process and the IG process are more appropriate. These two processes are found to be limits of compound Poisson processes (Lawless and Crowder 2004; Ye and Chen 2014), which make them appropriate for degradation modeling. This is because degradation phenomena such as corrosion and wear are often caused by a series of random external shocks with tiny magnitudes, and the shock arrival process may be approximated by a Poisson process. For applications of the Gamma process in degradation modeling, see Guida, Postiglione, and Pulcini (2012); Lawless and Crowder (2004); Tsai, Tseng, and Balakrishnan (2012); and Ye et al. (2014). For the IG process, see Peng et al. (2016); Peng et al. (2017); and Ye and Chen (2014), among others.

This article concerns statistical inference for the Gamma process and the IG process based on degradation

CONTACT Zhi-Sheng Ye yez@nus.edu.sg Department of Industrial Systems Engineering and Management, National University of Singapore, 1 Engineering Drive 2, Singapore 117576.
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data. Point estimation of parameters in these two models is generally easy. Maximum likelihood (ML) estimation is usually efficient, and the ML estimators can be obtained by directly maximizing the log-likelihood function (e.g., Lawless and Crowder 2004; Padgett and Tomlinson 2004) or through the EM algorithm (Tsai, Tseng, and Balakrishnan 2012; Wang and Xu 2010). The ML estimators of the parameters can then be used to obtain the ML estimators of reliability characteristics such as the lifetime cumulative distribution function (CDF) and quantiles. To quantify the uncertainties in the estimators interval estimation of the parameters is usually of more interest in practice. In the literature, the information matrix or the bootstrap is often used to obtain confidence intervals for the Gamma process and the IG process. For the Gamma process, Bagdonavicius and Nikulin (2001) and Park and Padgett (2006) constructed confidence intervals using the asymptotic normality of the ML estimators, while Lawless and Crowder (2004) suggested the bootstrap because of the complicated likelihood function. However, the bootstrap is often used to construct confidence intervals of parameters in the IG degradation model (Wang and Xu 2010; Ye and Chen 2014). A comprehensive review of these degradation models can be found in Ye and Xie (2015).

Theoretical support for most existing methods is based on an asymptotic argument. Therefore, a large sample size is usually required to achieve an accurate estimation when applying these methods. However, when the sample size is large, resampling methods such as the bootstrap is very time consuming. Yet our simulation experience suggests that the performance of the existing methods is not satisfactory when the sizes of the degradation data are small or moderate. In order to bridge this gap, this study seeks to propose accurate and efficient interval estimation methods for the Gamma and IG processes under both small and large sample sizes. The underlying idea is based on the method of generalized pivotal quantities (GPQs), which was first proposed by Weerahandi (1993) and further explored by Hannig, Iyer, and Patterson (2006). The GPQ method is a powerful tool for interval estimation in the presence of nuisance parameters. A confidence interval for the parameter of interest can be constructed based on the distribution of its GPQ. Under some mild assumptions, Hannig, Iyer, and Patterson (2006) showed that correct frequentist coverage is guaranteed by the GPQ method. Successful applications of the GPQ method in interval estimation can be found in Chen and Ye (2017); Cisewski and Hannig (2012); and Hannig and Iyer (2008), to name a few.

An appealing property of the GPQ method is that once GPQs for the model parameters are available, GPQs for a function of the parameters can be naturally obtained by simple plug-in (Hannig, Iyer, and Patterson 2006). In this article, we first show how to obtain GPQs for the parameters in the Gamma and IG degradation models. When exact GPQs for some parameters are not available, approximate GPQs are proposed. The (approximate) GPQs for model parameters can then be used to obtain approximate GPQs for lifetime CDF and quantiles. Performance of the GPQ-based confidence intervals is assessed both theoretically and by simulation.

The remainder of the article is organized as follows. Section 2 presents the format of degradation data and introduces some general notations. Sections 3 and 4 develop the GPQ-based interval estimation procedures for parameters as well as some reliability characteristics in the Gamma and IG processes, respectively. In the interest of clarity, most proofs of the main results are given in the Appendix. Good performance of the generalized confidence intervals is demonstrated through simulation in Section 5. Section 6 illustrates the proposed procedures by using degradation data from capacitors. A concluding remark is provided in Section 7.

2. Problem statement

Consider a stochastic degradation process $\{Y(t); t > 0\}$. A product is defined to have failed when the degradation process first reaches a degradation failure threshold D. The resulting failure time is denoted by T. Suppose the degradation processes of n independent and identical units are observed. Let $\mathbf{t}_i = \{t_{i,j};\ j = 0, 1, \dots, m_i\}$ be the ordered observation times for the ith unit and $\mathbf{Y}_i = \{Y_i(t_{i,j});\ j = 0, 1, \dots, m_i\}$ be the corresponding degradation levels, where m_i is the number of measurements for the ith unit, $t_{i,0} = 0$, and $Y_i(t_{i,0}) = 0$ for convenience. Let $\Delta Y_{i,j} = Y_i(t_{i,j}) - Y_i(t_{i,j-1})$, $j = 1, 2, \dots, m_i$. The sample data collected from the n units are denoted as $\mathbf{D} = \{(\mathbf{t}_i, \mathbf{Y}_i);\ i = 1, 2, \dots, n\}$.

Let τ be the largest real number such that $\Delta t_{i,j} \equiv t_{i,j} - t_{i,j-1} = k_{i,j}\tau$, where all $k_{i,j}, i = 1, 2, \dots, n;\ j = 1, 2, \dots, m_i$ are positive integers and they are, together as a set, coprime. That is, the greatest common divisor of all these integers is one. Define $Z_{i,j} = \Delta Y_{i,j}/k_{i,j}$, $K = \sum_{i=1}^{n}\sum_{j=1}^{m_i} k_{i,j}$, and $M = \sum_{i=1}^{n} m_i$. The respective arithmetic mean and geometric mean of $Z_{i,j}$ weighted by $k_{i,j}$ are defined as

$$\bar{Z} = \frac{1}{K}\sum_{i=1}^{n}\sum_{j=1}^{m_i} \Delta Y_{i,j} = \frac{1}{K}\sum_{i=1}^{n} Y_i(t_{i,m_i}),$$

$$\tilde{Z} = \left(\prod_{i=1}^{n}\prod_{j=1}^{m_i} Z_{i,j}^{k_{i,j}}\right)^{1/K}.$$

3. Gamma process

In this section, we first consider the Gamma process. Suppose $\{Y(t); t > 0\}$ follows a stationary Gamma process with shape function υt and rate parameter γ (i.e., $Y(t) \sim \mathrm{gam}(\upsilon t, \gamma)$) with probability density function (PDF)

$$f_{Y(t)}(y) = \frac{\gamma(\gamma y)^{\upsilon t-1}}{\Gamma(\upsilon t)} \exp(-\gamma y), \quad y > 0. \qquad [1]$$

Because $Y(t)$ is monotone increasing, the failure time distribution can be computed as $F_T(t) = P(Y(t) > D)$, which is equal to $1 - F_{\mathrm{gam}}(D; \upsilon t, \gamma)$. However, there is no closed form for the lifetime quantile t_p. Because $F_{\mathrm{gam}}(D; \upsilon t, \gamma)$ is decreasing in t, the quantile can be easily obtained by finding the root of $F_{\mathrm{gam}}(D; \upsilon t, \gamma) = 1 - p$ through the bisection method. In the next few subsections, interval estimation procedures for the shape parameter υ, rate parameter γ, and the lifetime CDF/quantile are developed.

3.1. Uncertainty quantification for the shape parameter υ

According to the definition of $Z_{i,j}$, we readily have $Z_{i,j} \sim \mathrm{gam}(\upsilon k_{i,j}\tau, k_{i,j}\gamma)$. Based on the arithmetic mean $\bar{Z}$, and the geometric mean $\tilde{Z}$ of $Z_{i,j}$ defined in Section 2, the ratio $\tilde{Z}/\bar{Z}$ is independent of the rate parameter γ. This fact can be seen by writing $\Delta Y_{i,j} = \gamma(\Delta Y_{i,j}/\gamma)$ and $Z_{i,j} = \gamma(Z_{i,j}/\gamma)$, and observe that $\tilde{Z}/\bar{Z}$ is a function of $(\Delta Y_{i,j}/\gamma)$ and $(Z_{i,j}/\gamma)$, which are two Gamma random variables free of γ. Define $W_2 = \ln(\bar{Z}/\tilde{Z})$ and $W_3 = 2K\upsilon\tau W_2$. Clearly, W_2 and W_3 depend on the parameter υ only. If the distribution of W_2 or W_3 is known (or approximately known), then the confidence interval for υ can be constructed. In this article, two interval estimation methods for υ are proposed based on approximating the distributions of W_2 and W_3.

The first method is based on the Cornish-Fisher expansion to W_2. The Cornish-Fisher expansion is a useful tool to approximate the quantiles of a distribution by using its cumulants (Fisher and Cornish 1960). For a random variable X with mean μ and variance σ^2, its pth quantile X_p can be approximated by $\mu + \sigma Z$, where Z is a function of the cumulants of X. For example, if the first five cumulants of X are used for the approximation, then

$$\begin{aligned} Z = z_p &+ \frac{1}{6}\tilde{c}_3\left(z_p^2 - 1\right) + \frac{1}{24}\tilde{c}_4\left(z_p^3 - 3z_p\right) \\ &- \frac{1}{36}\left(\tilde{c}_3\right)^2\left(2z_p^3 - 5z_p\right) \\ &+ \frac{1}{120}\tilde{c}_5\left(z_p^4 - 6z_p^2 + 3\right) - \frac{1}{24}\tilde{c}_3\tilde{c}_4\left(z_p^4 - 5z_p^2 + 2\right) \\ &+ \frac{1}{324}\left(\tilde{c}_3\right)^3\left(12z_p^4 - 53z_p^2 + 17\right), \end{aligned}$$

where c_i is the ith cumulant of X, $\tilde{c}_i = c_i/(c_2)^{i/2}$, and z_p is the pth quantile of a standard normal distribution. To apply the Cornish-Fisher expansion to W_2, we first show that the moment-generating function of W_2 can be expressed as

$$M_{W_2}(u) = \frac{\Gamma(\upsilon\tau K)\prod_{i,j} k_{i,j}^{k_{i,j}u/K}\Gamma(\upsilon\tau k_{i,j} - k_{i,j}u/K)}{K^u\Gamma(\upsilon\tau K - u)\prod_{i,j}\Gamma(\upsilon\tau k_{i,j})}, \quad u<\upsilon\tau K. \qquad [2]$$

The detailed procedure can be found in the Appendix. The cumulants of W_2 can be easily obtained based on Eq. [2]. Then, the pth quantile of W_2 can be approximated by $c_1(\upsilon) + [c_2(\upsilon)]^{1/2}Z(\upsilon, p)$. We express $c_i(\upsilon)$ and $Z(\upsilon, p)$ in this way because W_2 depends on υ only. However, Iliopoulos (2016) shows that W_2 is a monotone decreasing function of υ based on the observed data. Therefore, an equal-tail $100(1-\alpha)$ percent confidence interval $[\upsilon_L, \upsilon_U]$ for υ can be obtained by solving

$$\begin{aligned} \ln(\bar{Z}/\tilde{Z}) &= c_1(\upsilon_L) + [c_2(\upsilon_L)]^{1/2}Z(\upsilon_L, 1-\alpha/2), \\ \ln(\bar{Z}/\tilde{Z}) &= c_1(\upsilon_U) + [c_2(\upsilon_U)]^{1/2}Z(\upsilon_U, \alpha/2). \end{aligned}$$

The bisection method can be readily used to solve the above two equations.

The second interval estimation method for υ is based on an approximate distribution of W_3. Here, we use W_3 instead of W_2 because W_3 is an explicit function of υ. We first derive the moment-generating function of W_3 as

$$M_{W_3}(u) = \frac{\Gamma(\upsilon\tau K)\prod_{i,j} k_{i,j}^{2\upsilon\tau k_{i,j}u}\Gamma(\upsilon\tau k_{i,j}(1-2u))}{K^{2\upsilon\tau K u}\Gamma(\upsilon\tau K(1-2u))\prod_{i,j}\Gamma(\upsilon\tau k_{i,j})}, \quad u<1/2. \qquad [3]$$

When $\upsilon\tau$ is large, an asymptotic expansion to the Gamma functions involved in Eq. [3] shows that $W_3 \to_d \chi^2(M-1)$ as $\upsilon\tau \to \infty$, which is also a $\mathrm{gam}((M-1)/2, 1/2)$ distribution. Similarly, we can show that $W_3 \to_d \chi^2(2M-2)$ as $\upsilon\tau \to 0$, which is also a $\mathrm{gam}(M-1, 1/2)$ distribution. Based on these asymptotic results, we propose approximating W_3 by a Gamma random variable $\mathrm{gam}(a, b)$ when $\upsilon\tau$ is moderate. The values of a and b can be determined through moment-matching by matching the means and variances of W_3 and the Gamma distribution $\mathrm{gam}(a, b)$. The mean and variance of W_3 can be obtained through Eq. [3] as

$$\begin{aligned} \mathrm{E}[W_3] &= 2\upsilon\tau\Bigg\{K\psi(\upsilon\tau K) - K\ln K \\ &\qquad + \sum_{i,j} k_{i,j}\left[\ln k_{i,j} - \psi(\upsilon\tau k_{i,j})\right]\Bigg\}, \\ \mathrm{Var}[W_3] &= 4\upsilon^2\tau^2\Bigg[-K^2\psi_1(\upsilon\tau K) \\ &\qquad + \sum_{i,j} k_{i,j}^2\psi_1(\upsilon\tau k_{i,j})\Bigg], \end{aligned}$$

where $\psi(\cdot)$ and $\psi_1(\cdot)$ are the digamma function and the trigamma function, respectively. Therefore, the moment-matching yields

$$a = (\mathrm{E}[W_3])^2/\mathrm{Var}[W_3], \quad b = \mathrm{E}[W_3]/\mathrm{Var}[W_3].$$

After an estimate of υ (e.g., the ML estimate) is obtained, we can substitute υ with the estimate to obtain the values of a and b. Based on these values, an approximate distribution for W_3 is available, and then a confidence interval for υ can be constructed.

3.2. Uncertainty quantification for the rate parameter γ

The approximate distributions of W_2 and W_3 enable us to construct confidence intervals for the rate parameter γ by capitalizing on the idea of GPQ. Based on the Cornish-Fisher expansion to W_2, a GPQ $\mathcal{G}_\upsilon$ for υ can be the solution to $\ln(\bar{Z}/\tilde{Z}) = c_1(\mathcal{G}_\upsilon) + [c_2(\mathcal{G}_\upsilon)]^{1/2} Z[\mathcal{G}_\upsilon, F_{W_2}(W_2^*)]$, where W_2^* is a random copy of W_2. Because $F_{W_2}(W_2^*)$ follows the standard uniform distribution, $\mathcal{G}_\upsilon$ is actually the solution to $\ln(\bar{Z}/\tilde{Z}) = c_1(\mathcal{G}_\upsilon) + [c_2(\mathcal{G}_\upsilon)]^{1/2} Z(\mathcal{G}_\upsilon, U^*)$, where U^* follows the standard uniform distribution. However, another GPQ for υ can be $\mathcal{G}_\upsilon = W_3^*/[2K\tau \ln(\bar{Z}/\tilde{Z})]$, where W_3^* is treated as a $\mathrm{gam}(a, b)$ random variable while others are constant. Then $\mathcal{G}_\upsilon \sim \mathrm{gam}(a, 2bK\tau \ln(\bar{Z}/\tilde{Z}))$ is a GPQ for υ if we ignore the estimation uncertainties in a and b.

Based on the GPQs for the shape parameter, we are able to construct GPQs for the rate parameter γ. Recall that $K\bar{Z} \sim \mathrm{gam}(\upsilon\tau K, \gamma)$ and so $2K\bar{Z}\gamma \sim \chi^2(2\upsilon\tau K)$. Substituting υ by its GPQ $\mathcal{G}_\upsilon$ in the above relation yields a GPQ for γ as $\mathcal{G}_\gamma = W_4^*/(2K\bar{Z})$, where $W_4^* \sim \chi^2(2\mathcal{G}_\upsilon \tau K)$. A $100(1-\alpha)$ percent confidence interval for γ can then be easily constructed by first obtaining a $100(1-\alpha)$ percent prediction interval for W_4^* and then dividing the two end points by $2K\bar{Z}$. Alternatively, the confidence interval can be obtained directly from the distribution of $\mathcal{G}_\gamma$.

3.3. Uncertainty quantification for the CDF and quantiles

Based on the GPQs for υ and γ, confidence intervals for lifetime characteristics—such as lifetime CDF and quantiles—can be constructed. The probability of failure at time t, denoted as p_t, is given by $p_t = 1 - F_{\mathrm{gam}}(D; \upsilon t, \gamma)$. A GPQ for p_t is obtained by substituting υ and γ for $\mathcal{G}_\upsilon$ and $\mathcal{G}_\gamma$, which yields $\mathcal{G}_{p_t} = 1 - F_{\mathrm{gam}}(D; \mathcal{G}_\upsilon t, \mathcal{G}_\gamma)$. The closed-form distribution of $\mathcal{G}_{p_t}$ is difficult to obtain because of the complicated form of the Gamma CDF $F_{\mathrm{gam}}(\cdot)$. Nevertheless, the confidence interval based on $\mathcal{G}_{p_t}$ can be obtained by generating B realizations of $(\mathcal{G}_\upsilon, \mathcal{G}_\gamma)$ to give B realizations of $\mathcal{G}_{p_t}$. When generating a realization from $(\mathcal{G}_\upsilon, \mathcal{G}_\gamma)$, it should be noted that W_4^* in $\mathcal{G}_\gamma$ is dependent on $\mathcal{G}_\upsilon$. Therefore, we can generate a realization of $\mathcal{G}_\upsilon$ by either of the proposed methods, conditional on which realization of $\mathcal{G}_\gamma$ is generated.

By the same token, a GPQ $\mathcal{G}_{t_p}$ for the p-quantile t_p can be defined by replacing the parameters υ and λ by their respective GPQs. A confidence interval of t_p is then constructed by generating and sorting B samples of $\mathcal{G}_{t_p}$. Because of the complicated form of $F_T(t)$, t_p does not have a closed-form expression for fixed υ and γ. Its value has to be computed numerically. By noting that the p-quantile t_p is the solution of $F_{\mathrm{gam}}(D; \upsilon t, \gamma) = 1 - p$ and that $F_{\mathrm{gam}}(D; \upsilon t, \gamma)$ is decreasing in t, t_p can be easily obtained through a bisection search.

To better demonstrate the proposed inference method, Algorithm 1 summarizes the interval estimation procedure for the lifetime CDF p_t based on the Gamma process. Confidence intervals for the lifetime quantiles can be constructed in a very similar vein.

Algorithm 1 Constructing confidence intervals of p_t for the Gamma process.

Step 1. Generate B realizations of $(\mathcal{G}_\upsilon, \mathcal{G}_\gamma)$, denoted as $(\upsilon^{(b)}, \gamma^{(b)})$, $b = 1, \ldots, B$.
Step 2. For each $(\upsilon^{(b)}, \gamma^{(b)})$, compute a value of p_t and denote it as $p_t^{(b)}$, $b = 1, \ldots, B$.
Step 3. Use the $\alpha/2$ and $1-\alpha/2$ percentiles of $\left\{p_t^{(1)}, \ldots, p_t^{(B)}\right\}$ as the $100(1-\alpha)$*percent* confidence interval of p_t.

4. IG process

In this section, the IG process is considered for degradation data. Suppose $\{Y(t); t > 0\}$ follows a stationary IG process with rate parameter μ and shape parameter λ (i.e., $Y(t) \sim \mathcal{IG}(\mu t, \lambda t^2)$). The respective PDFs and CDFs of the IG distribution $\mathcal{IG}(a, b)$, $a, b > 0$ are given by

$$f_{\mathcal{IG}}(x; a, b) = \left(\frac{b}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{b(x-a)^2}{2a^2 x}\right], \quad x > 0 \text{ and} \qquad [4]$$

$$F_{\mathcal{IG}}(x; a, b) = \Phi\left(\sqrt{b/x}(x/a - 1)\right) + \exp(2b/a)\Phi\left(-\sqrt{b/x}(x/a + 1)\right), \quad x > 0, \qquad [5]$$

where $\Phi(\cdot)$ is the standard normal CDF. Because of the monotonic sample path, the failure time distribution can be obtained as $F_T(t) = P(Y(t) > D) = 1 - F_{\mathcal{IG}}(D; \mu t, \lambda t^2)$. Similar to the Gamma process case, there is no closed form for the lifetime quantile t_p, and the quantile can be obtained by solving the equation $1 - p =$

$F_{\mathcal{IG}}(D;\mu t,\lambda t^2)$ through the bisection method. In the next two subsections, interval estimation methods for the parameters and lifetime CDF/quantile are proposed.

4.1. Uncertainty quantification for the parameters

After the degradation data **D** are made available, we can specify the log-likelihood function, which is given in Eq. [A2] for reference, and obtain the MLE of μ and λ as

$$\hat{\mu}=\frac{\bar{Z}}{\tau},\quad 1/\hat{\lambda}=\frac{\tau^2}{M}\sum_{i,j}\left(\frac{k_{i,j}}{z_{i,j}}-\frac{1}{\bar{Z}}\right). \qquad [6]$$

We immediately have $\hat{\mu}\sim\mathcal{IG}(\mu,K\tau\lambda)$ because it is a scaling of the IG variable $\bar{Z}$. In Appendix B, we further show that $\hat{\mu}$ is independent of $\hat{\lambda}$ and $M/\hat{\lambda}\sim\chi^2_{M-1}/\lambda$. Therefore, a $100(1-\alpha)$ percent confidence interval for λ can be obtained as

$$I_{1-\alpha}(\lambda)=[\hat{\lambda}\chi^2_{M-1,\alpha/2}/M,\,\hat{\lambda}\chi^2_{M-1,(1-\alpha/2)}/M],$$

where $\chi^2_{M-1,\alpha}$ is the α-quantile of the $\chi^2(M-1)$ distribution.

When $X_1\sim\mathcal{IG}(a,b)$, $X_2\sim\chi^2(n)$, and X_1,X_2 are independent, Theorem 3 in Chhikara and Floks (1975) states that $|\sqrt{nb}(X_1/a-1)/\sqrt{X_1X_2}|$ follows $|t_n|$, a folded student-t distribution with n degrees of freedom. Substitute $\hat{\mu}$ and $M\lambda/\hat{\lambda}$ into the above result to obtain

$$\frac{\sqrt{(M-1)K\tau}}{\sqrt{M\hat{\mu}/\hat{\lambda}}}\left|\hat{\mu}/\mu-1\right|\sim|t_{M-1}|. \qquad [7]$$

Based on Eq. [7], a $100(1-\alpha)$ percent confidence interval for μ can be constructed as:

$$I_{1-\alpha}(\mu)=\left[\hat{\mu}\left[1-\frac{\sqrt{M\hat{\mu}}t_{M-1,\alpha/2}}{\sqrt{(M-1)K\tau\hat{\lambda}}}\right]^{-1},\right.$$
$$\left.\hat{\mu}\left[1+\frac{\sqrt{M\hat{\mu}}t_{M-1,\alpha/2}}{\sqrt{(M-1)K\tau\hat{\lambda}}}\right]^{-1}\right],$$

where $t_{M-1,\alpha/2}$ is the $\alpha/2$ quantile of t_{M-1}.

4.2. Uncertainty quantification for the CDF and quantiles

There are no pivotal quantities for the lifetime CDF and quantiles. Nevertheless, their confidence intervals can be constructed based on the confidence intervals for μ and λ. The method is based on the idea of equivariant confidence sets (Lehmann and Romano 2006), which is useful in finding confidence interval for a function of parameters. Because $\hat{\mu}$ and $\hat{\lambda}$ are independent, $I_{1-\alpha}(\mu)\times I_{1-\alpha}(\lambda)$ is a $(1-\alpha)^2$ confidence set for (μ,λ). Define

$$\underline{p_t}=\min_{(\mu,\lambda)}[1-F_{\mathcal{IG}}(D;\mu t,\lambda t^2)]\text{ and}$$
$$\overline{p_t}=\max_{(\mu,\lambda)}[1-F_{\mathcal{IG}}(D;\mu t,\lambda t^2)],$$

where $(\mu,\lambda)\in I_{1-\alpha}(\mu)\times I_{1-\alpha}(\lambda)$. Then $P(\underline{p_t}\le p_t\le\overline{p_t})\ge(1-\alpha)^2$. Therefore, $[\underline{p_t},\overline{p_t}]$ is a $(1-\alpha)^2$ confidence interval for p_t. To simplify the computation of $\underline{p_t}$ and $\overline{p_t}$, note that the partial derivative of $F_{\mathcal{IG}}(D;\mu t,\lambda t^2)$ with respect to μ is given by

$$\frac{\partial F_{\mathcal{IG}}(D;\mu t,\lambda t^2)}{\partial\mu}=-\frac{2\lambda t}{\mu^2}\exp\left(\frac{2\lambda t}{\mu}\right)\times\Phi\left(-\sqrt{\frac{\lambda t^2}{D}}\left(\frac{D}{\mu t+1}\right)\right),$$

which is obviously less than zero. Therefore, $F_{\mathcal{IG}}(D;\mu t,\lambda t^2)$ is decreasing in μ for all λ. The monotonicity allows us to compute $\underline{p_t}$ and $\overline{p_t}$ as

$$\underline{p_t}=\min_{\underline{\lambda}\le\lambda\le\bar{\lambda}}[1-F_{\mathcal{IG}}(D;\underline{\mu}t,\lambda t^2)],$$
$$\overline{p_t}=\max_{\underline{\lambda}\le\lambda\le\bar{\lambda}}\left[1-F_{\mathcal{IG}}(D;\bar{\mu}t,\lambda t^2)\right].$$

A $(1-\alpha)^2$ confidence interval for the quantile t_p can be obtained in a similar vein: we obtain $\underline{t_p}$ and $\overline{t_p}$ by minimizing and maximizing t_p over $I_{1-\alpha}(\mu)\times I_{1-\alpha}(\lambda)$, respectively. According to the triple product rule for partial derivatives, we know that $\partial t_p/\partial\mu=-\partial t_p/\partial F_T\times\partial F_T/\partial\mu$. Both partial derivatives on the right-hand side are positive. Therefore, t_p is decreasing in μ, and the lower and upper limits can be computed as

$$\underline{t_p}=\min_{\underline{\lambda}\le\lambda\le\bar{\lambda}}t_p(\bar{\mu},\lambda),\quad \overline{t_p}=\max_{\underline{\lambda}\le\lambda\le\bar{\lambda}}t_p(\underline{\mu},\lambda).$$

As we will see in the next section, the method of equivariant confidence sets for the lifetime CDF and quantiles is quite conservative in terms of coverage probabilities and average length. To improve its performance, we may consider elliptical confidence regions for (μ,λ) to reduce the area of the rectangle used above (Hong et al. 2010). Nevertheless, construction of such an elliptical region, and finding $(\underline{t_p},\overline{t_p})$ in the region, may not be easy. Yet the GPQ method can again be used to construct the confidence intervals for the lifetime CDF and quantiles.

A GPQ for the shape parameter λ can be $\mathcal{G}_\lambda=\hat{\lambda}W_5^*/M$, where W_5^* is a $\chi^2(M-1)$ random variable. Based on Eq. [7], an approximate GPQ for μ is $\mathcal{G}_\mu=\hat{\mu}(\frac{\sqrt{M\hat{\mu}/\hat{\lambda}}}{\sqrt{(M-1)K\tau}}W_6^*+1)^{-1}$, where W_6^* is a student-t random variable with $M-1$ degrees of freedom and W_6^* is independent of W_5^*. To avoid negative values, two modified GPQs for μ can be $|\mathcal{G}_\mu|$ and $\max(0,\mathcal{G}_\mu)$. These two modifications are asymptotically equivalent to $\mathcal{G}_\mu$ when $K \to \infty$. The $\mathcal{G}_\mu$ defined above is similar to the approximate GPQ proposed by Krishnamoorthy and Tian (2008) for the mean of a homogeneous IG population. Alternatively, based on the relation that $\sqrt{b}(X-a)/a\sqrt{X}$ is approximately $\mathcal{N}(0,1)$ when $X \sim \mathcal{IG}(a,b)$, another approximate GPQ for μ can be $\mathcal{G}_\mu = \hat{\mu}|\frac{\sqrt{\hat{\mu}M}}{\sqrt{\hat{\lambda}W_5^* K\tau}}W_7^* + 1|^{-1}$, where W_7^* is a standard normal random variable independent of W_5^*.

After the GPQs for μ and λ are specified, we can plug them into $F_T(t)$ and t_p to obtain their GPQs. Confidence intervals are then obtained through simulation (i.e., generating and then sorting B realizations of the respective GPQs). Algorithm 2 summarizes the proposed interval estimation methods for p_t, and Theorem 1 establishes the asymptotic behavior of these generalized confidence intervals.

Theorem 1. *In the IG process, suppose $K/M \to \upsilon$ when $K \to \infty$. Then the $100(1-\alpha)$ percent confidence intervals for the CDF p_t and quantile t_p based on (W_5^*, W_6^*), or based on (W_5^*, W_7^*), are asymptotically of level $1-\alpha$ when $K \to \infty$.*

Algorithm 2 Constructing confidence intervals of p_t for the IG process.

Step 1. Generate B realizations of $(\mathcal{G}_\mu, \mathcal{G}_\lambda)$, denoted as $(\mu^{(b)}, \lambda^{(b)})$, $b = 1, \ldots, B$.

Step 2. For each $(\mu^{(b)}, \lambda^{(b)})$, compute a value of p_t and denote it as $p_t^{(b)}$, $b = 1, \ldots, B$.

Step 3. Use the $\alpha/2$ and $1-\alpha/2$ percentiles of $\left\{p_t^{(1)}, \ldots, p_t^{(B)}\right\}$ as the $100(1-\alpha)\%$ confidence interval of p_t.

5. Simulation study

5.1. Degradation data with moderate sizes

In this subsection, simulation studies are carried out to assess the performance of the proposed methods when the size of degradation data is moderate. Confidence intervals for the lifetime CDF p_t and quantile t_p are considered as they are of primary interest in most applications. We set $\tau = 1$, the number of observations $m_i = 5$ for each unit i, and the observation intervals $\Delta t_{i,j} = 1$ for each i and j. For each parameter setting, the number of simulation replications is set at 10,000 for all the considered methods. Coverage probabilities and average lengths so obtained are used for comparison purposes. The R code for implementing the method is available from the authors upon request.

We first investigate the performance of the proposed method for the Gamma process. Bootstrap and normal approximation methods are used for comparison. For the bootstrap method, the percentile bootstrap is used and the number of bootstrap samples is set at 10,000. For the normal approximation method, we first use delta method to obtain the variance of the estimated CDF $\hat{p}_t$ as

$$Var(\hat{p}_t) = \left[\frac{\partial p_t}{\partial \upsilon}, \frac{\partial p_t}{\partial \gamma}\right] \Sigma_{\hat{\upsilon},\hat{\gamma}} \left[\frac{\partial p_t}{\partial \upsilon}, \frac{\partial p_t}{\partial \gamma}\right]',$$

where $\Sigma_{\hat{\upsilon},\hat{\gamma}}$ is the covariance matrix of $\hat{\upsilon}$ and $\hat{\gamma}$. In large samples, $\hat{p}_t$ is approximately $\mathcal{N}(p_t, Var(\hat{p}_t))$ distributed, and then the confidence interval for p_t can be obtained. However, when dealing with the quantile t_p, it is not easy to find the variance of $\hat{t}_p$ by directly using the delta method. This is because t_p is the solution of $F_{gam}(D; \upsilon t, \gamma) = 1 - p$ and it does not have a closed form. To overcome this difficulty, we approximate the lifetime distribution by a Birnbaum-Saunders distribution. As shown in Park and Padgett (2006), the lifetime T for a Gamma degradation process approximately follows a Birnbaum-Saunders distribution with CDF

$$F_T(t; D) = \Phi\left[\frac{1}{\alpha^*}\left(\sqrt{\frac{t}{\beta^*}} - \sqrt{\frac{\beta^*}{t}}\right)\right],$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution, $\alpha^* = 1/\sqrt{D\gamma}$, and $\beta^* = D\gamma/\upsilon$. This approximation works well according to their simulations. After the approximation, an application of the delta method yields the variance of t_p, and then the confidence interval for t_p can be obtained. We set $D = 4$ and consider three parameter settings (i.e., $(\upsilon, \gamma) = (2, 0.5), (3, 1)$, and $(4, 2)$). In addition, $n = 5, 10,$ and 20 is considered. Confidence intervals of 90 and 95 percent are obtained for the failure probability p_1 at $t = 1$ and the 0.1-quantile $t_{0.1}$. The simulation results are shown in Tables 1 and 2. As can be seen, the coverage probabilities of the bootstrap intervals are uniformly lower than the nominal values. The performance of the normal approximation is quite poor when estimating the quantile and when $(\upsilon, \gamma) = (4, 2)$. However, our method performs uniformly well because the coverage probabilities are close to the nominal values with all the parameter settings and sample size n. In addition, the performance of the proposed methods based on two different GPQs for υ is almost identical. Intervals based on the Cornish-Fisher expansion to W_2 tend to have slightly shorter average lengths.

Next, we assess the performance of the proposed methods for the IG process. We set $D = 5$ and $(\mu, \lambda) = (2, 0.5), (3, 1), (4, 2)$. Similarly, sample size $n = 5, 10,$ and 20 are considered; 90 percent and 95 percent confidence intervals for t_1 and $p_{0.1}$ are obtained. The simulation results are shown in Tables 3 and 4. As can be seen, the coverage probabilities by the method of equivariant confidence sets are uniformly larger than the

Table 1. Coverage probability (CP) and average length (AL) of p_1 in the Gamma process case by the proposed methods (PM1 for the Cornish-Fisher method based on W_2^*, and PM2 for the moment-matching method based on W_3^*), the bootstrap (BTR), and the normal approximation procedure (NAP).

				$n=5$				$n=10$				$n=20$			
$1-\alpha$	υ	γ		PM1	PM2	BTR	NAP	PM1	PM2	BTR	NAP	PM1	PM2	BTR	NAP
90%	2	0.5	CP	89.70	89.70	88.30	88.48	90.00	90.00	88.20	89.50	89.68	89.69	88.20	89.81
			AL	0.151	0.152	0.156	0.152	0.107	0.108	0.110	0.107	0.076	0.076	0.077	0.076
	3	1	CP	90.55	90.55	87.60	88.08	90.07	90.07	87.40	88.58	89.96	89.95	88.50	89.88
			AL	0.136	0.136	0.139	0.136	0.096	0.097	0.099	0.097	0.068	0.068	0.070	0.068
	4	2	CP	90.21	90.23	85.60	82.64	90.43	90.43	87.50	85.54	90.17	90.17	88.30	87.87
			AL	0.073	0.075	0.062	0.067	0.051	0.052	0.047	0.049	0.036	0.037	0.035	0.036
95%	2	0.5	CP	94.91	94.91	93.50	93.85	94.79	94.79	93.90	94.55	94.54	94.54	93.80	95.00
			AL	0.180	0.181	0.188	0.182	0.127	0.128	0.132	0.128	0.090	0.090	0.092	0.090
	3	1	CP	95.30	95.30	92.80	93.08	94.96	94.96	93.30	93.90	94.83	94.83	94.40	94.67
			AL	0.161	0.162	0.165	0.163	0.114	0.115	0.118	0.115	0.081	0.082	0.083	0.081
	4	2	CP	95.12	95.13	90.20	87.01	95.18	95.18	91.90	89.97	95.16	95.16	93.90	92.37
			AL	0.088	0.090	0.073	0.080	0.061	0.063	0.056	0.058	0.043	0.044	0.042	0.042

Table 2. Coverage probability (CP) and average length (AL) of $t_{0.1}$ in the Gamma process case by proposed methods (PM1 for the Cornish-Fisher method based on W_2^*, and PM2 for the moment-matching method based on W_3^*), the bootstrap (BTR), and the normal approximation procedure (NAP).

				$n=5$				$n=10$				$n=20$			
$1-\alpha$	υ	γ		PM1	PM2	BTR	NAP	PM1	PM2	BTR	NAP	PM1	PM2	BTR	NAP
90%	2	0.5	CP	90.33	90.32	85.00	88.11	90.58	90.58	87.40	88.85	90.55	90.55	86.90	89.43
			AL	0.189	0.191	0.198	0.220	0.133	0.135	0.139	0.156	0.094	0.096	0.097	0.110
	3	1	CP	90.55	90.56	85.50	85.53	90.31	90.31	87.00	85.88	90.00	90.00	88.50	84.65
			AL	0.274	0.279	0.276	0.279	0.194	0.197	0.196	0.199	0.137	0.139	0.139	0.141
	4	2	CP	90.39	90.40	86.10	85.49	90.31	90.31	87.20	86.22	90.02	90.02	88.90	85.10
			AL	0.401	0.409	0.386	0.384	0.281	0.286	0.280	0.275	0.198	0.201	0.200	0.195
95%	2	0.5	CP	95.24	95.24	91.00	93.36	95.36	95.36	92.60	93.78	95.28	95.29	92.50	94.50
			AL	0.224	0.226	0.237	0.262	0.158	0.161	0.166	0.186	0.112	0.114	0.116	0.131
	3	1	CP	95.51	95.51	90.60	91.34	95.16	95.15	92.00	91.60	95.11	95.10	94.00	90.80
			AL	0.325	0.332	0.328	0.332	0.231	0.235	0.234	0.237	0.163	0.166	0.166	0.168
	4	2	CP	95.20	95.21	91.50	91.00	95.10	95.10	92.30	91.85	94.91	94.91	94.50	91.48
			AL	0.479	0.488	0.460	0.458	0.335	0.341	0.334	0.328	0.236	0.240	0.238	0.233

nominal values, and the average length is also quite large. However, the coverage probabilities by the proposed methods are close to the nominal values in all scenarios. Though the proposed methods are based on two different GPQs for μ, their performances do not differ much.

5.2. Degradation data with big sizes

In the last subsection we focused on degradation data with moderate sizes. This kind of degradation data is not uncommon in practice; see the various degradation data sets in Meeker and Escobar (1998) for a few

Table 3. Coverage probability (CP) and average length (AL) of p_1 in the IG process case by proposed methods (PM1 based on W_6^*, PM2 based on W_7^*) and the method of equivariant confidence sets (ECS).

				$n=5$			$n=10$			$n=20$		
$1-\alpha$	μ	λ		PM1	PM2	ECS	PM1	PM2	ECS	PM1	PM2	ECS
90%	2	0.5	CP	91.22	90.01	96.51	88.83	88.82	95.21	90.14	90.71	95.13
			AL	0.085	0.081	0.133	0.060	0.058	0.086	0.042	0.041	0.058
	3	1	CP	91.42	88.84	97.73	90.71	88.94	97.15	89.97	91.62	96.95
			AL	0.104	0.100	0.165	0.072	0.071	0.108	0.051	0.050	0.072
	4	2	CP	91.32	91.26	98.47	89.77	89.39	98.41	90.43	89.29	98.07
			AL	0.122	0.118	0.195	0.085	0.083	0.131	0.060	0.059	0.090
95%	2	0.5	CP	95.86	95.43	98.47	94.43	94.39	97.75	94.67	95.64	97.98
			AL	0.104	0.097	0.160	0.072	0.070	0.103	0.050	0.050	0.069
	3	1	CP	95.93	94.55	98.95	94.82	93.65	98.35	95.05	95.75	98.74
			AL	0.127	0.119	0.198	0.087	0.085	0.127	0.061	0.059	0.085
	4	2	CP	95.58	95.54	99.34	95.15	94.42	99.48	94.75	95.02	99.61
			AL	0.149	0.140	0.233	0.102	0.099	0.153	0.072	0.071	0.104

Table 4. Coverage probability (CP) and average length (AL) of $t_{0.1}$ in the IG process case by proposed methods (PM1 based on W_6^*, PM2 based on W_7^*) and the method of equivariant confidence sets (ECS).

				$n=5$			$n=10$			$n=20$		
$1-\alpha$	μ	λ		PM1	PM2	ECS	PM1	PM2	ECS	PM1	PM2	ECS
90%	2	0.5	CP	91.25	90.14	96.48	88.78	87.98	95.11	90.11	90.84	95.18
			AL	0.848	0.832	1.229	0.565	0.563	0.780	0.384	0.385	0.509
	3	1	CP	91.42	88.78	97.01	90.41	88.95	95.81	90.05	91.21	95.48
			AL	0.463	0.448	0.673	0.310	0.308	0.427	0.212	0.213	0.282
	4	2	CP	90.85	91.32	96.11	90.46	89.07	96.38	89.68	89.67	96.15
			AL	0.295	0.282	0.429	0.195	0.195	0.270	0.136	0.135	0.182
95%	2	0.5	CP	95.61	95.58	98.44	94.35	94.38	97.72	94.73	95.62	98.09
			AL	1.028	0.993	1.459	0.680	0.671	0.917	0.460	0.459	0.594
	3	1	CP	96.61	94.27	98.58	94.87	93.88	98.18	94.85	95.21	98.14
			AL	0.561	0.536	0.800	0.373	0.366	0.502	0.254	0.253	0.329
	4	2	CP	94.67	96.14	98.35	95.21	94.77	98.81	95.12	94.78	97.81
			AL	0.357	0.337	0.510	0.235	0.232	0.316	0.163	0.161	0.212

examples. Our simulation study has shown that the proposed inference methods perform better than the existing methods. However, the size of the degradation data may be large in some cases. For example, the number of units n may be large. Another example is that sensors may generate the degradation data at a high frequency (Meeker and Hong 2014), and hence the number of observations m_i for each unit may be large. In this subsection, we assess the performance of the proposed inference method in the scenario of big degradation data. Specifically, we consider $m_i = 1{,}000$ for each unit i and $n = 5{,}000, 10{,}000, 50{,}000$. The 90 percent confidence interval of the lifetime CDF p_1 at $t = 1$ is of interest, and comparison among different methods is based on the coverage probabilities. The other simulation settings are similar to those in the last subsection.

Table 5 shows the simulation results for the Gamma process. As seen, the coverage probabilities based on the proposed methods and the bootstrap are all close to the nominal values. This is reasonable as the bootstrap often performs well in terms of a large sample. However, because of its resampling-based nature, the bootstrap usually requires a significant computation time, especially when the sample size is large. The average computation time to obtain one confidence interval of p_1 by using the R language on a standard Intel i5 processor is shown in Table 6. As we can see, the computation time for the proposed methods is much shorter compared to the bootstrap. In addition, the GPQ method based on W_3^* requires less computation time than the GPQ method based on W_2^*. This is because one needs to solve equations to obtain the GPQs for the parameter υ based on W_2^*. Therefore, we would suggest using the moment-matching method based on W_3^* in the big data scenario.

Table 6. Average computation time (in minutes) to obtain one confidence interval of p_1 in the Gamma process case by the proposed methods (PM1 for the Cornish-Fisher method based on W_2^*, and PM2 for the moment-matching method based on W_3^*) and the bootstrap (BTR).

$n=5{,}000$			$n=10{,}000$			$n=50{,}000$		
PM1	PM2	BTR	PM1	PM2	BTR	PM1	PM2	BTR
2.3	1.1	113.8	5.2	2.5	280.4	25.5	11.8	1760.1

Next, we examine the simulation results for the IG process, which are shown in Table 7. As we can see, the proposed GPQ methods can achieve very accurate coverage while the method of equivalent confidence sets is conservative. Because all three methods can be implemented in an affordable amount of time, we do not show the computation time of each method here. Nevertheless, based on their performance in terms of the coverage probabilities, the proposed methods are clearly preferred in real applications.

6. Applications

Metallized film capacitors are used as energy storage units for many important systems, such as the inertial

Table 5. Coverage probability of p_1 in the Gamma process case by the proposed methods (PM1 for the Cornish-Fisher method based on W_2^* and PM2 for the moment-matching method based on W_3^*), and the bootstrap (BTR).

			$n=5{,}000$			$n=10{,}000$			$n=50{,}000$		
$1-\alpha$	υ	γ	PM1	PM2	BTR	PM1	PM2	BTR	PM1	PM2	BTR
90%	2	0.5	89.96	89.94	90.06	90.00	90.04	90.02	90.09	89.95	90.05
	3	1	89.94	90.04	90.05	90.09	89.99	89.98	90.01	89.96	90.08
	4	2	90.07	89.99	90.04	90.06	90.07	90.06	90.00	89.95	90.06

Table 7. Coverage probability of p_1 in the IG process case by the proposed methods (PM1 based on W_6^*, PM2 based on W_7^*) and the method of equivariant confidence sets (ECS).

			$n = 5{,}000$			$n = 10{,}000$			$n = 50{,}000$		
$1-\alpha$	υ	γ	PM1	PM2	ECS	PM1	PM2	ECS	PM1	PM2	ECS
90%	2	0.5	90.01	90.00	95.01	90.02	90.01	94.89	89.99	90.00	94.78
	3	1	89.98	90.01	94.99	90.03	90.00	95.00	89.98	89.99	94.81
	4	2	90.00	89.97	95.03	89.97	90.02	94.91	90.02	89.98	94.72

confinement laser fusion facility. Hundreds of capacitors are connected in parallel to provide high-energy storage. Performance of a capacitor is characterized by its capacitance, and the capacitance decreases with use. When the cumulative capacitance loss exceeds 5 percent of the initial capacitance, or equivalently $D = 2.75\ \mu F$ in this example, the dielectric in the capacitor might be erratic, so a soft failure is defined according to industrial standards. The reliability of the capacitor pack is critical to the operation of the system in which it is installed. Specifically, reliability has significant impact on the preventive maintenance costs and system safety. Therefore, capacitors are designed to be highly reliable. To save time and capacitor samples during reliability evaluation, degradation tests are used in the design phase by measuring the capacitance loss over time. Figure 1 shows the capacitance losses of eight capacitors in a degradation test. The number of shots is used as the time scale, where each shot represents a charge/discharge cycle. The capacitance of each test unit is measured every 1,000 shots, based on when the capacitance loss is computed.

Table 8. AIC values of the Gamma and IG degradation models for the cumulative capacitance loss data.

	Gamma		IG process	
	Basic	Random γ	Basic	Random μ
Linear	− 161.6	− 165.2	− 175.8	− 174.0
Power-law	− 161.4	− 165.3	− 175.2	− 173.4

The sample paths are monotone. Therefore, the Gamma and IG processes are used to fit the data. In addition to the basic models, we further consider the random λ Gamma process (Lawless and Crowder 2004) and the random μ IG process (Ye and Chen 2014). To account for possible curvature in the degradation path, a power-law time-scale transformation to the above models is considered. Akaike information criterion (AIC) values of these models are displayed in Table 8 below.

Based on Table 8, the AIC values favor the stationary IG process without random effects for the data. The MLEs of the two parameters are $\hat{\mu} = 0.1311$ and $\hat{\lambda} = 0.0521$. Recall that $b(X-a)^2/(a^2X) \sim \chi^2(1)$ when $X \sim$

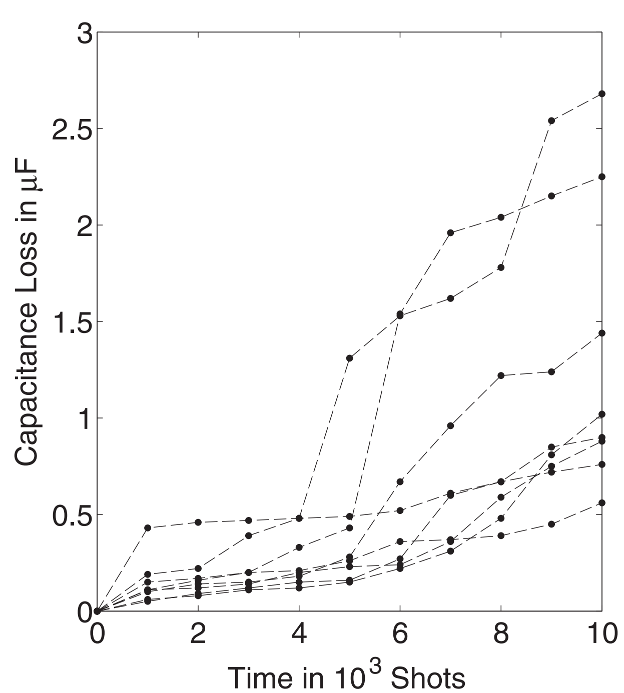

Figure 1. Cumulative capacitance loss data of eight capacitors.

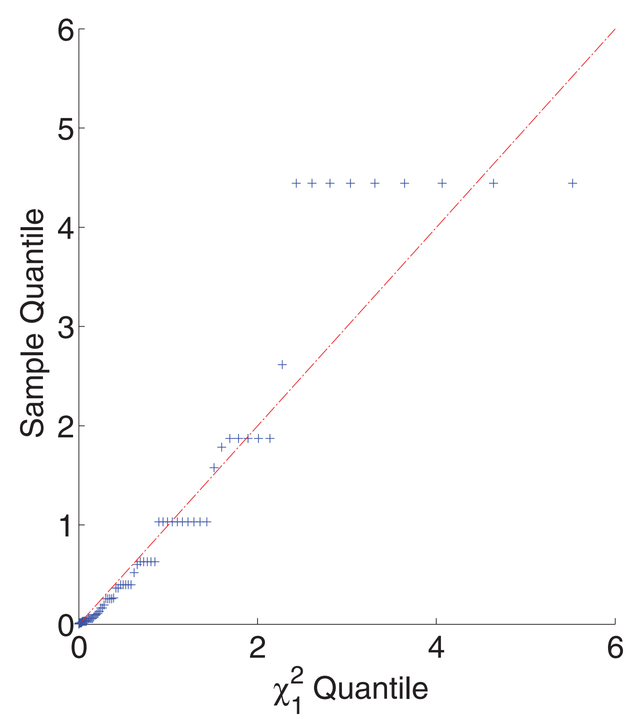

Figure 2. $\chi^2(1)$ q-q plot for the cumulative capacitance loss data.

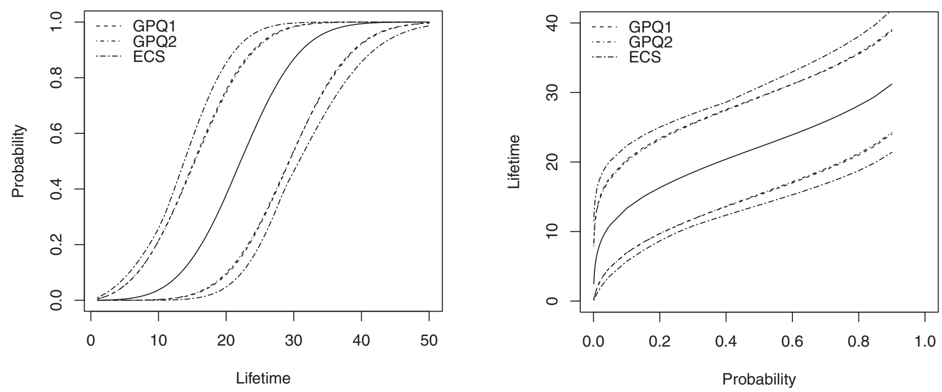

Figure 3. Ninety-five percent pointwise confidence bands of the lifetime CDF (left panel) and quantiles (right panel) for the capacitance loss data based on the GPQ method and the method of equivariant confidence sets (ECS). The solid lines in the center are the ML estimates.

$\mathcal{IG}(a, b)$. To assess the goodness of fit, we apply the above transformation to the degradation increments. The resulting $\chi^2(1)$ q-q plot is given in Figure 2. The plot shows that there is not obvious deviation from the χ^2 distribution and that some residuals share the same value (e.g., the nine points at the right tail). A careful check of the degradation data shows that this is because of the measurement limit. The data have been rounded off to the nearest two decimal places, leading to the same residuals for different degradation observations. Assuming the basic IG process for the capacitance loss, pointwise confidence bands for the lifetime CDF and quantiles can be obtained. The results are shown in Figure 3. Confidence bands based on (W_5^*, W_6^*) (GPQ1) and (W_5^*, W_7^*) (GPQ2) are almost the same, indicating similar performances between these two methods. In addition, the confidence bands based on the GPQ method are clearly narrower than those based on the method of equivariant confidence sets.

7. Conclusions

When the sample path of the degradation process is believed to be monotonic, the Gamma process and the IG process are two popular models for degradation data. This article has developed generalized confidence intervals for these two important models. We first considered the Gamma process and two methods were proposed to approximate the logarithm of the ratio between geometric and arithmetic means. The first method was based on the Cornish-Fisher expansion to W_2, while the second method was based on a Gamma approximation to W_3. These two approximations were used to construct confidence intervals of the shape parameter. However, the interval estimation for the scale parameter is based on an approximate GPQ. In both the IG and Gamma models, pivotal quantities for lifetime characteristics, such as lifetime quantile and lifetime CDF, are not available. By capitalizing on the pivotal quantities of the model parameters, GPQs for the lifetime characteristics were proposed and were shown to be asymptotically valid. We further considered interval estimation for the IG processes. Generalized pivotal quantities for parameters of interest were derived and used to construct their confidence intervals. Extensive simulations have demonstrated the effectiveness of the generalized confidence intervals compared with the bootstrap method and the normal approximation method based on the Fisher information matrix. The results showed that the performance of the proposed methods is satisfactory with both moderate and large sample sizes. The inference procedures were successfully applied to a real example.

About the authors

Piao Chen received the B.E. degree in industrial engineering from Shanghai Jiao Tong University, China, in 2013, and the Ph.D. degree in industrial systems engineering and management from the National University of Singapore, in 2017. He is currently a research scientist in the Institute of High Performance Computing, Singapore. His research interests include data analysis and industrial statistics.

Zhi-Sheng Ye received the joint B.E. degree in material science and engineering and economics from Tsinghua University, Beijing, China, in 2008, and the Ph.D. degree in industrial and systems engineering from the National University of Singapore, in 2012. He is currently an Assistant Professor with the Department of Industrial Systems Engineering and Management, National University of Singapore. His research interests include reliability engineering, complex systems modeling, and industrial statistics.

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Appendix A. Gamma process

The transformed degradation increment $Z_{i,j}$ follows $\text{gam}(\upsilon k_{i,j}\tau, k_{i,j}\gamma)$, and the $Z_{i,j}$'s are independent. Given the observed data $\mathbf{D}$, the log-likelihood of (υ, γ) can be specified as

$$l(\upsilon,\gamma)=\sum_{i,j}\Big[\upsilon k_{i,j}\tau\ln(k_{i,j}\gamma)+(\upsilon k_{i,j}\tau-1)\ln Z_{i,j} - \ln\Gamma(\upsilon k_{i,j}\tau)-k_{i,j}\gamma Z_{i,j}\Big]. \qquad [A1]$$

Setting the partial derivative $\partial l(\upsilon,\gamma)/\partial\gamma$ to zero gives $\gamma=\upsilon\tau/\bar{Z}$. Substitute this relation in Eq. [A1] to obtain the profile likelihood of υ. We can maximize the profile log-likelihood to obtain the MLE $\hat{\upsilon}$ and then $\hat{\gamma}=\hat{\upsilon}\tau/\bar{Z}$. The maximization is easy because there is only one decision variable.

Next, we will derive the MGF of $W_3=2K\upsilon\tau\ln(\bar{Z}/\tilde{Z})$. According to the definition of MGF, we know that

$$M_{W_3}(u)=E[\exp(uW_3)]=E[(\tilde{Z}/\bar{Z})^{-2K\upsilon\tau u}].$$

Because $Z_{i,j}$ are independent Gamma random variables and because the quantity W_3 as a function of $Z_{i,j}$ is free of the rate parameter γ, Property III in Pitman (1991; Section 6) implies that W_3 is independent of $\bar{Z}$. The independence ensures we compute $E[\tilde{Z}^{-2K\upsilon\tau u}]$ through

$$E[\tilde{Z}^{-2K\upsilon\tau u}]=E\left[\exp(uW_3)\bar{Z}^{-2K\upsilon\tau u}\right]=E\left[\exp(uW_3)\right]\times E\left[\bar{Z}^{-2K\upsilon\tau u}\right].$$

Therefore, the MGF of W_3 can be computed through

$$M_{W_3}(u)=E[\tilde{Z}^{-2K\upsilon\tau u}]/E\left[\bar{Z}^{-2K\upsilon\tau u}\right].$$

The two expectations on the right-hand side can be evaluated by using the fact that

$$EX^u=\frac{\Gamma(\upsilon+u)}{\Gamma(\upsilon)}\gamma^{-u}\text{ when }X\sim\text{gam}(\upsilon,\lambda).$$

After some algebraic manipulation, Eq. [3] follows. A similar procedure can be used to obtain the MGF of W_2 (e.g., Eq. [2]).

When $\upsilon\tau$ is large, all arguments of the Gamma function in Eq. [3] are large. We can apply Stirling's formula to approximate $\Gamma(x)$ as $\sqrt{2\pi}x^{x-1/2}e^{-x}[1+O(x^{-1})]$ (Abramowitz and Stegun 1972; Eq. [6.1.37]), which yields

$$M_{W_3}(u)=(1-2u)^{-(M-1)/2}[1+O((\tau\upsilon)^{-1})].$$

The first part on the right-hand side is exactly the MGF of $\chi^2(M-1)$. When $\upsilon\tau$ is small, we can apply series expansion to approximate $1/\Gamma(x)$ as $x(1+O(x))$ (Abramowitz and Stegun 1972; Eq. [6.1.34]). Then $M_{W_3}(u)$ can be expressed as

$$M_{W_3}(u)=(1-2u)^{-(M-1)}[1+O(\tau\upsilon)],$$

which is approximately equal to the MGF of a $\chi^2(2M-2)$ variable.

Appendix B. IG process

B.1. Pivotal quantities

When the process follows the IG process $Z_{i,j}\sim\mathcal{IG}(\mu\tau, k_{i,j}\tau^2\lambda)$, the log-likelihood function based on $\mathbf{D}$ can then be specified as

$$l(\mu,\lambda)=\frac{M}{2}\ln\left(\frac{\lambda\tau^2}{2\pi}\right)+\sum_{i,j}\left[\frac{1}{2}\ln\frac{k_{i,j}}{Z_{i,j}^3}-\frac{k_{i,j}\lambda(Z_{i,j}-\mu\tau)^2}{2\mu^2 Z_{i,j}}\right]. \qquad [B1]$$

Taking partial derivatives of $l(\mu,\lambda)$ with respect to μ and λ, setting the two derivatives to zero, and solving the two equations yields the MLE of μ and λ as

$$\hat{\mu}=\bar{Z}/\tau,\qquad 1/\hat{\lambda}=\frac{\tau^2}{M}\sum_{i,j}k_{i,j}\left(\frac{1}{Z_{i,j}}-\frac{1}{\bar{Z}}\right).$$

According to the definition of $\bar{Z}$, we know that $M\bar{Z}\sim\mathcal{IG}(K\tau\mu, K^2\tau^2\lambda)$. Therefore, $\hat{\mu}=M\bar{Z}/(M\tau)\sim\mathcal{IG}(\mu,K\tau\lambda)$. Conditional on $\hat{\mu}$, the joint distribution of $\mathbf{Z}=\{Z_{i,j}; i=1,2,\ldots,n, j=1,2,\ldots,m_i\}$ can be obtained as the quotient of the joint PDF of $\mathbf{Z}$ and the PDF of $\hat{\mu}$, which is given by

$$g(\mathbf{z};\lambda)=\left(\frac{\lambda\tau^2}{2\pi}\right)^{(M-1)/2}\frac{(\hat{\mu}\tau)^{3/2}}{\sqrt{K}}\left(\prod_{i,j}\frac{\sqrt{k_{i,j}}\tau}{z_{i,j}^{3/2}}\right)$$

$$\times \exp\left[-\frac{\lambda}{2}\sum_{i,j}k_{i,j}\tau^2\left(\frac{1}{z_{i,j}}-\frac{1}{\tau\hat{\mu}}\right)\right], \quad \text{[B2]}$$

where the domain is $\Omega = \{\mathbf{z} \in \mathcal{R}^M : z_{i,j} > 0, \sum_{i,j} z_{i,j} = K\hat{\mu}\}$. It is noted that

$$\exp(uM\lambda/\hat{\lambda})g(\mathbf{z};\lambda) = (1-2u)^{-(M-1)/2}g(\mathbf{z};(1-2u)\lambda).$$

Based on the relation, we know that

$$E[\exp(uM\lambda/\hat{\lambda})|\hat{\mu}] = \int_\Omega \exp(uM\lambda/\hat{\lambda})g(\mathbf{z};\lambda)d\mathbf{z} = (1-2u)^{-(M-1)/2}.$$

The right-hand side is the MGF of the $\chi^2(M-1)$ distribution. This means that conditional on $\hat{\mu}$, $M\lambda/\hat{\lambda}$ follows a $\chi^2(M-1)$ distribution independent of $\hat{\mu}$. Unconditionally, we still have $M\lambda/\hat{\lambda} \sim \chi^2(M-1)$, and $\hat{\lambda}$ is independent of $\hat{\mu}$.

B.2. Proof of Theorem 1

We present the proof for the GPQs based on (W_5^*, W_6^*). The proof for the GPQs based on (W_5^*, W_7^*) is simpler and similar. To prove Theorem 1, the following lemma is useful.

Lemma 1. *Let $W_5^* \sim \chi^2_{M-1}$, $W_6^* \sim t_{M-1}$, and W_5^*, W_6^* be independent. Then for each $(w_5^*, w_6^*) \in \mathcal{R} \times \mathcal{R}_+$, there exists a unique pair $(\hat{\mu}^*, \hat{\lambda}^*) \in \mathcal{R}_+^2$ satisfying $w_5^* = M\lambda/\hat{\lambda}^*$, $w_6^* = \frac{\sqrt{(M-1)K\tau}}{\sqrt{M\hat{\mu}^*/\hat{\lambda}^*}}(\hat{\mu}^*/\mu - 1)$. Moreover, $\sqrt{K}[(\hat{\mu}^*(W_5^*, W_6^*), \hat{\lambda}^*(W_5^*, W_6^*)) - (\mu, \lambda)]$ has the same asymptotic distribution as $\sqrt{K}[(\hat{\mu}, \hat{\lambda}) - (\mu, \lambda)]$.*

Proof. A unique $\hat{\lambda}^* > 0$ can be readily obtained based on $w_5^* = M\lambda/\hat{\lambda}^*$. For fixed $\mu > 0$, the equation $x^2/\mu - 1 = Cx$ always has a negative root and a positive root for any $C \in \mathcal{R}$. Letting $C = w_6^*\sqrt{w_5^*/\lambda}/\sqrt{(M-1)K\tau}$, we can see that the square root of the positive root is the unique solution of $\hat{\mu}^*$. This proves the first part of the lemma.

It is easy to see that $\hat{\lambda}^*$ has the same distribution as $\hat{\lambda}$ and that $\hat{\lambda}^* \to_p \lambda$. By noting that $W_6^*\sqrt{M/\hat{\lambda}^*}/\sqrt{(M-1)K\tau} \to_p 0$, we can see that $\sqrt{\hat{\mu}^*}/\mu - \sqrt{\hat{\mu}^*} \to_p 0$. A continuous mapping argument shows that $\hat{\mu}^* \to_p \mu$. Therefore, $\sqrt{K}(\hat{\mu}^*/\mu - 1) = W_6^*\sqrt{\mu/\lambda\tau} + o_p(1)$. We can see that $\sqrt{K}(\hat{\lambda}^* - \lambda)$ depends on W_5^*, $\sqrt{K}(\hat{\mu}^*/\mu - 1)$ depends on W_6^* asymptotically, and W_5^* and W_6^* are independent. Therefore, $\sqrt{K}(\hat{\mu}^*/\mu - 1)$ and $\sqrt{K}(\hat{\lambda}^* - \lambda)$ are asymptotically independent, and $\sqrt{K}(\hat{\mu}^* - \mu) \to_d \mathcal{N}(0, \mu^3/(\lambda\tau))$. We already know that $\hat{\mu}$ is independent of $\hat{\lambda}$. It is easy to check that $\sqrt{K}(\hat{\mu} - \mu) \to_d \mathcal{N}(0, \mu^3/(\lambda\tau))$. Therefore, the second part of the lemma holds. □

Let W_5^*, W_6^*, $\hat{\mu}^*$, and $\hat{\lambda}^*$ be the same as those defined in Lemma 1. Then we can rewrite $\mathcal{G}_\mu$ and $\mathcal{G}_\lambda$ as

$$\mathcal{G}_\mu = \hat{\mu}\left(\frac{\sqrt{M\hat{\mu}/\hat{\lambda}}W_6^*}{\sqrt{(M-1)K\tau}} + 1\right)^{-1}$$

$$= \hat{\mu}\left[\frac{\sqrt{\hat{\mu}/\hat{\lambda}}}{\sqrt{\hat{\mu}^*/\hat{\lambda}^*}}\left(\frac{\hat{\mu}^*}{\mu} - 1\right) + 1\right]^{-1},$$

$$\mathcal{G}_\lambda = \hat{\lambda}W_5^*/M = \frac{\hat{\lambda}}{\hat{\lambda}^*}\lambda.$$

Note that $\hat{\boldsymbol{\theta}}^* = (\hat{\mu}^*, \hat{\lambda}^*)$ is not an independent and identically distributed copy of $\hat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\lambda})$. Therefore, Theorem 1 of Hannig, Iyer, and Patterson (2006) cannot be directly applied here. Nevertheless, a simple modification of their proof yields that the conclusions in their Theorem 1 still hold when $\hat{\boldsymbol{\theta}}^*$ satisfies the following conditions: (a) conditional on $\hat{\boldsymbol{\theta}}$, the distributions of $\mathcal{G}_\mu$ and $\mathcal{G}_\lambda$ are free of $\boldsymbol{\theta}$; (b) $(\mathcal{G}_\mu, \mathcal{G}_\lambda) = \boldsymbol{\theta}$ when $\hat{\boldsymbol{\theta}}^* = \boldsymbol{\theta}$; and (c) $\sqrt{M}(\hat{\boldsymbol{\theta}}^* - \boldsymbol{\theta})$ and $\sqrt{M}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ are independent and have the same asymptotic distributions. With the help of Lemma 1, it is easy to see that $\hat{\boldsymbol{\theta}}^*$ satisfies all three conditions. The remaining work is to verify Condition A in Theorem 1 of Hannig, Iyer, and Patterson (2006) for both $\mathcal{G}_{p_t}$ and $\mathcal{G}_{t_p}$. Condition A1 is obvious as $\hat{\boldsymbol{\theta}}^*$ is asymptotically normal. Condition A2 can be verified by differentiating the IG distribution and checking the result. The detailed differentiation procedure is straightforward but tedious, and thus it is omitted.