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A systematic look at the gamma process capability indices

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ABSTRACT

Process capability indices (PCIs) are widely used to measure whether an in-control process conforms to manufacturing specifications. The normal distribution is assumed in most traditional applications of PCIs. Nevertheless, it is not uncommon that some quality characteristics have skewed distributions. In such cases, the gamma distribution is an appropriate model and percentile-based PCIs for the gamma process have been studied in the literature. In practical applications of PCIs, it is important to select an appropriate distribution between the normal and the gamma distributions based on historical data. In this study, we first construct a hypothesis test for model discrimination between the normal and the gamma distributions. Asymptotic distribution of the test statistic under the gamma process is derived. We then consider statistical inference for the percentile-based PCIs under the gamma process. The maximum likelihood method is used for point estimation and the method of generalized pivotal quantities is used for interval estimation. We demonstrate the proposed methods by a practical example.

1. Introduction

The process capability indices (PCIs) are widely used to assess whether an in-control process meets specification limits determined from engineering tolerances or customers' needs. They have become the common language for process quality between the customers and the suppliers (Müller & Haase, 2016). Numerous PCIs, such as the classical C_p, C_{pk}, C_{pm} and C_{pmk} (Hsu, Pearn, & Wu, 2008), have been proposed in the literature and industry. Among these PCIs, C_{pk} proposed by Boyles (1991) is probably the most frequently used due to its yield-based nature (Ryan, 2011, chap.7). To see this, assume that the process characteristic X follows a normal distribution $N(\mu, \sigma^2)$ with mean μ and standard deviation σ. Then C_{pk} is defined as

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\}, \tag{1}$$

where USL and LSL are the upper specification limit and the lower specification limit, respectively. For a given value of C_{pk}, the process yield is bounded between $2\Phi(3C_{pk})-1$ and $\Phi(3C_{pk})$, where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution (Boyles, 1991). Generally, a lower value of C_{pk} implies a higher faction of defectives of the process. Therefore, a customer would usually specify a minimum value of C_{pk} in the purchasing contract (Wu, Aslam, & Jun, 2012).

It is necessary to emphasize that C_{pk} defined in (1) is meaningful only if the process characteristic X is normally distributed. Otherwise, the bounds of the process yield based on the value of C_{pk} could be misleading. In fact, the normality assumption is the basis for many commonly-used PCIs (e.g., Wu, 2012; Yeong, Khoo, Lee, & Rahim, 2013). Although the normal distribution seems appropriate for some manufacturing systems, it is very likely that a process characteristic has a non-normal distribution (Ryan, 2011, chap.7). For example, many quality characteristics such as diameter and roundness are often non-normal in manufacturing process (Ryan, 2011, chap.7). In the procurement process of oil and gas companies, the distribution of cycle time data is often skewed (Aldowaisan, Nourelfath, & Hassan, 2015). Also, the distribution of some chemical process such as zinc plating is found far from normal (Pyzdek, 1992). In addition, the normality assumption comes into question in many service and transaction systems (Hosseinifard, Abbasi, & Niaki, 2014).

Since the PCIs based on the normality assumption could be meaningless (Ryan, 2011), several approaches have been suggested to handle the non-normal processes. The most straightforward way is to transform the non-normal data such that the transformed data will be approximately normal (e.g., Somerville & Montgomery, 1996). However, the transformation methods have drawbacks which are inherent in their utilization. First, transformation methods are computing-extensive (Tang & Than, 1999). For example, it is necessary to find the optimal parameter in the Box-Cox power transformation. Second, the approximate normality of the transformed data cannot be always guaranteed, and hence the PCIs may not be accurate. Finally, because of the problems associated

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with translating the computed results with regard to the original scales, the use of transformed data is often not appealing to practitioners (Ryan, 2011). On the other hand, a more appropriate way is to fit a distribution to the process characteristic data, and use percentiles of this distribution to modify classical PCIs (Clements, 1989; Rodriguez, 1992). For example, a percentile-based C_{pk} is defined as

$$C_{pk}^{*} = \min\left\{\frac{USL - X_{0.5}}{X_{0.9987} - X_{0.5}}, \frac{X_{0.5} - LSL}{X_{0.5} - X_{0.0013}}\right\}, \tag{2}$$

where X_β is the 100βth percentile of the fitted distribution; USL and LSL are the upper specification limit and the lower specification limit, respectively. Notice that C_{pk}^{*} and C_{pk} coincide for a normal process.

Obviously, an appropriate distribution for the process is the premise of the use of the percentile-based PCIs. Among all the possible distributions, the gamma distribution is a popular choice and it has been extensively used to fit a variety of process characteristic data (e.g., Hsu et al., 2008; Aldowaisan et al., 2015). This is because the gamma distribution belongs to the Pearson family, which usually provides a reasonable curve flexibility for the process characteristic data (Rodriguez, 1992; Ryan, 2011). The gamma distribution itself is a very flexible distribution; it involves the exponential distribution and the χ^2 distribution as special cases. In fact, the gamma distribution governs a wide class of non-normal applications in numerous disciplines. For example, it is well known that the gamma distribution is a useful lifetime model (e.g., Chen & Ye, 2017c; 2017b). In addition, the gamma distribution plays an important role in some genetic research (e.g., Agarwala, Flannick, Sunyaev, Consortium, & Altshuler, 2013). It is also extensively used in environmental science (e.g., Villarini, Seo, Serinaldi, & Krajewski, 2014; Chen & Ye, 2017a).

Given the importance of the PCIs and the wide applications of the gamma distribution in non-normal processes, we focus on the percentile-based C_{pk}^{*} under the gamma processes assumption in this study. Evidently, the first important task is to select the appropriate distribution between the normal and gamma distributions. In the literature, a histogram of the process data is often plotted for qualitative model discrimination (Hsu et al., 2008). This method is simple and useful when the sample size is large and the histogram is very skewed or symmetric. Otherwise, it provides limited information in selecting the appropriate distribution. Therefore, quantitative methods for discrimination between the normal and the gamma distributions are needed. In this study, we treat this discrimination problem as a hypothesis test problem. Such a treatment can be found in Dumonceaux, Antle, 1973, Kundu and Manglick (2004), and Kim and Yum (2008), to name a few. In this study, the test statistic is constructed as the logarithm of the ratio of maximized likelihoods (RML) (Cox, 1961; 1962). Under the gamma process assumption, the asymptotic distribution of the test statistic is provided, which can be used to determine the probability of correct selection easily. A simulation is conducted to assess the Type I error and Type II error of the hypothesis under the proposed decision rule.

Once the gamma distribution is selected, the next important task is to make statistical inference for the percentile-based C_{pk}^{*}, as the true parameters are unknown in reality. A point estimator of C_{pk}^{*} is generally easy to obtain by the maximum likelihood (ML) method. Nevertheless, a lower confidence limit is of more interest in practice (Chang & Wu, 2008; Kotz & Lovelace, 1998; Ryan, 2011). Because C_{pk}^{*} is defined as the minimum of two functions, its lower confidence limit is quite difficult to obtain even under the normal process assumption. In this study, we use the idea of generalized confidence interval (GCI) to obtain the lower confidence limit of C_{pk}^{*} for a gamma process. Since it was introduced in Weerahandi (1995), the method of GCI has been successfully applied in many statistical inference problems. See Krishnamoorthy and Mathew (2004) and Hannig, Iyer, and Patterson (2006), among others. Generally, accurate coverage can be guaranteed based on the method of GCI. We first construct generalized pivotal quantities (GPQs) for the gamma parameters. The GPQ for C_{pk}^{*} can then be naturally constructed, based on which the GCI can be readily obtained. The performance of the constructed GCI is examined by a simulation study.

The rest of this paper is organized as follows. In Section 2, a hypothesis test for discrimination between the gamma and the normal distributions is constructed. We also derive the asymptotic distribution of the test statistic under the gamma process assumption. A simulation study is then conducted to assess the Type I error and Type II error under the proposed decision rule. Section 3 introduces the inference methods for the percentile-based C_{pk}^{*}. A simulation study verifies the performance of GCI in constructing the lower confidence limit of C_{pk}^{*}. In Section 4, a practical example is provided to show the usefulness of the proposed model discrimination method and the inference method. Section 5 concludes the paper.

2. Model discrimination

Assume the process characteristic X follows either a normal distribution or a gamma distribution. Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) copies of X. We are interested in discriminating these two distributions based on the observed process characteristic data $x_1, \ldots, x_n$. The normal distribution $N(\mu, \sigma^2)$ with mean μ and standard deviation $\sigma > 0$ has a probability density function (PDF) as

$$f_N(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty, \tag{3}$$

and the gamma distribution $GA(k, \theta)$ has a PDF as

$$f_G(x; k, \theta) = \frac{\theta^k}{\Gamma(k)}x^{k-1}e^{-\theta x}, \quad x > 0, \tag{4}$$

where $k > 0$ is the shape parameter and $\theta > 0$ is the rate parameter. It is convenient to formulate this discrimination problem as a hypothesis test problem with null hypothesis H_0 and the alternative H_1 as Dumonceaux, Antle, 1973

$$H_0 : X \sim GA(k, \theta) \quad \text{and} \quad H_1 : X \sim N(\mu, \sigma^2). \tag{5}$$

The most commonly used test statistic for such a hypothesis is Cox's statistic (Cox, 1961; 1962), which is the logarithm of the ratio of the maximum likelihoods under both the null and alternative hypotheses. The widespread application of Cox's test can be found in Dumonceaux, Antle, 1973, Kundu and Manglick (2004), and Kim and Yum (2008), to name a few. For our problem, the RML is defined as

$$\mathrm{RML} = \frac{\max\prod_{i=1}^{n} f_G(X_i; k, \theta)}{\max\prod_{i=1}^{n} f_N(X_i; \mu, \sigma)} = \frac{\prod_{i=1}^{n} f_G(X_i; \hat{k}, \hat{\theta})}{\prod_{i=1}^{n} f_N(X_i; \hat{\mu}, \hat{\sigma})}. \tag{6}$$

Here, $(\hat{k}, \hat{\theta})$ are the ML estimators of (k, θ), given as solution to the system of equations

$$\theta = \frac{k}{\bar{X}} \quad \text{and} \quad n\log(\theta) - n\psi(k) + \sum_{i=1}^{n}\log X_i = 0, \tag{7}$$

with $\bar{X} = \sum_i X_i/n$ and $\psi(x) = d\log\Gamma(x)/dx$. In addition, $(\hat{\mu}, \hat{\sigma})$ are the ML estimators of (μ, σ), given by

$$\hat{\mu} = \bar{X} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2. \tag{8}$$

The test statistic is then constructed as the logarithm of the RML, i.e.,

$$T = \log(\mathrm{RML}) = \sum_{i=1}^{n} [\log f_G(X_i; \hat{k}, \hat{\theta}) - \log f_N(X_i; \hat{\mu}, \hat{\sigma})]. \tag{9}$$

Similar to the decision rule in Kundu and Manglick (2004) and Kim and Yum (2008), we select the gamma distribution when $T > 0$ and the normal distribution when $T < 0$ based on the observed data $x_1, \ldots, x_n$. Since both distributions have the same number of parameters, this decision rule is actually equivalent to the famous Akaike criteria (Akaike, 1974). To assess the performance of this decision rule, an important criteria is the probability of correct selection (PCS) under the null hypothesis (Kundu & Manglick, 2004). Therefore, it is desirable to know the distribution of T under the null hypothesis. Unfortunately, the distribution of T with finite sample is quite difficult to derive. On the other hand, since the distribution of T under the null hypothesis depend upon different values of n, k and θ, implementation of simulation-based method to obtain the PCS is restricted. In the next subsection, we show that the asymptotic distribution of T under the null hypothesis is normal and it only depends on the shape parameter k. With this good property, the PCS can be approximated easily.

2.1. Asymptotic distribution of T under H_0

Throughout this subsection, assume $X \sim GA(k_0, \theta_0)$ with k_0 and θ_0 the true parameters. Under this null hypothesis, the asymptotic distribution of T is derived. All the proofs of the lemmas and theorems in this subsection are given in the Appendix. Let $g(\cdot)$ be an arbitrary measurable function. Denote by respectively $\mathrm{E}(g(X))$ and $\mathrm{Var}(g(X))$ the mean and variance of $g(X)$. We first prove that $\mathrm{E}[\log f_N(X; \mu, \sigma)]$ has a unique maximum if $X \sim GA(k_0, \theta_0)$.

Lemma 1. *Under the null hypothesis $X \sim GA(k_0, \theta_0)$, $\mathrm{E}[\log f_N(X; \mu, \sigma)]$ has a unique maximum $\mathrm{E}[\log f_N(X; \tilde{\mu}, \tilde{\sigma})]$, where $\tilde{\mu} = k_0/\theta_0$ and $\tilde{\sigma}^2 = k_0/\theta_0^2$.*

With Lemma 1, we can show that the ML estimators in (7) and (8) converge almost surely (a.s.) to the parameters values which maximize the expected log-likelihood.

Lemma 2. *Under the null hypothesis $X \sim GA(k_0, \theta_0)$, we have that*

(i) *$\hat{k} \to k_0$ a.s. and $\hat{\theta} \to \theta_0$ a.s., where $\hat{k}$ and $\hat{\theta}$ are defined in (7). In addition, k_0 and θ_0 satisfy*

$$\mathrm{E}[\log f_G(X; k_0, \theta_0)] = \max_{k,\theta} \mathrm{E}[\log f_G(X; k, \theta)].$$

(ii) *$\hat{\mu} \to \tilde{\mu}$ a.s. and $\hat{\sigma} \to \tilde{\sigma}$ a.s., where $\hat{\mu}$ and $\hat{\sigma}$ are defined in (8), and*

$$\mathrm{E}[\log f_N(X; \tilde{\mu}, \tilde{\sigma})] = \max_{\mu,\sigma} \mathrm{E}[\log f_N(X; \mu, \sigma)]$$

Define $\tilde{T} = \sum_{i=1}^{n} [\log f_G(X_i; k_0, \theta_0) - \log f_N(X_i; \tilde{\mu}, \tilde{\sigma})]$. Let AM and AV be the mean and variance of $\log f_G(X; k_0, \theta_0) - \log f_N(X; \tilde{\mu}, \tilde{\sigma})$, respectively. Given that AV is finite, the central limit theorem ensures

$$n^{-1/2}(\tilde{T} - nAM) \to^d N(0, AV). \tag{10}$$

The following lemma shows that AM and AV are finite and they only depend on k_0.

Lemma 3. *Under the null hypothesis $X \sim GA(k_0, \theta_0)$, AM and AV only depend on k_0. In addition, we have*

$$AM(k_0) = (k_0 - 1)\psi(k_0) - k_0 - \log\Gamma(k_0) + \log k_0/2 + \log(2\pi)/2 + 1/2$$

and

$$AV(k_0) = (k_0 - 1)^2 \psi_1(k_0) + (k_0^2 - 1)[\psi(k_0 + 2) - \psi(k_0)] - 4(k_0^2 - k_0)[\psi(k_0 + 1) - \psi(k_0)] + \frac{(1 + k_0)(2k_0 + 3)}{2k_0} - 4.$$

Based on Lemmas 1–3, we are ready to state the main result about the asymptotic distribution of the test statistic T under H_0.

Theorem 1. *Under the null hypothesis $X \sim GA(k_0, \theta_0)$, $\sqrt{n}(T - \mathrm{E}(T))$ is asymptotically equivalent to $\sqrt{n}(\tilde{T} - nAM)$, i.e.,*

$$n^{-1/2}(T - \mathrm{E}(T)) \sim n^{-1/2}(\tilde{T} - nAM(k_0)) \to^d N(0, AV(k_0)).$$

Having the asymptotic distribution of T, the PCS under the null hypothesis can be approximated by

$$\mathrm{PCS} = P(T > 0 | H_0) \approx \Phi\left(\frac{\sqrt{n}AM(k_0)}{\sqrt{AV(k_0)}}\right), \tag{11}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Since this approximation to the PCS only depends on k_0, and both $AM(k_0)$ and $AV(k_0)$ have explicit expressions, its calculation is very convenient. The accuracy of the PCS based on the asymptotic approximation (11) is assessed by simulation in the next subsection.

2.2. Simulation study

In hypothesis testing, we need to control the Type I error and the Type II error. In our problem, the Type I error is the probability of choosing the normal distribution when X follows the gamma distribution, and the Type II error is the probability of choosing the gamma distribution when X follows the normal distribution. Our first task is to assess the Type I error. Because the rejection region is $\{T < 0\}$, the Type I error is actually equal to $1-$PCS under the null hypothesis. We consider $k_0 = 0.5$, 2, 5, 10, 50, 100 and $n = 20$, 50, 100, 500. θ_0 is set as 1 in all the cases. The empirical PCS is computed by Monte Carlo simulation. Specifically, for each combination of k_0 and n, we generate a sample of size n from $GA(k_0, 1)$, and then compute a T. By replicating this process 10,000 times, the empirical PCS is computed as the ratio of the cases $T > 0$. Both the empirical PCS and PCS based on (11) are shown in Table 1. As can be seen, the PCS based on (11) approximates the empirical PCS very well in all the scenarios and its accuracy improves with n. On the other hand, the PCS seems to decrease as the value of shape parameter k_0 increases. This is reasonable as the gamma distribution behaves like a normal distribution when k_0 is very large.

Table 1
Probability of correct selection by assuming the data are from $GA(k_0, 1)$ based on the asymptotic results and the empirical PCS (in parenthesis).

$k_0\downarrow n\rightarrow$	20	50	100	500
0.5	0.994	1.000	1.000	1.000
	(1.000)	(1.000)	(1.000)	(1.000)
2	0.906	0.982	0.999	1.000
	(0.932)	(0.990)	(1.000)	(1.000)
5	0.798	0.906	0.971	1.000
	(0.811)	(0.919)	(0.976)	(1.000)
10	0.721	0.819	0.907	0.998
	(0.727)	(0.829)	(0.909)	(0.999)
50	0.602	0.660	0.719	0.903
	(0.603)	(0.659)	(0.727)	(0.901)
100	0.573	0.613	0.658	0.821
	(0.576)	(0.613)	(0.665)	(0.821)

Our second task is to assess the Type II error. Similar to the Type I error case, if the asymptotic distribution of T under the alternative hypothesis can be derived, the Type II error should be approximated well. Unfortunately, the asymptotic mean and variance

Table 2
Power of the hypothesis by assuming the data are from $N(\mu_0, 1)$.

$\mu_0 \downarrow n \rightarrow$	20	50	100	500
3	0.649	0.792	0.896	0.999
4	0.618	0.726	0.830	0.991
5	0.583	0.686	0.776	0.967
6	0.584	0.650	0.737	0.934
7	0.564	0.633	0.700	0.908
8	0.559	0.624	0.684	0.872
9	0.552	0.600	0.666	0.847
10	0.545	0.594	0.643	0.817

of T do not have closed forms under the alternative hypothesis, and their computation involves numerical optimization as well as simulation. Therefore, we computed the power (1−Type II error) of the hypothesis defined in (5) by simulation. Assume the process characteristic $X \sim N(\mu_0, \sigma_0^2)$. We consider $\mu_0 = 3, 4, \ldots 9, 10$ and $n = 20, 50, 100, 500$. Also let $\sigma_0 = 1$ in all the scenarios. The power of the hypothesis (5) is shown in Table 2. As we can see, the power increases with the sample size n. When n is large enough, the power seems to be satisfactory for all considered values of μ_0.

3. Statistical inference

In this section, statistical inference for the percentile-based PCI C^*_{pk} under the gamma process is considered. Recall that C^*_{pk} is defined as

$$C^*_{pk} = \min\left\{\frac{USL - X_{0.5}}{X_{0.9987} - X_{0.5}}, \frac{X_{0.5} - LSL}{X_{0.5} - X_{0.0013}}\right\}, \tag{12}$$

where X_β is the β quantile of the gamma distribution; USL and LSL are respectively the upper and lower specification limits, and they are often determined by design engineers or customers. In the next subsection, we first consider the point estimation of C^*_{pk}.

3.1. Point estimation

Let $X_1, \ldots, X_n$ be an iid random sample from $GA(k, \theta)$. Recall that the ML estimator $\hat{k}$ is the solution to the equation

$$n\log(\hat{k}/\bar{X}) - n\psi(\hat{k}) + \sum_{i=1}^{n}\log X_i = 0, \tag{13}$$

where $\bar{X} = \sum_i X_i/n$ and $\psi(x) = d\log\Gamma(x)/dx$. The above function is numerically very well behaved, and hence numerical algorithms such as the Quasi-Newton algorithm can be used to obtain $\hat{k}$. Nevertheless, the Quasi-Newton algorithm may fail to converge when both k and n are very small (e.g., $k \le 0.5$ and $n \le 10$). When such a problem arises, we could use an ML-like estimator of k as

$$\hat{k} = \frac{(n-1)\sum_i X_i}{n\sum X_i\log(X_i) - \sum\log(X_i)\sum X_i}. \tag{14}$$

This estimator of k is proposed by Ye and Chen (2016), and is shown to be unbiased and strongly consistent. Once $\hat{k}$ is obtained, $\hat{\theta}$ is then given by $\hat{k}/\bar{X}$. Based on $\hat{k}$ and $\hat{\theta}$, the ML estimator of X_β, denoted as $\hat{X}_\beta$, can also be obtained. The ML estimator of C^*_{pk} is given by

$$\hat{C}^*_{pk} = \min\left\{\frac{USL - \hat{X}_{0.5}}{\hat{X}_{0.9987} - \hat{X}_{0.5}}, \frac{\hat{X}_{0.5} - LSL}{\hat{X}_{0.5} - \hat{X}_{0.0013}}\right\}. \tag{15}$$

3.2. Lower confidence limit

It would be unwise to use only a point estimate of C^*_{pk} in practice, as the ML estimator may have a large variance (Ryan, 2011, chap.7). Because many companies have established threshold values for PCIs, the lower confidence limit is of most interest (Chang & Wu, 2008; Kotz & Lovelace, 1998; Ryan, 2011). There are two difficulties in constructing the lower confidence limit for C^*_{pk} of a gamma process. On one hand, interval estimation of the gamma quantiles X_β is not an easy task. On the other hand, C^*_{pk} is defined as the minimum of two functions and this leads to the other difficulty. In this section, we use the idea of GCI to construct the lower confidence limit for C^*_{pk} of a gamma process.

The key step in implementing GCI is to find the GPQs for the parameters (Weerahandi, 1995). To illustrate the idea of GPQ, suppose now we are interested in finding a GPQ for the shape parameter k. In this case, the rate parameter θ is treated as a nuisance parameter. The GPQ of k, namely $R_k(U;\, u, k, \theta)$, is a function of a random variable U, its observed value u and the parameters. Following Weerahandi (1995), $R_k(U;\, u, k, \theta)$ should satisfy the following two conditions.

C1. $R_k(U;\, u, k, \theta)$ has a probability distribution free of unknown parameters;
C2. The observed pivotal, defined as $R_k(u;\, u, k, \theta)$, does not depend on the nuisance parameter θ.

Now we are ready to show the GPQs for the gamma parameters. Based on Bain and Engelhardt (1975), we have

$$U_1 = 2nk\log(\bar{X}/\tilde{X}) \sim c\chi^2(\nu), \tag{16}$$

where $\bar{X} = \sum_i X_i/n$, $\tilde{X} = \prod_i X_i^{1/n}$ and $\chi^2(\nu)$ is the χ^2 distribution with ν degrees of freedom. Although the χ^2 distribution is not exact except for the cases when $k \to 0$ and $k \to \infty$, the χ^2 approximation for U_1 generally works very well (e.g., Bhaumik, Kapur, & Gibbons, 2009). The values of c and ν can be obtained by matching the first two moments of U_1 and the scaled χ^2 distribution $c\chi^2(\nu)$. Bain and Engelhardt (1975) showed that

$$\mathrm{E}(U_1) = 2nk[\psi(nk) - \psi(k) - \log(n)] \tag{17}$$

and

$$\mathrm{Var}(U_1) = 4n^2k^2[\psi_1(k)/n - \psi_1(kn)], \tag{18}$$

where $\psi(x) = d\log\Gamma(x)/dx$ and $\psi_1(x) = d^2\log\Gamma(x)/dx^2$. Since the mean and variance of the distribution $c\chi^2(\nu)$ are respectively equal to $c\nu$ and $2c^2\nu$, we have

$$c = \mathrm{E}(U_1)/\nu \quad \text{and} \quad \nu = 2\mathrm{E}^2(U_1)/\mathrm{Var}(U_1). \tag{19}$$

Note that the expressions for c and ν involve k. The ML estimator of k can be substituted to obtain the values of c and ν. By this treatment, henceforth we assume that c and ν are known given the observed data $x_1, \ldots, x_n$. The GPQ of k can then be constructed as

$$R_k(U_1;\, u_1, k) = \frac{U_1}{2n\log(\bar{x}/\tilde{x})}, \tag{20}$$

where $\bar{x}$ and $\tilde{x}$ are the arithmetic and geometric mean based on $x_1, \ldots, x_n$, respectively. It is easy to check that the $R_k(U_1;\, u_1, k)$ satisfies the two conditions for a GPQ. On one hand, the distribution of $R_k(U_1;\, u_1, k)$ only depends on the random variable U_1, and hence is free of the parameters k and θ. On the other hand, $R_k(u_1; u_1, k) = k$ is independent of θ.

For the rate parameter θ, it is known that $U_2 = 2n\theta\bar{X} \sim \chi^2(2nk)$. Conditioning on U_1, we have

$$U_2|U_1 \sim \chi^2(U_1/\log(\bar{X}/\tilde{X})), \tag{21}$$

where $U_1 \sim c\chi^2(\nu)$. Given the observed data $x_1, \ldots, x_n$, the unconditional distribution of U_2 can be obtained by integrating U_1 out as

$$F_{U_2}(u_2) = \int_0^\infty F_{\chi^2}(u_2; u_1/\log(\bar{x}/\tilde{x}))f_{U_1}(u_1; c, \nu)du_1, \tag{22}$$

where $F_{\chi^2}(x; a)$ is the CDF of the $\chi^2(a)$ distribution and $f_{U_1}(x; c, \nu)$ is the PDF of U_1 with $U_1 \sim c\chi^2(\nu)$. Therefore, the distribution of U_2 is free of the unknown parameters. A GPQ for θ can then be constructed as

$$R_\theta(U_2; u_2, \theta) = \frac{U_2}{2n\bar{x}}. \tag{23}$$

The observed pivotal $R_\theta(u_2; u_2, \theta) = \theta$ and hence the second condition of a GPQ also holds.

Because $R_k(u_1; u_1, k) = k$ and $R_\theta(u_2; u_2, \theta) = \theta$, the constructed GPQs actually belong to an important subclass of GPQs, i.e., the fiducial generalized pivotal quantities (FGPQs) (Hannig et al., 2006). A good property of FGPQs is that they can be used to construct the FGPQ of functions of the parameters. For example, the FGPQ of the gamma quantile X_β can be constructed as

$$R_{X_\beta} = Q_G(\beta; R_k, R_\theta), \tag{24}$$

where $Q_G(\cdot; k, \theta)$ denotes the quantile function of $GA(k, \theta)$; R_k and R_θ are short for $R_k(U_1; u_1, k)$ and $R_\theta(U_2; u_2, \theta)$, respectively. As a result, the GPQ for C^*_{pk} is

$$R_{C^*_{pk}} = \min\left\{\frac{USL - R_{X_{0.5}}}{R_{X_{0.9987}} - R_{X_{0.5}}}, \frac{R_{X_{0.5}} - LSL}{R_{X_{0.5}} - R_{X_{0.0013}}}\right\}. \tag{25}$$

To obtain the lower confidence limit of C^*_{pk} based on $R_{C^*_{pk}}$, one way is to find the distribution of $R_{C^*_{pk}}$ and then use its α quantile as the $100(1-\alpha)\%$ lower confidence limit. Nevertheless, since $R_{C^*_{pk}}$ is defined as the minimum of two functions, its distribution is difficult to obtain. Alternatively, the Monte Carlo simulation method can be used. The algorithm for constructing the lower confidence limit of C^*_{pk} is summarized below. Its performance will be examined by a simulation in the next subsection.

Remark 1. Among all the popular PCIs, C_{pk} is probably the most commonly used (Ryan, 2011, chap.7). Therefore, we only consider the statistical inference for the percentile-based C^*_{pk} in this study. Nevertheless, the proposed inference methods are very general and they are also applicable to other percentile-based PCIs for a gamma process, such as C^*_p, C^*_{pm} and C^*_{pmk}. For detailed definition of C^*_p, C^*_{pm} and C^*_{pmk}, we refer readers to Ryan (2011).

3.3. Simulation study

Our first simulation compares the proposed gamma PCI C^*_{pk} with two existing PCIs for the non-normal process, e.g., C_s proposed by Wright (1995) and C_{jkp} proposed by Johnson, Kotz, and Pearn (1994). On one hand, C_s can be estimated as

$$\hat{C}_s = \min\left\{\frac{USL - \bar{X}}{3S_2}, \frac{\bar{X} - LSL}{3S_1}\right\}, \tag{26}$$

where $S_1 = \sqrt{\sum_{X_i \le \bar{X}}(X_i - \bar{X})^2/n_1}$ and $S_2 = \sqrt{\sum_{X_i > \bar{X}}(X_i - \bar{X})^2/n_2}$, with n_1 the number of observations that are less than or equal to $\bar{X}$ and n_2 the number of observations that are greater than $\bar{X}$. On the other hand, the estimator of C_{jkp} is given by

$$\hat{C}_{jkp} = \min\frac{1}{3\sqrt{2}}\left\{\frac{USL - \bar{X}}{S_+}, \frac{\bar{X} - LSL}{S_-}\right\}, \tag{27}$$

where $S_- = \sqrt{\sum_{X_i \le \bar{X}}(X_i - \bar{X})^2/n}$ and $S_+ = \sqrt{\sum_{X_i > \bar{X}}(X_i - \bar{X})^2/n}$.

We generate data from the log-normal distribution $LN(\mu, \sigma^2)$ such that the logarithm of the data are normally distributed. As a result, the classical PCI C_{pk} given by (1) can be computed. Another reason for using the log-normal distribution is that the log-normal distribution does fit well to many process characteristic datasets (Ryan, 2011). This is because the log-normal distribution belongs to the Johnson family, which can fit non-normal data with a wide variety of shapes (Rodriguez, 1992). We then estimate C^*_{pk}, C_s and C_{jkp} based on the log-normal distributed data, and check the closeness of these values to C_{pk}. This criteria is widely used as an evaluation of PCIs (Tang & Than, 1999; Wu & Swain, 2001), as PCIs are often used to compare different processes. The detailed simulation settings are as follows. We set $\sigma = 1$ and consider $\mu = 3, 5, 7$. Under each μ and σ, $(USL, LSL) = (e^{\mu+s\sigma}, e^{\mu-s\sigma})$ and $s = 2, 3, 4$ are considered, in which cases the classical $C_{pk} = 0.667, 1$ and 1.333, respectively. In addition, $n = 50, 100$ and 150 are considered. The average values of C^*_{pk}, C_s and C_{jkp} based on 10,000 replications are shown in Table 3. As can be seen, C^*_{pk} is generally closer to C_{pk} than C_s and C_{jkp}, indicating that the percentile-based PCI is a better choice for the non-normal process. This finding is also consistent with the results in Wu and Swain (2001). In addition, the closeness between C^*_{pk} and C_{pk} indicates that the minimum value of C^*_{pk} in a purchasing contract may be approximately specified as the minimum value of the classical C_{pk}. Another interesting finding is that although the data are from the log-normal distribution and we fit them by the gamma distribution, the performance of C^*_{pk} is still satisfactory. This fact gives a solid support for the choice of the gamma distribution in dealing with the non-normal data.

Our next simulation is to assess the performance of Algorithm 1. We consider k = 0.5, 2, 5, 10, 50, 100 and n = 20, 50, 100. Let $\theta = 1$ in all the scenarios. For each k, we set USL and LSL as the 0.99999 and 0.00001 quantiles of $GA(k, 1)$. In other words, the expected fraction of nonconforming items of the process is as low as 0.02%. Under this setting, the true value of C^*_{pk} ranges between 1.000 and 1.355. For each combination of k and n, we generate n random variables from $GA(k, 1)$ and use Algorithm 1 to obtain a 95% lower confidence limit of C^*_{pk}. By replicating the process $10,000$ times, the coverage probabilities are estimated as the ratio of the true value of C^*_{pk} larger than the lower confidence limit. As can be seen from Table 4, the coverage probabilities are close to the nominal value 95% and the average lower confidence limit increases with the sample size n, indicating the good performance of Algorithm 1.

Algorithm 1 Constructing the lower confidence limit of C^*_{pk}.

Step 1. Based on the observed data $x_1, \dots, x_n$, compute the ML estimator $\hat{k}$ by the Quasi-Newton algorithm. If the algorithm fails to converge, approximate $\hat{k}$ by (14);
Step 2. Compute the value of c and ν by using (19);
Step 3. Generate B random variables from $c\chi^2(\nu)$, denoted as $u_1^{(b)}$, $b = 1, \dots, B$. Then we could have B realizations of R_k, denoted as $r_k^{(b)} = u_1^{(b)}/2n/\log(\bar{x}/\tilde{x})$, $b = 1, \dots, B$;
Step 4. For each $u_1^{(b)}$, generate a random variable from $\chi^2(u_1^{(b)}/\log(\bar{x}/\tilde{x}))$, denoted as $u_2^{(b)}$, $b = 1, \dots, B$. Then B realizations of R_θ are given by $r_\theta^{(b)} = u_2^{(b)}/2n/\bar{x}$, $b = 1, \dots, B$;
Step 5. With B realizations of R_k and R_θ, we could have B realizations of R_{X_β} as $r_{X_\beta}^{(b)} = Q_G(\beta; r_k^{(b)}, r_\theta^{(b)})$, $b = 1, \dots, B$. Afterwards, B realizations of $R_{C^*_{pk}}$ are given by

$$r_{C^*_{pk}}^{(b)} = \min\left\{\frac{USL - r_{X_{0.5}}^{(b)}}{r_{X_{0.9987}}^{(b)} - r_{X_{0.5}}^{(b)}}, \frac{r_{X_{0.5}}^{(b)} - LSL}{r_{X_{0.5}}^{(b)} - r_{X_{0.0013}}^{(b)}}\right\}.$$

The α quantile of $r_{C^*_{pk}}^{(b)}$, $b = 1, \dots, B$, can be treated as the $100(1-\alpha)\%$ lower confidence limit of C^*_{pk}.

Table 3
Average values of C^*_{pk}, C_s and C_{jkp} based on 10000 replications; for each combination of μ and C_{pk}, n = 50 (left), 100 (middle) and 150 (right) are considered.

		C^*_{pk}			C_s			C_{jkp}		
μ	C_{pk}	50	100	150	50	100	150	50	100	150
3	0.667	0.693	0.611	0.563	0.499	0.424	0.453	0.762	0.686	0.716
	1.000	1.023	0.971	0.942	0.732	0.719	0.751	0.833	0.806	0.831
	1.333	1.262	1.287	1.237	0.957	1.043	1.001	0.983	0.976	0.983
5	0.667	0.695	0.643	0.603	0.525	0.507	0.496	0.793	0.769	0.753
	1.000	1.114	1.064	1.030	0.818	0.810	0.808	0.878	0.877	0.871
	1.333	1.358	1.353	1.274	1.053	1.047	1.062	1.042	1.048	1.059
7	0.667	0.697	0.743	0.693	0.617	0.509	0.518	0.825	0.797	0.759
	1.000	1.189	1.123	1.118	0.829	0.860	0.889	0.977	0.914	0.943
	1.333	1.414	1.435	1.317	1.149	1.125	1.071	1.076	1.122	1.072

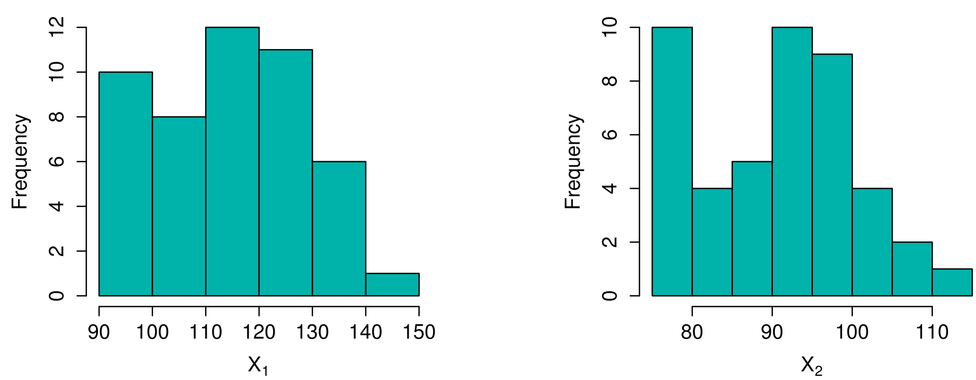

Fig. 1. Histogram plots of the 1.88 millimetre drill lifetimes from two suppliers.

Table 4
Coverage probabilities and average lower confidence limits (in parenthesis) of C^*_{pk}.

k	C^*_{pk}	$n = 20$		$n = 50$		$n = 100$	
0.5	1.000	0.976	(0.819)	0.975	(0.974)	0.973	(0.999)
2	1.029	0.963	(0.906)	0.959	(1.001)	0.964	(1.013)
5	1.132	0.955	(0.943)	0.957	(1.045)	0.952	(1.073)
10	1.213	0.954	(0.961)	0.957	(1.080)	0.953	(1.123)
50	1.328	0.962	(0.984)	0.958	(1.123)	0.955	(1.191)
100	1.355	0.966	(0.985)	0.963	(1.131)	0.956	(1.204)

4. Illustrative example

In this section, an example from a manufacturing process is used to illustrate the usefulness of our proposed methods. The factory uses drilling machines for production, and drill is one of the important components in the drilling machine. Drills of different sizes are needed in the production process. In this study, we use lifetime data from drills of size 1.88 millimetre for demonstration. Currently, the factory purchases the 1.88 millimetre drills from two different suppliers. To make subsequent purchasing decision, the factory wants to know which supplier is more reliable based on their PCIs. The lifetime of the drill is treated as the quality characteristic, and the lifetimes from the two suppliers are shown in Table 5. According to the process record, all the data are collected under normal manufacturing operations.

We first need to select an appropriate distribution for the drill lifetime. The histogram plots of these two lifetime datasets in Fig. 1 seem to be a bit skewed, indicating that the gamma distribution may be a good choice. However, model selection based on the histograms is very subjective. In fact, we are not sure whether the gamma distribution is indeed more appropriate than the normal distribution or to which extent it is more appropriate than the normal distribution. On the other hand, the proposed model discrimination method can be used here. For the drill lifetimes from the first supplier, the ML estimates of the parameters are $(\hat{k}_1, \hat{\theta}_1) = (72.37, 0.629)$ corresponding to the gamma distribution, and $(\hat{\mu}_1, \hat{\sigma}_1) = (115.1, 13.48)$ corresponding to the normal distribution. The test statistic T_1 defined in (9) is then equal to 0.0186, in favour of the gamma distribution. Based on (11), the PCS is approximately equal to 0.632 if the data is actually from a gamma distribution. In addition, the power of the test is equal to 0.604. Similarly, we have $(\hat{k}_2, \hat{\theta}_2) = (90.03, 0.985)$ and $(\hat{\mu}_2, \hat{\sigma}_2) = (91.42, 9.614)$ for drill lifetimes from the second supplier, leading to the test statistic $T_2 = 0.061 > 0$. The approximate PCS is 0.614 if the drill lifetime follows a gamma distribution and the power of the test is equal to 0.589. Therefore, all the aforementioned values of the PCS and power indicate that the gamma distribution is more appropriate for the drill lifetimes than the normal distribution.

After the gamma distribution has been selected, the next task is to compare the PCIs of the two suppliers in terms of the 1.88mm drill. Assume $LSL = 60$ determined by the company while no restriction is put on USL, i.e., $USL = \infty$. Based on (15), the estimated C^*_{pk} is 1.516 for the first supplier and 1.196 for the second supplier. For comparison purpose, here we also considered the Box-Cox power transformation to the lifetime data. Box and Cox (1964) proposed a general method to transform X to normality by

$$X^{(\lambda)} = \begin{cases} \dfrac{X^\lambda - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ \log(X), & \text{if } \lambda = 0. \end{cases} \tag{28}$$

The value of parameter λ is determined such that the likelihood function is maximized. For a detailed procedure, we refer readers to Tang and Than (1999). In terms of our lifetime data, it is found that $\lambda_1 = 0.637$ for the first dataset and $\lambda_2 = 0.486$ for the second

Table 5
Lifetimes (in minutes) of 1.88 millimetre drill from two suppliers.

X_1	135	98	114	137	138	144	99	93	115	106	132	122	94	98	127
	122	102	133	114	120	93	126	119	104	119	114	125	107	98	117
	111	106	108	127	126	135	112	94	127	99	120	120	121	122	96
	109	123	105												
X_2	105	105	95	87	112	80	95	97	77	103	78	87	107	96	79
	91	108	97	80	76	92	85	76	96	77	80	100	94	82	104
	91	95	93	99	99	94	84	99	91	85	86	79	89	89	100

dataset. As a result, the values of classical PCI C_{pk} are 1.493 and 1.183 for the two suppliers, which conform well with the values of gamma C^*_{pk}.

At last, we consider the lower confidence limits of C^*_{pk}. Based on Algorithm 1, they are respectively 1.271 and 0.986 for the two suppliers. Therefore, both the point estimates and the lower confidence limits of C^*_{pk} indicate the first supplier is more reliable. This valuable information can help the factory in making further purchasing decisions.

5. Conclusion

When the distribution of a process is skewed, the gamma distribution has appeared to be a popular alternative to the normal distribution. In this study, a hypothesis test for discrimination between the gamma and the normal distributions has been carried out. We have proved that the test statistic is asymptotically normally distributed under the gamma process hypothesis. According to our simulation, the PCS based on the asymptotic distribution is quite accurate even under small sample sizes. We then considered statistical inference for the PCIs of a gamma process. The ML method has been used for the point estimation. In practice, the interval estimation is more difficult but also more important. The method of GCIs has been successfully used for constructing the lower confidence limit of the percentile-based PCIs, and a simulation study has revealed its good performance. At last, a practical example has shown the usefulness of the proposed model discrimination and inference methodologies.

In this study, we mainly focused on the gamma distribution due to its flexibility and wide applications. Although the gamma distribution does provide an adequate fit to many process characteristic data, sometimes we may pursue the most accurate model. In fact, other distributions in the Pearson family of distributions or the Johnson family may also be good candidates for some process characteristic data (e.g., Rodriguez, 1992; Ye & Tang, 2016). When there are more than two candidates, we would suggest first selecting two models using some criterion and then carrying out a similar hypothesis test as in this study for discrimination between the selected two distributions. However, such a selection process may pose difficulties in statistical inference for the final model. On one hand, the number of parameters may be larger than two in the selected distribution. On the other hand, it is always a difficult problem to construct a lower confidence limit of the PCIs based on the selected distribution. This important problem is beyond the scope of this article and deserves further research.

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Appendix

A1. Proof of Lemma 1

Based on the PDF of $N(\mu, \sigma^2)$, we have

$$\begin{aligned}\mathrm{E}[\log f_N(X;\mu,\sigma)] &= \mathrm{E}\left[-\frac{(X-\mu)^2}{2\sigma^2} - \log(\sigma\sqrt{2\pi})\right] \\ &= -\frac{\mathrm{E}(X^2)}{2\sigma^2} + \frac{\mu\mathrm{E}(X)}{\sigma^2} - \frac{\mu^2}{2\sigma^2} \\ &\quad - \log(\sigma) - \log(\sqrt{2\pi}).\end{aligned}$$

Since $X\sim GA(k_0,\ \theta_0)$, we have $\mathrm{E}(X^2) = k_0(k_0+1)/\theta_0^2$ and $\mathrm{E}(X) = k_0/\theta_0$. Therefore,

$$\begin{aligned}\mathrm{E}[\log f_N(X;\mu,\sigma)] &= -\frac{k_0(1+k_0)}{2\theta_0^2\sigma^2} + \frac{k_0\mu}{\theta_0\sigma^2} - \frac{\mu^2}{2\sigma^2} \\ &\quad - \log(\sigma) - \log(\sqrt{2\pi}).\end{aligned}$$

Let $\Delta(\mu,\sigma) = \mathrm{E}[\log f_N(X;\mu,\sigma)]$. The first partial derivatives of $\Delta(\mu,\ \sigma)$ with respect to μ, σ are

$$\frac{\partial\Delta}{\partial\mu} = \frac{k_0}{\theta_0\sigma^2} - \frac{\mu}{\sigma}$$

and

$$\frac{\partial\Delta}{\partial\sigma} = \frac{k_0(1+k_0)}{\theta_0^2\sigma^3} - \frac{2k_0\mu}{\theta_0\sigma^3} + \frac{\mu^2}{\sigma^3} - \frac{1}{\sigma},$$

respectively. Setting these two equations to 0, the solutions of μ and σ are given as $\tilde{\mu} = k_0/\theta_0$ and $\tilde{\sigma}^2 = k_0/\theta_0^2$. In addition, it is easy to check that $\partial^2\Delta/\partial\mu^2 < 0$, $\partial^2\Delta/\partial\sigma^2 < 0$ and

$$\frac{\partial^2\Delta}{\partial\mu^2}\frac{\partial^2\Delta}{\partial\sigma^2} - \left(\frac{\partial^2\Delta}{\partial\mu\partial\sigma}\right)^2 > 0$$

at the point $(\tilde{\mu},\tilde{\sigma})$. Therefore, $(\tilde{\mu},\tilde{\sigma})$ is the global unique maximum point of $\mathrm{E}[\log f_N(X;\ \mu,\ \sigma)]$.

A2. Proof of Lemma 2

Since $X\sim GA(k_0,\ \theta_0)$, part (i) of Lemma 2 is obvious (e.g., White, 1982a). For part (ii), we need to verify Assumption A1-A3 in White (1982a). Because the PDFs of the gamma distribution and the normal distribution are well defined, the first two assumptions are satisfied. In addition, it is easy to see that $\mathrm{E}[\log f_N(X;\ \mu,\ \sigma)]$ exists given $(\mu,\ \sigma)$ in a compact set. Also, Lemma 1 ensures that $\mathrm{E}[\log f_N(X;\ \mu,\ \sigma)]$ has a unique maximum at $(\tilde{\mu},\tilde{\sigma})$. Therefore, Assumption A3 in White (1982a) is satisfied. Then part (ii) follows from Theorem 2.1 in White (1982a).

A3. Proof of Lemma 3

We first show that the distribution of $\varpi(k_0,\theta_0) = \log f_G(X;k_0,\theta_0) - \log f_N(X;\tilde{\mu},\tilde{\sigma})$ is independent of θ_0. Based on Lemma 1, we have $\tilde{\mu} = k_0/\theta_0$ and $\tilde{\sigma}^2 = k_0/\theta_0^2$. After a bit manipulation, it can be shown that

$$\varpi(k_0,\theta_0) = (k_0-1)\log(\theta_0 X) - \theta_0 X + \left(\frac{\theta_0 X - k_0}{2k_0}\right)^2 - \log\Gamma(k_0) + \frac{\log(2\pi k_0)}{2},$$

and hence $\varpi(k_0, \theta_0)$ only depends on k_0 and $\theta_0 X$. On the other hand, we know that $\theta_0 X \sim GA(k_0, 1)$ if $X \sim GA(k_0, \theta_0)$. Therefore, the distribution of $\varpi(k_0, \theta_0)$ is independent of θ_0. Let $\theta_0 = 1$ and $X \sim GA(k_0, 1)$. Then

$$\varpi(k_0,1) = (k_0-1)\log(X) + \frac{X^2}{2k_0} - 2X + \frac{k_0}{2} - \log\Gamma(k_0) + \frac{\log(2\pi k_0)}{2}.$$

Since $\mathrm{E}[\log(X)] = \psi(k_0)$, $\mathrm{E}(X^2) = k_0(k_0+1)$ and $\mathrm{E}(X) = k_0$ for $X \sim GA(k_0, 1)$, we have

$$AM(k_0) = \mathrm{E}[\varpi(k_0,1)] = (k_0-1)\psi(k_0) - k_0 - \log\Gamma(k_0) + \log k_0/2 + \log(2\pi)/2 + 1/2.$$

On the other hand, it can be shown that $\mathrm{Var}(X) = k_0$, $\mathrm{Var}(X^2) = 2k_0(k_0+1)(2k_0+3)$ and $\mathrm{Var}(\log X) = \psi_1(k_0)$, where $\psi_1(x) = d^2\log\Gamma(x)/dx^2$. In addition, $\mathrm{Cov}(\log X, X^2) = k_0(k_0+1)[\psi(k_0+2) - \psi(k_0)]$, $\mathrm{Cov}(\log X, X) = k_0[\psi(k_0+1) - \psi(k_0)]$ and $\mathrm{Cov}(X, X^2) = 2k_0(1+k_0)$, where $\mathrm{Cov}(g(\cdot), h(\cdot))$ denotes the covariance between two measurable functions $g(\cdot)$ and $h(\cdot)$. Then,

$$\begin{aligned} AV(k_0) &= \mathrm{Var}[\varpi(k_0,1)] = \mathrm{Var}[(k_0-1)\log X + X^2/(2k_0) - 2X] \\ &= (k_0-1)^2\mathrm{Var}(\log X) + \frac{\mathrm{Var}(X^2)}{3k_0^2} + 4\mathrm{Var}(X) \\ &\quad + \frac{(k_0-1)\mathrm{Cov}(\log X, X^2)}{k_0} - 4(k_0-1)\mathrm{Cov}(\log X, X) \\ &\quad - \frac{2\mathrm{Cov}(X, X^2)}{k_0} \\ &= (k_0-1)^2\psi_1(k_0) + (k_0^2-1)[\psi(k_0+2) - \psi(k_0)] \\ &\quad - 4(k_0^2-k_0)[\psi(k_0+1) - \psi(k_0)] \\ &\quad + \frac{(1+k_0)(2k_0+3)}{2k_0} - 4. \end{aligned}$$

A4. Proof of Theorem 1

It suffices to verify Assumptions 2.1–2.6 in White (1982b). Assumptions 2.1–2.3 are actually equivalent to Assumption A1–A3 in White (1982a) and hence are satisfied following the proof of Lemma 2. Since $\partial\log f_G(X;\mu,\sigma)/\partial\mu$ and $\partial\log f_G(X;\mu,\sigma)/\partial\sigma$ are well defined, Assumption 2.4 is satisfied. Define

$$A(k,\theta) = \mathrm{E}\begin{pmatrix} \dfrac{\partial^2\log f_G}{\partial k^2} & \dfrac{\partial^2\log f_G}{\partial k\partial\theta} \\ \dfrac{\partial^2\log f_G}{\partial\theta\partial k} & \dfrac{\partial^2\log f_G}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} -\psi_1(k) & \dfrac{1}{\theta} \\ \dfrac{1}{\theta} & -\dfrac{k}{\theta^2} \end{pmatrix}$$

and

$$B(k,\theta) = \mathrm{E}\begin{pmatrix} \dfrac{\partial\log f_G}{\partial k}\dfrac{\partial\log f_G}{\partial k} & \dfrac{\partial\log f_G}{\partial k}\dfrac{\partial\log f_G}{\partial\theta} \\ \dfrac{\partial\log f_G}{\partial k}\dfrac{\partial\log f_G}{\partial\theta} & \dfrac{\partial\log f_G}{\partial\theta}\dfrac{\partial\log f_G}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \psi_1(k) & -\dfrac{1}{\theta} \\ -\dfrac{1}{\theta} & \dfrac{k}{\theta^2} \end{pmatrix}.$$

Since $A(k, \theta)$ and $B(k, \theta)$ are well defined, Assumption 2.5 is satisfied. At last, it is readily seen that

$$A^{-1}(k_0,\theta_0) = -B^{-1}(k_0,\theta_0) = \frac{1}{1-k_0\psi_1(k_0)}\begin{pmatrix} k_0 & \theta_0 \\ \theta_0 & \theta_0^2\psi_1(k_0) \end{pmatrix},$$

and hence Assumption 2.6 is satisfied. Based on Lemma 3, the variance of $\log f_G(X;k_0,\theta_0) - \log f_N(X;\tilde{\mu},\tilde{\sigma})$ is finite. Theorem 1 then follows from the arguments in White (1982b).

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