Solutions

请同学先独立完成练习，答案仅供参考！

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Chp2

1. Show that the length of a free vector is not changed by rotation, i.e., that

$$\|v\| = \|Rv\|$$

Notice that $\|v\|^2 = v^T v \Rightarrow \|v\| = +\sqrt{v^T v}$. Therefore,

$$\begin{aligned} \|Rv\| &= +\sqrt{(Rv)^T Rv} = \sqrt{v^T R^T Rv} \\ &= \sqrt{v^T v} = \|v\| \end{aligned}$$

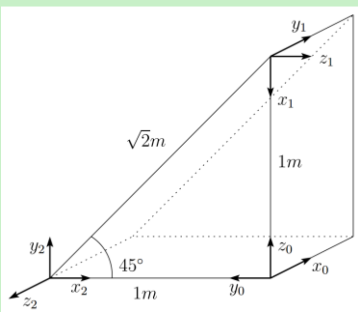
2. Show that the distance between points is not changed by rotation i.e.,

$$\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$$

This follows from Exercise 1 with $v = p_1 - p_2$

Chp2

3. Consider the diagram of right figure. Find the homogeneous transformations ${}^{0}T_{1}$, ${}^{0}T_{2}$, ${}^{1}T_{2}$ representing the transformations among the three frames shown. Show that ${}^{0}T_{2} = {}^{0}T_{1} \bullet {}^{1}T_{2}$.



$${}^{0}T_{2} = \begin{bmatrix} {}^{0}R_{2} & {}^{0}t_{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}T_{1} = \begin{bmatrix} {}^{0}R_{1} & {}^{0}t_{1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}T_{2} = \begin{bmatrix} {}^{1}R_{2} & {}^{1}t_{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chp3

1. Express the incremental rotation ${}^{B}\boldsymbol{R}_{\Delta}$ as an exponential series and verify $\boldsymbol{R}_B\langle t+\delta_t\rangle \approx \boldsymbol{R}_B\langle t\rangle + \delta_t \boldsymbol{R}_B\langle t\rangle[\boldsymbol{\omega}]_{\times}$.

$$\boldsymbol{R}_B\langle t+\delta_t\rangle = \boldsymbol{R}_B\langle t\rangle\,{}^{B}\boldsymbol{R}_{\Delta}$$

$${}^{B}\boldsymbol{R}_{\Delta} = e^{[{}^{B}\boldsymbol{\omega}]_{\times}\delta_t} = \sum_{k=0}^{\infty}\frac{\left([{}^{B}\boldsymbol{\omega}]_{\times}\delta_t\right)^k}{k!} \approx \boldsymbol{I} + [{}^{B}\boldsymbol{\omega}]_{\times}\delta_t$$

Chp3

2. Suppose that $\boldsymbol{R}(t) = \boldsymbol{R}_x(\theta(t))$. Compute $d\boldsymbol{R}(t)/dt$ directly using the chain rule. And show $d\boldsymbol{R}(t)/dt = [\omega]_\times \boldsymbol{R}(t)$

$$\frac{dR_x(\theta(t))}{dt} = \frac{dR_x(\theta)}{d\theta}\frac{d\theta(t)}{dt} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta(t))\omega_x & -\cos(\theta(t))\omega_x \\ 0 & \cos(\theta(t))\omega_x & -\sin(\theta(t))\omega_x \end{bmatrix}$$

$$S(\omega)R_x(\theta(t)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\omega_x \sin(\theta(t)) & -\omega_x \cos(\theta(t)) \\ 0 & \omega_x \cos(\theta(t)) & -\omega_x \sin(\theta(t)) \end{bmatrix}$$

Chp6

Write down Kalman filter for a robot moving on a one-dimensional straight line for the following tasks:

1) Dead Reckoning
2) Localizing with a map
3) Creating a map
4) Localization and mapping

Answer: 1) Dead Reckoning

We need to estimate the coordinate of robot x_v

$x_v\langle k+1\rangle = x_v\langle k\rangle + \delta_d + v_d$

v_d is a zero-mean Gaussian process with variance σ_d^2

a) Kalman filter prediction equations

$\hat{x}_v\langle k+1|k\rangle = \hat{x}_v\langle k|k\rangle + \delta_d$

$\hat{P}\langle k+1|k\rangle = \hat{P}\langle k|k\rangle + \sigma_d^2$

b) Kalman filter update equations

$\hat{x}_v\langle k+1|k+1\rangle = \hat{x}_v\langle k+1|k\rangle$

$\hat{P}\langle k+1|k+1\rangle = \hat{P}\langle k+1|k\rangle$

Answer: 2) Localizing with a map

Sensor model:
$z\langle k\rangle = x_i - x_v\langle k\rangle + w_r$ (Here we assume that x_i is always larger than x_v)
w_r is a zero-mean Gaussian process with variance σ_r^2.

a) Prediction step

$\hat{x}_v\langle k+1|k\rangle = \hat{x}_v\langle k|k\rangle + \delta_d$

$\hat{P}\langle k+1|k\rangle = \hat{P}\langle k|k\rangle + \sigma_d^2$

b) Update step

$v = z\langle k+1\rangle - (x_i - \hat{x}_v\langle k+1|k\rangle)$

$K = \hat{P}\langle k+1|k\rangle\left(\hat{P}\langle k+1|k\rangle + \sigma_r^2\right)^{-1}$

$\hat{x}\langle k+1|k+1\rangle = \hat{x}\langle k+1|k\rangle + Kv$

$\hat{P}\langle k+1|k+1\rangle = \hat{P}\langle k+1|k\rangle - K\hat{P}\langle k+1|k\rangle$

Answer: 3) Creating a map (with perfect localization)

We need to estimate the coordinates of the landmarks

$$x = (x_1, x_2, \cdots, x_M)^T$$

a). Prediction step

$\hat{x}\langle k+1|k\rangle = \hat{x}\langle k|k\rangle$

$\hat{P}\langle k+1|k\rangle = \hat{P}\langle k|k\rangle$

b). Update step

b1). When an old landmark i is observed

$H_x = (0 \cdots 1 \cdots 0), H_w = 1$

$S = H_x \hat{P}\langle k+1|k\rangle H_x^T + H_w \widehat{W} H_w^T = \hat{P}_{ii}\langle k+1|k\rangle + \sigma_r^2$

$K = \hat{P}\langle k+1|k\rangle H_x^T S\langle k+1\rangle^{-1} = \dfrac{1}{S}\left[0 \cdots \hat{P}_{ii}\langle k+1|k\rangle \cdots 0\right]^T$

Here we will $\hat{P}_{ii}\langle k+1|k\rangle$ to represent the element at row i and column i.

$\hat{x}\langle k+1|k+1\rangle = \hat{x}\langle k+1|k\rangle + Kv$

$\hat{P}\langle k+1|k+1\rangle = \hat{P}\langle k+1|k\rangle - KH_x\hat{P}\langle k+1|k\rangle$

Answer: 3) Creating a map

b). Update step

b2). When new landmark is observed

$$\hat{x}\langle k+1|k+1\rangle = \begin{pmatrix} \hat{x}\langle k+1|k\rangle \\ x_v\langle k+1\rangle + z \end{pmatrix}$$

$$\hat{P}\langle k+1|k+1\rangle = \begin{bmatrix} \hat{P}\langle k+1|k\rangle & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

Answer: 4) Localization and mapping

We need to estimate the coordinates of the vehicle and the landmarks

$$x = (x_v, x_1, x_2, \cdots, x_M)^T$$

a). Prediction equations

$$\hat{x}\langle k+1|k\rangle = \begin{bmatrix} \hat{x}_v\langle k|k\rangle + \delta_d \\ \hat{x}_{1:M}\langle k|k\rangle \end{bmatrix}$$

$$\hat{P}\langle k+1|k\rangle = \hat{P}\langle k|k\rangle + \begin{bmatrix} \sigma_d^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Answer: 4) Localization and mapping

b). Update step

b1). When an old landmark i is observed

$H_x = (1,0,\cdots,1,\cdots,0), H_w = 1$

$S = H_x \hat{P}\langle k+1|k\rangle H_x^T + H_w \widehat{W} H_w^T = \left(\hat{P}_{vv} + \hat{P}_{vi} + \hat{P}_{iv} + \hat{P}_{ii}\right) + \sigma_r^2$

$$K\langle k+1\rangle = \hat{P}\langle k+1|k\rangle H_x^T S\langle k+1\rangle^{-1} = \frac{1}{S}\begin{bmatrix} \hat{P}_{vv} + \hat{P}_{vi} \\ \hat{P}_{1v} + \hat{P}_{1i} \\ \vdots \\ \hat{P}_{Mv} + \hat{P}_{Mi} \end{bmatrix}$$

$\hat{x}\langle k+1|k+1\rangle = \hat{x}\langle k+1|k\rangle + Kv$

$\hat{P}\langle k+1|k+1\rangle = \hat{P}\langle k+1|k\rangle - KH_x\hat{P}\langle k+1|k\rangle$

Answer: 4) Localization and mapping

b). Update step

b2). When new landmark is observed

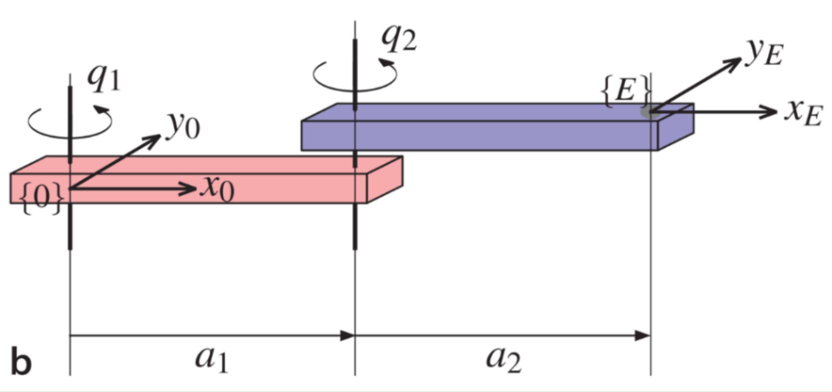
$$\hat{x}\langle k+1|k+1\rangle = \begin{pmatrix} \hat{x}\langle k+1|k\rangle \\ x_v\langle k+1\rangle + z \end{pmatrix}$$

$$\hat{P}\langle k+1|k+1\rangle = Y_z \begin{bmatrix} \hat{P}\langle k+1|k\rangle & 0 \\ 0 & \sigma_r^2 \end{bmatrix} Y_z^T$$

$$Y_z = \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & 0 \\ 0 & & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Chp7

1. Derive the forward kinematics for the 2-link robot. What is end-effector position (x, y) given link lengths (a_1, a_2) and joint angles (q_1, q_2)?



$$x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

Chp7

2. Derive the inverse kinematics for the 2-link robot. What are the joint angles (q_1, q_2) given the end-effector coordinates (x, y)?

$$x^2 = a_1^2 \cos^2\theta_1 + 2a_1a_2\cos\theta_1\cos(\theta_1+\theta_2) + a_2^2\cos^2(\theta_1+\theta_2)$$
$$y^2 = a_1^2 \sin^2\theta_1 + 2a_1a_2\sin\theta_1\sin(\theta_1+\theta_2) + a_2^2\sin^2(\theta_1+\theta_2)$$
$$\cos\theta_1\cos(\theta_1+\theta_2) + \sin\theta_1\sin(\theta_1+\theta_2) = \cos\theta_2$$

$$\theta_2 = \cos^{-1}\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}, \cos\theta_2 = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2} \coloneqq c_2, \sin\theta_2 = \pm\sqrt{1-c_2^2} \coloneqq s_2$$

$$x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) = a_1c_1 + a_2c_1c_2 - a_2s_1s_2$$
$$y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) = a_1s_1 + a_2s_1c_2 + a_2c_1s_2$$

$$\cos\theta_1 = \frac{a_1+a_2s_2(x+y)}{(a_1+a_2c_2)(a_1+a_2s_2)+a_2^2s_2^2}$$