第七章　应力和应变分析
强度理论（一）

第17讲

§ 7.1　应力状态概述

应力状态理论和强度理论的背景？

对于基本变形问题　会计算应力和变形，会校核强度和刚度

几个重要公式

基本变形

拉压　$\sigma = \dfrac{F_{\mathrm{N}}}{A}$　　$\Delta l = \dfrac{F_{\mathrm{N}} l}{EA}$　　$\varepsilon = \dfrac{\sigma}{E} = \dfrac{F_{\mathrm{N}}}{EA}$

扭转　$\tau = \dfrac{T\rho}{I_{\mathrm{p}}}$　　$\varphi = \dfrac{Tl}{GI_{\mathrm{p}}}$　　$\varphi' = \dfrac{T}{GI_{\mathrm{p}}}$　　$\gamma = \dfrac{\tau}{G}$

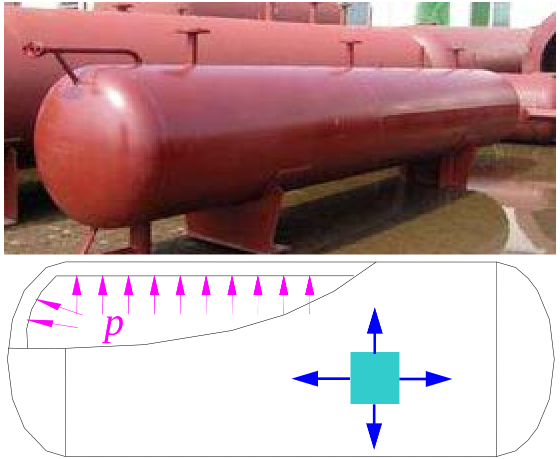
弯曲　$\sigma = \dfrac{My}{I_z}$　　$\tau = \dfrac{F_{\mathrm{S}} S_z^*}{I_z b}$　　$k = \dfrac{1}{\rho} = \dfrac{M}{EI_z}$

$EI_z w'' = M(x)$　　$\theta = w'$

两种应力状态：　单轴应力状态，　纯剪切应力状态

两类问题：静定问题，　超静定问题（简单型）

工程实际问题



双向受拉

弯+压

铸铁试样的扭转破坏断口

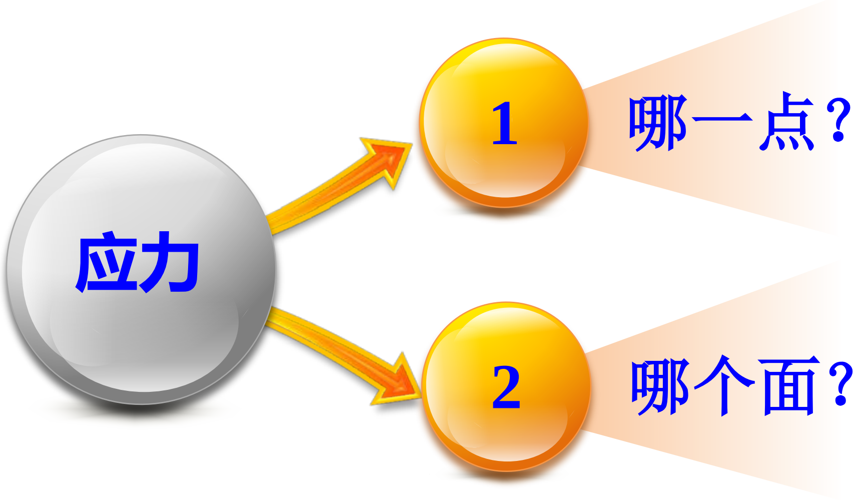
如何判断构件是否安全？

如何解释？

两个需解决的问题：

1. 各不同方位截面上的应力如何计算？ → 应力状态分析

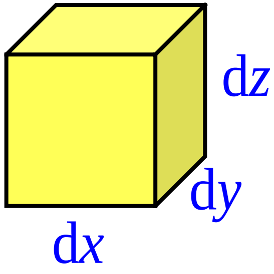
2. 强度条件怎么提？ → 强度理论



过一点所有不同方位截面上应力的全部情况称为这一点的应力状态。

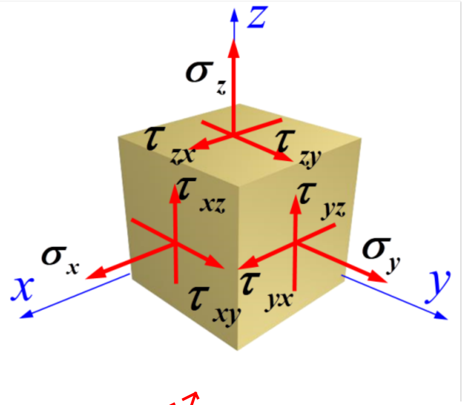
应力状态分析

微元 无穷小
直平行六面体



$dx, dy, dz \rightarrow 0$

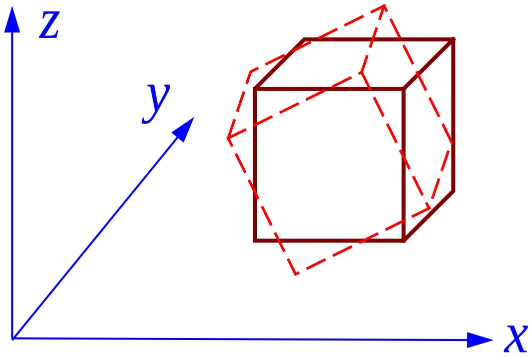
微元或单元（Element）



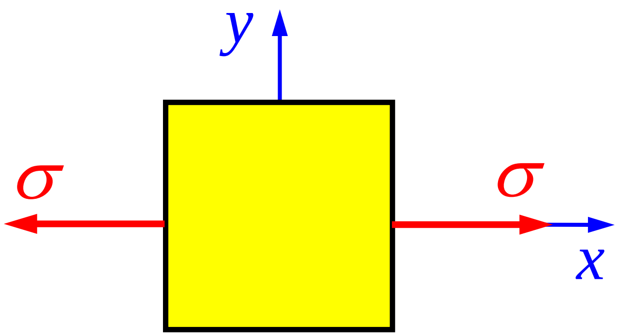
微元上的应力 一般情况六面体各面上皆有应力分量（正应力，切应力）

称为空间（三向）应力状态

一点可以用无穷个微元表示，确定它们之间各应力分量间的关系，即应力状态分析。



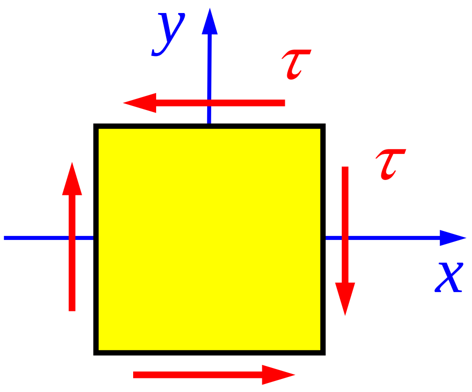
两种特殊的应力状态



单轴应力状态

Uniaxial stress

$$\sigma = E\varepsilon$$



纯剪切应力状态

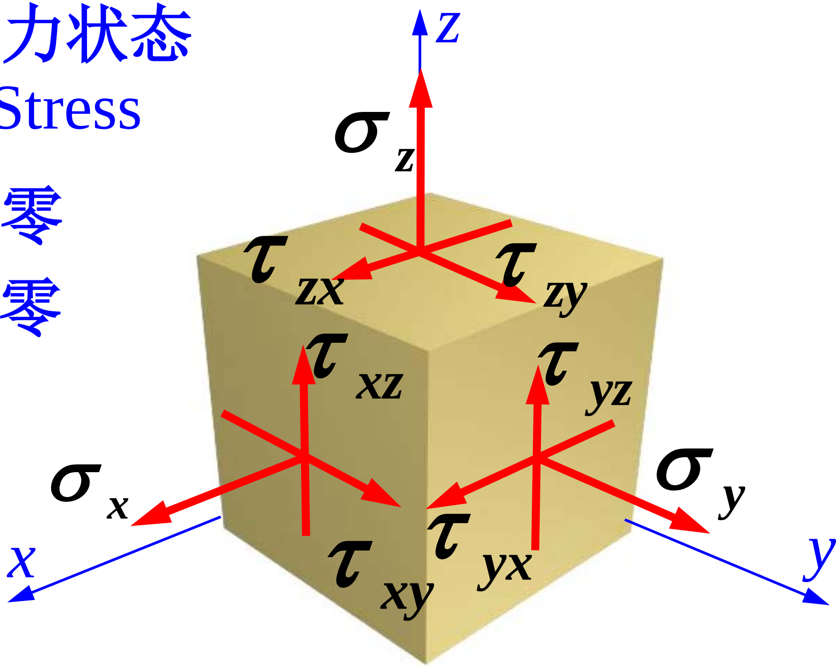
Pure shear

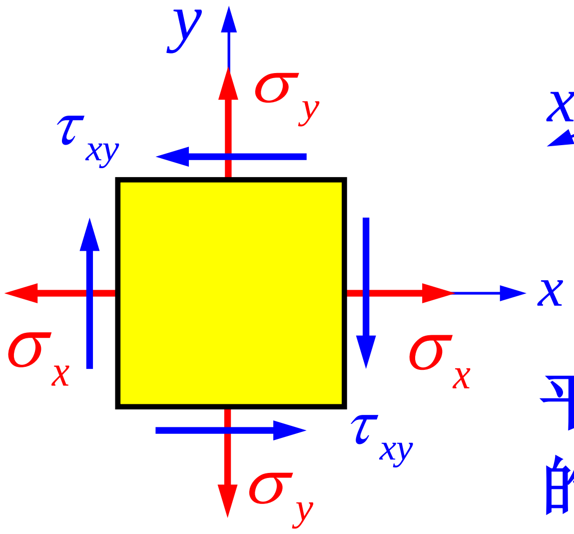
$$\tau = G\gamma$$

平面（二向）应力状态
State of Plane Stress

若单元体有一对平面上的应力等于零

平面（二向）应力状态：即不等于零的应力分量均处于同一坐标平面内





平面应力状态的单元体应力

§ 7.2　二向应力状态的实例

受内压作用的圆筒形压力容器

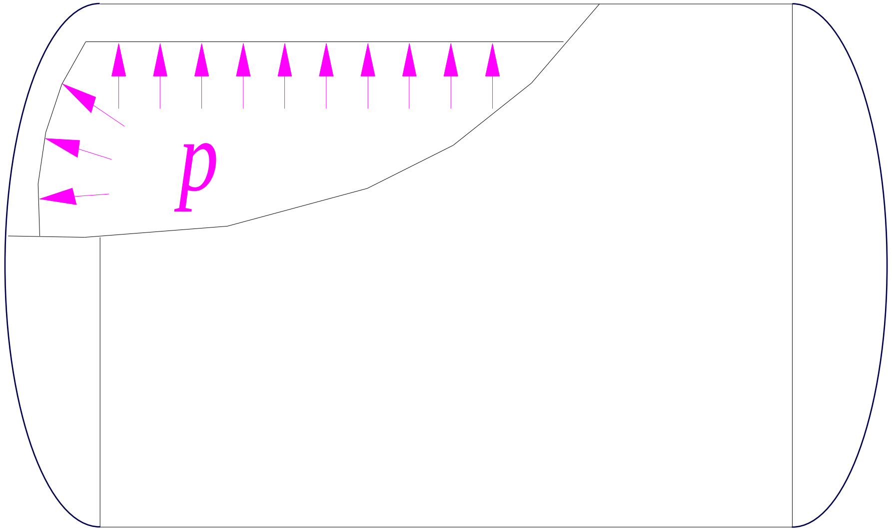
清洁能源，油气混合动力汽车

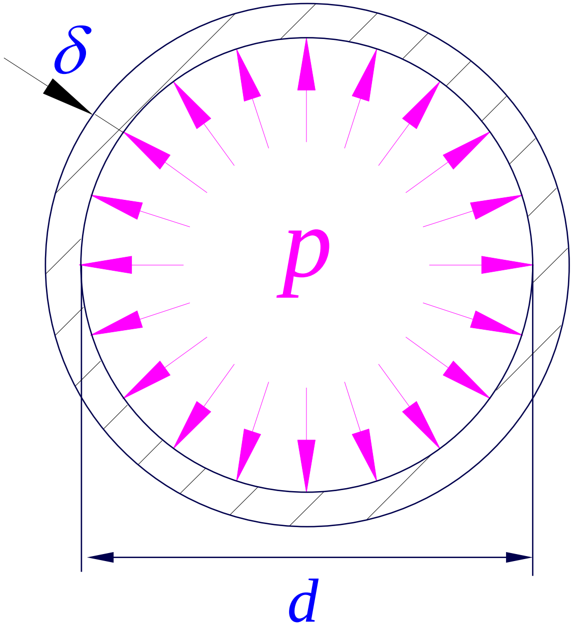
液化石油气罐

构件表面一般为自由表面，从构件表层取出的单元体就属于二向应力状态

一、受内压作用的圆筒形薄壁容器的应力

壁厚为 δ，内直径为 d， $\delta << d$ 薄壁：$\boxed{\delta < \dfrac{d}{20}}$

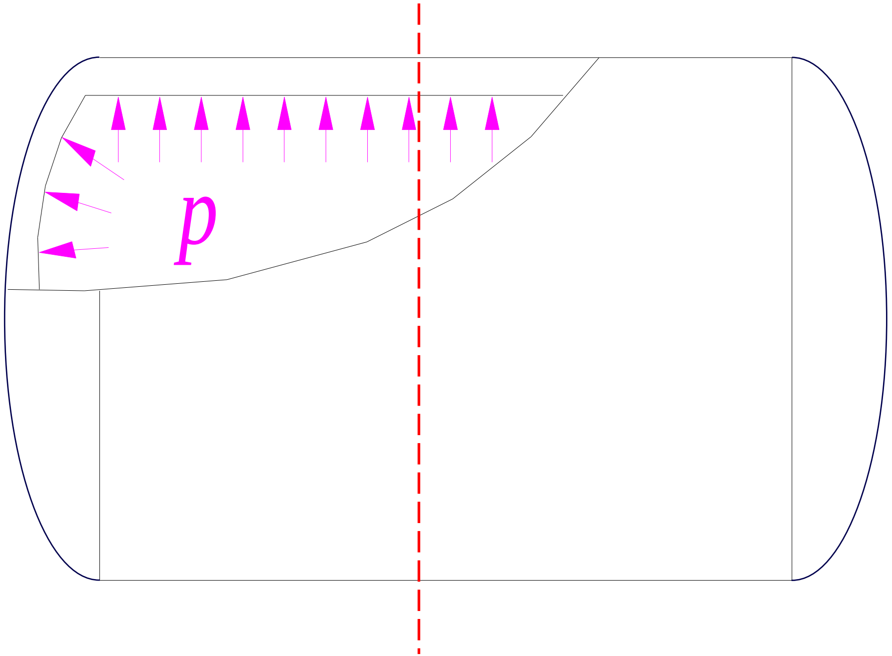


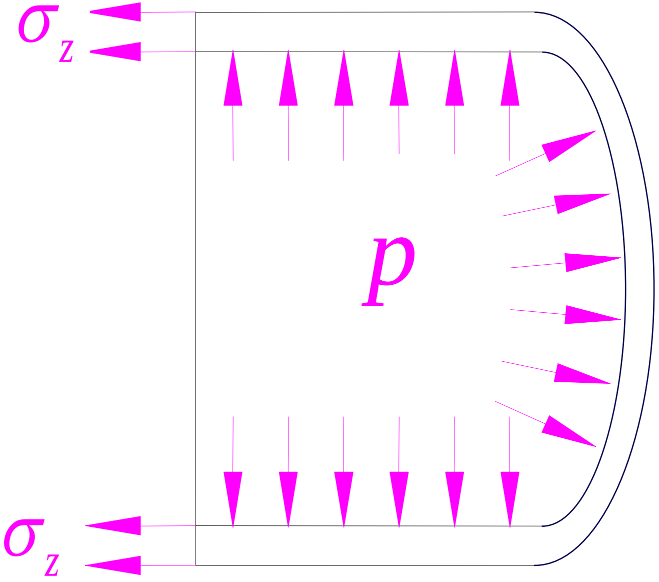


（1） 轴向应力

考虑截取部分的平衡，有

$$\sigma_z \cdot \pi d\delta = p \cdot \frac{\pi d^2}{4} \Rightarrow \sigma_z = \frac{F}{\pi d\delta} = \frac{pd}{4\delta}$$

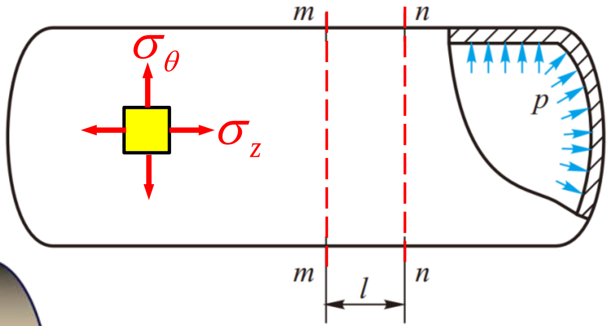




（2）环向应力（不考虑端部的影响）

截取长度为 l 的一段，再对半截开

由于**薄壁**容器，假设环向纤维的变形均相同，即σ_θ在截面上均匀分布



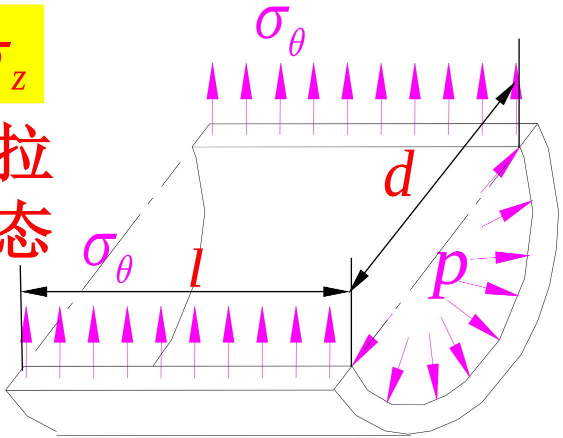
考虑截取部分的平衡，有

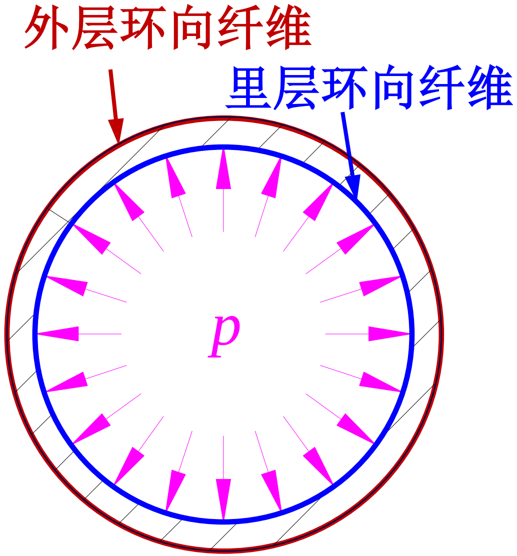
$$2\sigma_\theta \cdot l \cdot \delta - pld = 0$$

$$\sigma_z = \frac{pd}{4\delta}$$

$$\sigma_\theta = \frac{pd}{2\delta}$$

$$\boxed{\sigma_\theta = 2\sigma_z}$$





内直径d=600mm
厚度 δ=5mm

双向受拉应力状态

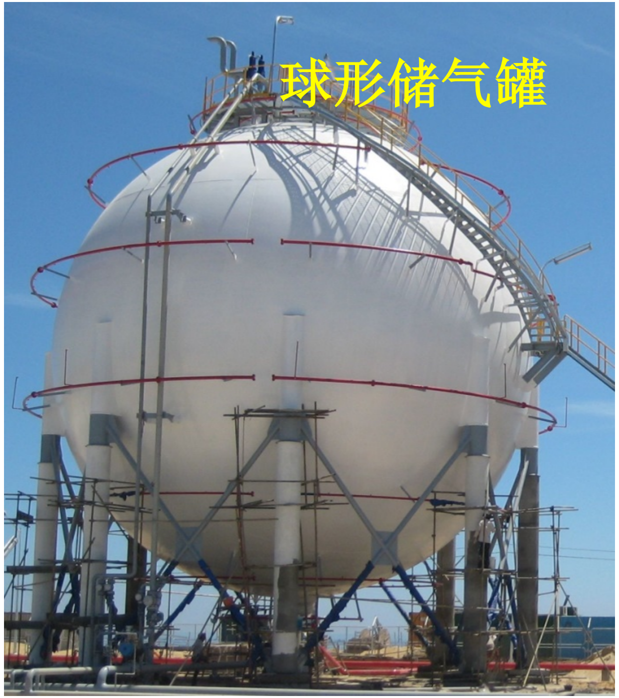
$$\sigma_\theta = \frac{pd}{2\delta} = 60p$$

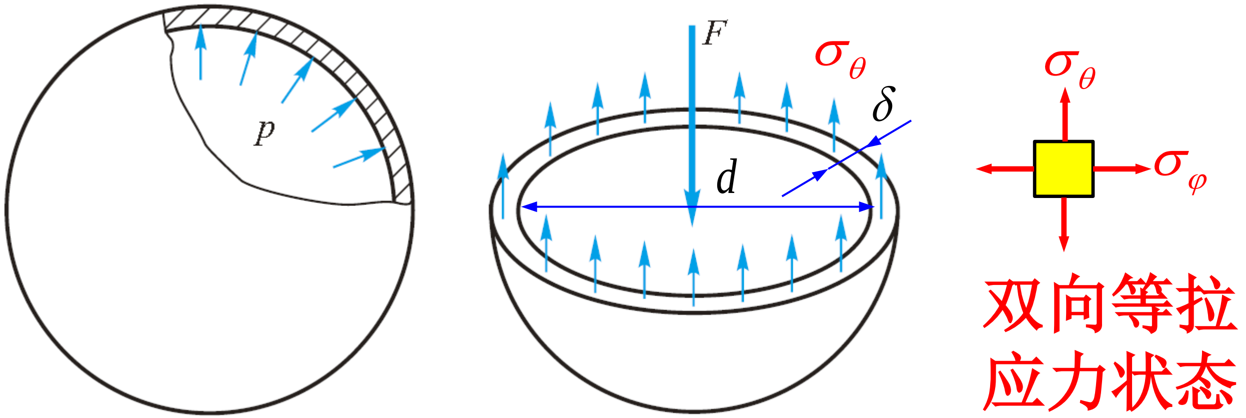
二、受内压作用的球形薄壁容器的应力

壁厚为 δ，内直径为 d，$\delta << d$

对半截开，由于薄壁容器，假设环向纤维的变形均相同，即σ_θ在截面上均匀分布

$$\sigma_\theta \cdot \pi d \cdot \delta - p\frac{\pi d^2}{4} = 0 \Rightarrow \sigma_\theta = \frac{pd}{4\delta}$$

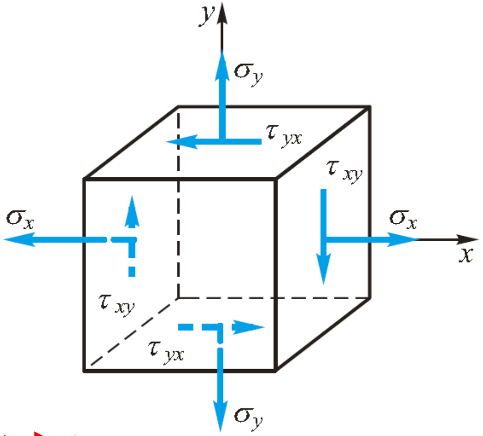


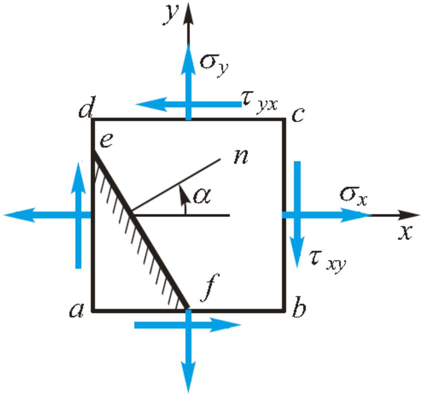


双向等拉应力状态

§ 7.3　二向应力状态分析——解析法

一般情况下，单元体的各面上的应力分量，不但有正应力还有切应力。





正负号规定

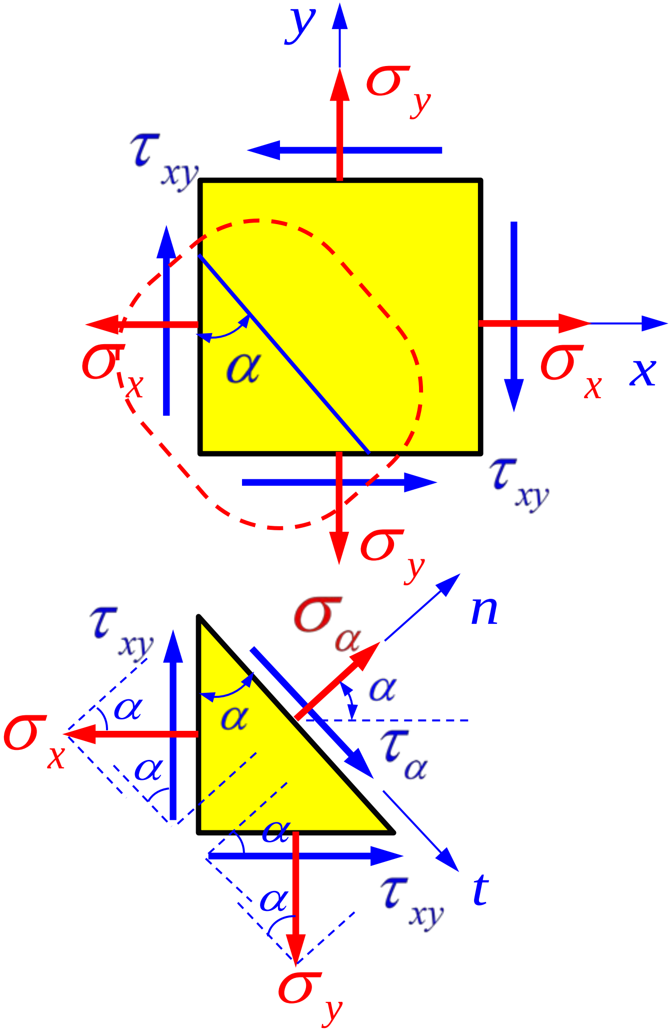
正应力：拉为正，压为负

切应力：对单元体内任意点的矩顺时针转向为正，反之为负

α 角：由 x 轴转到斜截面外法线 n 逆时针转向为正，反之为负

目标：

用一点某个微元上的应力表示其它无限多微元上的应力



一、斜截面上的应力

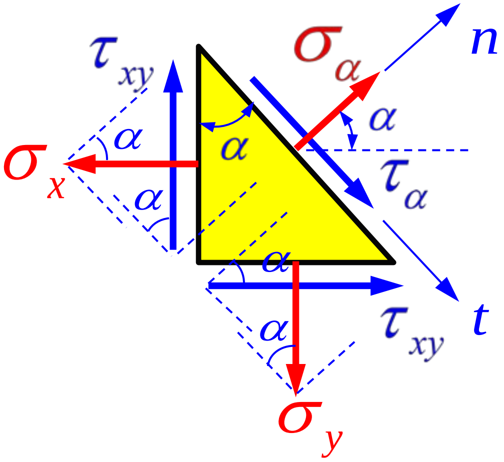
分析方法：

1. 解析方法
2. 应力圆方法

微元体的平衡方程（设斜截面面积为dA）

$$\sum F_n = 0:$$

$$\sigma_\alpha \mathrm{d}A - \sigma_x(\mathrm{d}A\cos\alpha)\cos\alpha$$
$$+\tau_{xy}(\mathrm{d}A\cos\alpha)\sin\alpha + \tau_{xy}(\mathrm{d}A\sin\alpha)\cos\alpha$$
$$-\sigma_y(\mathrm{d}A\sin\alpha)\sin\alpha = 0$$



$$\sum F_n = 0:$$

$$\sigma_\alpha \mathrm{d}A - \sigma_x(\mathrm{d}A\cos\alpha)\cos\alpha$$
$$+\tau_{xy}(\mathrm{d}A\cos\alpha)\sin\alpha + \tau_{xy}(\mathrm{d}A\sin\alpha)\cos\alpha$$
$$-\sigma_y(\mathrm{d}A\sin\alpha)\sin\alpha \;= 0$$

$$\sigma_\alpha = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sum F_t = 0:$$

$$\tau_\alpha \mathrm{d}A - \sigma_x(\mathrm{d}A\cos\alpha)\sin\alpha$$
$$-\tau_{xy}(\mathrm{d}A\cos\alpha)\cos\alpha + \tau_{xy}(\mathrm{d}A\sin\alpha)\sin\alpha$$
$$+\sigma_y(\mathrm{d}A\sin\alpha)\cos\alpha \;= 0$$

$$\tau_\alpha = \sigma_x\sin\alpha\cos\alpha - \sigma_y\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

最后，得到以下两个方程：

$$\sigma_\alpha = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_\alpha = \sigma_x \sin\alpha\cos\alpha - \sigma_y \sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

引入 $2\sin\alpha\cos\alpha = \sin 2\alpha$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

讨论：

（1）正应力的极值与方位角

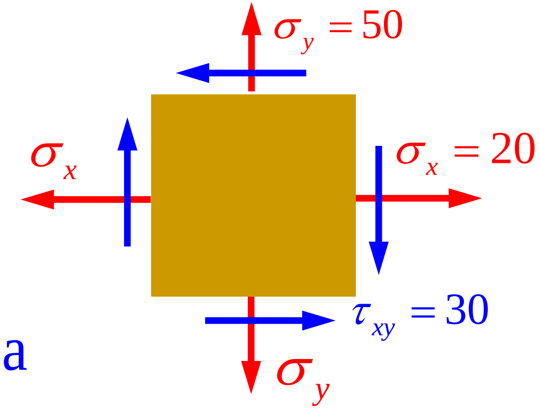
$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

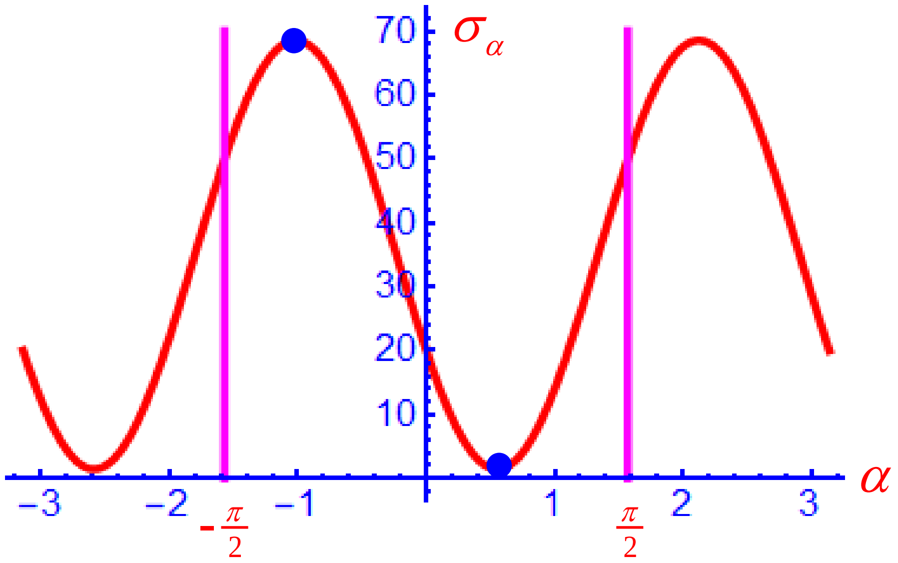
$\sigma_x = 20$

$\sigma_y = 50$

$\tau_{xy} = 30$

单位 : MPa





$[-90^\circ, 90^\circ]$区间内，有两个角度值，分别对应正应力的最大值和最小值！

求极值应力对应的角度：

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\boxed{令: \left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=\alpha_0} = 0}$$

$$-(\sigma_x-\sigma_y)\sin 2\alpha_0 - 2\tau_{xy}\cos 2\alpha_0 = 0$$

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

（$\tan 2\alpha$ 函数的周期是 $\frac{\pi}{2}$）

由此可得两个驻点：α_{01}, α_{02}　　$2\alpha_{01}, 2\alpha_{02} \in [-90°, 90°]$

$$\alpha_{02} = \begin{cases} \alpha_{01} - \frac{\pi}{2} & (\alpha_{01} > 0) \\ \alpha_{01} + \frac{\pi}{2} & (\alpha_{01} < 0) \end{cases}$$

$\alpha=\alpha_{01}$： $\sigma_{极值1} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha_{01} - \tau_{xy}\sin 2\alpha_{01}$

$\alpha=\alpha_{02}$： $\sigma_{极值2} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha_{02} - \tau_{xy}\sin 2\alpha_{02}$

将具体数值代入后，经计算，可知具体哪个角度对应的是最大值，哪个角度对应的是最小值！

例1 求图示单元体的正应力的极值与方位角。

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\times 30}{20-50} = 2$$

$$2\alpha_0 = \arctan(2) = 63.43°$$

$$\alpha_{01} = 31.72°$$

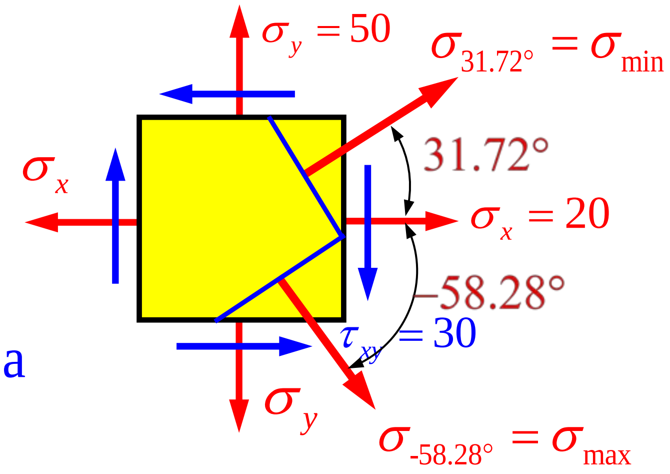
$$\alpha_{02} = 31.72° - 90° = -58.28°$$

$\sigma_x = 20$

$\sigma_y = 50$

$\tau_{xy} = 30$

单位：MPa



$$\sigma_{31.72°} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\times 31.72°) - \tau_{xy}\sin(2\times 31.72°) = 1.46\ \text{MPa}$$ 极小值

$$\sigma_{-58.28°} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(-2\times 58.28°) - \tau_{xy}\sin(-2\times 58.28°) = 68.54\ \text{MPa}$$ 极大值

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

求正应力极值和对应角度的另一种方法： 辅助角公式

$$y = a\cos\theta \pm b\sin\theta \longrightarrow y = \sqrt{a^2+b^2}\cos(\theta\pm\varphi) \quad \varphi = \arctan\left(\frac{b}{a}\right)$$

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\left[\frac{\frac{\sigma_x-\sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}\cos 2\alpha - \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}\sin 2\alpha\right]$$

$\cos 2\alpha_0$ $\sin 2\alpha_0$

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\times\cos 2(\alpha-\alpha_0)$$

函数 $\cos 2(\alpha-\alpha_0)\in[-1,1]$

极大值 $\sigma_{\alpha,\max} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$

极小值 $\sigma_{\alpha,\min} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} \times \cos 2(\alpha-\alpha_0)$$

$$\cos 2\alpha_0 = \frac{\frac{\sigma_x-\sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

【1】$2(\alpha-\alpha_0)=0$（即$\alpha=\alpha_0$）时有极大值：

$$\sigma_{\alpha,\max} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin 2\alpha_0 = -\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

【2】$2(\alpha-\alpha_0)=\pi$（即$\alpha=\alpha_0+\frac{\pi}{2}$）时有极小值：

$$\sigma_{\alpha,\min} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_x = 20\text{MPa}$

$\sigma_y = 50\text{MPa}$

$\tau_{xy} = 30\text{MPa}$

$$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = 15\sqrt{5}$$

$$\cos 2\alpha_0 = -\frac{1}{\sqrt{5}}, \quad \sin 2\alpha_0 = -\frac{2}{\sqrt{5}}$$

则角度 $\alpha_0+\frac{\pi}{2}=31.72°$ 对应正应力的极小值。

此时在[−90°, 90°]可找到唯一的角度 $\alpha_0=-58.28°$ 对应正应力的极大值。

极值应力的大小：
$$\begin{cases}\sigma_{\max}=\dfrac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\\ \sigma_{\min}=\dfrac{\sigma_x+\sigma_y}{2}-\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\end{cases}$$

极值应力与方位角间的对应关系再讨论

$$\mathrm{tg}2\alpha_0=-\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

$2\alpha_{01}\in[-90°,90°]$

定有一根

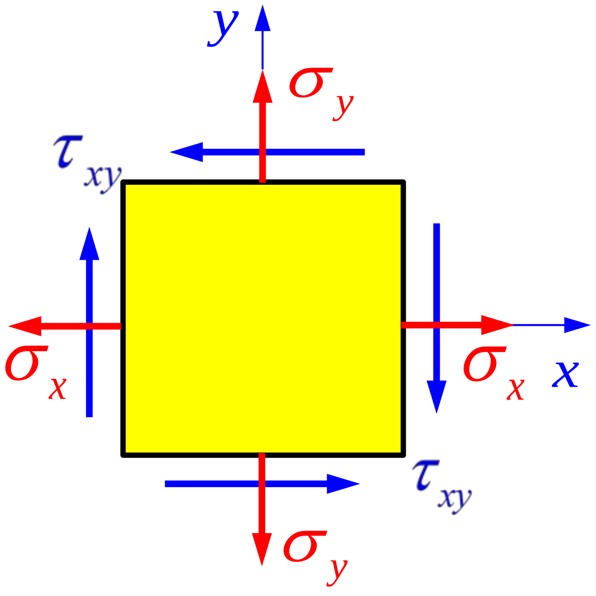
$\alpha_{01}\in[-45°,45°]$

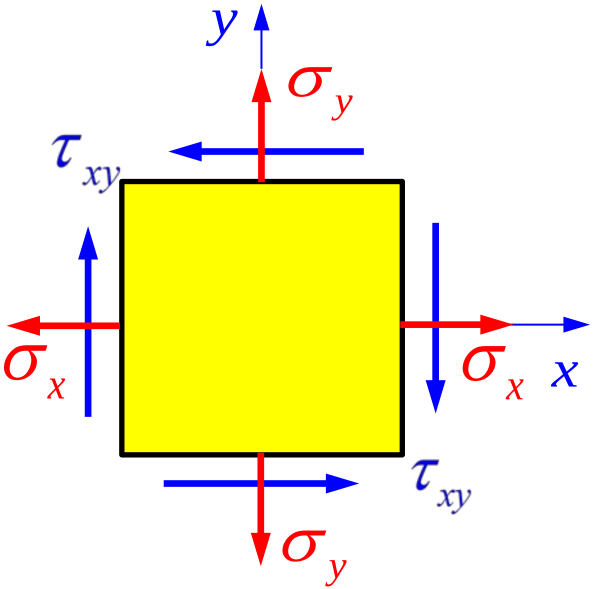
$$\alpha_{02}=\begin{cases}\alpha_{01}-\frac{\pi}{2} & (\alpha_{01}>0)\\ \alpha_{01}+\frac{\pi}{2} & (\alpha_{01}<0)\end{cases}$$

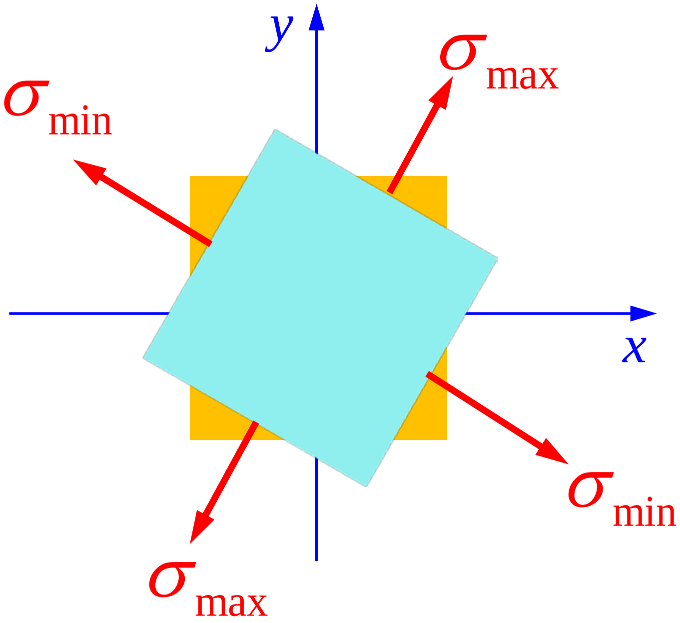
对应关系？

$\sigma_{\max}$

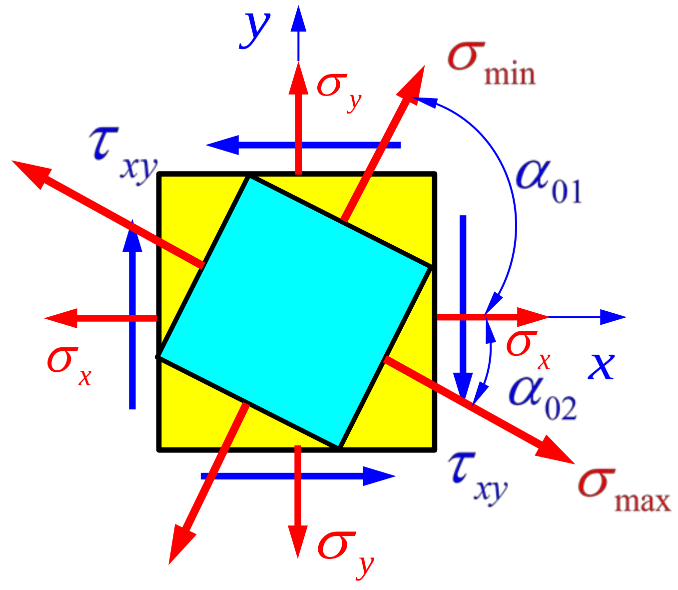
$\sigma_{\min}$

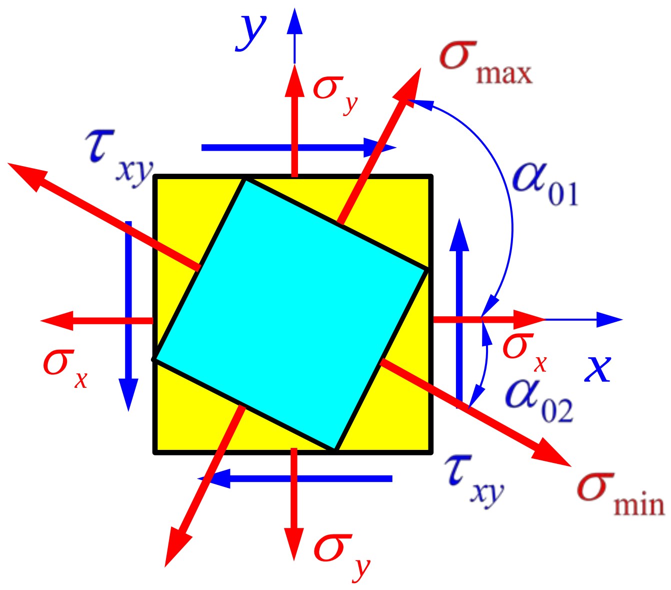






在[–90°,90°]区间内，总是有两个角度，分别对应最大正应力和最小正应力。有一种简单的判定方法！





$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$2\alpha_{01} \in [-90°, 90°]$

即 $\alpha_{01} \in [-45°, 45°]$

将 α_{02} 也限制在 $[-90°, 90°]$，取

$$\alpha_{02} = \begin{cases} \alpha_{01} + 90^0 & (\alpha_{01} < 0) \\ \alpha_{01} - 90^0 & (\alpha_{01} > 0) \end{cases}$$

（I）$\tau_{xy} > 0$ 时：

小于零的那个角（第四象限角）

对应最大正应力

（II）$\tau_{xy} < 0$ 时：

大于零的那个角（第一象限角）

对应最大正应力

重新计算例1 求图示单元体的正应力的极值与方位角。

（1）求正应力的极值

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{20+50}{2} + \sqrt{\left(\frac{20-50}{2}\right)^2 + 30^2}$$

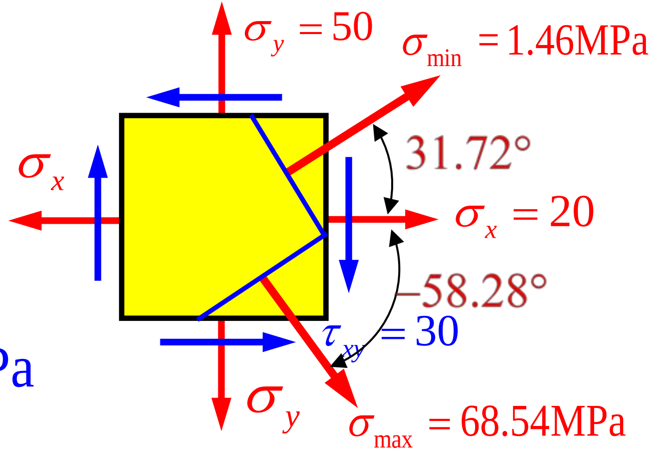
$$= 35 + 15\sqrt{5} = 68.54\text{MPa}$$

$\sigma_x = 20$

$\sigma_y = 50$

$\tau_{xy} = 30$

单位：MPa



$$\sigma_{\min} = 35 - 15\sqrt{5} = 1.46\text{MPa}$$

$\sigma_{\max} \longleftrightarrow \alpha_{02} = 31.72° - 90° = -58.28°$

$\sigma_{\min} \longleftrightarrow \alpha_{01} = 31.72°$

（2）求方位角

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\times 30}{20-50} = 2$$

$$2\alpha_0 = \arctan(2) = 63.43^0$$

二、主应力和主平面的概念

正应力为极值时的控制方程

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Leftrightarrow \frac{\sin 2\alpha_0}{\cos 2\alpha_0} = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha_0 + \tau_{xy}\cos 2\alpha_0 = 0$$

考察$\alpha = \alpha_0$ 面上的切应力：

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \Rightarrow \tau_{\alpha_0} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha_0 + \tau_{xy}\cos 2\alpha_0$$

比较得，$\tau_{\alpha_0} = 0$

一点处切应力等于零的截面称为主平面；

主平面上的正应力称为主应力。

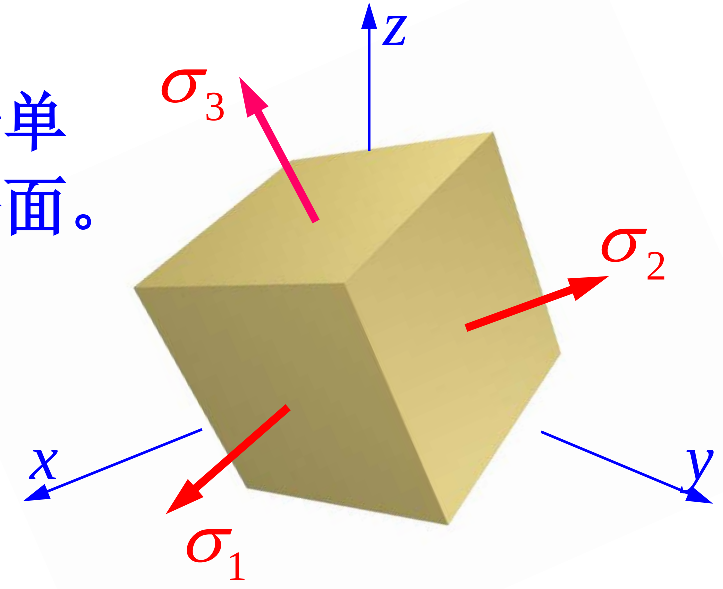
极值正应力就是主应力

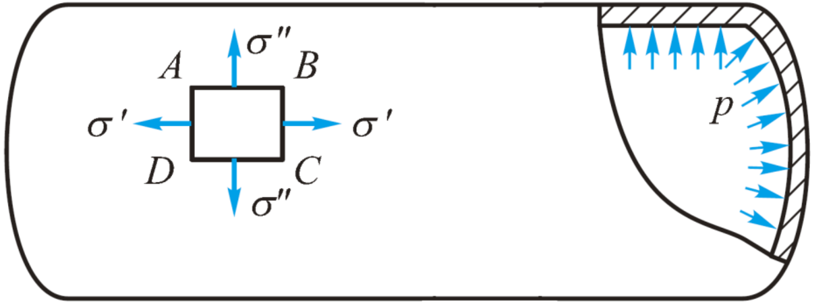
关于主应力的说明：

可以证明：一点处一定存在这样一个单元体，其三个相互垂直的面均为主平面。

主应力用σ_1、σ_2、σ_3表示

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$（按数值大小排序）





$$\sigma_1 = \frac{pd}{2\delta}, \quad \sigma_2 = \frac{pd}{4\delta}, \quad \sigma_3 = 0$$

$$\sigma' = \sigma_z = \frac{pd}{4\delta}, \quad \sigma'' = \sigma_\theta = \frac{pd}{2\delta}, \quad \sigma_r = 0$$

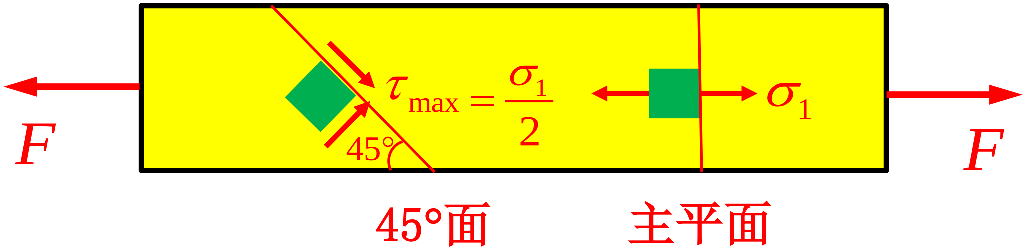
（2）切应力的极值

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \qquad \tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

令 $\left.\dfrac{d\tau_\alpha}{d\alpha}\right|_{\alpha=\alpha_1} = 0 \Rightarrow \tan 2\alpha_1 = \dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}$ 　　 $\tan 2\alpha_0 = -\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$

$$\tan 2\alpha_1 \tan 2\alpha_0 = -1$$

$$\alpha_1 = \alpha_0 + \frac{\pi}{4}$$

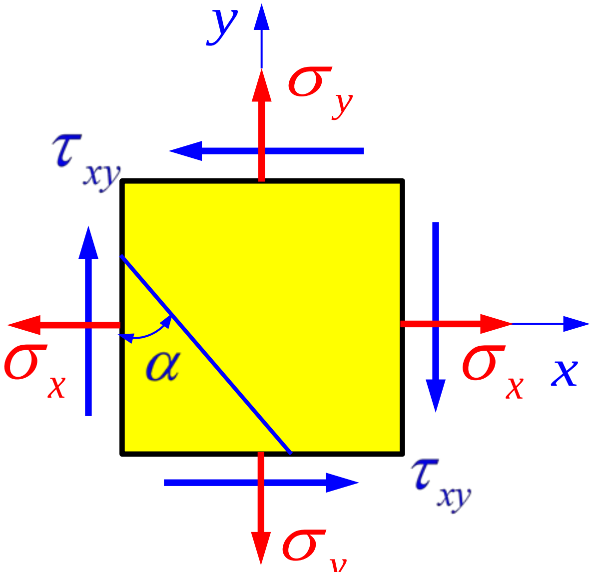


第一主应力所在平面逆时针转45°即为最大切应力所在平面（切应力为正）。

第一主应力所在平面顺时针转45°即为最小切应力所在平面（切应力为负）。

极值切应力面与主平面成45°角

小 结



斜截面上的正应力和切应力公式

$$\begin{cases} \sigma_\alpha = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_\alpha = \dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

主应力大小

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

主应力的方位角

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

α_{01}，控制 $2\alpha_{01} \in [-90°, 90°]$

$\alpha_{02} = \alpha_{01} \pm \frac{\pi}{2}$　α_{02}也限制在$[-90°, 90°]$

最大切应力的大小和方位角

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \alpha_1 = \alpha_0 + \frac{\pi}{4}$$

三、几种应力状态间的关系

划分方法：按不等于零的主应力个数。

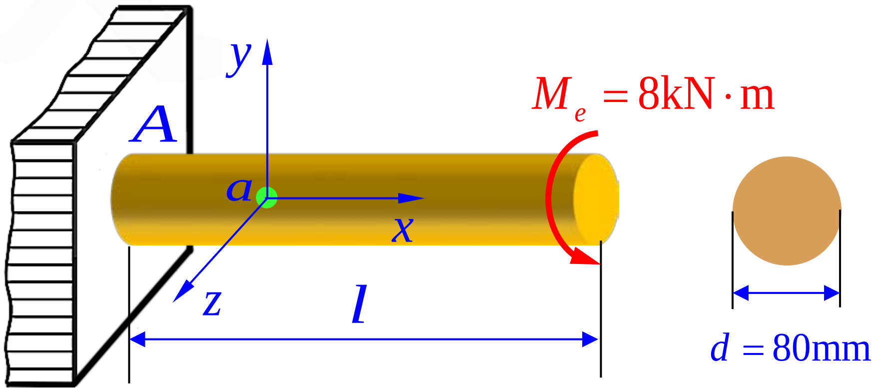
三向应力状态 → 二向应力状态 → 单向应力状态

二向应力状态 $\xrightarrow{\text{特例}}$ 纯剪切应力状态

例2 从图示受扭构件中a点取出单元体，并确定该单元体各面上的应力，然后计算出主应力的大小和方向。

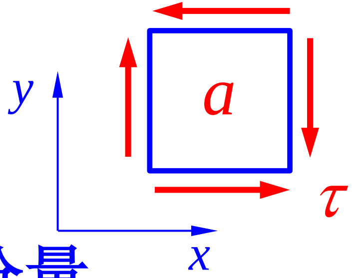
解：❶ 以a点为原点建立坐标系

（取z方向为a点切平面的外法线方向）



画出单元体

（沿z轴负方向看）



❷ 确定切应力分量

$$\tau = \frac{M}{W_P} = \frac{M}{\frac{1}{16}\pi d^3} = \frac{8\times10^3}{\frac{1}{16}\times\pi\times(80\times10^{-3})^3}$$

$$= 79.58\times10^6\,\mathrm{Pa} = 79.58\mathrm{MPa}$$

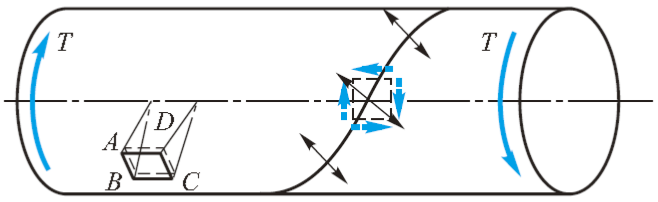
$\sigma_x = 0\mathrm{MPa}$， $\sigma_y = 0\mathrm{MPa}$， $\tau_{xy} = 79.58\mathrm{MPa}$

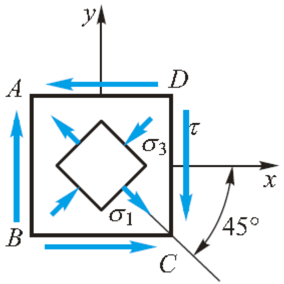
❸ 主应力的大小

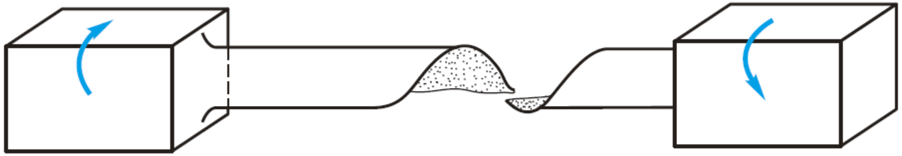
$$\left\{\begin{matrix}\sigma_1' \\ \sigma_2'\end{matrix}\right. = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$\left\{\begin{matrix}\sigma_1 \\ \sigma_3\end{matrix}\right. = 0 \pm 79.58 = \left\{\begin{matrix}79.58\ \mathrm{MPa} \\ -79.58\ \mathrm{MPa}\end{matrix}\right.$$

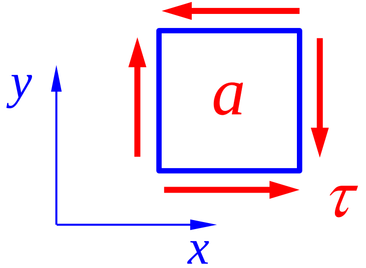
拉应力







铸铁试样的扭转破坏断口



主应力的方向

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

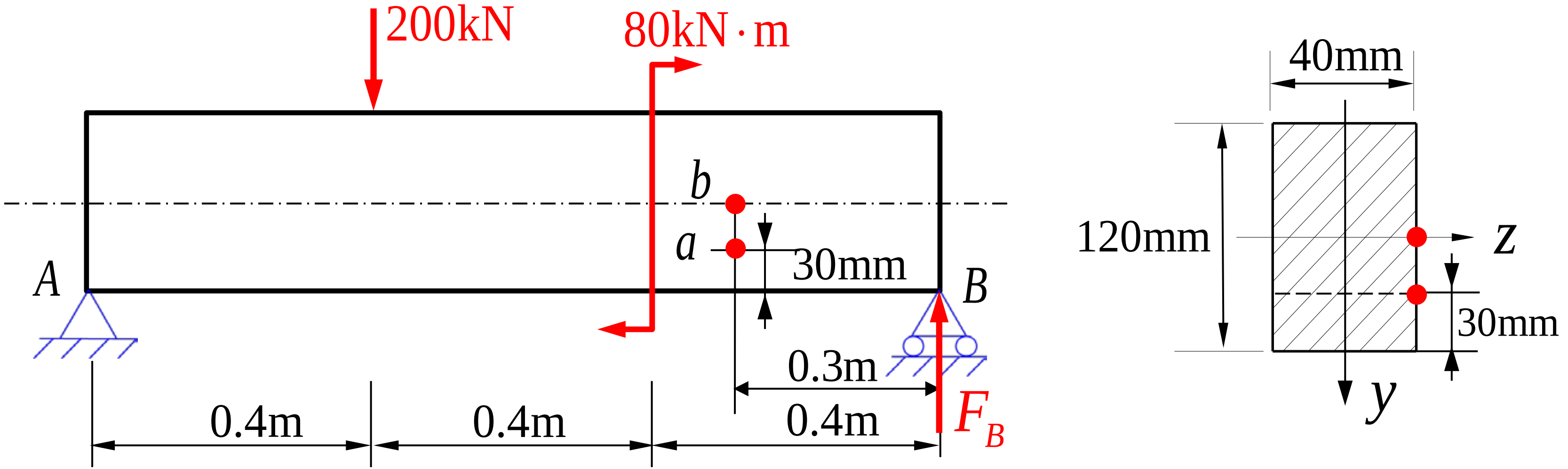
$$= -\frac{2\times 79.58}{0-0} = -\infty$$

$$2\alpha_0 = -90°$$

$$\alpha_{01} = -45°$$

$$\alpha_{02} = -45° + 90° = 45°$$

例3 从图示构件中a点和b点取出单元体，并确定该单元体各面上的应力，然后分别计算出主应力的大小和方向。



解：计算B支座的约束力

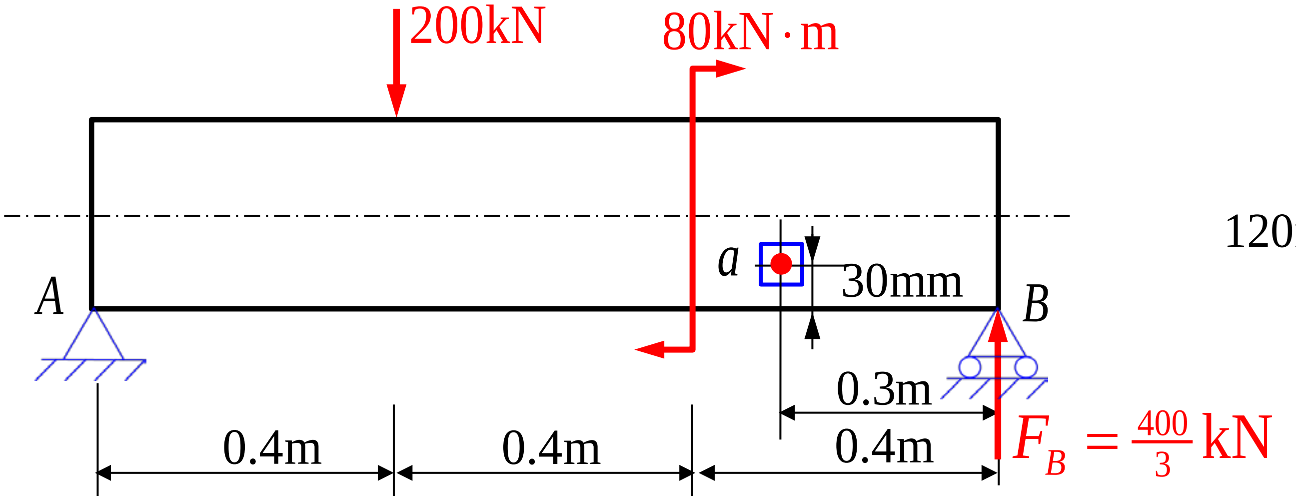
$$\sum M_A = 0,\ \ 200\times0.4+80-F_B\times1.2=0$$

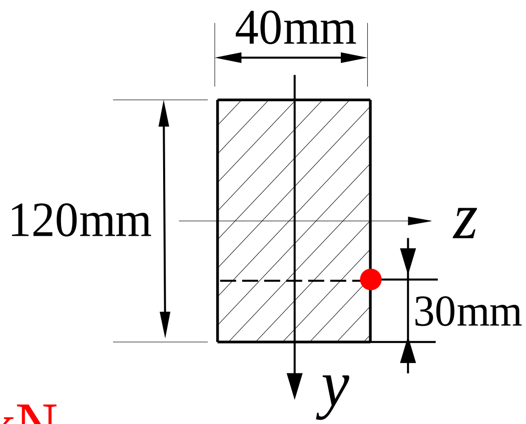
$$F_B = \frac{400}{3}\text{kN}$$

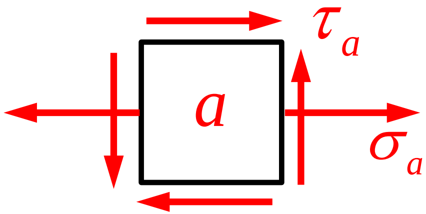
a点的右截面上的内力

$$M = F_B\times0.3=\frac{400}{3}\times0.3=40\text{kN}\cdot\text{m}$$

$$F_s = F_B = \frac{400}{3}\text{kN}\quad(\text{方向向上})$$





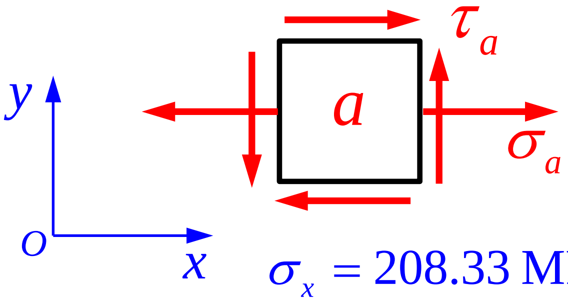


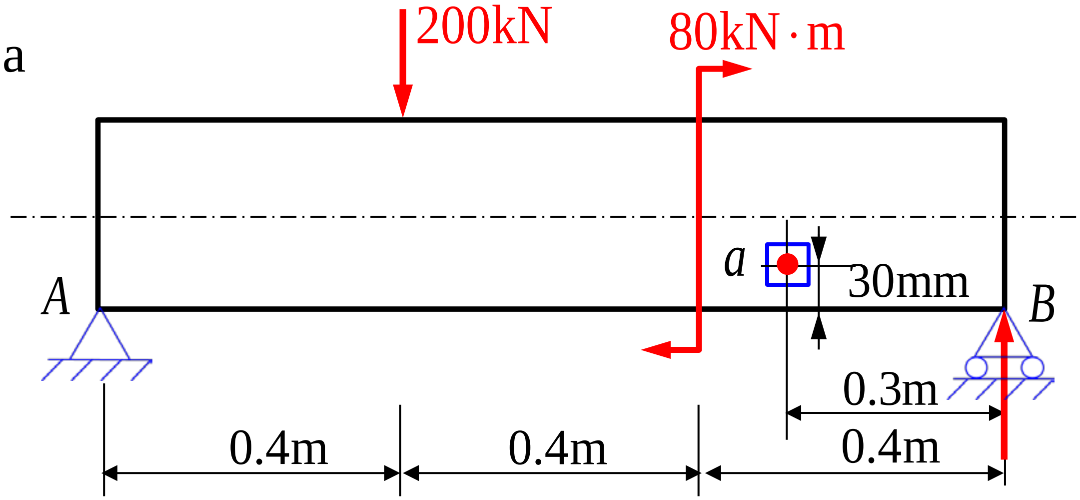
1. 在a点取出的单元体
   确定单元体的各应力分量

$$\sigma_a = \frac{My_a}{I_z} = \frac{40\times10^3\times30\times10^{-3}}{\frac{1}{12}\times40\times10^{-3}\times0.12^3} = 208.33\times10^6\,\mathrm{Pa} = 208.33\mathrm{MPa}$$

$$\tau_a = \frac{F_s S_z^*}{I_z b} = \frac{\frac{400}{3}\times10^3\times(40\times10^{-3}\times30\times10^{-3})\times45\times10^{-3}}{\frac{1}{12}\times40\times10^{-3}\times0.12^3\times40\times10^{-3}} = 31.25\times10^6\,\mathrm{Pa} = 31.25\mathrm{MPa}$$

$\sigma_a = 208.33\text{MPa} \quad \tau_a = 31.25\text{MPa}$





$\sigma_x = 208.33\ \text{MPa}$

$\sigma_y = 0\ \text{MPa}$

$\tau_{xy} = -31.25\ \text{MPa}$

（1）主应力的大小

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right. = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right. = 104.165 \pm 108.752 = \left\{\begin{matrix}212.92\ \text{MPa}\\ -4.59\ \text{MPa}\end{matrix}\right.$$

（2）主应力的方向

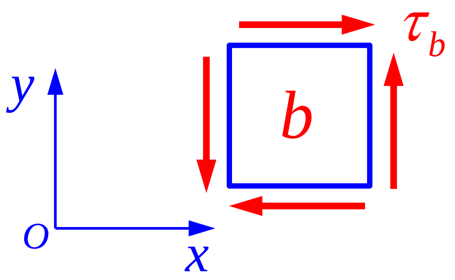
$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x-\sigma_y} = -\frac{2\times(-31.25)}{208.33} = 0.3$$

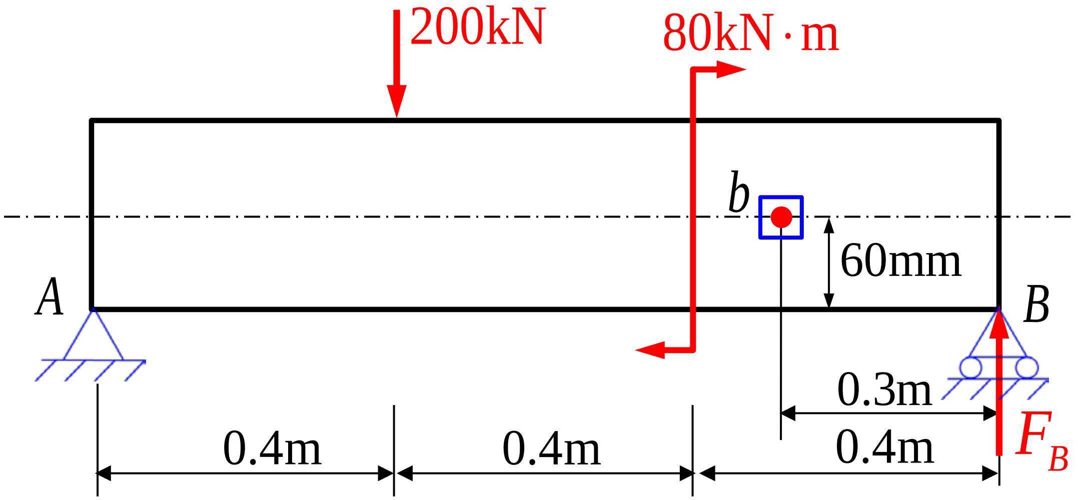
$$2\alpha_{01} = 16.7°$$

$$\alpha_{01} = 8.35°$$

$$\alpha_{02} = 8.35° - 90° = -81.65°$$

2. 从b点取出的单元体





$$F_B = \frac{400}{3}\,\mathrm{kN}$$

$$\tau_b = \frac{3}{2}\frac{F_s}{A} = \frac{3}{2}\times\frac{\frac{400}{3}\times10^3}{40\times10^{-3}\times120\times10^{-3}}$$

$$= 41.67\times10^6\,\mathrm{Pa} = 41.67\ \mathrm{MPa}$$

$$\sigma_x = 0\ \mathrm{MPa},\quad \sigma_y = 0\ \mathrm{MPa},\quad \tau_{xy} = -41.67\mathrm{MPa}$$

（1）主应力的大小

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right. = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right. = 0\pm41.67 = \left\{\begin{matrix}41.67\ \mathrm{MPa}\\ -41.67\ \mathrm{MPa}\end{matrix}\right.$$

（2）主应力的方向

$$\tan2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x-\sigma_y} = -\frac{2\times(-41.67)}{0-0} \to \infty$$

$$2\alpha_{01} = 90^\circ$$

$$\alpha_{01} = 45^\circ$$

$$\alpha_{02} = 45^\circ-90^\circ = -45^\circ$$

Thank you for your attention!

作业： Page 274-275: 7.7、7.8、7.10

对应第6版的题号 Page 267-268：7.7、7.8、7.10

下次课 讲应力圆