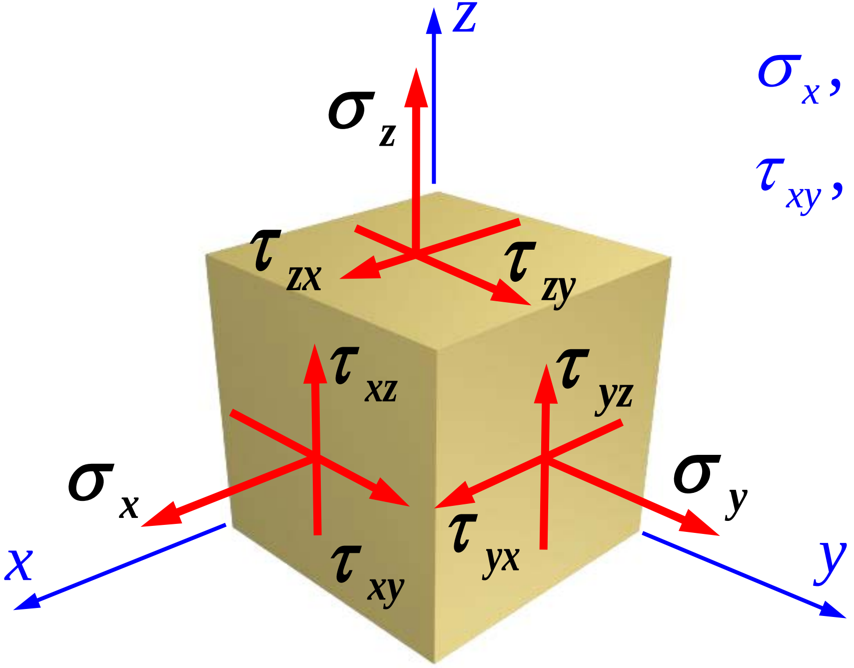
第七章　应力和应变分析
强度理论（三）

第19讲

§ 7.5 三向（空间）应力状态

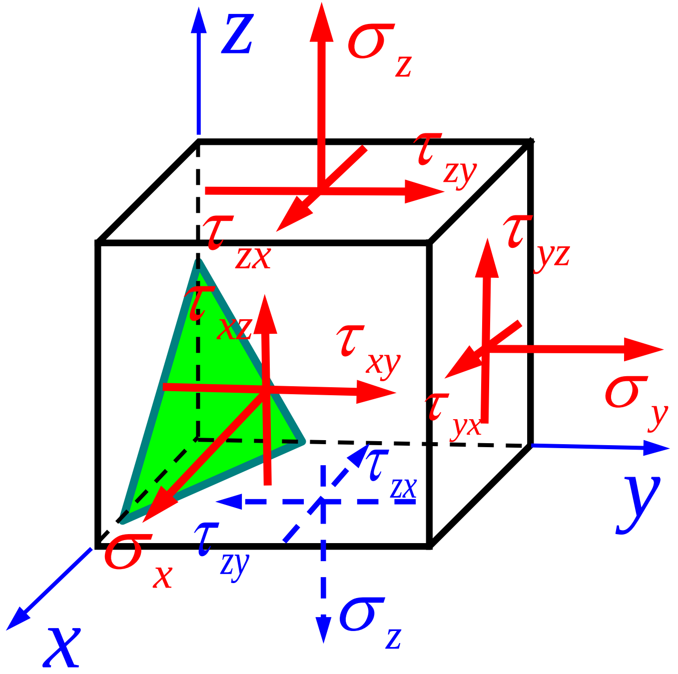
1. 空间应力状态的概念

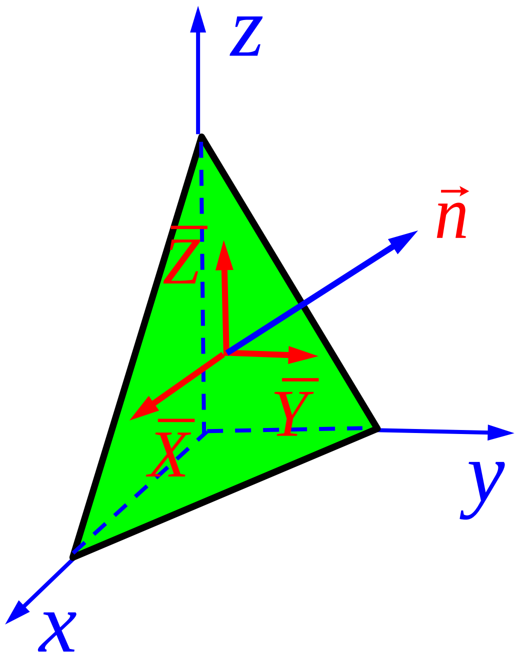


σ_x， σ_y， σ_z

τ_{xy}， τ_{yz}， τ_{zx}

2. 空间应力状态主应力的计算

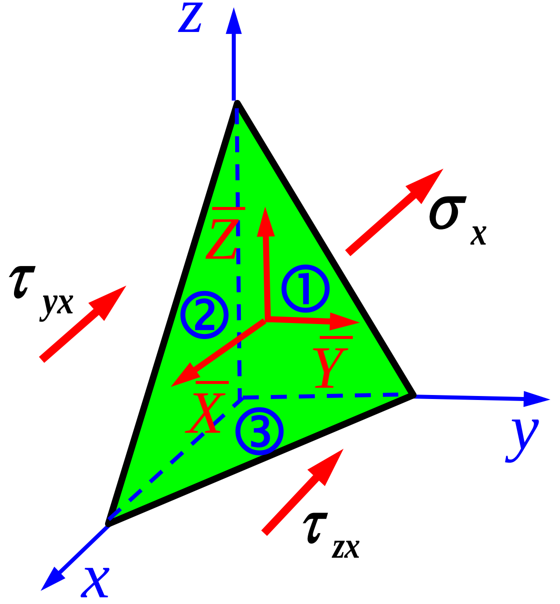


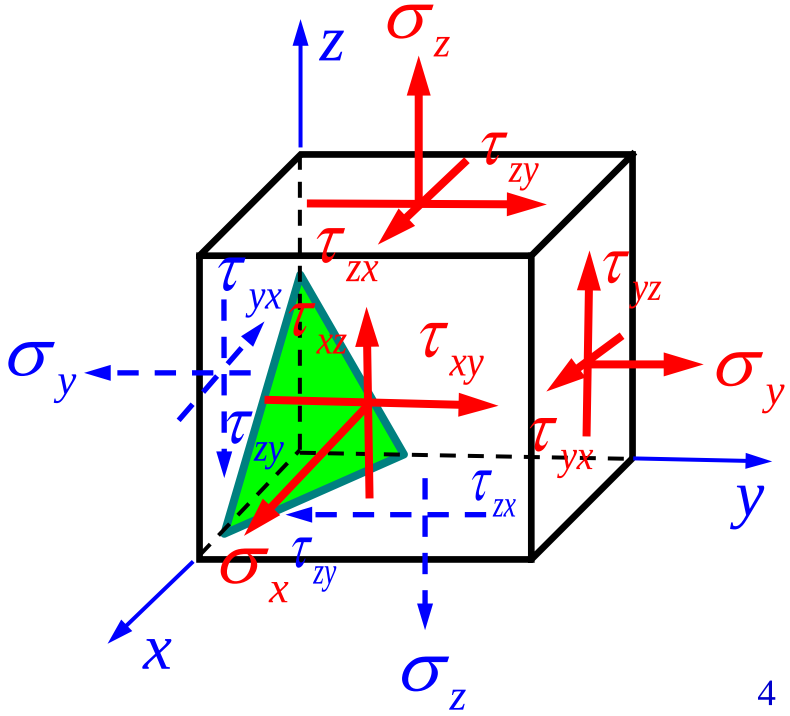


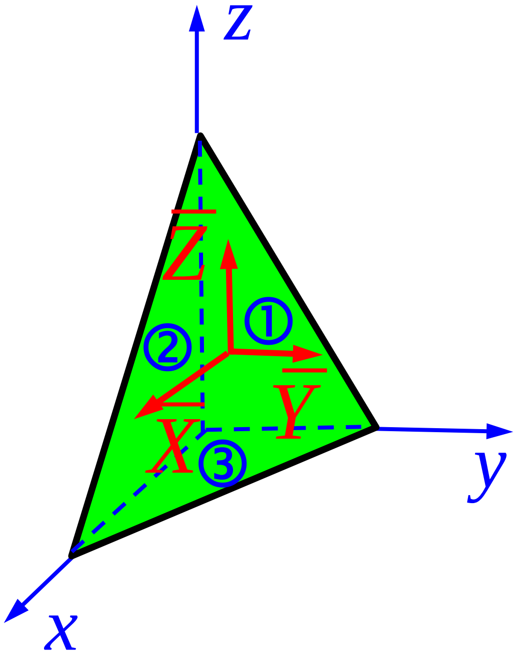
$$\sum F_x = 0 \qquad \bar{X}\mathrm{d}A - \sigma_x \cdot \mathrm{d}A\cos(\vec{n},\vec{x}) - \tau_{yx} \cdot \mathrm{d}A\cos(\vec{n},\vec{y}) - \tau_{zx} \cdot \mathrm{d}A\cos(\vec{n},\vec{z}) = 0$$

$l \qquad m \qquad n$

$$\boxed{\bar{X} = \sigma_x \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n}$$







$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$

$$\bar{X} = \sigma_x \cdot l + \tau_{yx} \cdot m + \tau_{zx} \cdot n$$
$$\bar{Y} = \tau_{xy} \cdot l + \sigma_y \cdot m + \tau_{zy} \cdot n$$
$$\bar{Z} = \tau_{xz} \cdot l + \tau_{yz} \cdot m + \sigma_z \cdot n$$

求得 $\bar{X}, \bar{Y}, \bar{Z}$ ➡ p（总应力）➡ $\sigma_\alpha, \tau_\alpha$

若α面为主平面，则该面上 $\bar{X}, \bar{Y}, \bar{Z}$ 的合矢量即为主应力（即方向垂直于该平面，也即与法线方向一致），有

$$\bar{X} = \sigma \cos(\vec{n}, \vec{x}) = \sigma \cdot l$$
$$\bar{Y} = \sigma \cos(\vec{n}, \vec{y}) = \sigma \cdot m$$
$$\bar{Z} = \sigma \cos(\vec{n}, \vec{z}) = \sigma \cdot n$$

$$\tau_{xy}=\tau_{yx},\quad \tau_{xz}=\tau_{zx},\quad \tau_{yz}=\tau_{zy}$$

$$\bar{X}=\sigma\cdot l=\sigma_x\cdot l+\tau_{yx}\cdot m+\tau_{zx}\cdot n \Longrightarrow (\sigma_x-\sigma)\cdot l+\tau_{xy}\cdot m+\tau_{xz}\cdot n=0$$

$$\bar{Y}=\sigma\cdot m=\tau_{xy}\cdot l+\sigma_y\cdot m+\tau_{zy}\cdot n \Longrightarrow \tau_{xy}\cdot l+(\sigma_y-\sigma)\cdot m+\tau_{yz}\cdot n=0$$

$$\bar{Z}=\sigma\cdot n=\tau_{xz}\cdot l+\tau_{yz}\cdot m+\sigma_z\cdot n \Longrightarrow \tau_{xz}\cdot l+\tau_{yz}\cdot m+(\sigma_z-\sigma)\cdot n=0$$

$$\begin{bmatrix}\sigma_x-\sigma & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y-\sigma & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma\end{bmatrix}\begin{Bmatrix}l\\ m\\ n\end{Bmatrix}=0 \qquad \begin{vmatrix}\sigma_x-\sigma & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y-\sigma & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma\end{vmatrix}=0$$

存在非零解

可见：求空面应力状态的主应力可以转化为求解矩阵的特征值

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \begin{vmatrix} \sigma_x-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y-\sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma \end{vmatrix}=0$$

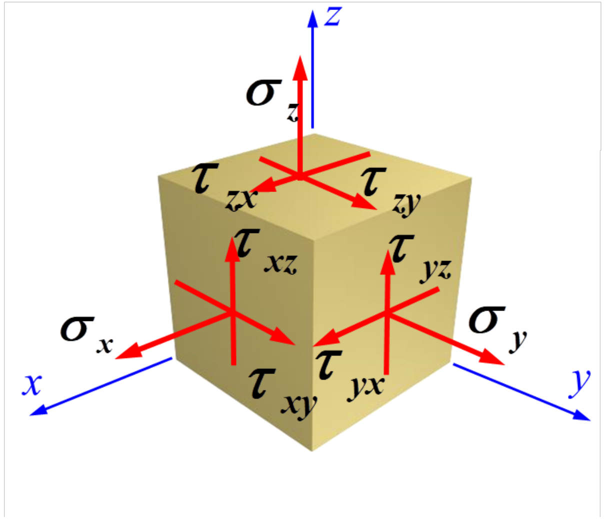
展开后是关于σ的一元三次方程，定有三个根，即三个主应力。

记为：$\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$\begin{bmatrix} \sigma_x-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y-\sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}=0$$

对每个主应力 $\sigma_i\ (i=1,2,3)$ ，利用左边的方程求出对应的$(l_i, m_i, n_i)\ \ (i=1,2,3)$，即得到每个主应力的方向。

空间应力状态分析—借助Matlab软件



$$\sigma_x = 100\text{MPa},\quad \tau_{xy} = 50\text{MPa},\quad \tau_{xz} = -70\text{MPa}$$

$$\sigma_y = 200\text{MPa},\quad \tau_{yz} = 30\text{MPa},\quad \sigma_z = 280\text{MPa}$$

$$A = \begin{bmatrix} 100 & 50 & -70 \\ 50 & 200 & 30 \\ -70 & 30 & 280 \end{bmatrix}$$

[V,D]=eig(A)

```
V =

   -0.8750    0.3873   -0.2906
    0.3643    0.9219    0.1316
   -0.3189   -0.0093    0.9477


D =

   53.6666         0         0
         0  220.7021         0
         0         0  305.6313
```

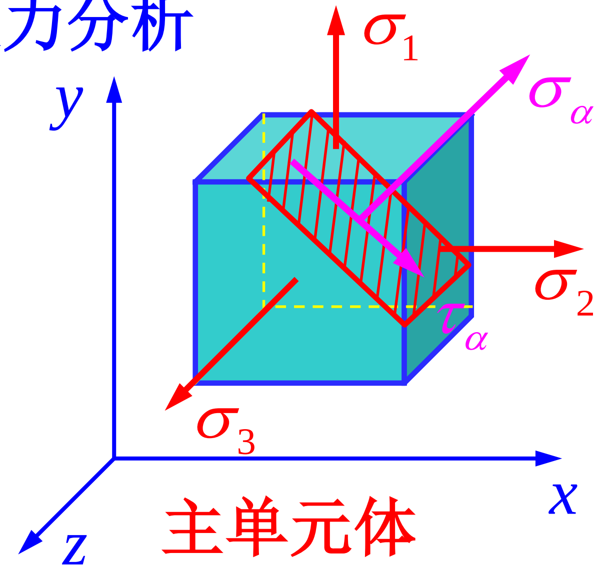
$$53.67\ \text{MPa} = \sigma_3$$

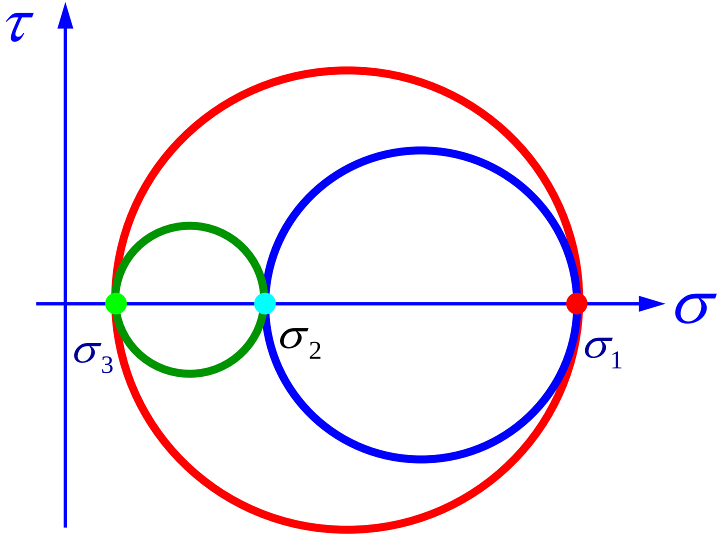
$$220.70\ \text{MPa} = \sigma_2$$

$$305.63\ \text{MPa} = \sigma_1$$

o

3. 三向应力分析





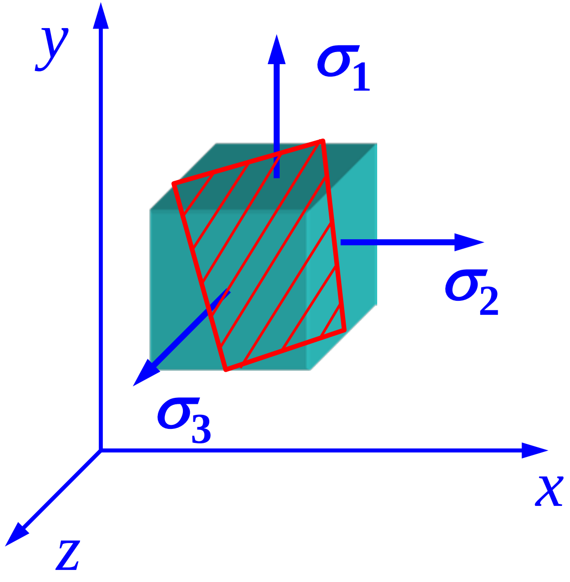
1. 所有平行于z轴的截面：该截面上的应力$(\sigma_\alpha,\tau_\alpha)$落在$(\sigma_1-\sigma_2)$的应力圆上；

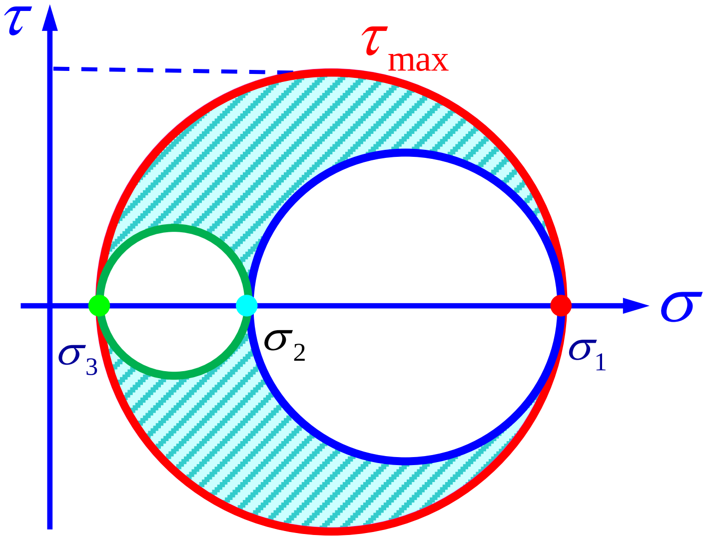
2. 所有平行于x轴的截面：该截面上的应力$(\sigma_\alpha,\tau_\alpha)$落在$(\sigma_1-\sigma_3)$的应力圆上；

3. 所有平行于y轴的截面：该截面上的应力$(\sigma_\alpha,\tau_\alpha)$落在$(\sigma_2-\sigma_3)$的应力圆上；

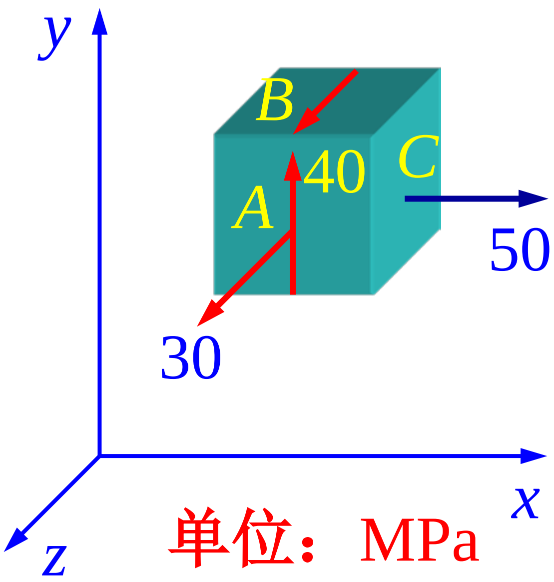
4. 单元体内任意一点任意截面上的应力，
可以证明，对应着应力圆上阴影区内的一点。

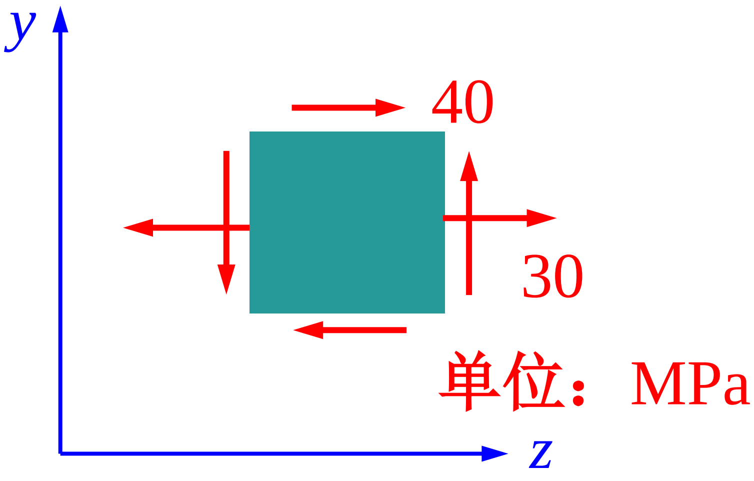
5. 整个单元体内的最大切应力：$\tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2}$





例1 求图示单元体的主应力和最大切应力。



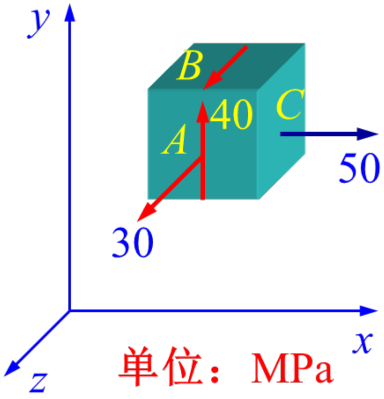


沿x负方向看

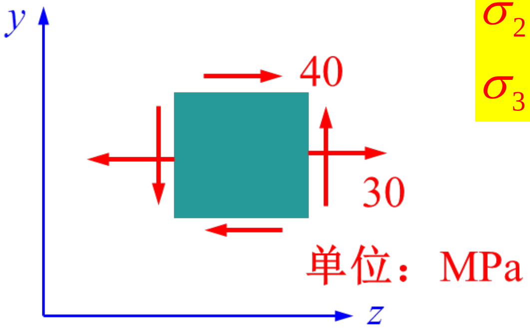
解：(1) 由上图知 yz 面为主平面之一

$$\sigma_1' = 50\text{MPa}$$

原问题可转化为平面的问题进行处理



$$\sigma_1' = 50\text{MPa}$$



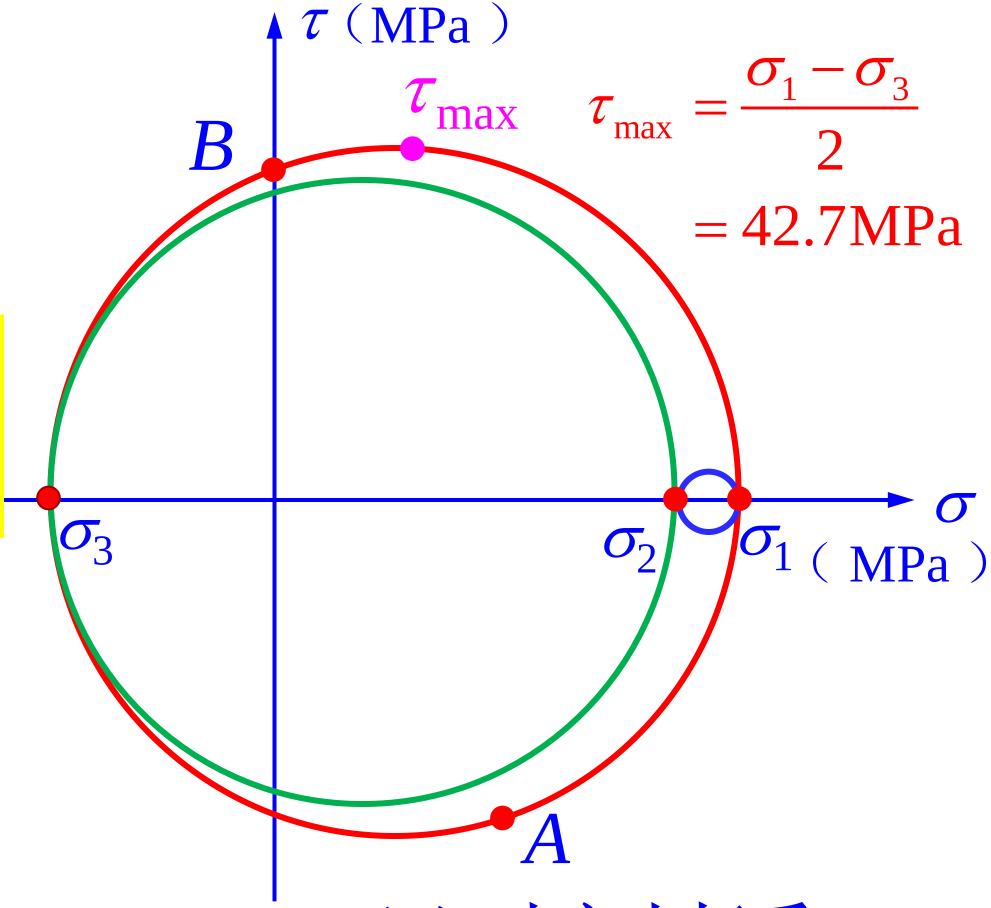
$$\sigma_1 = 57.7\text{MPa}$$

$$\sigma_2 = \sigma_1' = 50\text{MPa}$$

$$\sigma_3 = -27.7\text{MPa}$$

$$\sigma_{2,3}' = \frac{30+0}{2} \pm \sqrt{\left(\frac{30-0}{2}\right)^2 + (-40)^2}$$

$$= 15 \pm 42.7 = \begin{cases} 57.7\text{MPa} \\ -27.7\text{MPa} \end{cases}$$



$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

$$= 42.7\text{MPa}$$

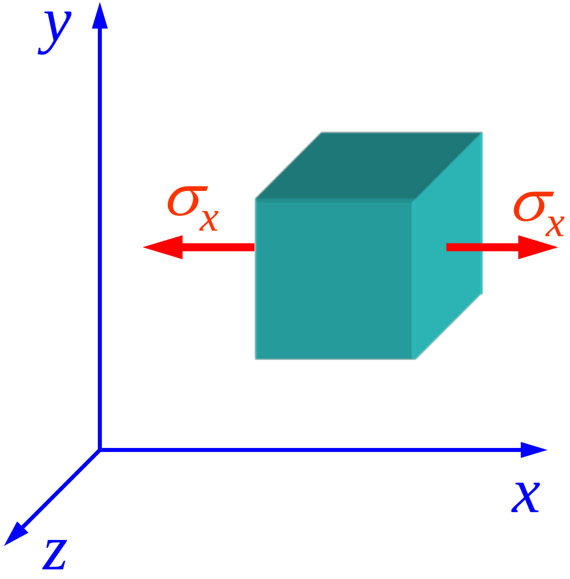
（2）建立坐标系

画应力圆

§ 7.8 广义胡克定律

空间应力状态下应力与应变间的关系?

一、单轴拉伸应力状态的本构关系



$$\sigma = E\varepsilon, \qquad \sigma_x = E\varepsilon_x,$$

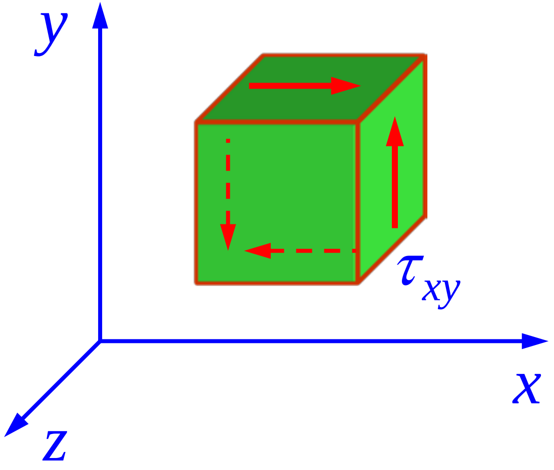
$$\varepsilon_x = \frac{\sigma_x}{E}, \quad \varepsilon_y = -\mu\frac{\sigma_x}{E}, \quad \varepsilon_z = -\mu\frac{\sigma_x}{E}$$

$$\gamma_{ij} = 0 \quad (i, j = x, y, z)$$

二、纯剪应力状态的本构关系

$$\tau = G\gamma \qquad \tau_{xy} = G\gamma_{xy}$$

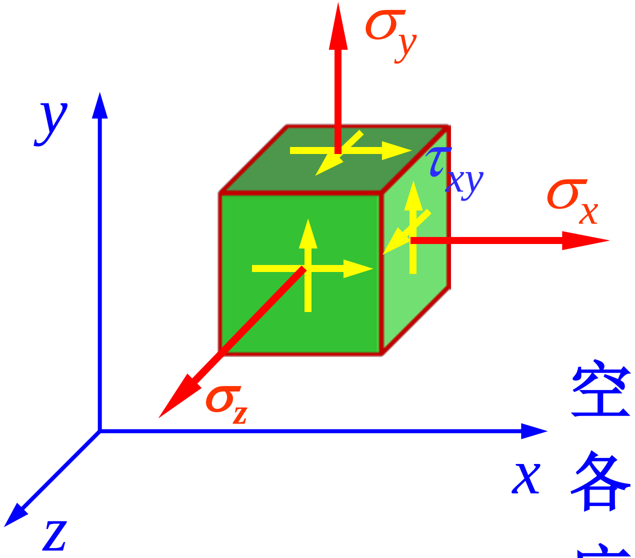
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$



规定：切应变以直角的改变量减小为正！

$$\gamma_{yz} = \gamma_{zx} = 0 \qquad \varepsilon_i = 0 \ \ (i = x, y, z)$$

三、空间应力状态的本构关系（各向同性材料）



依叠加原理，得 $\varepsilon_x = \dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_z}{E}$

$$= \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right]$$

空间应力状态下各向同性材料的应力-应变关系（广义胡克定律）

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right] \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G}, \quad \gamma_{yz} = \dfrac{\tau_{yz}}{G}, \quad \gamma_{zx} = \dfrac{\tau_{zx}}{G} \end{cases}$$

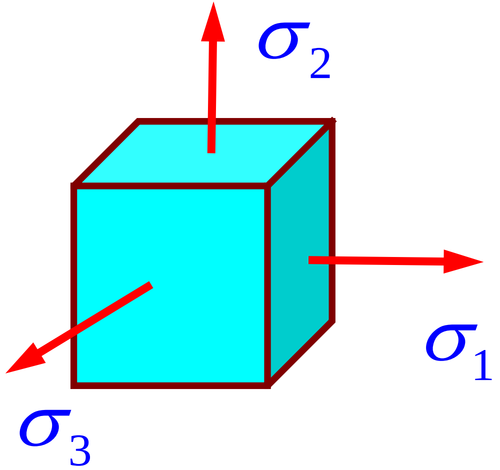
三个弹性常数之间的关系

$$\boxed{G = \frac{E}{2\left(1+\mu\right)}}$$

主单元体的本构关系（主应力-主应变）

主应变：相应切应变等于零的面上的正应变。

对于各向同性材料，主应力和主应变的方向一致。



$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right] \end{cases}$$

$$\begin{cases} \varepsilon_1 = \dfrac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right] \\ \varepsilon_2 = \dfrac{1}{E}\left[\sigma_2 - \mu\left(\sigma_3 + \sigma_1\right)\right] \\ \varepsilon_3 = \dfrac{1}{E}\left[\sigma_3 - \mu\left(\sigma_2 + \sigma_1\right)\right] \end{cases}$$

各向异性材料的本构关系

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
 & & a_{33} & a_{34} & a_{35} & a_{36} \\
 & \text{对} & & a_{44} & a_{45} & a_{46} \\
 & & \text{称} & & a_{55} & a_{56} \\
 & & & & & a_{66}
\end{bmatrix}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}
$$

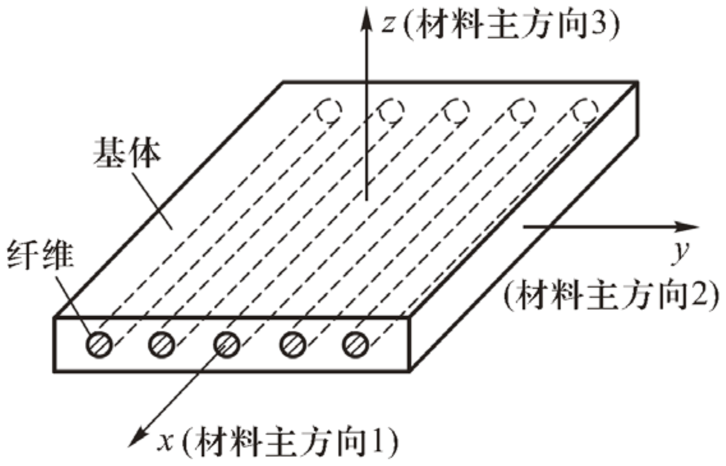
可以证明 $a_{ij} = a_{ji}$ 共21个独立的弹性常数

注意

对于各向异性材料，主应力的方向和主应变的方向将出现不一致的情况！

正交各向异性材料的本构关系

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

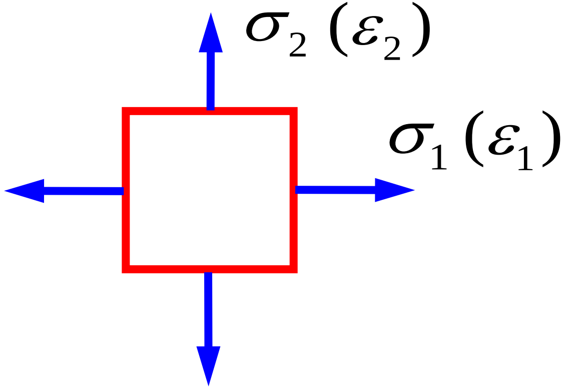


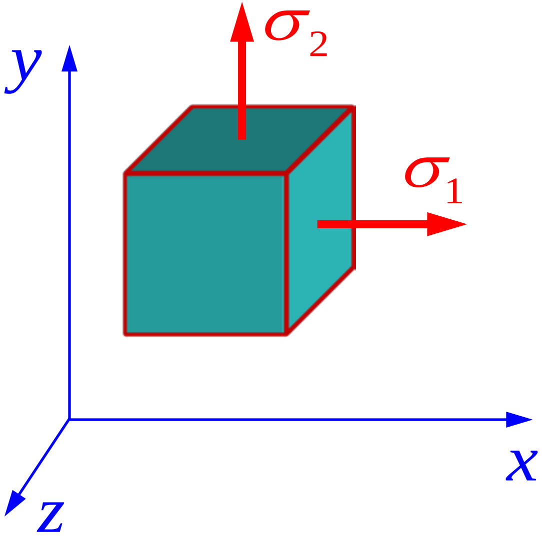
共9个独立的弹性常数。

长纤维增强复合材料就是典型的正交各向异性材料， 其纤维铺设的方向和与纤维相垂直的方向即为材料的主方向

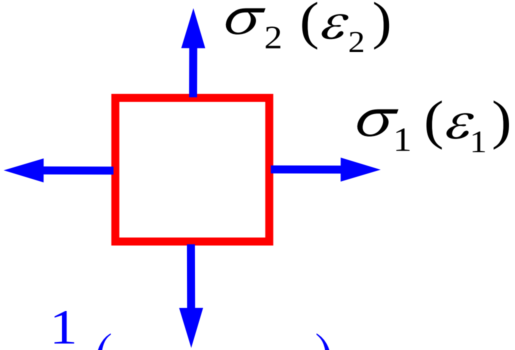
例2　构件表面某点的两个面内主应变分别为 $\varepsilon_1=240\times10^{-6}$，$\varepsilon_2=-160\times10^{-6}$，　E=210GPa，　μ=0.3，　求该点的主应力及另一主应变。

解：　因为自由面上各应力分量均为零
故为平面应力状态





$$\boxed{\begin{aligned}&\varepsilon_1 = 240\times10^{-6}, \quad E = 210\text{GPa}\\&\varepsilon_2 = -160\times10^{-6}, \ \mu = 0.3\end{aligned}}$$



利用主单元体的本构关系：

$$\begin{cases}\varepsilon_1 = \dfrac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right]\\[2ex] \varepsilon_2 = \dfrac{1}{E}\left[\sigma_2 - \mu\left(\sigma_3 + \sigma_1\right)\right]\\[2ex] \varepsilon_3 = \dfrac{1}{E}\left[\sigma_3 - \mu\left(\sigma_2 + \sigma_1\right)\right]\end{cases} \xrightarrow{\sigma_3 = 0} \begin{cases}\varepsilon_1 = \dfrac{1}{E}\left(\sigma_1 - \mu\sigma_2\right)\\[2ex] \varepsilon_2 = \dfrac{1}{E}\left(\sigma_2 - \mu\sigma_1\right)\\[2ex] \varepsilon_3 = -\dfrac{\mu}{E}\left(\sigma_2 + \sigma_1\right)\end{cases}$$

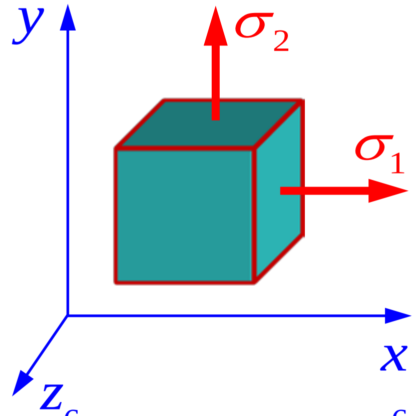
$$\sigma_1' = \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2) = \frac{210\times10^9}{1-0.3^2}(240 - 0.3\times160)\times10^{-6} = 44.3\text{MPa}$$

$$\sigma_2' = \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1) = \frac{210\times10^9}{1-0.3^2}(-160 + 0.3\times240)\times10^{-6} = -20.3\text{MPa}$$

主应力排序结果：

$$\sigma_1 = 44.3\text{MPa}$$
$$\sigma_2 = 0$$
$$\sigma_3 = -20.3\text{MPa}$$

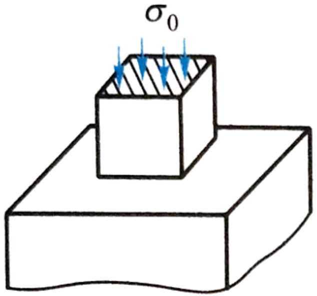
$$\sigma_1' = 44.3\text{MPa}$$
$$\sigma_2' = -20.3\text{MPa}$$
$$\sigma_3' = 0\text{MPa}$$

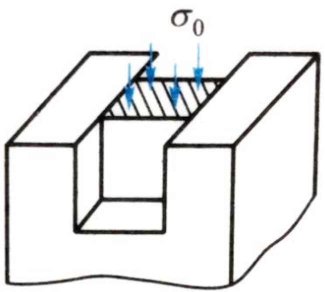


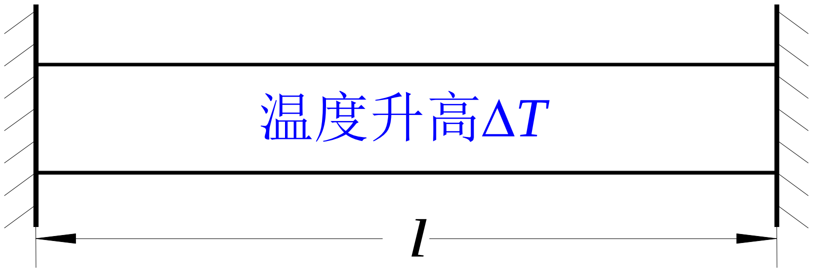
$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \mu(\sigma_3 + \sigma_1)] = -\frac{0.3}{210\times10^9}(-20.3 + 44.3)\times10^6 = -34.3\times10^{-6}$$

该方向上的应力为零，应变不一定为零！

该方向上的应变为零，应力不一定为零！





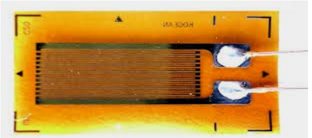


本构关系的应用—应力的量测

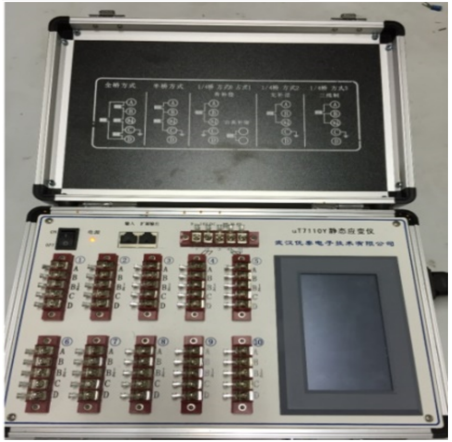
在工程实际中，应变是可以直接测量的。测出应变后，利用本构关系即可算出应力的大小。

应变如何测量？

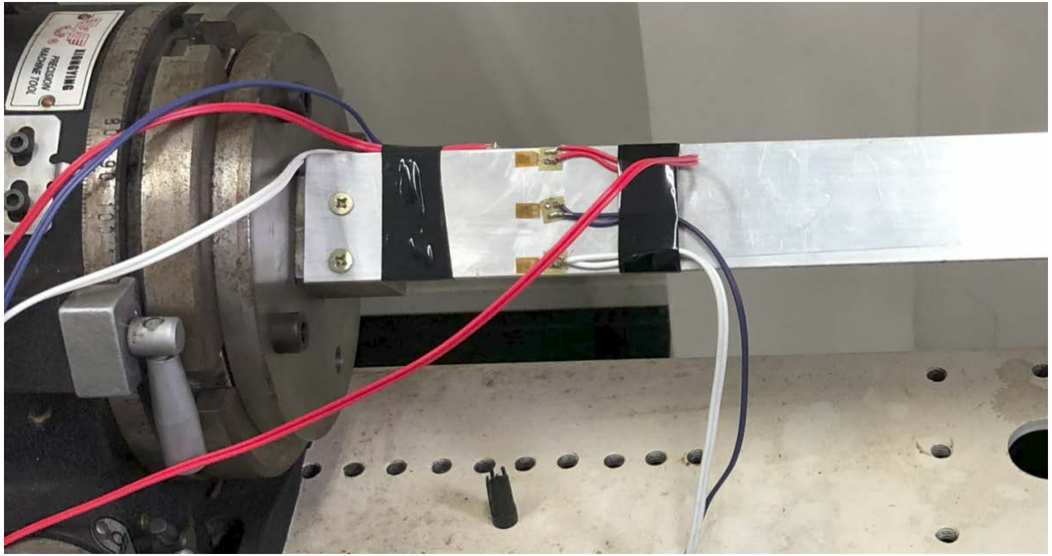
测试铝合金梁的应变



电阻应变片



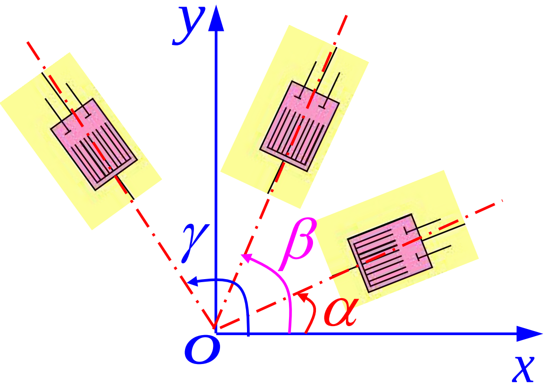
静态电阻应变仪

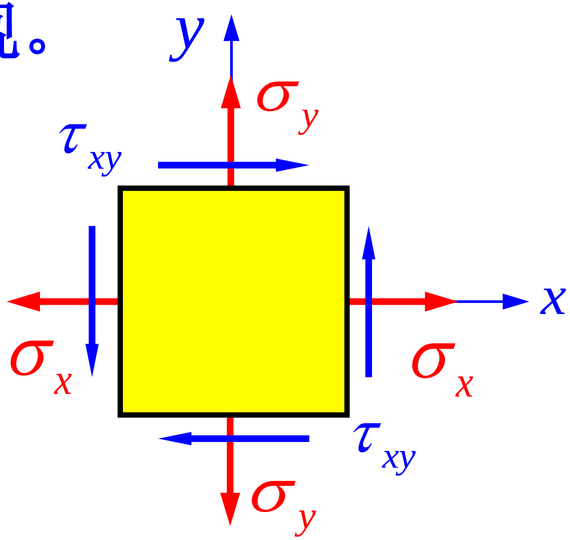


构件表面点应力的测量

通常情况，要确定平面应力状态的三个应力为σ_x，σ_y，τ_{xy}。

可通过在一点沿任意三个方向各贴一枚应变片来实现。





待确定的量共三个

$\sigma_x, \sigma_y, \tau_{xy}$

↑ 广义胡克定律

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$

↑ 应变分析

$\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma$ 通过测量获得

数据处理方法－ § 7.6 位移和应变分量　§ 7.7 平面应变状态分析

$$\varepsilon_\alpha = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\varepsilon_\beta = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\beta + \frac{\gamma_{xy}}{2}\sin 2\beta$$

$$\varepsilon_\gamma = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\gamma + \frac{\gamma_{xy}}{2}\sin 2\gamma$$

任意方向上的应变

应变分析，应变圆

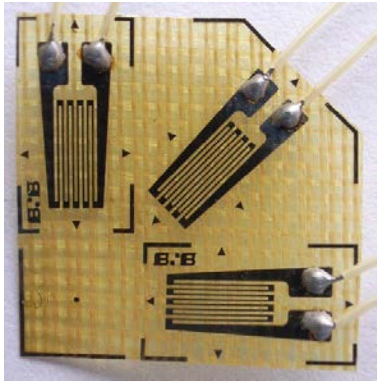
$\varepsilon_x, \varepsilon_y, \gamma_{xy} \longrightarrow \varepsilon_1, \varepsilon_2, \alpha_0$

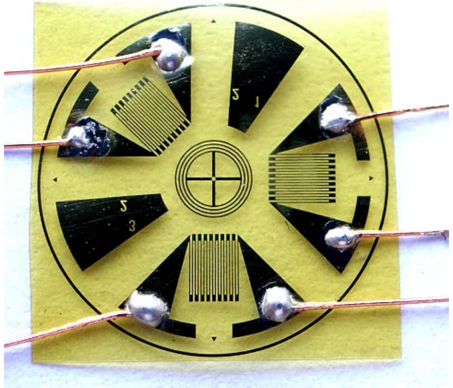
广义虎克定律 ↓　　可以绕过应变分析　　↓ 广义虎克定律

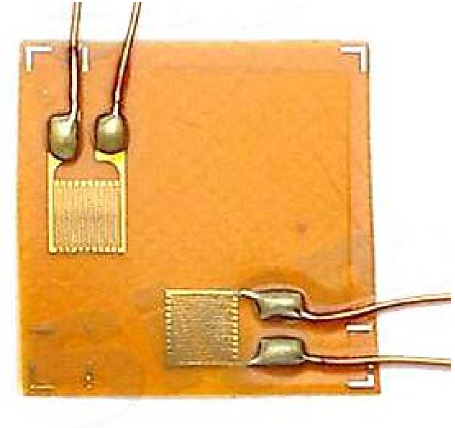
$\sigma_x, \sigma_y, \tau_{xy} \longrightarrow \sigma_1, \sigma_2, \alpha_0$

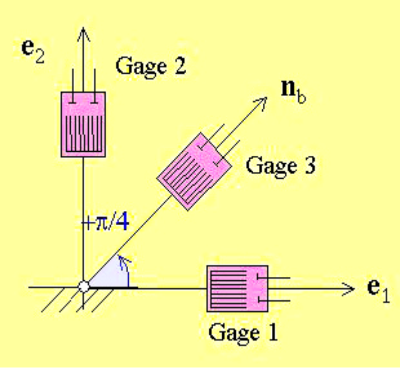
应力分析，应力圆

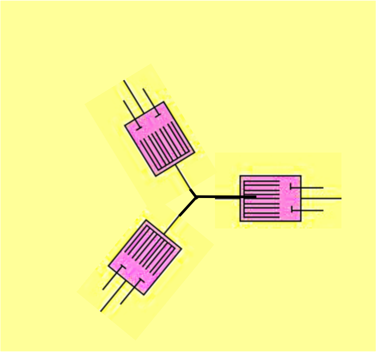
几种常见的应变花

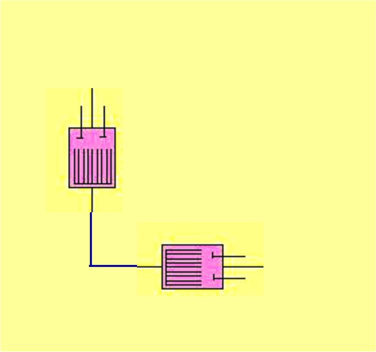








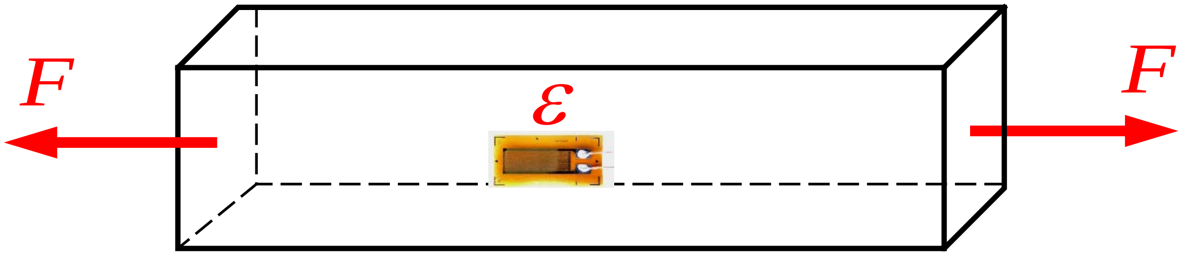


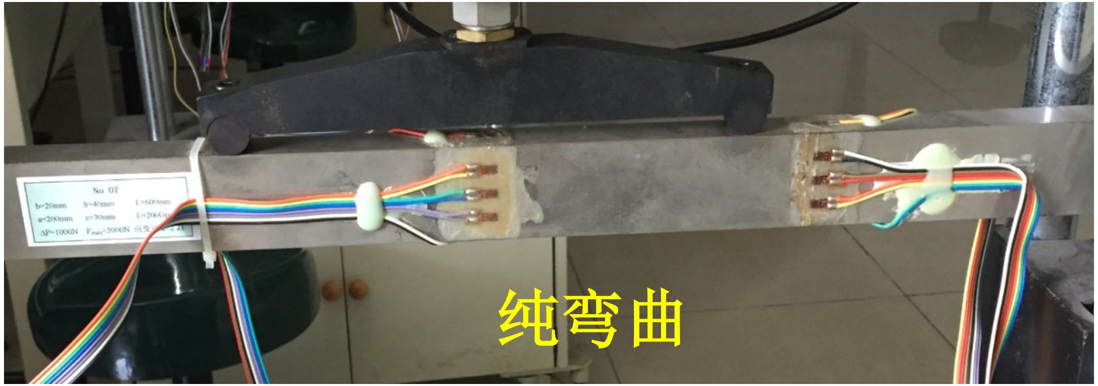


关于应变电测技术的具体实施

1. 单轴应力状态

$$\sigma = E\varepsilon$$





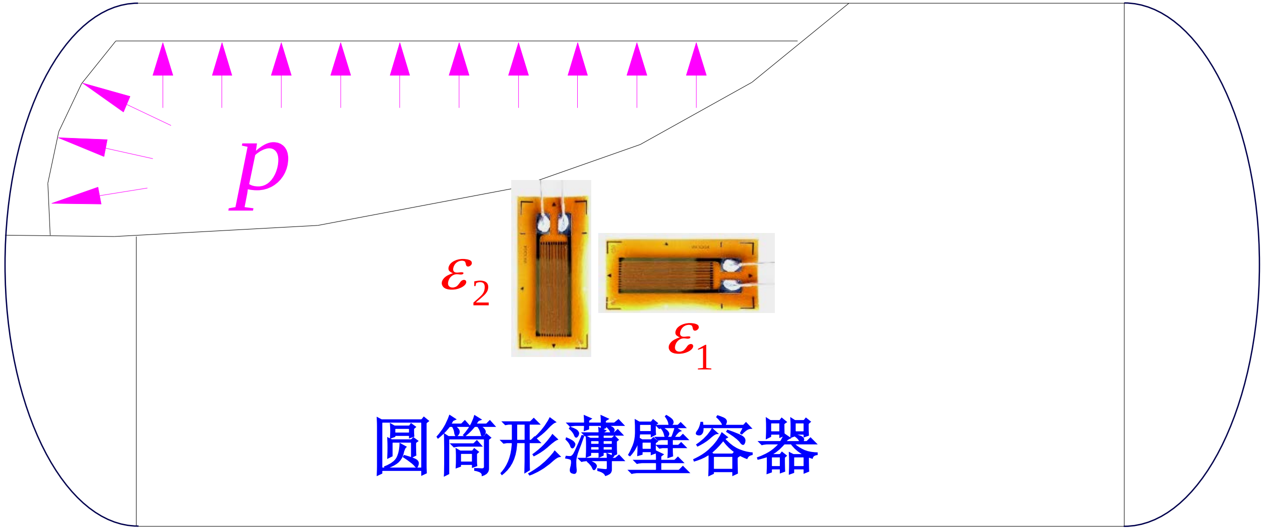
2. 两个主应力方向已知

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \mu\sigma_2)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \mu\sigma_1)$$

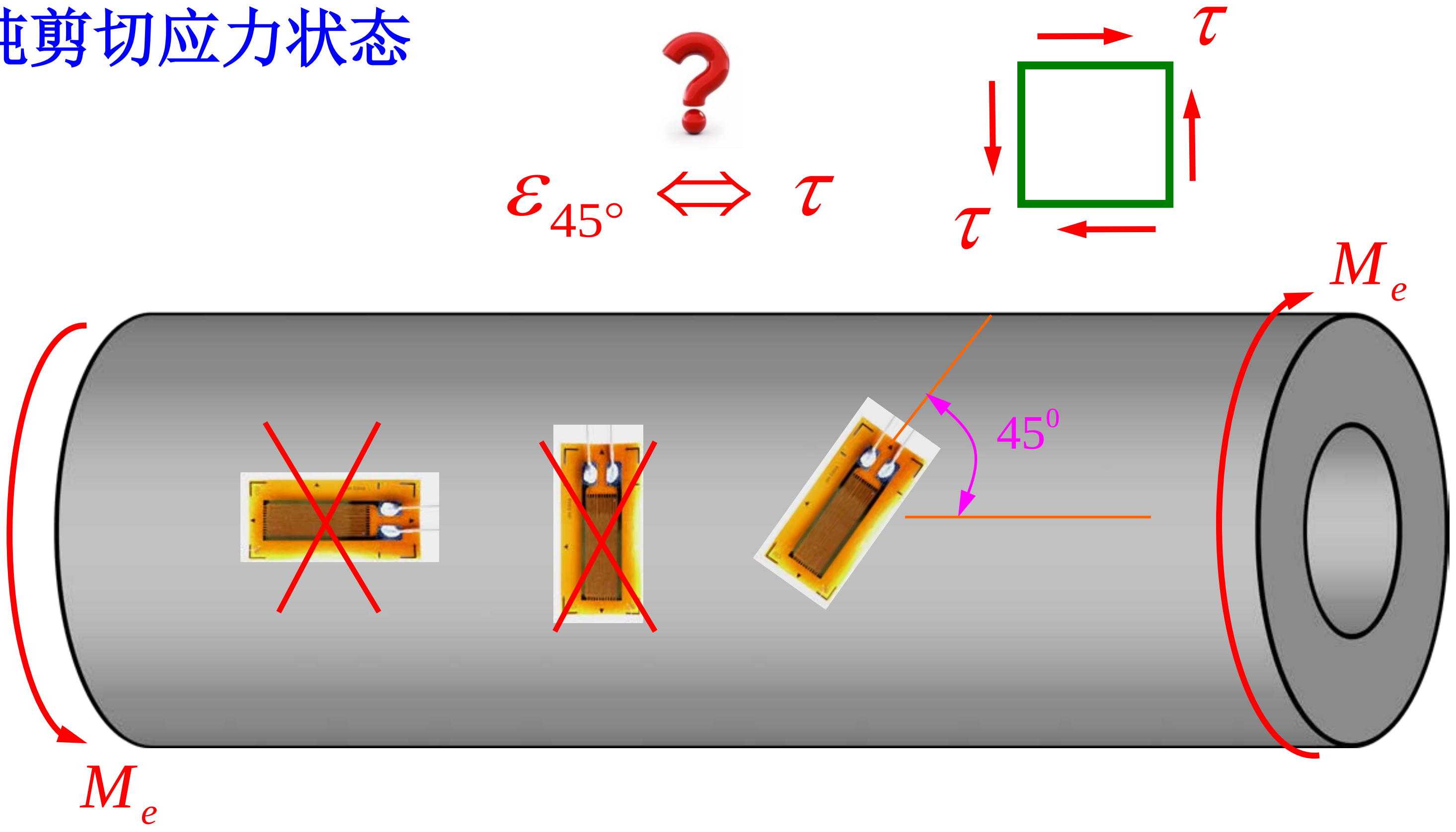
$$\sigma_1 = \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2)$$

$$\sigma_2 = \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1)$$

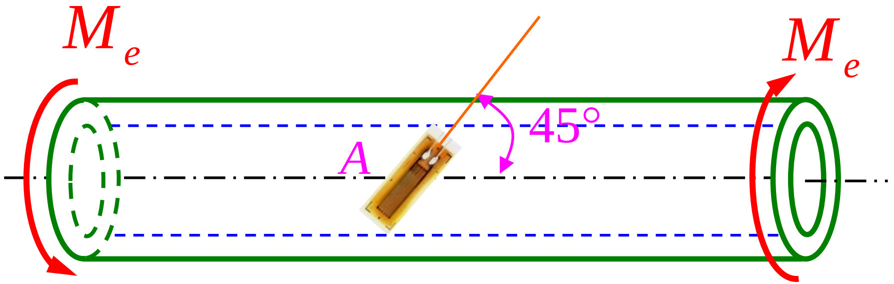


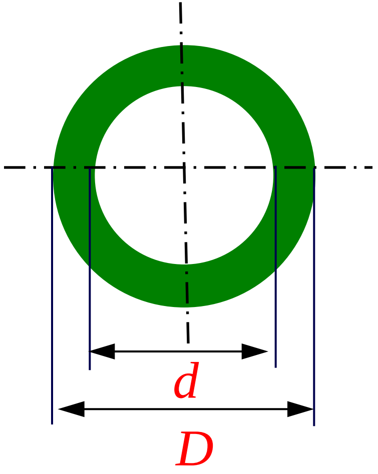
3. 纯剪切应力状态

$\varepsilon_{45°} \Leftrightarrow \tau$



例3　已知$D = 120$ mm，$d = 80$ mm的空心圆轴，两端受一对扭矩M_e作用，在轴的中部表面A点处，测得与其与母线成45°方向的线应变 $\varepsilon_{45^\circ} = 260\times10^{-6}$。已知材料的弹性模量$E$ =200GPa，μ =0.3 。 求扭矩M_e 。

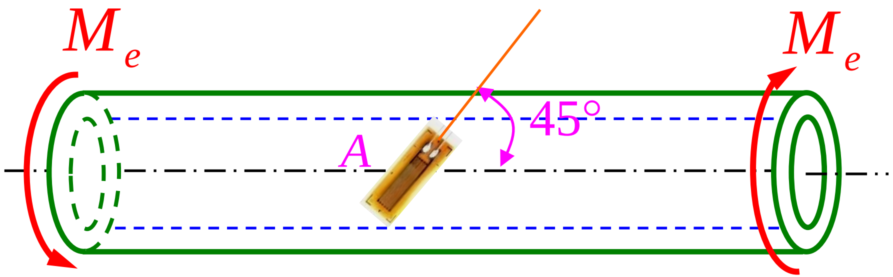


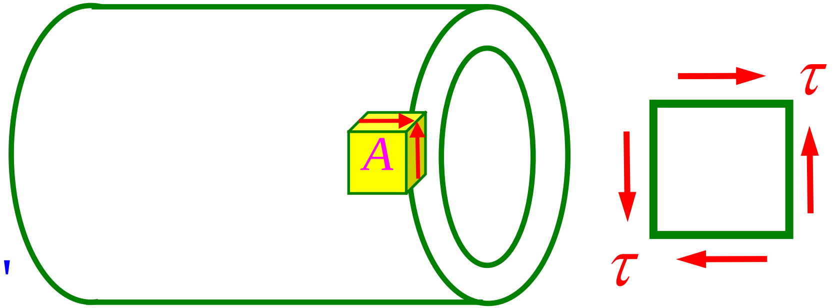


解：　画出A点单元体应力

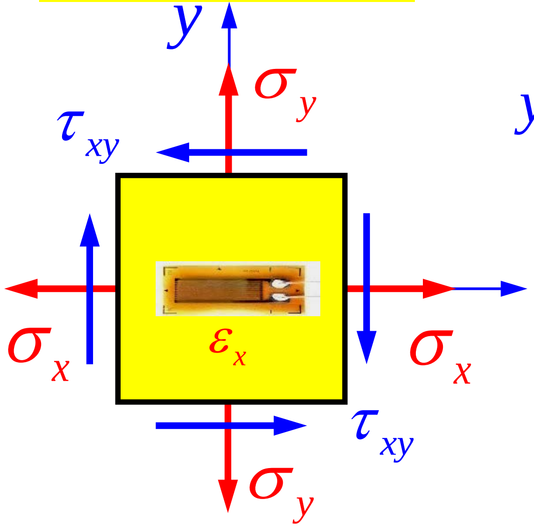
$$\tau = -\frac{M_e}{W_P} \qquad W_P = \frac{1}{16}\pi D^3\left[1-\left(\frac{d}{D}\right)^4\right]$$

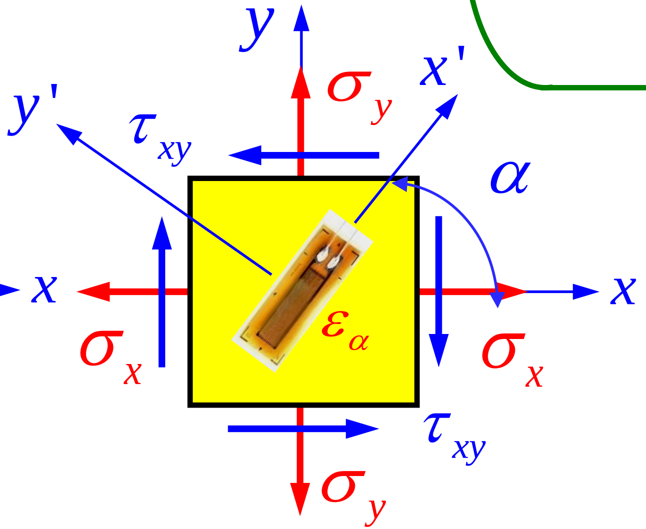
$$\varepsilon_{45^\circ} = \varepsilon_{45^\circ}(\tau) = ?$$





$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$



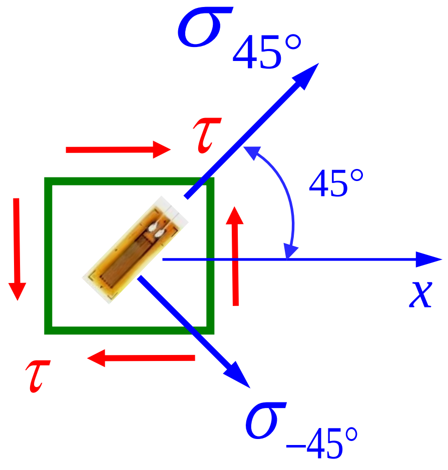


$$\varepsilon_\alpha = \frac{1}{E}(\sigma_\alpha - \mu\sigma_{\alpha+90^\circ})$$

或$\varepsilon_\alpha = \frac{1}{E}(\sigma_\alpha - \mu\sigma_{\alpha-90^\circ})$

$$\varepsilon_\alpha = \frac{1}{E}(\sigma_\alpha - \mu\sigma_{\alpha+90^0})$$

或$$\varepsilon_\alpha = \frac{1}{E}(\sigma_\alpha - \mu\sigma_{\alpha-90^0})$$



$$\alpha=45°: \quad \varepsilon_{45°} = \frac{1}{E}(\sigma_{45°} - \mu\sigma_{-45°})$$

$$\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sigma_x = 0; \quad \sigma_y = 0; \quad \tau_{xy} = \tau = -\frac{M_e}{W_p};$$

$$\sigma_{45°} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\times45°) - \tau_{xy}\sin(2\times45°) = -\tau_{xy} = \frac{M_e}{W_p}$$

$$\sigma_{-45°} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos[2\times(-45°)] - \tau_{xy}\sin[2\times(-45°)] = \tau_{xy} = -\frac{M_e}{W_p}$$

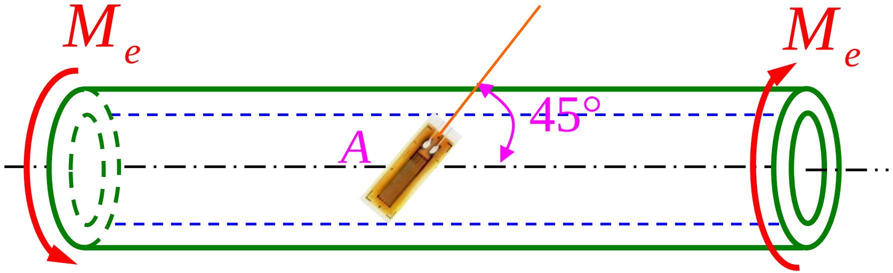
则 $$\varepsilon_{45°} = \frac{1}{E}(\sigma_{45°} - \mu\sigma_{-45°}) = \frac{1+\mu}{E}\frac{M_e}{W_p} \qquad M_e = \frac{E\varepsilon_{45°}}{1+\mu}W_p$$

$$M_e = \frac{E\varepsilon_{45^\circ}}{1+\mu} W_p$$

$$= \frac{E\varepsilon_{45^\circ}}{1+\mu} \times \frac{1}{16}\pi D^3 \left[1-\left(\frac{d}{D}\right)^4\right]$$

$$= \frac{200\times10^9 \times 260\times10^{-6}}{1+0.3} \times \frac{1}{16}\pi \times 0.12^3 \times \left[1-\left(\frac{80}{120}\right)^4\right]$$

$$= 1.0891\times10^4\ \mathrm{N\cdot m} = 10.891\ \mathrm{kN\cdot m}$$



D = 120 mm, d = 80 mm

E =200 GPa, μ =0.3

$\varepsilon_{45^\circ}=260\times10^{-6}$

谢谢各位！

作业：　P278-279: 7.27、7.29、7.31

对应第6版题号　P272-273: 7.27、7.29、7.30

下次课讲　强度理论