一

1

$$
\begin{aligned}
\frac{\mathrm{d}x_a(t)}{\mathrm{d}t} &= v_a(t)\cos(\alpha(t)) \\
\frac{\mathrm{d}y_a(t)}{\mathrm{d}t} &= v_a(t)\sin(\alpha(t)) \\
\frac{\mathrm{d}x_b(t)}{\mathrm{d}t} &= v_b(t)\cos(\beta(t)) \\
\frac{\mathrm{d}y_b(t)}{\mathrm{d}t} &= v_b(t)\sin(\beta(t))
\end{aligned}
$$

2

在海盗追击商船的过程中，海盗应选择每时每刻均沿两者直线方向的航向，此时$r(t)$减小最快。这是因为它导致 r(t)在任何时间 t 的最陡下降。如果海盗航向不沿两者直线方向，那么 r(t)的下降速度将会减慢。此时，$\theta(t)=\beta(t)$

3

$$
\begin{aligned}
\mathrm{d}r(t) &= -v_b(t)\mathrm{d}t + v_a(t)\cos(\beta(t)-\alpha(t))\mathrm{d}t \\
&= -\lambda v_a\mathrm{d}t + v_a\cos(\beta(t)-\alpha(t))\mathrm{d}t \\
&= v_a[(\cos(\beta(t)-\alpha(t))-\lambda)\mathrm{d}t]
\end{aligned}
$$

所以

$$
\begin{aligned}
\frac{\mathrm{d}r(t)}{\mathrm{d}t} &= v_a(\cos(\beta(t)-\alpha(t))-\lambda) \\
&= v_a(\cos(\theta(t)-\alpha(t))-\lambda)
\end{aligned}
$$

对于$\theta(t)$,有

$$
-\mathrm{d}\theta(t) = \sin(-\mathrm{d}\theta(t)) = \frac{v_a\mathrm{d}t\sin(\theta(t)-\alpha(t))}{r(t)}
$$

所以

$$
\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{v_a\sin(\alpha(t)-\theta(t))}{r(t)}
$$

综上

$$
\begin{cases}
\frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_a(\cos(\alpha(t)-\theta(t))-\lambda) \\
\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{\nu_a\sin(\alpha(t)-\theta(t))}{r(t)}
\end{cases}
$$

4

由上一问与条件$\alpha(t)\equiv 0$,我们可以得到微分方程组如下:

$$
\begin{cases}
\frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_a(\cos(\theta(t))-\lambda) \\
\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -\frac{v_a\sin(\theta(t))}{r(t)}
\end{cases}
$$

两式相除，得到

$$
\frac{\mathrm{d}r(t)}{\mathrm{d}\theta(t)} = \frac{r(t)}{\sin(\theta(t))}(\lambda-\cos(\theta(t)))
$$

所以

$$
\begin{aligned}
\frac{\mathrm{d}r(t)}{r(t)} &= \frac{\cos(\theta(t))-\lambda}{\sin(\theta(t))}\mathrm{d}\theta(t) \\
\ln(r(t)) &= \lambda\ln\left(\sin\left(\frac{\theta(t)}{2}\right)\right)-\lambda\ln\left(\cos\left(\frac{\theta(t)}{2}\right)\right)-\ln(\sin(\theta(t)))+C \\
r(t) &= C\frac{1}{\sin(\theta(t))}\left(\tan\frac{\theta(t)}{2}\right)^{\lambda}
\end{aligned}
$$

$t=0$ 时，$r(0)=\sqrt{x_0^2+y_0^2}, \sin\theta(0)=\frac{-y_0}{\sqrt{x_0^2+y_0^2}}, \cos\theta(0)=\frac{-x_0}{\sqrt{x_0^2+y_0^2}}, \tan\frac{\theta(0)}{2}=\frac{1-\cos\theta(0)}{\sin\theta(0)}=\frac{\sqrt{x_0^2+y_0^2+x_0}}{-y_0}$,所以

$$
C=-y_0\left(\frac{x_0-\sqrt{x_0^2+y_0^2}}{y_0}\right)^{\lambda}
$$

所以，$r(t)$与$\theta(t)$的关系为

$$
r(t)=-y_0\Big(\frac{x_0-\sqrt{x_0^2+y_0^2}}{y_0}\Big)^{\lambda}\frac{1}{\sin(\theta(t))}\Big(\tan(\frac{\theta(t)}{2})\Big)^{\lambda}
$$

当$\lambda=1$时，$r(t)$与$\theta(t)$的关系为

$$
\begin{aligned}
r(t) &= -y_0\Big(\frac{x_0 - \sqrt{x_0^2+y_0^2}}{y_0}\Big)\frac{1}{\sin(\theta(t))}\Big(\tan(\frac{\theta(t)}{2})\Big) \\
&= -y_0\Big(\frac{x_0 - \sqrt{x_0^2+y_0^2}}{y_0}\Big)\frac{1}{\sin(\theta(t))}\Big(\frac{\sin\theta(t)}{1+\cos(\theta(t))}\Big) \\
&= \frac{\sqrt{x_0^2+y_0^2}-x_0}{1+\cos(\theta(t))}
\end{aligned}
$$

所以

$$
\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -\frac{\nu_a\sin(\theta(t))}{r(t)} = -\frac{\nu_a\sin(\theta(t))}{\frac{\sqrt{x_0^2+y_0^2}-x_0}{1+\cos(\theta(t))}} = -\frac{\nu_a(1+\cos(\theta(t)))\sin(\theta(t))}{\sqrt{x_0^2+y_0^2}-x_0}
$$

所以

$$
\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -\frac{v_a(1+\cos(\theta(t)))\sin(\theta(t))}{\sqrt{x_0^2+y_0^2}-x_0}
$$

$$
\frac{\mathrm{d}\theta(t)}{(1+\cos(\theta(t)))\sin(\theta(t))} = -\frac{v_a}{\sqrt{x_0^2+y_0^2}-x_0}dt
$$

$$
\frac{1-2\cos^2(\frac{\theta(t)}{2})(\ln(\cos(\frac{\theta(t)}{2}))-\ln(\sin(\frac{\theta(t)}{2})))}{2\,(\cos(\theta(t))+1)} = -\frac{v_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C
$$

$$
\frac{1-(\cos(\theta(t))+1)(\ln(\frac{1}{\tan(\frac{\theta(t)}{2})}))}{2\,(\cos(\theta(t))+1)} = -\frac{\nu_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C
$$

$$
\frac{1}{2(\cos(\theta(t))+1)}+\frac{\ln(\tan(\frac{\theta(t)}{2}))}{2} = -\frac{\nu_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C
$$

$$
\frac{1}{2(\cos(\theta(t))+1)}+\frac{\ln(\sqrt{\frac{1-\cos(\theta(t))}{1+\cos(\theta(t))}})}{2} = -\frac{\nu_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C
$$

$$\frac{1}{2(\cos(\theta(t))+1)}+\frac{1}{4}\ln(\frac{1-\cos(\theta(t))}{1+\cos(\theta(t))})=-\frac{\nu_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C$$

将$t=0$代入上式，可以得到C的值。所以我们可以得到$\theta(t)$的表达式。同时，因为我们已经知道了$r(t)$与$\theta(t)$的关系，所以我们可以得到$r(t)$的表达式。

二

1

设运动的距离为 s,则$s=n\theta a$,所以设运动的距离为 s,则$s=n\theta a$,所以

$$\mathrm{d}x=(\mathrm{d}s)\cos\omega=na\cos\omega\mathrm{d}\theta$$

$$\mathrm{d}y=(\mathrm{d}s)\sin\omega=na\sin\omega\mathrm{d}\theta$$

所以:

$$\mathrm{d}x=(\mathrm{d}s)\cos\omega=na\cos\omega\mathrm{d}\theta$$

$$\mathrm{d}y=(\mathrm{d}s)\sin\omega=na\sin\omega\mathrm{d}\theta$$

所以:

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}\theta} & =na\cos\omega\\ \frac{\mathrm{d}y}{\mathrm{d}\theta} & =na\sin\omega\end{cases}$$

2

A点的坐标为$(a\cos\theta,a\sin\theta)$,所以B点的坐标为$(a\cos\theta-\rho\cos\omega,a\sin\theta-\rho\sin\omega)$,切线方向为AB
的方向，所以切线方程为:

$$y=\tan\omega(x-a\cos\theta)+a\sin\theta$$

3

已知 $x=a\cos\theta-\rho\cos\omega$, $y=a\sin\theta-\rho\sin\omega$,所以:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta - (\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\cos\omega - \rho\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\sin\omega) = na\cos\omega$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\cos\theta - (\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\sin\omega + \rho\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\cos\omega) = na\sin\omega$$

所以:

$$\begin{cases} \frac{\mathrm{d}\rho}{\mathrm{d}\theta} & = a\sin(\omega-\theta) - na \\ \\ \frac{\mathrm{d}\omega}{\mathrm{d}\theta} & = \frac{a}{\rho}(\cos(\omega-\theta)) \end{cases}$$

消去含θ的项，得:

$$(\frac{\mathrm{d}\rho}{\mathrm{d}\theta} + na)^2 + (\rho\frac{\mathrm{d}\omega}{\mathrm{d}\theta})^2 = a^2$$