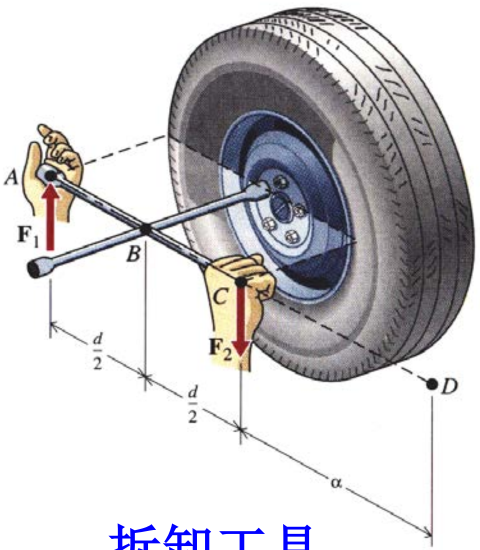
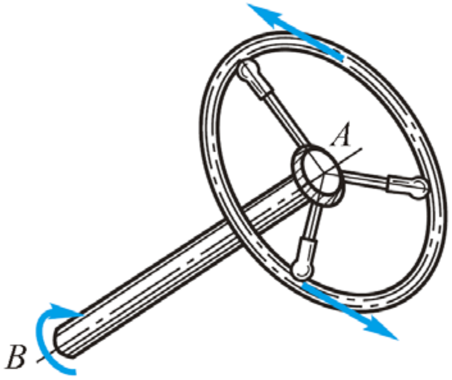
第三章　扭转（一）

第 8 讲

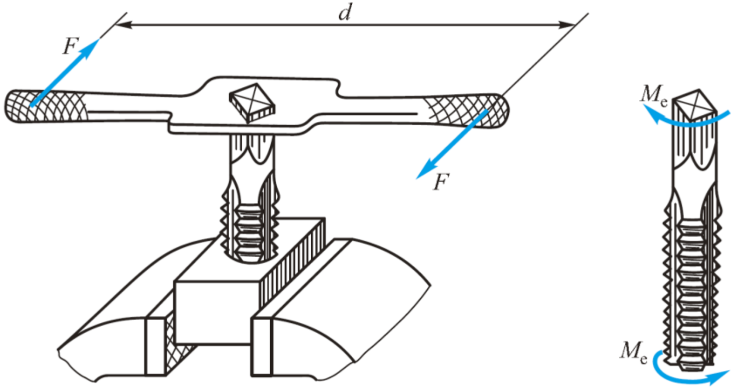
§ 3.1 扭转的概念及实例



拆卸工具

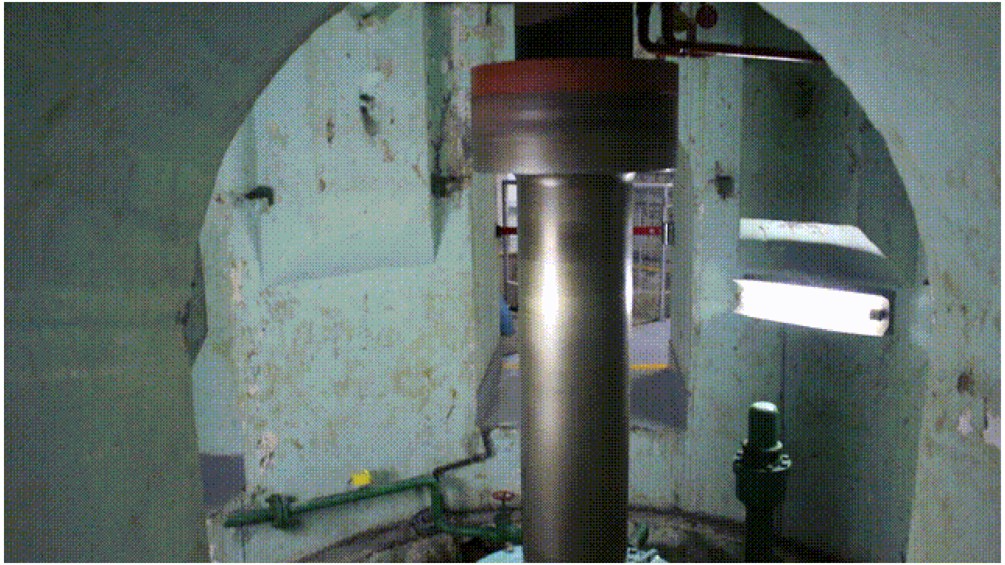


汽车转向轴

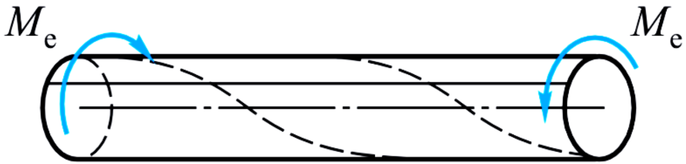


丝锥

传动轴



传动轴

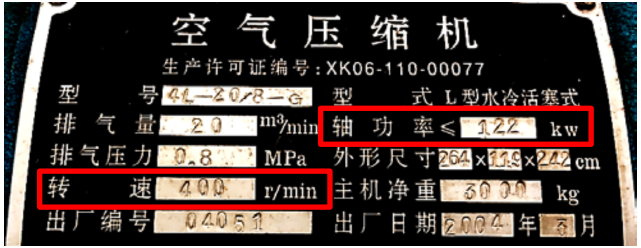


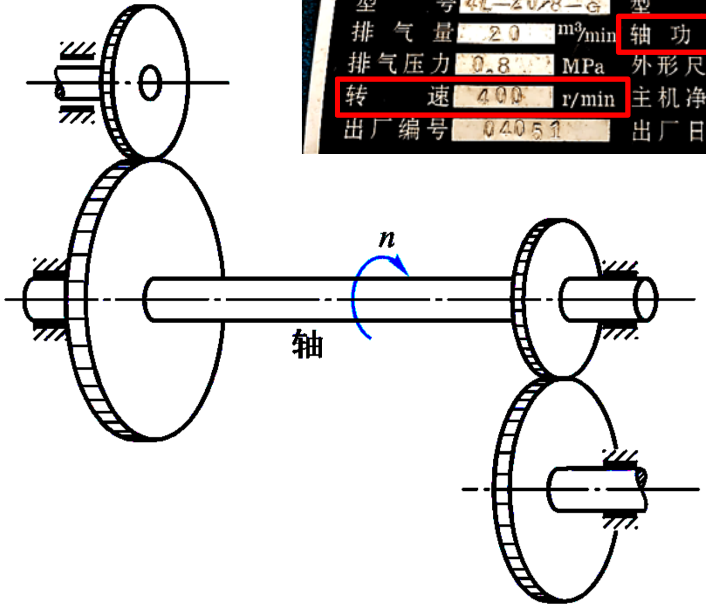
杆件的受力和变形特征：

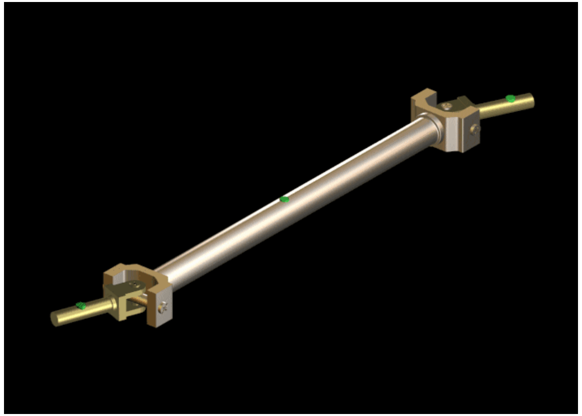
杆件受两个大小相等、方向相反、且作用平面垂直于杆件轴线的力偶。致使杆件的任意两个截面都发生绕轴线的相对转动，这就是扭转变形。

§ 3.2 传动轴的外力偶矩 扭矩 扭矩图

I、外力偶矩的计算







设某传动轴所传递的功率是 P (单位：kW)

轴的转速是 n (单位：rpm或r/min —revolution per minute)

外力偶矩的计算：

P(kW)的功率相当于每分钟作功：

$$W_1 = P \times 1000 \times 60 (\mathrm{J}) \qquad (1)$$

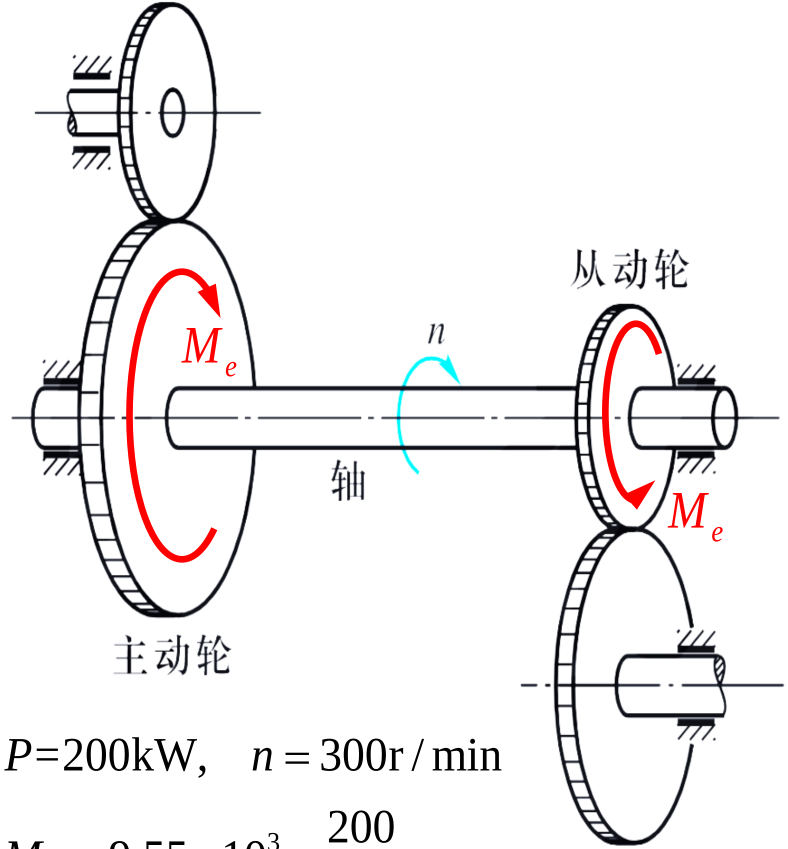
外力偶矩M_e每分钟所作的功：

$$W_2 = M_e \cdot 2\pi n (\mathrm{J}) \qquad (2)$$

(1)=(2) 得：$P \times 1000 \times 60 = M_e \cdot 2\pi n$

$$M_e = \frac{P \times 1000 \times 60}{2\pi n} = 9550 \frac{P}{n}$$

$$\{M_e\}_{\mathrm{N.m}} = 9.55 \times 10^3 \frac{\{P\}_{\mathrm{kW}}}{\{n\}_{r/\min}} \text{①} \quad \begin{cases} P — \mathrm{kW} \\ n — \mathrm{rpm} \\ M_e — \mathrm{N \cdot m} \end{cases}$$



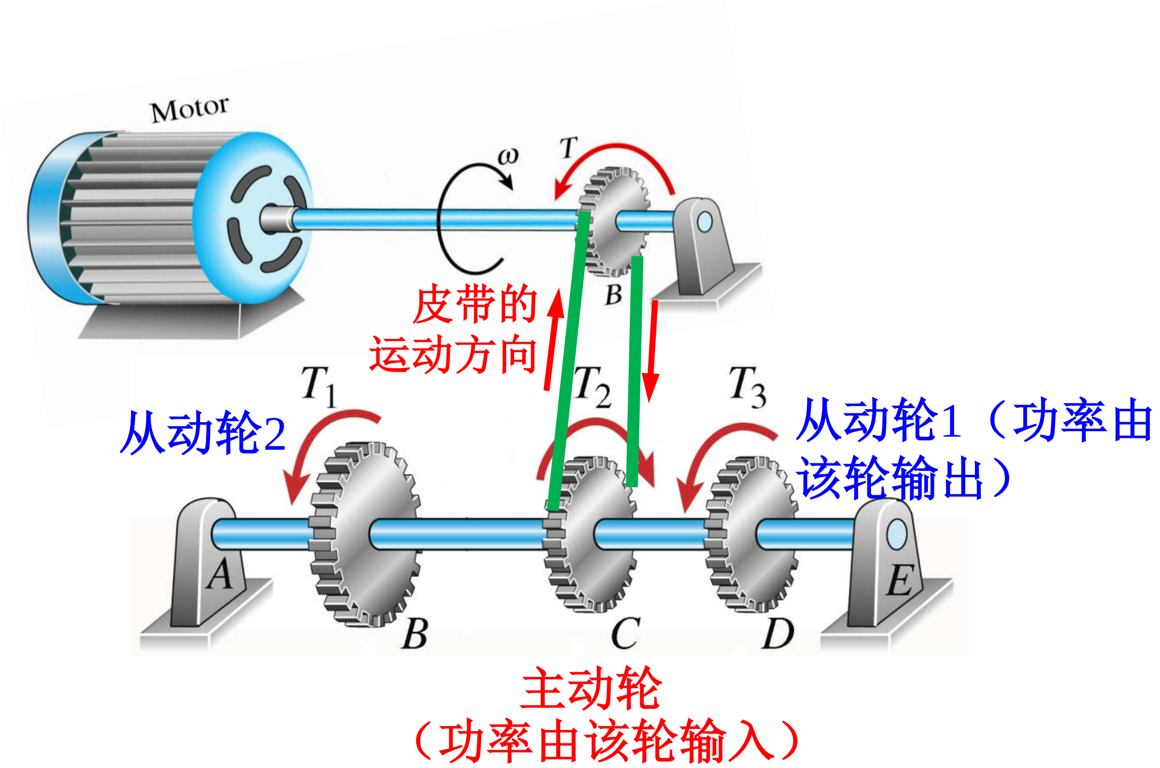
P=200kW，$n = 300\mathrm{r/min}$

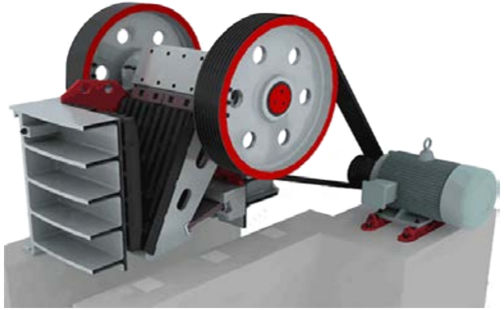
$$M_e = 9.55 \times 10^3 \times \frac{200}{300}$$

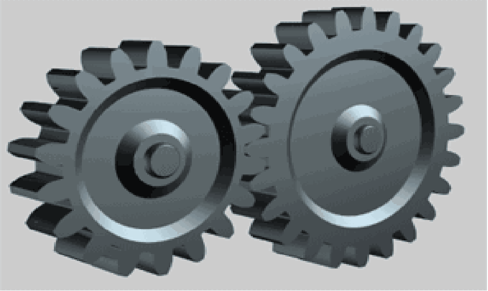
$$= 6.367 \times 10^3 \mathrm{N \cdot m} = 6.367 \mathrm{kN \cdot m}$$

① 数值方程式，GB 3101–93

传动轴上外力偶矩的方向



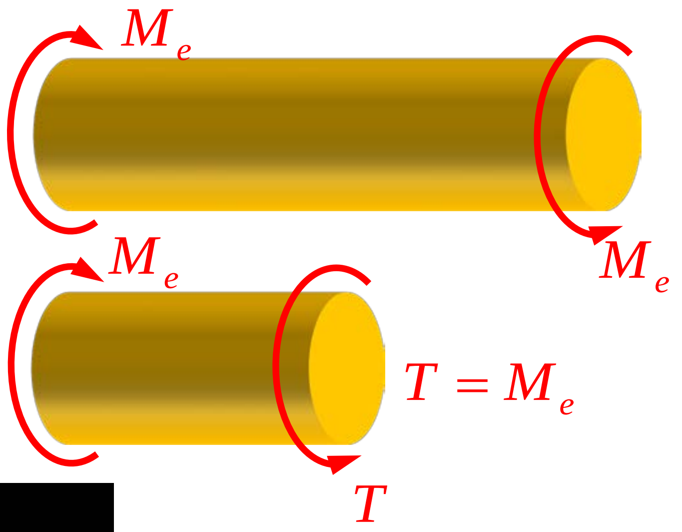




主动轮上的外力偶矩的方向与轴的转动方向相同

从动轮上的外力偶矩的方向与轴的转动方向相反

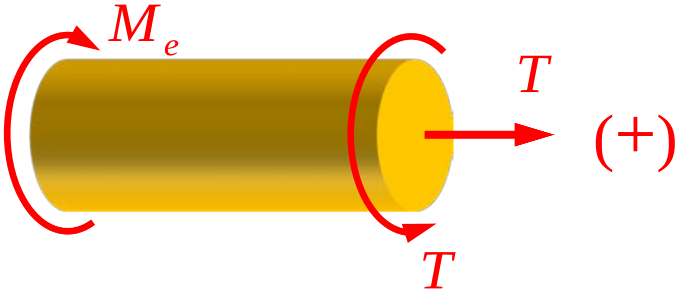
II、扭矩和扭矩图

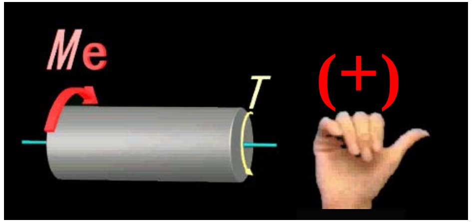


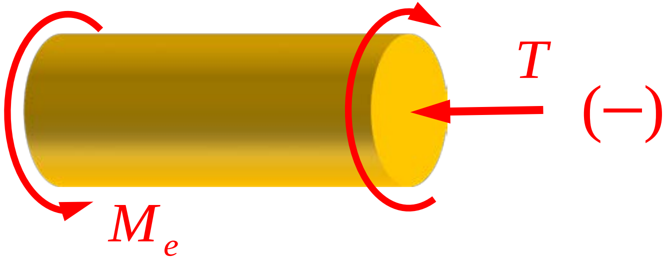
扭矩：T

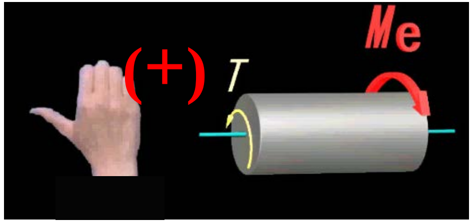
扭矩的正负号规定：

将扭矩按右手螺旋法则用力偶矢来表示，则当力偶矢的指向离开截面扭矩为正，反之为负。



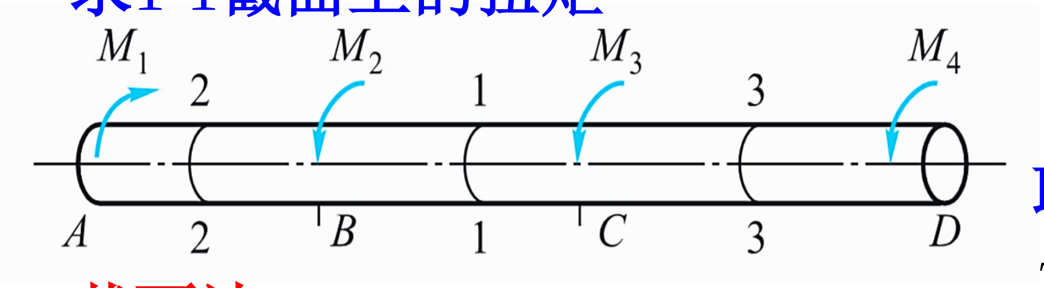




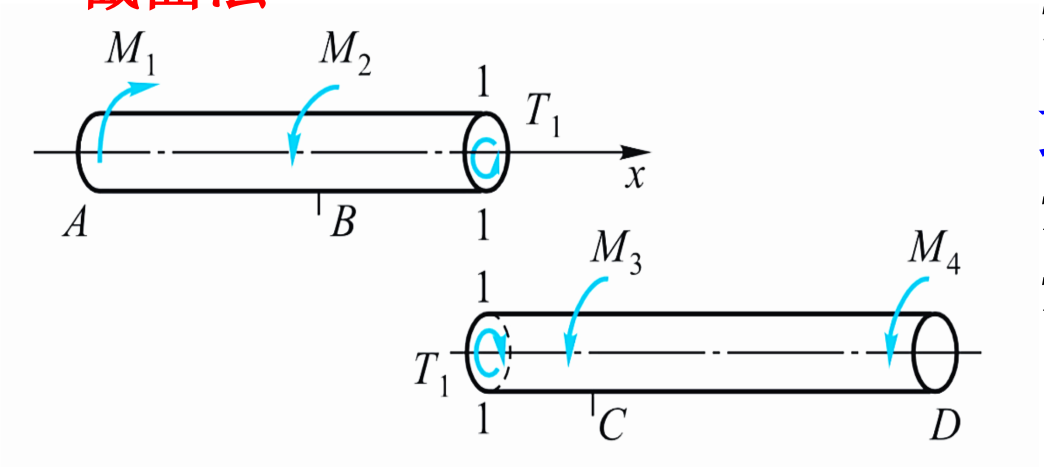


扭矩的计算

求1-1截面上的扭矩



截面法



$$M_1 = 6.0\,\mathrm{kN\cdot m},\quad M_2 = 1.0\,\mathrm{kN\cdot m}$$

$$M_3 = 2.0\,\mathrm{kN\cdot m},\quad M_4 = 3.0\,\mathrm{kN\cdot m}$$

取左段分析：

$$T_1 + M_2 - M_1 = 0$$

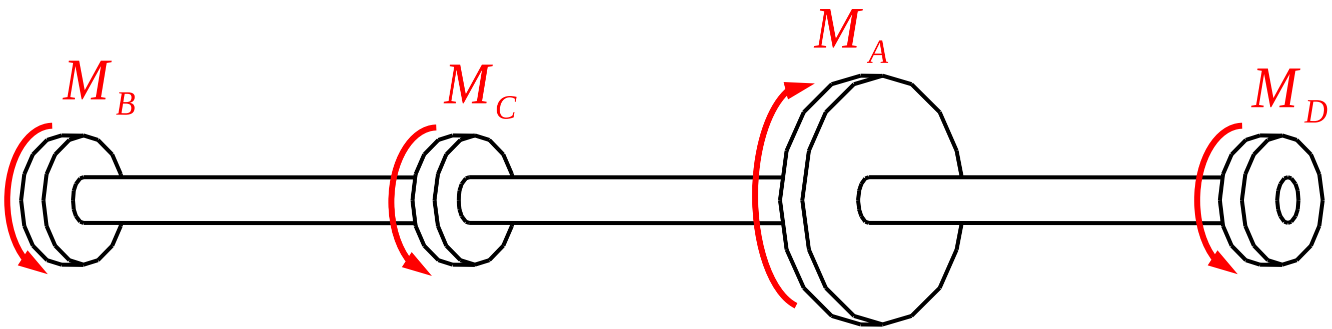
$$T_1 = M_1 - M_2 = 6.0 - 1.0 = 5.0\,\mathrm{kN\cdot m}$$

取右段分析：

$$T_1 - M_3 - M_4 = 0$$

$$T_1 = M_3 + M_4 = 2.0 + 3.0 = 5.0\,\mathrm{kN\cdot m}$$

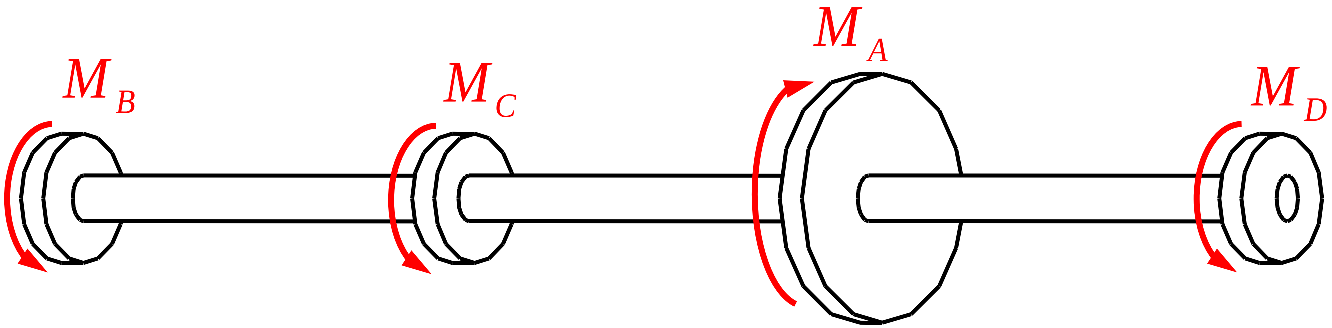
例1　图示传动轴，主动轮A 输入功率P_A=36.765 千瓦，从动轮B、C、D输出功率分别为 P_B=P_C=11.029千瓦，P_D=14.707千瓦，轴的转速为n=300转/分。作轴的扭矩图。



解： $M_e = 9550\dfrac{P}{n}(\mathrm{N}\cdot\mathrm{m})$

$P_A = 36.765\,\mathrm{kW}$;　$P_B = P_C = 11.029\mathrm{kW}$

$P_D = 14.707\,\mathrm{kW}$;　n=300 rpm

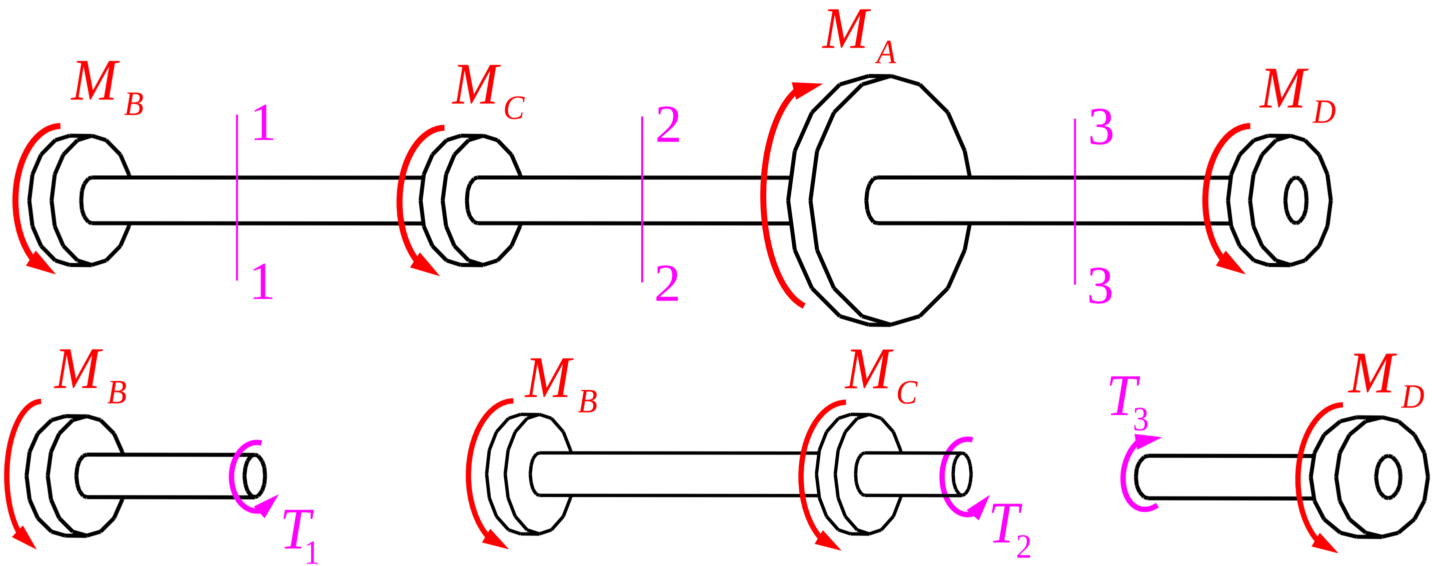


$$M_A = 9550\frac{P_A}{n} = 9550\times\frac{36.765}{300} = 1170\ \mathrm{N\cdot m}$$

$$M_B = M_C = 9550\frac{P_B}{n} = 9550\times\frac{11.029}{300} = 351\ \mathrm{N\cdot m}$$

$$M_D = 9550\frac{P_D}{n} = 9550\times\frac{14.707}{300} = 468\ \mathrm{N\cdot m}$$

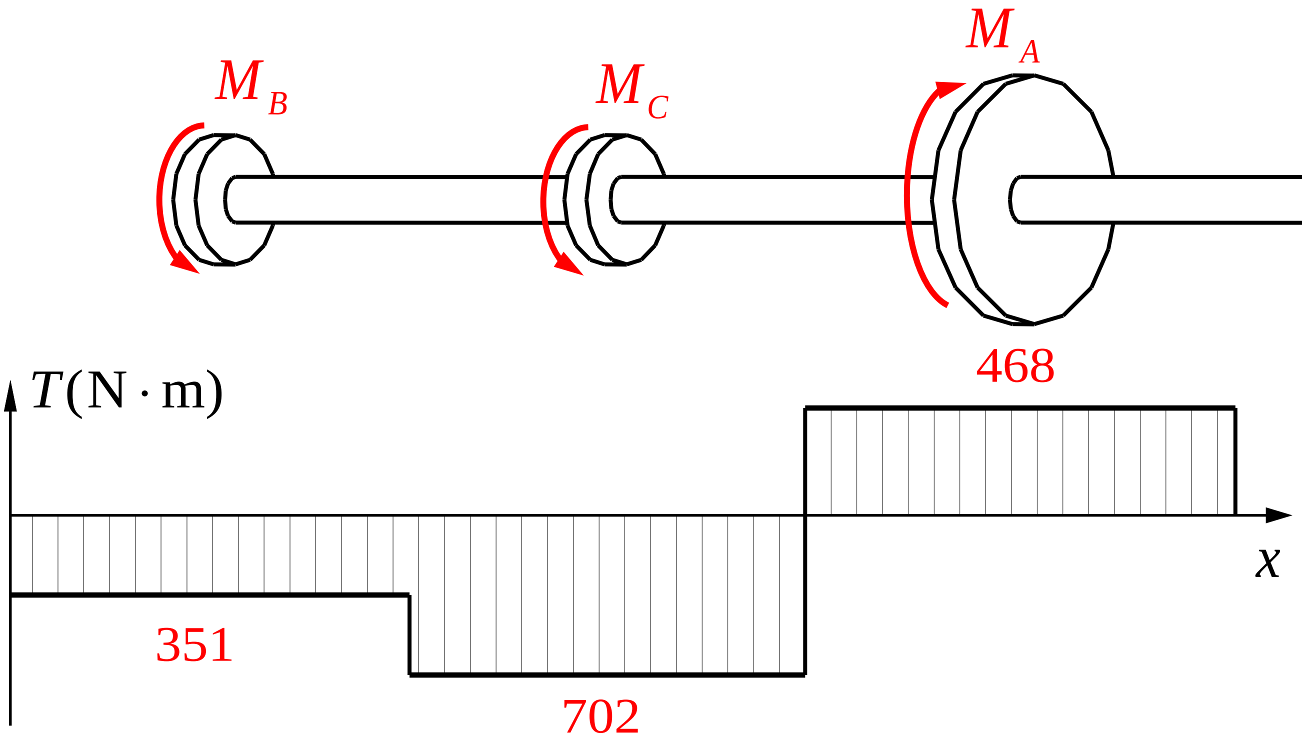
$M_A = 1170\ \mathrm{N\cdot m}$, $M_B = M_C = 351\ \mathrm{N\cdot m}$, $M_D = 468\ \mathrm{N\cdot m}$



$$T_1 = -M_B = -351\ \mathrm{N\cdot m}$$

$$T_2 = -(M_B + M_C) = -702\ \mathrm{N\cdot m}$$

$$T_3 = M_D = 468\ \mathrm{N\cdot m}$$



$T_1 = -351\ \mathrm{N \cdot m}$

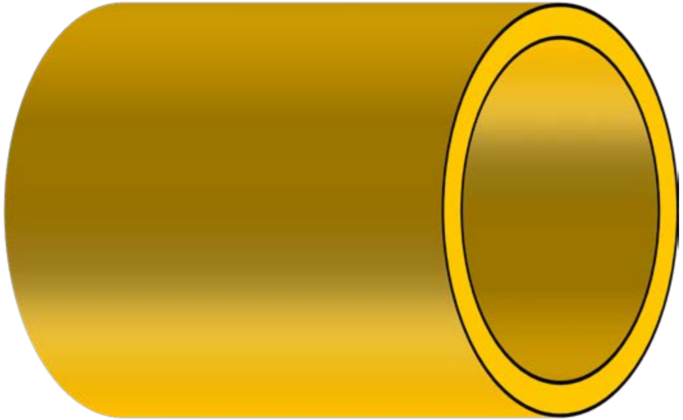
$T_2 = -702\ \mathrm{N \cdot m}$

$T_3 = 468\ \mathrm{N \cdot m}$

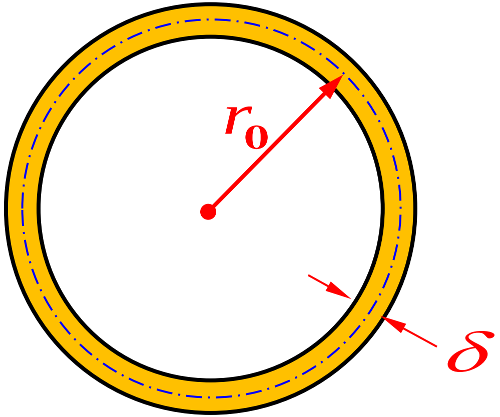
扭矩图

§ 3.3 纯剪切

一、薄壁圆筒的扭转应力分析



薄壁圆筒 $\delta \le \frac{r_0}{10}$



平均半径为 r_0，壁厚为 δ

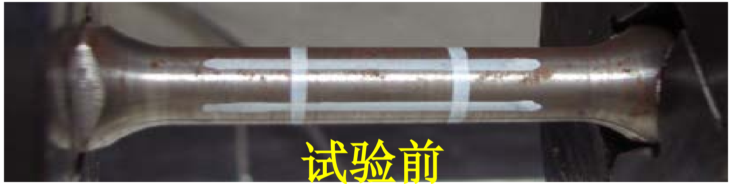
扭转试验

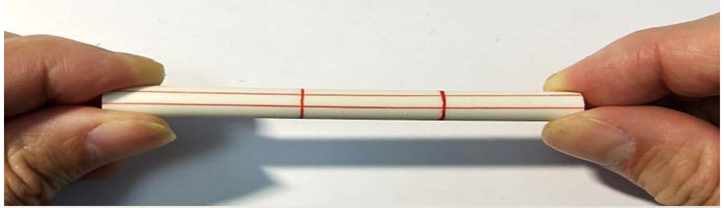
扭转试样

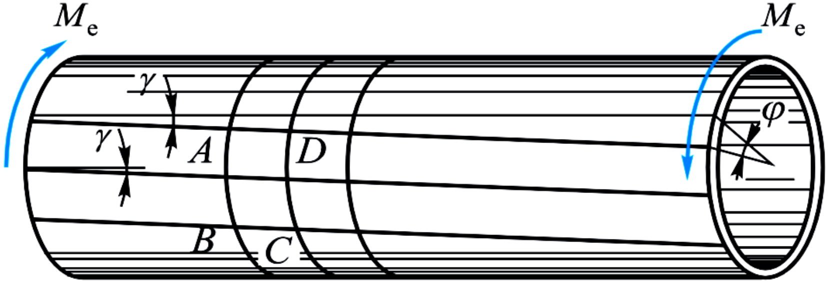
扭转试验机

扭转试验视频

扭转试验及现象观察



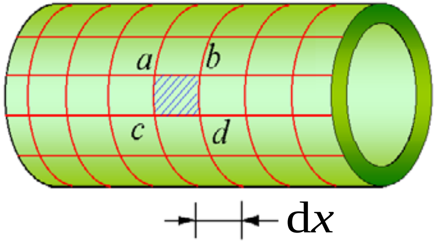


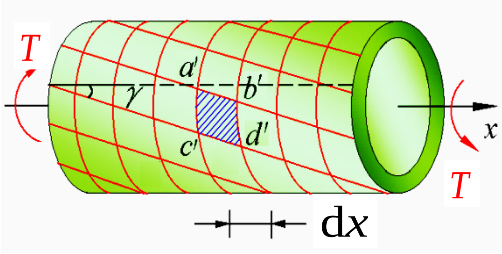


扭转变形示意图

观察到如下现象：

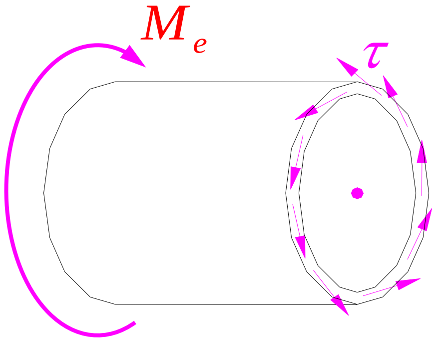
1. 圆周线的形状、大小及圆周线之间的距离没有改变；
2. 纵向线均倾斜了同一角度。

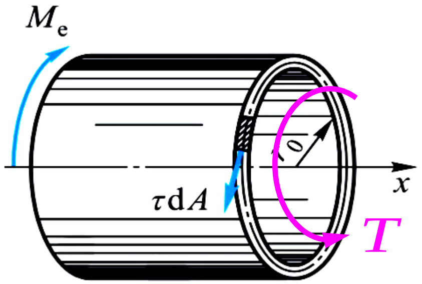


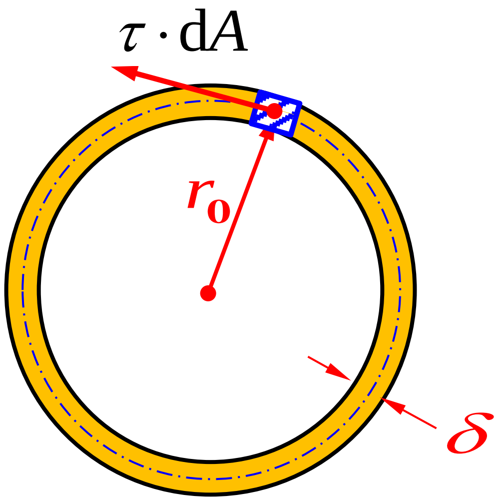


根据以上实验现象，可得结论：

1. 圆筒（杆）横截面上没有正应力，只有切应力；
2. 沿圆周各点的切应力数值上相等，方向垂直于半径；
3. 圆筒（杆）横截面上的切应力最终合成扭矩T。







横截面上切应力

$$\int_{环} r_0 \cdot \tau \mathrm{d}A = T$$

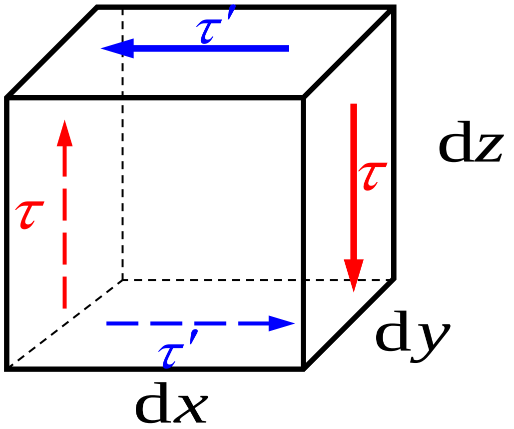
$$r_0 \cdot \tau \int_{环} \mathrm{d}A = T$$

$$r_0 \cdot \tau \cdot 2\pi r_0 \delta = T$$

$$\tau = \frac{T}{2\pi r_0^2 \delta}$$

利用精确理论分析可知，当$\delta \le r_0/10$时，上式的误差不超过4.52%（< 5%），是足够精确的。

二、切应力互等定理



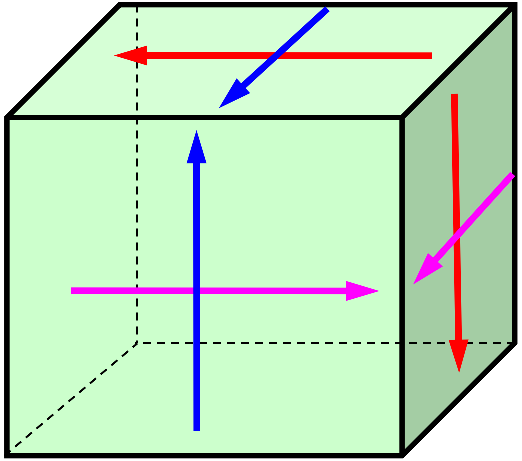
微元体（单元体）
考虑仅有切应力情形

切应力互等定理：

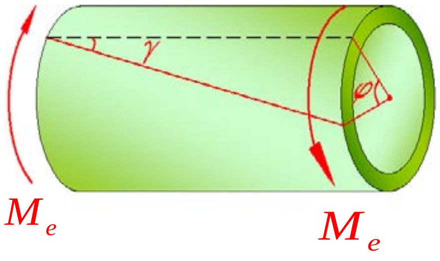
在相互垂直的两个平面上，切应力一定成对出现，其数值相等，方向同时指向或背离两平面的交线。

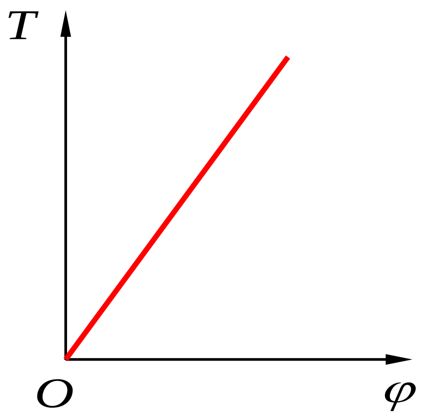
$$(\tau \cdot \mathrm{d}z\mathrm{d}y)\mathrm{d}x = (\tau' \cdot \mathrm{d}x\mathrm{d}y)\mathrm{d}z$$

$$\tau = \tau'$$



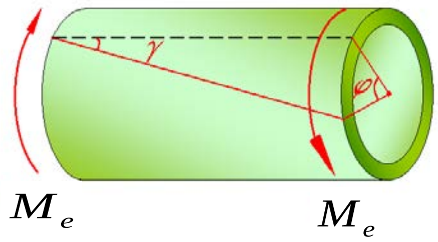
三、剪切胡克定律





由实验可观察到：

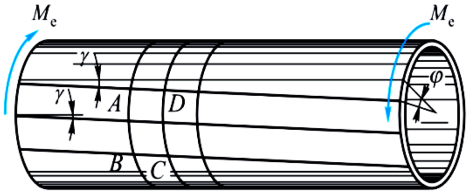
当力偶矩在某一范围内时，相对扭转角φ与外力偶矩 M_e（数值上等于扭矩T）成正比。



几何关系式（小变形情形）

$\varphi r = l\gamma$ （γ 称为切应变）

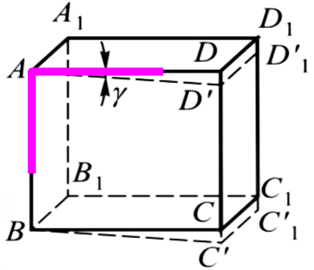
$\gamma \propto \varphi$



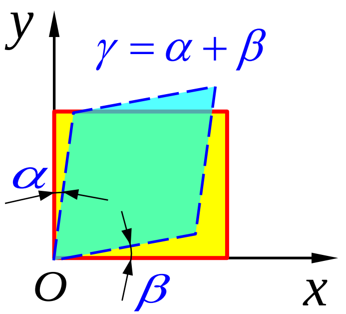
关于切应变的定义（见第一章 第7页）

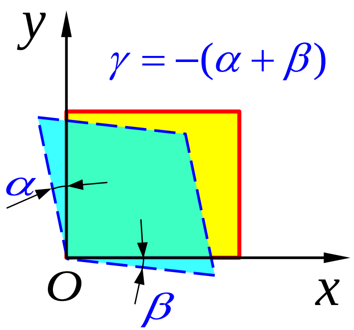
直角的改变量（通常规定直角的减小为正）

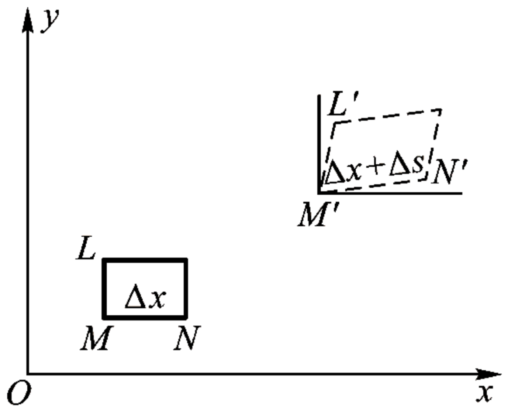
$$\gamma = \lim_{\substack{MN \to 0 \\ ML \to 0}} \left(\frac{\pi}{2} - \angle L'M'L' \right)$$



γ 为切应变





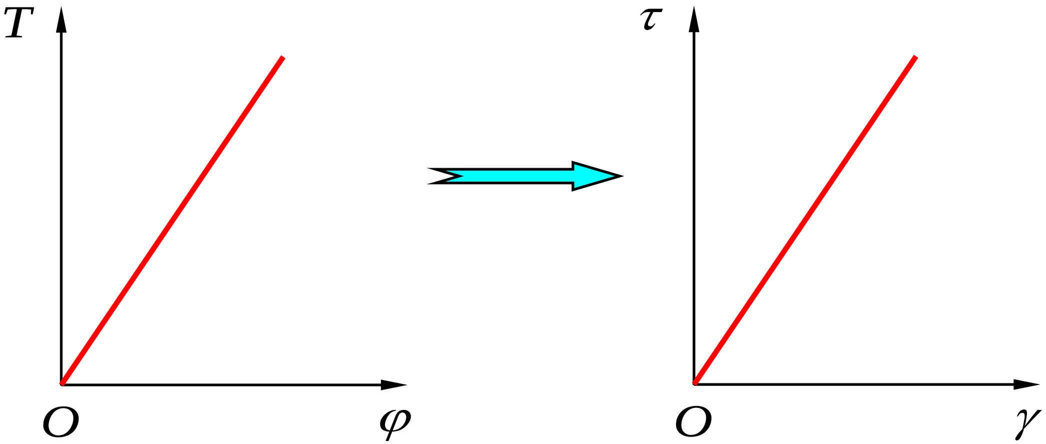


$$\varphi r = l\gamma$$

$$\gamma \propto \varphi$$

$$\tau = \frac{T}{2\pi r_0^2 \delta}$$

$$\tau \propto T$$



薄壁圆筒的扭转实验，证实了切应力与切应变之间存在线性关系，即当切应力不超过材料的剪切比例极限 τ_p 时，**切应力与切应变成正比** $\tau = G\gamma$ 称为**剪切胡克定律**

G 称为材料的切变模量。

$$\tau = G\gamma$$

G 的单位（国际单位制）：N/m^2（Pa）；

钢材的切变模量值约为 G＝80 GPa

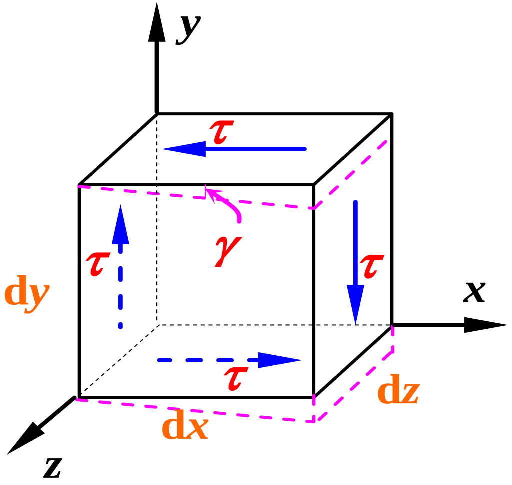
材料常数 $\begin{cases} \text{切变模量 } G \\ \text{杨氏模量 } E \\ \text{泊 松 比 } \mu \end{cases}$

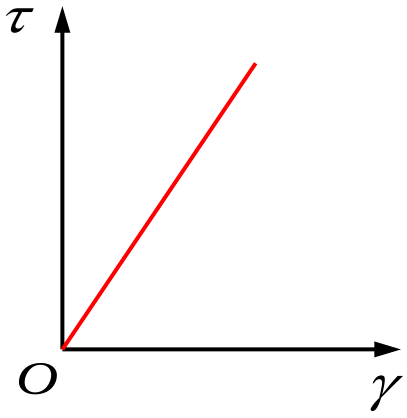
对于各向同性材料, 可以证明:

E、G 和 μ 三个弹性常数之间存在着如下关系

$$G = \frac{E}{2(1+\mu)}$$

四、剪切应变能





应变能密度（纯剪切）

$$v_\varepsilon = \frac{1}{2}\tau\gamma$$

$$\tau = G\gamma$$

$$v_\varepsilon = \frac{\tau^2}{2G} \qquad v_\varepsilon = \frac{1}{2}G\gamma^2$$

在线弹性范围内，单元体上外力做功

$$\mathrm{d}W = \tfrac{1}{2}(\tau\,\mathrm{d}y\mathrm{d}z)\cdot\gamma\,\mathrm{d}x = \tfrac{1}{2}\tau\gamma\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$v_\varepsilon = \frac{\mathrm{d}V_\varepsilon}{\mathrm{d}V} = \frac{\mathrm{d}W}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} = \frac{1}{2}\tau\gamma$$

扭转情形应变能计算：

$$V_\varepsilon = \iiint_V v_\varepsilon \mathrm{d}V$$

§3.4 等直圆杆扭转时的应力

一、横截面上的应力

从三方面考虑
- 几何关系
- 物理关系
- 静力关系

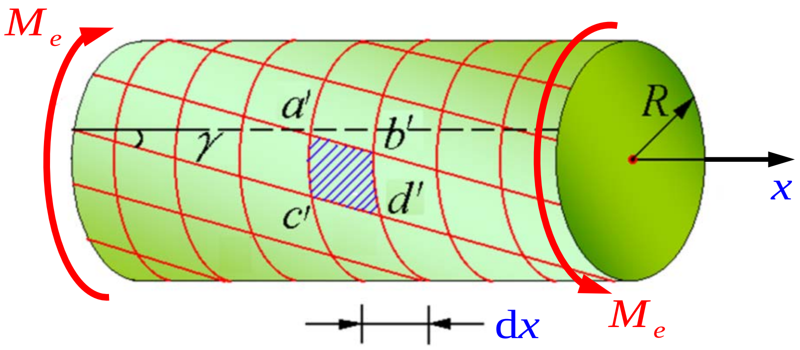
1. 几何关系

观察到下列现象：

1）各圆周线的形状、大小以及两圆周线间的距离没有变化；

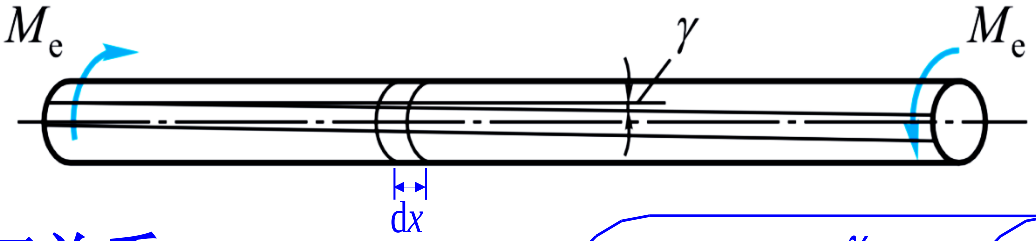
2）纵向线仍近似为直线，但都倾斜了同一角度；

3）变形前为平面的横截面变形后仍为平面，它像刚性平面一样绕杆的轴线旋转了一个角度。



平面假设

（只适用于等直圆杆）



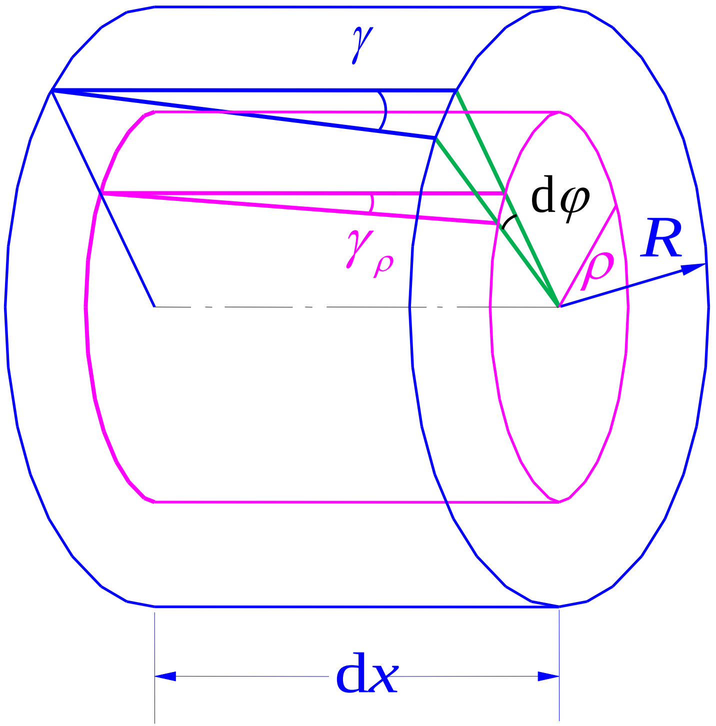
几何关系

$$\gamma_\rho \mathrm{d}x = \rho\, \mathrm{d}\varphi$$

$$\gamma_\rho = \rho \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

在外表面

$$\gamma = R \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$



2. 物理关系

根据剪切胡克定律，当切应力不超过材料的剪切比例极限时

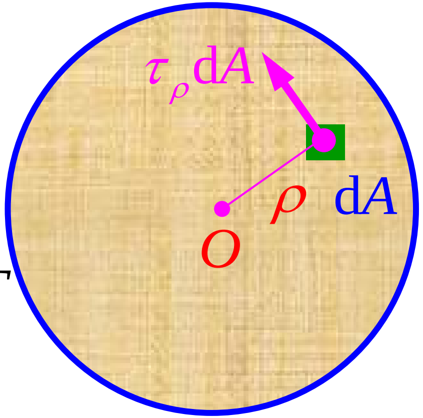
$$\tau_\rho = G\gamma_\rho \Longrightarrow \tau_\rho = G\rho\frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

切应力方向垂直于半径

$$\gamma_\rho = \rho\frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

3. 静力关系

$$\int_A \rho\cdot\tau_\rho \mathrm{d}A = T \Longrightarrow \int_A \rho\cdot G\rho\frac{\mathrm{d}\varphi}{\mathrm{d}x}\mathrm{d}A = T$$



$$G\frac{\mathrm{d}\varphi}{\mathrm{d}x}\boxed{\int_A \rho^2\mathrm{d}A} = T \Longrightarrow GI_\mathrm{p}\frac{\mathrm{d}\varphi}{\mathrm{d}x} = T \Longrightarrow \frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{T}{GI_\mathrm{p}}$$

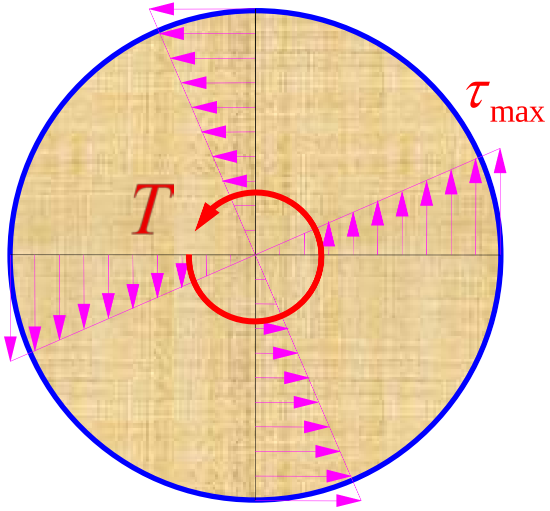
令 $I_\mathrm{p} = \int_A \rho^2\mathrm{d}A$

$$\tau_\rho = G\rho\frac{\mathrm{d}\varphi}{\mathrm{d}x} = G\rho\frac{T}{GI_\mathrm{p}} = \frac{T\rho}{I_\mathrm{p}}$$

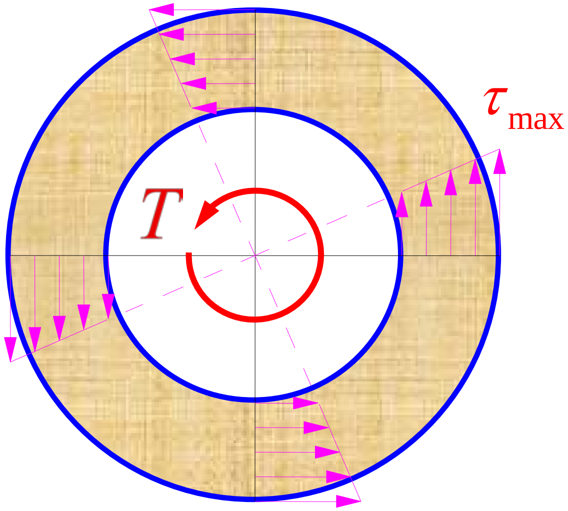
I_p – 极惯性矩

扭转切应力的分布

$$\boxed{\tau_\rho = \frac{T\rho}{I_p}} \qquad \tau_{max} = \frac{T\rho_{max}}{I_p} = \frac{T}{W_p} \qquad W_p = \frac{I_p}{\rho_{max}}$$ 抗扭截面系数



横截面

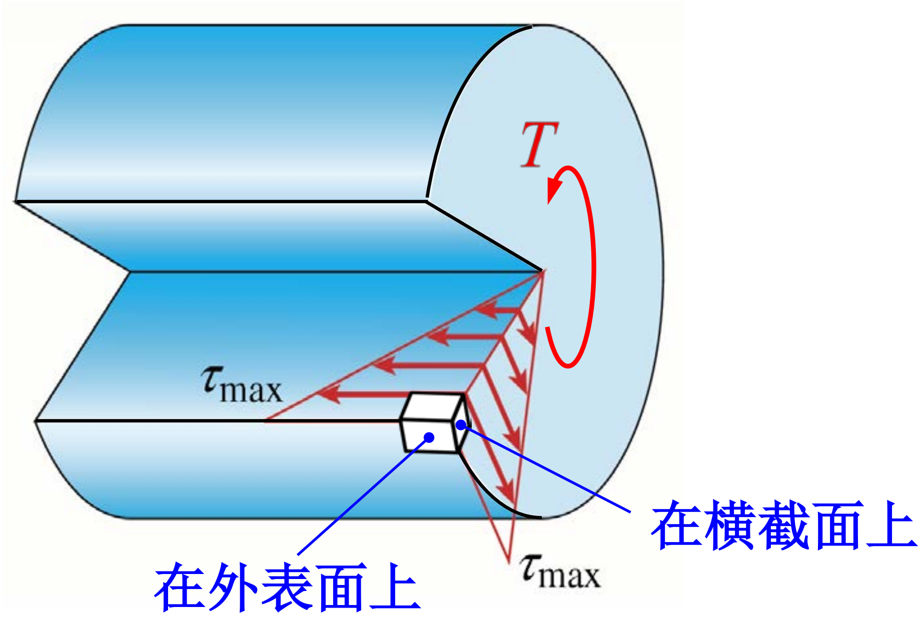


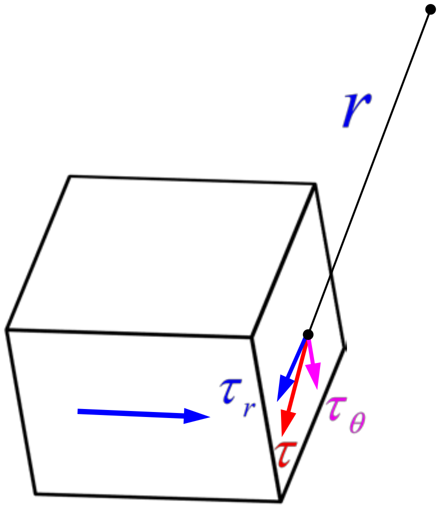
横截面

圆轴扭转时的强度条件

$$\tau_{max} = \frac{T}{W_p} \le [\tau]$$

切应力方向垂直于半径方向的说明





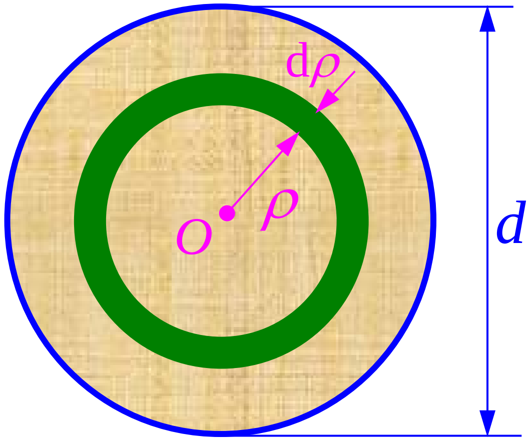
假设切应力τ的方向不垂直于半径方向

外直径为d 的实心圆截面：极惯性矩I_p和抗扭截面系数W_p

$$I_p = \int_A \rho^2 \mathrm{d}A = \int_0^{d/2} \rho^2 2\pi\rho \mathrm{d}\rho$$

$$= 2\pi \int_0^{d/2} \rho^3 \mathrm{d}\rho = 2\pi \frac{(d/2)^4}{4} = \frac{\pi d^4}{32}$$

$$W_p = \frac{I_p}{\rho_{max}} = \frac{I_p}{d/2} = \frac{\pi d^3}{16}$$



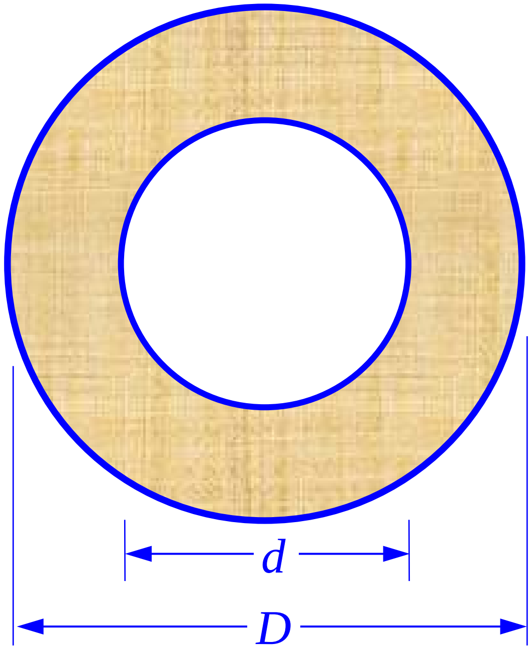
外直径为D、内直径为d 的空心圆截面：

$$I_p = \int_A \rho^2 \mathrm{d}A = \int_{d/2}^{D/2} \rho^2 2\pi\rho \mathrm{d}\rho$$

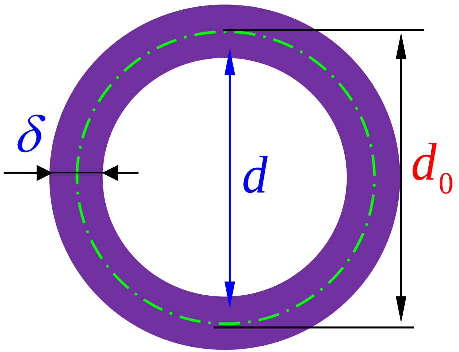
$$= \frac{\pi(D^4 - d^4)}{32} = \frac{\pi D^4(1-\alpha^4)}{32}$$

$$W_p = \frac{I_p}{\rho_{max}} = \frac{I_p}{D/2} \qquad \alpha = \frac{d}{D}$$

$$= \frac{\pi D^3}{16}(1-\alpha^4)$$



例2　一厚度为30mm、内直径为230mm 的空心圆管，承受扭矩T=180kN·m 。试求管中的最大切应力。1）用薄壁管的近似理论；2）用精确的扭转理论。



解：1 ）用薄壁管的近似理论

$$\tau_{薄壁} = \frac{T}{2\pi r_0^2 \delta} = \frac{180\times10^3}{2\,\pi\times0.13^2\times0.03} = 56.5\,\text{MPa}$$

$$d_0 = d+\delta = 230+30 = 260\text{mm} \qquad r_0 = 130\text{mm}$$

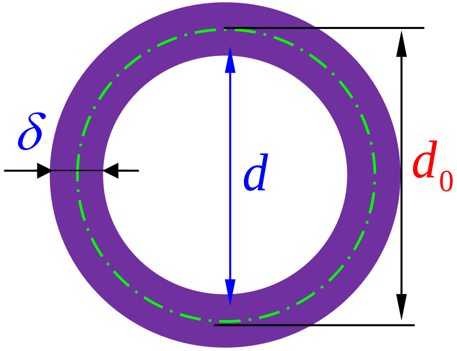
2）用精确的扭转理论

$$\tau_{\max} = \frac{T}{\dfrac{\pi D^3}{16}(1-\alpha^4)} = \frac{180\times10^3}{\dfrac{\pi\times0.29^3}{16}\left[1-\left(\dfrac{230}{290}\right)^4\right]} = 62.2\,\text{MPa}$$

$$D = d+2\delta = 290\text{mm}$$

相对误差 $err = \dfrac{56.5-62.2}{62.2}\times100\% = -9.16\%$

讨论：薄壁公式的适用条件



应用于工程计算时要求近似理论的解与精确解的误差 < 5%。

精确解： $\tau_{\max} = \dfrac{T}{W_{\mathrm{p}}}$

$$W_{\mathrm{p}} = \frac{\pi D^3}{16}\left(1-\alpha^4\right)$$

$$\alpha = \frac{d}{D}$$

令 $\beta = \dfrac{\delta}{d_0}$

$$(d = d_0 - \delta,\ D = d_0 + \delta)$$

$$\tau_{\max} = \frac{T}{W_{\mathrm{p}}} = \frac{T}{\frac{\pi D^3}{16}\left(1-\alpha^4\right)} = \frac{T}{\frac{\pi}{16}(d_0+\delta)^3\left[1-\left(\frac{d_0-\delta}{d_0+\delta}\right)^4\right]} = \frac{T}{\frac{\pi}{16}d_0^3(1+\beta)^3\left[1-\left(\frac{1-\beta}{1+\beta}\right)^4\right]}$$

$$\tau_{\max} = \frac{T}{\frac{\pi}{16} d_0^3 (1+\beta)^3 \left[1 - \left(\frac{1-\beta}{1+\beta}\right)^4\right]} = \frac{T(1+\beta)}{\frac{\pi}{16} d_0^3 \left[(1+\beta)^4 - (1-\beta)^4\right]}$$

$$\beta = \frac{\delta}{d_0}$$

$$= \frac{T(1+\beta)}{\frac{\pi}{16} d_0^3 \left[\left(1+4\beta+6\beta^2+4\beta^3+\beta^4\right) - \left(1-4\beta+6\beta^2-4\beta^3+\beta^4\right)\right]}$$

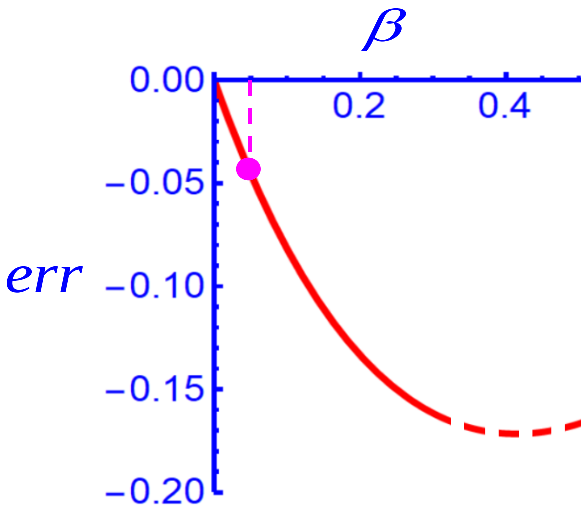
$$= \frac{T(1+\beta)}{\frac{\pi}{2} d_0^3 \beta(1+\beta^2)} = \frac{T(1+\beta)}{\frac{\pi}{2} d_0^2 \delta(1+\beta^2)} = \frac{T(1+\beta)}{2\pi r_0^2 \delta(1+\beta^2)} = \frac{1+\beta}{1+\beta^2}\tau_{薄壁}$$

$$err = \frac{\tau_{薄壁} - \tau_{\max}}{\tau_{\max}} = \frac{\tau_{薄壁}}{\tau_{\max}} - 1 = \frac{1+\beta^2}{1+\beta} - 1 = \frac{\beta^2 - \beta}{1+\beta} = \frac{\beta(\beta-1)}{1+\beta}$$

$$err = \frac{\beta(\beta-1)}{1+\beta}$$

当 $\beta = \frac{\delta}{d_0} = 0.05,$ 有 $err = -4.52\%$

$$\frac{\delta}{d_0} = 0.05 \leftrightarrow \delta = 0.05 d_0 = 0.1 r_0 = \frac{1}{10} r_0$$



即当 $\delta \le \frac{r_0}{10}$，可视为薄壁圆筒。

Thank you！

作业

P. 112：3.6、3.7

P. 113：3.9

对应第6版的题号：**P. 106-107：3.6，3.7，3.9**

下次课讲　扭转时的变形

非圆截面杆自由扭转