1

$$egin{aligned} rac{\mathrm{d}x_a(t)}{\mathrm{d}t} &= v_a(t)\cos(lpha(t)) \ rac{\mathrm{d}y_a(t)}{\mathrm{d}t} &= v_a(t)\sin(lpha(t)) \ rac{\mathrm{d}x_b(t)}{\mathrm{d}t} &= v_b(t)\cos(eta(t)) \ rac{\mathrm{d}y_b(t)}{\mathrm{d}t} &= v_b(t)\sin(eta(t)) \end{aligned}$$

2

在海盗追击商船的过程中,海盗应选择每时每刻均沿两者直线方向的航向,此时r(t)减小最快。这是因为它导致 r(t)在任何时间 t 的最陡下降。如果海盗航向不沿两者直线方向,那么 r(t)的下降速度将会减慢。此时, $\theta(t)=\beta(t)$

3

$$dr(t) = -v_b(t)dt + v_a(t)\cos(\beta(t) - \alpha(t))dt$$

$$= -\lambda v_a dt + v_a \cos(\beta(t) - \alpha(t))dt$$

$$= v_a[(\cos(\beta(t) - \alpha(t)) - \lambda)dt]$$

所以

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_a(\cos(\beta(t) - \alpha(t)) - \lambda)$$
$$= v_a(\cos(\theta(t) - \alpha(t)) - \lambda)$$

对于 $\theta(t)$,有

$$-\mathrm{d}\theta(t) = \sin(-\mathrm{d}\theta(t)) = \frac{v_a \mathrm{d}t \sin(\theta(t) - \alpha(t))}{r(t)}$$

所以

$$rac{\mathrm{d} heta(t)}{\mathrm{d}t} = rac{v_a \sin(lpha(t) - heta(t))}{r(t)}$$

综上

$$\begin{cases} \frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_a(\cos(\alpha(t) - \theta(t)) - \lambda) \\ \\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{\nu_a \sin(\alpha(t) - \theta(t))}{r(t)} \end{cases}$$

4

由上一问与条件 $\alpha(t) \equiv 0$,我们可以得到微分方程组如下:

$$\begin{cases} \frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_a(\cos(\theta(t)) - \lambda) \\ \\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -\frac{v_a\sin(\theta(t))}{r(t)} \end{cases}$$

两式相除,得到

$$\frac{\mathrm{d}r(t)}{\mathrm{d}\theta(t)} = \frac{r(t)}{\sin(\theta(t))} (\lambda - \cos(\theta(t)))$$

所以

$$\frac{\mathrm{d}r(t)}{r(t)} = \frac{\cos(\theta(t)) - \lambda}{\sin(\theta(t))} \mathrm{d}\theta(t)$$

$$\ln(r(t)) = \lambda \ln\left(\sin\left(\frac{\theta(t)}{2}\right)\right) - \lambda \ln\left(\cos\left(\frac{\theta(t)}{2}\right)\right) - \ln(\sin(\theta(t))) + C$$

$$r(t) = C \frac{1}{\sin(\theta(t))} \left(\tan\frac{\theta(t)}{2}\right)^{\lambda}$$

$$t=0$$
 时, $r(0)=\sqrt{x_0^2}+y_0^2,\sin heta(0)=rac{-y_0}{\sqrt{x_0^2+y_0^2}},\cos heta(0)=rac{-x_0}{\sqrt{x_0^2+y_0^2}}, anrac{ heta(0)}{2}=rac{1-\cos heta(0)}{\sin heta(0)}=rac{\sqrt{x_0^2+y_0^2+x_0}}{-y_0}$,所以

$$C = -y_0 igg(rac{x_0 - \sqrt{x_0^2 + y_0^2}}{y_0}igg)^{\lambda}$$

所以, r(t)与 $\theta(t)$ 的关系为

$$r(t) = -y_0 \Big(rac{x_0 - \sqrt{x_0^2 + y_0^2}}{y_0}\Big)^\lambda rac{1}{\sin(heta(t))} \Big(an(rac{ heta(t)}{2})\Big)^\lambda$$

当 $\lambda = 1$ 时,r(t)与 $\theta(t)$ 的关系为

$$egin{aligned} r(t) &= -y_0 \Big(rac{x_0 - \sqrt{x_0^2 + y_0^2}}{y_0}\Big) rac{1}{\sin(heta(t))} \Big(an(rac{ heta(t)}{2})\Big) \ &= -y_0 \Big(rac{x_0 - \sqrt{x_0^2 + y_0^2}}{y_0}\Big) rac{1}{\sin(heta(t))} \Big(rac{\sin{ heta(t)}}{1 + \cos(heta(t))}\Big) \ &= rac{\sqrt{x_0^2 + y_0^2} - x_0}{1 + \cos(heta(t))} \end{aligned}$$

所以

$$rac{\mathrm{d} heta(t)}{\mathrm{d} t} = -rac{
u_a \sin(heta(t))}{r(t)} = -rac{
u_a \sin(heta(t))}{\sqrt{x_0^2 + y_0^2 - x_0}} = -rac{
u_a (1 + \cos(heta(t))) \sin(heta(t))}{\sqrt{x_0^2 + y_0^2} - x_0}$$

所以

$$rac{\mathrm{d} heta(t)}{\mathrm{d}t} = -rac{v_a(1+\cos(heta(t)))\sin(heta(t))}{\sqrt{x_0^2+y_0^2}-x_0}$$

$$rac{\mathrm{d} heta(t)}{(1+\cos(heta(t)))\sin(heta(t))} = -rac{v_a}{\sqrt{x_0^2+y_0^2-x_0}}dt$$

$$\frac{1 - 2\cos^2(\frac{\theta(t)}{2})(\ln(\cos(\frac{\theta(t)}{2})) - \ln(\sin(\frac{\theta(t)}{2})))}{2(\cos(\theta(t)) + 1)} = -\frac{v_a}{\sqrt{x_0^2 + y_0^2} - x_0}t + C$$

$$rac{1-(\cos(heta(t))+1)(\ln(rac{1}{ an(rac{ heta(t)}{2})}))}{2\left(\cos(heta(t))+1
ight)} = -rac{
u_a}{\sqrt{x_0^2+y_0^2}-x_0}t+C$$

$$rac{1}{2(\cos(heta(t))+1)} + rac{\ln(an(rac{ heta(t)}{2}))}{2} = -rac{
u_a}{\sqrt{x_0^2+y_0^2}-x_0}t + C$$

$$rac{1}{2(\cos(heta(t))+1)} + rac{\ln(\sqrt{rac{1-\cos(heta(t))}{1+\cos(heta(t))}})}{2} = -rac{
u_a}{\sqrt{x_0^2+y_0^2}-x_0}t + C$$

$$\frac{1}{2(\cos(\theta(t))+1)} + \frac{1}{4}\ln(\frac{1-\cos(\theta(t))}{1+\cos(\theta(t))}) = -\frac{\nu_a}{\sqrt{x_0^2+y_0^2}-x_0}t + C$$

将t=0代入上式,可以得到C的值。所以我们可以得到 $\theta(t)$ 的表达式。同时,因为我们已经知道了r(t)与 $\theta(t)$ 的关系,所以我们可以得到r(t)的表达式。

1

设运动的距离为 s,则s=n heta a ,所以设运动的距离为 s,则s=n heta a ,所以

$$dx = (ds)\cos\omega = na\cos\omega d\theta$$

$$dy = (ds)\sin\omega = na\sin\omega d\theta$$

所以:

$$dx = (ds)\cos\omega = na\cos\omega d\theta$$

$$dy = (ds)\sin\omega = na\sin\omega d\theta$$

所以:

$$\left\{egin{array}{ll} rac{\mathrm{d}x}{\mathrm{d} heta} &= na\cos\omega \ rac{\mathrm{d}y}{\mathrm{d}a} &= na\sin\omega \end{array}
ight.$$

2

A点的坐标为 $(a\cos\theta,a\sin\theta)$,所以B点的坐标为 $(a\cos\theta-\rho\cos\omega,a\sin\theta-\rho\sin\omega)$,切线方向为AB

的方向, 所以切线方程为:

$$y = \tan \omega (x - a\cos \theta) + a\sin \theta$$

3

已知 $x = a\cos\theta - \rho\cos\omega$, $y = a\sin\theta - \rho\sin\omega$,所以:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta - \left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\cos\omega - \rho\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\sin\omega\right) = na\cos\omega$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\cos\theta - \left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\sin\omega + \rho\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\cos\omega\right) = na\sin\omega$$

所以:

$$\left\{egin{array}{ll} rac{\mathrm{d}
ho}{\mathrm{d} heta} &= a\sin(\omega- heta)-na \ & \ rac{\mathrm{d}\omega}{\mathrm{d} heta} &= rac{a}{
ho}(\cos(\omega- heta)) \end{array}
ight.$$

消去含 θ 的项,得:

$$(rac{\mathrm{d}
ho}{\mathrm{d} heta}+na)^2+(
horac{\mathrm{d}\omega}{\mathrm{d} heta})^2=a^2$$