第三部分

我选择的问题是(请将不做的题目编号删除)： $\boxed{\text{H}}$

Part 1.

(0) 解：对于方程 $\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t)=0$

令 $u(x,t)=X(x)T(t)$，得 $X''(x)T(t)-\frac{1}{c^2}X(x)T''(t)=0$

即 $\frac{X''}{X}=\frac{T''}{c^2T}$，即对应的常微分方程

令 $\frac{X''}{X}=\frac{T''}{c^2T}=\lambda$

① $\lambda=0$，得 $X(x)=C_1x+C_2$ ；$T(t)=C_1't+C_2'$

② $\lambda>0$，解得 $X(x)=C_1e^{\sqrt{\lambda}x}+C_2e^{\sqrt{\lambda}x}$ ；$T(t)=C_1'e^{c\sqrt{\lambda}t}+C_2'e^{c\sqrt{\lambda}t}$

③ $\lambda<0$，解得 $X(x)=C_1\cos\sqrt{-\lambda}x+C_2\sin\sqrt{-\lambda}x$ ；$T(t)=C_1'\cos(c\sqrt{-\lambda}t)+C_2'\sin(c\sqrt{-\lambda}t)$

(1) 解：对于方程 $\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(\vec{r},t)=0$

令 $u(\vec{r},t)=A(\vec{r})T(t)$，代入方程

有 $\frac{T''}{c^2T}=\frac{\nabla^2A(\vec{r})}{A(\vec{r})}$，若等式成立，则比值必为常数，记为 $-k^2$

$\therefore \nabla^2A+k^2A=0$

(2) 解：由(1)，$\nabla^2A+k^2A=0$

由令 $A(\vec{r})=u(\vec{r})e^{ikz}=u(x,y,z)e^{ikz}$，代入，有

$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}+2ik\frac{\partial u}{\partial z}-k^2u+k^2u=0$，在傍轴下，$\frac{\partial^2u}{\partial z^2}=0$

即 $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+2ik\frac{\partial u}{\partial z}=0$

即 $\nabla_\perp^2u(\vec{r})+2ik\frac{\partial u(\vec{r})}{\partial z}=0$

Bonus 1:

(1) 解：由 $\hat{p}=-i\hbar\nabla$，$\hat{L}=r\times\hat{p}$

可得 $\hat{L}$ 分量形式

$\hat{L}_x=y\hat{p}_z-z\hat{p}_y=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$

$\hat{L}_y=z\hat{p}_x-x\hat{p}_z=-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$

$\hat{L}_z=x\hat{p}_y-y\hat{p}_x=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$

算符对易式：

$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$

$[\hat{A},\hat{B}+\hat{C}]=[\hat{A},\hat{B}]+[\hat{A},\hat{C}]$

$[\hat{A},\hat{B}\hat{C}]=\hat{B}[\hat{A},\hat{C}]+[\hat{A},\hat{B}]\hat{C}$

故 $[\hat{L}_x,\hat{L}_x]=\cancel{\hbar^2}\,0$ $\quad[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z$

$[\hat{L}_y,\hat{L}_y]=0$ $\quad[\hat{L}_y,\hat{L}_z]=i\hbar\hat{L}_x$

$[\hat{L}_z,\hat{L}_z]=0$ $\quad[\hat{L}_z,\hat{L}_x]=i\hbar\hat{L}_y$

又 $\hat{L}^2=\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$

故 $[\hat{L}^2,\hat{L}_z]=[\hat{L}_x^2,\hat{L}_z]+[\hat{L}_y^2,\hat{L}_z]$

$=-([\hat{L}_z,\hat{L}_x^2]+[\hat{L}_z,\hat{L}_y^2])$

$=-[\hat{L}_x[\hat{L}_z,\hat{L}_x]+[\hat{L}_z,\hat{L}_x]\hat{L}_x]$

$-[\hat{L}_y[\hat{L}_z,\hat{L}_y]+[\hat{L}_z,\hat{L}_y]\hat{L}_y]$

$=-\hat{L}_x i\hbar\hat{L}_y-i\hbar\hat{L}_y\hat{L}_x+i\hbar\hat{L}_y\hat{L}_x+i\hbar\hat{L}_x\hat{L}_y=0$

故 $\hat{L}^2$ 的本征函数可同时取为 $\hat{L}_z$ 的本征态

故 $\hat{L}^2$、$\hat{L}_z$ 具有同一本征态

引入球坐标 $\begin{cases}x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}$

可得 $\hat{L}_z=-i\hbar x\left(\frac{\partial r}{\partial r}\frac{\partial}{\partial y}+\frac{\partial\theta}{\partial\theta}\frac{\partial}{\partial y}+\frac{\partial\varphi}{\partial\varphi}\frac{\partial}{\partial y}\right)+i\hbar y\left(\frac{\partial r}{\partial r}\frac{\partial}{\partial x}+\frac{\partial\theta}{\partial\theta}\frac{\partial}{\partial x}+\frac{\partial\varphi}{\partial\varphi}\frac{\partial}{\partial x}\right)=-i\hbar\frac{\partial}{\partial\varphi}$

令 $-i\hbar\frac{d}{d\varphi}\psi(\varphi)=L_z\psi(\varphi)$

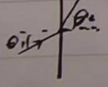
解得 $\psi(\varphi)=ce^{iL_z\varphi}$

故球坐标新形式 $\langle r|l\rangle\sim e^{il\varphi}$

Part 2.

(1) a. 解：易知 $\theta_1=\theta_2$, $x_1+\tan\theta_1=x_2$. 在傍轴条件下, $\theta_1\approx\tan\theta_1$

又 $\begin{cases} x_2 = Ax_1+B\theta_1 \\ \theta_2 = Cx_1+D\theta_1 \end{cases}$ 得 $\begin{cases} A=1 \quad C=0 \\ B=1 \quad D=1 \end{cases}$ ∴传播矩阵 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

b. 解:  由折射定律 $n_1\sin\theta_1 = n_2\sin\theta_2$. 傍轴条件下, $n_1\theta_1 = n_2\theta_2$

$\begin{cases} x_2 = Ax_1+B\theta_1 \\ \theta_2 = Cx_1+D\theta_1 \end{cases} \Rightarrow \begin{cases} A=1 \quad C=0 \\ B=0 \quad D=\frac{n_1}{n_2} \end{cases}$ ∴传播矩阵 $\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$

c. 解: $u = \frac{x_1}{\tan\theta_1} \approx \frac{x_1}{\theta_1}$ 又 $\begin{cases} x_2 = Ax_1+B\theta_1 \\ \theta_2 = Cx_1+D\theta_1 \end{cases}$

由 $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, $v=\frac{x_2}{-\theta_2}$

得 $\begin{cases} \frac{\theta_1}{x_1}+\frac{-\theta_2}{x_2}=\frac{1}{f} \\ x_1=x_2 \end{cases}$ 得 $\begin{cases} A=1 \quad C=-\frac{1}{f} \\ B=0 \quad D=1 \end{cases}$

∴矩阵为 $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

(2) 解: 入射光与出射光平行, 故 $\theta_1=\theta_2$

考虑偏移量 $\Delta x_i = d_i\tan\theta_i$, $\theta_1 = n_i\sin\theta_i$

利用傍轴近似, $\tan\theta_i \approx \sin\theta_i \approx \theta_i$

∴ $M_i = \begin{pmatrix} 1 & \frac{d_i}{n_i} \\ 0 & 1 \end{pmatrix}$ ∴ $M = M_N \cdots M_1 = \begin{pmatrix} 1 & \sum_{i=1}^{N}\frac{d_i}{n_i} \\ 0 & 1 \end{pmatrix}$

(3) 解: 由(1). a, 自由传播矩阵 $M_1 = \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}$ $M_3 = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}$

由(1) c. 透镜折射矩阵 $M_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

故 $M = M_3M_2M_1 = \begin{pmatrix} 1-\frac{d_2}{f} & d_1+d_2-\frac{d_1d_2}{f} \\ -\frac{1}{f} & 1-\frac{d_1}{f} \end{pmatrix}$

Bonus 2

(1) 解：设一个周期后，表达式为

$$\begin{pmatrix} x_{m+1} \\ \theta_{m+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_m \\ \theta_m \end{pmatrix}$$

① 经过 R_1 反射

由 (1).c，$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$. 又规定凹面镜的 $R<0$

$\therefore M_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{R_1} & 1 \end{pmatrix}$

② 传播 $M_2 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

③ R_2 反射. $M_3 = \begin{pmatrix} 1 & 0 \\ \frac{2}{R_2} & 1 \end{pmatrix}$

④ 反传播 $M_4 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

解得 $M = M_4M_3M_2M_1 =$

$$\begin{pmatrix} 1+2d\left(\frac{2}{R_1}+\frac{1}{R_2}\right)+\frac{4d^2}{R_1R_2} & 2d\left(1+\frac{d}{R_2}\right) \\ \frac{4d}{R_1R_2}+2\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{2d}{R_2}+1 \end{pmatrix}$$

Bonus 2

(2) 解：修正 M_2、M_4，由 (2)

$$M_2' = M_4' = \begin{pmatrix} 1 & \frac{l}{n}+d-l \\ 0 & 1 \end{pmatrix}$$

$$M' = M_4'M_3'M_2'M_1' = \begin{pmatrix} 1+2d'\left(\frac{1}{R_2}+\frac{2}{R_1}\right)+\frac{4d'^2}{R_1R_2} & 2d'\left(1+\frac{d'}{R_2}\right) \\ 2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{4d'}{R_1R_2} & \frac{2d'}{R_2}+1 \end{pmatrix}$$

其中 $d' = d-\frac{n-1}{n}l$

Part1 的第三部分代码见附件。

Part3 因为时间限制未能完成。

下面提供一种测定拓扑荷数的思路：不同拓扑荷数的 L-G 光束的光阑衍射图样不同，通过将衍射图样与已测定拓扑荷数的光束衍射图样对比可得结论

参考资料：工程光学；光电子学；量子力学讲义；数学物理方法；

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