电子电路基础

第四讲~过渡过程的经典解法
~part1

课程纲要

- 4.1 一阶电路的响应
- 4.1.1 一阶电路的零输入和零状态响应
- 4.1.2 一阶电路的全响应
- 4.1.3 一阶电路的阶跃响应和冲激响应

First-Order Circuits
Chapter 7

7.1 The Source-Free (零输入响应) RC Circuit

7.2 The Source-Free (零输入响应) RL Circuit

7.3 Singularity Function (奇异函数)

7.4 Step Response (阶跃响应) of an RC Circuit

7.5 Step Response (阶跃响应) of an RL Circuit

补充：冲激响应

动态电路的概念

- 电感、电容的“电压电流约束关系”与时间 t 有关，所以电感、电容又称为**动态元件**，含电感、电容的电路也被称为**动态电路**

- 动态电路的一个**特征**是：当电路结构发生变化时（一般通过“开关”的切换来实现），可能使电路从原来的工作状态，转变到一个新的工作状态，这种转变往往需要一定的时间，这一过程被称为动态电路的**过渡过程**。

- 开关切换的动作也被称为“**换路**”，一般认为换路是在t=0时刻进行的，把换路前的最终时刻记为 $t = 0^-$，把换路后的最初时刻记为 $t = 0^+$

- 分析动态电路的方法之一：根据KCL、KVL、元件电压电流关系，写出微分方程，并求解；这一方法在**时域**中进行，称为**经典解法**

比如电路中只有电容电阻，没有电阻

- ①无**外加激励**电源（零**输入**），仅由动态元件初始储能所产生的响应：**零输入响应**；②动态元件**初始储能**为零（零**状态**），由外加激励电源引起的响应：**零状态响应**；③实际上往往既有输入，也有初始储能：**全响应**

常微分方程

·求出通解后待定系数求解

- 齐次线性微分方程

$$\frac{d^n y}{dx^n}+A_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+A_n y=0$$

1）求特征方程

$$z^n+A_1 z^{n-1}+\cdots+A_n=0.$$

获得n个根 $z_1, \ldots, z_n$

若没有重根，n个独立解 $e^{z_i x}$

若z是 m_z 重根，m_z 个独立解

$$y=x^k e^{zx} \qquad k\in\{0,1,\ldots,m_z-1\}$$

依旧可获得总共 n个独立解

2）写出通解

n个独立解的线性组合就是方程的通解

- 非齐次线性微分方程

$$y''+py'+qy=f(x)$$

1）求出对应齐次方程的通解

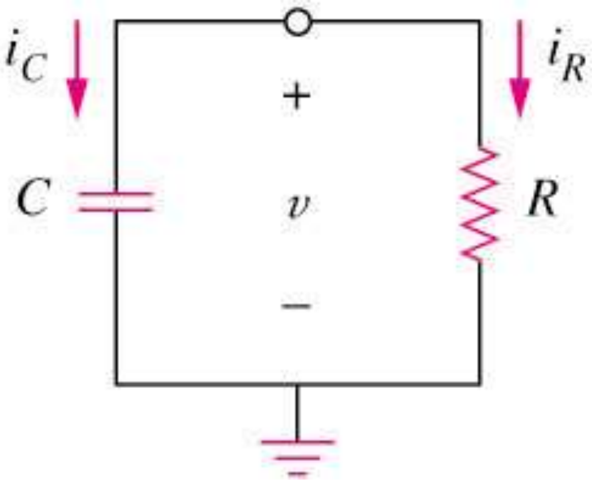
2）求出非齐次方程的一个特解（电路分析中一般选取 t = ∞ 稳态时刻的解为特解）

·无穷大稳态电路好求解

3）两者相加，获得非齐次方程的通解

7.1 一阶电路的概念

- A **first-order circuit** is characterized by a first-order differential equation.
  - 数学上：一阶电路的方程为一阶微分方程
  - 直观上：化简后的电路，仅含一个储能元件（**L**，或**C**）



By KCL

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

- Apply Kirchhoff's laws to purely resistive circuit results in algebraic equations.
- Apply the laws to RC and RL circuits produces differential equations.

用电流法和节点法都可以，因为考虑是电容，所以本图片用电流，得微分方程方便解

抓住不能突变的点，比如电容转化为电压的方程。0+时刻的方程由0-时刻获得

- 有两种一阶电路：*RC* & *RL*
- 0+时刻的电路状态（电容电压 & 电感电流）获得
  - 0-时刻的电路状态
  - 电感电流不能突变；电容电压不能突变
- 0+时刻，**根据有无独立源**，分两种情况：
  - Source-free circuits, 无独立源。初始时，能量储藏在储能元件 *L* 或 *C* 中，然后逐渐被电阻消耗；**逐渐衰减的过程**（独立源在 $t = 0$ 时刻突然**撤掉**）**【即零输入响应】**
  - By independent sources，有独立源。（独立源在 $t = 0$ 时刻突然**加入,**该情况也称为step-response）**逐渐到达另一个稳态的过程【全响应】**
- 因此，一阶电路有四种组合：
  - ①source-free *RC*；②source-free *RL*；
  - ③step-response of *RC*; ④step-response of *RL*

The source-free *RC* circuit

- A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.
- It is a series combination of a resistor & a initially charged capacitor.
- Our objective is to determine the circuit response (the voltage v(t) across the capacitor)
  - 在 t = 0+ 初始时刻，假设 C 两端的初始电压为 V_0；
  - 那么 C 的初始储能为 0.5 CV_0^2

Applying KCL at the top node of the circuit in Fig. 7.1 yields

$$i_C + i_R = 0 \tag{7.3}$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \quad \boxed{\text{一阶微分方程}} \tag{7.4a}$$

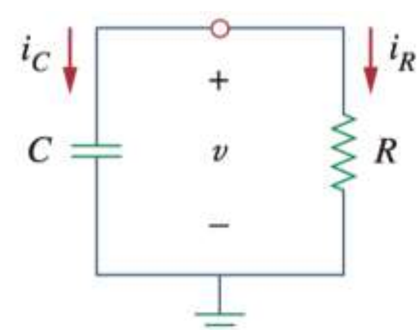


Figure 7.1
A source-free *RC* circuit.

- 求解此一阶微分方程，即可得到 C 两端电压 $v(t)$，问题解决！

$$C\frac{dv}{dt}+\frac{v}{R}=0 \Longrightarrow \frac{dv}{v}=-\frac{1}{RC}dt$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC}+\ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A}=-\frac{t}{RC}$$

Taking powers of e produces

$$v(t)=Ae^{-t/RC}$$

But from the initial conditions, $v(0)=A=V_0$. Hence,

$$v(t)=V_0e^{-t/RC} \tag{7.7}$$

抓住关键参数（电容电压、电感电流），列出微分方程，剩下的就是数学问题了

通用方法：
特征方程 C*s + 1/R = 0
特征根：s_1 = -1/RC
通解：$A*e^{(-t/RC)}$

·**通解后待定系数，使用0+时刻（本质是已知）**

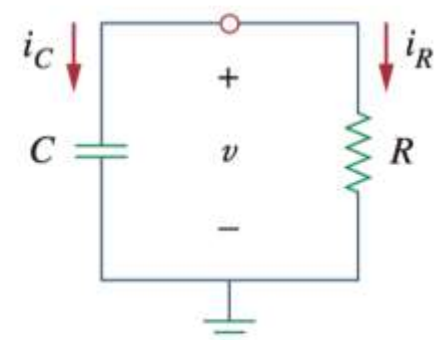


Figure 7.1
A source-free *RC* circuit.

- 结论：source-free *RC* 电路的电压响应是**初始电压值的指数衰减**。

$$v(t) = V_0 e^{-t/RC}$$

电压响应仅与初始储能V_0、电路的物理特性 RC 有关，与外部电压和电流源无关，故也称为电路的 natural response (自然响应、零输入响应)

The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

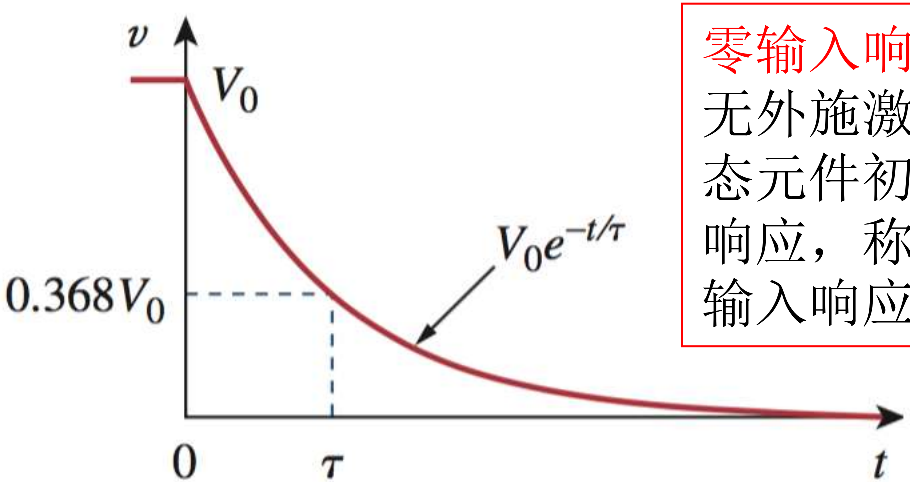


Figure 7.2
The voltage response of the *RC* circuit.

零输入响应：**动态电路**中无外施激励电源，仅由动态元件初始储能所激励的响应，称为动态电路的零输入响应

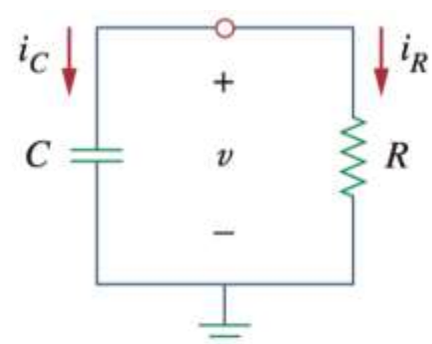


Figure 7.1
A source-free *RC* circuit.

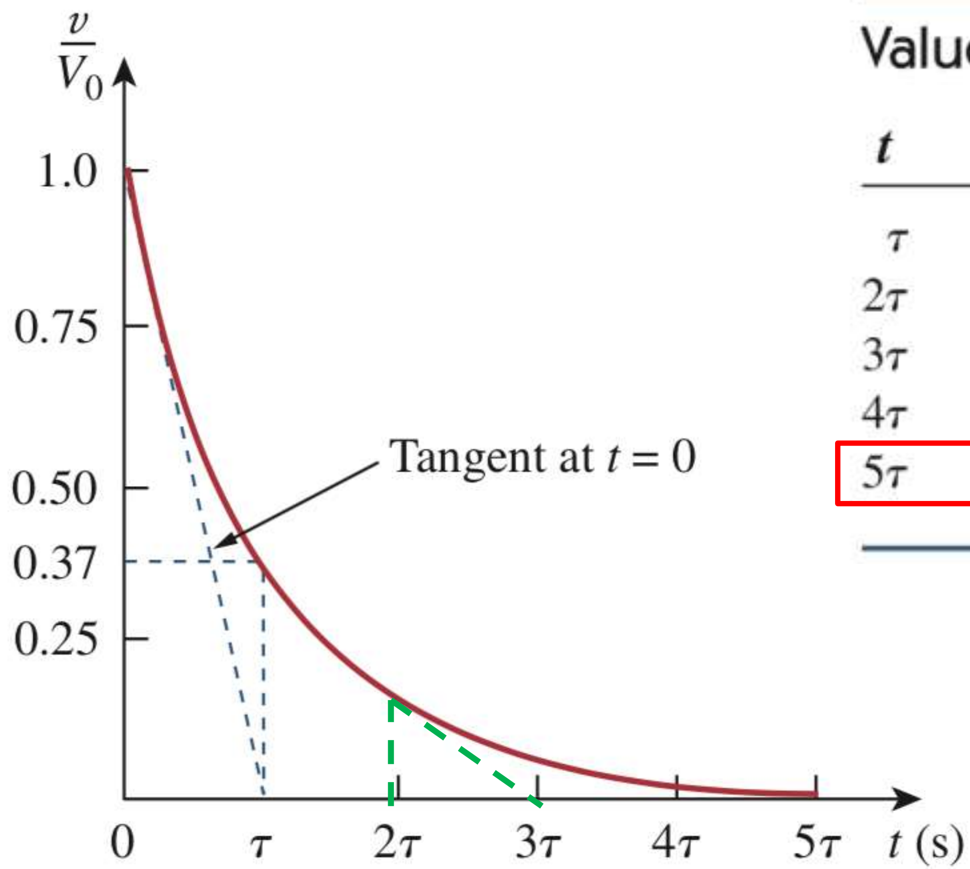


Figure 7.3
Graphical determination of the time constant τ from the response curve.

如何从实测曲线推出时间常数？

TABLE 7.1

Values of $v(t)/V_0 = e^{-t/\tau}$.

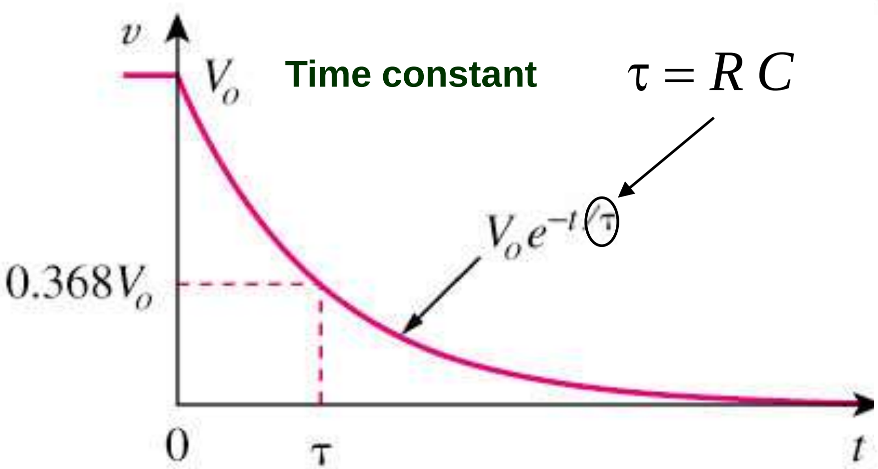
t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

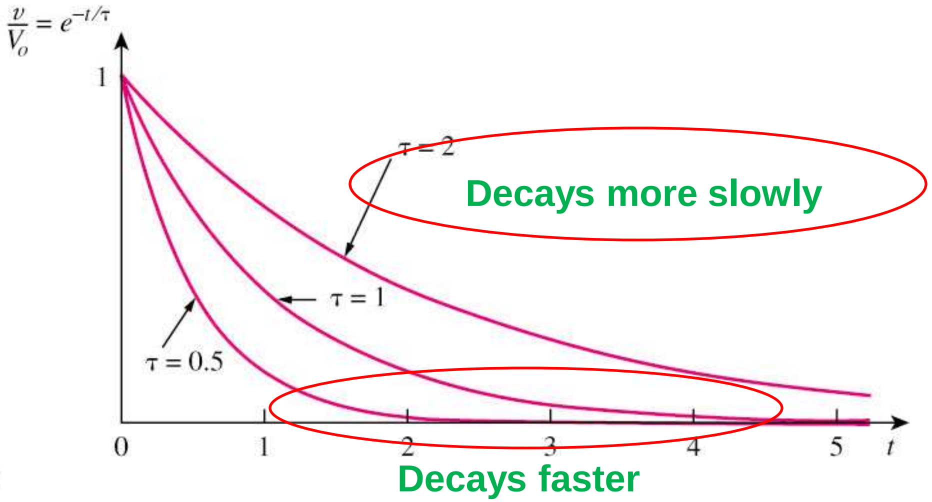
自然响应曲线的衰减特性

- 5个时间常数 τ 之后才几乎衰减完 (衰减>99%)。因此，我们说：经过5倍的时间常数，电容被**完全放电（或充电）** → 电路达到最终态或**稳态**
- 在曲线上任意一点作**切线**，在横坐标方向经过 τ 后与 t 轴相交

7.1 The Source-Free *RC* Circuit (2)

定义时间常数：$\boxed{\tau = RC}$ ⇒ $\boxed{v(t) = V_0 e^{-t/\tau}}$





- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- 时间常数越小，衰减越快.
- 不管 τ 的具体值是多少，都是经过5倍的 τ 后衰减完

- 电路的其他量可以从$v(t)$导出

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau} \tag{7.11}$$

The energy absorbed by the resistor up to time t is

$$\begin{aligned} w_R(t) &= \int_0^t p(\lambda) d\lambda = \int_0^t \frac{V_0^2}{R} e^{-2\lambda/\tau} d\lambda \\ &= -\frac{\tau V_0^2}{2R} e^{-2\lambda/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = RC \end{aligned} \tag{7.12}$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2}CV_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

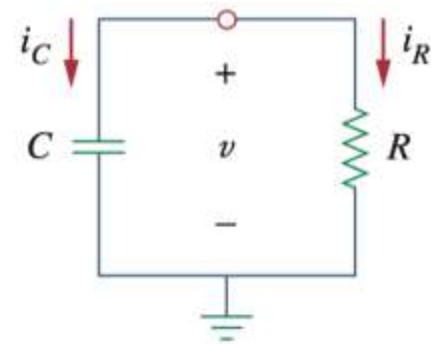


Figure 7.1
A source-free RC circuit.

Soure-free *RC*电路总结

The Key to Working with a Source-Free *RC* Circuit Is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

- 求解source-free *RC* 电路的方法：
  - 第一步：求出0+时刻电容两端的初始电压值 V_0；（根据0-时刻电路和电容电压不能突变的特性）
  - 第二步：求出 时间常数*RC*；
    - ***R* 为 *C* 两端的等效电阻**（即戴维南电阻）；
  - 然后可以直接写出 *v*(*t*) 的表达式：
  - 电路中其他的量可由 *v*(*t*) 导出

$$\boxed{v(t) = V_0 e^{-t/\tau}}$$

复杂电阻网络可以化简和等效

Example 7.1

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

Solution:

We first need to make the circuit in Fig. 7.5 conform with the standard *RC* circuit in Fig. 7.1. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x.

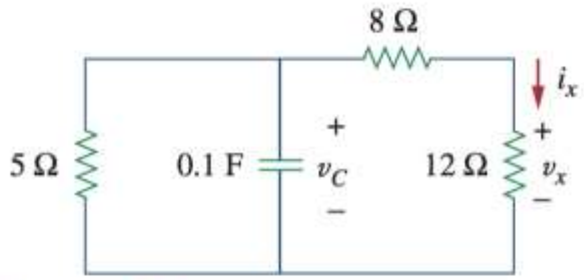


Figure 7.5
For Example 7.1.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$

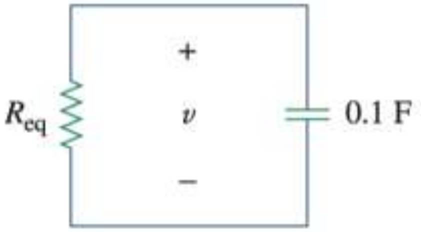


Figure 7.6
Equivalent circuit for the circuit in Fig. 7.5.

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\text{ V}, \qquad v_C = v = 15e^{-2.5t}\text{ V}$$

From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\text{ V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\text{ A}$$

注意：所有的量都具有相同的时间常数

Example 7.2

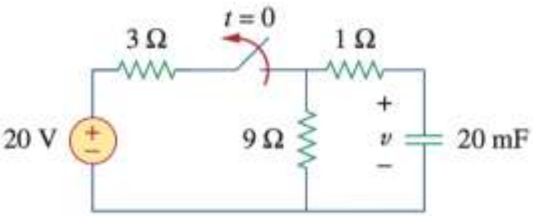


Figure 7.8
For Example 7.2.

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

①求电容两端的初始电压

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

电容两端电压不能突变

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10 \ \Omega$$

The time constant is

②求时间常数

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

or

$$v(t) = 15e^{-5t} \text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

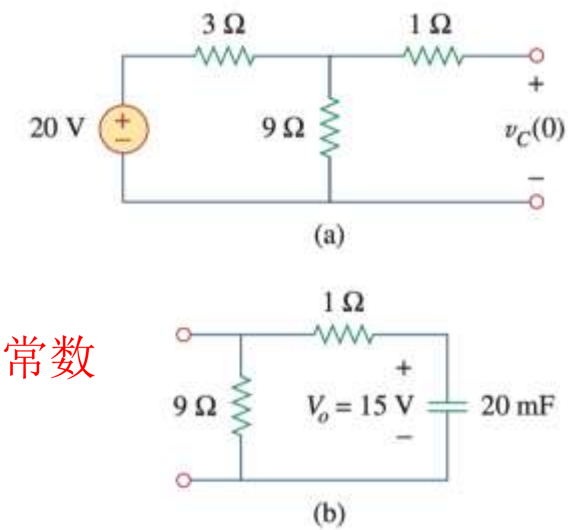


Figure 7.9
For Example 7.2: (a) $t < 0$, (b) $t > 0$.

The source-free *RL* circuit

- Consider the series connection of a resistor & a inductor.
- Our goal is to determine the circuit response (the current $i(t)$ through the inductor)
  - Why choose $i(t)$? 因为流经电感的电流不能突变
  - 在 $t = 0$ 初始时刻，假设流经电感的初始电流为 I_0;
  - 那么 L 的初始储能为 $0.5\,LI_0^2$

Applying KVL around the loop in Fig. 7.11,

$$v_L + v_R = 0$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

一阶微分方程

$$L\frac{di}{dt} + Ri = 0$$

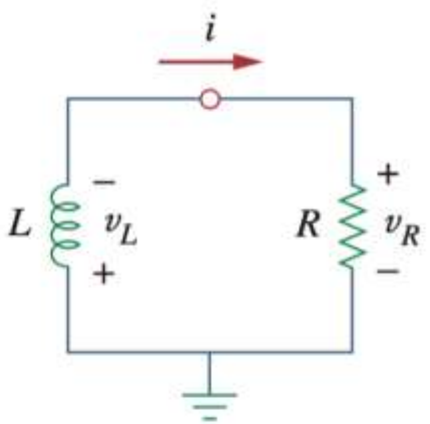


Figure 7.11
A source-free *RL* circuit.

- 求解此一阶微分方程，即可得到流经 L 的电流 $i(t)$，问题解决！

$$L\frac{di}{dt} + Ri = 0 \Longrightarrow \frac{di}{dt} + \frac{R}{L}i = 0$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{7.17}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{7.18}$$

抓住关键参数（电容电压、电感电流），列出微分方程，剩下的就是数学问题了

通用方法：
特征方程 L*s + R = 0
特征根：$s_1 = -R/L$
通解：$A*e^{(-Rt/L)}$

·**初始状态求A**

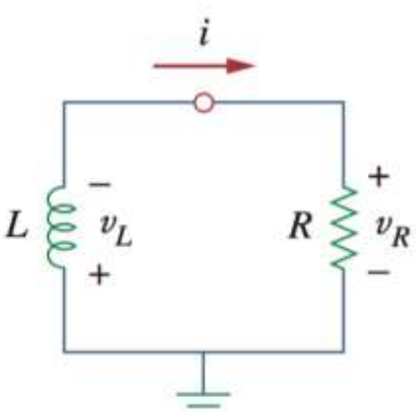


Figure 7.11
A source-free *RL* circuit.

- 结论：source-free *RL* 电路的电流响应是**初始电流值的指数衰减**。（与*RC*类似）

$$i(t) = I_0 e^{-Rt/L}$$

电流响应仅与初始储能 I_0、电路的物理特性 ***L/R*** 有关，与外部电压和电流源无关，故也是电路的 natural response (自然响应)

定义时间常数： $\boxed{\tau = \frac{L}{R}}$ ⇒ $\boxed{i(t) = I_0 e^{-t/\tau}}$

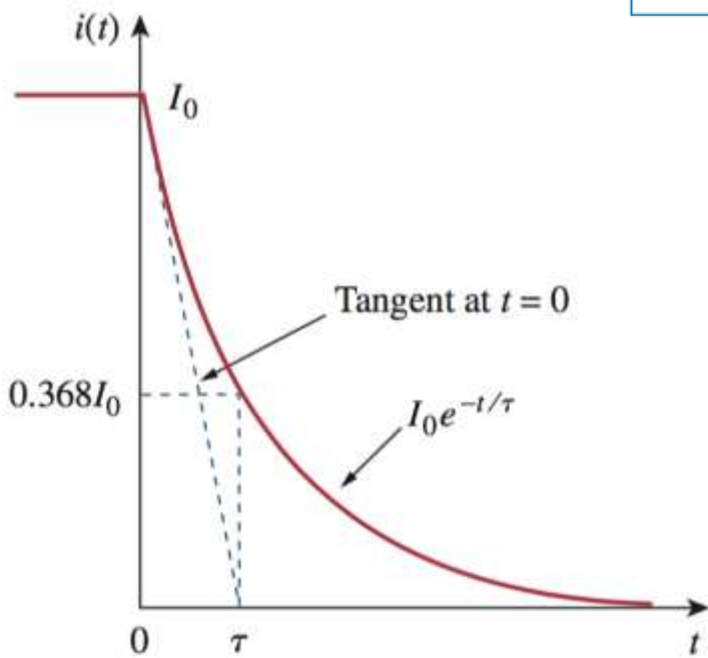


Figure 7.12
The current response of the *RL* circuit.

*时间常数的另一物理意义：
$i(t)$在 t=0 时刻的切线与 t 轴的交点

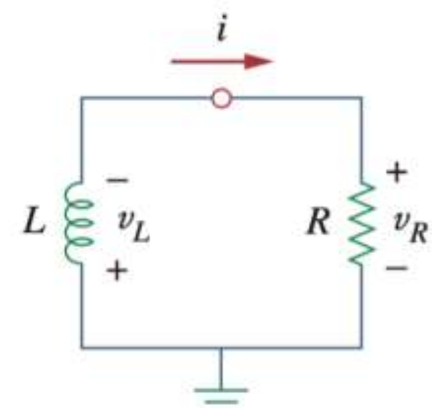


Figure 7.11
A source-free *RL* circuit.

- 电路的其他量可以从 *i*(*t*) 导出

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{7.21}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{7.22}$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p(\lambda)d\lambda = \int_0^t I_0^2 e^{-2\lambda/\tau}\, d\lambda = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

or

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}) \tag{7.23}$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (7.14). Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

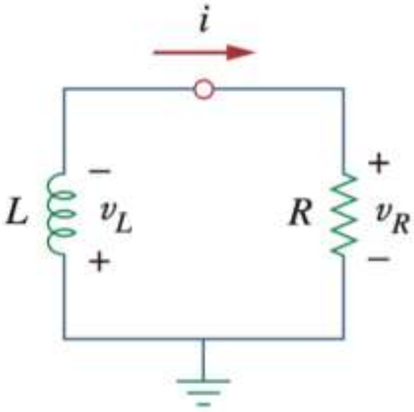


Figure 7.11
A source-free *RL* circuit.

Soure-free *RL* 电路总结

The Key to Working with a Source-Free *RL* Circuit Is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

- 求解source-free *RL* 电路的方法：
  - 第一步：求出0+时刻流经电感的初始电流值 I_0； （根据0-时刻电路和电感电流不能突变的特性）
  - 第二步：求出时间常数 L/R；
    - *R* 为 *L* 两端的等效电阻（即戴维南电阻）；
  - 然后可以直接写出 $i(t)$ 的表达式：
  - 电路中其他的量可由 $i(t)$ 导出

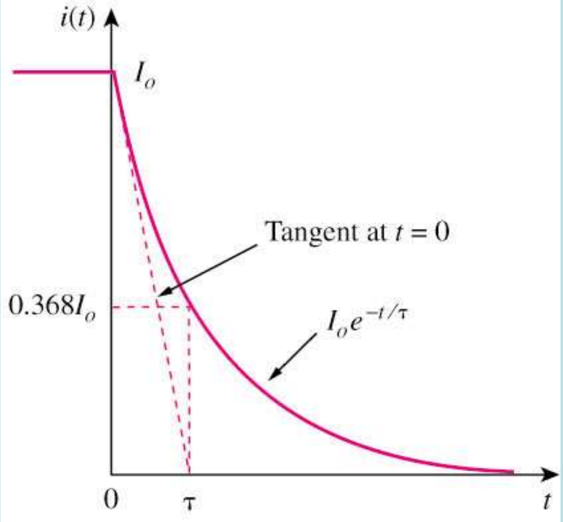
$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

7.2 The Source-Free *RL* Circuit

Comparison between a *RL* and *RC* circuit

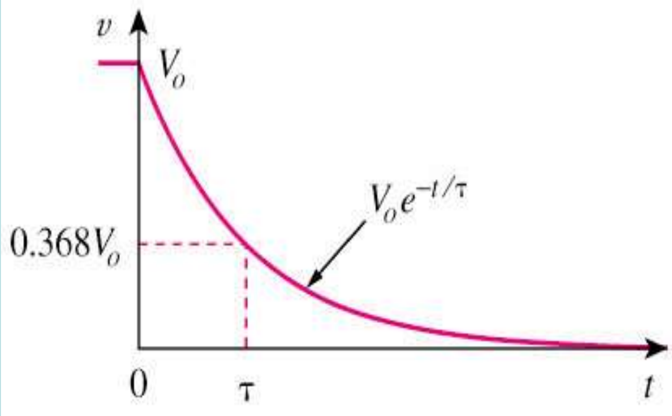
A RL source-free circuit

$$i(t) = I_0 \, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$



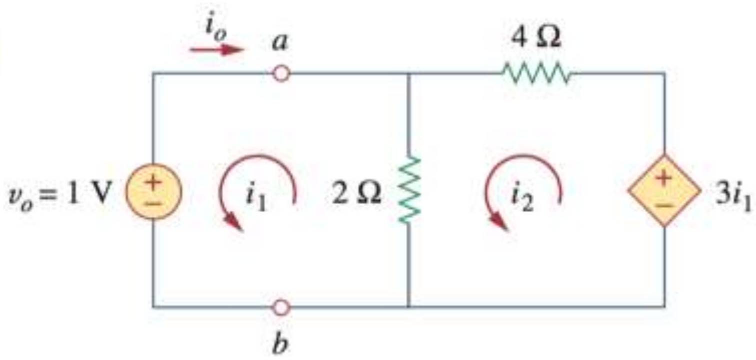
A RC source-free circuit

$$v(t) = V_0 \, e^{-t/\tau}$$ where $$\tau = RC$$



Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig. 7.13.

Example 7.3



$$2(i_1 - i_2) + 1 = 0$$

$$6i_2 - 2i_1 - 3i_1 = 0$$

$$\Downarrow$$

$$i_o = -i_1 = 3 \text{ A}$$

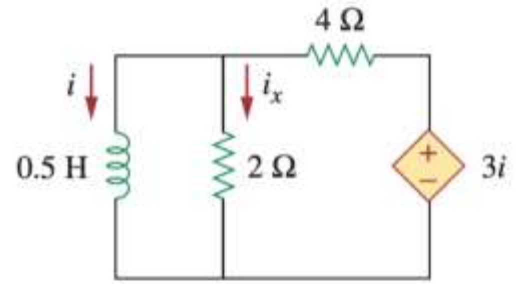


Figure 7.13
For Example 7.3.

Hence,

戴维南等效？——就是加激励

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\ \Omega$$

①求电感两端的等效电阻

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\ \text{s}$$

②写出时间常数 *L/R*

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Example 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

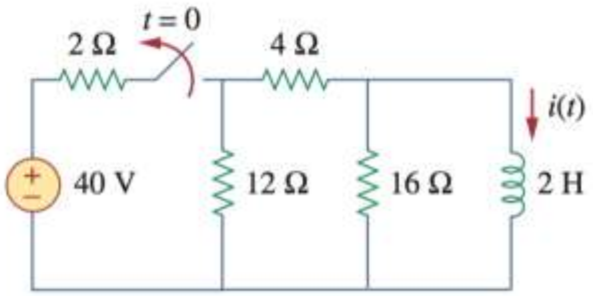


Figure 7.16
For Example 7.4.

Example 7.5

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.

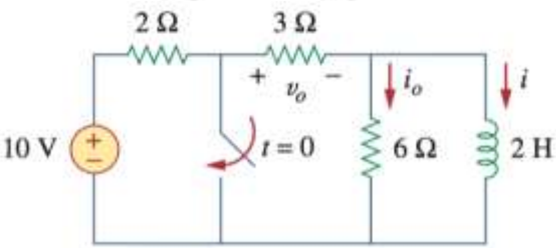


Figure 7.19
For Example 7.5.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

i与i0的关系由并联得出

$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

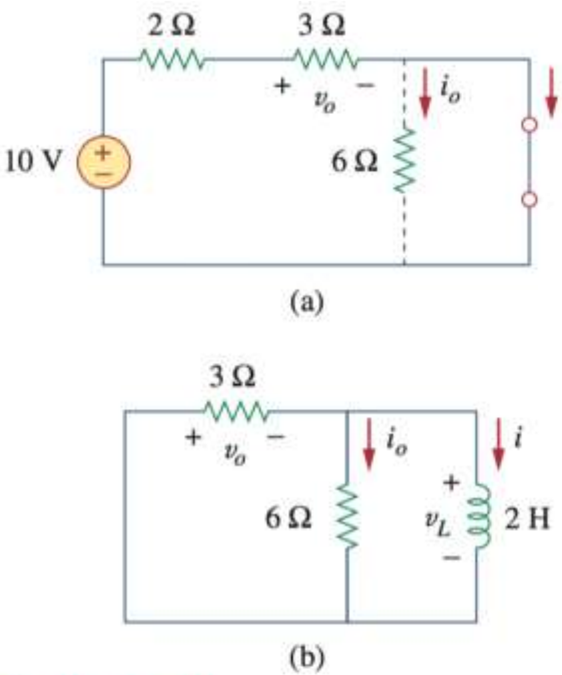


Figure 7.20
The circuit in Fig. 7.19 for: (a) $t < 0$, (b) $t > 0$.

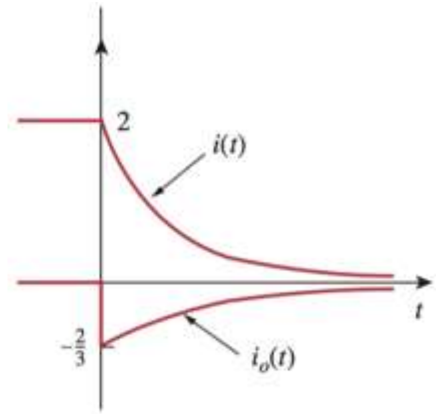
For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals,

$$R_{Th} = 3 \parallel 6 = 2\ \Omega$$

so that the time constant is

问电路

$$\tau = \frac{L}{R_{Th}} = 1\text{ s}$$



Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\text{ A}, \qquad t > 0$$

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\text{ V}, \qquad t > 0$$

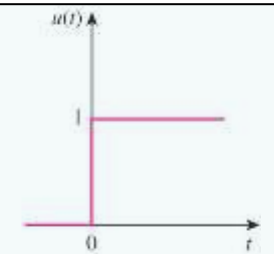
$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\text{ A}, \qquad t > 0$$

7.3 Singularity Functions 奇异函数

Singularity functions are functions that either are discontinuous or have discontinuous derivatives. 函数本身不连续，或者其导数不连续

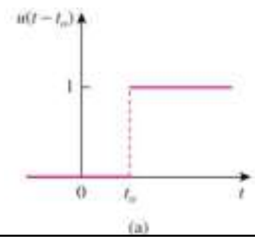
- The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t. (undefined at t=0)

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$

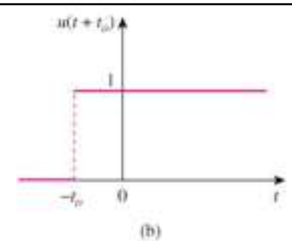


单位阶跃函数

$$u(t-t_o)=\begin{cases}0, & t<t_o\\ 1, & t>t_o\end{cases}$$



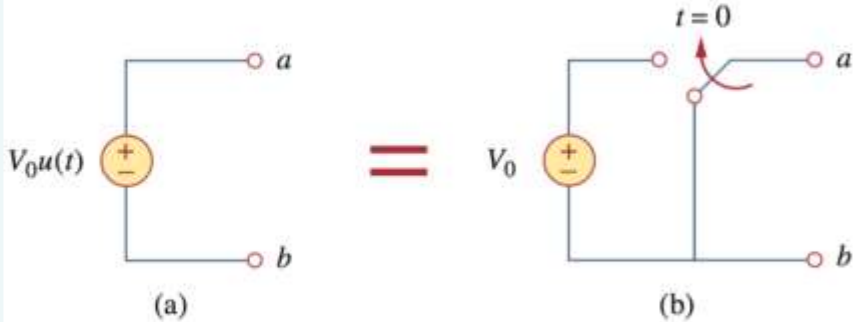
$$u(t+t_o)=\begin{cases}0, & t<-t_o\\ 1, & t>-t_o\end{cases}$$



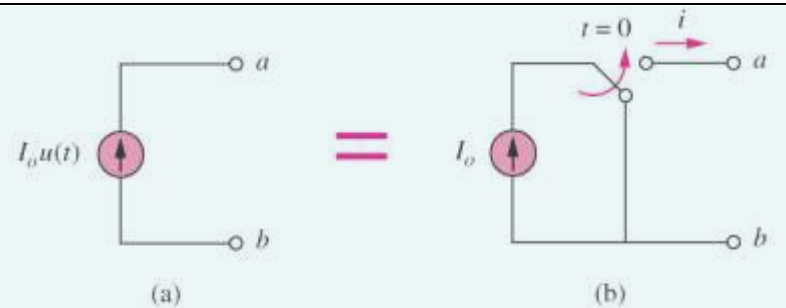
7.3 Unit-Step Function

Represent an **abrupt change** for:

1. voltage source.



2. for current source:



Unit impulse /delta function

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

单位冲激函数
（delta函数）

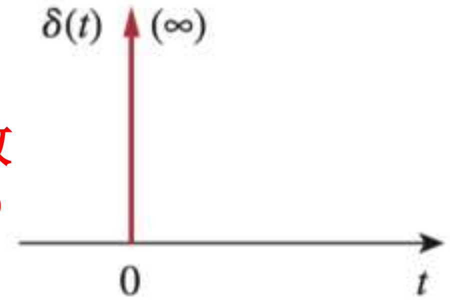


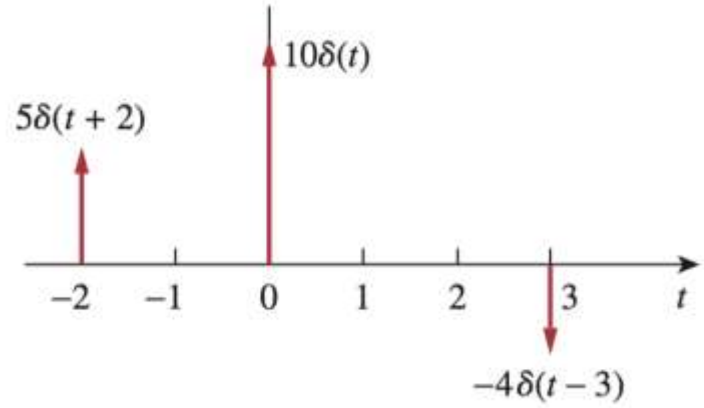
Figure 7.27
The unit impulse function.

The **unit impulse function** $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

- 单位冲激函数是单位阶跃函数的导数
- 单位冲激函数在物理上是不能实现的，但在数学上非常有用；
- 单位冲激函数可看成**面积为1**的时间非常短的脉冲函数；

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$

- 冲激函数的面积也可以不是1，如右图所示，数字表示面积值，也称为冲激函数的强度(strength)



- 冲激函数对其他函数的作用：

$$\int_a^b f(t)\delta(t-t_0)\,dt = \int_a^b f(t_0)\delta(t-t_0)\,dt$$

$$= f(t_0)\int_a^b \delta(t-t_0)\,dt = f(t_0)$$

$$\boxed{\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0)}$$

$$a < t_0 < b$$

若 t_0=0： $\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0)$

- 冲激函数有把某一函数在某一时刻的值“筛”出来的本领，该性质称为“筛分”性质，也称**取样**性质。

$$f*g=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau \qquad f*\delta=f$$

Unit ramp function

- 单位斜坡函数是单位阶跃函数的积分

$$r(t)=\int_{-\infty}^{t} u(\lambda)d\lambda = tu(t)$$

$$\boxed{r(t)=\begin{cases}0, & t\le 0\\ t, & t\ge 0\end{cases}}$$

单位斜坡函数

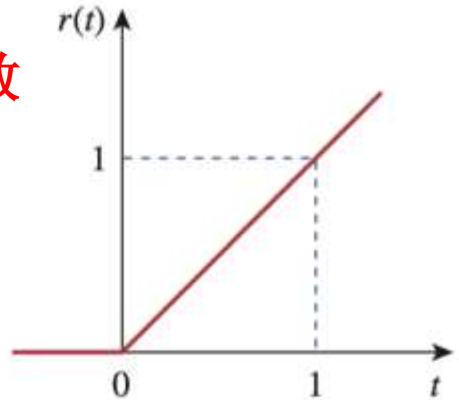


Figure 7.29
The unit ramp function.

The **unit ramp function** is zero for negative values of t and has a unit slope for positive values of t.

- 三个奇异函数的关系： 单位斜坡函数 → 单位阶跃函数 → 单位冲激函数

$$\delta(t)=\frac{du(t)}{dt}, \qquad u(t)=\frac{dr(t)}{dt}$$

或

$$u(t)=\int_{-\infty}^{t}\delta(\lambda)d\lambda, \qquad r(t)=\int_{-\infty}^{t}u(\lambda)d\lambda$$

Example 7.6

Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

Gate functions are used along with switches to pass or block another signal.

Solution:

The type of pulse in Fig. 7.31 is called the *gate function*. It may be regarded as a step function that switches on at one value of t and switches off at another value of t. The gate function shown in Fig. 7.31 switches on at $t = 2$ s and switches off at $t = 5$ s. It consists of the sum of two unit step functions as shown in Fig. 7.32(a). From the figure, it is evident that

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

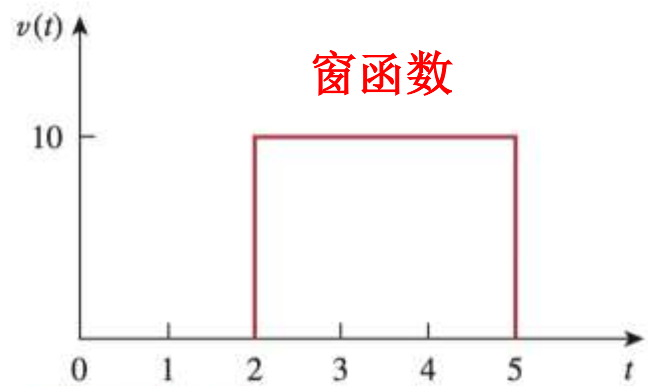
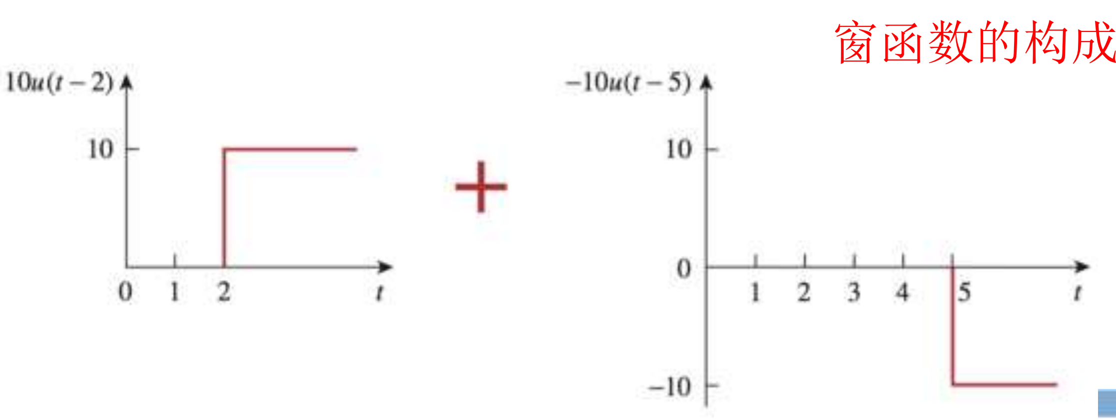
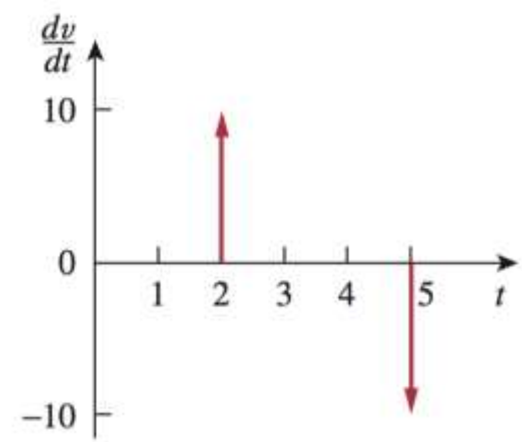


Figure 7.31
For Example 7.6.





Example 7.7

Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

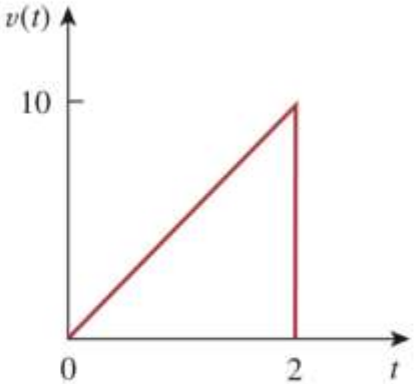
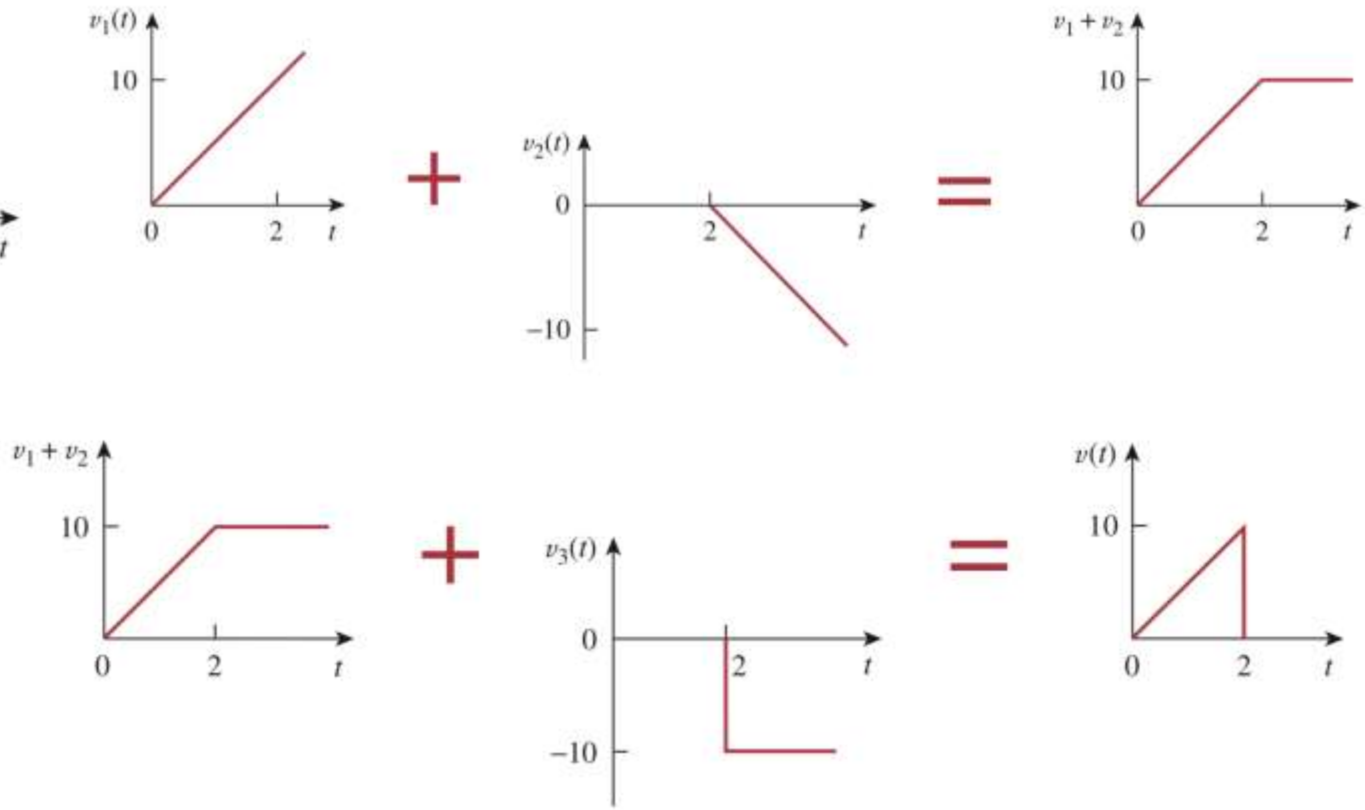


Figure 7.35
For Example 7.7.

锯齿函数



$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

Step response of an *RC* Circuit

- *RC* 电路的阶跃响应：若一直流源 (电压源或电流源) 突然加到 *RC* 电路中，则该直流源可以用一个阶跃函数表示，因此，我们把该响应称为阶跃响应 (step response)

> The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

- 阶跃响应，是电路对突然施加的 dc 电流源或电压源的响应

- 我们的目标：求得 C 两端的电压值 $v(t)$
  - 假设 C 两端的初始电压值为 V_0

$$v(0^-) = v(0^+) = V_0$$

  - 电源施加后，应用 KCL：

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0 \quad \xrightarrow{t>0} \quad \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

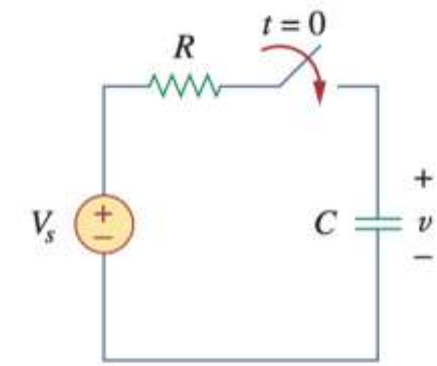
通用方法：
特征方程 s + 1/RC = 0
特征根：s_1 = -1/RC
齐次通解：$A*e^{(-t/RC)}$
特解(t=∞)：Vs
通解：$Vs + A*e^{(-t/RC)}$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{7.43}$$



(a)

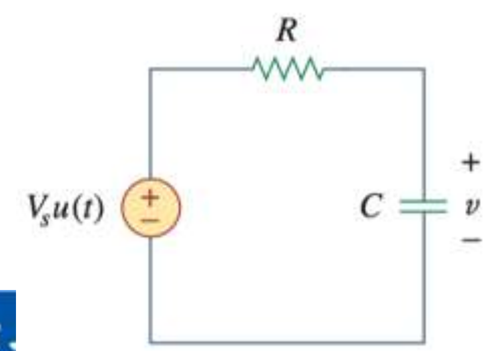
Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{7.44}$$



(b)

- 我们的目标：求得 C 两端的电压值 $v(t)$

$$\ln \frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{7.44}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

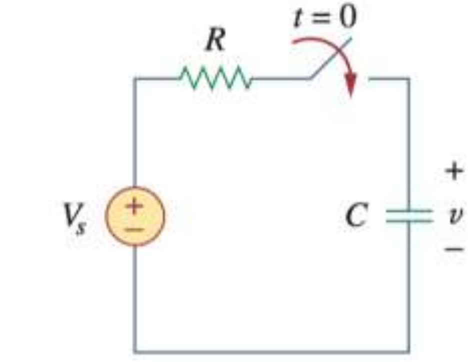
$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

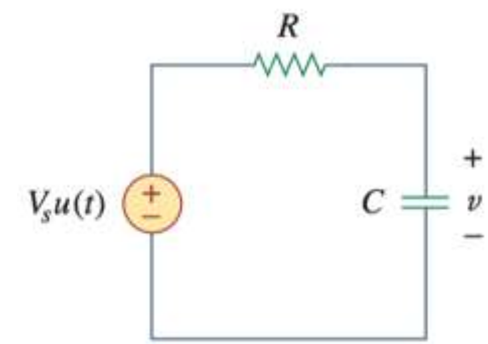
$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \tag{7.45}$$

Thus,

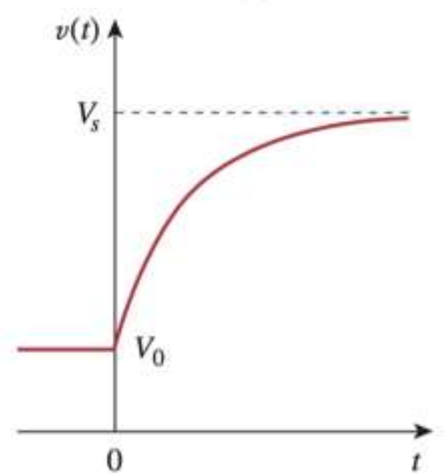
$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{7.46}$$



(a)



(b)



- **观察**
  - 电容两端电压不能突变；$t = 0$时刻为转折点
  - 最终趋向于稳态：$v = V_s$
  - 初始态与最终态的差额自然衰减，衰减常数为 τ

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

电路的全响应(观察1)

- 若电容在初始状态是未充电的，则 $V_0 = 0$，此时的响应称为**零状态响应**

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

或

$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

零状态响应：电路在零初始状态下（动态元件初始储能值为零），由外施激励引起的响应

- 观察1a（中文教材）：
- **全响应=零输入响应+零状态响应**

回顾：零输入响应：动态电路中无外施激励电源，仅由动态元件初始储能所激励的响应，称为动态电路的零输入响应

$$v(t) = V_0 e^{-t/RC}$$

$$v(t) = \begin{cases} V_0, & t < 0 \\ \underline{V_s} + \underline{(V_0 - V_s)e^{-t/\tau}}, & t > 0 \end{cases}$$

电路的全响应（观察2）

- 观察1b：（英文教材） 零输入响应 就是 自然响应
- 全响应=自然响应（V_0）+ 强制响应（外部有强加的V_s）

$$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

$$v_n = V_o e^{-t/\tau} \qquad v_f = V_s(1 - e^{-t/\tau})$$

若 t→无穷大

$V_n=0$ $V_f=V_s$

- 观察2：对应于“齐次微分方程的通解”+“∞时的特解”
- 全响应=瞬态响应（临时量）+ 稳态响应（永久量）

$$\text{Complete response} = \underset{\text{temporary part}}{\text{transient response}} + \underset{\text{permanent part}}{\text{steady-state response}}$$

瞬态响应自然衰减 $v_t = (V_o - V_s)e^{-t/\tau}$ $v_{ss} = V_s$

RC 阶跃响应总结

Whichever way we look at it, the complete response in Eq. (7.45) may be written as ①先写最终态（稳态） ②再初始态与最终态（稳态）的差值

③最后添加自然衰减项

$$\boxed{v(t) = \underline{v(\infty)} + \underline{[v(0) - v(\infty)]e^{-t/\tau}}} \tag{7.53}$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady-state value. Thus, to find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

1. 三要素法：① C 的初始电压值；② C 的稳态电压值；③时间常数；
2. 若电路是在 $t = t_0$ 时刻切换，则

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau}$$

Example 7.10

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.

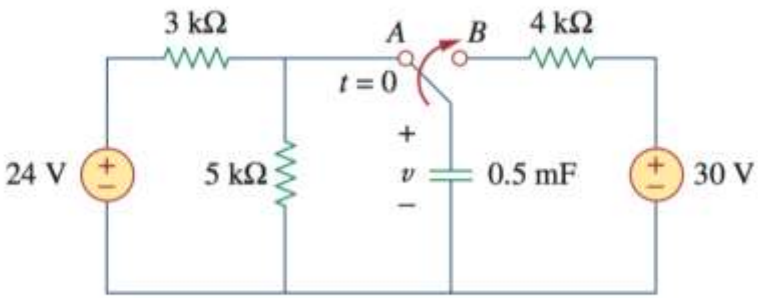


Figure 7.43
For Example 7.10.

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$ ①求初始值

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4\text{ k}\Omega$, and the time constant is

③求时间常数 $$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

②求稳态值

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V} \end{aligned}$$

At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.9 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

求0+初始值，要分析0-时刻，并利用电容电压不能突变原则

Example 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

All Time 节点+电流

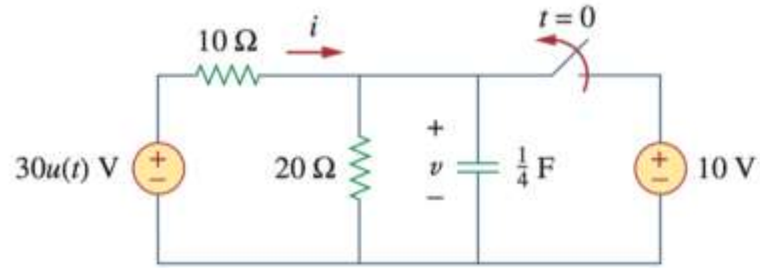
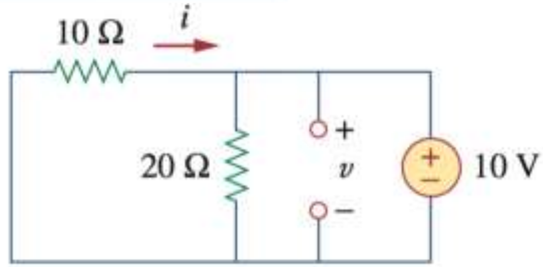
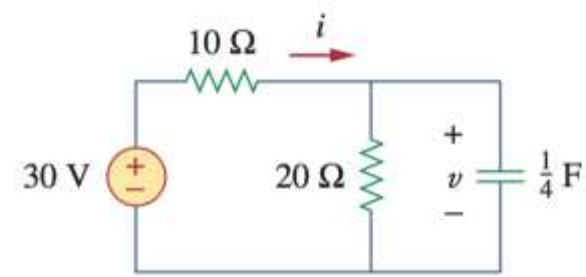


Figure 7.45 For Example 7.11.



(a)



(b)

①求初始值

$$v = 10\text{ V}, \qquad i = -\frac{v}{10} = -1\text{ A}$$

②求稳态值

$$v(\infty) = \frac{20}{20+10}(30) = 20\text{ V}$$

③求时间常数

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\ \Omega$$

and the time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\text{ s}$$

* $t < 0$时刻的 i 要单独计算

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V} \end{aligned}$$

$$\begin{aligned} i &= \frac{v}{20} + C\frac{dv}{dt} \\ &= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A} \end{aligned}$$

$$v = \begin{cases} 10\text{ V}, & t < 0 \\ (20 - 10e^{-0.6t})\text{ V}, & t \ge 0 \end{cases}$$

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$

RL 阶跃响应~与 RC 类似

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

1. 三要素法：① L 的初始电流值；② L 的稳态电流值；③时间常数；
2. 若电路是在 $t = t_0$ 时刻切换，则

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

求0+初始值，要分析0-时刻，并利用电感电流不能突变原则

Example 7.12

Find $i(t)$ in the circuit of Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.

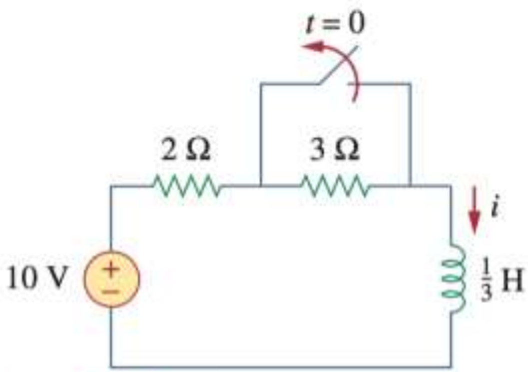


Figure 7.51
For Example 7.12.

先看开关变化前后
是否是**source free**

Solution:

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

①求初始电流值 $$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

②求稳态电流值 $$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{Th} = 2 + 3 = 5\ \Omega$$

For the time constant,

③求时间常数 $$\tau = \frac{L}{R_{Th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

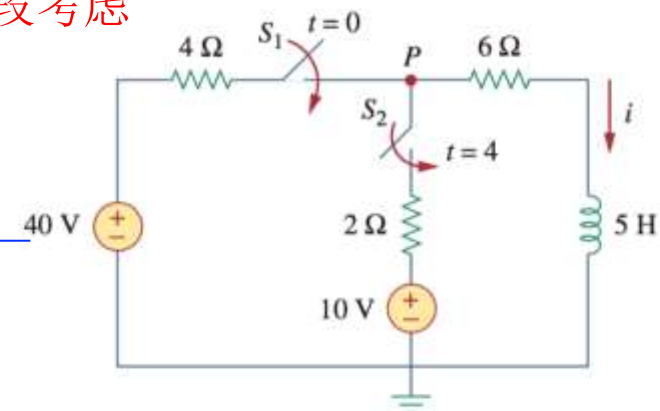
Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

Example 7.13

At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.

①分三个阶段考虑



We need to consider the three time intervals $t \le 0$, $0 \le t \le 4$, and $t \ge 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$

②求初始电流

For $0 \le t \le 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series. (Remember, at this time, S_2 is still open.) Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A}, \qquad R_{Th} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

③第二阶段的稳态、时间常数

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \le t \le 4 \end{aligned}$$

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

④第三阶段的初始态、稳态、时间常数

To find $i(\infty)$, let v be the voltage at node P in Fig. 7.53. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6} \quad \Rightarrow \quad v = \frac{180}{11} \text{ V}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727 \text{ A}$$

$$R_{Th} = 4 \| 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\ \Omega \qquad \tau = \frac{L}{R_{Th}} = \frac{5}{\frac{22}{3}} = \frac{15}{22} \text{ s}$$

$$i(t) = \begin{cases} 0, & t \le 0 \\ 4(1 - e^{-2t}), & 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \ge 4 \end{cases}$$

⇧

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \ge 4$$

中文教材内容补充

- 单位阶跃响应：电路对于**单位阶跃函数**输入的零状态响应
- 单位冲激响应：电路对于**单位冲激函数**激励的零状态响应
  - 零状态：储能元件（L、C）没有初始储能；
  - 把一个单位冲激电流加到初始电压为零的电容 C 上，则：

单位冲击等于初始值

$$u_C = \frac{1}{C}\int_{0_-}^{0_+}\delta_i(t)\mathrm{d}t = \frac{1}{C}$$

    - 相当于在 t_{0+} 时刻，电容有了初始电压 1/C；（注意：此时电容两端的电压有突变：t_{0-}时刻为0，t_{0+}时刻为1/C）
    - $t > 0$时，输入为零。因此，**单位冲激响应可当作电容两端初始电压为 1/C 的零输入响应来处理；**
  - 把一个单位冲激电压加到初始电流为零的电感 L 上，则：

$$i_L = \frac{1}{L}\int_{0_-}^{0_+}\delta_u(t)\mathrm{d}t = \frac{1}{L}$$

    - 相当于在 t_{0+}时刻，电感有了初始电流1/L；（注意：此时流经电感的电流有突变：t_{0-}时刻为0，t_{0+}时刻为1/L）
    - $t > 0$时，输入为零。因此，**单位冲激响应可当作流经电感的初始电流为1/L的零输入响应来处理；**

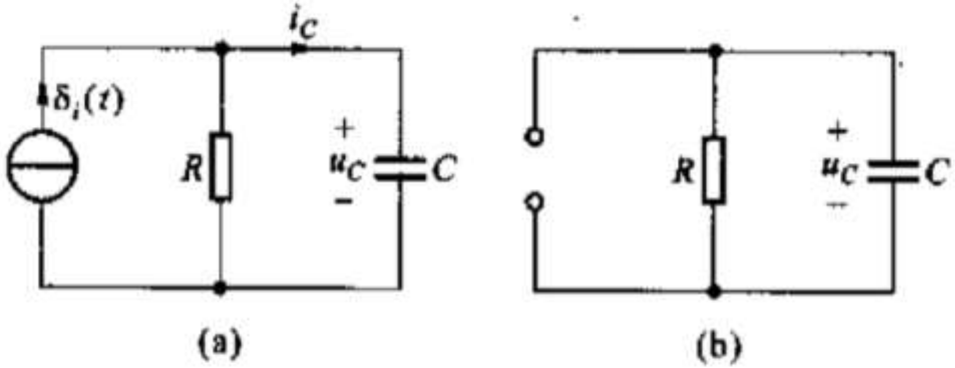


图 7－35　*RC* 电路的冲激响应

$$u_C = u_C(0_+)e^{-\frac{t}{\tau}} = \frac{1}{C}e^{-\frac{t}{\tau}}$$

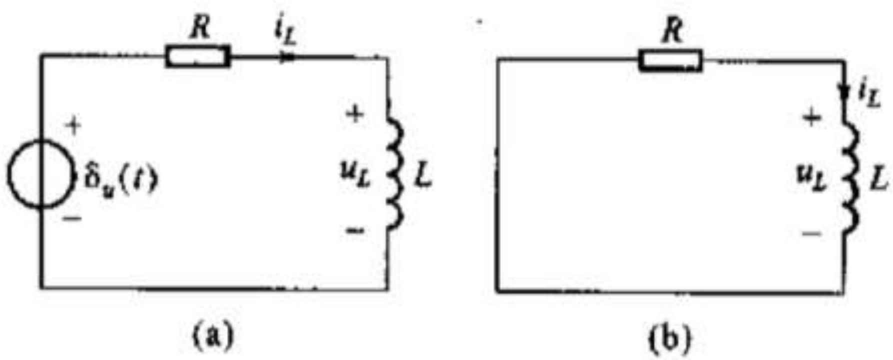


图 7－36　*RL* 电路的冲激响应

$$i_L = \frac{1}{L}e^{-\frac{t}{\tau}}$$

应用——简化的闪光灯电路

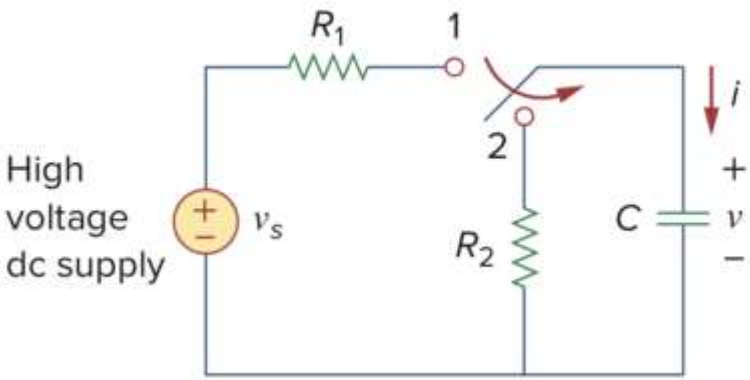


Figure 7.75
Circuit for a flash unit providing slow charge in position 1 and fast discharge in position 2.

电容两端电压不能突变

→ 切换到2时，因R_2阻值较小，可产生**瞬间大电流**（瞬间闪光）

$$t_{\text{charge}} = 5R_1C$$

$$t_{\text{discharge}} = 5R_2C$$

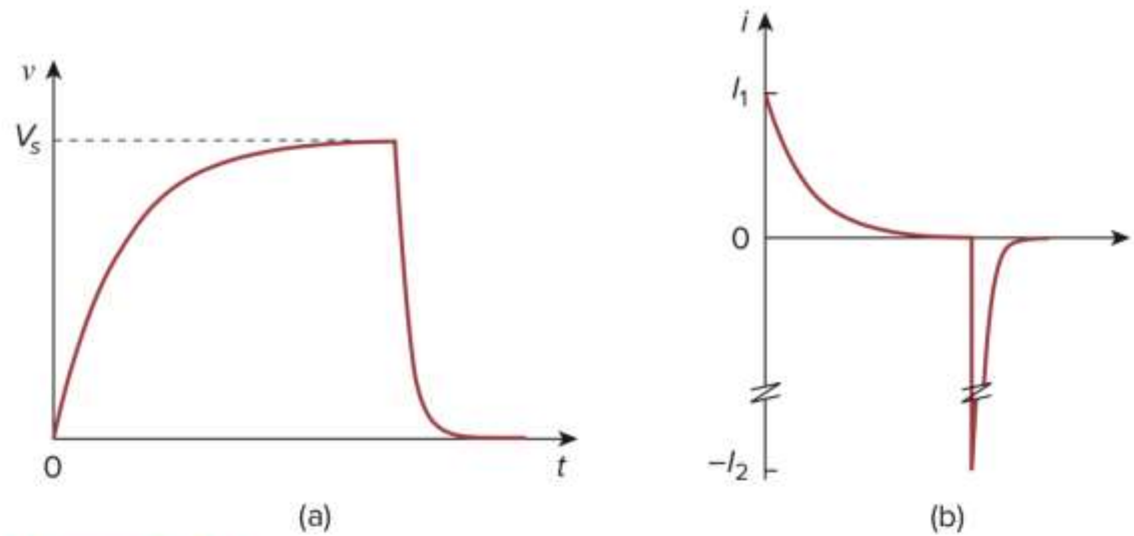


Figure 7.76
(a) Capacitor voltage showing slow charge and fast discharge, (b) capacitor current showing low charging current $I_1 = V_s/R_1$ and high discharge current $I_2 = V_s/R_2$.

小结

- 动态电路（含有L、C储能元件的电路）的分析
  - 同样基于KCL、KVL，以及“元件的电压电流约束关系”（依次为：电阻、电容、电感）

$$v = iR \qquad i = C\frac{dv}{dt} \qquad v = L\frac{di}{dt}$$

  - 但现在要求解的是**微分方程**。需要 1）确定初始条件；2）写出通解；
  - 初始条件（0+时刻）由0-时刻的电容电压、电感电流确定（利用电容电压不能突变、电感电流不能突变的原则）

小结

- 一阶电路的“source-free（零输入）”响应
  - 先确定初始条件：0+时刻，电容两端的电压 V_0，流过电感的电流 I_0
  - 再计算时间常数（R 为C或者L两端的等效电阻）

$$\boxed{\tau = RC} \qquad \boxed{\tau = \frac{L}{R}}$$

  - 再写出电容电压、电感电流的表达式

$$\boxed{v(t) = V_0 e^{-t/\tau}} \qquad \boxed{i(t) = I_0 e^{-t/\tau}}$$

  - 计算电路中其他的量

- 奇异函数：
  - 单位斜坡函数、单位阶跃函数、单位冲激函数，为依次求导关系；
  - 单位阶跃函数可用来表征“开关”的切换

小结

- 一阶电路的“全响应”，两种表述：
  - 表述1: 全响应 = 零输入响应（有储能，无输入） + 零状态响应（无储能，有输入）
  - 表述2: 全响应 = 稳态响应 + 瞬态响应

- **表述2的实际求解电路应用【重点】**：
  - 第一步：求初始值；
  - 第二步：求稳态值；
  - 第三步：求时间常数；
  - 然后，可以写出电容电压、电感电流的表达式：

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

作业

Practice Problem 7.2

If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

Answer: $8e^{-2t}$ V, 5.333 J.

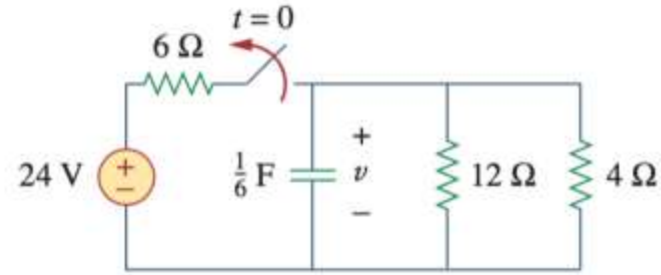


Figure 7.10
For Practice Prob. 7.2.

The Key to Working with a Source-Free *RC* Circuit Is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

Find i and v_x in the circuit of Fig. 7.15. Let $i(0) = 7$ A.

Practice Problem 7.3

Answer: $7e^{-2t}$ A, $-7e^{-2t}$ V, $t > 0$.

含受控源情况下求时间常数

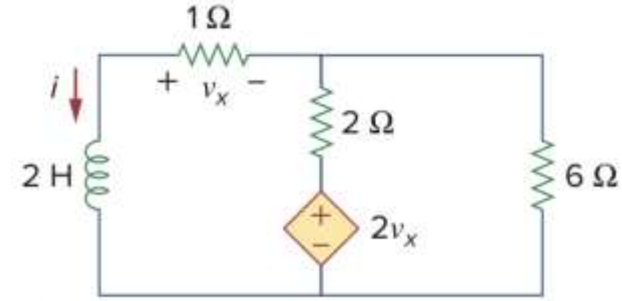


Figure 7.15
For Practice Prob. 7.3.

Practice Problem 7.10

Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.

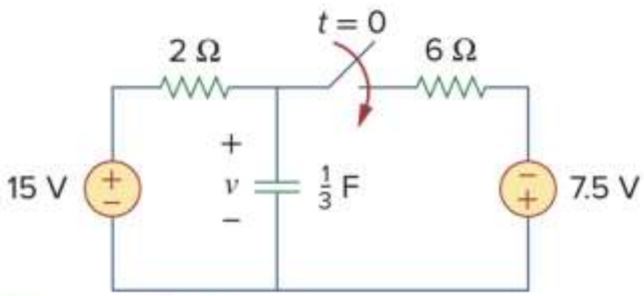


Figure 7.44
For Practice Prob. 7.10.

Answer: $(9.375 + 5.625e^{-2t})$ V for all $t > 0$, 11.444 V.

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

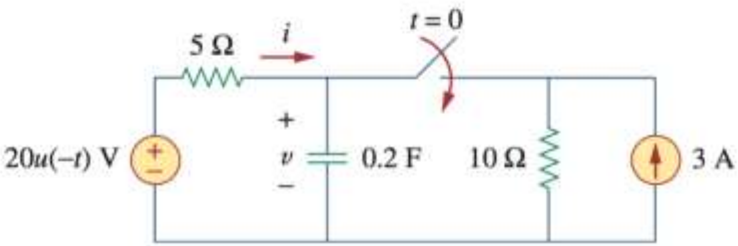


Figure 7.47
For Practice Prob. 7.11.

熟悉阶跃函数在电路描述中的应用

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

Answer: $i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-1.5t})\text{ A}, & t > 0, \end{cases}$

$v = \begin{cases} 20\text{ V}, & t < 0 \\ 10(1 + e^{-1.5t})\text{ V}, & t > 0 \end{cases}$

电流根据可以节点电压相除而得

Practice Problem 7.13

Switch S_1 in Fig. 7.54 is closed at $t = 0$, and switch S_2 is closed at $t = 2$s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

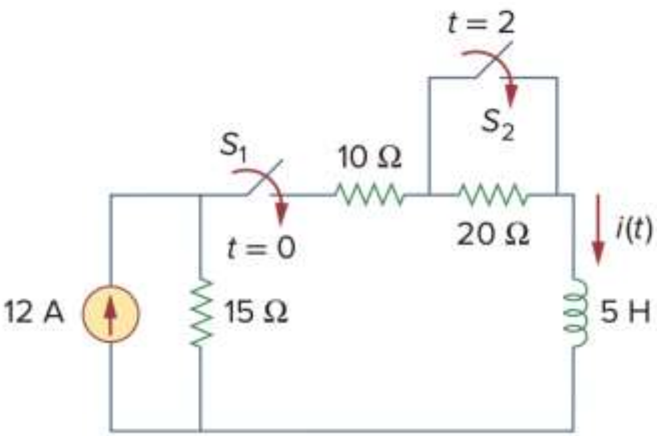


Figure 7.54
For Practice Prob. 7.13.

Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ 4(1 - e^{-9t}), & 0 < t < 2 \\ 7.2 - 3.2e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 4$ A, $i(3) = 7.178$ A.

多开关多阶段分析