Problem Set 5 Soln

Problem 1

(a) Consider a square wave $y(t)$ with period $T=4$. For $-2<t<2$,

$$y(t)=\begin{cases}1, & |t|<\frac{1}{4}\\ 0, & \frac{1}{4}<|t|<2\end{cases}$$

The Fourier Series coefficients for $y(t)$ is $b_k=\dfrac{\sin(k\pi/8)}{k\pi},\ b_0=\dfrac{1}{8}$

Consider a signal $z(t)=y(t)-\frac{1}{8}$, of which the Fourier Series coefficients $c_k=\begin{cases}0, & k=0\\ \frac{\sin(k\pi/8)}{k\pi}, & k\neq 0\end{cases}$ So $x(t)=z(t+1)$

For $-3<t<1$,

$$x(t)=\begin{cases}7/8, & -5/4<t<-3/4\\ -1/8, & -3<t<-5/4,\ -3/4<t<1\end{cases}$$

(b) Consider a square wave $y(t)$ with period $T=4$. For $-2<t<2$,

$$y(t)=\begin{cases}\frac{1}{2}, & |t|<\frac{1}{4}\\ 0, & \frac{1}{4}<|t|<2\end{cases}$$

So $x(t)=y(t+2)$ For $-4<t<0$,

$$x(t)=\begin{cases}1/2, & -9/4<t<-7/4\\ 0, & -4<t<-9/4,\ -7/4<t<0\end{cases}$$

(c)

$$x(t)=je^{j\frac{\pi}{2}t}+2je^{j\pi t}-je^{-j\frac{\pi}{2}t}-2je^{-j\pi t}=-2\sin(\frac{\pi}{2}t)-4\sin(\pi t)$$

(d) The Fourier series coefficients of $\sum\delta(t-nT)$ is $a_k=\dfrac{1}{T}$

$$x(t)=\sum_{k=-\infty}^{+\infty}e^{j\frac{2\pi}{T}kt}+e^{jw_0t}\sum_{k=-\infty}^{+\infty}e^{j\frac{2\pi}{T}2kt}=4\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{j\frac{2\pi}{T}kt}+2e^{jw_0t}\sum_{k=-\infty}^{+\infty}\frac{2}{T}e^{j\frac{2\pi}{T/2}kt}$$

Observe the equation above, we can conclude

$$x(t)=4\sum_{n=-\infty}^{+\infty}\delta(t-4n)+2e^{j\frac{\pi}{2}t}\sum_{n=-\infty}^{+\infty}\delta(t-2n)$$

Problem 2

- From $a_k=a_{-k}$, we can deduce that $x(t)=x(-t)$
- From $a_k=a_{k+2}$, we can see that $x(t)=x(t)e^{-j(4\pi/T)t}$, so $x(t)$ is always zero except $t=0,\pm1.5,\pm3...$
- Since $\int_{-0.5}^{0.5}x(t)dt=1$, $x(t)=\delta(t)\quad(-0.5<t<0.5)$
- Since $\int_{1}^{2}x(t)dt=2$, $x(t)=2\delta(t-1.5)\quad(1<t<2)$

Overall, we can conclude that

$$x(t)=\sum_{k=-\infty}^{+\infty}(\delta(t-3k)+2\delta(t-3k-1.5))$$

Problem 3

Since $x(t)$ and a_1 are real, we can conclude that $a_1=a_{-1}$, $a_2^*=a_{-2}$

Assume $x(t)=a_1(e^{jw_0t}+e^{-jw_0t})+a_2e^{2jw_0t}+a_2^*e^{-2jw_0t}=2a_1\cos(w_0t)+A_2\cos(2w_0+\varphi)$, so

$x(t-3)=2a_1\cos(w_0t-3w_0)+A_2\cos(2w_0+\varphi-6w_0)=2a_1\cos(w_0t-\pi)+A_2\cos(2w_0+\varphi-2\pi)=-2a_1\cos(w_0t)+A_2\cos(2w_0+\varphi)$

Since $x(t-3)=-x(t)$, we can obtain

$$x(t) = 2a_1\cos(w_0 t)$$

According to eq(5), $|a_1|^2 + |a_{-1}|^2 = \frac{1}{2}$, so $a_1 = \frac{1}{2}$

Overall, we can conclude that $x(t) = \cos(\pi t/3)$

Problem 4

(a) The Fourier transform of $tx(t)$ is $j\frac{d}{dw}X(jw)$, so we can obtain

$$te^{-|t|} \stackrel{\mathscr{F}}{\longleftrightarrow} j\frac{d}{dw}\left\{\frac{2}{1+w^2}\right\} = \frac{-4jw}{(1+w^2)^2}$$

(b) From (a) we can obtain

$$\int_{-\infty}^{+\infty} te^{-|t|}e^{-jwt}\,dt = \frac{-4jw}{(1+w^2)^2}$$

Let $w = -t$ and $t = w$, equation above can be transformed as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} -j2\pi we^{-|w|}e^{jwt}\,dw = \frac{4t}{(1+t^2)^2}$$

So it's easy to see that

$$\frac{4t}{(1+t^2)^2} \stackrel{\mathscr{F}}{\longleftrightarrow} -j2\pi we^{-|w|}$$

Problem 5

(a)

(1) If $x(t) = -x(-t)$, then $Re\left\{X(jw)\right\} = 0$, so **(a)** and **(d)** satisfy (1).

(2) If $x(t) = x(-t)$, then $Im\left\{X(jw)\right\} = 0$, so **(e)** and **(f)** satisfy (2).

(3) Condition (3) equals that there exists a real α such that $x(t+\alpha)$ is even, so **(a), (b), (e)** and **(f)** satisfy (3).

(4) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}\,dw$, $x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)dw = 0$, so **(a), (d)** and **(f)** satisfy (4).

(5) $\frac{dx(t)}{dt}|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} jwX(jw)dw = 0$, so **(b), (c), (e)** and **(f)** satisfy (5).

(6) Only discrete signals has periodic Fourier transform, so **(b)** satisfy (6).

(b) $x(t) = t^3$

Problem 6

(a) System function

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{2}{-w^2+6jw+8} = \frac{1}{jw+2} - \frac{1}{jw+4}$$

So the impulse response of this system

$$h(t) = (e^{-2t} - e^{-4t})u(t)$$

(b) Since $x(t) = te^{-2t}u(t)$,

$$X(jw) = \frac{1}{(jw+2)^2}$$

$$Y(jw) = X(jw)H(jw) = \frac{0.25}{jw+2} - \frac{0.25}{jw+4} - \frac{0.5}{(jw+2)^2} + \frac{1}{(jw+2)^3}$$

So $y(t) = [\frac{1}{4}e^{-2t} - \frac{1}{4}e^{-4t} - \frac{1}{2}te^{-2t}u(t) + \frac{1}{2}t^2e^{-2t}]u(t)$

(c) System function

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{-2w^2-2}{-w^2+\sqrt{2}jw+1} = 2 - \frac{\sqrt{2}(1-j)}{jw+\frac{\sqrt{2}}{2}(1-j)} - \frac{\sqrt{2}(1+j)}{jw+\frac{\sqrt{2}}{2}(1+j)}$$

So the impulse response of this system

$$h(t) = 2\delta(t) - 2\sqrt{2}u(t)e^{-t/\sqrt{2}}(\cos\frac{\sqrt{2}}{2}t + \sin\frac{\sqrt{2}}{2}t)$$

Problem 7

(a) The Fourier coefficients of $x_1(t)$

$$a_k = \frac{1}{10}\int_0^1 e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\frac{10}{-j2\pi k}e^{-j\frac{\pi}{5}kt}\Big|_0^1 = \frac{1}{j2\pi k}(1 - e^{-j\pi k/5}) \qquad k \neq 0$$

And $a_0 = \frac{1}{10}$

(b) The Fourier coefficients of $x_2(t)$

$$b_k = \frac{1}{10}\int_0^2 e^{-j\frac{2\pi}{10}kt}dt = \frac{1}{10}\frac{10}{-j2\pi k}e^{-j\frac{\pi}{5}kt}\Big|_0^2 = \frac{1}{j2\pi k}(1 - e^{-j2\pi k/5}) \qquad k \neq 0$$

And $b_0 = \frac{1}{5}$

(c) Since $x_3(t) = x_1(t) - x_1(t-2)$

$$c_k = a_k - e^{-2j\frac{2\pi}{10}k}a_k = \frac{1}{j2\pi k}(1 - e^{-j2\pi k/5})(1 - e^{-j\pi k/5}) \quad k \neq 0$$

And $c_0 = 0$

(d) Since $x_3(t) = \frac{dx_4(t)}{dt}$

$$d_k = \frac{10}{j2\pi k}c_k = -\frac{5}{2\pi^2 k^2}(1 - e^{-j2\pi k/5})(1 - e^{-j\pi k/5}) \quad k \neq 0$$

And $d_0 = \frac{1}{5}$

Problem 8

(a) $e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$, so

$$e^{-|t|} \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1+jw} + \frac{1}{1-jw} = \frac{2}{1+w^2}$$

Meanwhile,

$$\cos(2t) = \overset{\mathscr{F}}{\longleftrightarrow} \pi(\delta(w-2) + \delta(w+2))$$

Therefore, by the multiplication property,

$$e^{-|t|}\cos(2t) \overset{\mathscr{F}}{\longleftrightarrow} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi(\delta(\theta-2) + \delta(\theta+2))\frac{2}{1+(w-\theta)^2}d\theta = \frac{1}{1+(w-2)^2} + \frac{1}{1+(w+2)^2}$$

(b) $x_2(t) = \frac{\sin(2\pi t)}{\pi(t-1)} = \frac{\sin(2\pi(t-1))}{\pi(t-1)}$. Since

$$\frac{\sin 2\pi t}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} \begin{cases} 1, & |w| < 2\pi \\ 0, & |w| > 2\pi \end{cases}$$

We can obtain

$$X_2(jw) = e^{-jw}(u(w+2\pi) - u(w-2\pi))$$

(c) $x_3(t) = t^2(u(t) - u(t-1))$. The Fourier transform of $u(t) - u(t-1)$ is

$$A(jw) = (\frac{1}{jw} + \pi\delta(w))(1 - e^{-jw}) = \frac{1-e^{-jw}}{jw}$$

So

$$X_3(jw) = -\frac{d^2A}{dw^2} = \frac{j}{w}e^{-jw} + \frac{2}{w^2}e^{-jw} + \frac{2j}{w^3}(1 - e^{-jw})$$

(d)

$$X_4(jw) = \int_{-\infty}^{+\infty} (1-|t|)u(t+1)u(t-1)e^{-jwt}dt = \int_{-1}^{1} (1-|t|)e^{-jwt}dt = \int_{-1}^{1} e^{-jwt}dt - \int_{0}^{1} te^{-jwt}dt + \int_{-1}^{0} te^{-jwt}dt$$
$$= \frac{1}{jw}(e^{jw} - e^{-jw}) + \frac{1}{jw}e^{-jw} + \frac{1}{w^2}(1-e^{-jw}) - \frac{1}{jw}e^{jw} + \frac{1}{w^2}(1-e^{jw}) = \frac{2(1-\cos w)}{w^2}$$

Problem 9

$$y_1(5) = \int_{-\infty}^{+\infty} x_1(\tau)h(5-\tau)d\tau = A\int_{5-T}^{5} x_1(\tau)d\tau = 0$$

According to the figure of $x_1(t)$, we can conclude that $5-T=1 \longrightarrow T=4$

$$y_2(9) = A\int_{5}^{9} \sin(\pi\tau/3)d\tau = \frac{9A}{2\pi} = 9$$

So $A = 2\pi$

$$y_2(t) = \begin{cases} 0, & t<0 \\ 6(1-\cos(\pi t/3), & 0 \le t < 4 \\ 6(\cos[\pi(t-4)/3] - \cos(\pi t/3)), & t \ge 4 \end{cases}$$