Signals and Systems

Lecture3: Laplace, Z-Transform

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Outline

Making Continuous-Time Pole-Zero Diagrams

How to display the poles and zeros of a rational system function H(s) in a pole-zero diagram.

$$H(s) = \frac{s-1}{s^2+3s+2},$$

Using the function **roots**.

```
>> b = [1 -1];
>> a = [1 3 2];
>> zs = roots(b)
zs =
     1
>> ps = roots(a)
ps =
    -2
    -1
```

A simple pole-zero plot can be made by placing an 'x' at each pole location and an 'o' at each zero location in the complex s-plane, i.e.,

```
>> plot(real(zs),imag(zs),'o');
>> hold on
>> plot(real(ps),imag(ps),'x');
>> grid
>> axis([-3 3 -3 3]);
```

Making Continuous-Time Pole-Zero Diagrams

Problem 1

(a). Each of the following system functions corresponds to a stable LTI system. Use `roots` to find the poles and zeros of each system function and make an appropriately labeled pole-zero diagram using `plot` as shown above.

(i) $H(s) = \dfrac{s+5}{s^2+2s+3}$

(ii) $H(s) = \dfrac{2s^2+5s+12}{s^2+2s+10}$

(iii) $H(s) = \dfrac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$

Several different signals can have the same rational expression for their Laplace transform while having different regions of convergence. For example the causal and anticausal LTI systems with impulse responses

$$h_c(t) = e^{-\alpha t}u(t), \qquad h_{ac}(t) = -e^{-\alpha t}u(-t),$$

Making Continuous-Time Pole-Zero Diagrams

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Making Continuous-Time Pole-Zero Diagrams

Problem 1

have a rational system function with the same numerator and denominator polynomials,

$$H_c(s) = \frac{1}{s+\alpha}, \quad \Re e(s) > -\alpha,$$
$$H_{ac}(s) = \frac{1}{s+\alpha}, \quad \Re e(s) < -\alpha.$$

However, they have different system functions, since their regions of convergence are different.

(b). For each of the rational expressions in Part (a), determine the ROC corresponding to the stable system.

(c). For the causal LTI system whose input and output satisfy the following differential equation

$$\frac{dy(t)}{dt} - 3y(t) = \frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} + 5x(t),$$

find the poles and zeros of the system and make an appropriately labeled pole-zero diagram.

Making Continuous-Time Pole-Zero Diagrams

Function **pzplot** (Computer Explorations Toolbox)

- a pole-zero diagram for the LTI system whose numerator and denominator polynomials have the coefficients in the vectors **a** and **b**, respectively.
- The function will return the values of the poles and zeros in addition to making the plot.
- An optional argument, ROC, can be used to indicate the region of convergence on the diagram.
- $\gg [\mathrm{ps}, \mathrm{zs}] = \mathrm{pzplot}(\mathrm{b}, \mathrm{a}, 1);$

(d) Examine how **pzplot** can determine the region of convergence of a rational transform from knowledge of a single point within the ROC.

Outline

Making Discrete-Time Pole-Zero Diagrams

In this excercise you will learn how to display the poles and zeros of a discrete-time rational system function H(z) in a pole-zero diagram. The poles and zeros of a rational system function can be computed using roots. The function roots requires the coefficient vector to be in descending order of the independent variable. For example, consider the LTI system with system function

$$H(z) = \frac{z^2 - z}{z^2 + 3z + 2}, \tag{1}$$

Making Discrete-Time Pole-Zero Diagrams

The poles and zeros can be computed by executing

```
>> b = [1 -1 0];
>> a = [1 3 2];
>> zs = roots(b)
zs =
     0
     1
>> ps = roots(a)
ps =
    -2
    -1
```

Making Discrete-Time Pole-Zero Diagrams

It is often desirable to write discrete-time system functions in terms of increasing order of z^{-1}. The coefficients of these polynomials are easily obtained from the linear constant-coefficient difference equation and are also in the form that filter or freqz requires. However, if the numerator and denominator polynomials do not have the same order, some poles or zeros at z = 0 may be overlooked. For example, Eq.(1), could be rewritten as

$$H(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}, \tag{2}$$

Making Discrete-Time Pole-Zero Diagrams

If you were to obtain the coefficients from Eq. (10.4), you would get the following

```
>> b = [1 -1];
>> a = [1 3 2];
>> zs = roots(b)
zs =
     1
>> ps = roots(a)
ps =
    -2
    -1
```

Note that the zero at $z = 0$ does not appear here. In order to find the complete set of poles and zeros when working with a system function in terms of z^{-1}, you must append zeros to the coefficient vector for the lower-order polynomial such that the coefficient vectors are the same length.

Making Discrete-Time Pole-Zero Diagrams

Problem 2

For the exercises in this problem, you will need the M-file dpzplot.m. The function **dpzplot(b,a)** plots the poles and zeros of discrete-time systems. The inputs to **dpzplot** are in the same format as **filter**, and **dpzplot** will automatically append an appropriate number of zeros to a or to b if the numerator and denominator polynomials are not of the same order. Also, **dpzplot** will include the unit circle in the plot as well as an indication of the number of poles or zeros at the origin-if there are more than one.

Making Discrete-Time Pole-Zero Diagrams

Problem 2

- (a). Use **dpzplot** to plot the poles and zeros for $H(z)$ in Eq. (1).
- (b). Use **dpzplot** to plot the poles and zeros for a filter which satisfies the difference equation

$$y[n] + y[n-1] + 0.5y[n-2] = x[n]. \tag{3}$$

- (c). Use **dpzplot** to plot the poles and zeros for a filter which satisfies the difference equation

$$y[n]-1.25y[n-1]+0.75y[n-2]-0.125y[n-3] = x[n]+0.5x[n-1]. \tag{4}$$

Outline

Problem 3

The functions **impulse** and **lsim** can be used to simulate the impulse response and output, respectively, of LTI systems whose input and output satisfy linear constant-coefficient differential equations. These functions assume that the LTI system associated with the differential equation

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}, \tag{9.11}$$

is causal, i.e., the impulse response $h(t)$ is nonzero only for $t \geq 0$. However, there are multiple LTI systems which satisfy any linear constant-coefficient differential equation, and which system is chosen depends upon the auxiliary conditions used to constrain the system output $y(t)$. For instance, initial rest conditions lead to a causal system, while final rest conditions lead to an anticausal system[1]. (An anticausal system has an impulse response $h(t)$ which is nonzero only for $t \leq 0$.) Various sets of auxiliary conditions lead to other LTI systems which satisfy Eq. (9.11).

In many applications, one is interested only in systems which are stable. The causal system associated with Eq. (9.11) is not necessarily stable. For instance, if the system function $H(s)$ has a pole in the right-half plane, then the causal system will be unstable. In this exercise, you will learn how to use **lsim** to calculate the response of the stable LTI system whose input and output satisfy Eq. (9.11).

Problem 3

Using `lsim` to calculate the response of the anticausal LTI system satisfying Eq. (9.11) requires a number of steps. The first step is to find a causal system whose impulse response $h_c(t)$ is a time-reversal of the impulse response for the desired anticausal system $h_{ac}(t)$, i.e., $h_c(t) = h_{ac}(-t)$. This causal system can be simulated using `lsim`. A differential equation for the causal system can be derived by substituting $-t$ for t in Eq. (9.11), which yields

$$\sum_{k=0}^{K}(-1)^k a_k \frac{d^k y(-t)}{dt^k} = \sum_{m=0}^{M}(-1)^m b_m \frac{d^m x(-t)}{dt^m}\,. \tag{9.12}$$

The input to this system is $r(t) = x(-t)$ and the output is $w(t) = y(-t)$. The auxiliary conditions for this causal system are initial rest conditions. The next step is to obtain the input $r(t)$ to the causal system by time-reversing the input $x(t)$ to the anticausal system. The third step is to use `lsim` to simulate the response $w(t)$ of the causal LTI system satisfying Eq. (9.12) when the input is $r(t)$. The final step is to time-reverse the simulated response $w(t)$ to obtain $y(t)$ for the original anticausal system. These steps are carried out in the following problems.

(a). If $H_{ac}(s)$ is the system function associated with Eq. (9.12) and $H_c(s)$ is the system function associated with Eq. (9.11), how are the poles of these two system functions related? How is $H_c(s)$ related to $H_{ac}(s)$?

Problem 3

For the following problems, consider the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)\,. \tag{9.13}$$

(b). Determine the system function $H(s)$ associated with Eq. (9.13) and all of the possible regions of convergence of $H(s)$. For each region of convergence, determine the impulse response of the corresponding LTI system.

(c). For each region of convergence determined in Part (b), what is the corresponding auxiliary condition for the differential equation.

(d). Define `a` and `b` to contain the coefficients of the denominator and numerator polynomials of $H(s)$. For the causal system which has system function $H(s)$, use `impulse` to verify the analytic expression for the impulse response. Store in the vector `h` the values of the impulse response at the times in `t=[-5:0.01:5]` returned by `impulse`. Plot `h` versus `t`. Note that since `hs=impulse(b,a,ts)` returns samples of the response to $\delta(t-$`ts(1)`$)$, you will have to select the appropriate time samples to input to `impulse` and append the result to an appropriate number of zeros.

(e). Repeat Part (d) for the anticausal system. Remember that `impulse(b,a,ts)` assumes that the coefficients in `a` and `b` correspond to a causal system. You will need to define a new set of coefficients for a time-reversed system (see Eq. (9.12)) and then appropriately flip the impulse response computed by `impulse` (since the output of Eq. (9.12) is time-reversed).

Problem 3

(f). Analytically calculate the output of the anticausal LTI system satisfying Eq. (9.13) when the input is $x(t) = e^{5t/2}u(-t)$.

(g). Use `lsim` to verify the output of the anticausal system derived in Part (f) at the time samples `t`. Like `impulse`, the function `lsim(b,a,x,ts)` also assumes that the vectors `a` and `b` correspond to a causal system, so you must work with the time-reversed differential equation. The coefficients of this differential equation should already have been computed in Part (e). Note that the input to the time-reversed system must be time-reversed, i.e., $r(t) = x(-t)$.

Problem 3

Now consider the third-order differential equation

$$\frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} + 24\,\frac{dy(t)}{dt} - 26\,y(t) = \frac{d^2x(t)}{dt^2} + 7\,\frac{dx(t)}{dt} + 21\,x(t)\,. \tag{9.14}$$

(h). Determine the system function $H(s)$ associated with Eq. (9.14) and plot the poles and zeros of this system function as shown in Tutorial 9.1. Determine all of the possible regions of convergence. For which region of convergence is the system stable?

(i). Use **`residue`** to determine the partial fraction expansion of $H(s)$. For each region of convergence determined in Part (h), analytically determine the associated impulse response.

(j). For the causal system associated with Eq. (9.14), use **`impulse`** to verify the corresponding impulse response computed in Part (i). Use the vector **`t`** for the time samples. What are the auxiliary conditions on $y(t)$ associated with this causal system?

(k). For the anticausal system associated with Eq. (9.14), use **`impulse`** to verify the corresponding impulse response computed in Part (i). Use the vector **`t`** for the time samples. Remember that you must first determine the coefficients of the time-reversed differential equation, and then derive the anticausal impulse response of Eq. (9.14) from the causal impulse response of the time-reversed equation. What are the auxiliary conditions on $y(t)$ associated with this anticausal system?

Problem 3

The following problems consider the numerical implementation of the stable LTI system whose inputs and outputs satisfy Eq. (9.14). The impulse response $h_s(t)$ of this system, as calculated in Part (i), should be nonzero for all time t. This system is called noncausal since it is neither causal nor anticausal, and a simple time-reversal of the differential equation will not suffice for implementation. Instead, the differential equation must be decomposed into a causal and an anticausal component, each of which are computed separately. This is known as a parallel realization of the system.

(l). A parallel realization an LTI system is easily visualized by decomposing the system function as $H(s) = H_1(s) + H_2(s)$, so that the output $y(t)$ can be computed by separately computing the response to $H_1(s)$ and $H_2(s)$. Note that the regions of

Problem 3

convergence of $H_1(s)$ and $H_2(s)$ must be properly specified. Given the partial fraction expansion you found in Part (i), determine $H_1(s)$ and $H_2(s)$ for the stable system such that $h_1(t)$ is causal and $h_2(t)$ anticausal. Make sure to specify the regions of convergence of both $H_1(s)$ and $H_2(s)$.

(m). Determine $h_1(t)$ and $h_2(t)$, the inverse transforms of $H_1(s)$ and $H_2(s)$. You should be able to determine these impulse responses from the expression for $h_s(t)$.

(n). Define $y_1(t)$ to be the output of the system defined by $H_1(s)$ and its region of convergence. Specify the differential equation corresponding to $H_1(s)$ along with the associated auxiliary conditions on $y_1(t)$.

(o). Define $y_2(t)$ to be the output of the system defined by $H_2(s)$ and its region of convergence. Specify the differential equation corresponding to $H_2(s)$ along with the associated auxiliary conditions on $y_2(t)$.

(p). Use `lsim` to determine the response of the stable system to the input

$$x(t) = \begin{cases} 1, & -3 \leq t \leq 2, \\ 0, & \text{otherwise}, \end{cases} \tag{9.15}$$

on the interval $-10 \leq t \leq 10$ for the time samples in `t=[-10:0.01:10]`. Plot $y_1(t)$, $y_2(t)$ and $y(t)$ on the interval $-10 \leq t \leq 10$.

Outline

Assignments

Issued: April 11, 2024 Due: May 5, 2024

You should hand in the Matlab code (.m files), graphics, audio files and a brief description of your reasoning as well as comments if any. Please make sure that your Matlab code can be run on Matlab R2007b or higher version. You should pack all of your files into a .rar or .zip file, titled as xxxxxxx(your student ID) xxxx(your name) Lab pre, and then upload to blackboard before 11:59pm of the due day.