Laplace&Z Transform

$H(s)=\int_{-\infty}^{\infty}h(t)e^{st}\,dt$ $\quad e^{st}\rightarrow H(s)e^{st}$ **特征函数**

因果信号的收敛域在最右极点的右边；反因果信号的收敛域在最左极点的左边；非因果信号的收敛域在两个极点中间，不包括任何一个极点。信号收敛则收敛域包含实轴。同表达式的拉氏变换由于收敛域不同对应不同的信号。

$X(z)=\sum_{n=-\infty}^{+\infty}x[n]z^{-n}$ $\quad z^{n}\rightarrow H(z)z^{n}$ **特征函数**

因果信号的收敛域在最外侧极点的外侧；反因果信号的收敛域在最内测极点的内测；非因果信号的收敛域在两个极点之间，不包括任何一个极点。信号收敛则收敛域包含单位圆。同表达式的Z变换由于收敛域不同对应不同的信号。

Fundamental Mode

CT: $e^{-at}u(t)\rightarrow\frac{1}{s+a}$

DT: $-a^{n}u[n]=\frac{z}{z-a}=\frac{1}{1-az^{-1}}$

Fourier series

CTFS

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k\,e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty}a_k\,e^{jk(2\pi/T)t}$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt$$

卷积： $(u(t+t_0)-u(t-t_0))*(u(t+t_0)-u(t-t_0))$

$=(2t_0-|t|)u(t+2t_2)u(2t_0-t)$

有限长序列$[a,b]$ $[c,d]$卷积后非零区间为$[a+b,c+d]$，方法如下图。

```
x(n)  1 2 3
h(n)  1 1 1 1
-------------
      1 2 3
        1 2 3
          1 2 3
            1 2 3
-------------
      1 3 6 6 5 3
```

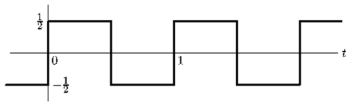
连续时间傅里叶级数 Properties

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0=2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t)+By(t)$	Aa_k+Bb_k
Time Shifting	3.5.2	$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0}=a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}=e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t),\alpha>0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	Ta_kb_k
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty}a_lb_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0a_k=jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t}x(t)\,dt$ (finite valued and periodic only if $a_0=0$)	$\left(\frac{1}{jk\omega_0}\right)a_k=\left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases}a_k=a^*_{-k}\\ \Re e\{a_k\}=\Re e\{a_{-k}\}\\ \Im m\{a_k\}=-\Im m\{a_{-k}\}\\ \lvert a_k\rvert=\lvert a_{-k}\rvert\\ \sphericalangle a_k=-\sphericalangle a_{-k}\end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real]; $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{a_k\}$; $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T|x(t)|^2dt=\sum_{k=-\infty}^{+\infty}|a_k|^2$$



方波的傅里叶级数为：$a_k=\begin{cases}\frac{1}{j\pi k} & if\ k\ is\ odd\\ 0 & otherwise\end{cases}$

因此三角波可以视为方波的积分：

$$a_k=\begin{cases}\frac{1}{j\pi k}\times\frac{1}{j2\pi k} & if\ k\ is\ odd\\ 0 & otherwise\end{cases}$$

冲激序列 FS：$x_\delta(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{j2\pi kt/T}$

可见其**无穷长**

方波卷积为三角波，周期性方波卷积为周期性三角波，三角波宽度为方波两倍，最大值为方波宽度乘方波高度的平方。

DTFS

$$x[n]=\sum_{k=<N>}a_k\,e^{jk\omega_0 n}=\sum_{k=<N>}a_k\,e^{jk(2\pi/N)n}$$

$$a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\omega_0 n}=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

离散时间傅里叶级数 Properties

离散时间傅里叶级数周期为 N

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$, $y[n]$ Periodic with period N and fundamental frequency $\omega_0=2\pi/N$	a_k, b_k Periodic with period N
Linearity	$Ax[n]+By[n]$	Aa_k+Bb_k
Time Shifting	$x[n-n_0]$	$a_ke^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n]=\begin{cases}x[n/m], & \text{if } n \text{ is a multiple of } m\\ 0, & \text{if } n \text{ is not a multiple of } m\end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N\rangle}x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle}a_lb_{k-l}$
First Difference	$x[n]-x[n-1]$	$(1-e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n}x[k]$ (finite valued and periodic only if $a_0=0$)	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}a_k=a^*_{-k}\\ \Re e\{a_k\}=\Re e\{a_{-k}\}\\ \Im m\{a_k\}=-\Im m\{a_{-k}\}\\ \lvert a_k\rvert=\lvert a_{-k}\rvert\\ \sphericalangle a_k=-\sphericalangle a_{-k}\end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$x_e[n]=\mathcal{E}v\{x[n]\}$ [x[n] real]; $x_o[n]=\mathcal{O}d\{x[n]\}$ [x[n] real]	$\Re e\{a_k\}$; $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

将输入输出表示为傅里叶级数综合方程后，一个 LTI 系统仅对每个系数进行伸缩，换言之，**输入级数为 0，输出对应级数也为 0。**将 LTI 系统理解为一个滤波器。对连续也如此。

CTFT

$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\,d\omega;\ X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$

Duality: $\omega\rightarrow t;\ t\rightarrow\omega$; Reverse: $t=-t$ or $\omega=-\omega$

Periodic Signal: $X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi a_k\,\delta(\omega-k\omega_0)$

由傅里叶级数综合方程变换而来。

Differential Equation:

$$\sum_{k=0}^{N}a_k\frac{d^ky(t)}{dt^k}=\sum_{k=0}^{M}\frac{d^kx(t)}{dt^k};H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{\sum_{k=0}^{M}b_k(j\omega)^k}{\sum_{k=0}^{N}a_k(j\omega)^k}$$

连续信号傅里叶变换性质

Property	Aperiodic signal	Fourier transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega)=X^*(-j\omega)$ $\mathcal{Re}\{X(j\omega)\}=\mathcal{Re}\{X(-j\omega)\}$ $\mathcal{Im}\{X(j\omega)\}=-\mathcal{Im}\{X(-j\omega)\}$ $\|X(j\omega)\|=\|X(-j\omega)\|$ $\sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

常见连续信号傅里叶变换

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta(\omega-k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1=1$ $a_k=0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1=a_{-1}=\frac{1}{2}$ $a_k=0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1=-a_{-1}=\frac{1}{2j}$ $a_k=0$, otherwise
$x(t)=1$	$2\pi\delta(\omega)$	$a_0=1,\quad a_k=0,\ k\neq 0$ (this is the Fourier series representation for any choice of $T>0$)
Periodic square wave $x(t)=\begin{cases}1, & \|t\|<T_1\\ 0, & T_1<\|t\|\leq\frac{T}{2}\end{cases}$ and $x(t+T)=x(t)$	$\sum_{k=-\infty}^{+\infty}\frac{2\sin k\omega_0T_1}{k}\delta(\omega-k\omega_0)$	$\frac{\omega_0T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0T_1}{\pi}\right)=\frac{\sin k\omega_0T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k=\frac{1}{T}$ for all k
$x(t)\begin{cases}1, & \|t\|<T_1\\ 0, & \|t\|>T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega)=\begin{cases}1, & \|\omega\|<W\\ 0, & \|\omega\|>W\end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	—
$\delta(t-t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\mathcal{Re}\{a\}>0$	$\frac{1}{a+j\omega}$	—
$te^{-at}u(t)$, $\mathcal{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^n}$	—

DTFT $x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}\,d\omega \qquad X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}$

Periodic Signal $x[n]=\sum_{k=<N>}a_k e^{jk(2\pi/N)n} \qquad X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}2\pi a_k\,\delta\left(\omega-\frac{2\pi k}{N}\right)$

差分方程： $\sum_{k=0}^{N}a_k\,y[n-1]=\sum_{k=0}^{M}b_k\,x[n-k]\rightarrow H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{\sum_{k=0}^{M}b_k e^{-j\omega}}{\sum_{k=0}^{M}a_k e^{-jk\omega}}$

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$X(e^{j\omega})$ periodic with $Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\ 0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$X(e^{j\omega})=X^*(e^{-j\omega})$ $\mathcal{Re}\{X(e^{j\omega})\}=\mathcal{Re}\{X(e^{-j\omega})\}$ $\mathcal{Im}\{X(e^{j\omega})\}=-\mathcal{Im}\{X(e^{-j\omega})\}$ $\|X(e^{j\omega})\|=\|X(e^{-j\omega})\|$ $\sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{Ev}\{x[n]\}$ [$x[n]$ real] $x_o[n]=\mathcal{Od}\{x[n]\}$ [$x[n]$ real]	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty}\|x[n]\|^2=\frac{1}{2\pi}\int_{2\pi}\|X(e^{j\omega})\|^2d\omega$		

Properties

特别注意乘积性质频谱卷积只在一个周期内做卷积

可以推导出以下性质：

1. $X(e^{j0})=\sum_{n=-\infty}^{\infty}x[n]$

2. 对于时移的偶信号，其幅角为$-\omega_0 n$

3. $x(0)=\int_{-\pi}^{\pi}X(e^{j\omega})d$

4. 对于简单的频率值，如π，$X(e^{j\pi})=\sum_{n=-\infty}^{\infty}x(n)(-1)^n$

5. 为了求一个信号傅里叶变换的实部，可以转化为原信号的偶信号的傅里叶变换

6. 遇到平方项可以试试 Parseval 定理

7. 离散时间傅里叶变换是周期的，周期为2π

8. 实信号的傅里叶变换实部偶对称，虚部奇对称，那么反过来可以通过傅里叶变换证明一个信号是实信号。

9. 系统给定系统的输入输出可以确定系统函数：

$$x[n]=\left(\frac{1}{2}\right)^n u[n]-\frac{1}{4}\left(\frac{1}{2}\right)^{n-1}n[n-1]\rightarrow\frac{1}{1-\frac{1}{2}e^{-j\omega}}-\frac{1}{4}e^{-j\omega}\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$y[n]=\left(\frac{1}{3}\right)^n u[n]\rightarrow\frac{1}{1-\frac{1}{3}e^{-j\omega}}$ *故系统函数为* $H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}$

常见离散信号傅里叶变换

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$	(a) $\omega_0=\frac{2\pi m}{N}$, $a_k=\begin{cases}1, & k=m, m\pm N, m\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0=\frac{2\pi m}{N}$, $a_k=\begin{cases}\frac{1}{2}, & k=\pm m, \pm m\pm N, \pm m\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0=\frac{2\pi r}{N}$, $a_k=\begin{cases}\frac{1}{2j}, & k=r, r\pm N, r\pm 2N,\dots\\ -\frac{1}{2j}, & k=-r, -r\pm N, -r\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n]=1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$	$a_k=\begin{cases}1, & k=0,\pm N,\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$
Periodic square wave $x[n]=\begin{cases}1, & \lvert n\rvert\le N_1\\ 0, & N_1<\lvert n\rvert\le N/2\end{cases}$ and $x[n+N]=x[n]$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]}, k\ne 0,\pm N,\pm 2N,\dots$; $a_k=\frac{2N_1+1}{N}, k=0,\pm N,\pm 2N,\dots$
$\sum_{k=-\infty}^{+\infty}\delta[n-kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert<1$	$\frac{1}{1-ae^{-j\omega}}$	—
$x[n]\begin{cases}1, & \lvert n\rvert\le N_1\\ 0, & \lvert n\rvert>N_1\end{cases}$	$\frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$	
$\frac{\sin Wn}{\pi n}=\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right)$, $0<W<\pi$	$X(\omega)=\begin{cases}1, & 0\le\lvert\omega\rvert\le W\\ 0, & W<\lvert\omega\rvert\le\pi\end{cases}$ $X(\omega)$ periodic with period 2π	
$\delta[n]$	1	
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$	
$\delta[n-n_0]$	$e^{-j\omega n_0}$	
$(n+1)a^n u[n], \quad \lvert a\rvert<1$	$\frac{1}{(1-ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n], \quad \lvert a\rvert<1$	$\frac{1}{(1-ae^{-j\omega})^r}$	—

$$x[n]=\cos\left(\frac{5\pi}{2}n-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}n-\frac{\pi}{4}\right)$$

对于频率大于2π的信号首先要将频率变换到2π内:进一步

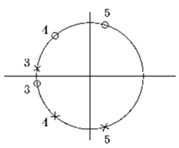
$$x[n]=\frac{1}{2}\left(e^{j\left(\frac{\pi}{2}n-\frac{1}{4}\right)}+e^{-j\left(\frac{\pi}{2}n-\frac{1}{4}\right)}\right)$$

DT 2 DT: $\Omega=\omega T$, 以$T=0.001$采样，得到：

$x_1(t)=\cos(3000t)$ $\quad x_1[t]=\cos(3n)$

$x_2(t)=\cos(4000t)$ $\quad x_2[t]=\cos(4n)$

$x_3(t)=\cos(5000t)$ $\quad x_3[t]=\cos(5n)$

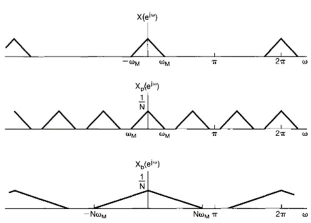


由于ω的周期性，超过2π的角频率不能代表频率快慢。越靠近π的频率越高。

DT Sampling and Decimation

$$x_p[n]=\sum_{k=-\infty}^{\infty}x[kN]\delta[n-kN] \quad x_b(e^{j\omega})=\sum_{k=-\infty}^{\infty}x_p[kN]e^{-j\omega k}=X_p(e^{j\omega/N})$$

当频谱占满$[-\pi,\pi]$代表没有信息被浪费，Decimation 只拉开$[-\pi,\pi]$的部分，频谱依然是按照2π重复的。Decimation 可以减少对空间的浪费。Decimation 也就是 down-sampling，那么 up-sampling 反之，就是将数据点拉开，向相邻点间补零，在频域上的效果就是压缩频带。



Modulation

复指数信号调制

$$y(t)=x(t)e^{-j\omega_c t}=x(t)\cos\omega_c t+jx(t)\sin\omega_c t \qquad Y(j\omega)=X(j(\omega-\omega_c)$$

余弦信号调制

$$y(t)=\cos(\omega t)x(t) \qquad Y(j\omega)=\frac{1}{2}\left[X(j(\omega-\omega_c))+X(j(\omega+\omega_c))\right]$$

Synchronous demodulation（乘以$\cos(\omega t)$）：

$$x_r(t)=\cos(\omega t)y(t)=\frac{1}{2}(1+\cos(2\omega t))x(t)$$

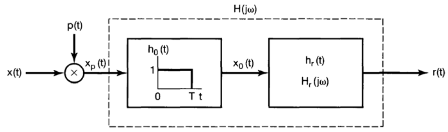
Asynchronous demodulation（使用 envelop detection）：

Sampling $\quad X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X(j(\omega-k\omega_s))$

Sampling Theory $\omega_s>2\omega_M;\ \omega_s=\frac{2\pi}{T}$

Zero hold

在时域上对用冲激串采样的信号卷积上窗函数，注意是移位的窗函数，即可获得零阶保持信号，等效为在频域上乘上一个 sinc 函数，那么只需要把它除掉就可以还原。



$$H_0(j\omega)=e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

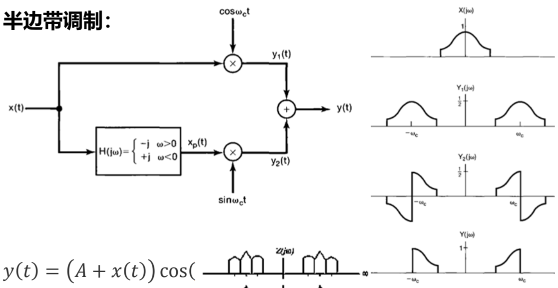
$$H_r(j\omega)=\frac{e^{j\omega T/2}H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

Aliasing

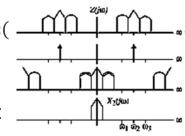
采样等于在频域上将信号按照$k\omega$的间隔进行周期延拓，那么如果某个频率超过了采样频率的一半，将其加减$k\omega$就有可能落到$[-\omega_0,\omega_0]$间，这样就可以知道其与哪个频率（变换前的）发生了混叠。混叠会导致频谱改变，使其失真。

A harmonic is **a wave or signal whose frequency is an integral (whole number) multiple of the frequency of the same reference signal or wave**

半边带调制：



$y(t)=(A+x(t))\cos($



Frequency-Division Multiplexing

解调信号与信道中心频率一致即可。如右图：

FM modulation $y(t)=A\cos(\omega_c t+\theta_c(t)) \qquad \frac{d\theta_c(t)}{dt}=\omega_c+k_f x(t)$

Examples A CT signal $x_c(t)$ is converted to DT signal $x_d[n]$ as follows:

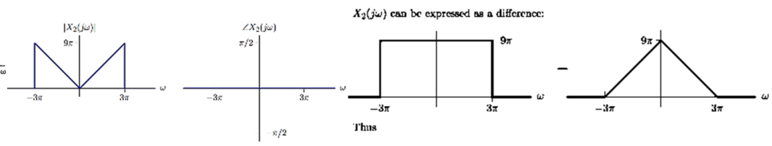
$$x_d[n]=\begin{cases}x_c(nT) & n\ even\\ -x_c(nT) & n\ odd\end{cases}$$

$$p_d(t)=\sum_{n=-\infty}^{\infty}(-1)^n\delta(t-nT)=\sum_{n=-\infty}^{\infty}2\delta(t-2nT)-\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$P_d(j\omega)=\sum_{n=-\infty}^{\infty}2\frac{2\pi}{2T}\delta\left(\omega-\frac{2\pi n}{2T}\right)-\sum_{n=-\infty}^{\infty}\frac{2\pi}{T}\delta\left(\omega-\frac{2\pi n}{T}\right)=\sum_{n=-\infty}^{\infty}\frac{2\pi}{T}\delta\left(\omega-\frac{\pi n}{T}\right)$$

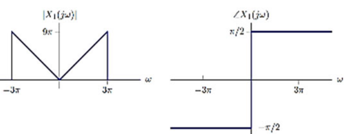
可见，要将采样函数表征为不同冲激串的叠加。

1. 给出振幅和相位求傅里叶变换



$$x_2(t)=9\pi\frac{\sin 3\pi t}{\pi t}-6\pi\frac{\sin^2\frac{3\pi}{2}t}{\pi^2t^2}$$

注意到$X_1(j\omega)=3j$，可以看成窗函数乘上一个$3j\omega$，因此：



$$x_1(t)=3\frac{d}{dt}x_{1a}(t)=3\frac{d}{dt}\left(\frac{\sin 3\pi t}{\pi t}\right)$$

使用正弦调制等效于乘以 j 和-j 的冲激,乘 j 等于虚部变负实部，实部变虚部

3. 通过原函数变换出函数，并求解傅里叶变换

$$x_3[n]=\begin{cases}t^2 & 0<t<1\\ 0 & otherwise\end{cases}$$

$$x_3(t)=t^2\big(u(t)-u(t-1)\big)\qquad u(t)\to\frac{1}{j\omega}+\pi\delta(\omega)$$

$$u(t)-u(t-1)\to\left(1-e^{-j\omega}\right)\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)=\frac{1-e^{-j\omega}}{j\omega}$$

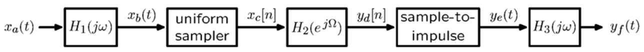
利用导数性质两次可以获得t^2：$t^2f(t)=-\frac{d^2}{d\omega^2}F(j\omega)$

因此：$x_3(t)=\frac{j}{\omega}e^{-j\omega}+\frac{2}{\omega^2}e^{-j\omega}+\frac{2j}{\omega^3}\left(1-e^{-j\omega}\right)$

$x_4(t)=(1-|t|)u(t+1)u(t-1)=\big(u(t+0.5)-u(t-0.5)\big)$*卷积自身*

4. DT Processing of CT signal

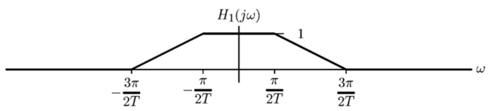
Consider the following system for DT processing of CT signals:

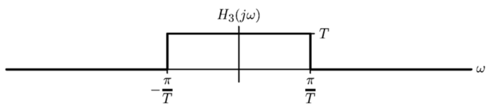


where $x_c[n]=x_b(nT)$ and

$$y_e(t)=\sum_{n=-\infty}^{\infty}y_d[n]\delta(t-nT)\,.$$

The frequency responses $H_1(j\omega)$ and $H_3(j\omega)$ are given below.





这个系统会造成混叠，$H_1(j\omega)$不能保证$\omega>\frac{\pi}{2T}$的频谱完全为 0

对于$x_a(t)=cos\left(\frac{\pi}{2T}t\right)+sin\left(\frac{5\pi}{4T}t\right)$，若$H_2(e^{j\omega})=1$，其傅里叶变换为：

$$\pi\left(\delta\left(\omega-\frac{\pi}{2T}\right)+\delta\left(\omega+\frac{\pi}{2T}\right)\right)+\frac{\pi}{j}\left(\delta\left(\omega-\frac{5\pi}{4T}\right)-\delta\left(\omega+\frac{5\pi}{4T}\right)\right)$$

在离散采样后频谱为：

$$\pi\left(\delta\left(\omega-\frac{\pi}{2T}\right)+\delta\left(\omega+\frac{\pi}{2T}\right)\right)+\frac{\pi}{4j}\left(-\delta\left(\omega-\frac{3\pi}{4T}\right)+\delta\left(\omega+\frac{3\pi}{4T}\right)\right)$$

因此为了实现良好的抗混叠效果，$H_2(e^{j\omega})$应从$\frac{\pi}{2}$截断。

5. Discrete-Time Response

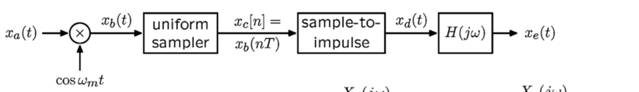
在一个周期内：$H\left(e^{j\Omega}\right)=\begin{cases}1\ if|\Omega|<\frac{2}{\pi}\\ 0\ otherwie\end{cases}$

x[n]周期为 3，x[0]=1，x[1]=x[2]=0

则**通过这个系统等于在频域上与系统函数相乘**，得到$H\left(e^{j\Omega}\right)=\frac{2\pi}{3}$

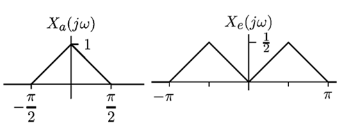
求时域表达式即除2π为高度为$\frac{1}{3}$周期为 1 的冲激。

This signal passes through the following system



where $x_d(t)=\sum_{n=-\infty}^{\infty}x_c[n]\delta(t-nT)$ and

$$H(j\omega)=\begin{cases}T & \text{if } |\omega|<\frac{\pi}{T}\\ 0 & \text{otherwise}.\end{cases}$$



调整 T 使得$\omega_m=5\pi$时$X_e(j\omega)$如图：

首先考虑 LPF 区间大于$[-\pi,\pi]$则$0<T<1$,其次考虑将-5π处信号搬移到$-\frac{\pi}{2}$或$\frac{\pi}{2}$

因此:$-5\pi+k\frac{2\pi}{T}=-\frac{\pi}{2}$或$-5\pi+k\frac{2\pi}{T}=\frac{\pi}{2}$，因此:$T=k\frac{4}{9}$或$T=k\frac{4}{11}$

$$T=\frac{4}{11}\ or\ \frac{8}{11}\ or\ \frac{4}{9}\ or\ \frac{8}{9}$$

Enter label of corresponding frequency response magnitude (**A**-**F** or **none**) in box.

$H_1(z)=\frac{z^3}{z^3-0.5}\to$ F

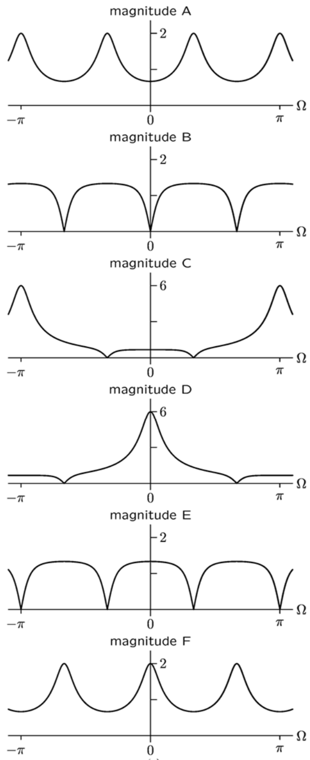
$H_2(z)=\frac{z^3-1}{z^3-0.5}\to$ B

$H_3(z)=\frac{z^3+z^2+z}{z^3-0.5}\to$ D

$H_4(z)=\frac{z^3}{z^3+0.5}\to$ A

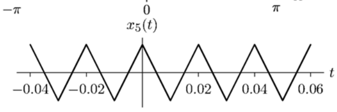
$H_5(z)=\frac{z^3+1}{z^3+0.5}\to$ E

$H_6(z)=\frac{z^3-z^2+z}{z^3+0.5}\to$ C



找特殊值带入即可。

6. 判断是否是 LTI 系统



1)通过 LTI 不能产生$y(t)=sin(2\pi 100t)$。因**半波奇对称的偶数谐波分量为 0**。

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}\,dt=\int_{0}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}\,dt+\int_{-\frac{T}{2}}^{0}x(t)e^{-jk\omega_0 t}\,dt$$

$$\int_{-\frac{T}{2}}^{0}x(t)e^{-jk\omega_0 t}\,dt=\int_{0}^{\frac{T}{2}}x\left(t-\frac{T}{2}\right)e^{-jk\omega_0 t+jk\pi t}\,dt=\int_{0}^{\frac{T}{2}}x\left(t-\frac{T}{2}\right)e^{-jk\omega_0 t+jk\pi}\,dt$$

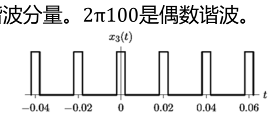
$$=(-1)^k\int_{0}^{\frac{T}{2}}x\left(t-\frac{T}{2}\right)e^{-jk\omega_0 t}\,dt=(-1)^{k+1}\int_{0}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}\,dt$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}\,dt=\int_{0}^{\frac{T}{2}}\left(1+(-1)^{k+1}\right)x(t)e^{-jk\omega_0 t}\,dt$$

可见对于偶数 k, FS 为 0 因此不会有偶数次谐波分量。$2\pi 100$是偶数谐波。

2)$x_3(t)$*周期为*$0.02s$,每个冲激持续$0.004s$

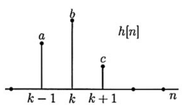
通过 LTI 能产生$y(t)=sin(2\pi 100t)$



等效为时域上窗函数卷积冲激串，等效为时域上冲激串与 sinc 相乘，在200π有冲激，只需要再说明 sinc 非零即可。

7. 有一系统函数如图

where k is an unknown integer and a, b, and c are unknown real numbers. It is known that $h[n]$ satisfies the following conditions:



(i) Let $H(e^{j\omega})$ be the Fourier transform of $h[n]$. $H(e^{j\omega})e^{j\omega}$ is real and even.

(ii) If $x[n]=(-1)^n$ for all n, then $y[n]=0$.

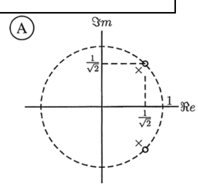
(iii) If $x[n]=\left(\frac{1}{2}\right)^n u[n]$ for all n, then $y[2]=\frac{9}{2}$.

Provide a labeled sketch of the output $y[n]$ when the input $x[n]$ is shown below. Your answer should not include a, b, c, nor k.

第一个条件表明 h(n)左移一个单位是偶函数则 a=c;$x[n]=(-1)^n=e^{j\pi n}$FT 后与$H(e^{j\omega})$相乘得到 2a=b；条件三可以直接卷积得到 a=2。

8. 频域响应：

方法：频率轴(圆或虚轴)一点到零点的连线比上极点连线：$H(z)=\frac{z-q_1}{z-p_1}$



Notch: 为了滤去特定频率, 表现为在频率轴上的某个零点, 右图 notch 滤去$\omega=\frac{\pi}{4}$