Signals and Systems

Lecture 15: Relations among Fourier Representations

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Adapted from the materials provided on the MIT OpenCourseWare

Fourier Representations

We've seen a variety of Fourier representations:

- CT Fourier series
- CT Fourier transform
- DT Fourier series
- DT Fourier transform

Today: relations among the four Fourier representations.

Four Fourier Representations

We have discussed four closely related Fourier representations.

DT Fourier Series

$$a_k = a_{k+N} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

DT Fourier transform

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

CT Fourier Series

$$a_k = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

CT Fourier transform

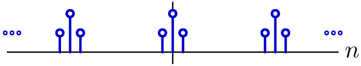
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

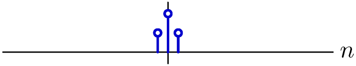
Four Types of Time

discrete vs. continuous ($\updownarrow$) and periodic vs aperiodic ($\leftrightarrow$)

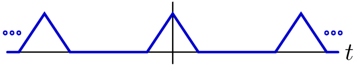
DT Fourier Series



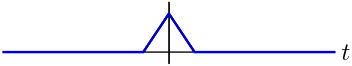
DT Fourier transform



CT Fourier Series



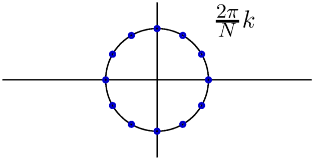
CT Fourier transform



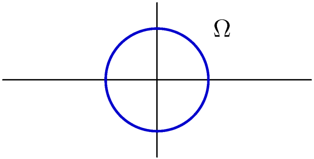
Four Types of Frequency

discrete vs. continuous ($\leftrightarrow$) and periodic vs aperiodic ($\updownarrow$)

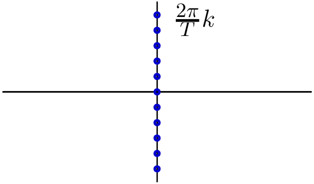
DT Fourier Series



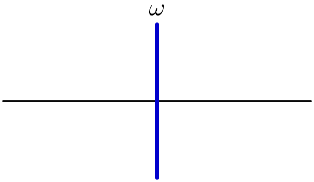
DT Fourier transform



CT Fourier Series



CT Fourier transform



Relations among Fourier Representations

Different Fourier representations are related because they apply to signals that are related.

DTFS (discrete-time Fourier series):	periodic DT
DTFT (discrete-time Fourier transform):	aperiodic DT
CTFS (continuous-time Fourier series):	periodic CT
CTFT (continuous-time Fourier transform):	aperiodic CT

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{periodic DT}\\ \text{DTFS}\end{array}} \arrow[r, shift left, "N \to \infty"] \arrow[d, shift right, "\text{interpolate}"'] & \boxed{\begin{array}{c}\text{aperiodic DT}\\ \text{DTFT}\end{array}} \arrow[l, shift left, "\text{periodic extension}"] \arrow[d, shift right, "\text{interpolate}"'] \\
\boxed{\begin{array}{c}\text{periodic CT}\\ \text{CTFS}\end{array}} \arrow[r, shift left, "T \to \infty"] \arrow[u, shift right, "\text{sample}"'] & \boxed{\begin{array}{c}\text{aperiodic CT}\\ \text{CTFT}\end{array}} \arrow[l, shift left, "\text{periodic extension}"] \arrow[u, shift right, "\text{sample}"']
\end{tikzcd}
$$

Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

Series: represent periodic signal as weighted sum of harmonics

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$

The Fourier transform of a sum is the sum of the Fourier transforms:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

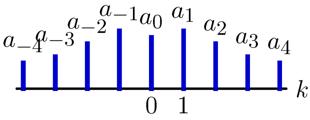
Therefore periodic signals can be equivalently represented as Fourier transforms (with impulses!).

Relation between Fourier Series and Transform

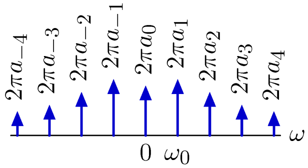
A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \quad \leftrightarrow$$

Fourier Series



Fourier Transform

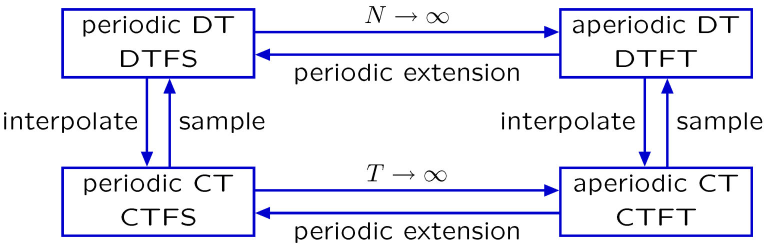


Relations among Fourier Representations

Explore other relations among Fourier representations.

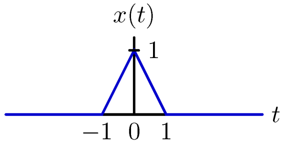
Start with an aperiodic CT signal. Determine its Fourier transform.

Convert the signal so that it can be represented by alternate Fourier representations and compare.



Start with the CT Fourier Transform

Determine the Fourier transform of the following signal.

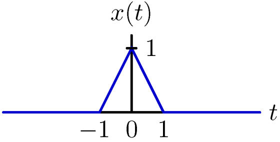


Could calculate Fourier transform from the definition.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

Start with the CT Fourier Transform

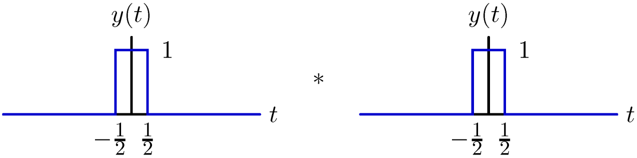
Determine the Fourier transform of the following signal.



Could calculate Fourier transform from the definition.

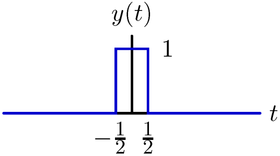
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

Easier to calculate $x(t)$ by convolution of two square pulses:

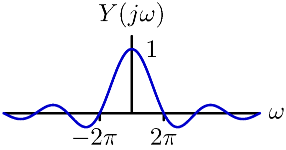


Start with the CT Fourier Transform

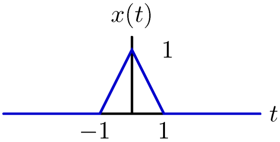
The transform of $y(t)$ is $\frac{2\sin(\omega/2)}{\omega}$



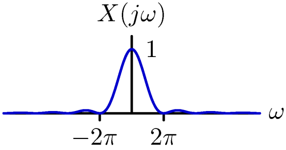
$\leftrightarrow$



so the transform of $x(t) = (y * y)(t)$ is $X(j\omega) = Y(j\omega) \times Y(j\omega)$.



$\leftrightarrow$

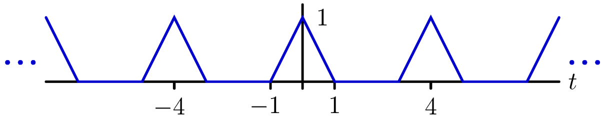


Relation between Fourier Transform and Series

What is the effect of making a signal periodic in time?

Find Fourier transform of periodic extension of $x(t)$ to period $T = 4$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$

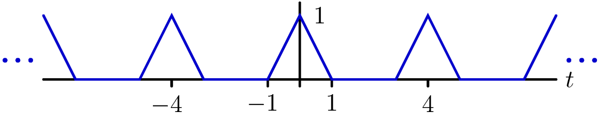


Could calculate $Z(j\omega)$ for the definition ... ugly.

Relation between Fourier Transform and Series

Easier to calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k)$$



$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k) = (x * p)(t)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t+4k)$$

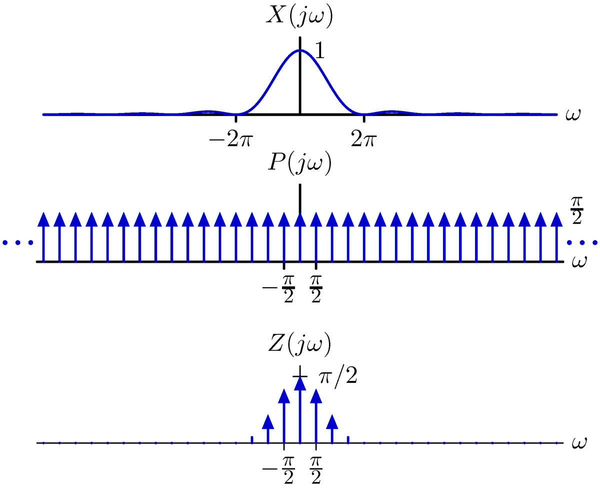
Then

$$Z(j\omega) = X(j\omega) \times P(j\omega)$$

We already know $P(j\omega)$: it's also an impulse train!

Relation between Fourier Transform and Series

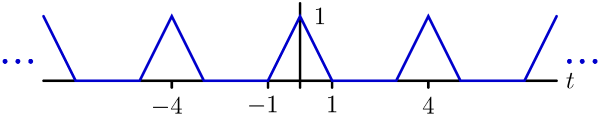
Convolving in time corresponds to multiplying in frequency.

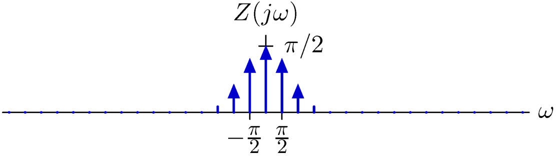


Relation between Fourier Transform and Series

The Fourier transform of a periodically extended function is a discrete function of frequency ω.

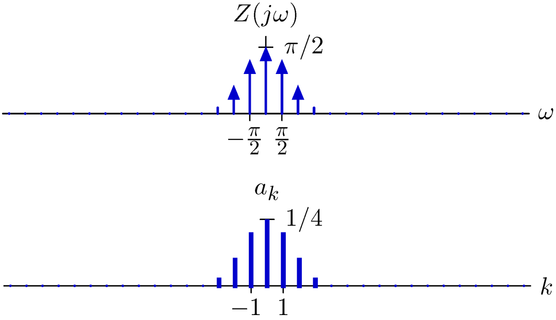
$$z(t) = \sum_{k=-\infty}^{\infty} x(t+4k)$$





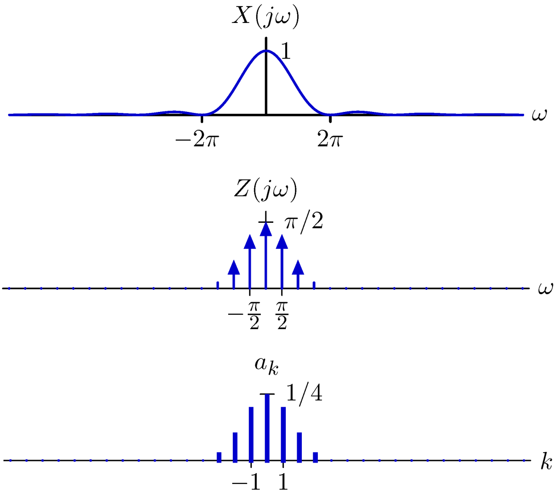
Relation between Fourier Transform and Series

The weight (area) of each impulse in the Fourier transform of a periodically extended function is 2π times the corresponding Fourier series coefficient.



Relation between Fourier Transform and Series

The effect of periodic extension of $x(t)$ to $z(t)$ is to sample the frequency representation.



Relation between Fourier Transform and Series

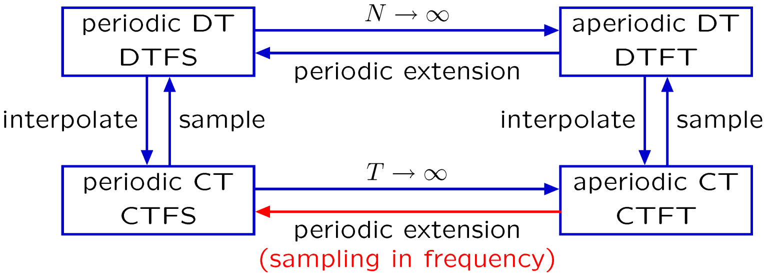
Periodic extension of a CT signal produces a discrete function of frequency.

Periodic extension

= convolving with impulse train in time

= multiplying by impulse train in frequency

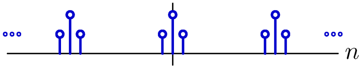
$\rightarrow$ sampling in frequency



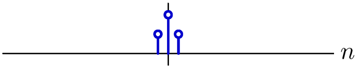
Four Types of Time

discrete vs. continuous (↕) and periodic vs aperiodic (↔)

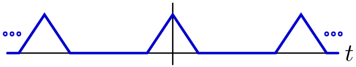
DT Fourier Series



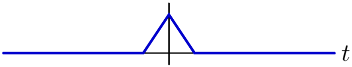
DT Fourier transform



CT Fourier Series



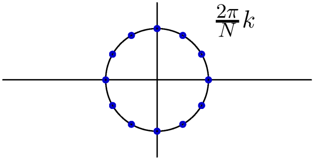
CT Fourier transform



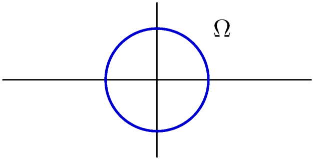
Four Types of Frequency

discrete vs. continuous ($\leftrightarrow$) and periodic vs aperiodic ($\updownarrow$)

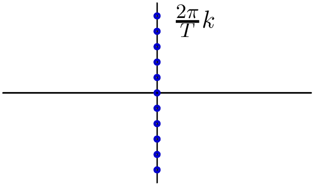
DT Fourier Series



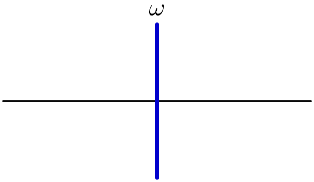
DT Fourier transform



CT Fourier Series



CT Fourier transform



Relations among Fourier Representations

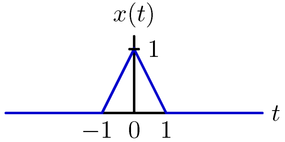
Compare to sampling in time.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{matrix}\text{periodic DT}\\ \text{DTFS}\end{matrix}} \arrow[r, shift left, "N \to \infty"] \arrow[d, shift right, "\text{interpolate}"'] & \boxed{\begin{matrix}\text{aperiodic DT}\\ \text{DTFT}\end{matrix}} \arrow[l, shift left, "\text{periodic extension}"] \arrow[d, shift right, "\text{interpolate}"'] \\
\boxed{\begin{matrix}\text{periodic CT}\\ \text{CTFS}\end{matrix}} \arrow[r, shift left, "T \to \infty"] \arrow[u, shift right, "\text{sample}"'] & \boxed{\begin{matrix}\text{aperiodic CT}\\ \text{CTFT}\end{matrix}} \arrow[l, shift left, "\text{periodic extension}"] \arrow[u, shift right, red, "\text{sample}"']
\end{tikzcd}
$$

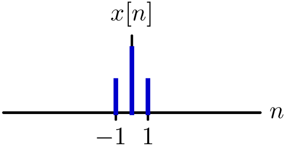
Relations between CT and DT transforms

Sampling a CT signal generates a DT signal.

$$x[n] = x(nT)$$



Take $T = \frac{1}{2}$.

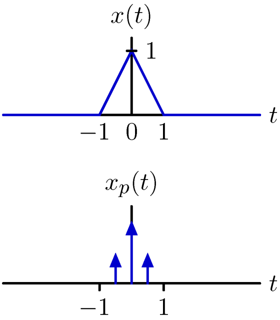


What is the effect on the frequency representation?

Relations between CT and DT transforms

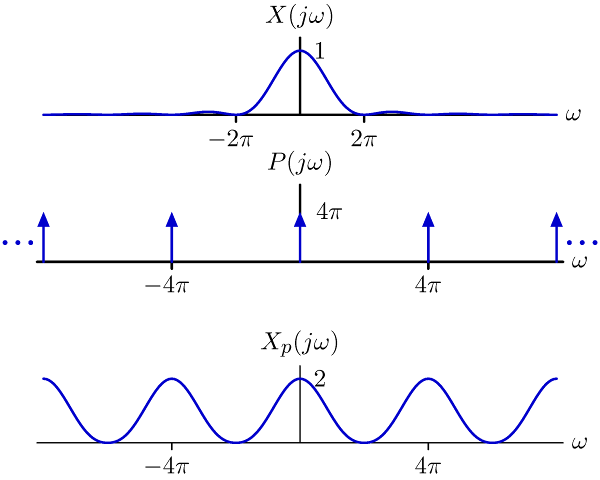
We can generate a signal with the same shape by multiplying $x(t)$ by an impulse train with $T = \frac{1}{2}$.

$$x_p(t) = x(t) \times p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)$$



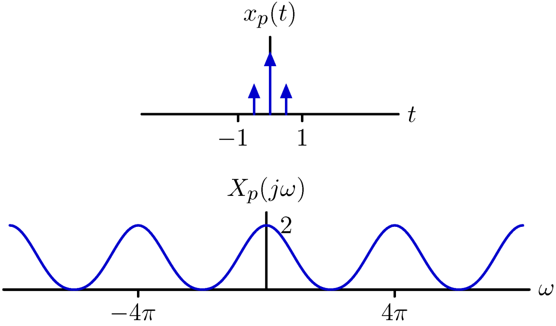
Relations between CT and DT transforms

Multiplying $x(t)$ by an impulse train in time is equivalent to convolving $X(j\omega)$ by an impulse train in frequency (then $\div 2\pi$).



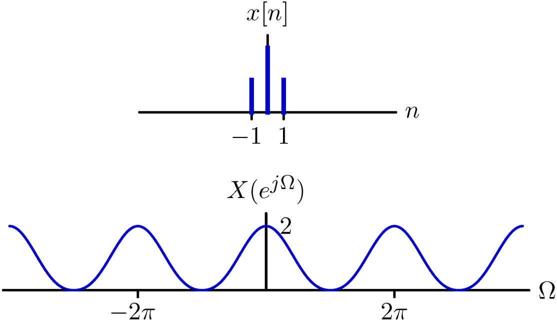
Relations between CT and DT transforms

The Fourier transform of the "sampled" signal $x_p(t)$ is periodic in ω with period 4π.



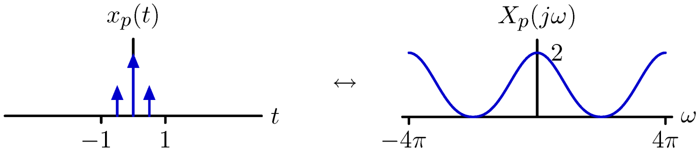
Relations between CT and DT transforms

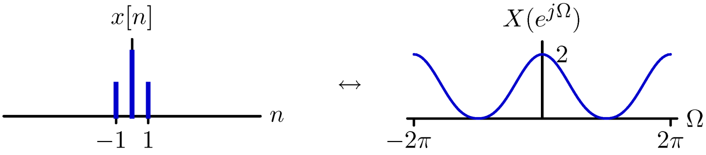
The Fourier transform of the "sampled" signal $x_p(t)$ has the same shape as the DT Fourier transform of $x[n]$.



DT Fourier transform

The CT Fourier transform of a "sampled" signal ($x_p(t)$) is equal to the DT Fourier transform of the samples ($x[n]$) where $\Omega = \omega T$, i.e., $X(j\omega) = X(e^{j\Omega})\Big|_{\Omega=\omega T}$.





$\Omega = \omega T = \frac{1}{2}\omega$

DT Fourier transform

Compare the definitions:

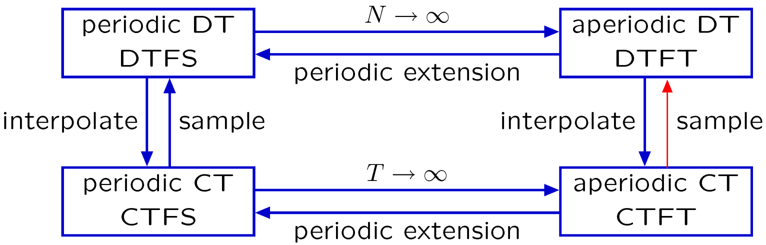
$$X(e^{j\Omega}) = \sum_n x[n]e^{-j\Omega n}$$

$$\begin{aligned}
X_p(j\omega) &= \int x_p(t)e^{-j\omega t}dt \\
&= \int \sum_n x[n]\delta(t-nT)e^{-j\omega t}dt \\
&= \sum_n x[n] \int \delta(t-nT)e^{-j\omega t}dt \\
&= \sum_n x[n]e^{-j\omega nT}
\end{aligned}$$

$$\Omega = \omega T$$

Relation between CT and DT Fourier transforms

The CT Fourier transform of a “sampled” signal ($x_p(t)$) is equal to the DT Fourier transform of the samples ($x[n]$) where $\Omega = \omega T$, i.e., $X(j\omega) = X(e^{j\Omega})\Big|_{\Omega=\omega T}$.



Relation Among Fourier Representations

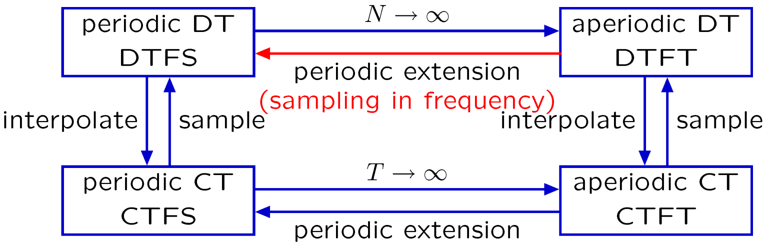
Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension

$=$ convolving with impulse train in time

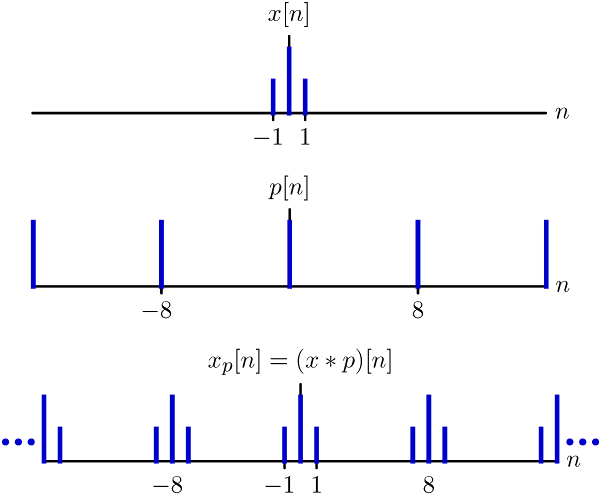
$=$ multiplying by impulse train in frequency

$\rightarrow$ sampling in frequency



Relation Between DT Fourier Transform and Series

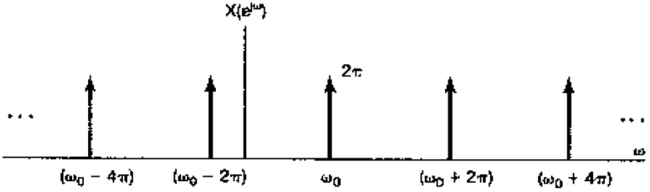
Periodic extension of a DT signal is equivalent to convolution of the signal with an impulse train.



DTFT of Periodic Signals

$$x[n] = e^{j\omega_0 n} \Longleftrightarrow X(e^{j\omega}) = ?$$

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$



DTFT of Periodic Signals

DTFT of Periodic Signals $x[n] = x[n+N]$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

From the last page: $e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right]$$

Linearity of DTFT

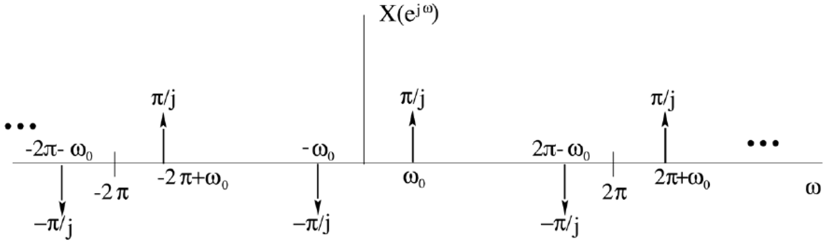
$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

DTFT of Periodic Signals

Ex. DT sine function

$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

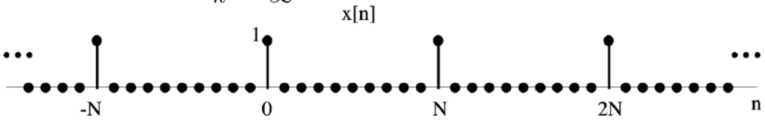
$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



DTFT of Periodic Signals

Ex. DT periodic impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad \omega_0 = 2\pi/N$$

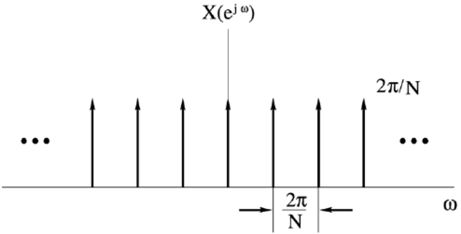


$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

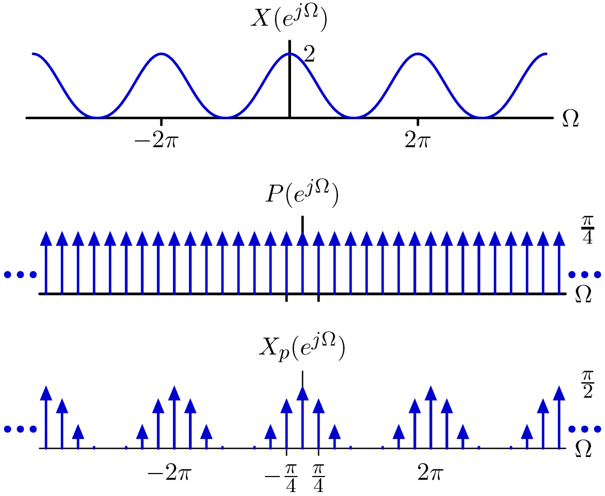
$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



— Also periodic impulse train – in the frequency domain!

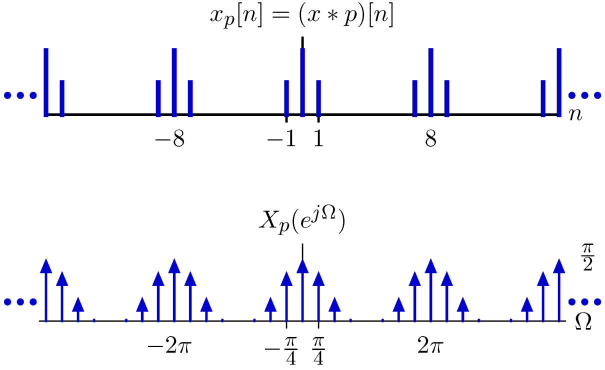
Relation Between DT Fourier Transform and Series

Convolution by an impulse train in time is equivalent to multiplication by an impulse train in frequency.



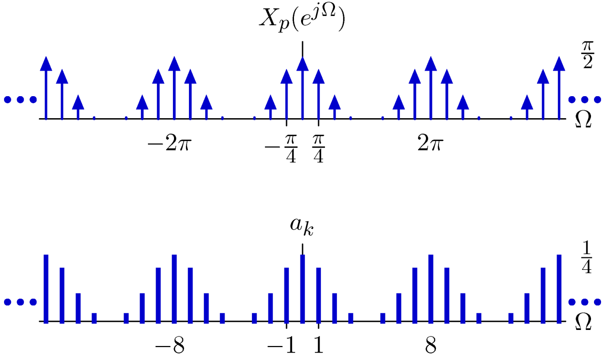
Relation Between DT Fourier Transform and Series

Periodic extension of a discrete signal ($x[n]$) results in a signal ($x_p[n]$) that is both periodic and discrete. Its transform ($X_p(e^{j\Omega})$) is also periodic and discrete.



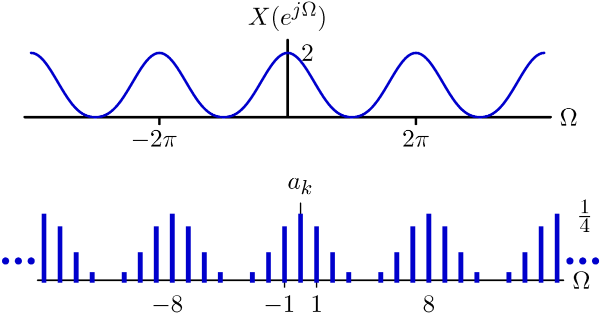
Relation Between DT Fourier Transform and Series

The weight of each impulse in the Fourier transform of a periodically extended function is 2π times the corresponding Fourier series coefficient.



Relation Between DT Fourier Transform and Series

The effect of periodic extension was to sample the frequency representation.



Relation between Fourier Transforms and Series

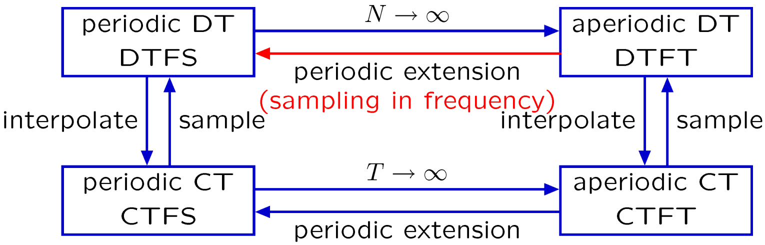
Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension

$=$ convolving with impulse train in time

$=$ multiplying by impulse train in frequency

$\rightarrow$ sampling in frequency



Duality

Fourier Transform is highly symmetric

CTFT: *Both* time and frequency are continuous and in general *aperiodic*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Same except for these differences

$$\Downarrow$$

Suppose $f(\cdot)$ and $g(\cdot)$ are two functions related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}\,d\tau$$

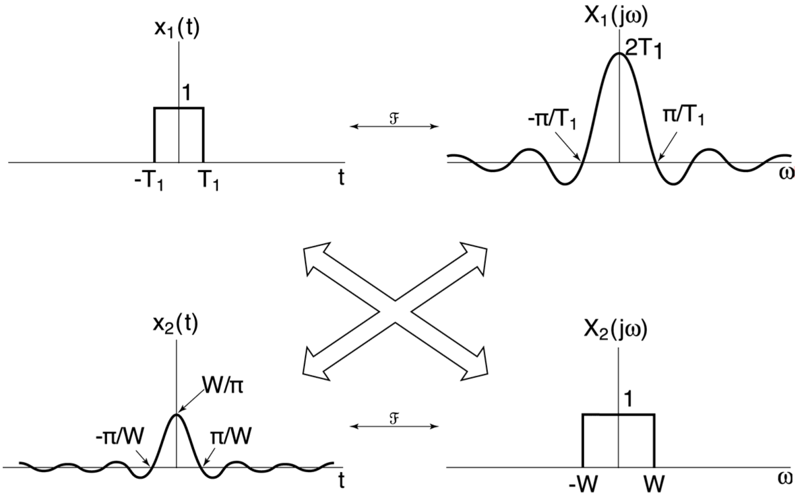
Then

Let $\tau = t$ and $r = \omega$: $\quad x_1(t) = g(t) \longleftrightarrow X_1(j\omega) = f(\omega)$

Let $\tau = -\omega$ and $r = t$: $\quad x_2(t) = f(t) \longleftrightarrow X_2(j\omega) = 2\pi g(-\omega)$

Example of CTFT duality

Square pulse in either time or frequency domain



Duality

DTFS

Discrete & *periodic* in time $\longleftrightarrow$ *Periodic* & *discrete* in frequency

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = x[n+N], \quad \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk\omega_0 n} = a_{k+N}$$

Duality in DTFS

Suppose $f[\cdot]$ and $g[\cdot]$ are two functions related by

$$f[m] = \frac{1}{N} \sum_{r=<N>} g[r] e^{-jr\omega_0 m}$$

$$\Rightarrow g[r] = \sum_{m=<N>} f[m] e^{jr\omega_0 m}$$

Then

Let $m = n$ and $r = -k$: $\quad x_1[n] = f[n] \longleftrightarrow a_k = \frac{1}{N} g[-k]$

Let $r = n$ and $m = k$: $\quad x_2[n] = g[n] \longleftrightarrow a_k = f[k]$

Duality

Duality between CTFS and DTFT

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x(t+T), \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Periodic in time $\leftrightarrow$ *Discrete* in frequency

DTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X\left(e^{j(\omega+2\pi)}\right)$$

Discrete in time $\leftrightarrow$ *Periodic* in frequency

CTFS-DTFT Duality

Suppose $f(\cdot)$ is a CT signal and $g[\cdot]$ a DT sequence related by

$$f(\tau) = \sum_{m=-\infty}^{+\infty} g[m]e^{jm\tau} = f(\tau + 2\pi)$$

Then

$$x(t) = f(t) \longleftrightarrow a_k = g[k]$$
(periodic with period 2π)

$$x[n] = g[n] \longleftrightarrow X(e^{j\omega}) = f(-\omega)$$

Magnitude and Phase of FT, and Parseval Relation

CT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

Parseval Relation:

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}\frac{1}{2\pi}\underbrace{|X(j\omega)|^2}_{\text{Energy density in }\omega} d\omega$$

DT:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Parseval Relation:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \int_{2\pi}\frac{1}{2\pi}|X(e^{j\omega})|^2 d\omega$$

Assignments

- Reading Assignment: Ch. 5