Problem Set 4

Problem 1 Solution

The overall system function for this interconnection is

$$Q_1(z) = H_0(z) + \frac{H_1(z)}{1 + H_1(z)G(z)} \tag{1}$$

Problem 2 Solution

We can obtain the overall function for the system

$$Q(z) = \frac{H(z)}{1 + H(z)G(z)} = \frac{0.5}{1 - (\frac{1}{4} + \frac{b}{2})z^{-1}} \tag{2}$$

which has a pole at $z = \frac{1}{4} + \frac{b}{2}$. So the system is stable when $|\frac{1}{4} + \frac{b}{2}| < 1$, that is,

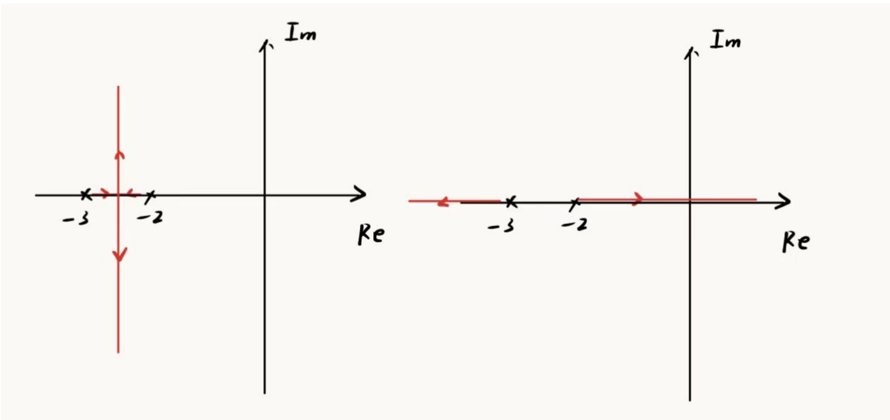
$$-\frac{5}{2} < b < \frac{3}{2} \tag{3}$$

Problem 3 Solution

First we can depict the root locus of the system. The left is for $K > 0$ and the right is for $K < 0$. The system is stable only if all poles lie on the left of jw-axis, so figure below

shows that the system is always stable if $K > 0$. When $K < 0$, the poles is on the left of jw-axis initially but gradually move rightward with K decreasing from 0 to $-\infty$.

Let $s = 0$, we can get $K = -6$, so the system is stable if $K > -6$.



Problem 4 Solution

(a) From the expression of $H_p(s)$ and $H_c(s)$ we can obtain

$$Y(s) = \frac{H(s)}{1 + H(s)G(s)}X(s) = \frac{K\frac{\alpha}{s+\alpha}}{1 + K\frac{\alpha}{s+\alpha}}X(s) = \frac{\alpha K}{s + \alpha(K+1)}X(s) \tag{4}$$

The system is stable if pole $s = -\alpha(K+1) < 0$, so when can choose appropriate K to stabilize the system.

Since

$$E(s) = \frac{Y(s)}{H(s)} = \frac{s+\alpha}{s + \alpha(K+1)}X(s) \tag{5}$$

Apply the inverse Laplace transform to eq(5)

$$e(t) = (\delta(t) - \alpha K e^{-\alpha(K+1)t}u(t)) * x(t) \tag{6}$$

If $x(t)=\delta(t)$, then

$$e(t)=\delta(t)-\alpha K e^{-\alpha(K+1)t}u(t) \tag{7}$$

Since $-\alpha(K+1)<0$, we can get $e(t)\to 0$ if $x(t)=\delta(t)$.

If $x(t)=u(t)$, then

$$e(t)=u(t)-\alpha K e^{-\alpha(K+1)t}u(t)*u(t)=u(t)-\alpha K\int_0^t e^{-\alpha(K+1)\tau}d\tau=u(t)(\frac{1}{K+1}-\frac{K}{K+1}e^{-\alpha(K+1)t}) \tag{8}$$

So we can not get $e(t)\to 0$ if $x(t)=u(t)$.

(b) From the expression of $H_p(s)$ and $H_c(s)$ we can obtain

$$Q(s)=\frac{H(s)}{1+H(s)G(s)}=\frac{\alpha(K_1 s+K_2)}{s^2+\alpha(K_1+1)s+\alpha K_2} \tag{9}$$

The system is stable if $\alpha(K_1+1)>0$ and $\alpha K_2>0$ so we can choose K_1 and K_2 so to stabilize the system.

$$E(s)=\frac{Y(s)}{H(s)}=\frac{s(s+\alpha)}{s^2+\alpha(K_1+1)s+\alpha K_2}X(s) \tag{10}$$

Apply the inverse Laplace transform to eq(10)

$$e(t)=(\delta(t)-ae^{-\omega_1 t}u(t)-be^{-\omega_2 t}u(t))*x(t) \tag{11}$$

where w_1,w_2 is the roots of $s^2+\alpha(K_1+1)s+\alpha K_2=0$ and $a=\alpha\frac{K_2-K_1w_1}{w_2-w_1},\ b=\alpha\frac{K_1w_2-K_2}{w_2-w_1}$

If $x(t)=u(t)$, then

$$e(t)=(\delta(t)-ae^{-\omega_1 t}u(t)-be^{-\omega_2 t}u(t))*u(t)=u(t)-\int_0^t ae^{-\omega_1\tau}+be^{-\omega_2\tau}d\tau=u(t)(1-\frac{a}{w_1}-\frac{b}{w_2}+\frac{a}{w_1}e^{-w_1 t}+\frac{b}{w_2}e^{-w_2 t}) \tag{12}$$

Since $1-\frac{a}{w_1}-\frac{b}{w_2}=1-\alpha K_2/(w_1w_2)=0$, so $e(t)=u(t)(\frac{a}{w_1}e^{-w_1 t}+\frac{b}{w_2}e^{-w_2 t})\to 0$

(c) If we use PI control, then

$$Q(s)=\frac{H(s)}{1+H(s)G(s)}=\frac{K_1 s+K_2}{s^3-2s^2+(K_1+1)s+K_2} \tag{13}$$

Since coefficient $-2<0$, the system is not stable.

If we use PID control, then

$$Q(s)=\frac{H(s)}{1+H(s)G(s)}=\frac{K_3s^2+K_1 s+K_2}{s^3+(K_3-2)s^2+(K_1+1)s+K_2} \tag{14}$$

So the system is stable if $K_3>2\quad K_1,K_2>0$.

Problem 5 Solution

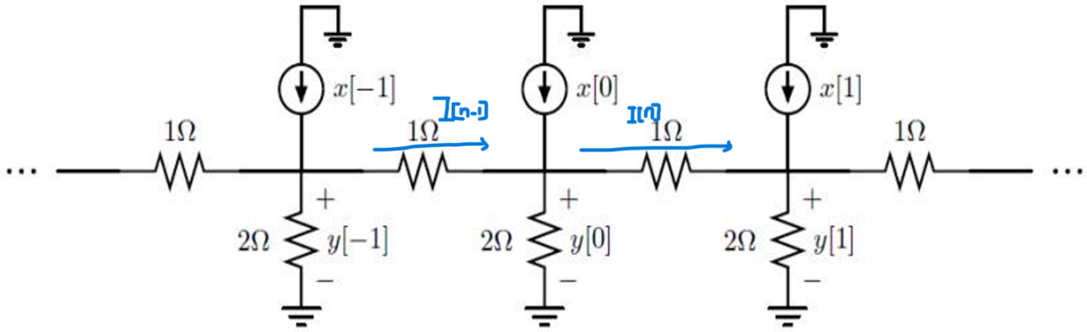
From the block diagram we can know $H(z)=\frac{K}{z^2+z-2}$ and $G(z)=1$, so the overall system function

$$Q(z)=\frac{H(z)}{1+H(z)G(z)}=\frac{K}{z^2+z+K-2} \tag{15}$$

(a) The system is stable only if $K-2>0$, that is, $K>2$

(b) The feedback system has real-valued poles if $\Delta=1-4(K-2)\geq 0$, so $K\leq\frac{9}{4}$

Problem 6 Solution



From the figure above we can deduce that

$$\begin{aligned} x[n] + I[n-1] &= I[n] + y[n]/2 \\ I[n-1] &= y[n-1] - y[n] \end{aligned} \tag{16}$$

Simplifying the equations we can get

$$x[n] = -y[n+1] + \frac{5}{2}y[n] - y[n-1] \tag{17}$$

(a) From eq(12) we can see that the system is linear and time-invariant.

(b) Apply the Laplace transform to eq(12) we can obtain

$$\begin{aligned} X(z) &= -zY(z) + \frac{5}{2}Y(z) - z^{-1}Y(z) \\ H(z) &= \frac{1}{-z + \frac{5}{2} - z^{-1}} = \frac{2}{3}\left(\frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - 2z^{-1}}\right) \end{aligned} \tag{18}$$

So $h[n] = \frac{2}{3}((\frac{1}{2})^n u[n] + (2)^n u[-n-1])$

(c) We have calulated in **(b)** that

$$H(z) = \frac{1}{-z + \frac{5}{2} - z^{-1}} = \frac{-2z}{2z^2 - 5z + 1} \tag{19}$$

$ROC: \frac{1}{2} < |z| < 2$

(d) poles: $z = 2,\ \frac{1}{2}$, zeros: $z = 0$