习题三

1. 解：$n+1$ 次交换后A箱中白球个数仅与上一次交换后A箱中白球个数有关

$\therefore P(X_{n+1}=j \mid X_0=i_0, \ldots, X_{n-1}=i_{n-1}, X_n=i) = P(X_{n+1}=j \mid X_n=i)$

又 $P(X_{n+1} \mid X_n=i)$ 不依赖于 n $\therefore \{X_n\}$ 是时齐马尔可夫链

状态空间：$I=\{0,1,\ldots,m\}$

一步转移概率：

$$p_{ij}=\begin{cases}\dfrac{(m-i)^2}{m^2}, & j=i+1\\ \dfrac{i^2}{m^2}, & j=i-1\\ \dfrac{2i(m-i)}{m^2}, & j=i\end{cases}$$

3. 解：状态空间 $I=\{0,1,2,\ldots\}$

一步转移概率：$p_{ij}=\begin{cases}p, & j=i+1\\ 1-p, & j=0\\ 0, & 其它\end{cases}$

5. 解：(1) 若 $Y_0=1$，则 $\max\{X_1,X_2\}=1$ $\therefore X_1=X_2=1$

$Y_1=6$，则 $\max\{X_2,X_3\}=6$ $\therefore X_3=6$ $\therefore Y_2=\max\{X_3,X_4\}=6$

$\therefore P\{Y_2=1 \mid Y_0=1, Y_1=6\}=0$

$P(Y_2=1 \mid Y_1=6)=P(X_3=1, X_4=1 \mid \max\{X_2,X_3\}=6)$

$$=\frac{P(X_3=1, X_4=1, X_2=6)}{P(\max\{X_2,X_3\}=6)}=\frac{(\frac{1}{6})^3}{\frac{11}{36}}=\frac{1}{66}$$

(2) $P(Z_2=12 \mid Z_0=2, Z_1=7)=P(X_3+X_4=12 \mid X_1+X_2=2, X_2+X_3=7)=P(X_4=6)=\frac{1}{6}$

$$P(Z_2=12 \mid Z_1=7)=\frac{P(X_2+X_3=7, X_3+X_4=12)}{P(X_2+X_3=7)}=\frac{(\frac{1}{6})^3}{\frac{6}{36}}=\frac{1}{36}$$

(3) $P(Y_2=1 \mid Y_0=1, Y_1=6) \neq P(Y_2=1 \mid Y_1=6)$ $\therefore \{Y_n\}$ 不具有马尔可夫性

同理 $\{Z_n\}$ 也不具有马尔可夫性

7. 解：(1) $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 2 & \frac{2}{3} & \frac{1}{3} & 0 \end{array}$

(2) $$P^2 = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix}$$

$\therefore P(X_0=0, X_2=0, X_4=1) = P(X_0=0)\, p_{00}^{(2)} p_{01}^{(2)} = \frac{1}{2} \times \frac{4}{9} \times \frac{1}{9} = \frac{2}{81}$

$P(X_2=1) = \sum_{i=0}^{2} P(X_0=i) P(X_2=1 \mid X_0=i) = \frac{1}{2} p_{01}^{(2)} + \frac{1}{4} p_{11}^{(2)} + \frac{1}{4} p_{21}^{(2)} =$

$\frac{1}{2} \times \frac{1}{9} + \frac{1}{4} \times \frac{4}{9} + \frac{1}{4} \times \frac{4}{9} = \frac{5}{18}$

8. 解：(1) $$P^2 = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{7}{9} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

$\therefore P(X_2=0 \mid X_0=0) = p_{00}^{(2)} = \frac{5}{9}$

$P(X_0=0 \mid X_2=0) = \frac{P(X_0=0, X_2=0)}{P(X_2=0)} = \frac{P(X_0=0) P(X_2=0 \mid X_0=0)}{P(X_2=0)}$

而 $P(X_2=0) = \frac{1}{3}\left(p_{00}^{(2)} + p_{10}^{(2)} + p_{20}^{(2)}\right) = \frac{1}{3} \times \frac{13}{9} = \frac{13}{27}$

$\therefore P(X_0=0 \mid X_2=0) = \frac{\frac{1}{3} \times \frac{5}{9}}{\frac{13}{27}} = \frac{5}{13}$

(2) $P(X_1=0) = \frac{1}{3}(p_{00} + p_{10} + p_{20}) = \frac{1}{3}$

$P(X_1=0, X_3=0, X_4=1, X_6=1) = P(X_1=0)\, p_{00}^{(2)} p_{01}^{(1)} p_{11}^{(2)} = \frac{1}{3} \times \frac{5}{9} \times \frac{2}{3} \times \frac{7}{9} = \frac{70}{729}$

(3) $f_{11}^{(n)} = P(X_n=1, X_{n-1} \neq 1, \ldots, X_1 \neq 1 \mid X_0=1)$

$f_{11}^{(1)} = 0,\ f_{11}^{(2)} = \frac{7}{9}$，当 $n \geq 3$，$f_{11}^{(n)} = \frac{2}{3} \times \left(\frac{1}{3}\right)^{n-2} \times \frac{2}{3} = \frac{4}{3^n}$

$f_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = \frac{7}{9} + 4 \frac{1}{1-\frac{1}{3}} \times \frac{1}{27} = 1$

$\mu_1 = \sum_{n=1}^{\infty} n f_{11}^{(n)} = \frac{14}{9} + 4 \sum_{n=3}^{\infty} \frac{n}{3^n} = \frac{14}{9} + 4 \times \frac{\frac{1}{2} \times 3 + \frac{1}{4}}{9} = \frac{21}{9} = \frac{7}{3}$