Signals and Systems

Lecture1: Introduction to Signals and Systems

Instructor: Prof. Xiaojin Gong
Zhejiang University

02/26/2024

Partly adapted from the materials provided on the MIT OpenCourseWare

Outline

Course Introduction

This subject deals with mathematical methods used to describe signals and to analyze and synthesize systems

- Signals are variables that carry information
- Systems process input signals to produce output signals

Course Introduction

Why is it important?

Used almost everywhere

- ISEE: communications, circuit design, video/image, sonar/radar, speech processing
- Commercial electronics
- Aeronautics and astronautics
- Seismology
- Biomedical engineering
- Energy generation and distribution
- Chemical process control
- Financial analysis

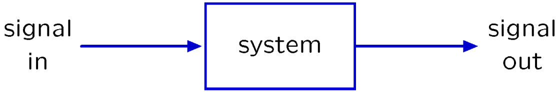
What is it all about?

- Existing system analysis
- System design
    - Signal enhancement and restoration
    - Information extraction
    - Signal design
- System control

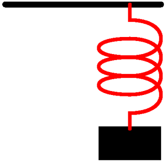
Outline

The Signals and Systems Abstraction

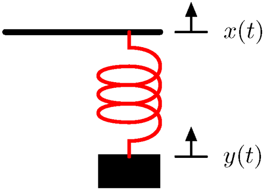
Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.

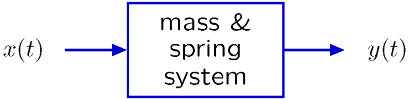


Example: Mass and Spring

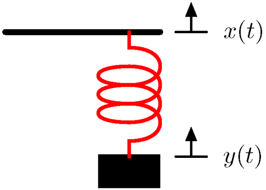


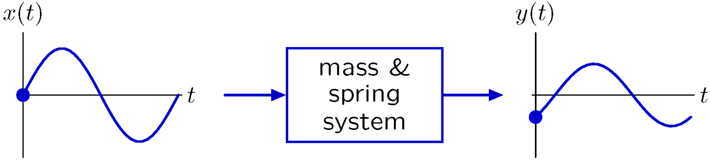
Example: Mass and Spring



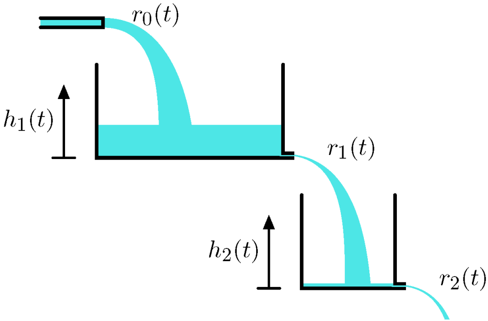


Example: Mass and Spring

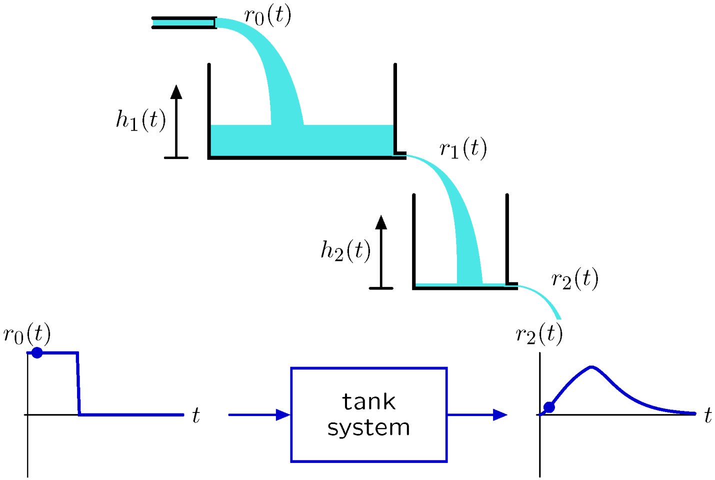




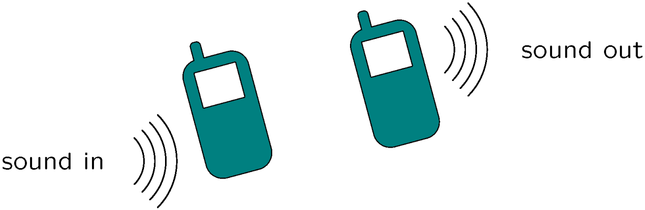
Example: Tanks

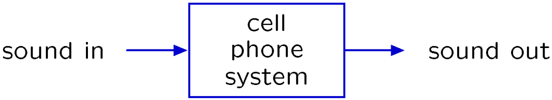


Example: Tanks

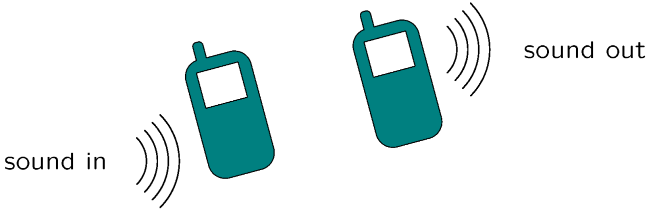


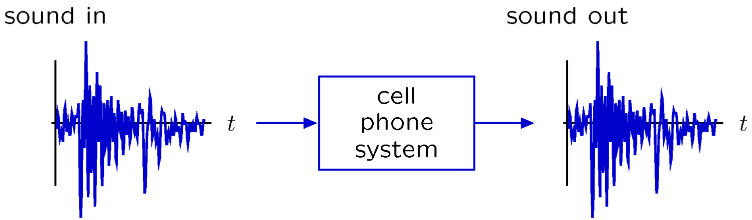
Example: Cell Phone System





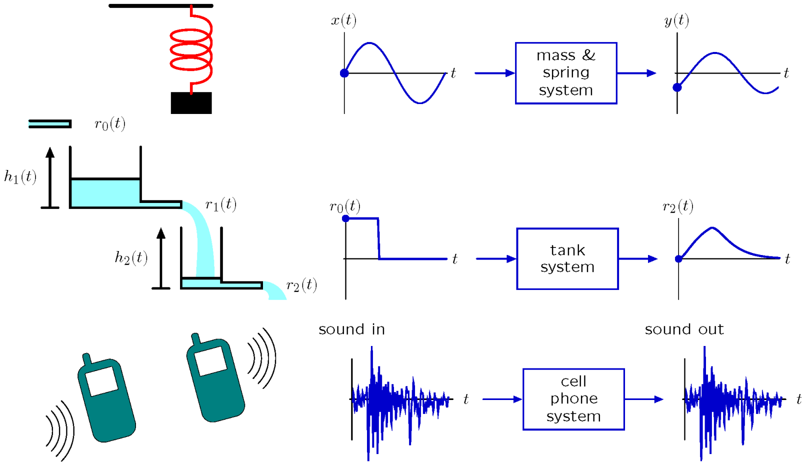
Example: Cell Phone System





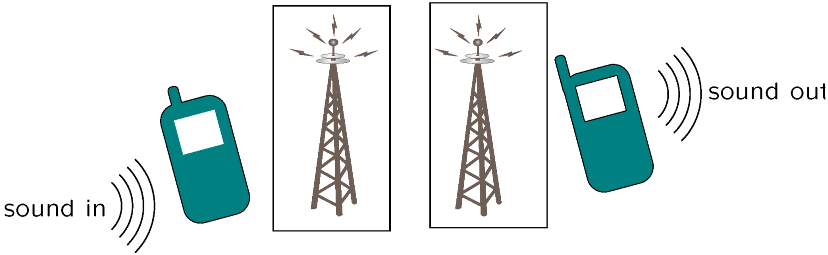
Signals and Systems: Widely Applicable

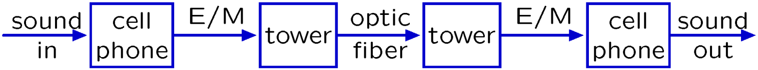
The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...



Signals and Systems: Modular

The representation does not depend upon the physical substrate.



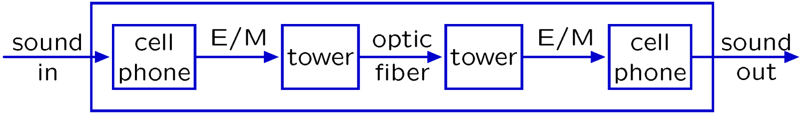


focuses on the flow of **information**, abstracts away everything else

Signals and Systems: Hierarchical

Representations of component systems are easily combined.

Example: cascade of component systems



Composite system

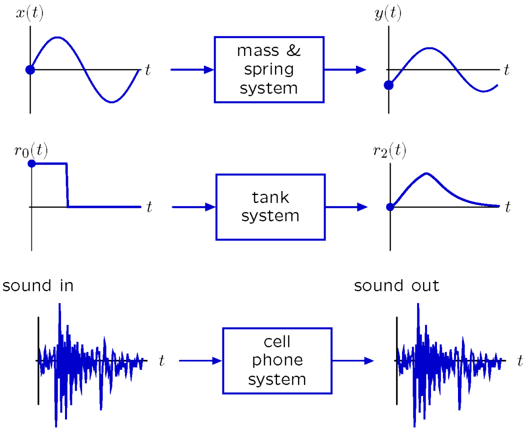
Component and composite systems have the same form, and are analyzed with same methods.

Outline

Signals

Signals are mathematical functions.

- independent variable = time
- dependent variable = voltage, flow rate, sound pressure



Signals

Generic time

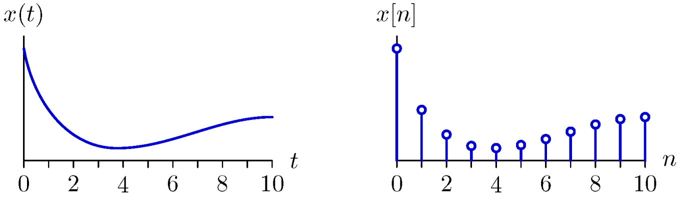
The term time is often used generically to represent the independent variable of a signal. The independent variable may be

- continuous or discrete
- 1-D, 2-D, $\cdots$ N-D

For this course: Focus on a single (1-D) independent variable which we call time.

Signals: Continuous Time and Discrete Time

continuous "time" (CT) and discrete "time" (DT)



Many physical systems operate in continuous time.

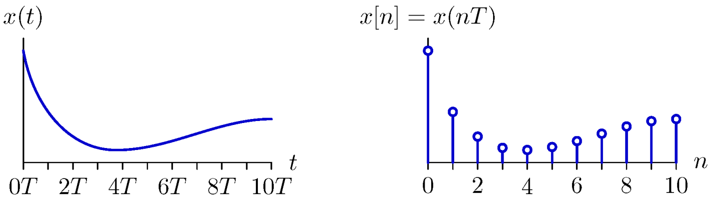
- mass and spring
- leaky tank

Digital computations are done in discrete time.

- state machines: given the current input and current state, what is the next output and next state.

Signals: Continuous Time and Discrete Time

Sampling: converting CT signals to DT



$T =$ sampling interval

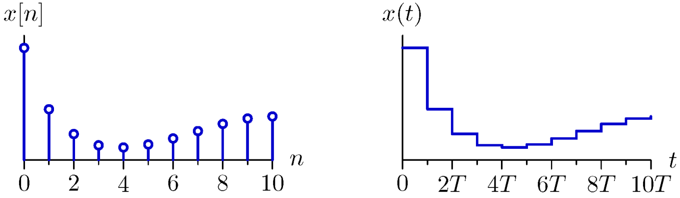
Important for computational manipulation of physical data.

- digital representations of audio signals (e.g., MP3)
- digital representations of pictures (e.g., JPEG)

Signals: Continuous Time and Discrete Time

Reconstruction: converting DT signals to CT

zero-order hold



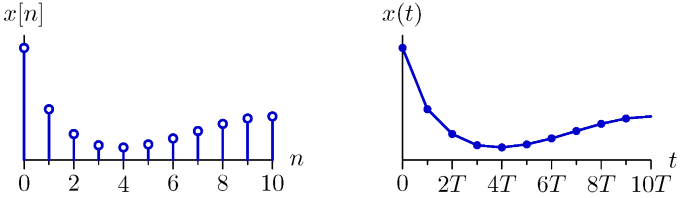
$T =$ sampling interval

commonly used in audio output devices such as CD players

Signals: Continuous Time and Discrete Time

Reconstruction: converting DT signals to CT

piecewise linear



$T =$ sampling interval

commonly used in rendering images

Signals: Real and Complex

Signals can be real, imaginary, or complex. An important class of signals are the complex exponentials:

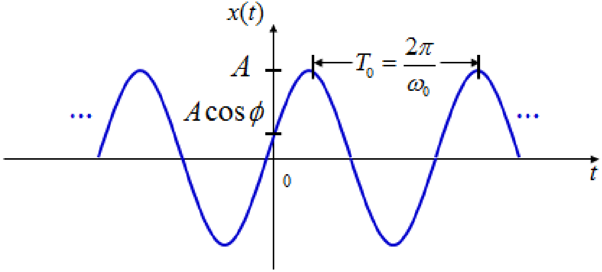
- the CT signal $x(t) = e^{st}$ where s is a complex number,
- the DT signal $x[n] = z^n$ where z is a complex number.

Q. Why do we deal with complex signals?

A. They are often analytically simpler to deal with than real signals.

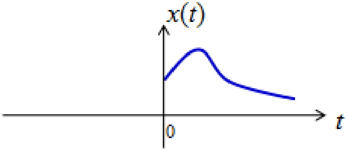
Signals: Periodic and Aperiodic

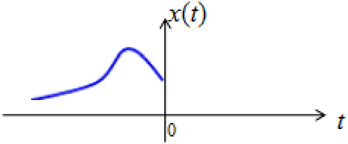
Periodic signals have the property that $x(t+T) = x(t)$ for all t. The smallest value of T that satisfies the definition is called the period.



Signals: Causal and Anti-causal

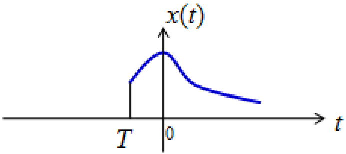
A causal signal is zero for $t < 0$ and anti-causal signal is zero for $t > 0$

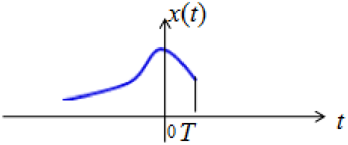




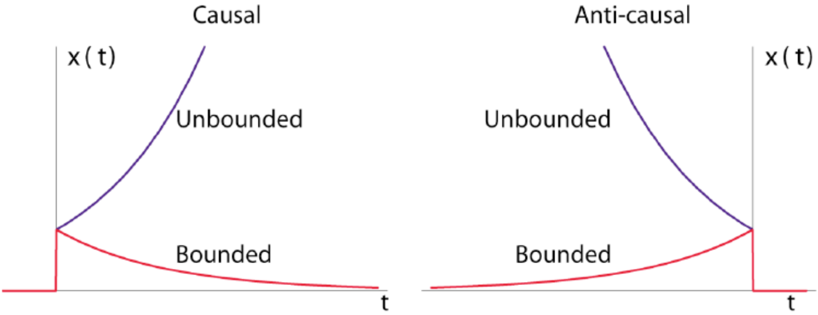
Signals: Right- and Left-sided

A right-sided signal is zero for $t < T$ and left-sided signal is zero for $t > T$ where T can be positive or negative.





Signals: Bounded and Unbounded



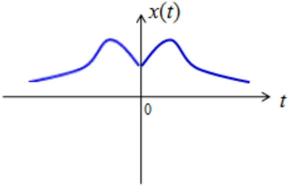
Whether the output signal of a system is bounded or unbounded determines the stability of the system.

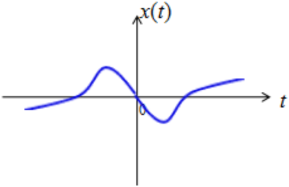
Signals: Even and Odd

$$x_e(t) = x_e(-t) \qquad x_o(t) = -x_o(-t)$$

- Any signal can be broken into a sum of even and odd signals

$$Ev\{x(t)\} = \frac{x(t) + x(-t)}{2} \qquad Od\{x(t)\} = \frac{x(t) - x(-t)}{2}$$





Signals: Energy and Power

- Energy and Power Signals
  - CT signals

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

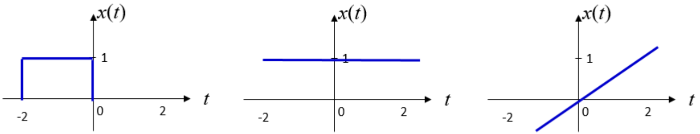
  - DT signals

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

Signals: Energy and Power

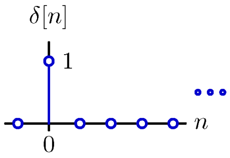
- Energy and Power Signals
  - Finite total energy $E_\infty < \infty \Rightarrow P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T} = 0$
  - Finite average power $P_\infty < \infty \Rightarrow E_\infty = \infty$
  - Neither E_∞ nor P_∞ are finite



Elementary Building-Block Signals

Elementary DT signal: $\delta[n]$.

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}$$



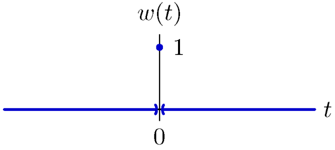
- shortest possible duration (most "transient")
- useful for constructing more complex signals

What CT signal serves the same purpose?

Elementary CT Building-Block Signal

Consider the analogous CT signal.

$$w(t) = \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ 0 & t > 0 \end{cases}$$



Is this a good choice as a building-block signal?

Elementary CT Building-Block Signal

Consider the analogous CT signal.

$$w(t) = \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ 0 & t > 0 \end{cases}$$

$$w(t) \longrightarrow \boxed{\int_{-\infty}^{t} (\,\cdot\,)\, dt} \longrightarrow 0$$

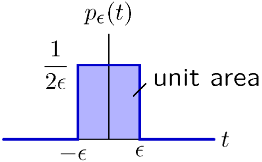
Is this a good choice as a building-block signal? No

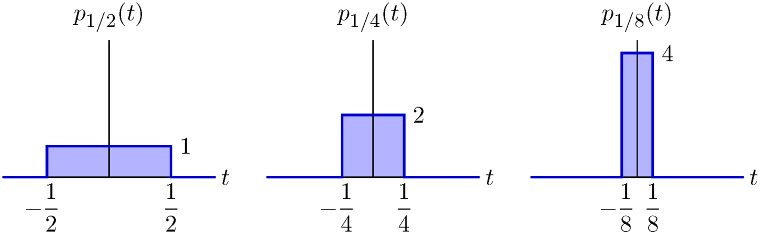
The integral of $w(t)$ is zero!

Unit-Impulse Signal

The unit-impulse signal acts as a pulse with unit area but zero width.

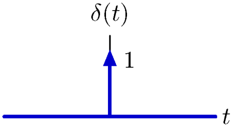
$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$





Unit-Impulse Signal

The unit-impulse function is represented by an arrow with the number **1**, which represents its area or "weight."

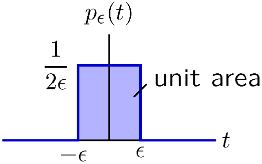


It has two seemingly contradictory properties:

- it is nonzero only at $t = 0$, and
- its definite integral $(-\infty, \infty)$ is one!

Both of these properties follow from thinking about $\delta(t)$ as a limit:

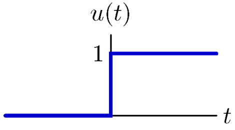
$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$



Unit-Impulse and Unit-Step Signals

The indefinite integral of the unit-impulse is the unit-step.

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1; & t \geq 0 \\ 0; & \text{otherwise} \end{cases}$$



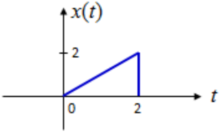
Equivalently

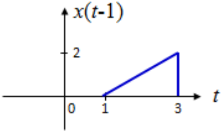
Transformations of Time

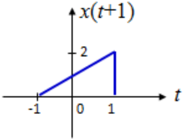
- Time shift

$$x(t) \Rightarrow x(t-t_0) \quad t_0 > 0 \quad \text{delayed, move to right}$$

$$t_0 < 0 \quad \text{advanced, move to left}$$

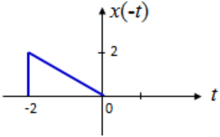


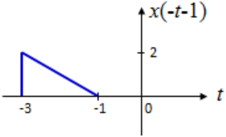


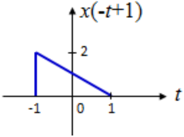


- Time reversal

$$x(t) \Rightarrow x(-t)$$



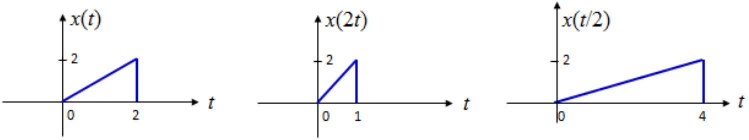




Transformations of time

- Time scaling

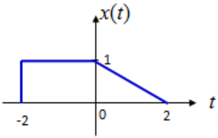
$$x(t) \Rightarrow x(\alpha t) \begin{cases} |\alpha| < 1 & \text{stretched} \\ |\alpha| > 1 & \text{compressed} \end{cases}$$



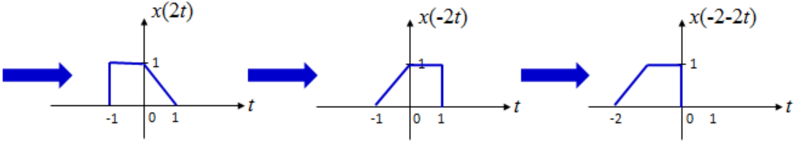
Time scaling of signals in DT?

Check Yourself

Given $x(t)$, sketch $x(-2-2t)$.



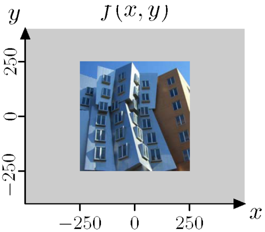
① $x(t) \Rightarrow x(2t) \Rightarrow x(-2t) \Rightarrow x(-2-2t)$



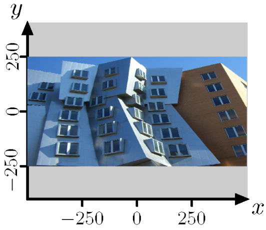
② $x(t) \Rightarrow x(2t) \Rightarrow x(2t-2) \Rightarrow x(-2t-2)$

③ ...

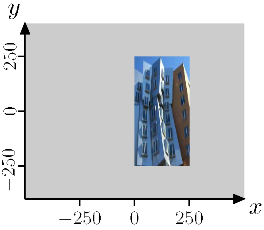
Check Yourself



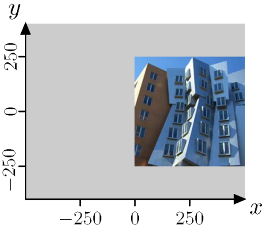
How many images match the expressions beneath them?



$f_1(x,y) = f(2x, y)$?

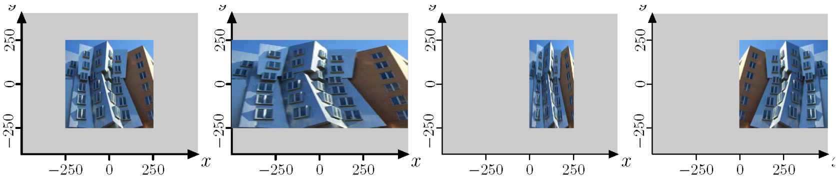


$f_2(x,y) = f(2x - 250, y)$?



$f_3(x,y) = f(-x - 250, y)$?

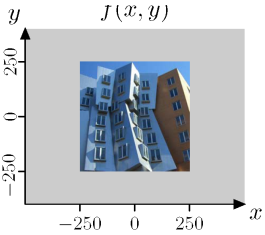
Check Yourself



$f(x,y)$ $\quad$ $f_1(x,y) = f(2x,y)$? $\quad$ $f_2(x,y) = f(2x-250,y)$? $\quad$ $f_3(x,y) = f(-x-250,y)$?

$$
\begin{array}{lll}
x = 0 & \rightarrow f_1(0,y) = f(0,y) & \surd \\
x = 250 & \rightarrow f_1(250,y) = f(500,y) & \times \\
 & & \\
x = 0 & \rightarrow f_2(0,y) = f(-250,y) & \surd \\
x = 250 & \rightarrow f_2(250,y) = f(250,y) & \surd \\
 & & \\
x = 0 & \rightarrow f_3(0,y) = f(-250,y) & \times \\
x = 250 & \rightarrow f_3(250,y) = f(-500,y) & \times
\end{array}
$$

Check Yourself



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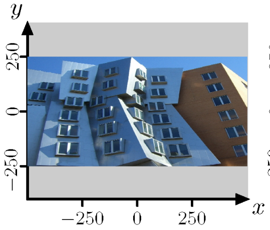
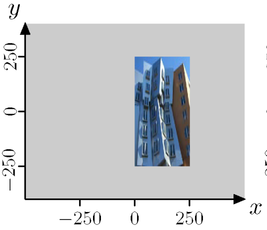
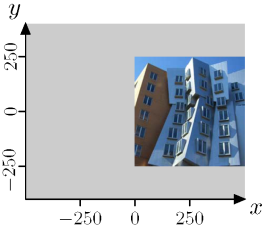


$\cancel{f_1(x,y) = f(2x,y)}$?



$f_2(x,y) = f(2x-250, y)$?

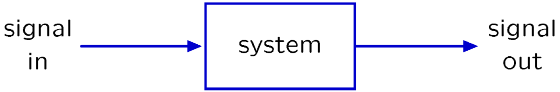


$\cancel{f_3(x,y) = f(-x-250, y)}$?

Outline

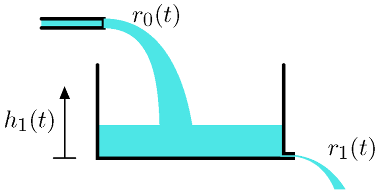
The Signals and Systems Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Example System: Leaky Tank

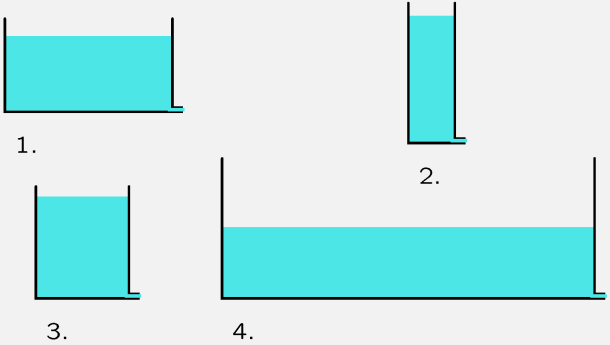
Formulate a mathematical description of this system.



What determines the leak rate?

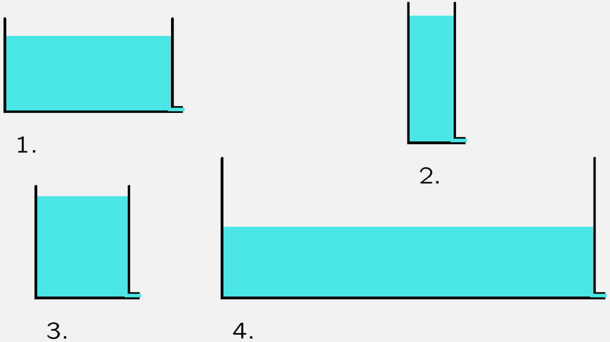
Check Yourself

The holes in each of the following tanks have equal size. Which tank has the largest leak rate $r_1(t)$?



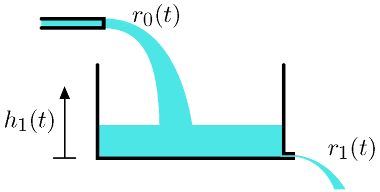
Check Yourself

The holes in each of the following tanks have equal size. Which tank has the largest leak rate $r_1(t)$? 2



Example System: Leaky Tank

Formulate a mathematical description of this system.

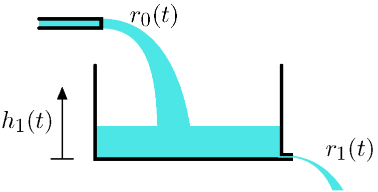


Assume linear leaking: $r_1(t) \propto h_1(t)$

What determines the height $h_1(t)$?

Example System: Leaky Tank

Formulate a mathematical description of this system.



Assume linear leaking: $r_1(t) \propto h_1(t)$

Assume water is conserved: $\frac{dh_1(t)}{dt} \propto r_0(t) - r_1(t)$

Solve: $\frac{dr_1(t)}{dt} \propto r_0(t) - r_1(t)$

Check Yourself

What are the dimensions of constant of proportionality C?

$$\frac{dr_1(t)}{dt} = C\Big(r_0(t) - r_1(t)\Big)$$

Check Yourself

What are the dimensions of constant of proportionality C?
inverse time (to match dimensions of dt)

$$\frac{dr_1(t)}{dt} = C\Big(r_0(t) - r_1(t)\Big)$$

Analysis of the Leaky Tank

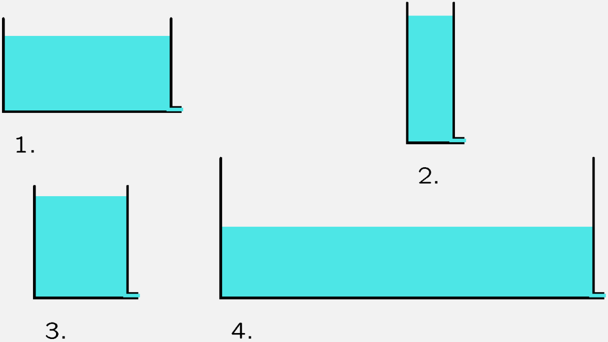
Call the constant of proportionality $1/\tau$.

Then τ is called the **time constant** of the system.

$$\frac{dr_1(t)}{dt} = \frac{r_0(t)}{\tau} - \frac{r_1(t)}{\tau}$$

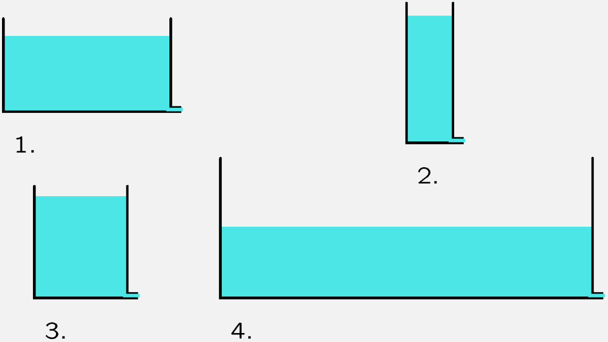
Check Yourself

Which of the following tanks has the largest time constant τ?



Check Yourself

Which of the following tanks has the largest time constant τ?   4



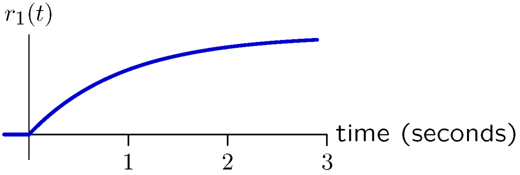
Analysis of the Leaky Tank

Call the constant of proportionality $1/\tau$.

Then τ is called the **time constant** of the system.

$$\frac{dr_1(t)}{dt} = \frac{r_0(t)}{\tau} - \frac{r_1(t)}{\tau}$$

Assume that the tank is initially empty, and then water enters at a constant rate $r_0(t) = 1$. Determine the output rate $r_1(t)$.



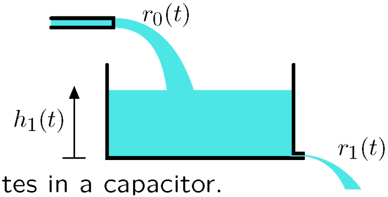
Explain the shape of this curve mathematically.

Explain the shape of this curve physically.

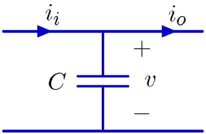
Leaky Tanks and Capacitors

Although derived for a leaky tank, this sort of model can be used to represent a variety of physical systems.

Water accumulates in a leaky tank.



Charge accumulates in a capacitor.



$$\frac{dv}{dt} = \frac{i_i - i_o}{C} \propto i_i - i_o \qquad \text{analogous to} \qquad \frac{dh}{dt} \propto r_0 - r_1$$

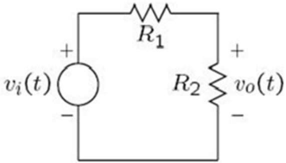
Systems: with and without memory

- **Memoryless sytem**

  The output of a memoryless sytem at a given time depends only on the input at the same time.

Ex.#1 $y[n] = (2x[n] - x^2[n])^2$

Ex.#2



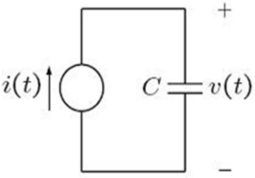
$$v_o(t) = \frac{R_2}{R_1 + R_2} v_i(t).$$

Systems: with and without memory

- **Systems with memory**

Ex.#1 $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Ex.#2



$$\begin{aligned} i(t) &= C\frac{dv(t)}{dt}, \\ v(t) &= \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau. \end{aligned}$$

Systems: Causal and Noncausal

A system is **causal** if the output at time t_0 depends only on the input for $t \le t_0$, i.e., the system cannot anticipate the input.

- Causality

A CT system $x(t) \to y(t)$ is causal if

When $x_1(t) \to y_1(t) \qquad x_2(t) \to y_2(t)$

and $x_1(t) = x_2(t) \qquad \textit{for all } t \le t_0$

Then $y_1(t) = y_2(t) \qquad \textit{for all } t \le t_0$

Systems: Causal and Noncausal

Ex.#1 $y(t) = x(t-1)$

Ex.#2 $y(t) = x(t+1)$

Ex.#3 $y[n] = x[-n]$

Ex.#4 $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

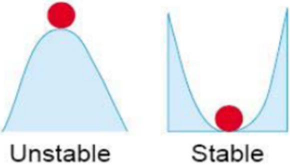
Ex.#5 $y(t) = x(t)\cos(t+1)$

Systems: Causal and Noncausal

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

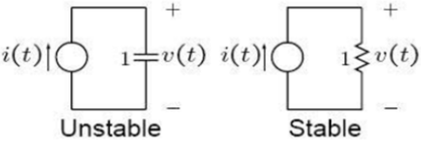
Systems: Stable and Non-stable

Stability can be defined in a variety of ways.
Definition 1: a stable system is one for which an incremental input leads to an incremental output.



An incremental force leads to only an incremental displacement in the stable system but not in the unstable system.

Definition 2: A system is BIBO stable if every bounded input leads to a bounded output. We will use this definition.



For the resistor, if $i(t)$ is bounded then so is $v(t)$, but for the capacitance this is not true. Consider $i(t) = u(t)$ then $v(t) = tu(t)$ which is unbounded.

Time-invariant Systems

Informally, a system is time-invariant (TI) if its behavior does not depend on the choice of $t = 0$. Then two identical experiments will yield the same results, regardless the starting time.

- Mathematically (in DT): A system is TI if for *any* input $x[n]$ and *any* time shift n_0,

$$\begin{array}{ll} \text{If} & x[n] \rightarrow y[n] \\ \text{then} & x[n - n_0] \rightarrow y[n - n_0] \,. \end{array}$$

- Similarly for CT time-invariant system,

$$\begin{array}{ll} \text{If} & x(t) \rightarrow y(t) \\ \text{then} & x(t - t_o) \rightarrow y(t - t_o) \,. \end{array}$$

Time-invariant Systems

Ex.#1 $y(t) = x^2(t+1)$

Ex.#2 $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

Ex.#3 $y(t) = \sin[x(t)]$

Ex.#4 $y[n] = nx[n]$

Ex.#5 $y(t) = x(2t)$

Time-invariant Systems

- If the input to a TI System is periodic, then the output is periodic with the same period.

"Proof": Suppose $x(t+T) = x(t)$

and $x(t) \quad \rightarrow \quad y(t)$

Then by TI

$$\underset{\uparrow}{x(t+T)} \rightarrow \underset{\uparrow}{y(t+T).}$$

These are the same input!

So these must be the same output, i.e., $y(t) = y(t+T)$

Linear Systems

A (CT) system is linear if it has the superposition property:

If $\quad x_1(t) \to y_1(t) \quad$ and $\quad x_2(t) \to y_2(t)$

then $\quad ax_1(t) + bx_2(t) \;\to\; ay_1(t) + by_2(t)$

$y[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, <u>not</u> TI, Noncausal

Can you find systems with other combinations ?

- e.g. Linear, TI, Noncausal
  Linear, not TI, Causal

Linear Systems

- Superposition

If $$x_k[n] \rightarrow y_k[n]$$

Then $$\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$$

- For linear systems, zero input $\rightarrow$ zero output

"Proof" $$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$$

Outline

Assignments

- Reading Assignment: Chapter 1.0-1.6