Signals and Systems

Lecture 2: Discrete-Time Systems & Continuous-Time Systems

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Partly adapted from the materials provided on the MIT OpenCourseWare

Review

- Introduction to signals
  - Classification
  - Building block signals
  - Transformation of time
- Introduction to systems
  - Classification
  - Linear time-invariant (LTI) system

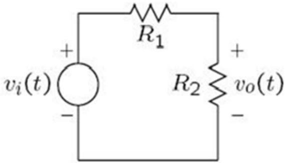
Systems: with and without memory

- **Memoryless sytem**

  The output of a memoryless sytem at a given time depends only on the input at the same time.

Ex.#1 $y[n] = (2x[n] - x^2[n])^2$

Ex.#2



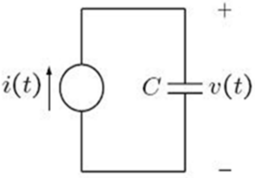
$$v_o(t) = \frac{R_2}{R_1 + R_2} v_i(t).$$

Systems: with and without memory

- **Systems with memory**

Ex.#1 $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Ex.#2



$$\begin{aligned} i(t) &= C\frac{dv(t)}{dt}, \\ v(t) &= \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau. \end{aligned}$$

Systems: Causal and Noncausal

A system is **causal** if the output at time t_0 depends only on the input for $t \le t_0$, i.e., the system cannot anticipate the input.

- Causality

A CT system $x(t) \to y(t)$ is causal if

When $x_1(t) \to y_1(t) \qquad x_2(t) \to y_2(t)$

and $x_1(t) = x_2(t) \qquad \textit{for all } t \le t_0$

Then $y_1(t) = y_2(t) \qquad \textit{for all } t \le t_0$

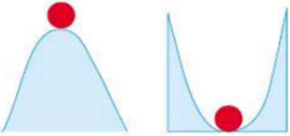
Systems: Causal and Noncausal

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

Systems: Stable and Non-stable

Stability can be defined in a variety of ways.

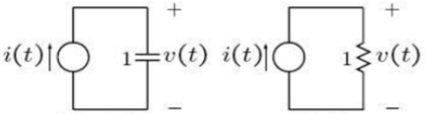
Definition 1: a stable system is one for which an incremental input leads to an incremental output.



Unstable Stable

An incremental force leads to only an incremental displacement in the stable system but not in the unstable system.

Definition 2: A system is BIBO stable if every bounded input leads to a bounded output. We will use this definition.



Unstable Stable

For the resistor, if $i(t)$ is bounded then so is $v(t)$, but for the capacitance this is not true. Consider $i(t) = u(t)$ then $v(t) = tu(t)$ which is unbounded.

Time-invariant Systems

Informally, a system is time-invariant (TI) if its behavior does not depend on the choice of $t = 0$. Then two identical experiments will yield the same results, regardless the starting time.

- Mathematically (in DT): A system is TI if for *any* input $x[n]$ and *any* time shift n_0,

$$\text{If} \quad x[n] \rightarrow y[n]$$
$$\text{then} \quad x[n - n_0] \rightarrow y[n - n_0] \,.$$

- Similarly for CT time-invariant system,

$$\text{If} \quad x(t) \rightarrow y(t)$$
$$\text{then} \quad x(t - t_o) \rightarrow y(t - t_o) \,.$$

Time-invariant Systems

Ex.#1 $y(t) = x^2(t+1)$

Ex.#2 $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

Ex.#3 $y(t) = \sin[x(t)]$

Ex.#4 $y[n] = nx[n]$

Ex.#5 $y(t) = x(2t)$

Linear Systems

A (CT) system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$y[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, <u>not</u> TI, Noncausal

Can you find systems with other combinations ?

- e.g. Linear, TI, Noncausal
  Linear, not TI, Causal

Outline

Discrete-Time Systems

We start with discrete-time (DT) systems because they

- are conceptually simpler than continuous-time systems
- illustrate same important modes of thinking as continuous-time
- are increasingly important (digital electronics and computation)

Multiple Representations of Discrete-Time Systems

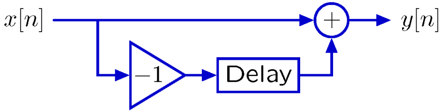
Systems can be represented in different ways to more easily address different types of issues.

Verbal description: 'To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.'

Difference equation:

$$y[n] = x[n] - x[n-1]$$

Block diagram:



We will exploit particular strengths of each of these representations.

Difference Equations

Difference equations are mathematically precise and compact.

Example:

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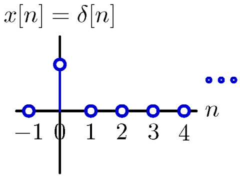
Let $x[n]$ equal the "unit sample" signal $\delta[n]$,

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$



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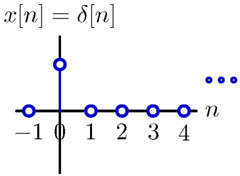
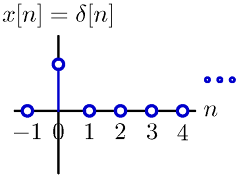


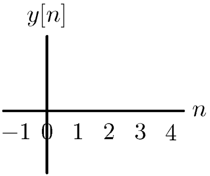
We will use the unit sample as a "primitive" (building-block signal) to construct more complex signals.

Discrete-Time Systems

Difference equations are convenient for step-by-step analysis.

Find $y[n]$ given $x[n] = \delta[n]$: $\quad y[n] = x[n] - x[n-1]$





Step-By-Step Solutions

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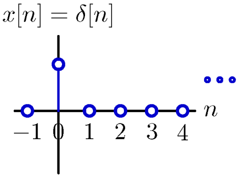

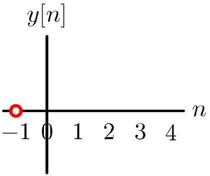

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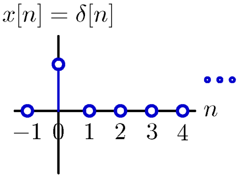

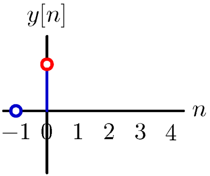

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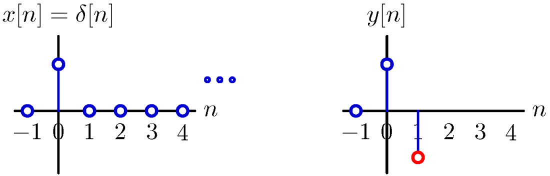

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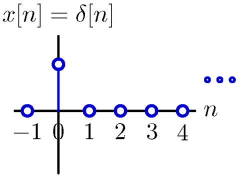

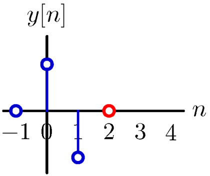

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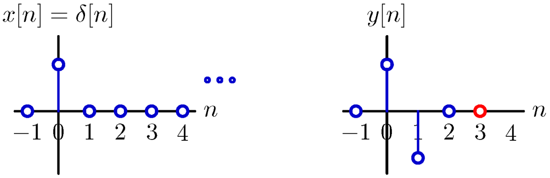

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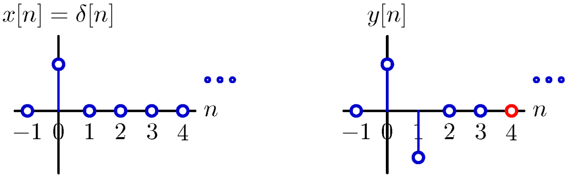

Step-By-Step Solutions

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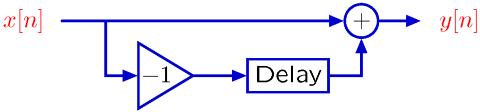
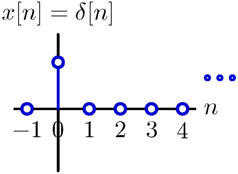
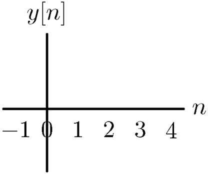
Represent $y[n] = x[n] - x[n-1]$ with a block diagram:







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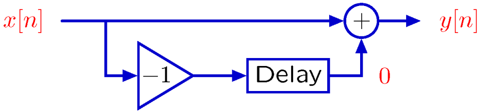

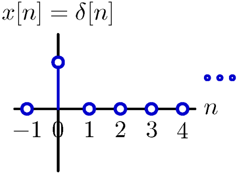

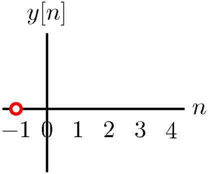

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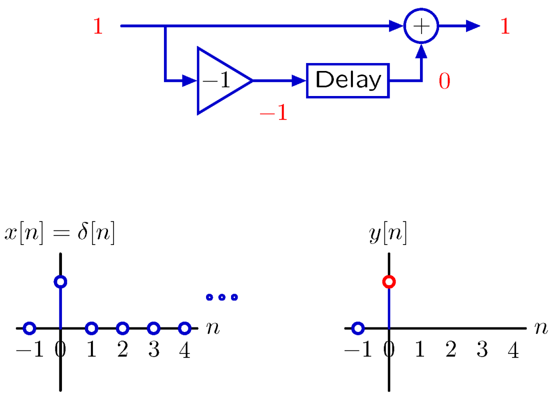

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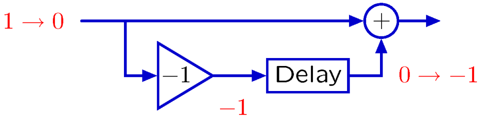

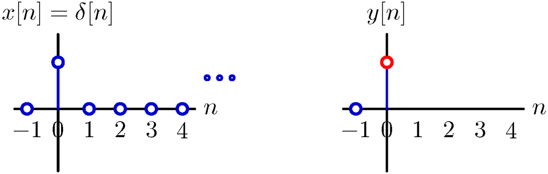

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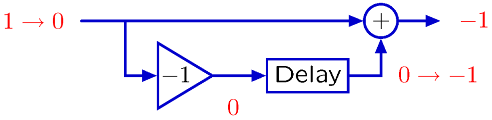

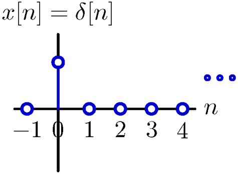

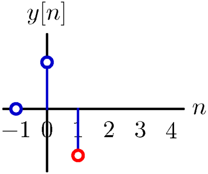

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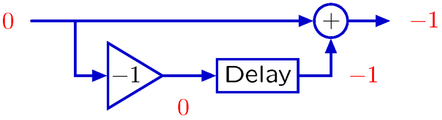

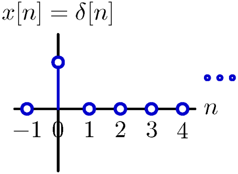

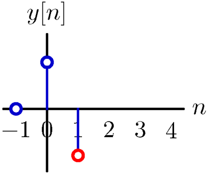

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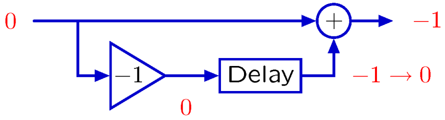

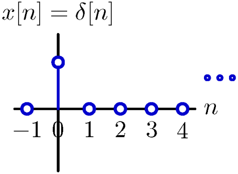

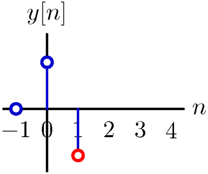

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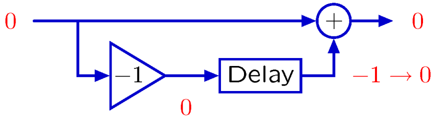

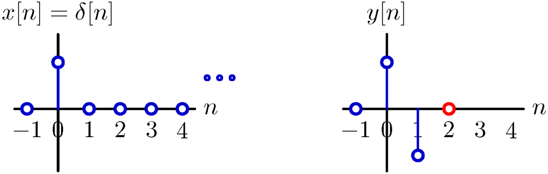

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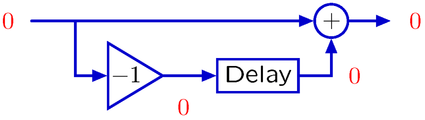

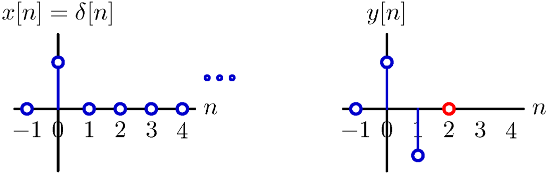

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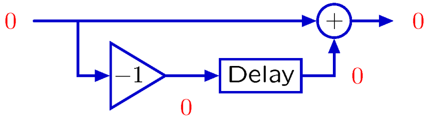

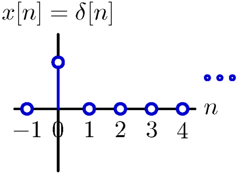

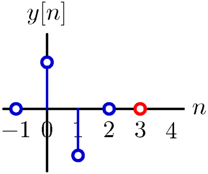

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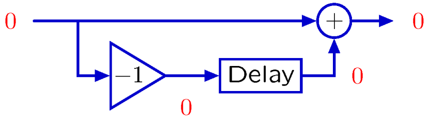
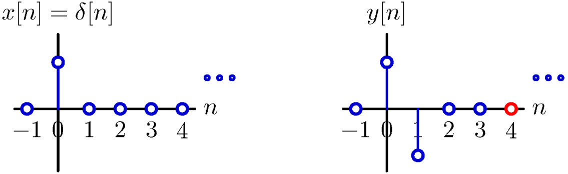




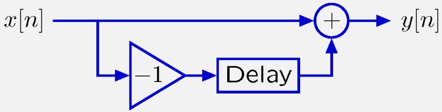
Check Yourself

DT systems can be described by difference equations and/or block diagrams.

Difference equation:

$$y[n] = x[n] - x[n-1]$$

Block diagram:



In what ways are these representations different?

Check Yourself

In what ways are difference equations different from block diagrams?

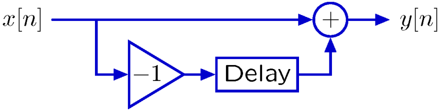
Difference equation:

$$y[n] = x[n] - x[n-1]$$

Difference equations are "declarative."

They tell you rules that the system obeys.

Block diagram:



Block diagrams are "imperative."

They tell you what to do.

Block diagrams contain **more** information than the corresponding difference equation (e.g., what is the input? what is the output?)

From Samples to Signals

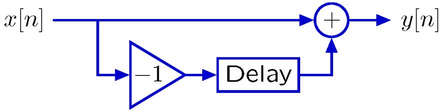
Lumping all of the (possibly infinite) samples into a single object — the signal — simplifies its manipulation.

This lumping is an **abstraction** that is analogous to

- representing coordinates in three-space as points
- representing lists of numbers as vectors in linear algebra
- creating an object in Python

From Samples to Signals

Operators manipulate signals rather than individual samples.



Nodes represent whole signals (e.g., X and Y).

The boxes **operate** on those signals:

- Delay = shift whole signal to right 1 time step
- Add = sum two signals
- -1: multiply by -1

Signals are the primitives.

Operators are the means of combination.

Operator Notation

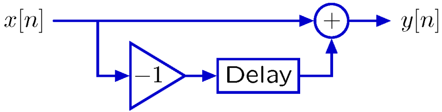
Symbols can now compactly represent diagrams.

Let $\mathcal{R}$ represent the right-shift **operator**:

$$Y = \mathcal{R}\{X\} \equiv \mathcal{R}X$$

where X represents the whole input signal ($x[n]$ for all n) and Y represents the whole output signal ($y[n]$ for all n)

Representing the difference machine



with $\mathcal{R}$ leads to the equivalent representation

$$Y = X - \mathcal{R}X = (1 - \mathcal{R})\, X$$

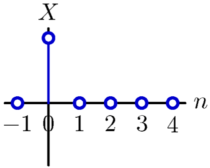
Check Yourself

Let $Y = \mathcal{R}X$. Which of the following is/are true:

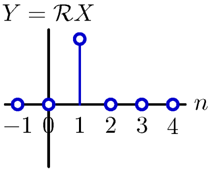
1. $y[n] = x[n]$ for all n
2. $y[n+1] = x[n]$ for all n
3. $y[n] = x[n+1]$ for all n
4. $y[n-1] = x[n]$ for all n
5. none of the above

Check Yourself

Consider a simple signal:



Then



Clearly $y[1] = x[0]$. Equivalently, if $n = 0$, then $y[n+1] = x[n]$.

The same sort of argument works for all other n.

Check Yourself

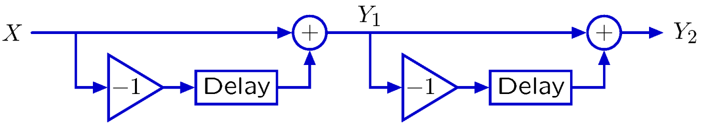
Let $Y = \mathcal{R}X$. Which of the following is/are true:

1. $y[n] = x[n]$ for all n
2. $y[n+1] = x[n]$ for all n
3. $y[n] = x[n+1]$ for all n
4. $y[n-1] = x[n]$ for all n
5. none of the above

Operator Representation of a Cascaded System

System operations have simple operator representations.

Cascade systems $\rightarrow$ multiply operator expressions.



Using operator notation:

$$Y_1 = (1 - \mathcal{R})\, X$$

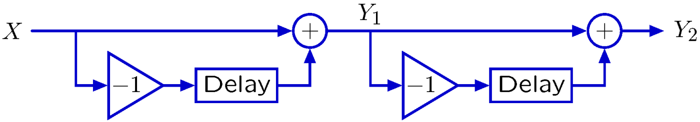
$$Y_2 = (1 - \mathcal{R})\, Y_1$$

Substituting for Y_1:

$$Y_2 = (1 - \mathcal{R})(1 - \mathcal{R})\, X$$

Operator Algebra

Operator expressions can be manipulated as polynomials.



Using difference equations:

$$\begin{aligned} y_2[n] &= y_1[n] - y_1[n-1] \\ &= (x[n] - x[n-1]) - (x[n-1] - x[n-2]) \\ &= x[n] - 2x[n-1] + x[n-2] \end{aligned}$$

Using operator notation:

$$\begin{aligned} Y_2 &= (1-\mathcal{R})\, Y_1 = (1-\mathcal{R})(1-\mathcal{R})\, X \\ &= (1-\mathcal{R})^2 X \\ &= (1 - 2\mathcal{R} + \mathcal{R}^2)\, X \end{aligned}$$

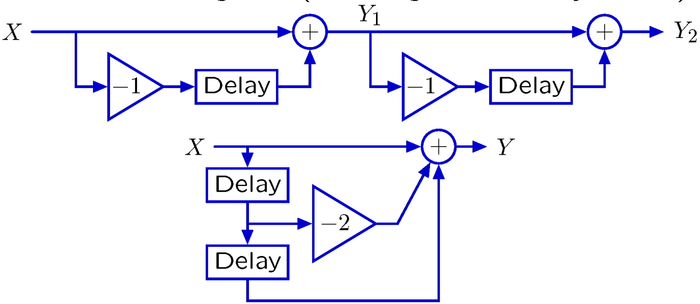
Operator Approach

Applies your existing expertise with polynomials to understand block diagrams, and thereby understand systems.

Operator Algebra

Operator notation facilitates seeing relations among systems.

"Equivalent" block diagrams (assuming both initially at rest):



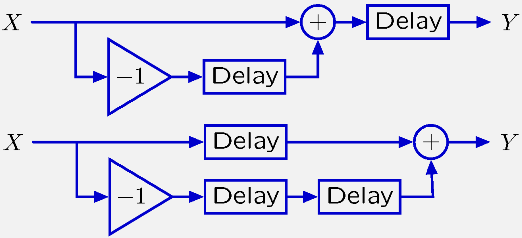
Equivalent operator expressions:

$$(1-\mathcal{R})(1-\mathcal{R}) = 1 - 2\mathcal{R} + \mathcal{R}^2$$

The operator equivalence is much easier to see.

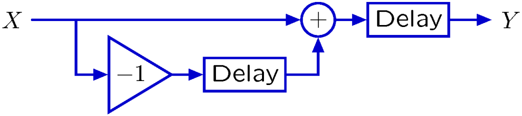
Check Yourself

Operator expressions for these "equivalent" systems (if started "at rest") obey what mathematical property?

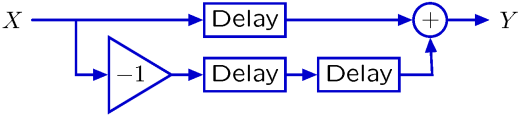


1. commutate
2. associative
3. distributive
4. transitive
5. none of the above

Check Yourself



$$Y = \mathcal{R}(1 - \mathcal{R})X$$

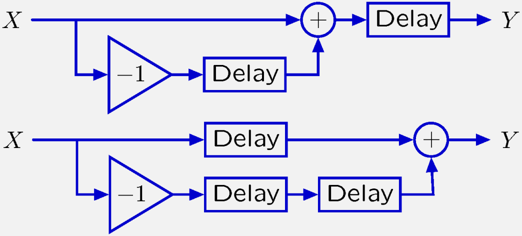


$$Y = (\mathcal{R} - \mathcal{R}^2)X$$

Multiplication by $\mathcal{R}$ distributes over addition.

Check Yourself

Operator expressions for these "equivalent" systems (if started "at rest") obey what mathematical property? 3

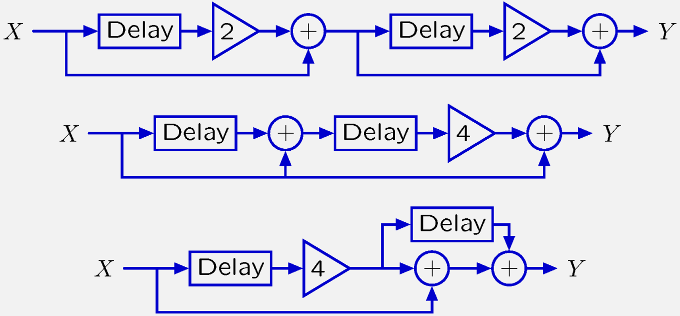


1. commutate
2. associative
3. distributive
4. transitive
5. none of the above

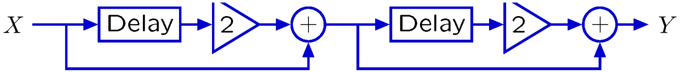
Check Yourself

How many of the following systems are equivalent to

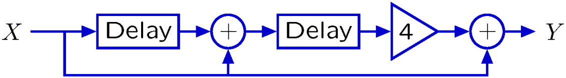
$$Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\,X \quad ?$$



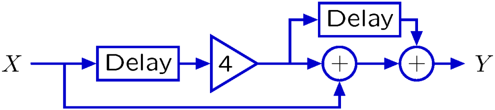
Check Yourself



$$Y = (2\mathcal{R}+1)(2\mathcal{R}+1)\,X$$



$$Y = (4\mathcal{R}^2+4\mathcal{R}+1)\,X$$



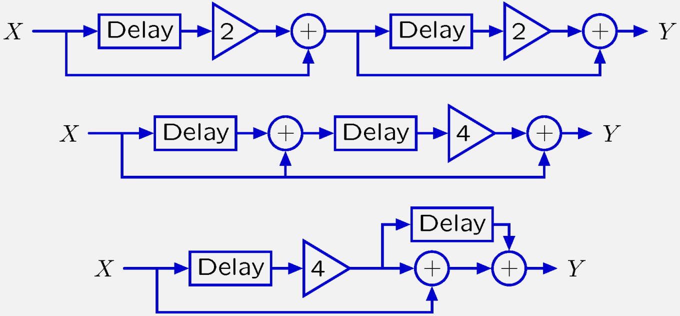
$$Y = (4\mathcal{R}^2+4\mathcal{R}+1)\,X$$

All implement $Y = (4\mathcal{R}^2+4\mathcal{R}+1)\,X$

Check Yourself

How many of the following systems are equivalent to

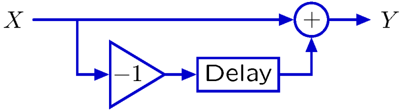
$$Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\,X \quad ? \qquad 3$$



Operator Algebra: Explicit and Implicit Rules

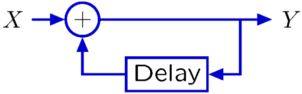
Recipes versus constraints.

Recipe: subtract a right-shifted version of the input signal from a copy of the input signal.



$$Y = (1 - \mathcal{R})\,X$$

Constraint: the difference between Y and $\mathcal{R}Y$ is X.

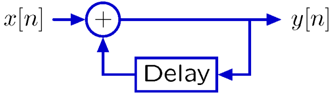


$$Y = \mathcal{R}Y + X$$
$$(1 - \mathcal{R})\,Y = X$$

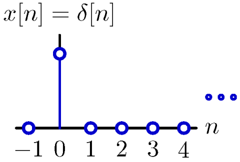
But how does one solve such a constraint?

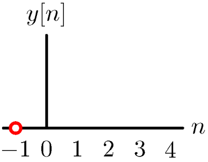
Example: Accumulator

Try step-by-step analysis: it always works. Start "at rest."



Find $y[n]$ given $x[n] = \delta[n]$: $\quad y[n] = x[n] + y[n-1]$





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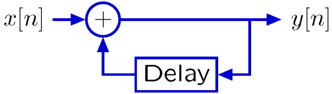
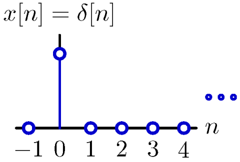
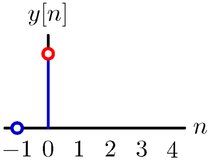


Find $y[n]$ given $x[n] = \delta[n]$: $\quad y[n] = x[n] + y[n-1]$

$$y[0] = x[0] + y[-1] \quad = 1 + 0 = 1$$





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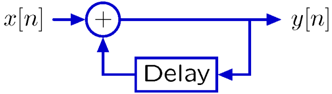

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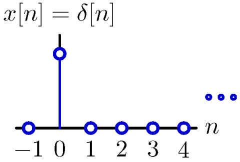

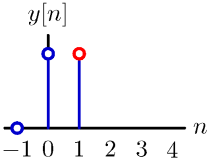

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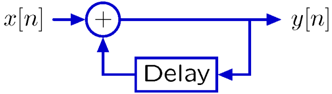

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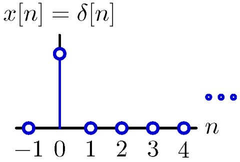
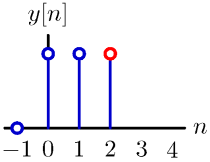
$$
\begin{aligned}
y[0] &= x[0] + y[-1] &&= 1 + 0 = 1 \\
y[1] &= x[1] + y[0] &&= 0 + 1 = 1 \\
y[2] &= x[2] + y[1] &&= 0 + 1 = 1
\end{aligned}
$$





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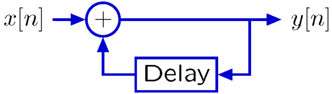

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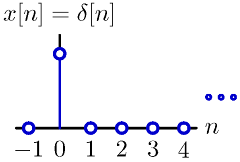
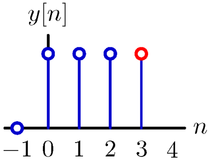
$$y[2] = x[2] + y[1] \quad = 0 + 1 = 1$$
$$\ldots$$





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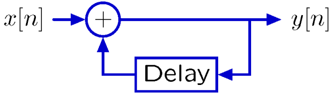

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$$
\begin{aligned}
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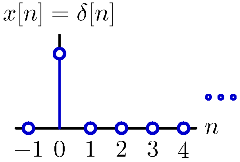
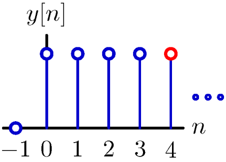
y[1] &= x[1] + y[0] &&= 0 + 1 = 1\\
y[2] &= x[2] + y[1] &&= 0 + 1 = 1\\
&\dots
\end{aligned}
$$

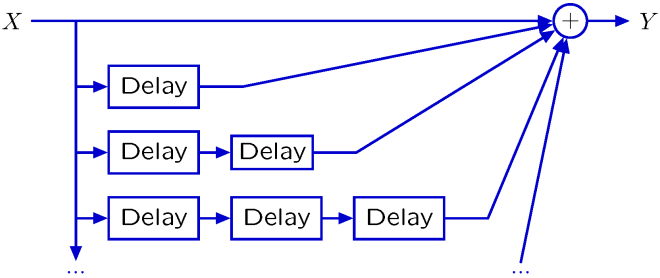




Persistent response to a transient input!

Example: Accumulator

The response of the accumulator system could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous.



$$Y = (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)\, X$$

Example: Accumulator

These systems are equivalent in the sense that if each is initially at rest, they will produce identical outputs from the same input.

$$(1-\mathcal{R})\,Y_1 = X_1 \qquad \Leftrightarrow ? \qquad Y_2 = (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_2$$

Proof: Assume $X_2 = X_1$:

$$\begin{aligned} Y_2 &= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_2 \\ &= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_1 \\ &= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,(1-\mathcal{R})\,Y_1 \\ &= ((1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)-(\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots))\,Y_1 \\ &= Y_1 \end{aligned}$$

It follows that $Y_2 = Y_1$.

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Proof: Assume $X_2 = X_1$:

$$\begin{aligned}
Y_2 &= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_2 \\
&= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,X_1 \\
&= (1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,(1-\mathcal{R})\,Y_1 \\
&= ((1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)-(\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots))\,Y_1 \\
&= Y_1
\end{aligned}$$

It follows that $Y_2 = Y_1$.

It also follows that $(1-\mathcal{R})$ and $(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)$ are reciprocals.

Example: Accumulator

The reciprocal of $1-\mathcal{R}$ can also be evaluated using synthetic division.

$$
\begin{array}{r|llllll}
\multicolumn{1}{r}{} & 1 & +\mathcal{R} & +\mathcal{R}^2 & +\mathcal{R}^3 & +\cdots & \\ \cline{2-7}
1-\mathcal{R} & 1 & & & & & \\
\multicolumn{1}{r}{} & 1 & -\mathcal{R} & & & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & \mathcal{R} & & & & \\
\multicolumn{1}{r}{} & & \mathcal{R} & -\mathcal{R}^2 & & & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & \mathcal{R}^2 & & & \\
\multicolumn{1}{r}{} & & & \mathcal{R}^2 & -\mathcal{R}^3 & & \\ \cline{4-5}
\multicolumn{1}{r}{} & & & & \mathcal{R}^3 & & \\
\multicolumn{1}{r}{} & & & & \mathcal{R}^3 & -\mathcal{R}^4 & \\ \cline{5-7}
\multicolumn{1}{r}{} & & & & & \cdots &
\end{array}
$$

Therefore

$$
\frac{1}{1-\mathcal{R}} = 1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \mathcal{R}^4 + \cdots
$$

Outline

Representations of Continuous-Time Systems

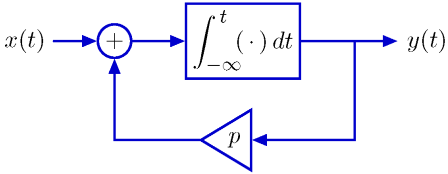
Verbal descriptions: preserve the rationale.

"Your account will grow in proportion to the current interest rate plus the rate at which you deposit."

Differential equations: mathematically compact.

$$\frac{dy(t)}{dt} = x(t) + py(t)$$

Block diagrams: illustrate signal flow paths.

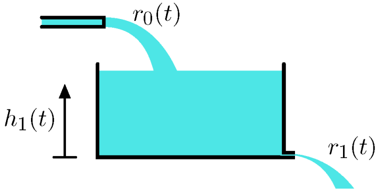


Operator representations: analyze systems as polynomials.

$$(1 - p\mathcal{A})Y = \mathcal{A}X$$

Differential Equations

Differential equations are mathematically precise and compact.



$$\frac{dr_1(t)}{dt} = \frac{r_0(t) - r_1(t)}{\tau}$$

Solution methodologies:

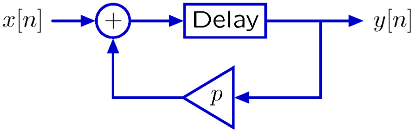
- general methods (separation of variables; integrating factors)
- homogeneous and particular solutions
- inspection

Today: new methods based on **block diagrams** and **operators**, which provide new ways to think about systems' behaviors.

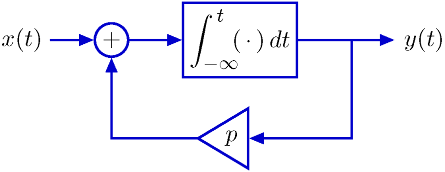
Block Diagrams

Block diagrams illustrate signal flow paths.

DT: adders, scalers, and delays – represent systems described by linear difference equations with constant coefficents.



CT: adders, scalers, and integrators – represent systems described by a linear differential equations with constant coefficients.



Operator Representation

CT Block diagrams are concisely represented with the $\mathcal{A}$ **operator**.

Applying $\mathcal{A}$ to a CT signal generates a new signal that is equal to the integral of the first signal at all points in time.

$$Y = \mathcal{A}X$$

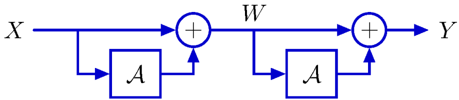
is equivalent to

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

for **all** time t.

Evaluating Operator Expressions

As with $\mathcal{R}$, $\mathcal{A}$ expressions can be manipulated as polynomials.



$$w(t) = x(t) + \int_{-\infty}^{t} x(\tau)d\tau$$

$$y(t) = w(t) + \int_{-\infty}^{t} w(\tau)d\tau$$

$$y(t) = x(t) + \int_{-\infty}^{t} x(\tau)d\tau + \int_{-\infty}^{t} x(\tau)d\tau + \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau_2} x(\tau_1)d\tau_1 \right) d\tau_2$$

$$W = (1 + \mathcal{A})\,X$$

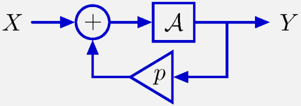
$$Y = (1 + \mathcal{A})\,W = (1 + \mathcal{A})(1 + \mathcal{A})\,X = (1 + 2\mathcal{A} + \mathcal{A}^2)\,X$$

Evaluating Operator Expressions

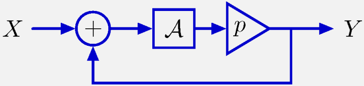
Expressions in $\mathcal{A}$ can be manipulated using rules for polynomials.

- Commutativity: $\mathcal{A}(1-\mathcal{A})X = (1-\mathcal{A})\mathcal{A}X$
- Distributivity: $\mathcal{A}(1-\mathcal{A})X = (\mathcal{A}-\mathcal{A}^2)X$
- Associativity: $\Big((1-\mathcal{A})\mathcal{A}\Big)(2-\mathcal{A})X = (1-\mathcal{A})\Big(\mathcal{A}(2-\mathcal{A})\Big)X$

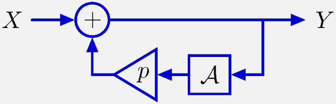
Check Yourself



$$\dot{y}(t) = \dot{x}(t) + p\ddot{y}(t)$$

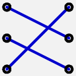
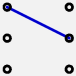
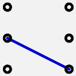
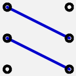


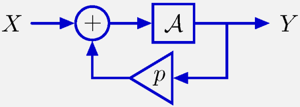
$$\dot{y}(t) = x(t) + py(t)$$



$$\dot{y}(t) = px(t) + py(t)$$

Which best illustrates the left-right correspondences?

1. 
2. 
3. 
4. 
5. none



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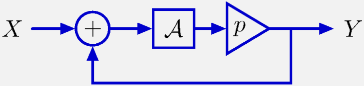

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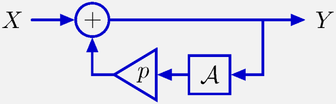
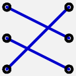
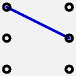
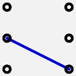


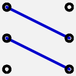
$$\dot{y}(t) = px(t) + py(t)$$

Which best illustrates the left-right correspondences? 4

1. 

2. 

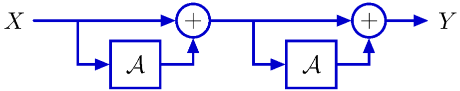
3. 

4. 

5. none

Impulse Response of Acyclic CT System

If the block diagram of a CT system has no feedback (i.e., no cycles), then the corresponding operator expression is "imperative."



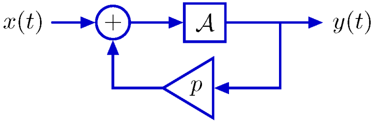
$$Y = (1+\mathcal{A})(1+\mathcal{A})\,X = (1+2\mathcal{A}+\mathcal{A}^2)\,X$$

If $x(t) = \delta(t)$ then

$$y(t) = (1+2\mathcal{A}+\mathcal{A}^2)\,\delta(t) = \delta(t) + 2u(t) + tu(t)$$

CT Feedback

Find the impulse response of this CT system with feedback.



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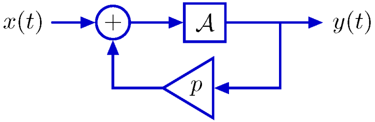


Method 1: find differential equation and solve it.

$$\dot{y}(t) = x(t) + py(t)$$

Linear, first-order difference equation with constant coefficients.

Try $y(t) = Ce^{\alpha t}u(t)$.

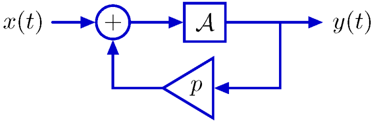
Then $\dot{y}(t) = \alpha Ce^{\alpha t}u(t) + Ce^{\alpha t}\delta(t) = \alpha Ce^{\alpha t}u(t) + C\delta(t)$.

Substituting, we find that $\alpha Ce^{\alpha t}u(t) + C\delta(t) = \delta(t) + pCe^{\alpha t}u(t)$.

Therefore $\alpha = p$ and $C = 1 \quad \rightarrow \quad y(t) = e^{pt}u(t)$.

CT Feedback

Find the impulse response of this CT system with feedback.



Method 2: use operators.

$$Y = \mathcal{A}\,(X + pY)$$
$$\frac{Y}{X} = \frac{\mathcal{A}}{1 - p\mathcal{A}}$$

Now expand in ascending series in $\mathcal{A}$:

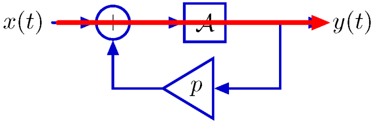
$$\frac{Y}{X} = \mathcal{A}(1 + p\mathcal{A} + p^2\mathcal{A}^2 + p^3\mathcal{A}^3 + \cdots)$$

If $x(t) = \delta(t)$ then

$$y(t) = \mathcal{A}(1 + p\mathcal{A} + p^2\mathcal{A}^2 + p^3\mathcal{A}^3 + \cdots)\,\delta(t)$$
$$= (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots)\,u(t) = e^{pt}u(t)\,.$$

CT Feedback

We can visualize the feedback by tracing each cycle through the cyclic signal path.



$$y(t) = (\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots)\,\delta(t)$$

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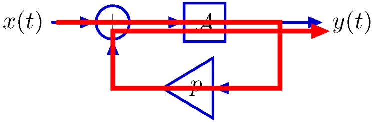

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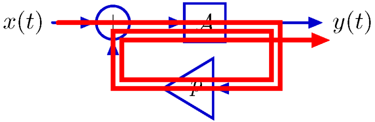

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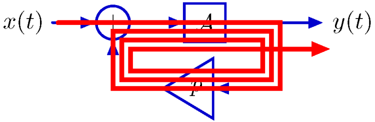

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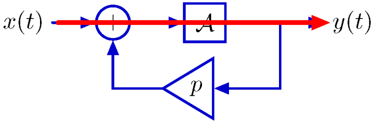
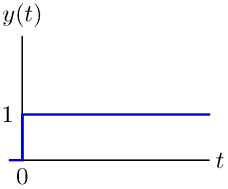


$$
\begin{aligned}
y(t) &= \left(\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots\right)\delta(t) \\
&= \left(1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots\right)u(t)
\end{aligned}
$$



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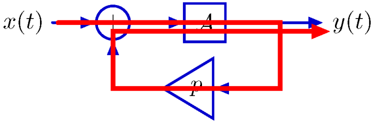

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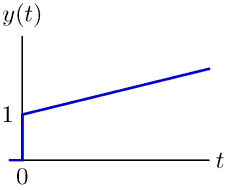

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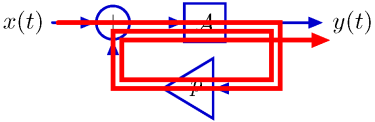

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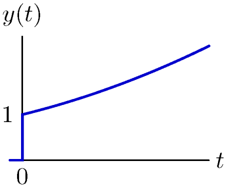

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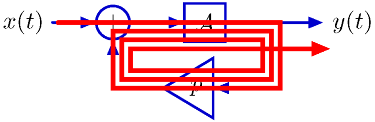

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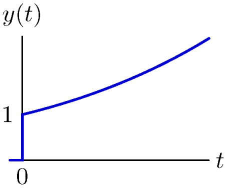

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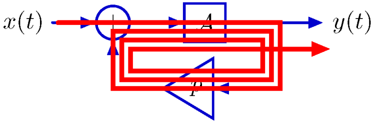

$$y(t) = (\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots)\,\delta(t)$$
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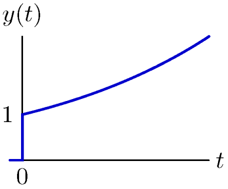
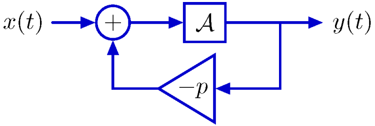


CT Feedback

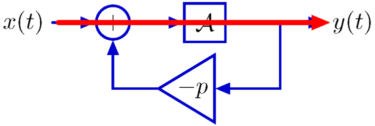
Making p negative makes the output converge (instead of diverge).



$$y(t) = (\mathcal{A} - p\mathcal{A}^2 + p^2\mathcal{A}^3 - p^3\mathcal{A}^4 + \cdots)\,\delta(t)$$
$$= (1 - pt + \frac{1}{2}p^2t^2 - \frac{1}{6}p^3t^3 + \cdots)\,u(t)$$

CT Feedback

Making p negative makes the output converge.



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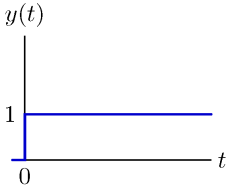

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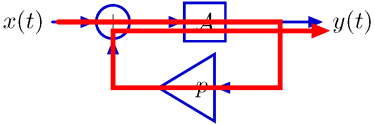

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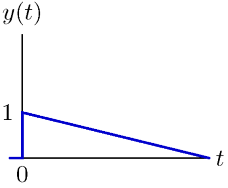

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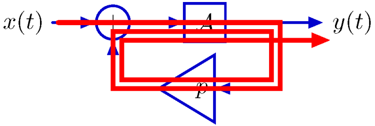

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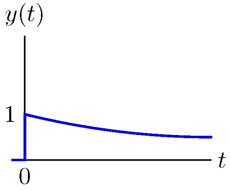

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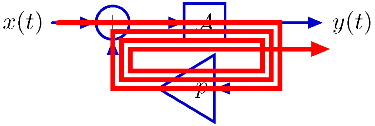

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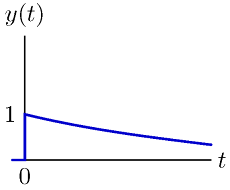

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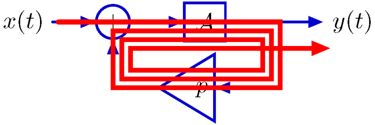

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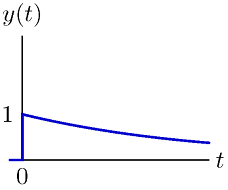

Outline

Assignments

- Reading Assignment: Supplementary notes
- Homework 1: Due by Mar. 7, 2024