Signals and Systems

Lecture 4: Laplace Transform

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Zhejiang University

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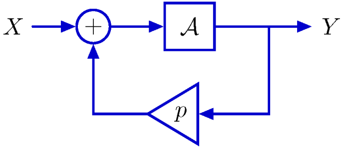
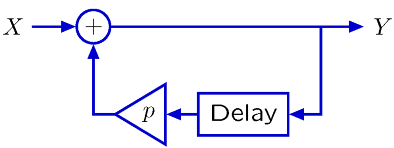
Partly adapted from the materials provided on the MIT OpenCourseWare

Review



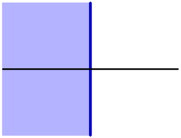


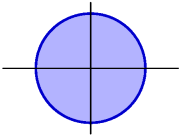
$$\frac{\mathcal{A}}{1 - p\mathcal{A}}$$

$$\frac{1}{1 - p\mathcal{R}}$$

$$e^{pt}u(t)$$

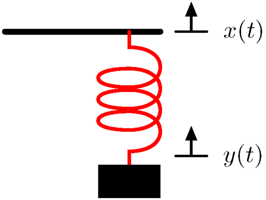
$$p^n u[n]$$



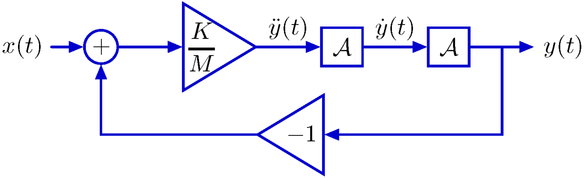


Mass and Spring System

Use the $\mathcal{A}$ operator to solve the mass and spring system.



$$F = K\big(x(t) - y(t)\big) = M\ddot{y}(t)$$



$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2}$$

Mass and Spring System

Factor system functional to find the poles.

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2} = \frac{\frac{K}{M}\mathcal{A}^2}{(1 - p_0\mathcal{A})(1 - p_1\mathcal{A})}$$

$$1 + \frac{K}{M}\mathcal{A}^2 = 1 - (p_0 + p_1)\mathcal{A} + p_0 p_1 \mathcal{A}^2$$

The sum of the poles must be zero.
The product of the poles must be K/M.

$$p_0 = j\sqrt{\frac{K}{M}} \quad p_1 = -j\sqrt{\frac{K}{M}}$$

Mass and Spring System

Alternatively, find the poles by substituting $\mathcal{A} \rightarrow \frac{1}{s}$.
The poles are then the roots of the denominator.

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2}$$

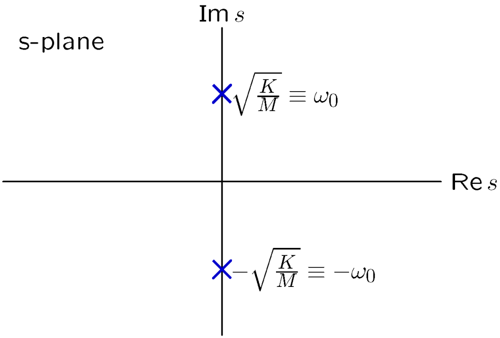
Substitute $\mathcal{A} \rightarrow \frac{1}{s}$:

$$\frac{Y}{X} = \frac{\frac{K}{M}}{s^2 + \frac{K}{M}}$$

$$s = \pm j\sqrt{\frac{K}{M}}$$

Mass and Spring System

The poles are complex conjugates.



The corresponding fundamental modes have complex values.

fundamental mode 1: $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

fundamental mode 2: $e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$

Mass and Spring System

Real-valued inputs always excite combinations of these modes so that the imaginary parts cancel.

Example: find the impulse response.

$$\begin{aligned}\frac{Y}{X} &= \frac{\frac{K}{M}\mathcal{A}^2}{1+\frac{K}{M}\mathcal{A}^2} = \frac{\frac{K}{M}}{p_0-p_1}\left(\frac{\mathcal{A}}{1-p_0\mathcal{A}} - \frac{\mathcal{A}}{1-p_1\mathcal{A}}\right)\\ &= \frac{\omega_0^2}{2j\omega_0}\left(\frac{\mathcal{A}}{1-j\omega_0\mathcal{A}} - \frac{\mathcal{A}}{1+j\omega_0\mathcal{A}}\right)\\ &= \frac{\omega_0}{2j}\underbrace{\left(\frac{\mathcal{A}}{1-j\omega_0\mathcal{A}}\right)}_{\text{makes mode 1}} - \frac{\omega_0}{2j}\underbrace{\left(\frac{\mathcal{A}}{1+j\omega_0\mathcal{A}}\right)}_{\text{makes mode 2}}\end{aligned}$$

The modes themselves are complex conjugates, and their coefficients are also complex conjugates. So the sum is a sum of something and its complex conjugate, which is real.

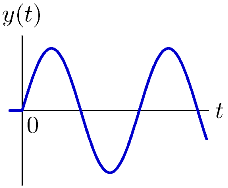
Mass and Spring System

The impulse response is therefore real.

$$\frac{Y}{X} = \frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 - j\omega_0\mathcal{A}}\right) - \frac{\omega_0}{2j}\left(\frac{\mathcal{A}}{1 + j\omega_0\mathcal{A}}\right)$$

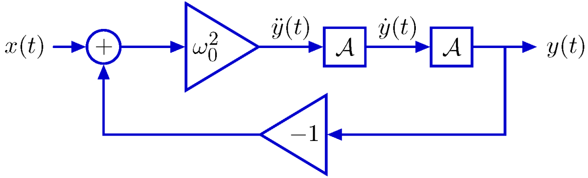
The impulse response is

$$h(t) = \frac{\omega_0}{2j} e^{j\omega_0 t} - \frac{\omega_0}{2j} e^{-j\omega_0 t} = \omega_0 \sin \omega_0 t\,; \qquad t > 0$$



Mass and Spring System

Alternatively, find impulse response by expanding system functional.



$$\frac{Y}{X} = \frac{\omega_0^2\mathcal{A}^2}{1+\omega_0^2\mathcal{A}^2} = \omega_0^2\mathcal{A}^2 - \omega_0^4\mathcal{A}^4 + \omega_0^6\mathcal{A}^6 - + \cdots$$

If $x(t) = \delta(t)$ then

$$y(t) = \omega_0^2 t - \omega_0^4\frac{t^3}{3!} + \omega_0^6\frac{t^5}{5!} - + \cdots \ , \ t \ge 0$$

Mass and Spring System

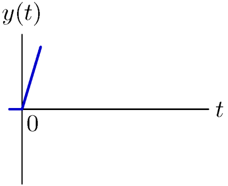
Look at successive approximations to this infinite series.

$$\frac{Y}{X} = \frac{\omega_0^2 \mathcal{A}^2}{1 + \omega_0^2 \mathcal{A}^2} = \omega_0^2 \mathcal{A}^2 \sum_{l=0}^{\infty} \left(-\omega_0^2 \mathcal{A}^2\right)^l$$

If $x(t) = \delta(t)$ then

$$y(t) = \sum_{l=0}^{\infty} \omega_0^2 \left(-\omega_0^2\right)^l \mathcal{A}^{2l+2} \delta(t)$$

$$= \omega_0^2 t$$



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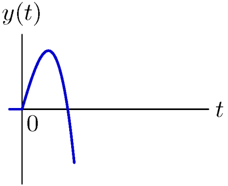
$$= \omega_0^2 t - \omega_0^4\frac{t^3}{3!}$$



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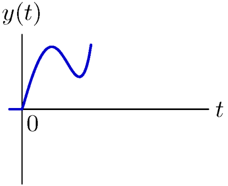

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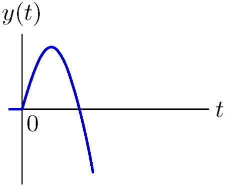

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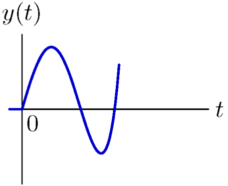

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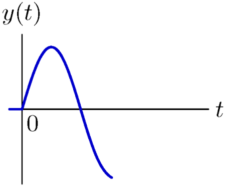

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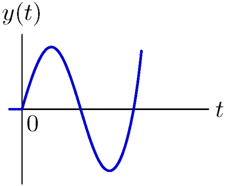

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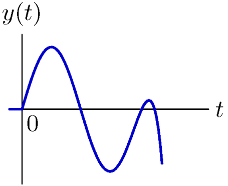

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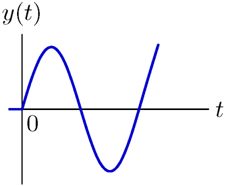

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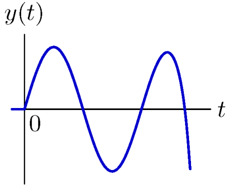

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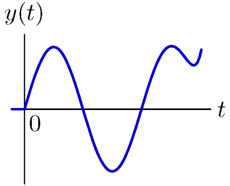

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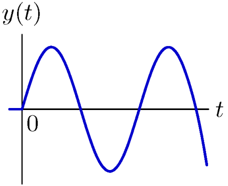

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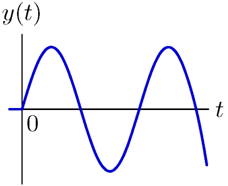
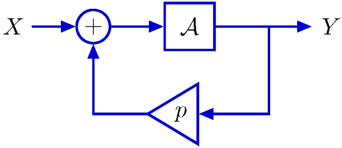
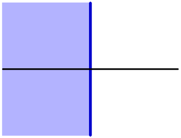


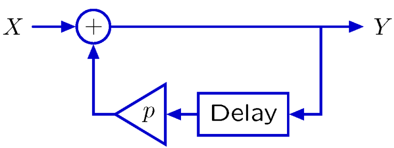
Important similarities and important differences.



$$\frac{\mathcal{A}}{1-p\mathcal{A}}$$

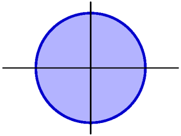
$$e^{pt}u(t)$$





$$\frac{1}{1-p\mathcal{R}}$$

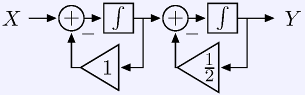
$$p^n u[n]$$



Concept Map: Continuous-Time Systems

Multiple representations of CT systems.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\, u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

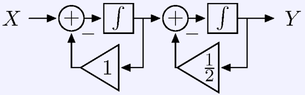
$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Continuous-Time Systems

Two interpretations of $\int$.

$$X \rightarrow \boxed{\int} \rightarrow \mathcal{A}X$$

Block Diagram



System Functional

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$$\dot{x}(t) \rightarrow \boxed{\int} \rightarrow x(t)$$

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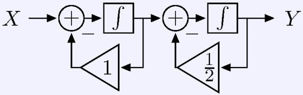
Concept Map: Continuous-Time Systems

Relation between System Functional and System Function.

Block Diagram



System Functional

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$$\mathcal{A} \rightarrow \frac{1}{s}$$

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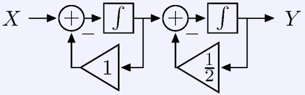
Check Yourself

How to determine impulse response from system functional?

Block Diagram



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$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Check Yourself

How to determine impulse response from system functional?

Expand functional using **partial fractions**:

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2+3\mathcal{A}+\mathcal{A}^2} = \frac{\mathcal{A}^2}{(1+\frac{1}{2}\mathcal{A})(1+\mathcal{A})} = \frac{2\mathcal{A}}{1+\frac{1}{2}\mathcal{A}} - \frac{2\mathcal{A}}{1+\mathcal{A}}$$

Recognize forms of terms: each corresponds to an exponential.

Alternatively, expand each term in a **series**:

$$\frac{Y}{X} = 2\mathcal{A}\Big(1 - \frac{1}{2}\mathcal{A} + \frac{1}{4}\mathcal{A}^2 - \frac{1}{8}\mathcal{A}^3 + - \cdots\Big) - 2\mathcal{A}\Big(1 - \mathcal{A} + \mathcal{A}^2 - \mathcal{A}^3 + - \cdots\Big)$$

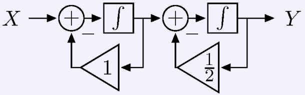
Let $X = \delta(t)$. Then

$$\begin{aligned} Y &= 2\Big(1 - \frac{1}{2}t + \frac{1}{8}t^2 - \frac{1}{48}t^3 + - \cdots\Big)\, u(t) - 2\Big(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + - \cdots\Big) \\ &= 2\Big(e^{-t/2} - e^{-t}\Big)\, u(t) \end{aligned}$$

Check Yourself

How to determine impulse response from system functional?

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

series partial fractions

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

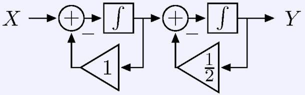
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Continuous-Time Systems

Today: new relations based on Laplace transform.

Block Diagram



System Functional

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Impulse Response

$$h(t)=2(e^{-t/2}-e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t)+3\dot{y}(t)+y(t)=2x(t)$$

System Function

$$\frac{Y(s)}{X(s)}=\frac{2}{2s^2+3s+1}$$

Laplace Transform: Definition

Laplace transform maps a function of time t to a function of s.

$$X(s) = \int x(t)e^{-st}dt$$

There are two important variants:

Unilateral (18.03)

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Bilateral (6.003)

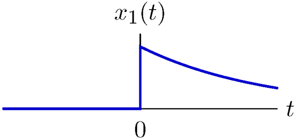
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Both share important properties — will discuss differences later.

Laplace Transforms

Example: Find the Laplace transform of $x_1(t)$:

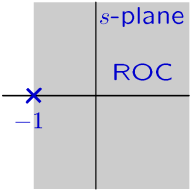
$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_0^{\infty} e^{-t}e^{-st}dt = \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^{\infty} = \frac{1}{s+1}$$

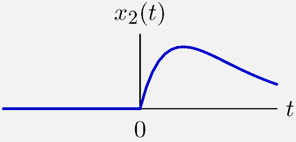
provided $\mathrm{Re}(s+1) > 0$ which implies that $\mathrm{Re}(s) > -1$.

$$\frac{1}{s+1} \ ; \quad \mathrm{Re}(s) > -1$$



Check Yourself

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Which of the following is the Laplace transform of $x_2(t)$?

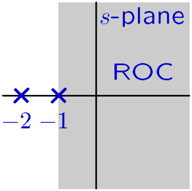
1. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\mathrm{Re}(s) > -1$
2. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\mathrm{Re}(s) > -2$
3. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\mathrm{Re}(s) > -1$
4. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\mathrm{Re}(s) > -2$
5. none of the above

Check Yourself

$$X_2(s) = \int_0^\infty (e^{-t} - e^{-2t}) e^{-st} dt$$
$$= \int_0^\infty e^{-t} e^{-st} dt - \int_0^\infty e^{-2t} e^{-st} dt$$
$$= \frac{1}{s+1} - \frac{1}{s+2} = \frac{(s+2)-(s+1)}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$

These equations converge if $\text{Re}(s+1) > 0$ and $\text{Re}(s+2) > 0$, thus $\text{Re}(s) > -1$.

$$\frac{1}{(s+1)(s+2)} \;; \quad \text{Re}(s) > -1$$



Check Yourself

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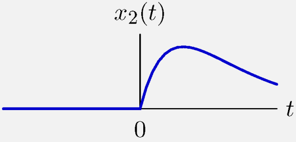

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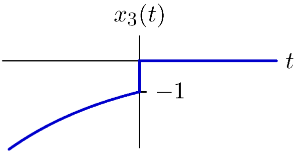
5. none of the above

Regions of Convergence

Left-sided signals have left-sided Laplace transforms (bilateral only).

Example:

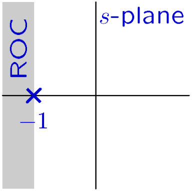
$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{0} -e^{-t}e^{-st}dt = \left.\frac{-e^{-(s+1)t}}{-(s+1)}\right|_{-\infty}^{0} = \frac{1}{s+1}$$

provided $\mathrm{Re}(s+1) < 0$ which implies that $\mathrm{Re}(s) < -1$.

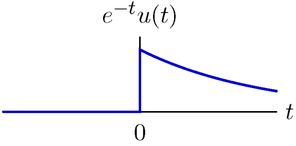
$$\frac{1}{s+1} \; ; \quad \mathrm{Re}(s) < -1$$



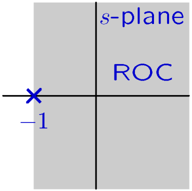
Left- and Right-Sided ROCs

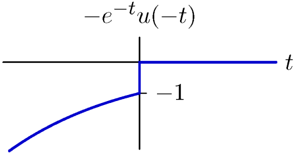
Laplace transforms of left- and right-sided exponentials have the same form (except $-$); with left- and right-sided ROCs, respectively.

time function | Laplace transform



$$\frac{1}{s+1}$$





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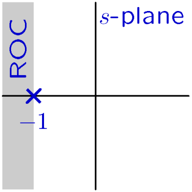

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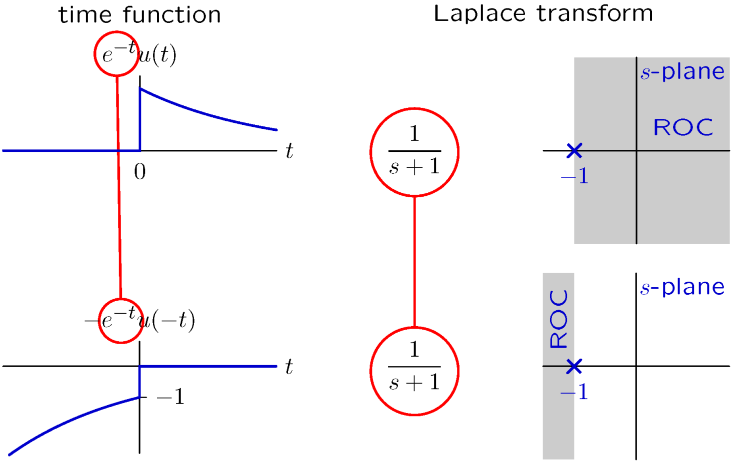
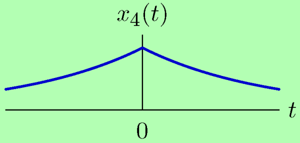
Laplace transforms of left- and right-sided exponentials have the same form (except $-$); with left- and right-sided ROCs, respectively.



Check Yourself

Find the Laplace transform of $x_4(t)$.

$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $\quad -\infty < \mathrm{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $\quad -1 < \mathrm{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $\quad -\infty < \mathrm{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $\quad -1 < \mathrm{Re}(s) < 1$
5. none of the above

Check Yourself

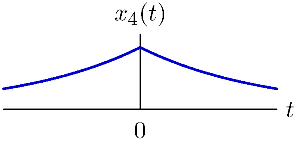
$$\begin{aligned}
X_4(s) &= \int_{-\infty}^{\infty} e^{-|t|}e^{-st}dt \\
&= \int_{-\infty}^{0} e^{(1-s)t}dt + \int_{0}^{\infty} e^{-(1+s)t}dt \\
&= \left.\frac{e^{(1-s)t}}{(1-s)}\right|_{-\infty}^{0} + \left.\frac{e^{-(1+s)t}}{-(1+s)}\right|_{0}^{\infty} \\
&= \underbrace{\frac{1}{1-s}}_{\mathrm{Re}(s)<1} + \underbrace{\frac{1}{1+s}}_{\mathrm{Re}(s)>-1} \\
&= \frac{1+s+1-s}{(1-s)(1+s)} = \frac{2}{1-s^2}\ ;\quad -1<\mathrm{Re}(s)<1
\end{aligned}$$

The ROC is the intersection of $\mathrm{Re}(s)<1$ and $\mathrm{Re}(s)>-1$.

Check Yourself

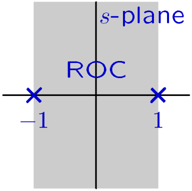
The Laplace transform of a signal that is both-sided a vertical strip.

$$x_4(t) = e^{-|t|}$$



$$X_4(s) = \frac{2}{1 - s^2}$$

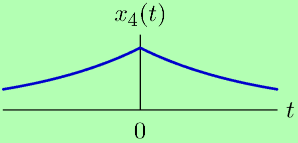
$$-1 < \mathrm{Re}(s) < 1$$



Check Yourself

Find the Laplace transform of $x_4(t)$. 2

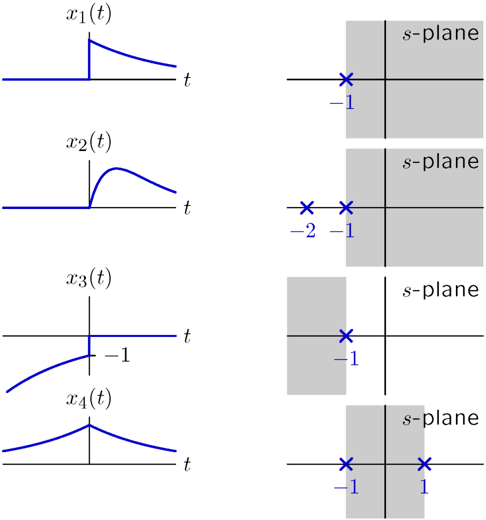
$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \text{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \text{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \text{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \text{Re}(s) < 1$
5. none of the above

Time-Domain Interpretation of ROC

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$



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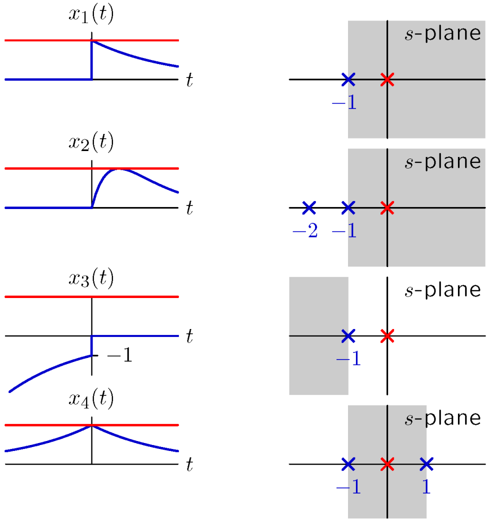

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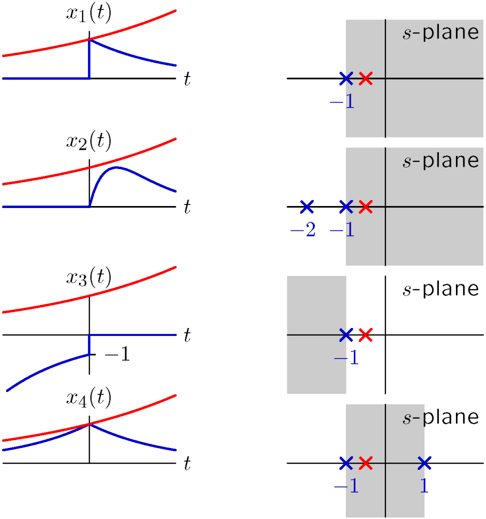

Time-Domain Interpretation of ROC

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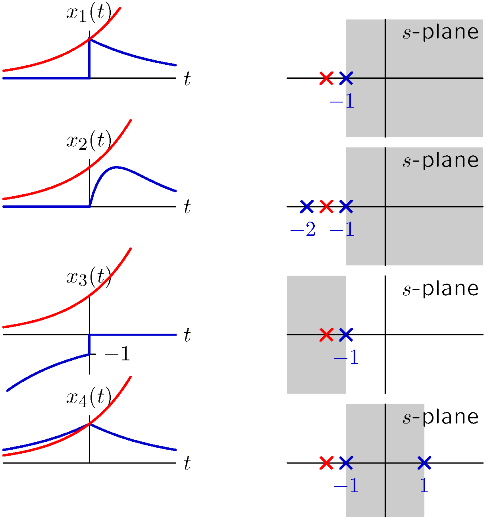


Check Yourself

The Laplace transform $\frac{2s}{s^2-4}$ corresponds to how many of the following signals?

1. $e^{-2t}u(t)+e^{2t}u(t)$
2. $e^{-2t}u(t)-e^{2t}u(-t)$
3. $-e^{-2t}u(-t)+e^{2t}u(t)$
4. $-e^{-2t}u(-t)-e^{2t}u(-t)$

Check Yourself

Expand with partial fractions:

$$\frac{2s}{s^2-4} = \underbrace{\frac{1}{s+2}}_{\text{pole at } -2} + \underbrace{\frac{1}{s-2}}_{\text{pole at } 2}$$

pole	function	right-sided; ROC	left-sided (ROC)
-2	e^{-2t}	$e^{-2t}u(t)$; $\text{Re}(s) > -2$	$-e^{-2t}u(-t)$; $\text{Re}(s) < -2$
2	e^{2t}	$e^{2t}u(t)$; $\text{Re}(s) > 2$	$-e^{2t}u(-t)$; $\text{Re}(s) < 2$

1.	$e^{-2t}u(t) + e^{2t}u(t)$	$\text{Re}(s) > -2 \cap \text{Re}(s) > 2$	$\text{Re}(s) > 2$
2.	$e^{-2t}u(t) - e^{2t}u(-t)$	$\text{Re}(s) > -2 \cap \text{Re}(s) < 2$	$-2 < \text{Re}(s) < 2$
3.	$-e^{-2t}u(-t) + e^{2t}u(t)$	$\text{Re}(s) < -2 \cap \text{Re}(s) > 2$	none
4.	$-e^{-2t}u(-t) - e^{2t}u(-t)$	$\text{Re}(s) < -2 \cap \text{Re}(s) < 2$	$\text{Re}(s) < -2$

Check Yourself

The Laplace transform $\frac{2s}{s^2-4}$ corresponds to how many of the following signals? 3

1. $e^{-2t}u(t) + e^{2t}u(t)$
2. $e^{-2t}u(t) - e^{2t}u(-t)$
3. $-e^{-2t}u(-t) + e^{2t}u(t)$
4. $-e^{-2t}u(-t) - e^{2t}u(-t)$

Solving Differential Equations with Laplace Transforms

Solve the following differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation.

$$\mathcal{L}\left\{\dot{y}(t) + y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

The Laplace transform of a sum is the sum of the Laplace transforms (prove this as an exercise).

$$\mathcal{L}\left\{\dot{y}(t)\right\} + \mathcal{L}\left\{y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

What's the Laplace transform of a derivative?

Laplace transform of a derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t) = \dot{x}(t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \underbrace{\dot{x}(t)}_{\dot{v}}\underbrace{e^{-st}}_{u}dt$$

$$= \underbrace{x(t)}_{v}\underbrace{e^{-st}}_{u}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x(t)}_{v}(\underbrace{-se^{-st}}_{\dot{u}})dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s) = s\int_{-\infty}^{\infty} x(t)e^{-st}dt = sX(s)$$

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$\mathcal{L}\left\{\dot{y}(t)\right\} + \mathcal{L}\left\{y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

Let $Y(s)$ represent the Laplace transform of $y(t)$.

Then $sY(s)$ is the Laplace transform of $\dot{y}(t)$.

$$sY(s) + Y(s) = \mathcal{L}\left\{\delta(t)\right\}$$

What's the Laplace transform of the impulse function?

Laplace transform of the impulse function

Let $x(t) = \delta(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ e^{-st}\Big|_{t=0}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ 1\ dt \\ &= 1 \end{aligned}$$

Sifting property: $\delta(t)$ **sifts** out the value of e^{-st} at $t = 0$.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\left\{\delta(t)\right\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$$y(t) = e^{-t}u(t)$$

Notice that we solved the differential equation $\dot{y}(t)+y(t) = \delta(t)$ without computing homogeneous and particular solutions.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$$y(t) = e^{-t}u(t) \qquad \text{(why not } y(t) = -e^{-t}u(-t)\text{ ?)}$$

Notice that we solved the differential equation $\dot{y}(t)+y(t) = \delta(t)$ without computing homogeneous and particular solutions.

Solving Differential Equations with Laplace Transforms

Summary of method.

Start with differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation:

$$sY(s) + Y(s) = 1$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

Take inverse Laplace transform (by recognizing form of transform):

$$y(t) = e^{-t}u(t)$$

Solving Differential Equations with Laplace Transforms

Recognizing the form ...

Is there a more systematic way to take an inverse Laplace transform?

Yes ... and no.

Formally,

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

but this integral is not generally easy to compute.

This equation can be useful to prove theorems.

We will find better ways (e.g., partial fractions) to compute inverse transforms for common systems.

Solving Differential Equations with Laplace Transforms

Example 2:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$

Laplace transform:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1$$

Solve:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Inverse Laplace transform:

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

These forward and inverse Laplace transforms are easy if

- differential equation is linear with constant coefficients, and
- the input signal is an impulse function.

Properties of Laplace Transforms

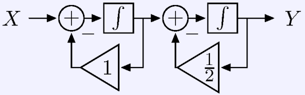
The use of Laplace Transforms to solve differential equations depends on several important properties.

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\frac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\frac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(s)}{s}$	$\supset \Big(R \cap \big(\operatorname{Re}(s) > 0$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

Concept Map: Continuous-Time Systems

Where does Laplace transform fit in?

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Continuous-Time Systems

Where does Laplace transform fit in?

1. Link from differential equation and system function:

Start with differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

Take the Laplace transform of a each term:

$$2s^2 Y(s) + 3sY(s) + Y(s) = 2X(s)$$

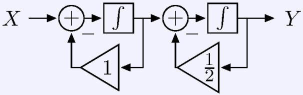
Solve for system function:

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Continuous-Time Systems

Where does Laplace transform fit in?

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Continuous-Time Systems

This same development shows an even more important relation.

2. Link between system function and impulse response:

Differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

If $x(t) = \delta(t)$ then $y(t)$ is the impulse response $h(t)$.

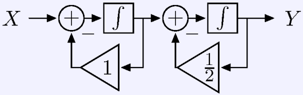
If $X(s) = 1$ then $Y(s) = H(s)$.

System function is Laplace transform of the impulse response!

Concept Map: Continuous-Time Systems

Where does Laplace transform fit in?

Block Diagram



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$\mathcal{L}$

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System Function

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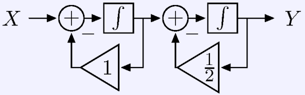
Concept Map: Continuous-Time Systems

Where does Laplace transform fit in? many more connections

Block Diagram



System Functional

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System Function

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LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\dfrac{d}{dt} \leftrightarrow s, \quad \dfrac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\dfrac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$ $\begin{array}{l} \longleftarrow \text{roots of numerator} \Rightarrow \textit{zeros} \\ \longleftarrow \text{roots of denominator} \Rightarrow \textit{poles} \end{array}$

Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s
(in general, LTIs described by LCCDEs), i. e.

$$X(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)}\ , \quad \mathrm{N}(s), \mathrm{D}(s) \text{ — polynomials in } s$$

- Roots of $\mathrm{N}(s)$ = *zeros* of $X(s)$

- Roots of $\mathrm{D}(s)$ = *poles* of $X(s)$

- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$
has a rational Laplace transform.

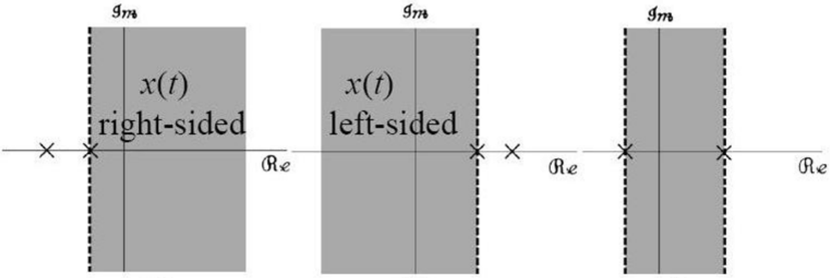
ROC for Rational Transforms

If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Suppose $X(s)$ is rational, then

(a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.

(b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



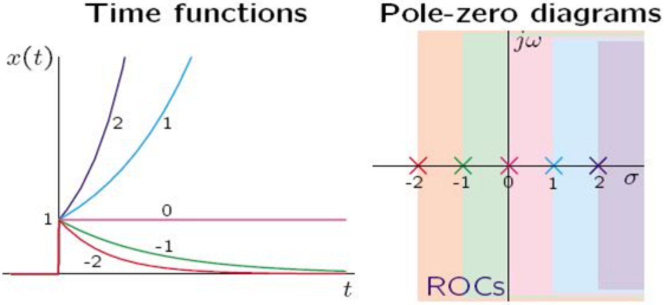
(a) (b) (c)

Relation Between Time Functions and Pole-zero Diagrams

Consider the causal exponential time function and its Laplace transform

$$x(t) = e^{\alpha t}u(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \frac{1}{s-\alpha} \text{ for } \sigma > \alpha$$

The following shows both the time functions and the pole-zero diagrams for 5 different values of α.



Relation Between Time Functions and Pole-zero Diagrams

Pole characteristics	Time function
On real axis	Exponential
On imaginary axis	Sinusoid
In complex s plane	Exponentially modulated sinusoid
Negative real part	Bounded
Positive real parts	Unbounded
Far from origin of s plane	Rapid time course

CT System Function Properties

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

H(s) = “system function”

$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

Question:

If the ROC of $H(s)$ is a right-half plane, is the system causal?

Ex.

$$\begin{aligned} H(s) &= \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided} \\ h(t) &= \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t \to t+T} = e^{-t}u(t)|_{t \to t+T} \\ &= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t < 0 \end{aligned}$$

Non-causal

Properties of CT Rational System Function

a) However, if $H(s)$ is ***rational***, then

The system is causal $\Leftrightarrow$ The ROC of $H(s)$ is to the right of the rightmost pole

b) If $H(s)$ is rational and is the system function of a causal system, then

The system is stable $\Leftrightarrow$ $j\omega$-axis is in ROC $\Leftrightarrow$ all poles are in LHP

Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then

$$x(0^+) = \lim_{s \to \infty} sX(s)\,.$$

Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then

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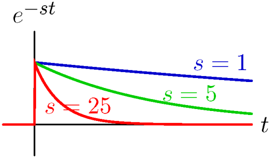
Consider $\displaystyle\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to\infty}\int_{0}^{\infty} x(t)\,se^{-st}dt.$

As $s \to \infty$ the function e^{-st} shrinks towards 0.



Area under e^{-st} is $\dfrac{1}{s}$ $\quad\to\quad$ area under se^{-st} is 1 $\quad\to\quad$ $\displaystyle\lim_{s\to\infty} se^{-st} = \delta(t)$!

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\int_0^{\infty} x(t)se^{-st}dt \to \int_0^{\infty} x(t)\delta(t)dt = x(0^+)$$

(the 0^+ arises because the limit is from the right side.)

Final Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$

$$x(\infty) = \lim_{s \to 0} sX(s)\,.$$

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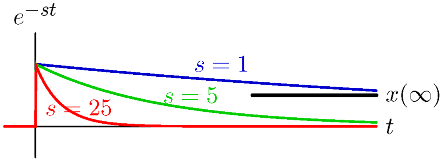
Consider $\lim_{s \to 0} sX(s) = \lim_{s \to 0} s \int_{-\infty}^{\infty} x(t) e^{-st} dt = \lim_{s \to 0} \int_{0}^{\infty} x(t)\, se^{-st} dt.$

As $s \to 0$ the function e^{-st} flattens out. But again, the area under se^{-st} is always 1.



As $s \to 0$, area under se^{-st} monotonically shifts to higher values of t (e.g., the average value of se^{-st} is $\frac{1}{s}$ which grows as $s \to 0$).

In the limit, $\lim_{s \to 0} sX(s) \to x(\infty)\,.$

Assignments

- Reading Assignment: Chap. 9.1-9.3, 9.5-9.9
- Homework 2: Due by Mar. 21, 2024