Problem Set 2

Problem 1 Solution

(a) From the given block diagram, we can deduce that

$$(X - \frac{1}{2}AY)A - \frac{3}{2}AY = Y$$

Simplify eq. (1), we can get

$$\frac{Y}{X} = \frac{A}{1 + \frac{3}{2}A + \frac{1}{2}A^2}$$

(b) Consider the equation

$$1 + \frac{3}{2}A + \frac{1}{2}A^2 = \frac{1}{2}(2 + 3A + A^2) = (1 + A)(1 + \frac{1}{2}A)$$

So the poles of the system is -1 and $-\frac{1}{2}$

(c)

$$\frac{Y}{X} = \frac{A}{1 + \frac{3}{2}A + \frac{1}{2}A^2} = \frac{2A}{1 + A} - \frac{A}{1 + \frac{1}{2}A}$$

So the impulse response of the system is

$$y(t) = (2e^{-t} - e^{-\frac{1}{2}t})u(t)$$

Problem 2 Solution

(a) The output $y[n]$ can be viewed as the superposition of $y_1[n] = u[n]$ and $y_2[n] = \delta[n]$, while the input is $x[n] = u[n]$.

We can see that

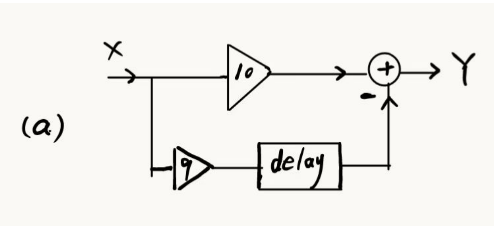
$$Y_1 = X, \ \frac{X}{Y_2} = \frac{1}{1 - R}$$

So

$$Y = Y_1 + 9Y_2 = X + 9(1 - R)X = 10X - 9RX$$

Finally, we can get the difference equation of the system

$$y[n] = 10x[n] - 9x[n-1] \quad (n \geq 0)$$



(b) The input $x[n]$ can be viewed as the superposition of $x_1[n] = u[n]$ and $x_2[n] = 9\delta[n]$

We can see that

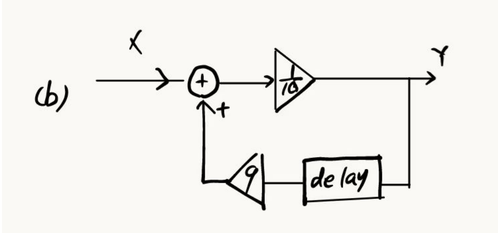
$$Y = X_1, \quad Y = \frac{1}{9(1 - R)}X_2$$

So

$$X = X_1 + X_2 = Y + 9(1 - R)Y = 10Y - 9RY$$

Finally, we can get the difference equation of the system

$$x[n] = 10y[n] - 9y[n-1]$$



(c) The differential equation in **(b)** can be obtained by replacing y with x and x with y in the differential equation in **(a)**.

Problem 3 Solution

(a) $X_1(s) = \int_{-\infty}^{+\infty} e^{-2(t-3)}u(t-3)e^{-st}dt = e^6\int_3^{+\infty} e^{-(2+s)t}dt = \frac{e^{-3s}}{s+2} \quad (Re\{s\} > -2)$

(b) $X_2(s) = \int_{-\infty}^{+\infty} |t|e^{-|t|}e^{-st}dt = \int_0^{+\infty} te^{-t}(e^{st}+e^{-st})dt = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2} \quad (-1 < Re\{s\} < 1)$

(c) $X_3(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_0^1 te^{-st}dt + \int_1^2 e^{-st}dt + \int_2^3 (3-t)e^{-st}dt = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s})$ (entire s-plane)

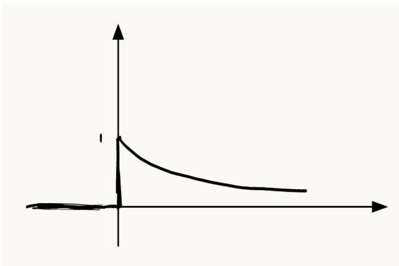
Problem 4 Solution

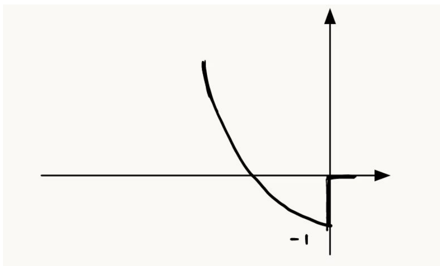
(a) $X_1(s) = \frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$, which has a pole at $s = -1$

So there are two signals of which the Laplace transforms are coincide with $X_1(s)$

$$\begin{aligned} x_1(t) &= e^{-t}u(t) + te^{-t}u(t) = (t+1)e^{-t}u(t) \qquad &(ROC: Re\{s > -1\} \quad or \\ x_2(t) &= -e^{-t}u(-t) - te^{-t}u(-t) = -(t+1)e^{-t}u(-t) \quad &(ROC: Re\{s < -1\}) \end{aligned}$$

Sketch the signals



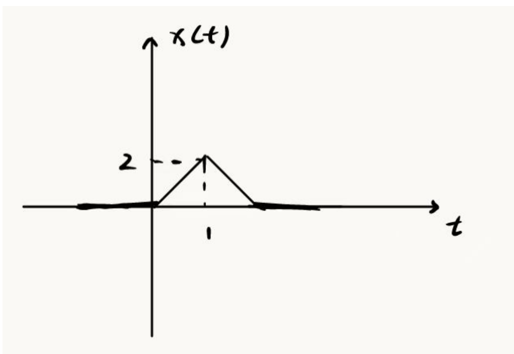


(b) $X_2(s) = \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - 2\frac{e^{-s}}{s^2}$

So there are two signals of which the Laplace transforms are coincide with $X_2(s)$

$$\begin{aligned} x_1(t) &= tu(t) + (t-2)u(t-2) - 2(t-1)u(t-1) \quad or \\ x_2(t) &= -tu(-t) - (t-2)u(-t+2) + 2(t-1)u(-t+1) \end{aligned}$$

Sketch the signals and we can see that these two signals are **the same** actually.



It's easy to see that the ROC is the entire s-plane.

Problem 5 Solution

Consider a even function $x(t)$, and the Laplace transform can be represented as

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} x(t)e^{-st}dt + \int_{0}^{+\infty} x(t)e^{-st}dt = \int_{+\infty}^{0} -x(-t)e^{st}dt + \int_{0}^{+\infty} x(t)e^{-st}dt$$

Since $x(t)$ is an even signal, so (14) can be simplified as

$$X(s) = \int_{0}^{+\infty} x(t)(e^{st} + e^{-st})dt = X(-s)$$

1. **Figure1** can represent Laplace transform of signal $\mathbf{x(t) = A(e^t u(-t) + e^{-t} u(t))}$
2. **Figure2** can't represent Laplace transform because there is only one pole at $s = -1$, conflicting to $H(s) = H(-s)$
3. **Figure3** can represent Laplace transform of signal $\mathbf{x(t) = A(2\delta(t) - e^t u(-t) - e^{-t} u(t))}$
4. **Figure4** cannot represent Laplace transform because there is no intersection of $Re\{s < 0\}$ and $Re\{s > 0\}$

The Laplace transform of an even signal is also even and the ROC is Symmetric about the jw-axis without containing any poles.

Problem 6 Solution

(a)

1. $x(0) = \lim\limits_{s\to\infty} sX(s) = \lim\limits_{s\to\infty} 1 - e^{-sT} = 1$

   $x(\infty) = \lim\limits_{s\to 0} sX(s) = \lim\limits_{s\to 0} 1 - e^{-sT} = 0$
2. $x(0) = \lim\limits_{s\to\infty} sX(s) = \lim\limits_{s\to\infty} \frac{s^3+2s^2}{[(s+1)^2+1]^2} = 0$

   $x(\infty) = \lim\limits_{s\to 0} sX(s) = \lim\limits_{s\to 0} \frac{s^3+2s^2}{[(s+1)^2+1]^2} = 0$

(b)

1. $x(t) = u(t) - u(t-T)$, so $x(0) = 1$ and $x(\infty) = 0$
2. $x(t) = te^{-t}cos(t)u(t)$, so $x(0) = 0$ and $x(\infty) = 0$

Problem 7 Solution

(a) $X_1(z) = \sum\limits_{n=-\infty}^{+\infty} x_1[n]z^{-n} = \sum\limits_{3}^{+\infty}(\frac{1}{2z})^n = \frac{1}{4z^2(2z-1)}, \quad |z| > \frac{1}{2}$

(b) $X_2(z) = \sum\limits_{n=-\infty}^{+\infty} x_2[n]z^{-n} = \sum\limits_{0}^{+\infty}(1+n)(\frac{1}{3z})^n = \sum\limits_{0}^{+\infty}(\frac{1}{3z})^n - z(\sum\limits_{0}^{+\infty}(\frac{1}{3z})^n)' = \frac{z^2}{(z-\frac{1}{3})^2}, \quad |z| > \frac{1}{3}$

(c) $X_3(z) = \sum\limits_{n=-\infty}^{+\infty} x_3[n]z^{-n} = -z + \frac{1}{z} + \frac{1}{z^2}, \quad 0 < |z| < +\infty$

Problem 8 Solution

(a) $X_1(z) = \frac{1}{z(z-1)^2} = \frac{1}{z} + 2 - \frac{2}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2}$, which has poles at $s = 0$ and $s = 1$

So there are two possible signals

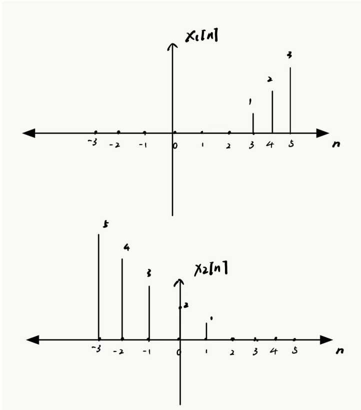
- $x_1[n] = \delta[n-1] + 2\delta[n] - 2u[n] + nu[n] = \delta[n-1] + 2\delta[n] + (n-2)u[n], \quad |z| > 1$
- $x_2[n] = \delta[n-1] + 2\delta[n] + 2u[-n-1] - nu[-n-1] = \delta[n-1] + 2\delta[n] - (n-2)u[-n-1], \quad 0 < |z| < 1$

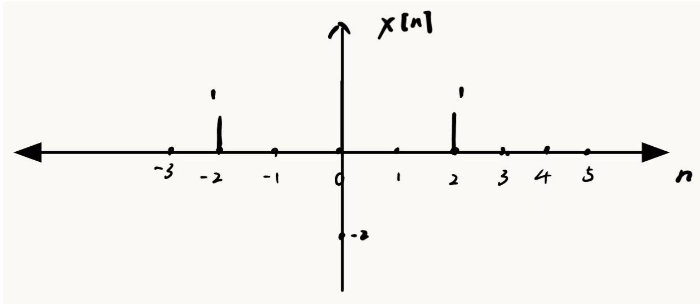
The sketch of signals is demonstrated in left picture.

(b) $X_2(z) = z^2 - 2 + \frac{1}{z^2}$, which has poles at $s = 0$ and $s = \infty$

So the signal is $x[n] = \delta[n+2] + \delta[n-2] - 2\delta[n], \quad 0 < |z| < +\infty$

The sketch of signals is demonstrated in right picture.





Problem 9 Solution

(a) Since $x_e[n] = \frac{1}{2}(x[n] + x[-n])$, we can represent $X_e(z)$ as

$$X_e(z) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}(x[n] + x[-n])z^{-n} = \frac{1}{2}\sum_{n=-\infty}^{+\infty}x[n](z^{-n} + z^{n}) = \frac{1}{2}(X(z) + X(\frac{1}{z}))$$

So the Z transform of $x_e[n]$ exist only both z and $\frac{1}{z}$ lie on the ROC , the condition of which automatically be met if

$$\mathbf{r_0 < 1 < r_1}$$

(b) Assume $x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n]$, so $x_e[n] = x[n]$

The Z transform of $x_e[n]$ is

$$X_e(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{1}{1 - 0.5z}$$

$ROC : 0.5 < |z| < 2$

Problem 10 Solution

(a) **True**. The ROC of a system which is stable and causal is a right-half plane and includes the entire jw-axis, so $H(s)$ converge when $Re\,\{s = 3\}$, which means $h(t)e^{-3t}$ is absolutely integrable.

(b) **False**. The ROC is the region in the s-plane to the right of the rightmost pole, but we can't testify that $-1 + j$ is the rightmost pole.

(c) **True**. $H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}$, where P and Q are all polynomials with real coefficients. So the differential equation relating inputs $x(t)$ and outputs $y(t)$ can be obtained by the inverse Laplace transform of $Y(s)Q(s) - X(s)P(s) = 0$, which has only real coefficients.

(d) **False**. Since $H(s)$ has exactly two zeros at infinity, we can obtain that $\lim_{s\to\infty} H(s) = 0$.

(e) **True**. Since $h(t)$ is real, it's poles and zeros must occur in conjugate pairs, so at least $H(s)$ has four zeros. We have already known that $H(s) \to 0$ when $s \to 0$, so $H(s)$ must have more than four poles to meet the condition.

(f) **False**. We can't assure that there must be zeros at $jw-$axis.

(g) **False**. We can't determine the exact expression of $h(t)$, so we have no idea about the output of $e^{3t}\,sin(t)$

Problem 11 Solution

(a) $F = M\frac{d^2y(t)}{dt^2}$, $F = K(x(t) - y(t))$, $M = K = 1$. Combine all the conditions, we can obtain

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$

(b) Apply Laplace transform to **(a)**

$$\mathscr{L}\left\{\frac{d^2y(t)}{dt^2} + y(t)\right\} = s^2Y(s) + Y(s) = \mathscr{L}\left\{u(t)\right\} = \frac{1}{s}$$

So $Y(s) = \frac{1}{s(s^2+1)}$ Finally we can calculate that $y(t) = (1 - cost)u(t)$

(c) Replace $\frac{d^2y(t)}{dt^2}$ by

$$\frac{\frac{y[n+2]-y[n+1]}{T} - \frac{y[n+1]-y[n]}{T}}{T}$$

The difference equation can be transformed as

$$\frac{\frac{y[n+2]-y[n+1]}{T} - \frac{y[n+1]-y[n]}{T}}{T} + y[n] = x[n]$$

which is

$$y[n+2] - 2y[n+1] + (1+T^2)y[n] = T^2x[n]$$

Apply z transform to the equation, we can get

$$y[n] = \frac{1}{2}[(1+T)^n + (1-T)^n - 2]u[n]$$

(d) Similar to **(c)** , we can obtain

$$(1+T^2)y[n] - 2y[n-1] + y[n-2] = T^2x[n]$$

Apply z transform to the equation, we can get

$$y[n] = -u[n] + \frac{1}{2}(1-T)^{n-1}u[n] + \frac{1}{2}(1+T)^{n-1}u[n]$$

(e) Replace $\frac{d^2y(t)}{dt^2}$ by $\frac{y[n+1]-2y[n]+y[n-1]}{T^2}$, we can obtain

$$y[n+1] - (2-T^2)y[n] + y[n-1] = T^2x[n]$$

Apply z transform to the equation, we can get

$$Y(z) = \frac{z^2}{z-1}\frac{T^2}{z^2+(T^2-2)z+1} = \frac{1}{T^2}(\frac{1}{1-z^{-1}} + \frac{w_1-1}{w_1-w_2}\frac{1}{1-w_1z^{-1}} + \frac{w_2-1}{w_1-w_2}\frac{1}{1-w_2z^{-1}})$$

So

$$y[n] = \frac{1}{T^2}u[n](1 + \frac{w_1-1}{w_1-w_2}w_1^n + \frac{w_2-1}{w_1-w_2}w_2^n) = \frac{1}{T^2}u[n](1 + \frac{w_1^{n+1}-w_2^{n+1}-(w_1^n-w_2^n)}{w_1-w_2})$$

w_1 and w_2 are the roots of $z^2 + (T^2-2)z + 1$.