Signals and Systems

Lecture 11: Continuous-time Fourier Series

Instructor: Prof. Ting Zhang
Zhejiang University

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Adapted from the materials provided on the MIT OpenCourseWare

Fourier Representations

Fourier series represent **signals** in terms of **sinusoids**.

$\rightarrow$ leads to a new representation for **systems** as **filters**.

Fourier Series: Review

Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$ ("analysis" equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ ("synthesis" equation)

Simplifying Math By Using Complex Numbers

Our biggest simplification comes from **Euler's formula**, which relates complex exponentials to trigonometric functions (Leonhard Euler, 1748).

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$.

Richard Feynman called this "the most remarkable formula in mathematics."

Difference: Negative k

The complex exponential form of the series has positive and negative k's.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Only positive values of k are used in the trig form.

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

Q: Why? What does negative k mean?

The negative k's are required by Euler's formula.

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$

$$\cos(k\omega_o t) = \mathrm{Re}\{e^{jk\omega_o t}\} = \frac{1}{2}\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right)$$

$$\sin(k\omega_o t) = \mathrm{Im}\{e^{jk\omega_o t}\} = \frac{1}{2j}\left(e^{jk\omega_o t} - e^{-jk\omega_o t}\right)$$

The negative k do not indicate negative frequencies. They are the mathematical result of representing sinusoids with complex exponentials.

CTFS: Complex Exponential Form

Represent periodic signal f(t) as a sum of harmonically-related *complex exponentials:*

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}$$

Q: How to find coefficients?

We can "sift" out the component at $l\omega_o$ by multiplying both sides by $e^{-jl\omega_o t}$ and integrating over a period.

Let's try it!

$$e^{j2\pi} = 1; e^{j\pi} = -1; e^{j\pi/2} = j;$$

$$\int_T f(t)e^{-j\omega_o lt}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt} e^{-j\omega_o lt} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_o (k-l)t} dt = \begin{cases} Ta_l & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

Solving for a_l provides an explicit formula for the coefficients:

$$a_k = \frac{1}{T}\int_T f(t)e^{-j\omega_o kt}dt\,; \quad \text{where } \omega_o = \frac{2\pi}{T}.$$

Q: why can we do this?

Fourier Series: Orthogonal Decomposition

Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt \qquad \text{("analysis" equation)}$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \qquad \text{("synthesis" equation)}$$

We can think of Fourier series as an **orthogonal decomposition**.

Orthogonal Decompositions

Vector representation of 3-space: let $\bar{r}$ represent a vector with components $\{x$, y, and $z\}$ in the $\{\hat{x}$, $\hat{y}$, and $\hat{z}\}$ directions, respectively.

$$\begin{aligned} x &= \bar{r} \cdot \hat{x} \\ y &= \bar{r} \cdot \hat{y} \\ z &= \bar{r} \cdot \hat{z} \end{aligned} \qquad \text{(“analysis” equations)}$$

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \text{(“synthesis” equation)}$$

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Fourier series: let $x(t)$ represent a signal with harmonic component $\{a_0,\ a_1,\ \ldots,\ a_k\}$ for harmonics $\{e^{j0t},\ e^{j\frac{2\pi}{T}t},\ \ldots,\ e^{j\frac{2\pi}{T}kt}\}$ respectively.

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y &= \bar{r} \cdot \hat{y} \qquad \text{(“analysis” equations)} \\
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$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \text{(“synthesis” equation)}$$

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Orthogonal Decompositions

Integrating over a period **sifts** out the k^{th} component of the series.

Sifting as a dot product:

$$x = \bar{r} \cdot \hat{x} \equiv |\bar{r}||\hat{x}| \cos\theta$$

Sifting as an inner product:

$$a_k = e^{j\frac{2\pi}{T}kt} \cdot x(t) \equiv \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

where

$$a(t) \cdot b(t) = \frac{1}{T}\int_T a^*(t)b(t)dt\,.$$

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where

$$a(t) \cdot b(t) = \frac{1}{T}\int_T a^*(t)b(t)dt\,.$$

The complex conjugate (*) makes the inner product of the k^{th} and m^{th} components equal to 1 iff $k = m$:

$$\frac{1}{T}\int_T \left(e^{j\frac{2\pi}{T}kt}\right)^* \left(e^{j\frac{2\pi}{T}mt}\right)dt = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt}e^{j\frac{2\pi}{T}mt}dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

Check Yourself

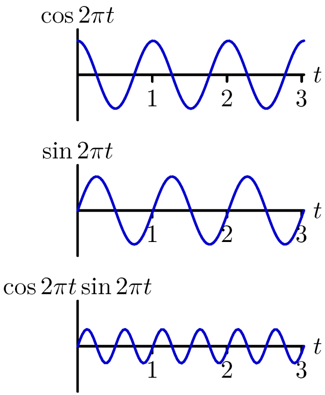
How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$?

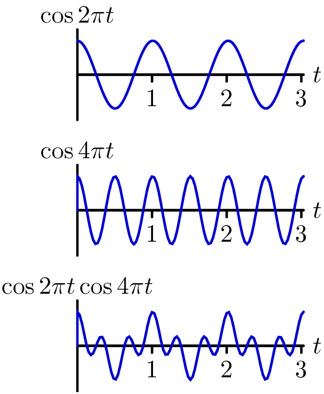


$\int_0^3 dt = 0$ therefore YES

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$?

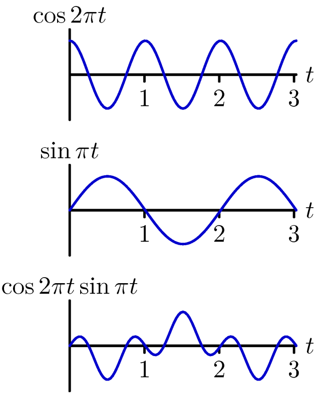


$\int_0^3 dt = 0$ therefore YES

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?



$\int_0^3 dt \neq 0$ therefore NO

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp e^{2\pi t}$?

$$e^{2\pi t} = \cos 2\pi t + j \sin 2\pi t$$

$\cos 2\pi t \perp \sin 2\pi t$ but not $\cos 2\pi t$

Therefore NO

Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$? $\surd$
2. $\cos 2\pi t \perp \cos 4\pi t$? $\surd$
3. $\cos 2\pi t \perp \sin \pi t$? $\times$
4. $\cos 2\pi t \perp e^{j2\pi t}$? $\times$

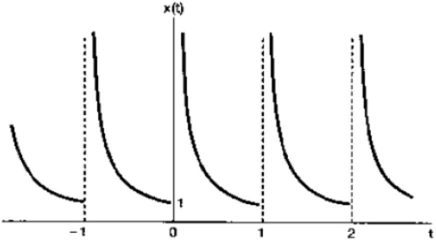
Convergence: The Dirichlet Conditions

- **Condition 1:**

  Over any period, x(t) must be absolutely integrable

$$\int_T |x(t)|dt < \infty$$

Counter example



$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

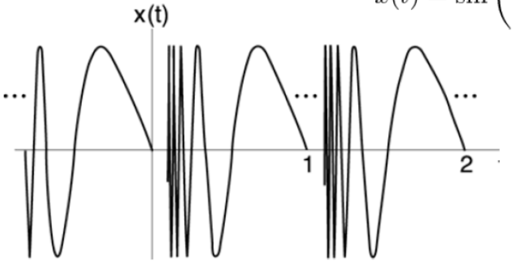
Convergence: The Dirichlet Conditions

- **Condition 2:**

  In any finite interval of time, x(t) is of bounded variation; i.e. there are no more than a finite number of maxima and minima during any single period of the signal.

Counter example

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \qquad 0 < t \leq 1$$

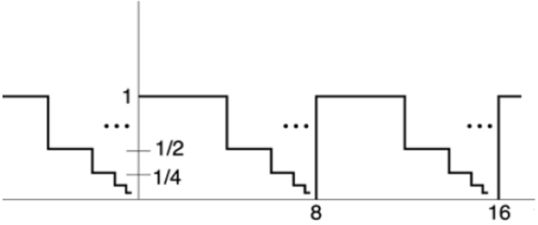


Convergence: The Dirichlet Conditions

- **Condition 3:**

  In any finite interval of time, there are only a finite number of discontinuities. Each of these discontinuities is finite.

Counter example



Review: Fourier Series

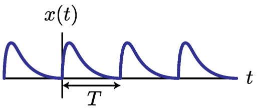
Representing periodic signals as sums of **sinusoids**.

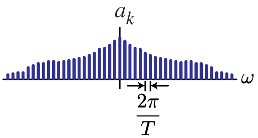
$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

("analysis" equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

("synthesis" equation)





Filtering

Notion of a filter.

LTI systems

- cannot create new frequencies.
- can scale magnitudes and shift phases of existing components.

Filtering

The output of an LTI system is a "filtered" version of the input.

Input: Fourier series → sum of complex exponentials.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

Complex exponentials: eigenfunctions of LTI systems.

$$e^{j\frac{2\pi}{T}kt} \rightarrow H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

Output: same eigenfunctions, amplitudes/phases set by system.

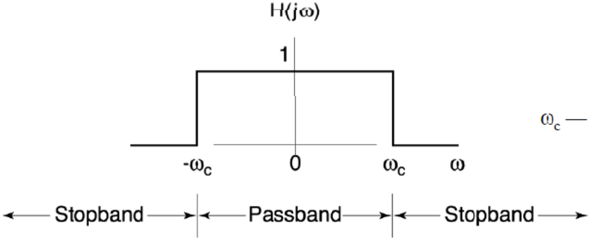
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

Filtering

Idealized Filters

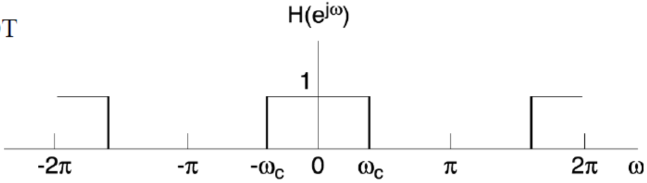
Lowpass filter

CT



ω_c — cutoff frequency

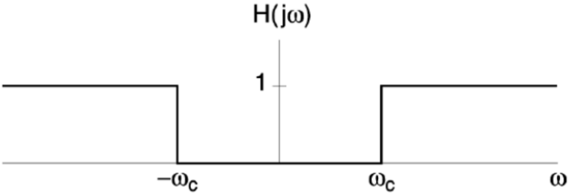
DT



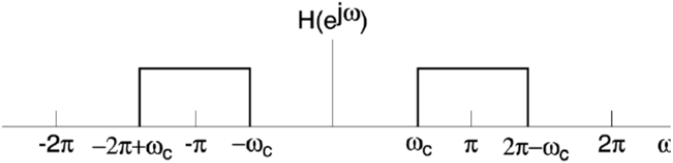
Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

Highpass

CT



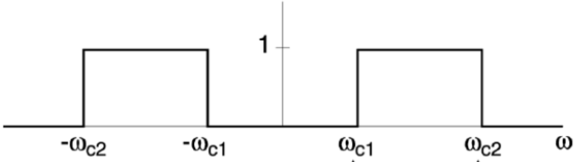
DT



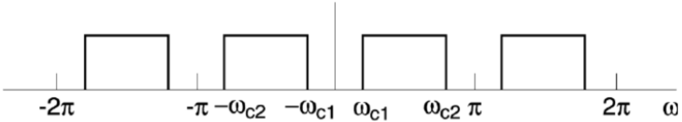
Filtering

Bandpass

CT

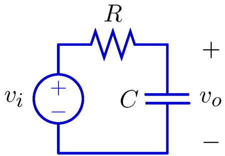


DT



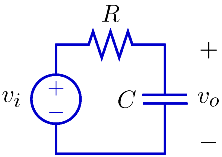
Filtering

Example: Low-Pass Filtering with an RC circuit

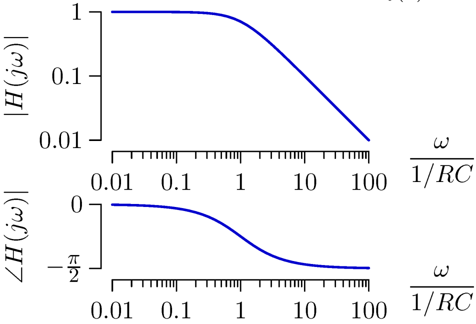


Lowpass Filter

Calculate the frequency response of an RC circuit.

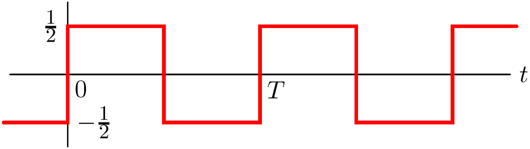


$$
\begin{aligned}
\text{KVL:} \quad & v_i(t) &&= Ri(t) + v_o(t) \\
\text{C:} \quad & i(t) &&= C\dot{v}_o(t) \\
\text{Solving:} \quad & v_i(t) &&= RC\dot{v}_o(t) + v_o(t) \\
& V_i(s) &&= (1 + sRC)V_o(s) \\
& H(s) &&= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

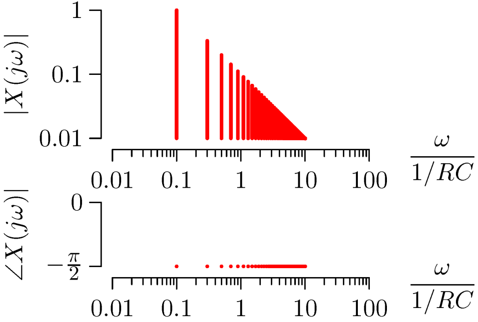


Lowpass Filtering

Let the input be a square wave.

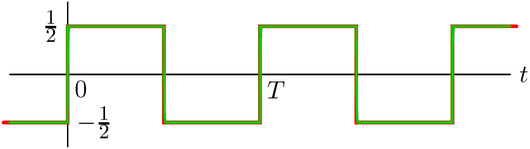


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

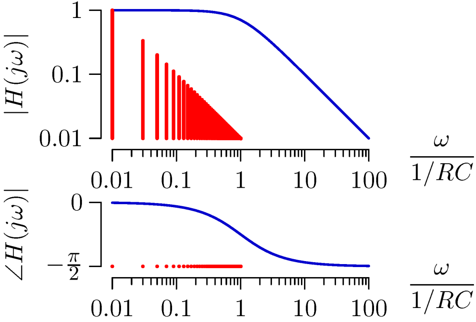


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

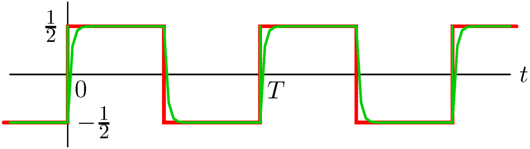


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$

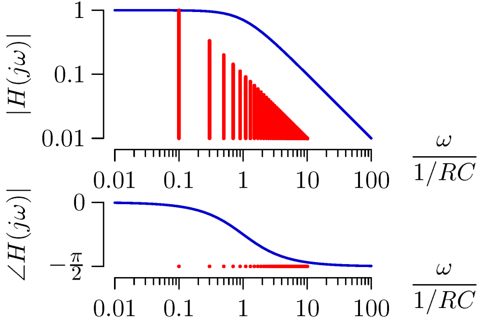


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

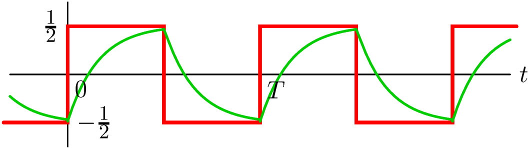


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

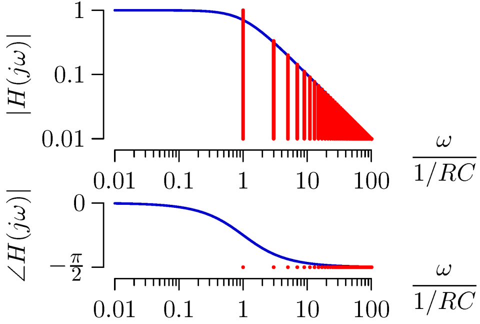


Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

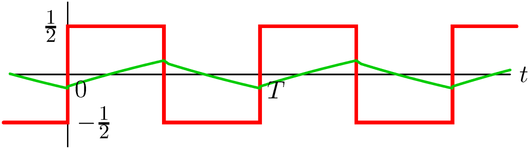


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

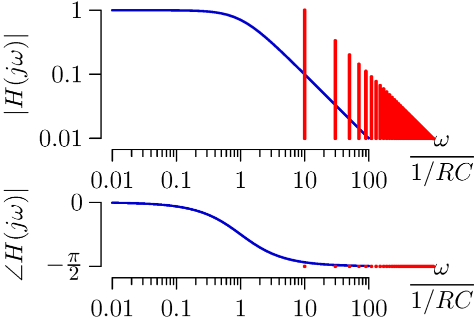


Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



Properties of CTFS

- Linearity, time reversal, symmetry, time shift, time derivative → making it easier to find the Fourier coefficients of a new signal
- The Complex Exponential Form is much easier in math
- Get more familiar with the CE form

Properties of CTFS: Linearity

Consider $y(t) = Ax_1(t) + Bx_2(t)$, where $x_1(t)$ and $x_2(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?

Q: Is $y(t)$ periodic in T?

$$Y[k] = \frac{1}{T}\int_T y(t)e^{-j\frac{2\pi kt}{T}}\,dt \;=\; \frac{1}{T}\int_T (Ax_1(t) + Bx_2(t))e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= A\frac{1}{T}\int_T x_1(t)e^{-j\frac{2\pi kt}{T}}\,dt + B\frac{1}{T}\int_T x_2(t)e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= AX_1[k] + BX_2[k]$$

If $y(t) = Ax_1(t) + Bx_2(t)$, then $Y[k] = AX_1[k] + BX_2[k]$

Q: Why is this property useful?

Properties of CTFS: Time flip (reversal)

- Consider $y(t) = x(-t)$, where $x(t)$ is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y(t)$ must also be periodic in T

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k(-t)}{T}} = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi(-k)t}{T}}$$

Let $m = -k$

$$y(t) = x(-t) = \sum_{m=\infty}^{-\infty} X[-m]e^{j\frac{2\pi mt}{T}} = \sum_{m=-\infty}^{\infty} X[-m]e^{j\frac{2\pi mt}{T}}$$

Since we know

$$y(t) = \sum_{m=-\infty}^{\infty} Y[m]e^{j\frac{2\pi mt}{T}} \Longrightarrow Y[k] = X[-k]$$

If $y(t) = x(-t), Y[k] = X[-k] \quad = X^*[k]$

Q: What happens when the signal is real?

Properties of CTFS: Symmetric and Antisymmetric Parts

Any arbitrary signal $f(t)$ can be written into two parts:

$$f(t) = f_S(t) + f_A(t)$$

$f_S(t)$ is **symmetric** about $t = 0, if \ f_S(t) = f_S(-t)$ for all t.

$f_A(t)$ is **antisymmetric** about $t = 0, if \ f_A(t) = -f_A(-t)$ for all t.

$$f_S(t) = \frac{f(t) + f(-t)}{2} \qquad f_A(t) = \frac{f(t) - f(-t)}{2}$$

If $f(t) = f_S(t) + f_A(t)$ is a real-valued signal and periodic in time with fundamental period T, what are the Fourier Series coefficients of $f_S(\cdot)$ and $f_A(\cdot)$, in terms of F[k]?

Properties of CTFS: Symmetric and Antisymmetric Parts

If $f(t) = f_S(t) + f_A(t)$ is a real valued signal and periodic in time with fundamental period T, what are the Fourier coefficients of $f_S(\cdot)$ and $f_A(\cdot)$, in terms of $\mathrm{F[k]}$?

If $f(t)$ is real valued periodic signal, $F[k] = F^*[-k]$

$$f_S(t) = \frac{f(t)+f(-t)}{2} \xRightarrow[\text{time flip}]{\text{Linearity}} F_S[k] = \frac{F[k]+F[-k]}{2} = \frac{F[k]+F^*[k]}{2} = \frac{2Re(F[k])}{2} = Re(F[k])$$

$$f_A(t) = \frac{f(t)-f(-t)}{2} \xRightarrow[\text{time flip}]{\text{Linearity}} F_A[k] = \frac{F[k]-F[-k]}{2} = \frac{F[k]-F^*[k]}{2} = \frac{2j\cdot Im(F[k])}{2} = j\cdot Im(F[k])$$

The real part of $\mathrm{F[k]}$ comes from the symmetric part of the signal,
the imaginary part of $\mathrm{F[k]}$ comes from the antisymmetric part of the signal

Properties of CTFS: Time Shift

- Consider $y(t) = x(t - t_0)$, where x is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

$$Y[k] = \frac{1}{T}\int_T y(t)e^{-j\frac{2\pi kt}{T}}\,dt = \frac{1}{T}\int_T x(t-t_0)e^{-j\frac{2\pi kt}{T}}\,dt$$

let $u = t - t_0$,
then $t = u + t_0$,
$dt = du$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi k(u+t_0)}{T}}\,du$$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}e^{-j\frac{2\pi kt_0}{T}}\,du$$

$$= e^{-j\frac{2\pi kt_0}{T}}\frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}\,du = e^{-j\frac{2\pi kt_0}{T}}X[k]$$

Each coefficient $Y[k]$ in the series for $y(t)$ is a constant $e^{-jk\omega_0\tau}$ times the corresponding coefficient $X[k]$ in the series for $x(t)$.

Properties of CTFS: Time Derivative

Consider $y(t) = \frac{d}{dt}x(t)$, where $x(t)$ and $y(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Start with the synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$

Then, from the definition of y(·), we have:

$$y(t) = \dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}\right) = \sum_{k=-\infty}^{\infty}\left(j\frac{2\pi k}{T} X[k]\right) e^{j\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} Y[k] e^{j\frac{2\pi kt}{T}}$$

From this form, we can see that $Y[k] = j\frac{2\pi k}{T} X[k]$.

Properties of CTFS: Summary

- Linearity — If $y(t) = Ax_1(t) + Bx_2(t)$, then $Y[k] = AX_1[k] + BX_2[k]$
- Time reversal — If $y(t) = x(-t), Y[k] = X[-k]$
- Time shift — If $y(t) = x(t - t_0)$, then $Y[k] = e^{-j\frac{2\pi k t_0}{T}} X[k]$
- Time derivative — If $y(t) = \dot{x}(t)$, then $Y[k] = j\frac{2\pi k}{T} X[k]$

These can help us find the Fourier coefficients of a new signal without explicitly integrating!

Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.