Signals and Systems

Lecture 7: Bode Diagrams

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Partly adapted from the materials provided on the MIT OpenCourseWare

Review: Convolution

Last time, we saw how a linear, time-invariant (LTI) system can be characterized by its unit-sample/impulse response.

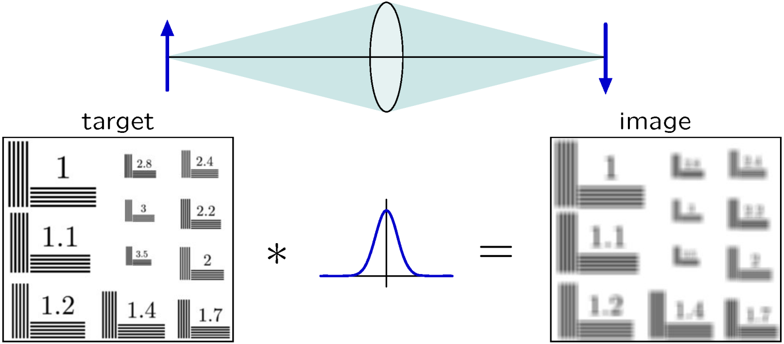
$$\text{DT:}\quad y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\text{CT:}\quad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Characterizing a system by its unit-sample/impulse response is especially insightful for some systems.

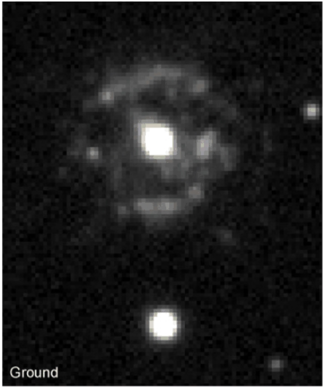
Microscope

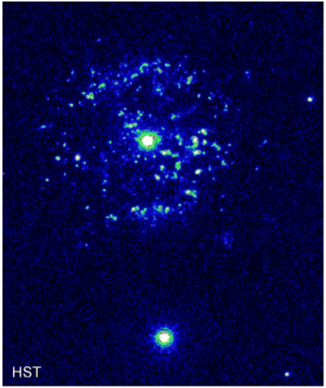
Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).



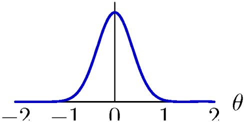
Blurring is inversely related to the diameter of the lens.

Hubble Space Telescope

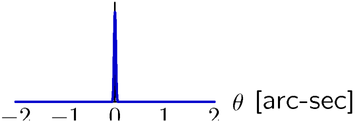




optical + atmospheric blurring



optical blurring

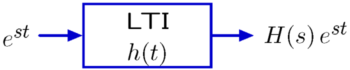


Review: Frequency Response

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)\,e^{st}$$



Eternal sinusoids are sums of complex exponentials.

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

Review: Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

The response to a sum is the sum of the responses,

$$\begin{aligned}
y(t) &= \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\
&= \mathrm{Re}\left\{H(j\omega_0)e^{j\omega_0 t}\right\} \\
&= \mathrm{Re}\left\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right\} \\
&= |H(j\omega_0)|\mathrm{Re}\left\{e^{j\omega_0 t + j\angle H(j\omega_0)}\right\}
\end{aligned}$$

$$y(t) = |H(j\omega_0)| \cos\left(\omega_0 t + \angle\left(H(j\omega_0)\right)\right).$$

Review: Frequency Response

The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated at $s = j\omega$.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

Review: Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

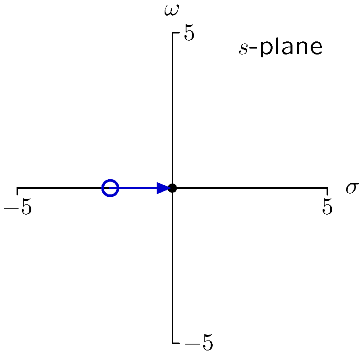
$$|H(s_0)| = |K|\frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

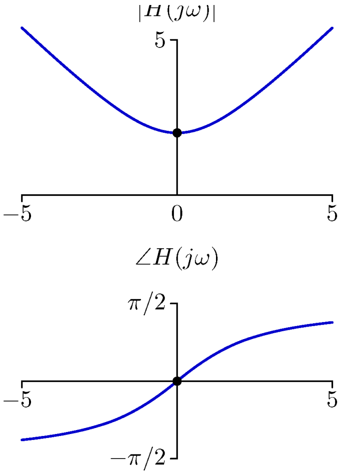
The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots$$

Review: Vector Diagrams

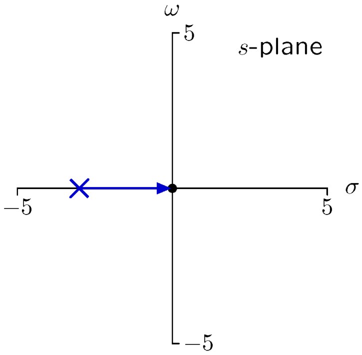
$$H(s) = s - z_1$$

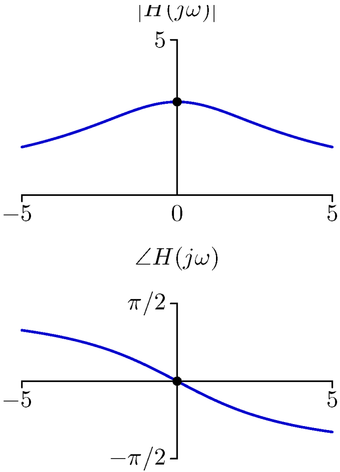




Review: Vector Diagrams

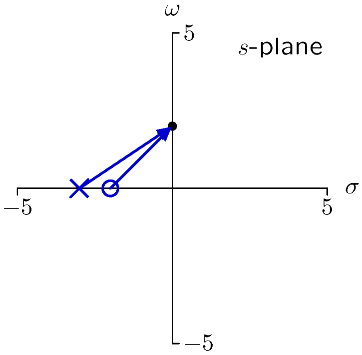
$$H(s) = \frac{9}{s - p_1}$$

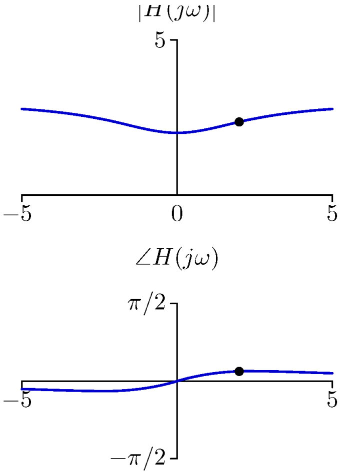




Review: Vector Diagrams

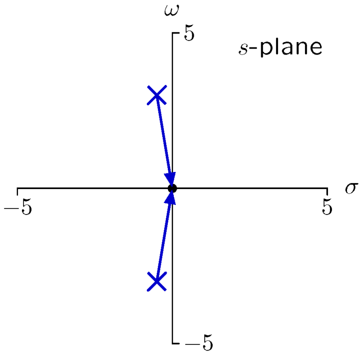
$$H(s) = 3\frac{s - z_1}{s - p1}$$

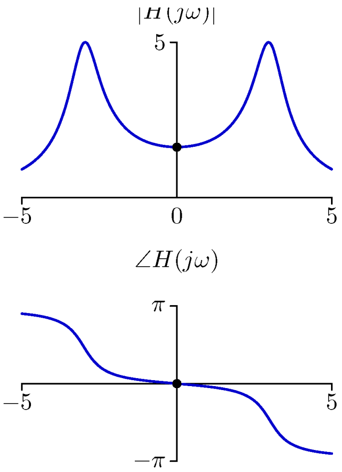




Review: Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$





Review: Frequency Response Summary

LTI systems can be characterized by responses to eternal sinusoids.

Many systems are naturally described by their frequency response.
– audio systems
– mass, spring, dashpot system

Frequency response is easy to calculate from the system function.

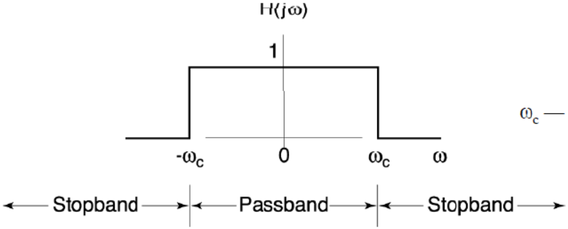
Frequency response lives on the $j\omega$ axis of the Laplace transform.

Filtering

Ideal*ized* Filters

Lowpass filter

CT

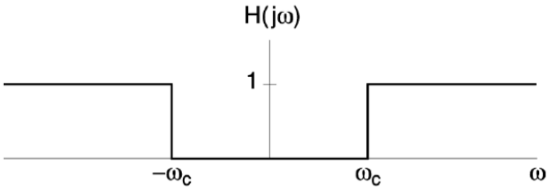


ω_c — cutoff frequency

Filtering

Highpass

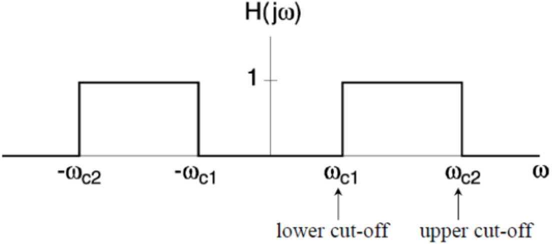
CT



Filtering

Bandpass

CT



Poles and Zeros

Thinking about systems as collections of poles and zeros is an important design concept.

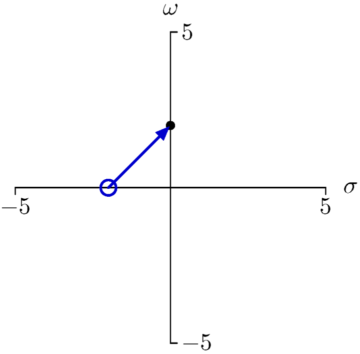
- simple: just a few numbers characterize entire system
- powerful: complete information about frequency response

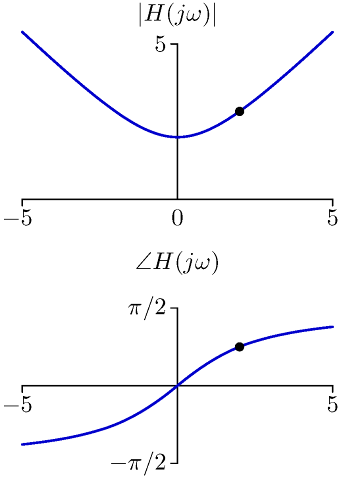
Today: poles, zeros, frequency responses, and Bode plots.

Asymptotic Behavior: Isolated Zero

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$





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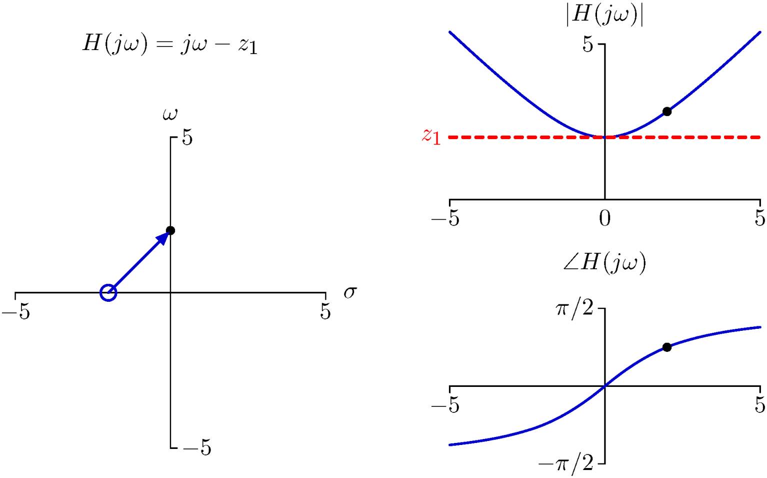

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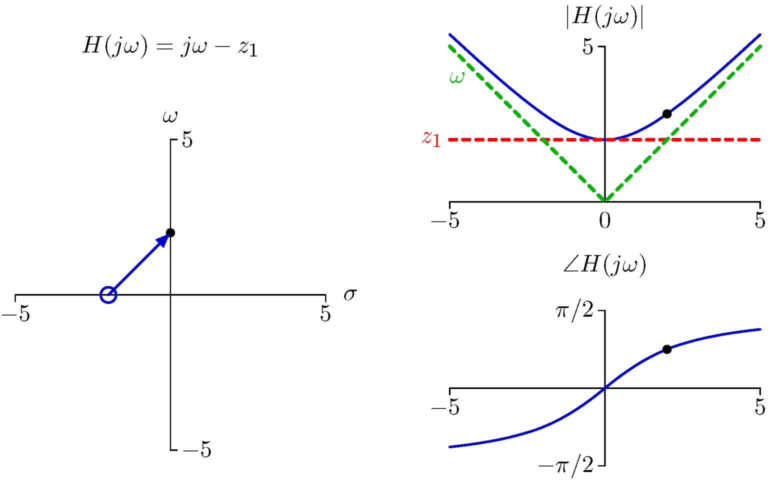
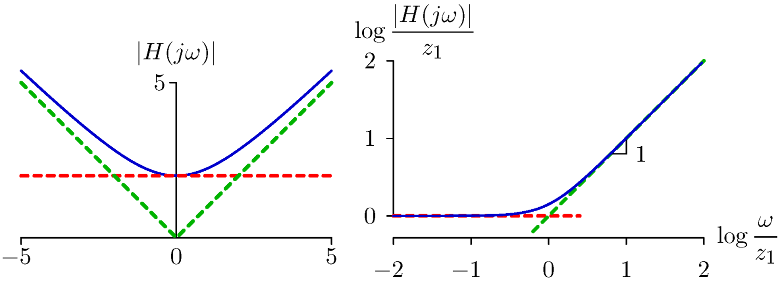


Asymptotic Behavior: Isolated Zero

Two asymptotes provide a good approxmation on log-log axes.

$$H(s) = s - z_1$$



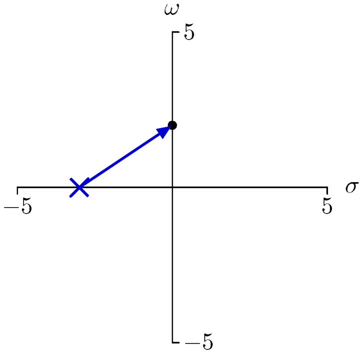
$$\lim_{\omega \to 0} |H(j\omega)| = z_1$$

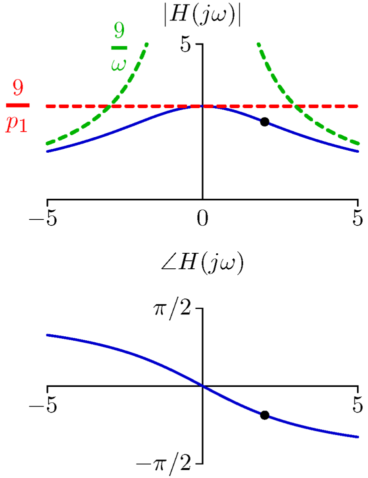
$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

Asymptotic Behavior: Isolated Pole

The magnitude response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$





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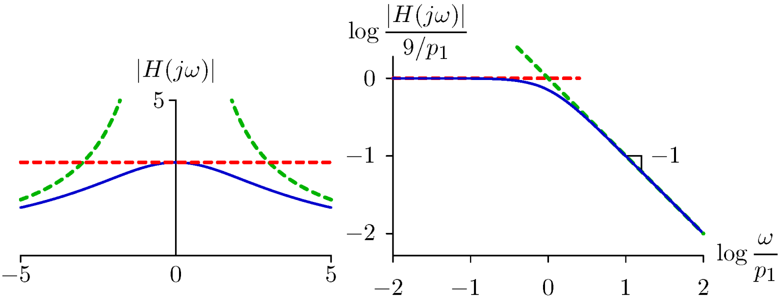

$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{p_1}$$

$$\lim |H(j\omega)| = \frac{9}{\ }$$

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \qquad \text{and} \qquad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

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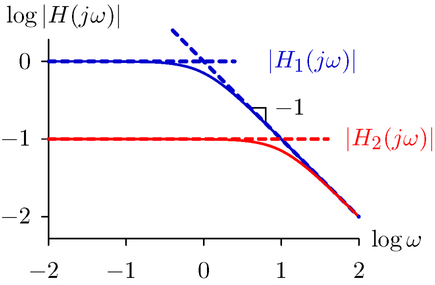

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The former can be transformed into the latter by 3

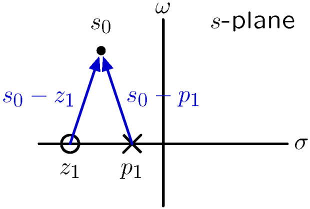
1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions !

Asymptotic Behavior of More Complicated Systems

Constructing $H(s_0)$.

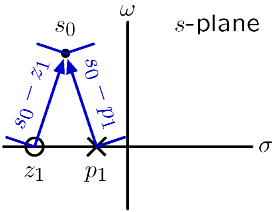
$$H(s_0) = K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \quad \begin{array}{l} \leftarrow \text{ product of vectors for zeros} \\ \\ \leftarrow \text{ product of vectors for poles} \end{array}$$



Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$



Bode Plot

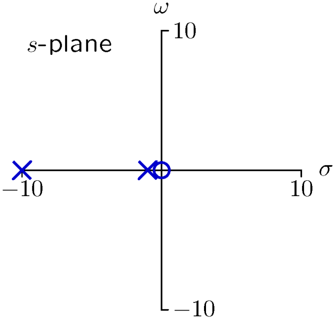
The log of the magnitude is a sum of logs.

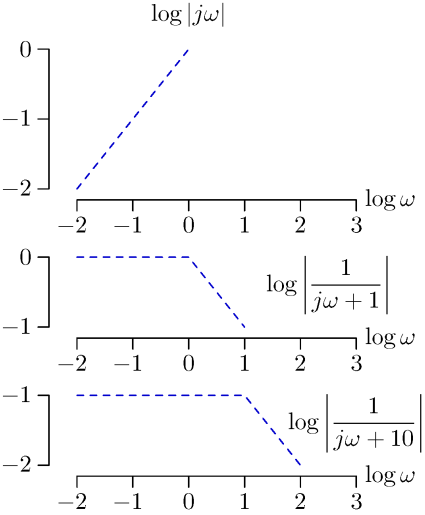
$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q} \left| s_0 - z_q \right|}{\prod_{p=1}^{P} \left| s_0 - p_p \right|}$$

$$\log |H(j\omega)| = \log |K| + \sum_{q=1}^{Q} \log \left| j\omega - z_q \right| - \sum_{p=1}^{P} \log \left| j\omega - p_p \right|$$

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





Bode Plot: Adding Instead of Multiplying

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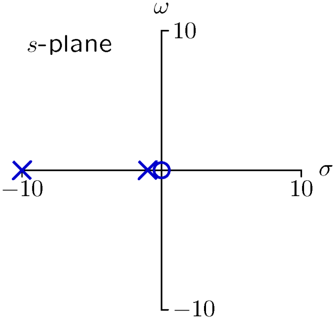

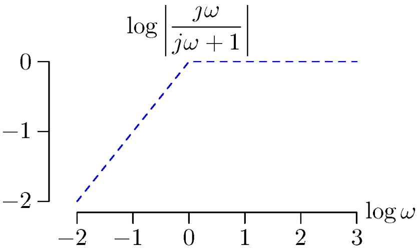

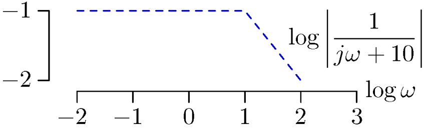

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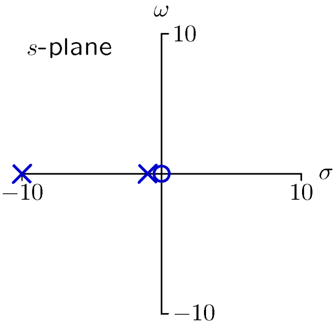

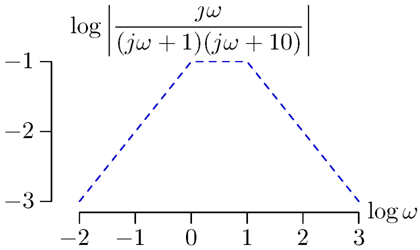

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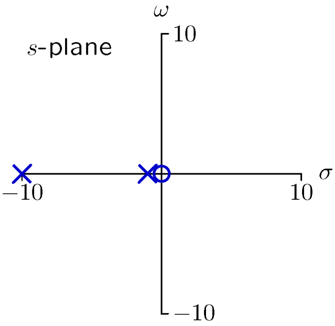
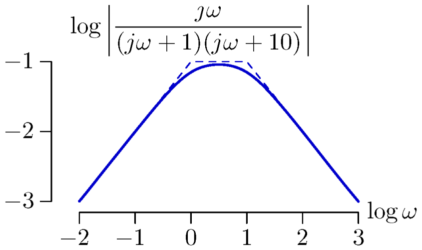
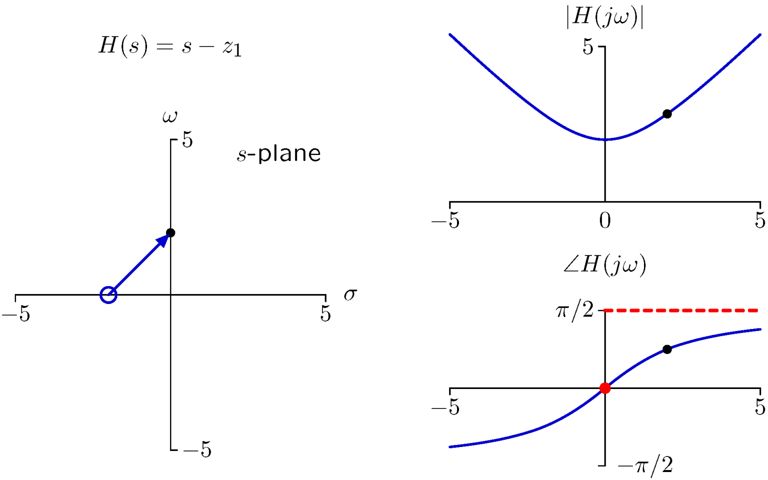




Asymptotic Behavior: Isolated Zero

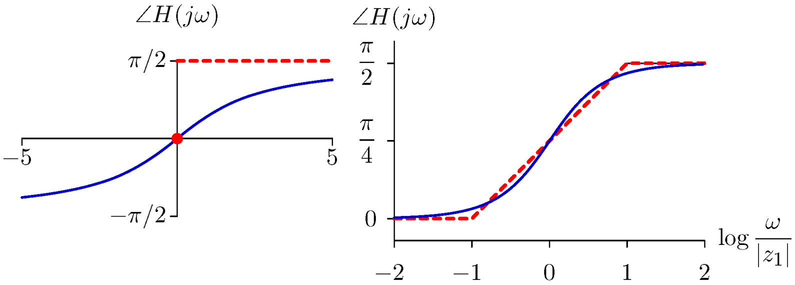
The angle response is simple at low and high frequencies.



Asymptotic Behavior: Isolated Zero

Three straight lines provide a good approxmation versus log ω.

$$H(s) = s - z_1$$



$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = \pi/2$$

Asymptotic Behavior: Isolated Pole

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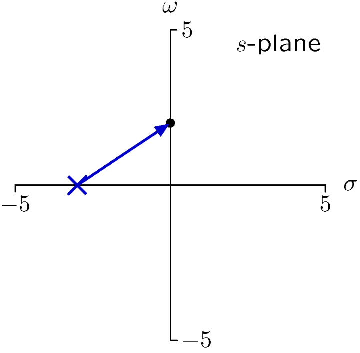

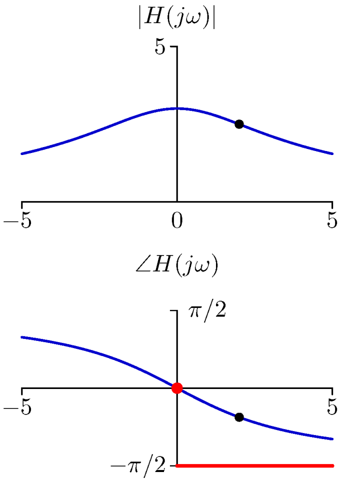

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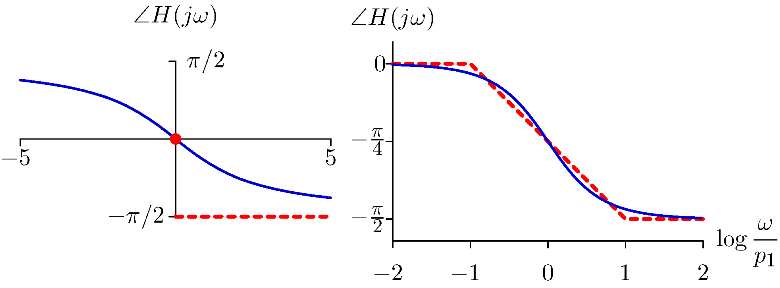


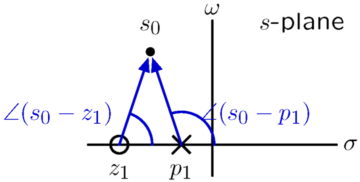
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\pi/2$$

Bode Plot

The angle of a product is the sum of the angles.

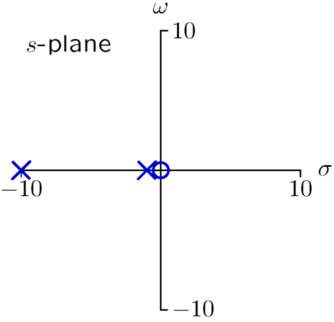
$$\angle H(s_0) = \angle \left(K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right) = \angle K + \sum_{q=1}^{Q} \angle \left(s_0 - z_q\right) - \sum_{p=1}^{P} \angle \left(s_0 - p_p\right)$$

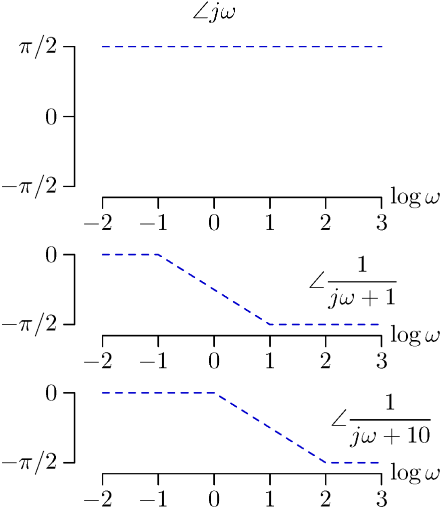


The angle of K can be 0 or π for systems described by linear differential equations with constant, real-valued coefficients.

Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$





Bode Plot

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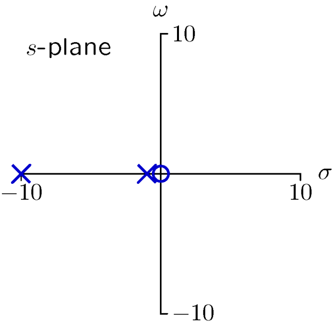
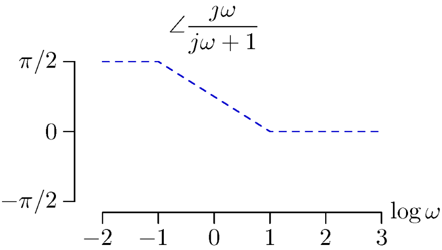
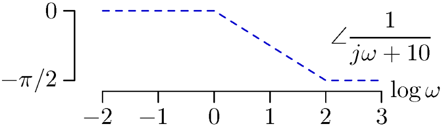
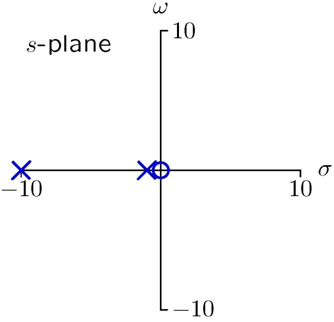


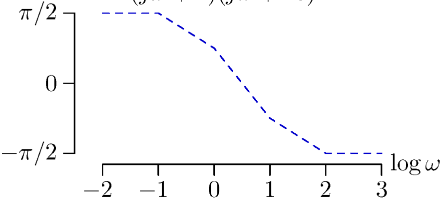




Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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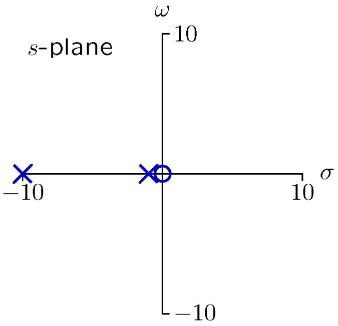
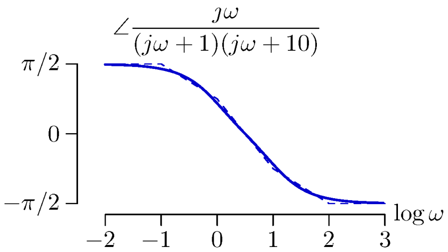




From Frequency Response to Bode Plot

The magnitude of $H(j\omega)$ is a product of magnitudes.

$$|H(j\omega)| = |K| \frac{\prod_{q=1}^{Q} \left|j\omega - z_q\right|}{\prod_{p=1}^{P} \left|j\omega - p_p\right|}$$

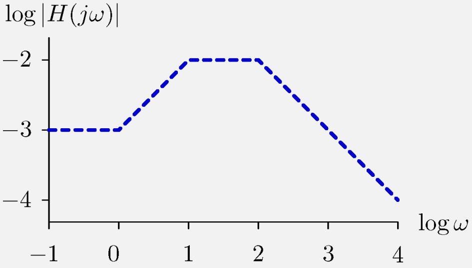
The log of the magnitude is a sum of logs.

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log\left|j\omega - z_q\right| - \sum_{p=1}^{P} \log\left|j\omega - p_p\right|$$

The angle of $H(j\omega)$ is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle\left(j\omega - z_q\right) - \sum_{p=1}^{P} \angle\left(j\omega - p_p\right)$$

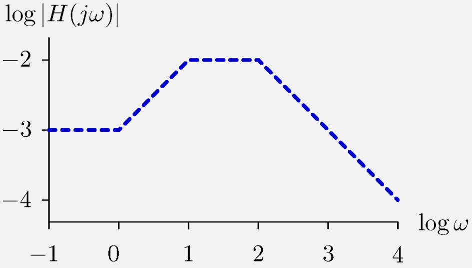
Check Yourself



Which corresponds to the Bode approximation above?

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

Check Yourself



Which corresponds to the Bode approximation above? 2

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$

2. $\dfrac{s+1}{(s+10)(s+100)}$

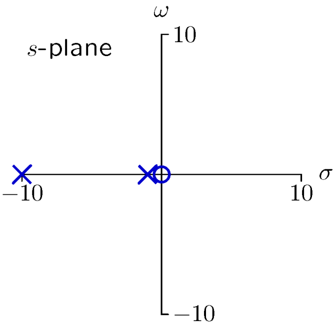
3. $\dfrac{(s+10)(s+100)}{s+1}$

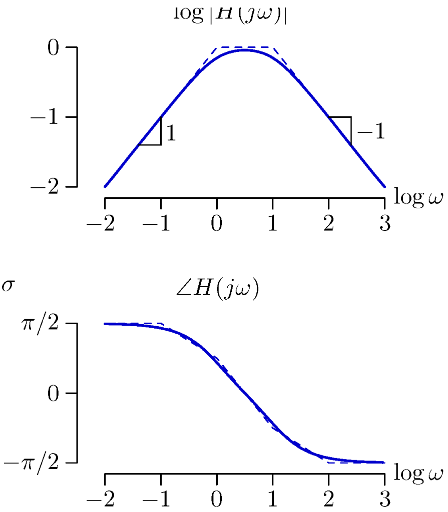
4. $\dfrac{s+100}{(s+1)(s+10)}$

5. none of the above

Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$





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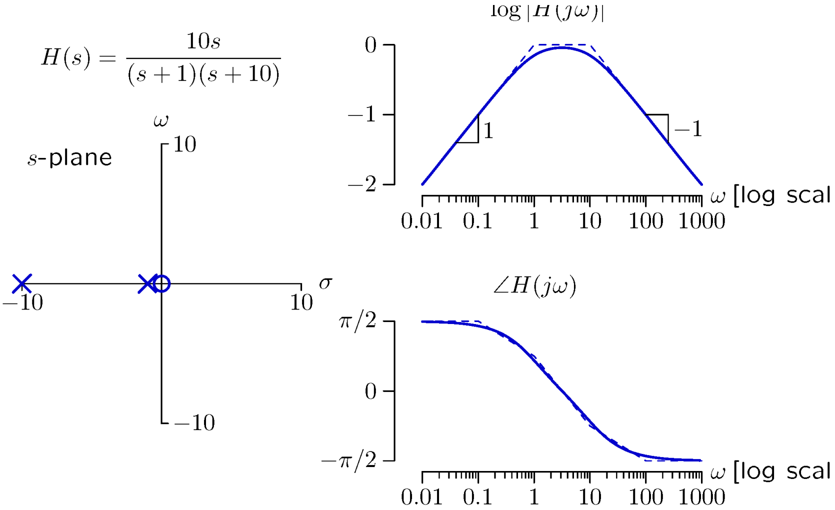

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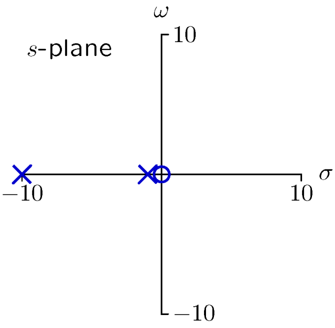

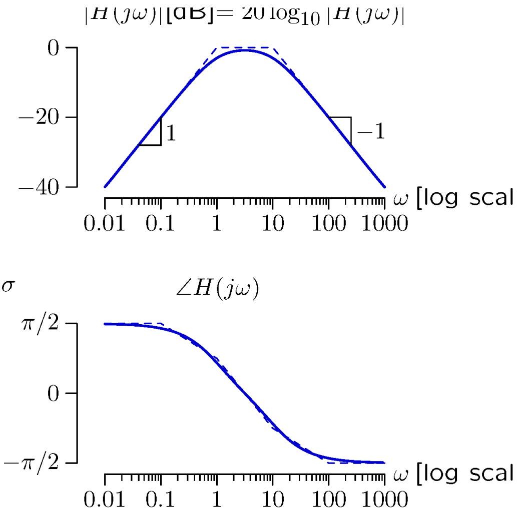

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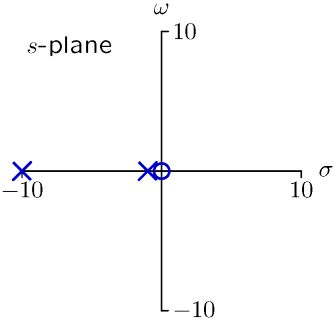
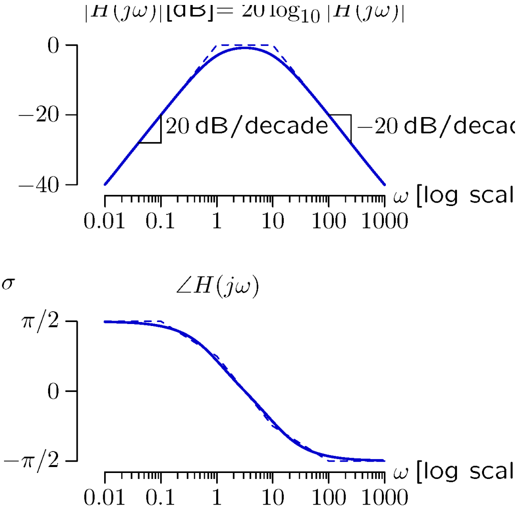
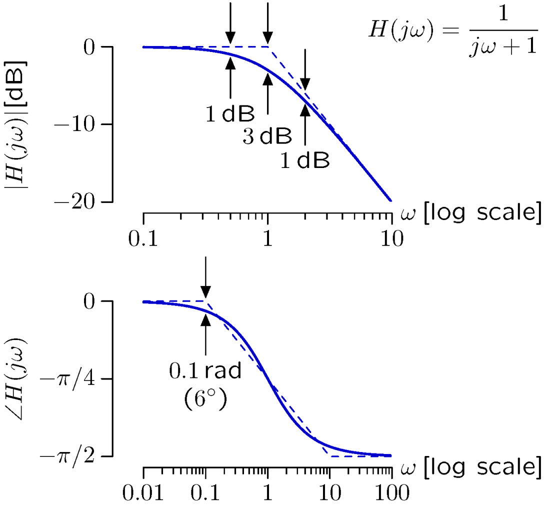




Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

$$H(j\omega) = \frac{1}{j\omega + 1}$$

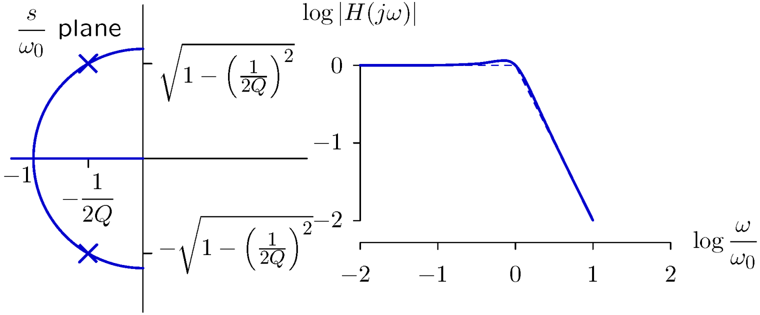


X	$20 \log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB

Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



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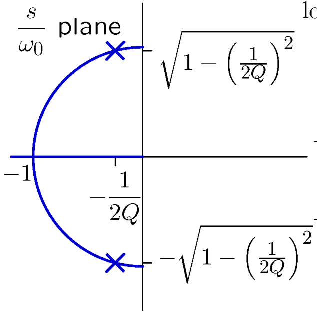

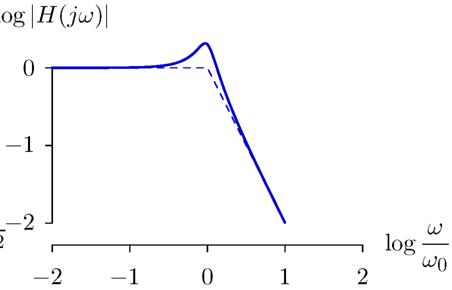

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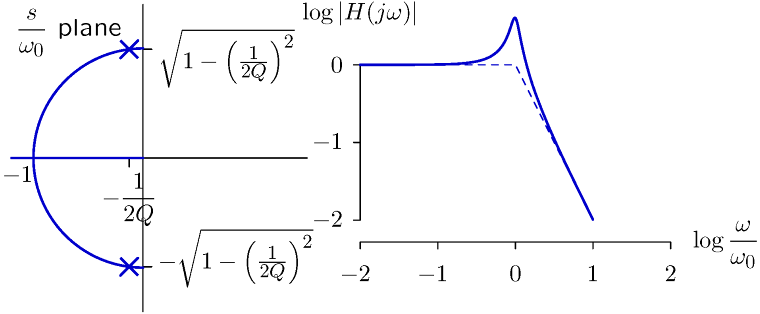

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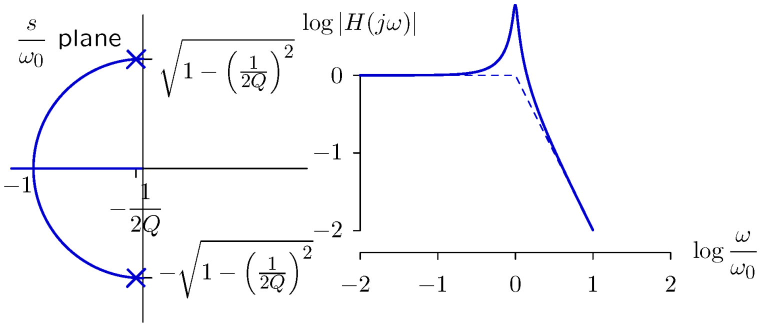

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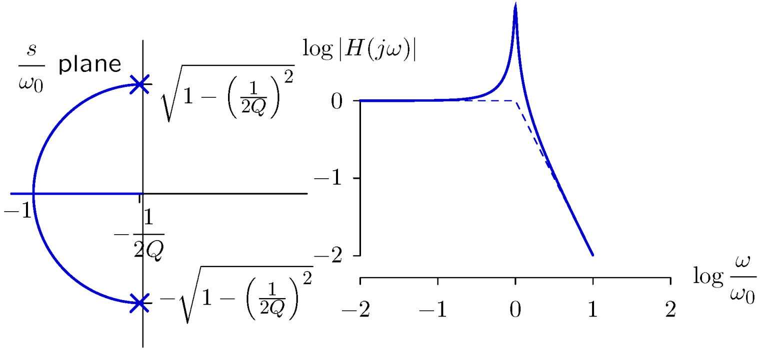
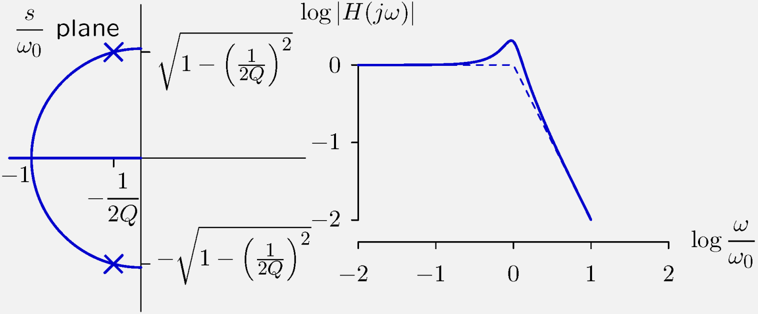


Check Yourself

Find dependence of peak magnitude on Q (assume $Q > 3$).

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

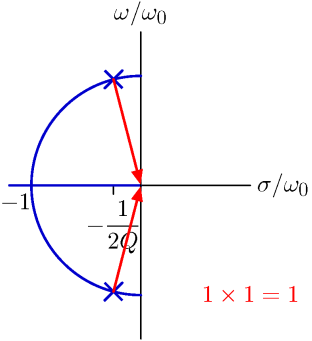


Check Yourself

Find dependence of peak magnitude on Q (assume $Q > 3$).

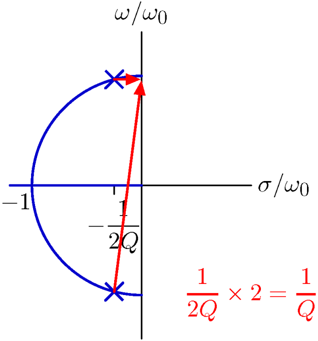
Analyze with vectors.

low frequencies



$1 \times 1 = 1$

high frequencies



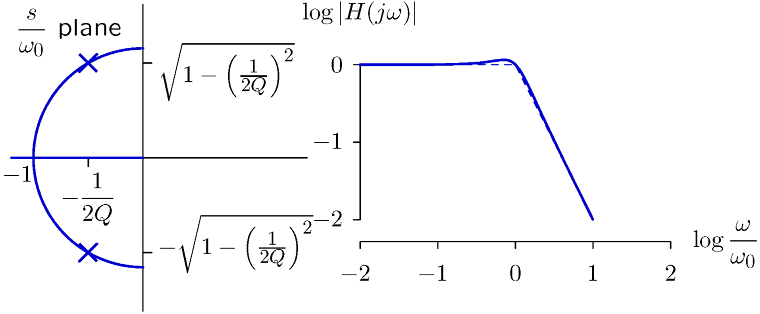
$\frac{1}{2Q} \times 2 = \frac{1}{Q}$

Peak magnitude increases with Q!

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



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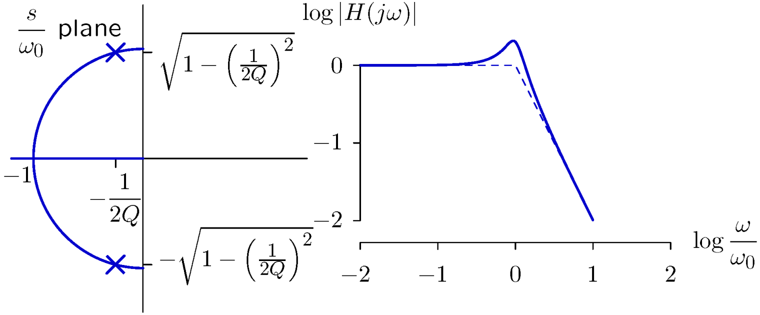

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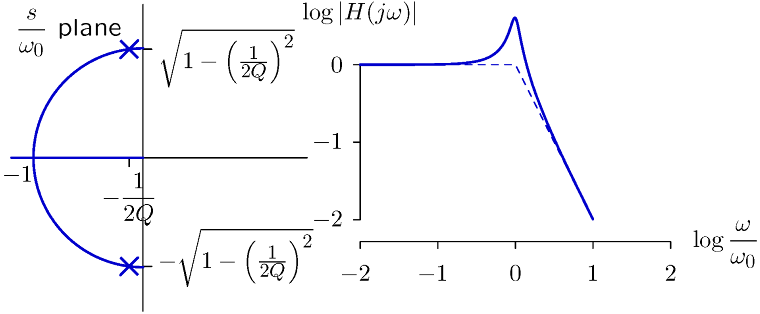

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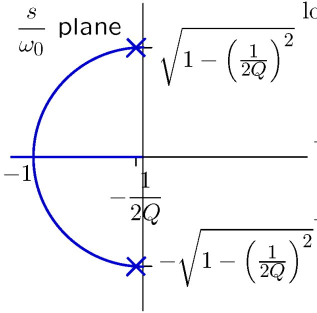

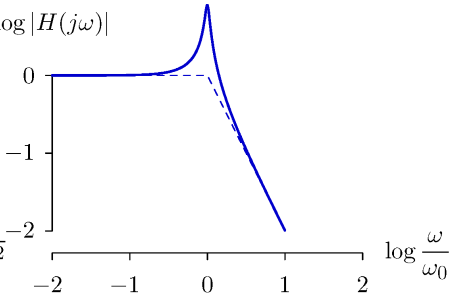

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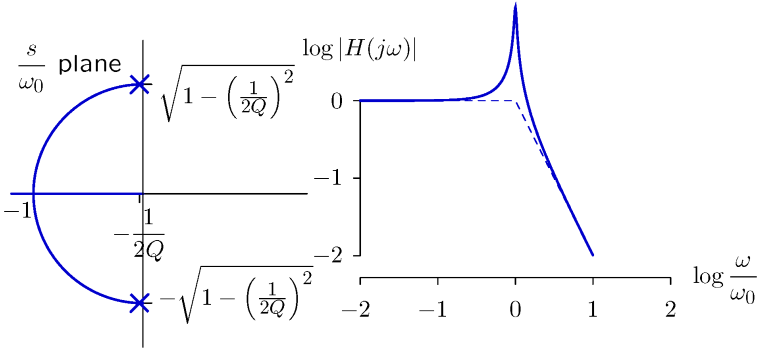
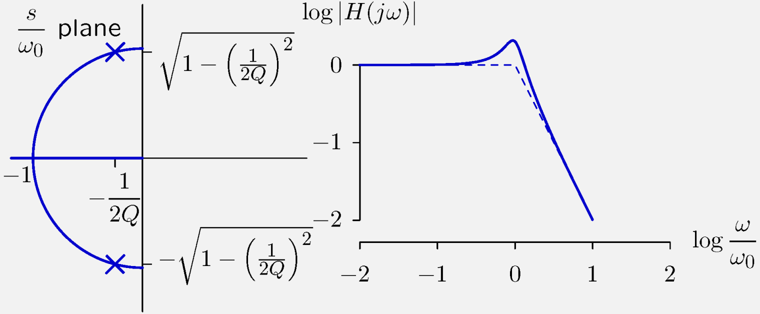


Check Yourself

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

Let ω_l (or ω_h) represent the lowest (or highest) frequency for which the magnitude is greater than the peak value divided by $\sqrt{2}$. The 3dB bandwidth is then $\omega_h - \omega_l$.

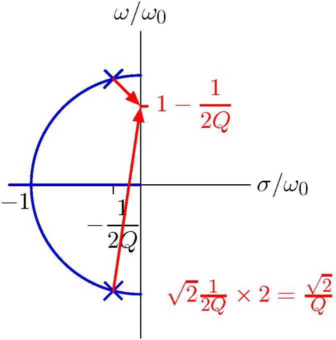


Check Yourself

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

Analyze with vectors.

low frequencies | high frequencies

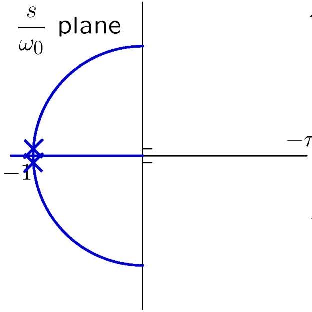


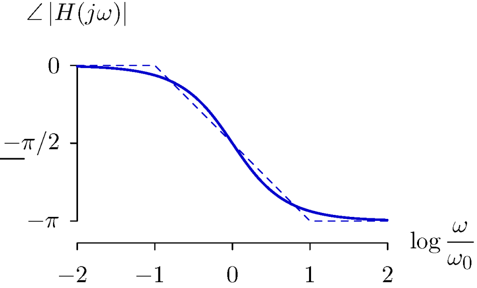
Bandwidth approximately $\frac{1}{Q}$

Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$





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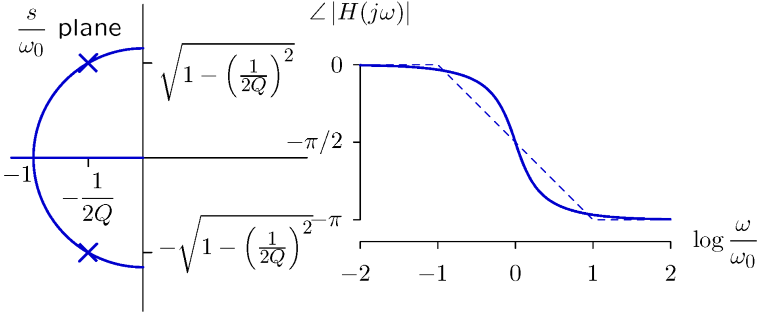

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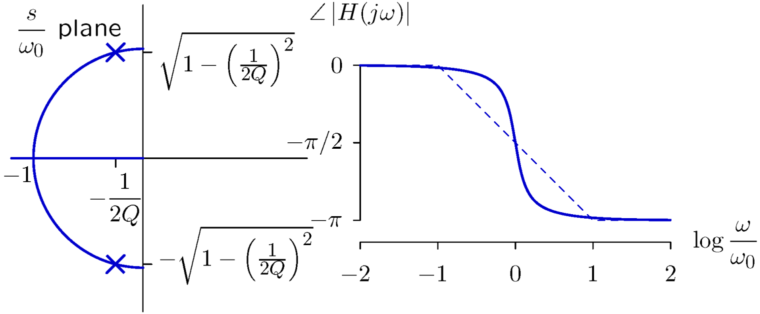

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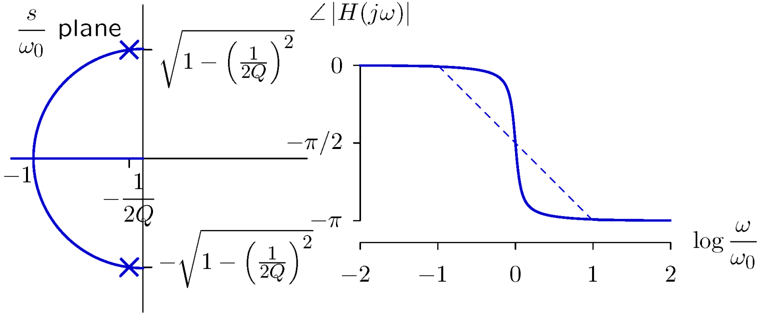

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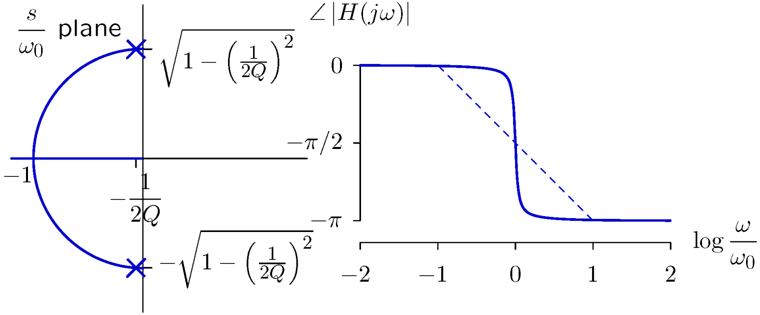
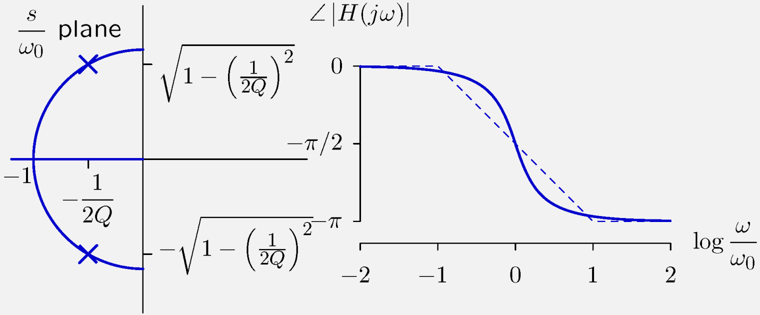


Check Yourself

Estimate change in phase that occurs over the 3dB bandwidth.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

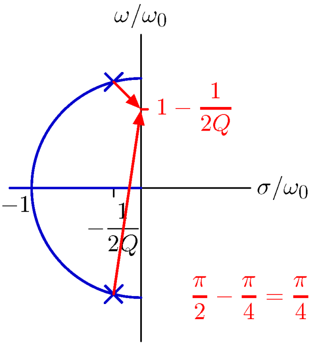


Check Yourself

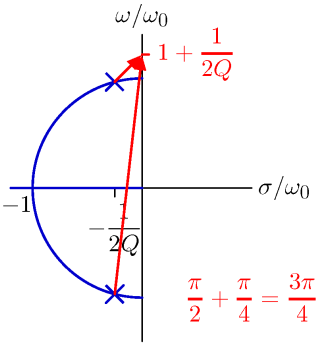
Estimate change in phase that occurs over the 3dB bandwidth.

Analyze with vectors.

low frequencies



high frequencies



Change in phase approximately $\frac{\pi}{2}$.

Summary

The frequency response of a system can be quickly determined using Bode plots.

Bode plots are constructed from sections that correspond to single poles and single zeros.

Responses for each section simply sum when plotted on logarithmic coordinates.

Assignments

- Reading Assignment: Chap. 9.4, 10.4, 3.8-3.11, 6.0-6.3, 6.5