Signals and Systems

Lecture 6: Convolution & Frequency Response

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Zhejiang University

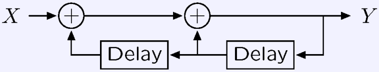
03/27/2024

Partly adapted from the materials provided on the MIT OpenCourseWare

Multiple Representations of DT Systems

Relations among representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

Convolution

Representing a system by a single signal.

Responses to Arbitrary Signals

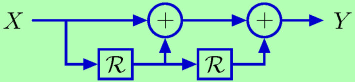
Although we have focused on responses to simple signals ($\delta[n], \delta(t)$) we are generally interested in responses to more complicated signals.

How do we compute responses to a more complicated input signals?

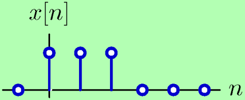
No problem for difference equations / block diagrams.
$\rightarrow$ use step-by-step analysis.

Check Yourself

Example: Find $y[3]$



when the input is

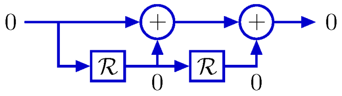


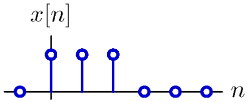
1. 1 2. 2 3. 3 4. 4 5. 5

0. none of the above

Responses to Arbitrary Signals

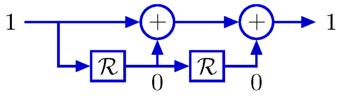
Example.





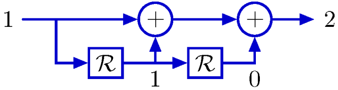
Responses to Arbitrary Signals

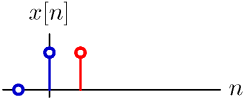
Example.

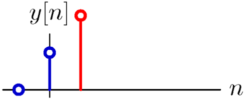


Responses to Arbitrary Signals

Example.

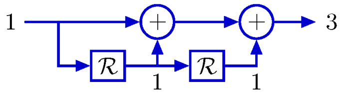


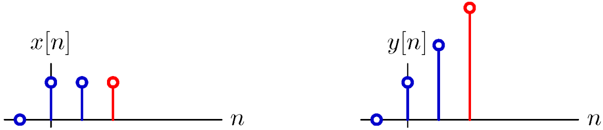




Responses to Arbitrary Signals

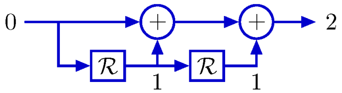
Example.

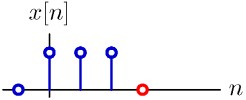


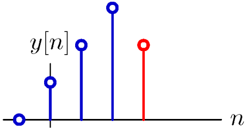


Responses to Arbitrary Signals

Example.

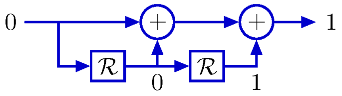


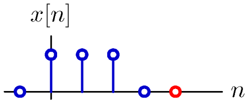


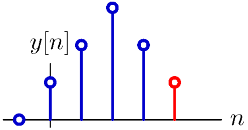


Responses to Arbitrary Signals

Example.

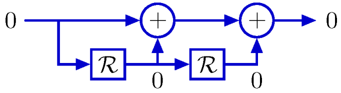


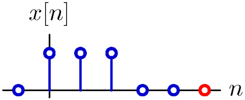


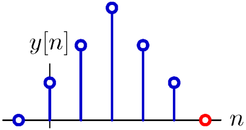


Responses to Arbitrary Signals

Example.

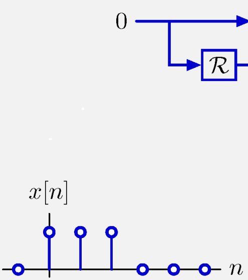






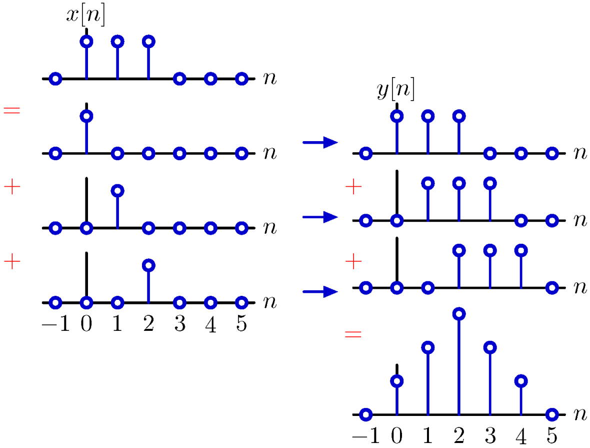
Check Yourself

What is $y[3]$? 2



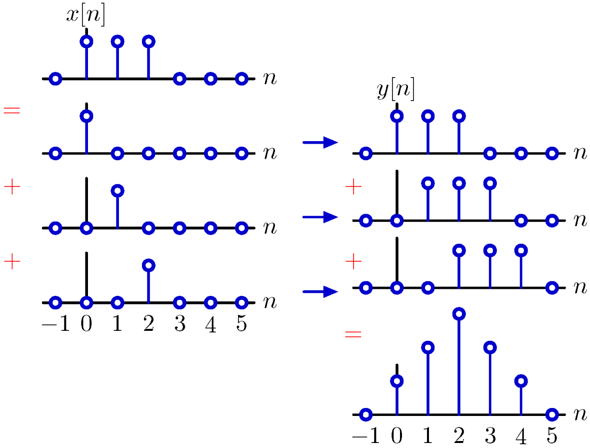
Alternative: Superposition

Break input into additive parts and sum the responses to the parts.



Superposition

Break input into additive parts and sum the responses to the parts.



Superposition works if the system is **linear**.

Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

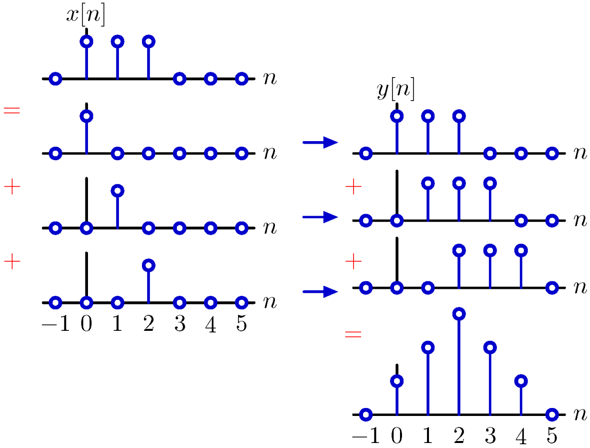
the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β.

Superposition

Break input into additive parts and sum the responses to the parts.



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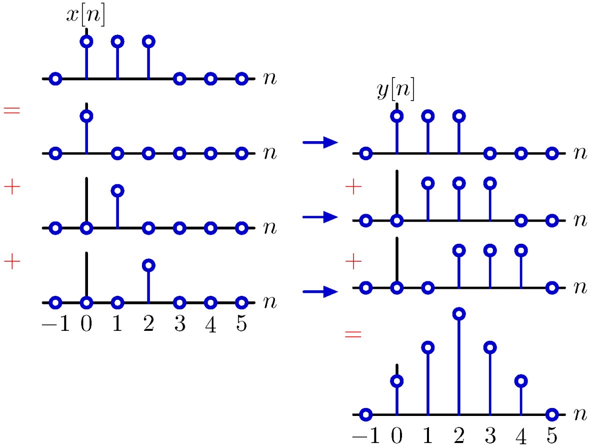
Superposition

Break input into additive parts and sum the responses to the parts.



Reponses to parts are easy to compute if system is **time-invariant**.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

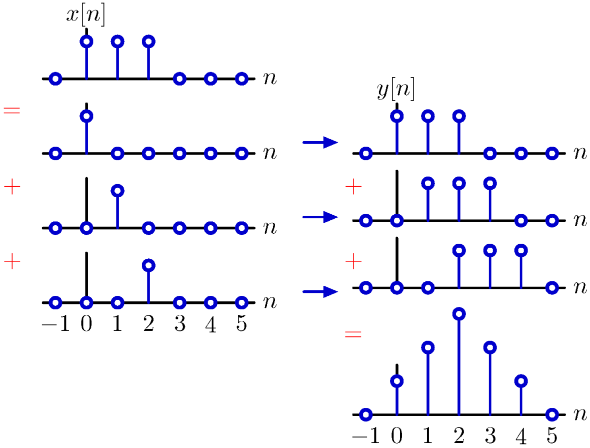
the system is time invariant if

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0.

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition is easy if the system is **linear** and **time-invariant**.

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution

Response of an LTI system to an arbitrary input.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called **convolution**.

Notation

Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x * h)[n] = x[n] * h[n]$$

Notation

Do not be fooled by the confusing notation.

Confusing (but conventional) notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

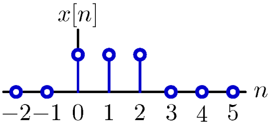
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $x * h$.

Structure of Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



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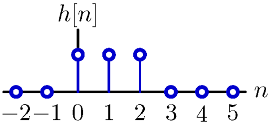
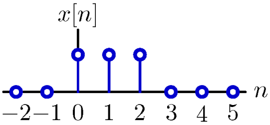


Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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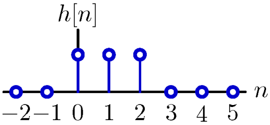

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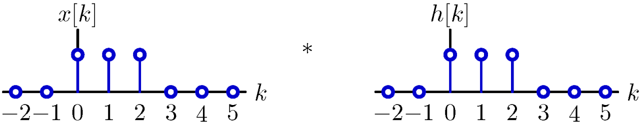
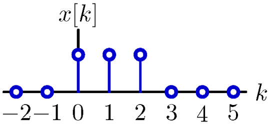


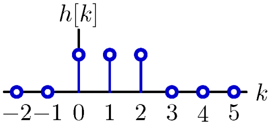
Structure of Convolution

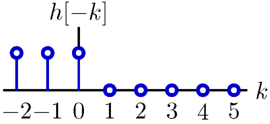
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0 - k]$$



flip

$*$





Structure of Convolution

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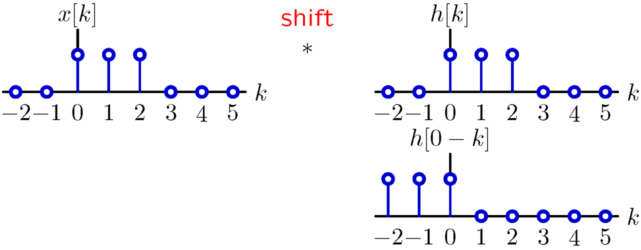

Structure of Convolution

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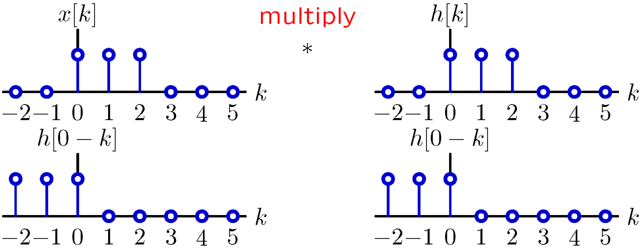

Structure of Convolution

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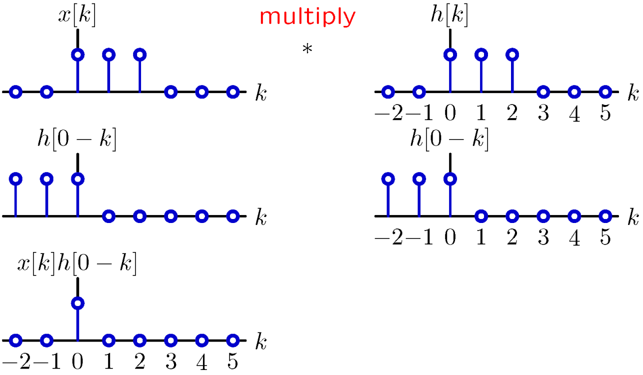

Structure of Convolution

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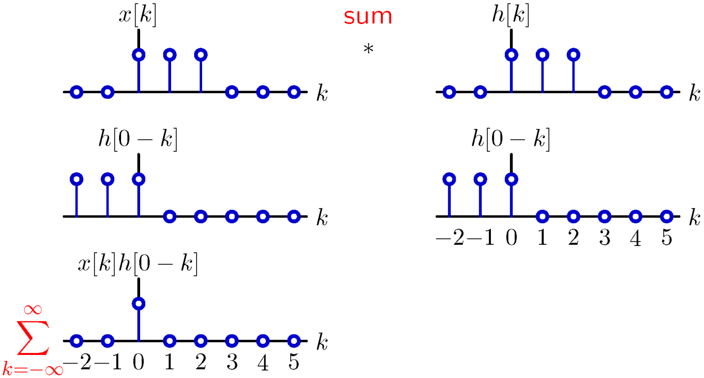
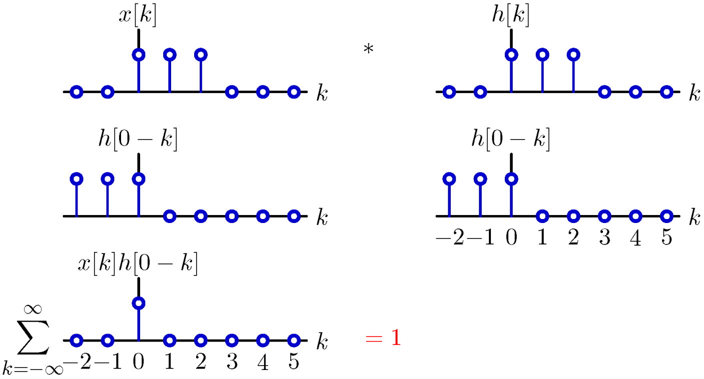


Structure of Convolution

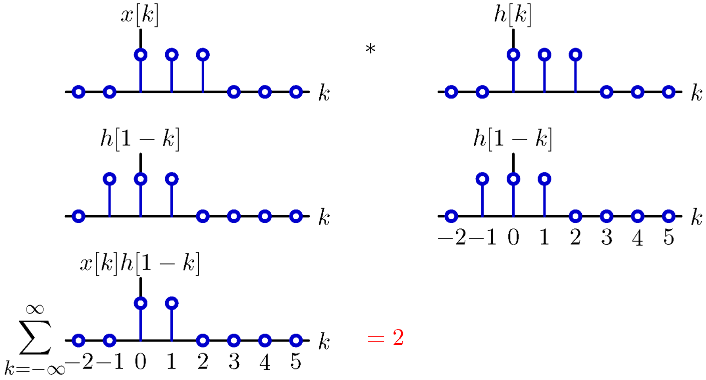
$$y[{\color{red}0}] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



$= 1$

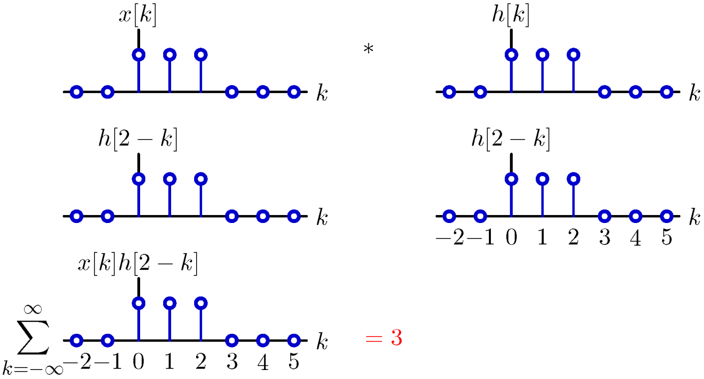
Structure of Convolution

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



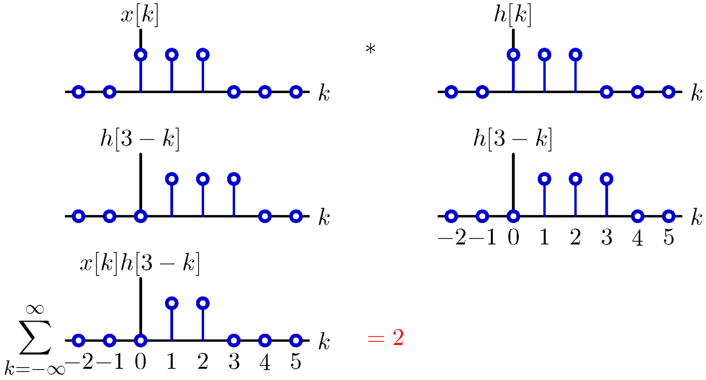
Structure of Convolution

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



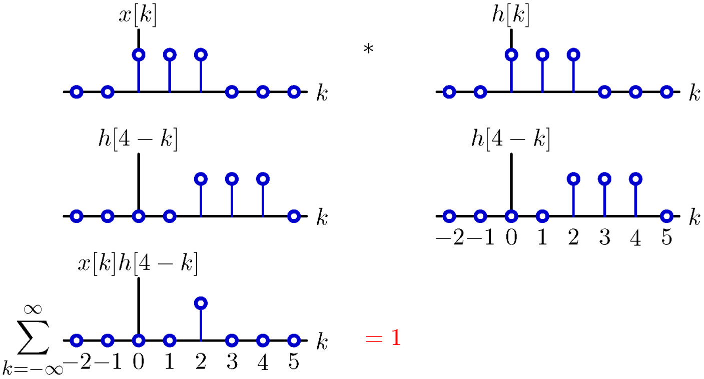
Structure of Convolution

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



Structure of Convolution

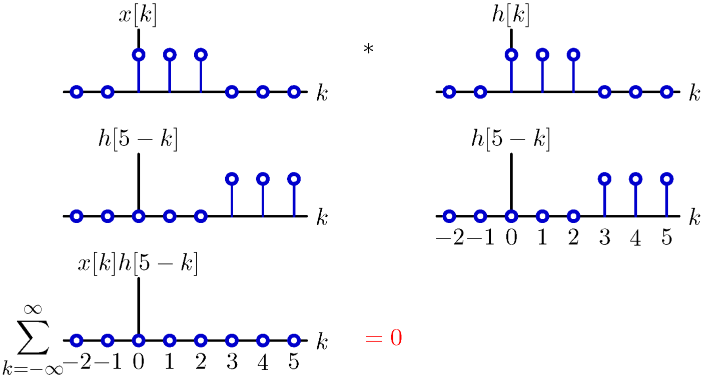
$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$



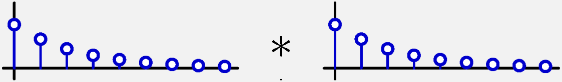
$= 1$

Structure of Convolution

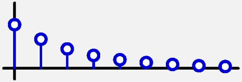
$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

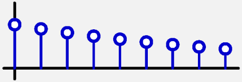


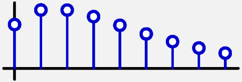
Check Yourself

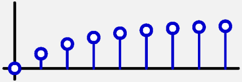


Which plot shows the result of the convolution above?

1. 

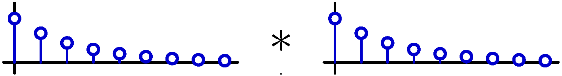
2. 

3. 

4. 

5. none of the above

Check Yourself



Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^k u[k]\right) \times \Bigg(\left(\frac{2}{3}\right)^{n-k} u[n-k$$

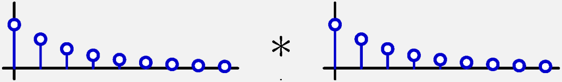
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{k=0}^{n} 1$$

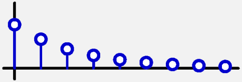
$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

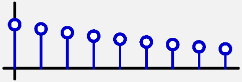
$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

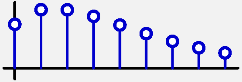
Check Yourself

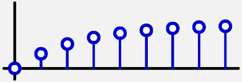


Which plot shows the result of the convolution above? 3

1. 

2. 

3. 

4. 

5. none of the above

Convolution

Representing an LTI system by a single signal.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Unit-sample response $h[n]$ is a complete description of an LTI system.

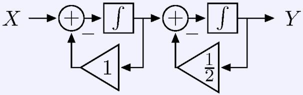
Given $h[n]$ one can compute the response $y[n]$ to any arbitrary input signal $x[n]$:

$$y[n] = (x * h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Multiple Representations of CT Systems

Relations among representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\, u(t)$$

Differential Equation

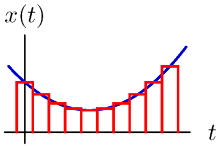
$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

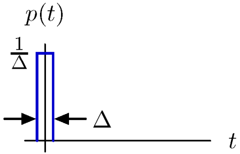
CT Convolution

The same sort of reasoning applies to CT signals.



$$x(t) = \lim_{\Delta \to 0} \sum_k x(k\Delta) p(t - k\Delta)\Delta$$

where



As $\Delta \to 0$, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:

$$x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

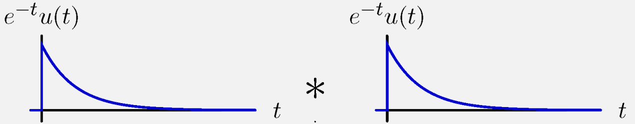
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

CT Convolution

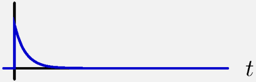
Convolution of CT signals is analogous to convolution of DT signals.

DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Which plot shows the result of the convolution above?

1.


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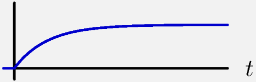

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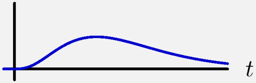

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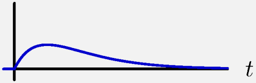
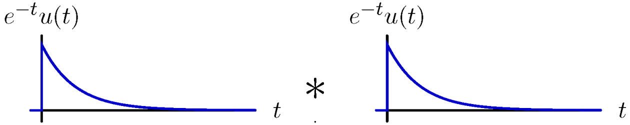


5. none of the above

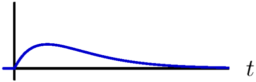
Check Yourself

Which plot shows the result of the following convolution?

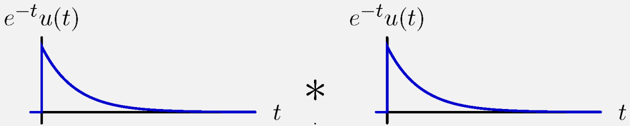


$$\left(e^{-t}u(t)\right) * \left(e^{-t}u(t)\right) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

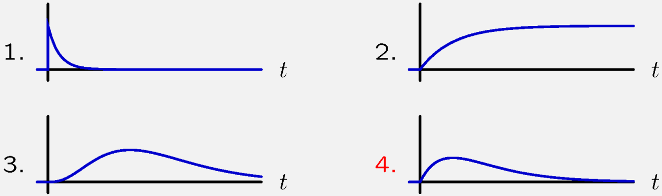
$$= \int_0^t e^{-\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_0^t d\tau = te^{-t}u(t)$$



Check Yourself



Which plot shows the result of the convolution above? 4



5. none of the above

Convolution

Convolution is an important **computational tool.**

Example: characterizing LTI systems

- Determine the unit-sample response $h[n]$.
- Calculate the output for an arbitrary input using convolution:

$$y[n] = (x * h)[n] = \sum x[k]h[n-k]$$

Properties of Convolution

- Commutativity:

$$x(t) * h(t) = h(t) * x(t)$$

- Sifting property:

$$x(t) * \delta(t) = x(t)$$

- An integrator:

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

- Step response:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

Properties of Convolution

The use of Laplace Transforms to solve differential equations depends on several important properties.

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t - T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\dfrac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s + \alpha)$	shift R by $-\alpha$
Differentiate in t	$\dfrac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\dfrac{X(s)}{s}$	$\supset \big(R \cap \big(\operatorname{Re}(s) > 0$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)\, d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

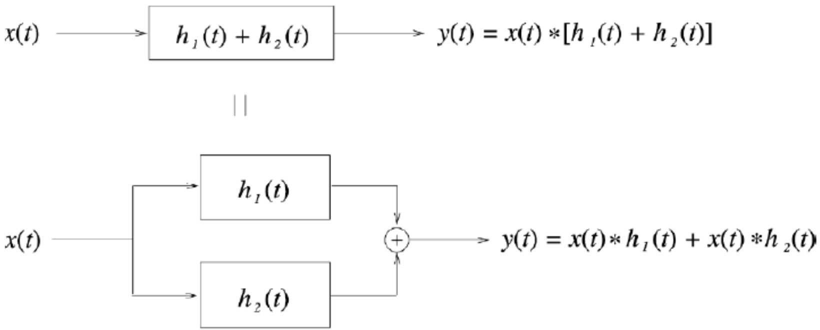
Properties of Convolution

The use of Z Transforms to solve differential equations depends on several important properties.

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

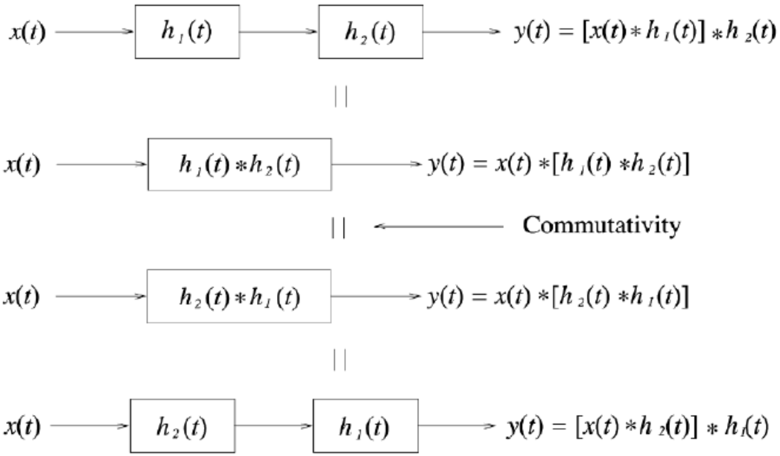
Properties of LTI Systems

DISTRIBUTIVITY



Properties of LTI Systems

ASSOCIATIVITY



Properties of LTI Systems

- LTI systems with and without memory

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\text{Memoryless} \iff h(t) = 0, t \neq 0$$

Properties of LTI Systems

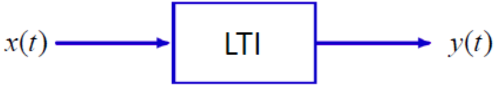
- Invertibility of LTI systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t) = x(t)$$

$$h(t) * h_1(t) = \delta(t)$$

Properties of LTI Systems

- Causality for LTI systems



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\text{CT LTI system is causal} \Leftrightarrow h(t) = 0, t < 0$$

Properties of LTI Systems

- Stability for LTI systems

$$\text{CT LTI system is stable} \Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$$

Sufficient condition:

$$\text{For } |x(t)| \le x_{\max} < \infty.$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \right| \le x_{\max} \left| \int_{-\infty}^{+\infty} h(t-\tau) d\tau \right| < \infty.$$

Necessary condition:

$$\text{Suppose} \quad \int_{-\infty}^{+\infty} |h(\tau)| d\tau = \infty$$

$$\text{Let } \; x(t) = h^*(-t)/|h^*(-t)|, \; \text{ then } \; |x(t)| \equiv 1 \; \text{ bounded}$$

$$\text{But } \; y(0) = \int_{-\infty}^{+\infty} x(\tau) h(-\tau) d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau) h(-\tau)}{|h(-\tau)|} d\tau = \int_{-\infty}^{+\infty} |h(-\tau)| d\tau = \infty$$

Properties of LTI Systems

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

H(s) = "system function"

$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty$ $\Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow$ $h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

Question:

If the ROC of $H(s)$ is a right-half plane, is the system causal?

Ex.

$$\begin{aligned} H(s) &= \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided} \\ h(t) &= \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t \to t+T} = e^{-t}u(t)|_{t \to t+T} \\ &= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t < 0 \qquad \text{Non-causal} \end{aligned}$$

Properties of LTI Systems

- LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

- A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

Impulse Response: Summary

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.

Frequency Response

Today we will investigate a different way to characterize a system: the **frequency response**.

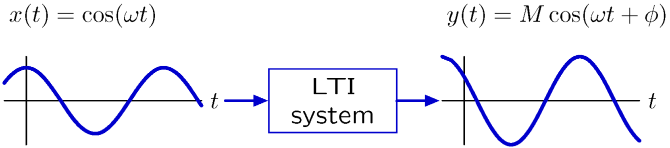
Many systems are naturally described by their responses to sinusoids.

Example: audio systems

Frequency Response Preview

If the input to a linear, time-invariant system is an eternal sinusoid, then the output is also an eternal sinusoid:

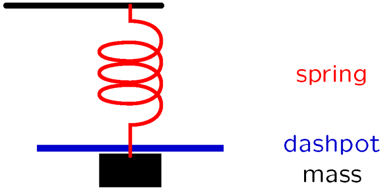
- same frequency
- possibly different amplitude, and
- possibly different phase angle.



The **frequency response** is a plot of the magnitude M and angle ϕ as a function of frequency ω.

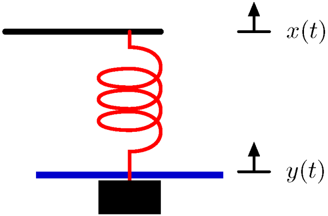
Example

Mass, spring, and dashpot system.



Demonstration

Measure the frequency response of a mass, spring, dashpot system.



Frequency Response

Calculate the frequency response.

Methods

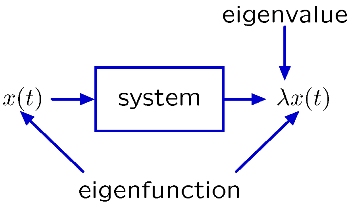
- solve differential equation
  $\rightarrow$ find particular solution for $x(t) = \cos\omega_0 t$
- find impulse response of system
  $\rightarrow$ convolve with $x(t) = \cos\omega_0 t$

New method

- use eigenfunctions and eigenvalues

Eigenfunctions

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.



Check Yourself: Eigenfunctions

Consider the system described by

$$\dot{y}(t) + 2y(t) = x(t).$$

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. e^{-t} for all time
2. e^{t} for all time
3. e^{jt} for all time
4. $\cos(t)$ for all time
5. $u(t)$ for all time

Check Yourself: Eigenfunctions

$$\dot{y}(t) + 2y(t) = x(t)$$

1. $e^{-t}: \quad -\lambda e^{-t} + 2\lambda e^{-t} = e^{-t} \rightarrow \lambda = 1$

2. $e^{t}: \quad \lambda e^{t} + 2\lambda e^{t} = e^{t} \rightarrow \lambda = \dfrac{1}{3}$

3. $e^{jt}: \quad j\lambda e^{jt} + 2\lambda e^{jt} = e^{jt} \rightarrow \lambda = \dfrac{1}{j+2}$

4. $\cos t: \quad -\lambda \sin t + 2\lambda \cos t = \cos t \rightarrow$ not possible!

5. $u(t): \quad \lambda\delta(t) + 2\lambda u(t) = u(t) \rightarrow$ not possible!

Check Yourself: Eigenfunctions

Consider the system described by

$$\dot{y}(t) + 2y(t) = x(t).$$

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. e^{-t} for all time $\surd$ $\lambda = 1$
2. e^{t} for all time $\surd$ $\lambda = \frac{1}{3}$
3. e^{jt} for all time $\surd$ $\lambda = \frac{1}{j+2}$
4. $\cos(t)$ for all time X
5. $u(t)$ for all time X

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s)\, e^{st}$$

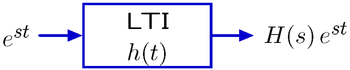
$$e^{st} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t)\end{array}} \longrightarrow H(s)\, e^{st}$$

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s)\, e^{st}$$



Eternal sinusoids are sums of complex exponentials.

$$\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s)\, e^{st}$$

$$e^{st} \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ h(t)\end{array}} \longrightarrow H(s)\, e^{st}$$

Eternal sinusoids are sums of complex exponentials.

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

Furthermore, the eigenvalue associated with e^{st} is $H(s)$!

Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos \omega_0 t$ (for all time). Then

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and the response to a sum is the sum of the responses.

$$y(t) = \frac{1}{2}\left(H(j\omega_0)\, e^{j\omega_0 t} + H(-j\omega_0)\, e^{-j\omega_0 t}\right)$$

Conjugate Symmetry

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

The system function is the Laplace transform of the impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $h(t)$ is a real-valued function of t for physical systems.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$H(-j\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt \equiv \big(H(j\omega)\big)^*$$

Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

The response to a sum is the sum of the responses,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\ &= \mathrm{Re}\left\{H(j\omega_0)e^{j\omega_0 t}\right\} \\ &= \mathrm{Re}\left\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right\} \\ &= |H(j\omega_0)|\mathrm{Re}\left\{e^{j\omega_0 t + j\angle H(j\omega_0)}\right\} \end{aligned}$$

$$y(t) = |H(j\omega_0)|\cos\left(\omega_0 t + \angle\left(H(j\omega_0)\right)\right).$$

Frequency Response

The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated at $s = j\omega$.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

Rational System Functions

Eigenvalues are particularly easy to evaluate for systems represented by linear differential equations with constant coefficients.

Then the system function is a ratio of polynomials in s.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

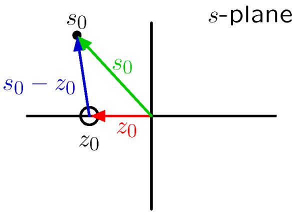
$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} \equiv \frac{N(s)}{D(s)}$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



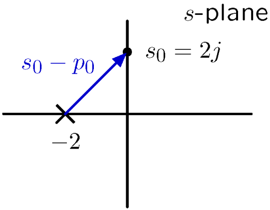
Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here z_0) to s_0, the point of interest in the s-plane.

Vector Diagrams

Example: Find the response of the system described by

$$H(s) = \frac{1}{s+2}$$

to the input $x(t) = e^{2jt}$ (for all time).



The denominator of $H(s)|_{s=2j}$ is $2j+2$, a vector with length $2\sqrt{2}$ and angle $\pi/4$. Therefore, the response of the system is

$$y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-\frac{j\pi}{4}}e^{2jt}.$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

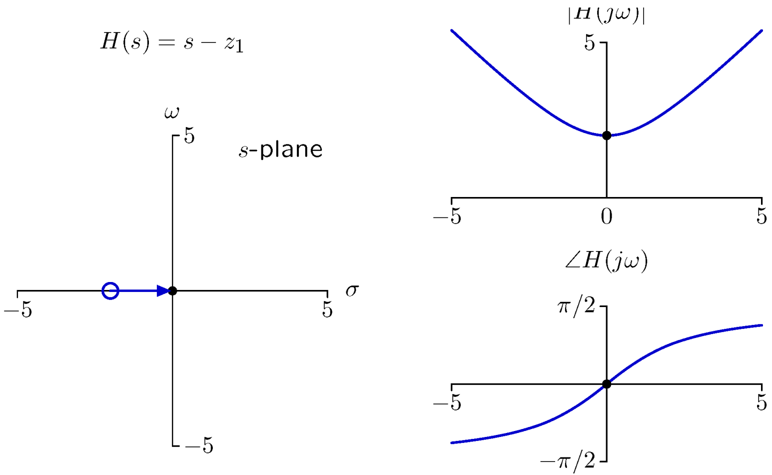
$$|H(s_0)| = |K|\frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots$$

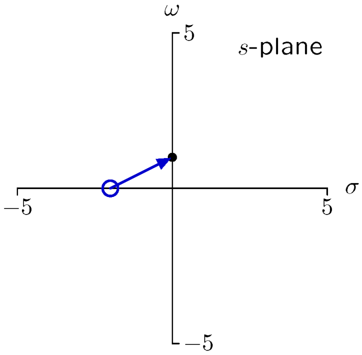
Vector Diagrams

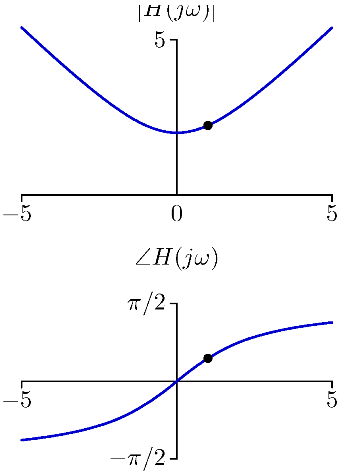
$$H(s) = s - z_1$$



Vector Diagrams

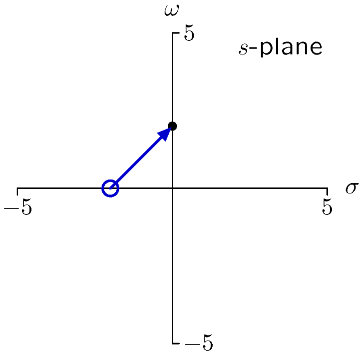
$$H(s) = s - z_1$$

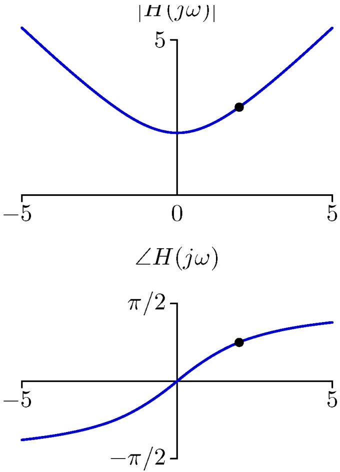




Vector Diagrams

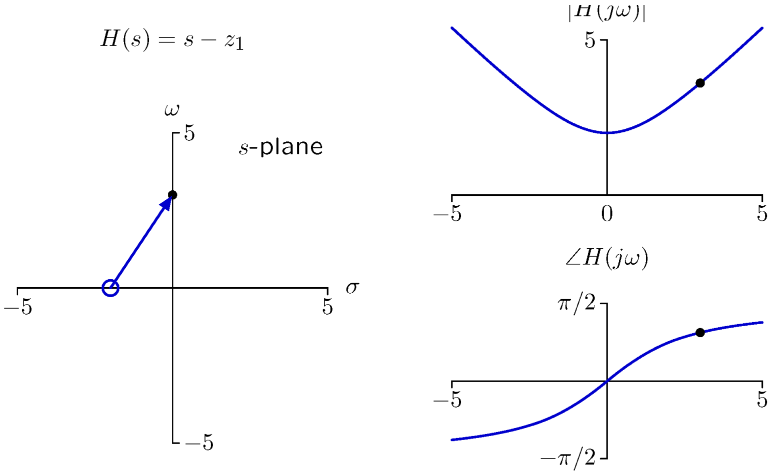
$$H(s) = s - z_1$$





Vector Diagrams

$$H(s) = s - z_1$$



Vector Diagrams

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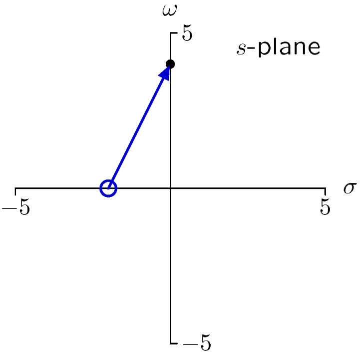

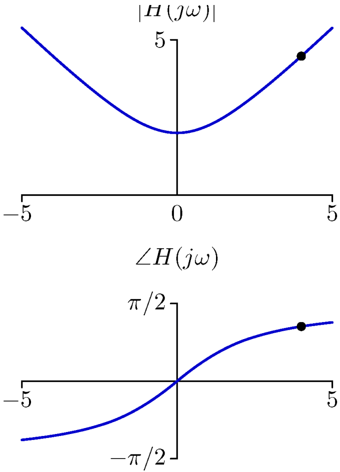

Vector Diagrams

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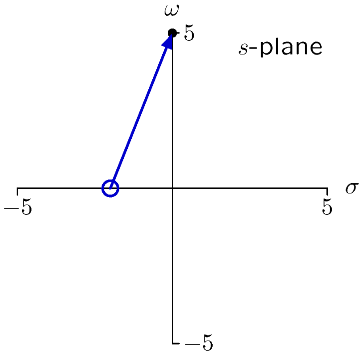

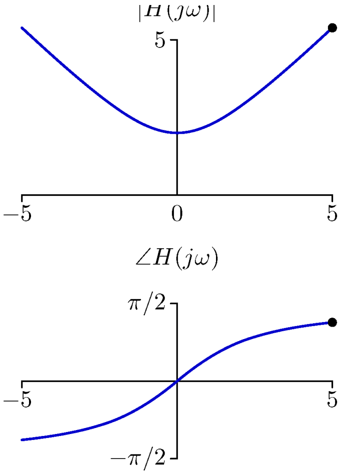

Vector Diagrams

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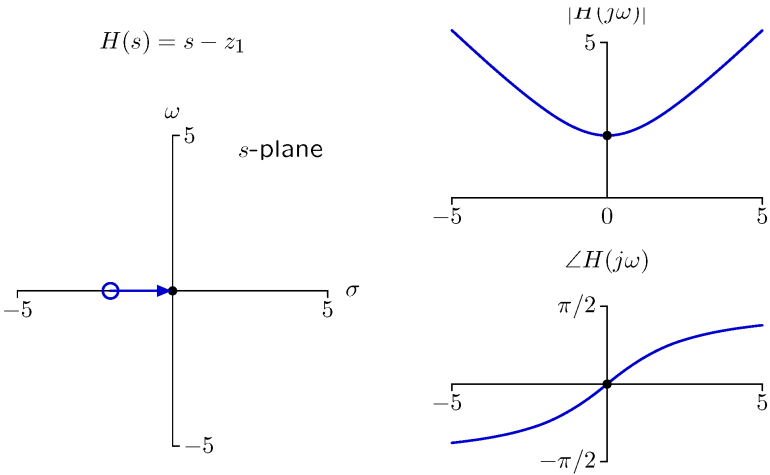

Vector Diagrams

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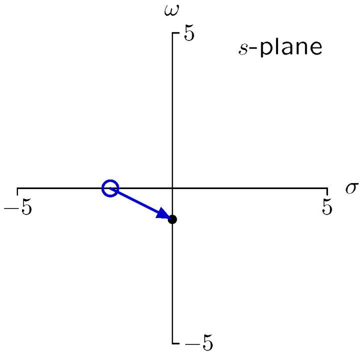

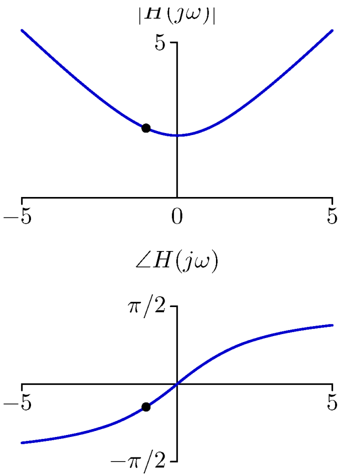

Vector Diagrams

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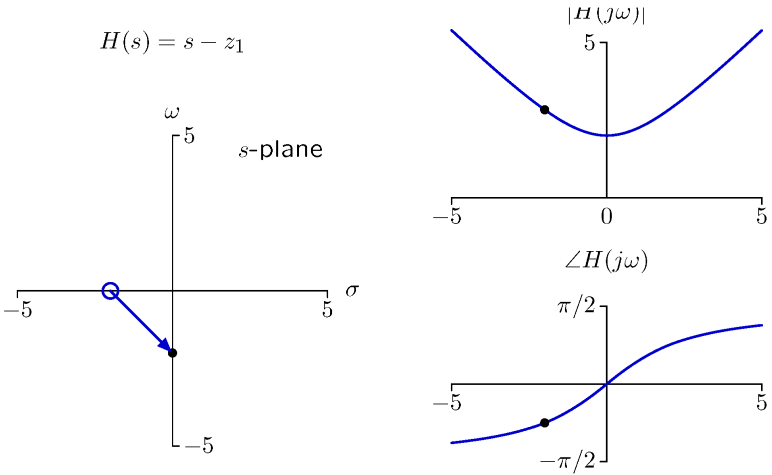

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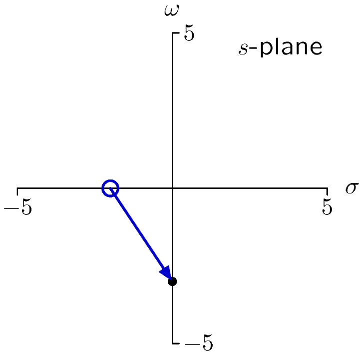

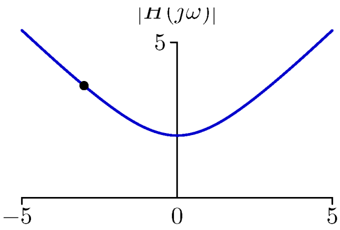

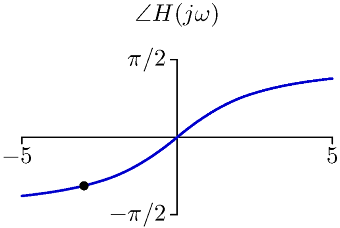

Vector Diagrams

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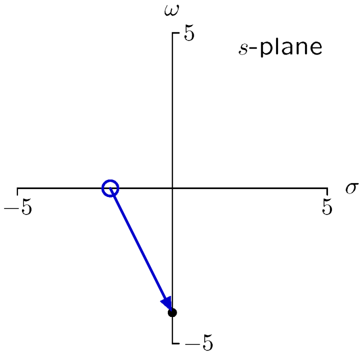

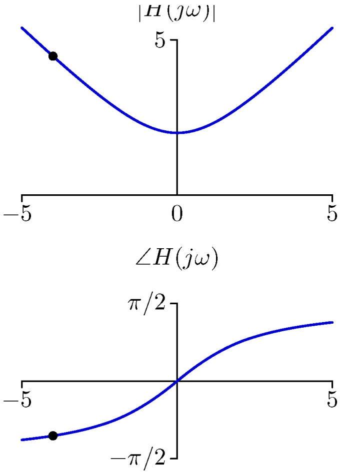

Vector Diagrams

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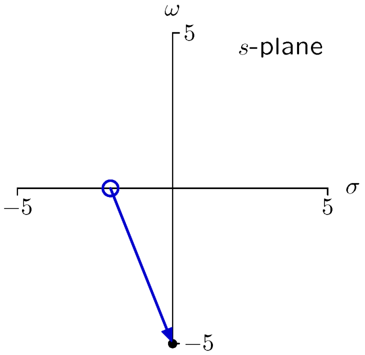
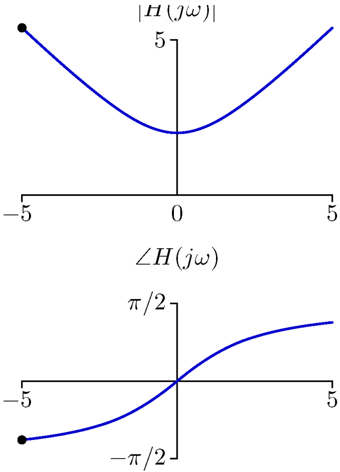
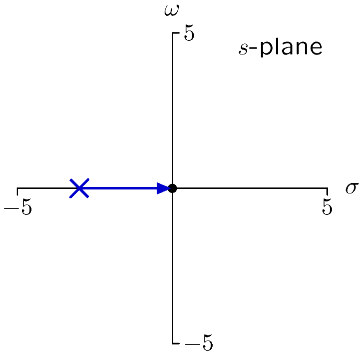
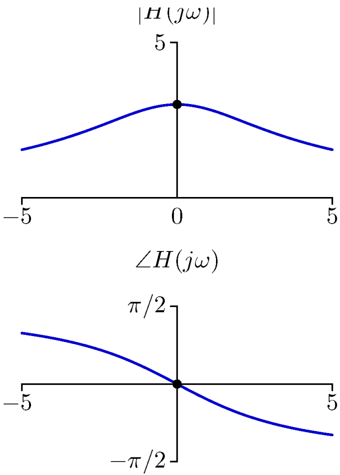




Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$





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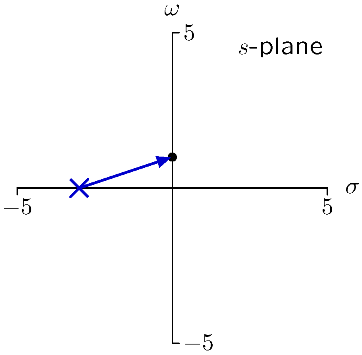

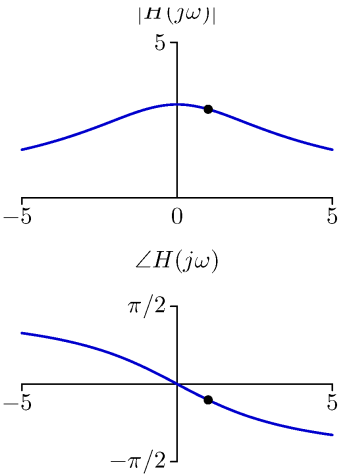

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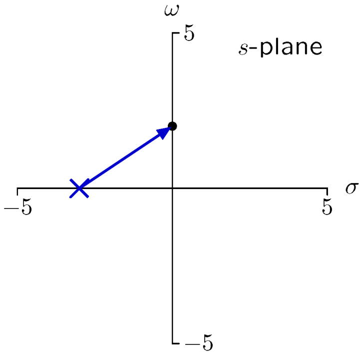

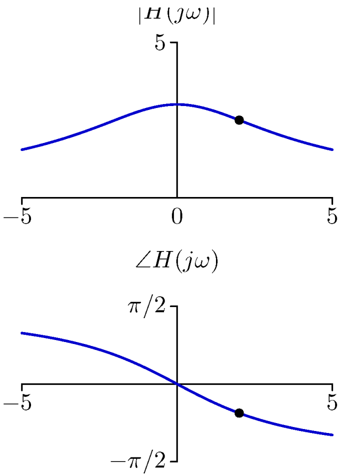

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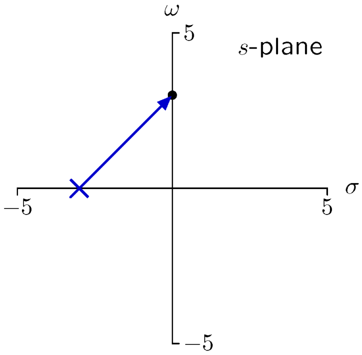

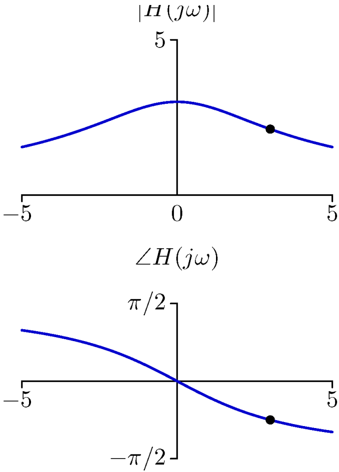

Vector Diagrams

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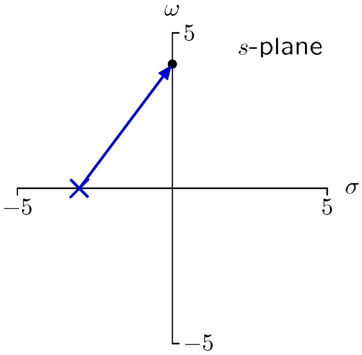

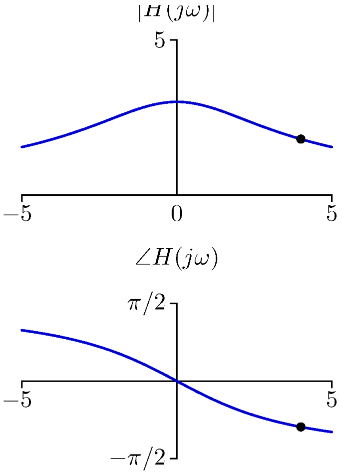

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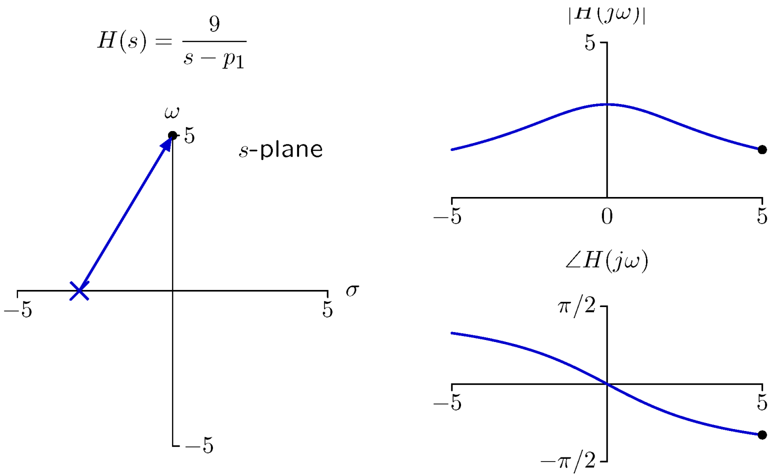
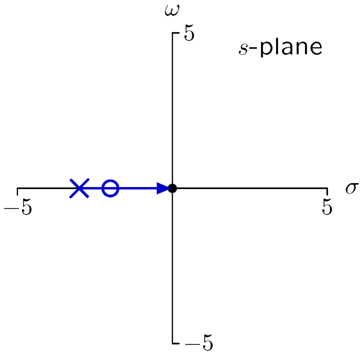
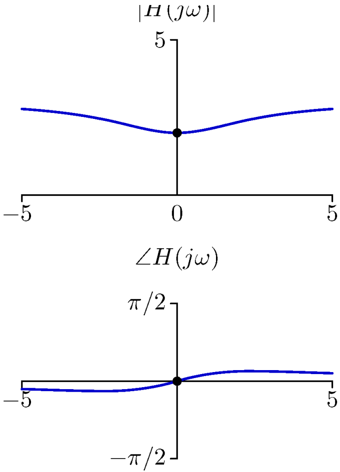


Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$





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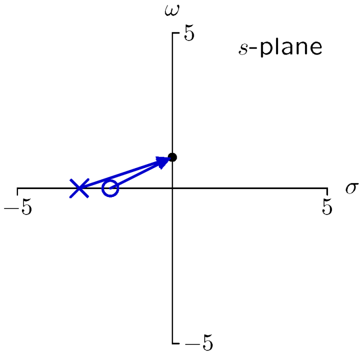

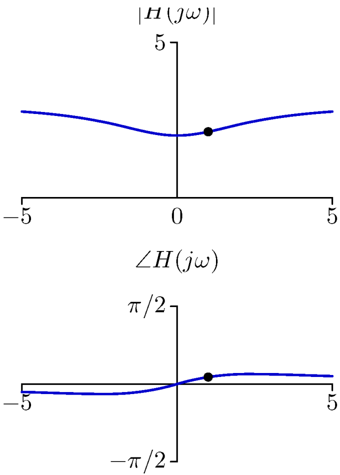

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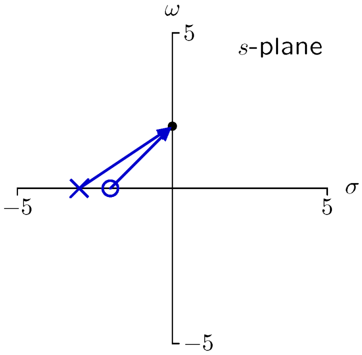

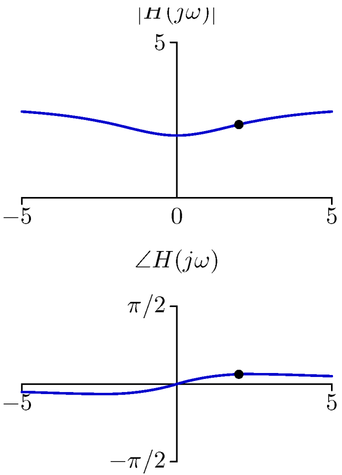

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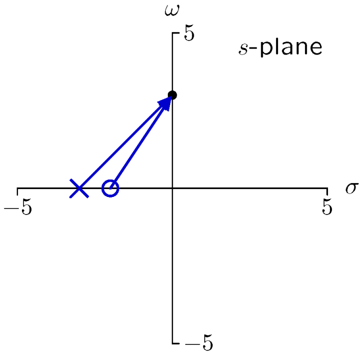

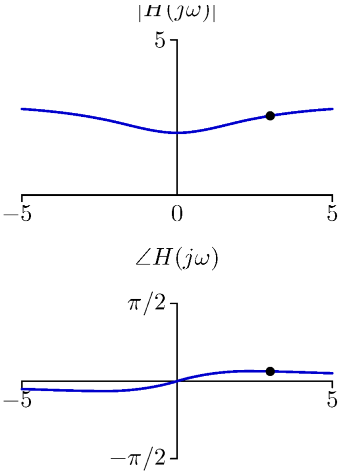

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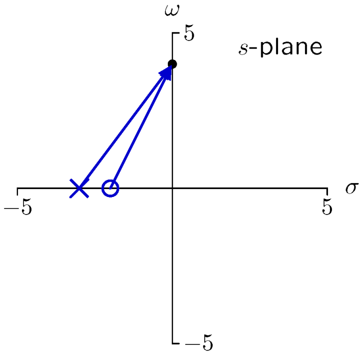

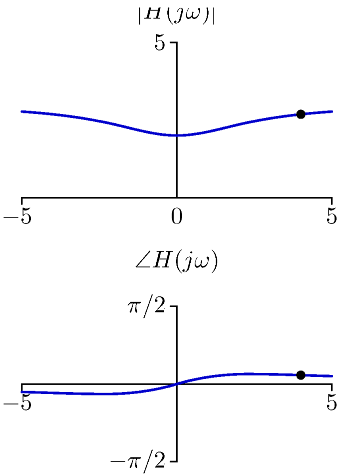

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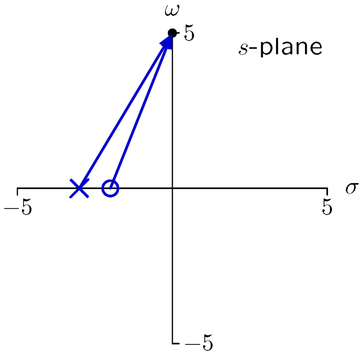
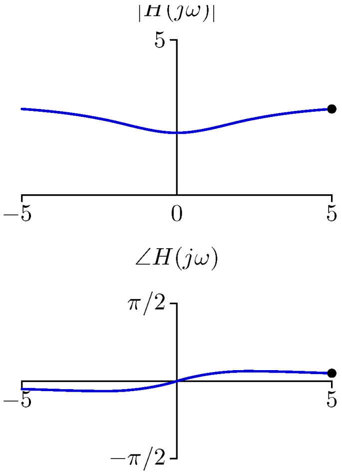
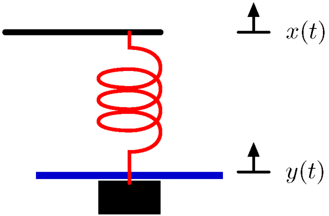




Example: Mass, Spring, and Dashpot



$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

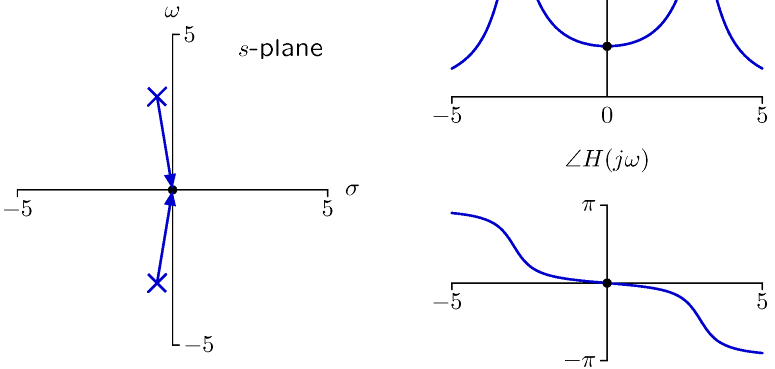
$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

$$(s^2M + sB + K)\ Y(s) = KX(s)$$

$$H(s) = \frac{K}{s^2M + sB + K}$$

Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$



Vector Diagrams

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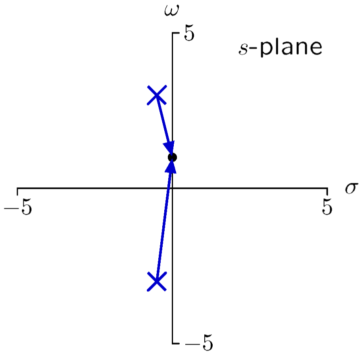

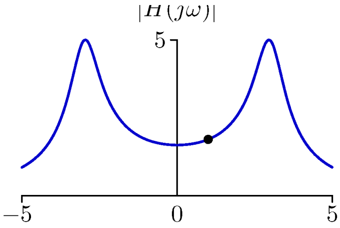

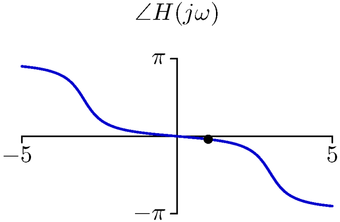

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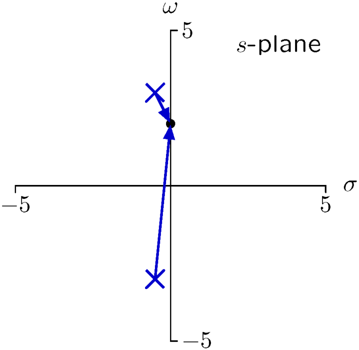
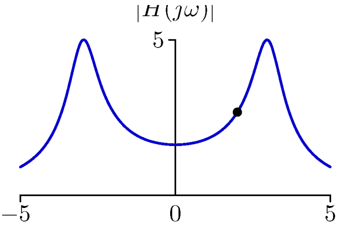
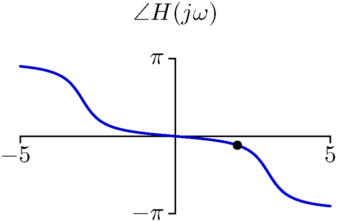
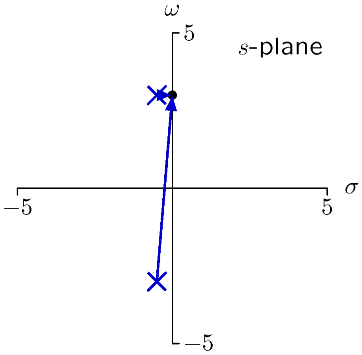


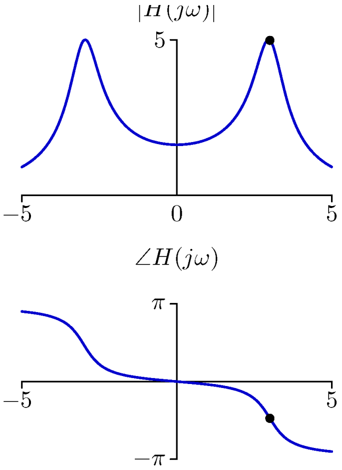




Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$





Vector Diagrams

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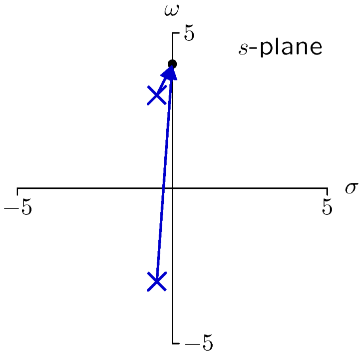

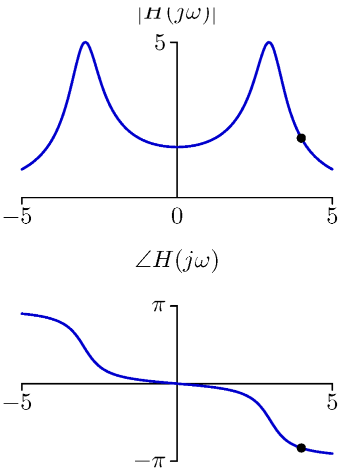

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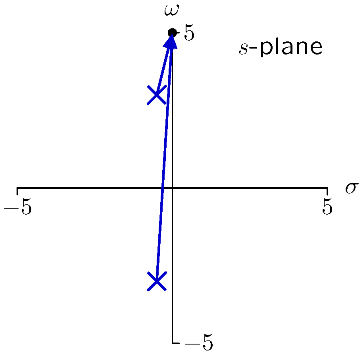
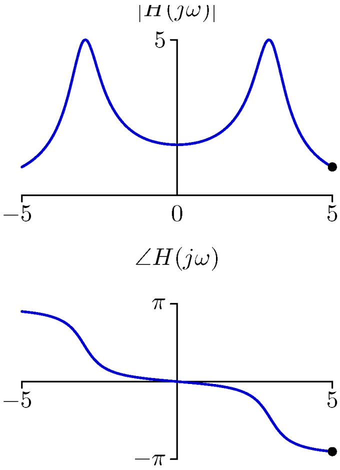
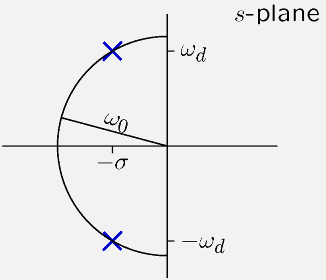




Check Yourself

Consider the system represented by the following poles.

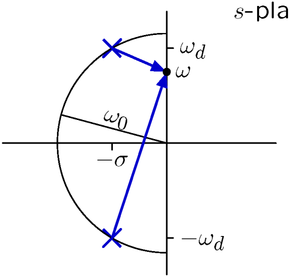


Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$.

1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

Check Yourself: Frequency Response

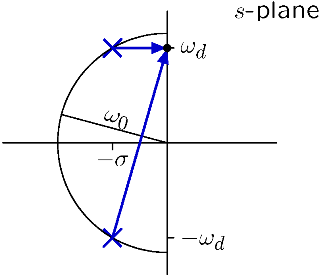
Analyze with vectors.



The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Check Yourself: Frequency Response

Analyze with vectors.

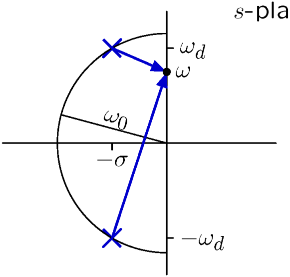


The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Decreasing ω from ω_d to $\omega_d - \epsilon$ decreases the product since length of bottom vector decreases as ϵ while length of top vector increases only ϵ^2.

Check Yourself: Frequency Response

More mathematically ...



The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

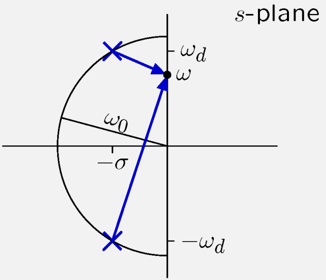
Maximum occurs where derivative of squared lengths is zero.

$$\frac{d}{d\omega}\left((\omega+\omega_d)^2+\sigma^2\right)\left((\omega-\omega_d)^2+\sigma^2\right)=0$$

$$\rightarrow \quad \omega^2=\omega_d^2-\sigma^2=\omega_0^2-2\sigma^2\,.$$

Check Yourself

Consider the system represented by the following poles.

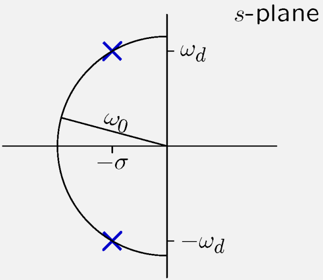


Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$. 3

1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

Check Yourself

Consider the system represented by the following poles.

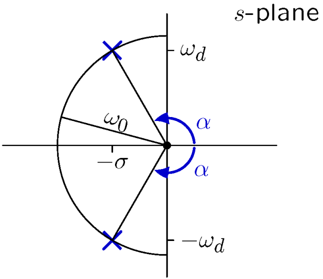


Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$.

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

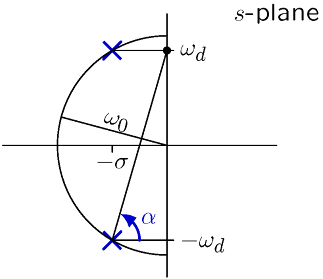
Check Yourself

The phase is 0 when $\omega = 0$.



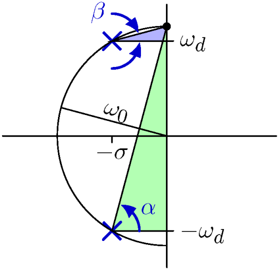
Check Yourself

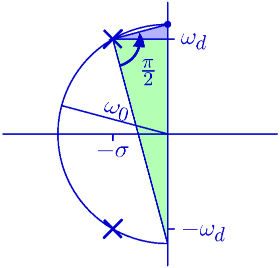
The phase is less than $\pi/2$ when $\omega = \omega_d$.



Check Yourself

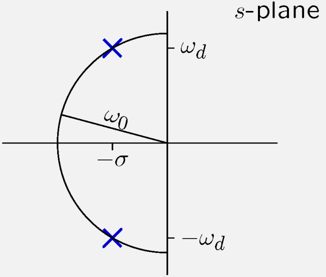
The phase at $\omega = \omega_0$ is $-\pi/2$.





Check Yourself

Consider the system represented by the following poles.



Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$. 3

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

Frequency Response: Summary

LTI systems can be characterized by responses to eternal sinusoids.

Many systems are naturally described by their frequency response.
– audio systems
– mass, spring, dashpot system

Frequency response is easy to calculate from the system function.

Frequency response lives on the $j\omega$ axis of the Laplace transform.

Assignments

- Reading Assignment: Chap. 2, 9.4, 10.4
- Homework 3: Due by Apr. 5, 2024