内积

定义

设$\alpha=(a_1, a_2, \ldots, a_n)^T$, $\beta=(b_1, b_2, \ldots, b_n)^T$,

则称实数$\sum_{i=1}^{n} a_i b_i$为向量α与β的**内积**

(inner/dot/scalar product).

记为$\langle\alpha, \beta\rangle$, 即

$$\langle\alpha, \beta\rangle = \sum_{i=1}^{n} a_i b_i = \alpha^T \beta.$$

性质

1. 2. 内积的基本性质
   (1) **对称性**: $\langle\alpha, \beta\rangle = \langle\beta, \alpha\rangle$;
   (2) **线性性**:
   $\langle k_1\alpha_1 + k_2\alpha_2, \beta\rangle = k_1\langle\alpha_1, \beta\rangle + k_2\langle\alpha_2, \beta\rangle$;
   (3) $\langle\alpha, \alpha\rangle \geq 0$; 且$\langle\alpha, \alpha\rangle = 0 \Leftrightarrow \alpha = 0$.
   (4) (Cauchy-Schwartz Inequality)
   $$|\langle\alpha, \beta\rangle| \leq \sqrt{\langle\alpha, \alpha\rangle}\sqrt{\langle\beta, \beta\rangle}.$$
2. $||\alpha+\beta|| \leq ||\alpha|| + ||\beta||$
3. $<\alpha, \beta>=0$, 则称α, β正交
4. $\alpha_1, \alpha_2, \ldots, \alpha_s$正交 $\Rightarrow$ 线性无关.
5. $E+A^TA$正交，利用内积求解

标准正交基

$a_1, a_2, \ldots, a_n$是V的一组基，两两正交且都为单位向量（模 $=1$）

施密特正交化

求解标准正交基

1. 先任意求解出一组基$a_1, a_2, \ldots, a_n$
2.
$$\beta_1 = \alpha_1,$$
$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1,$$
$$\cdots\ \cdots\ \cdots$$
$$\beta_s = \alpha_s - \frac{\langle \alpha_s, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \ldots - \frac{\langle \alpha_s, \beta_{s-1} \rangle}{\langle \beta_{s-1}, \beta_{s-1} \rangle}\beta_{s-1}$$
再将$\beta_1, \beta_2, \ldots, \beta_s$单位化得:
$$\eta_1 = \frac{\beta_1}{\|\beta_1\|},\ \eta_2 = \frac{\beta_2}{\|\beta_2\|},\ \ldots,\ \eta_s = \frac{\beta_s}{\|\beta_s\|}.$$

正交矩阵

$Q^TQ = E$或者$Q^{-1} = Q^T$

性质

定理2.10. n阶实方阵Q为正交阵 $\Leftrightarrow$ Q的列向量组构成$\mathbf{R}^n$的标准正交基.

注: 正交阵还具有以下性质

(1) Q为正交阵$\Rightarrow |Q| = \pm 1$.

(2) Q为正交阵$\Rightarrow Q^{-1} = Q^{\mathrm{T}}$也是正交阵.

(3) A, B为正交阵 $\Rightarrow AB$为正交阵.

(4) Q为正交阵$\Rightarrow \|Qx\| = \|x\|$.

$$\|Qx\|^2 = (Qx)^{\mathrm{T}}(Qx) = x^{\mathrm{T}}Q^{\mathrm{T}}Qx$$
$$= x^{\mathrm{T}}Ex = x^{\mathrm{T}}x = \|x\|^2$$