Basic Concepts of Monte Carlo Event Generator

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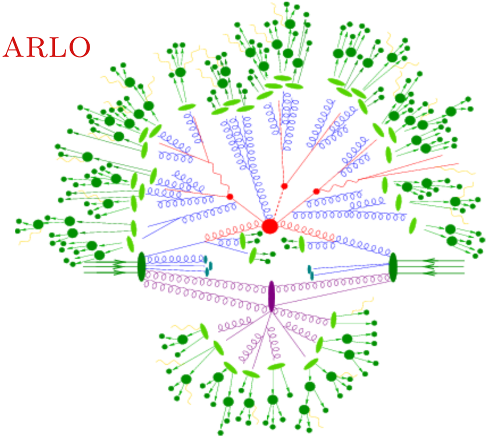

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pp Event Generator

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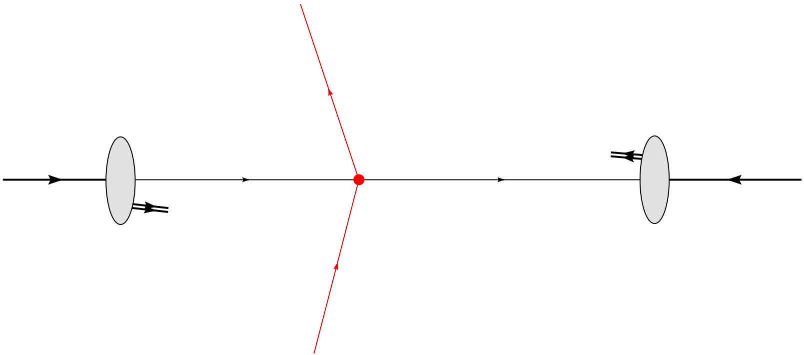
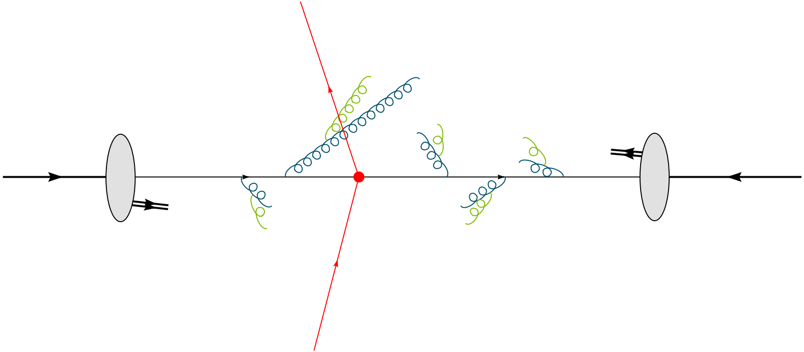


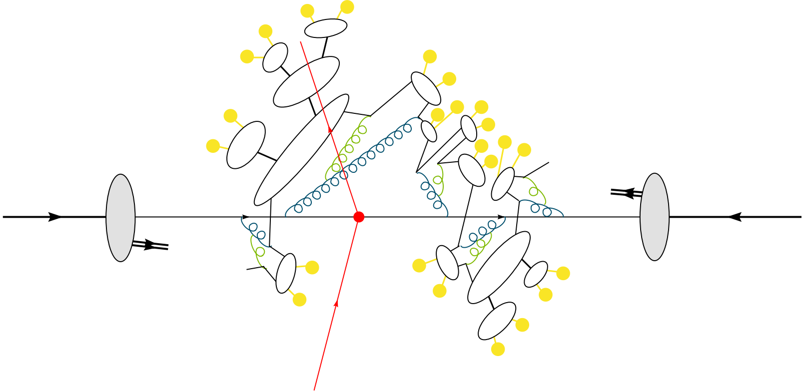
An example of Monte Carlo event

pp Event Generator



An example of Monte Carlo event

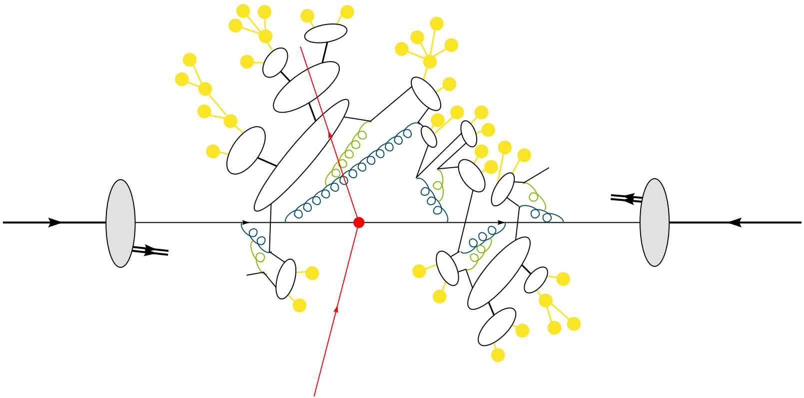
pp Event Generator



An example of Monte Carlo event

1 Why Monte Carlo?

pp Event Generator



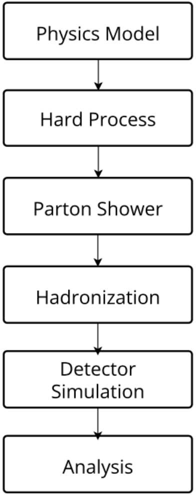
Workflow of Monte Carlo simulation

The purpose of Monte Carlo Generator is modelling the hard scattering, parton shower, hadronization and decays of unstable particles.

Some famous event generators:

- MadGraph5
- Whizard
- Sherpa
- Pythia
- Herwig
- ...

Scale: $\mathcal{O}(1)$ TeV $\rightarrow$ $\mathcal{O}(1)$ GeV



Consider a process $a + b \to f_n$, where a and b are two incoming particles and f_n is the final state with n particles. The cross section can be calculated by

$$\sigma = \int dx_1 f_a^A(x_1, Q^2) \int dx_2 f_b^B(x_2, Q^2) \int d\hat{\sigma}(\hat{s}, Q^2),$$

where A and B are the true beam particles. The probability of finding a particle a inside A is described by the structure function $f_a^A(x, Q^2)$, with a fraction x of the momentum, if the hard-collision process probes the particle at a factorization scale Q^2. For the collision energy, we have:

$$\hat{s} = (p_a + p_b)^2 = x_1 x_2 s = x_1 x_2 (p_A + p_B)^2$$

The differential cross section is

$$d\hat{\sigma} = \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(\hat{s}, m_a^2, m_b^2)}} d\Phi_n \approx \frac{|\mathcal{M}|^2}{2\hat{s}} d\Phi_n,$$

where $\lambda(a^2, b^2, c^2) = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$. The last expression is obtained when m_a and m_b are negligible.
The n-body phase space is

$$d\Phi_n = (2\pi)^4 \delta^{(4)} \left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2p_i^0}.$$

This is a $3n - 4$ dimensions integration. When $n = 2$, $3n - 4 = 2$. But we have dim-14 integration when $n = 6$!
$\rightarrow$ Monte Carlo integration!

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2 Monte Carlo Methods

Probability density

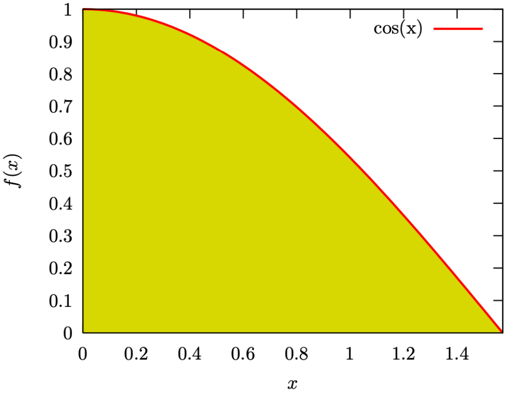
$$dP = f(x)dx$$

is probability to find value x.

$$F(x) = \int_{x_0}^{x} f(x')dx'$$

is called probability distribution.

Example: $f(x) = \cos(x)$.



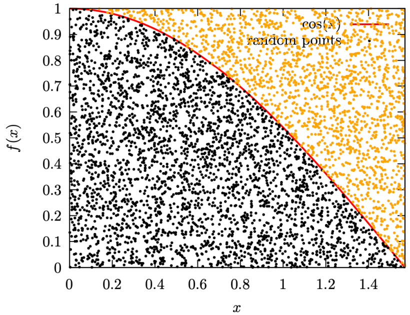
Hit and miss method:

- Throw N random points into (x, y) region.
- Count hits N_{hit}: whenever $y < f(x)$.

The integral is estimated:

$$I \approx V \cdot \frac{N_{\text{hit}}}{N}$$

Example: $f(x) = \cos(x)$.



- Accepted value of x is considered as an event.
- $f(x)$ is a "histogram" of the variable x.
- It is inefficient when the variance of f is large.

Simple MC Integration

Mean value theorem of integration:

$$\begin{aligned} I &= \int_{x_0}^{x_1} f(x)dx \\ &= (x_1 - x_0)\langle f(x)\rangle \\ &\approx (x_1 - x_0)\frac{1}{N}\sum_{i=1}^{N} f(x_i) \end{aligned}$$

- We can choose x_i randomly.
- A flat distribution of events x_i is obtained.
- Events are weighted with weight $f(x)$. $\rightarrow$ unweighting

The error on the integral can be estimated by the central limit theorem:

$$\begin{aligned} I &= \int f dV \\ &\approx V\langle f\rangle \pm V\sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{N}} \\ &\approx V\langle f\rangle \pm V\frac{\sigma}{\sqrt{N}}, \end{aligned}$$

where

$$\begin{aligned} \langle f\rangle &= \frac{1}{N}\sum_{i=1}^{N} f(x_i) \\ \langle f^2\rangle &= \frac{1}{N}\sum_{i=1}^{N} f^2(x_i) \end{aligned}$$

We can reduce error by reducing variance of integrand $\rightarrow$ important Sampling

$$
\begin{aligned}
I &= \int f dV \\
&= \int \frac{f}{g} g dV \\
&= \left\langle \frac{f}{g} \right\rangle \pm \frac{1}{\sqrt{N}} \sqrt{\left\langle \frac{f^2}{g^2} \right\rangle - \left\langle \frac{f}{g} \right\rangle^2}
\end{aligned}
$$

- Idea: divide out the singular structure.
- We need to sample flat in gdV.
- We know $\int gdV$ and its inverse.

We can choose

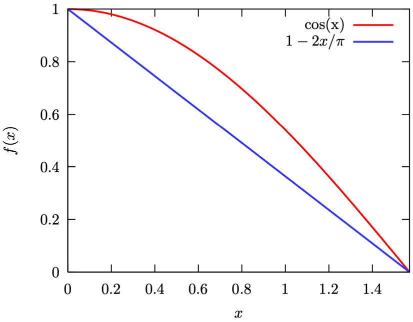
$$g(x) = 1 - \frac{2}{\pi}x$$

So we have

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\cos x}{1-\frac{2}{\pi}x}\left(1-\frac{2}{\pi}x\right)dx \\ &= \int_0^1 \left.\frac{\cos x}{1-\frac{2}{\pi}x}\right|_{x=x(\xi)} d\xi, \end{aligned}$$

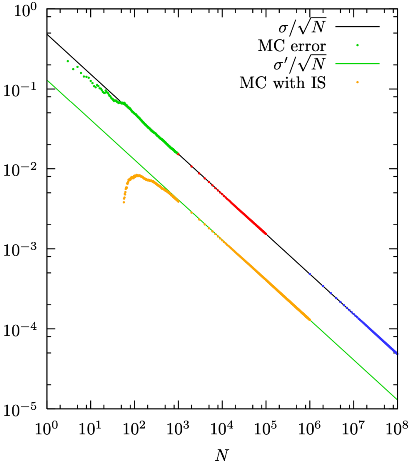
where

$$x = \frac{\pi}{2}\left(1-\sqrt{1-\xi}\right)$$



Important Sampling

- Much better convergence
- More accepted events
- Reduced variance → better efficiency

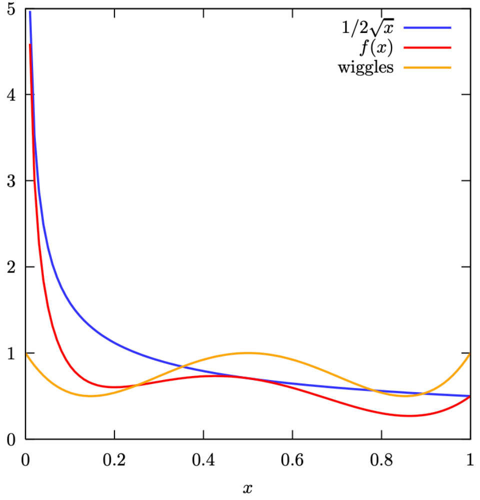


A more interesting example with divergent integrands:

$$f(x) = \frac{p(x)}{2\sqrt{x}},$$

with some wiggles

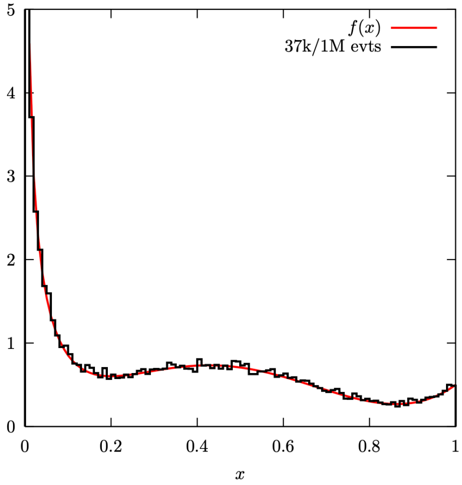
$$p(x) = 1 - 8x + 40x^2 - 64x^3 + 32x^4$$



Important Sampling

We use hit and miss method:

- Choose random number r.
- The weight is $w = f(x)$.
- If $r < w/w_{\max}$ then “hit”.
- $w_{\max} = 20$.
- MC efficiency: $\text{hit}/N \approx 3.7\%$.



Important Sampling

2 Monte Carlo Methods

$$I = \int_0^1 \frac{p(x)}{2\sqrt{x}} dx = \int_0^1 p(x(\rho)) d\rho$$

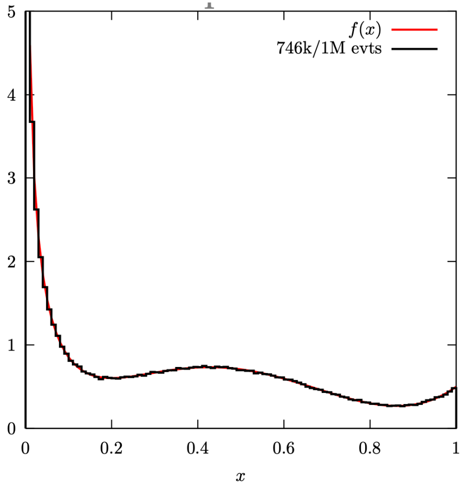
with

$$\rho = \sqrt{x}, d\rho = \frac{dx}{2\sqrt{x}}$$

We have

$$I = \int_0^1 1 - 8\rho^2 + 40\rho^4 - 64\rho^6 + 32\rho^8 d\rho$$

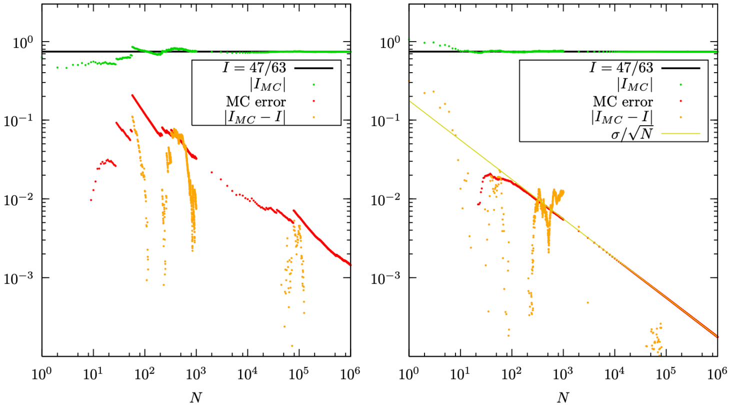
MC efficiency: 74.6%.



Important Sampling

2 Monte Carlo Methods

Crude MC vs Important Sampling:



100x more events needed to reach same accuracy.

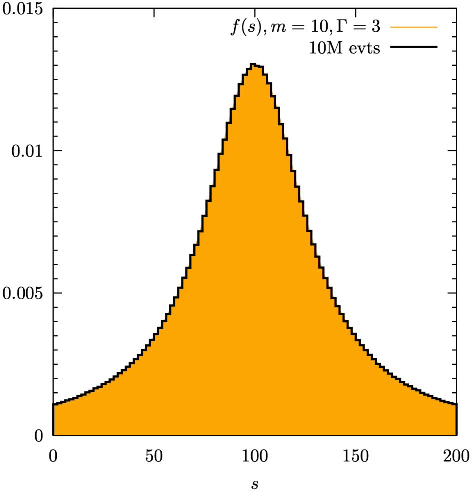
Application: "mapping of divergencies".

Breit-Wigner resonance:

$$
\begin{aligned}
I &= \int_{s_0}^{s_1} \frac{ds}{(s-m^2)^2 + m^2\Gamma^2} \\
&= \frac{1}{m\Gamma} \int_{y_0}^{y_1} \frac{dy}{y^2+1} \\
&= \frac{1}{m\Gamma} \arctan \frac{s-m^2}{m\Gamma} \bigg|_{s_0}^{s_1}
\end{aligned}
$$

with

$$
y = \frac{s-m^2}{m\Gamma}
$$



VEGAS Algorithm [G.P. Lepage, 1980]

How to find the $g(x)$?

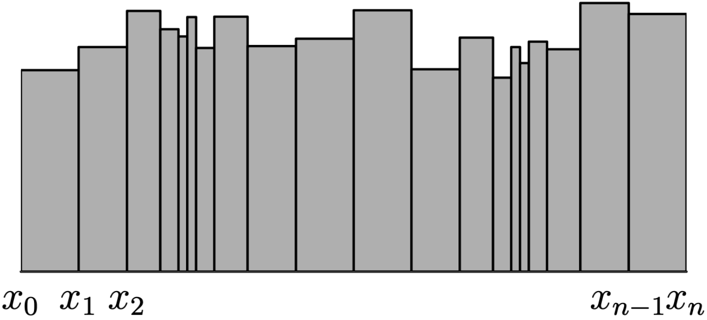
1. Divide the integration interval into N bins B_i with bin width $\Delta x_i = x_i - x_{i-1}$.
2. $g(x)$ is a constant if $x \in B_i$ and given by

$$g(x) = \frac{1}{N\Delta x_i}$$

3. Start with equidistant grid and compute the first estimate of the integral.
4. Rebin the bin widths based on the variance distribution.
5. The set of updated parameters Δx_i defines the integration for the next iteration.
6. For d dimensions integration, $g(x) = \prod_{k=1}^{d} g(x_k)$.

VEGAS Algorithm [G.P. Lepage, 1980]

Rebinning:



[T. Ohl, VAMP]

$\rightarrow$ Sample frequently where the density is high.

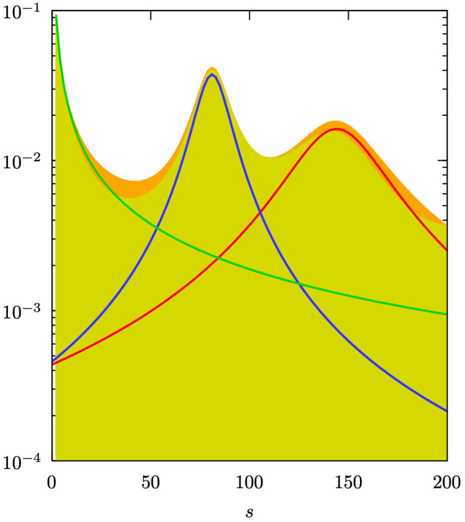
$f(s)$ has mulitple peaks is normal in particle physics.

$$\begin{aligned} I &= \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g(s) ds \\ &= \sum_i \alpha_i \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g_i(s) ds \end{aligned}$$

The weights α_i should

$$\sum_i \alpha_i = 1$$

α_i can be optimized iteratively.



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The general cross section formula is

$$\sigma = \int f_a^A(x_1, Q^2) f_b^B(x_2, Q^2) \frac{|\mathcal{M}|^2 \Theta(\text{cuts})}{F} dx_1 dx_2 d\Phi_n,$$

We first convert the integration variables:

$$\frac{1}{F} dx_1 dx_2 d\Phi_n = J(\vec{x}) \prod_{i=1}^{3n-2} dx_i$$

So we have:

$$f(\vec{x}) = J(\vec{x}) f_A f_B |\mathcal{M}|^2 \Theta(\text{cuts})$$

Now we can calculate the cross section:

$$
\begin{aligned}
\sigma &= \int f(\vec{x}) d^{3n-2}\vec{x} \\
&= \frac{1}{N}\sum_{i=1}^{N} \frac{f(\vec{x})}{g(\vec{x})} \\
&= \frac{1}{N}\sum_{i=1}^{N} w_i
\end{aligned}
$$

We have generated events $\vec{x}$ with weight w_i.

Reweighting

Now we have weighted events, but we want to unweight them because events have same probability in nature!

1. Choose a suitable weight $w_{\max}$.
2. Accept events with probability

$$P_i = \frac{w_i}{w_{\max}}$$

3. When $\max w_i = \bar{w}_{\max} > w_{\max}$,

$$P_i = \frac{w_i}{\bar{w}_{\max}} = \frac{w_i}{w_{\max}} \frac{w_{\max}}{\bar{w}_{\max}}$$

4. Reject events with probability $w_{\max}/\bar{w}_{\max}$

NLO calculation

A general formula for NLO cross section:

$$\sigma_{NLO} = \int d\Phi_n \mathcal{B} + \int d\Phi_n \mathcal{V} + \int d\Phi_{n+1} \mathcal{R}$$

The divergencies coming from virtual corrections are canceled by the divergencies coming from real corrections. → KLN thoerem

In pratice, we add a subtraction term to the formula:

$$\begin{aligned}\sigma_{NLO} &= \int d\Phi_n \mathcal{B} + \int d\Phi_n \mathcal{V} + \int d\Phi_{n+1} \mathcal{S} \\ &\quad + \int d\Phi_{n+1} \mathcal{R} - \int d\Phi_{n+1} \mathcal{S} \\ &= \int d\Phi_n \left(\mathcal{B} + \mathcal{V} + \int d\Phi_1 \mathcal{S} \right) \\ &\quad + \int d\Phi_{n+1} \left(\mathcal{R} - \mathcal{S} \right)\end{aligned}$$

→ FKS substraction

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- MC method is the only choice for phase space integration.
- MC techniques aim to generate events efficiently with small variance.
- NLO calculation is automated in modern MC generators.

Further topics:

- Matrix elements calculation.
- Parton Distribution Function.

Q&A

Thank you for listening!

Your feedback will be highly appreciated!