Parton Shower

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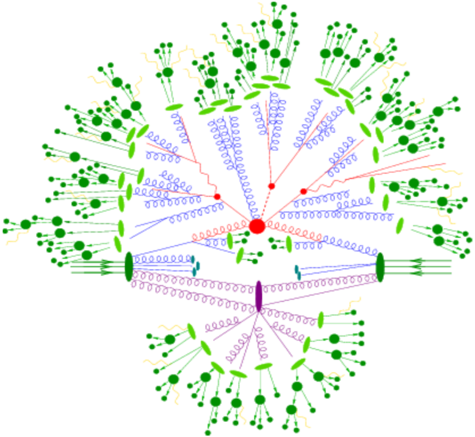

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Why Parton Shower?

- Hard collision occurs at scale $Q \sim \mathcal{O}(100)$ GeV or $\mathcal{O}(1)$ TeV.
- We may see several well separated jets in an event, but jets-inside-jets-inside-jets.
- The large phase space allows to produce partons perturbatively until $Q \sim 1$ GeV.
- An event may contain more than one hundred partons.
- Performing matrix element calculation is not a good idea.

Parton showers attempt to describe how a basic hard process is dressed up by emissions at successively "softer" and/or more "collinear" resolution scales. [C. Bierlich *et al.*, 2022]

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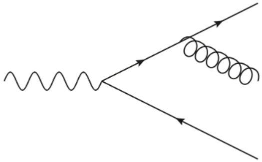
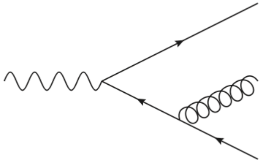
2 Theory for Parton Shower

Let's consider $e^+e^- \to q\bar{q}g$. We write moementa in terms of

$$x_i = \frac{2p_i \cdot q}{Q^2}, \ (i = 1, 2, 3)$$

where $0 < x_i < 1$, $x_1 + x_2 + x_3 = 2$,
$q = (Q, 0, 0, 0)$, $Q = E_{cm}$.





Differential cross section is

$$\frac{d\sigma}{d\cos\theta dx_3} = \sigma_{q\bar{q}} C_F \frac{\alpha_S}{2\pi}\left[\frac{2}{\sin^2\theta}\frac{1+(1-x_3)^2}{x_3} - x_3\right],$$

where θ is the angle between quark and gluon.

- Collinear limit: $\theta \to 0$.
- Soft limit: $x_3 \to 0$.

Seperate into two independent collinear regions:

$$\frac{2}{\sin^2\theta} = \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = \frac{1}{1-\cos\theta} + \frac{1}{1-\cos\tilde{\theta}}$$

where $\tilde{\theta}$ is the angle between anti-quark and gluon.

We know:

$$\frac{d\cos\theta}{1-\cos\theta} \approx \frac{d\theta^2}{\theta^2}$$

So we can rewrite $d\sigma$:

$$d\sigma \approx \sigma_{q\bar{q}} \sum \frac{d\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} C_F \frac{1+(1-z)^2}{z} dz,$$

where

$$P(z) = C_F \frac{1+(1-z)^2}{z}$$

is DGLAP splitting function.

DGLAP Splitting functions

- $q \to qg$:

$$P_{q\to qg}(z) = C_F \frac{1+(1-z)^2}{z}$$

- $g \to gg$:

$$P_{g\to gg}(z) = C_A \frac{[1-z(1-z)]^2}{z(1-z)}$$

- $g \to qq$:

$$P_{g\to qq}(z) = T_R[z^2 + (1-z)^2]$$

$C_F = (N_c^2 - 1)/2N_c$, $C_A = N_c$ and $T_R = 1/2$.

In MC event generators, the variable θ^2 is replaced by

- $t = q^2 = z(1-z)\theta^2 E^2$: virtuality of off-shell propagator.
- $p_\perp^2 = z^2(1-z)^2\theta^2 E^2$: gluon transverse momentum w.r.t. parent qurak.

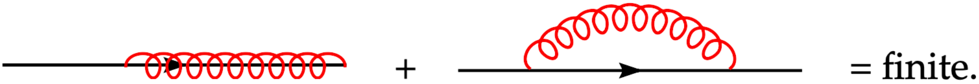
$$\frac{d\theta^2}{\theta^2} = \frac{dq^2}{q^2} = \frac{dp_\perp^2}{p_\perp^2} = \frac{d\rho}{\rho}$$

We can write the emission probability:

$$\mathcal{P} = \frac{\sigma_{q\bar{q}g}}{\sigma_{q\bar{q}}} = \int_{\rho_{\min}}^{\rho_{\max}} \frac{d\rho}{\rho} \frac{\alpha_S}{2\pi} \int_{z_{\min}}^{z_{\max}} P(z) dz$$

Resolution

- We need to introduce resolution ρ_{min}.
- Emissions below ρ_{min} are unresolvable.
- Unresolvable emission + virtual corrections can get a finite result.



No-emission Probability

Unitarity / probability conservation: $\mathcal{P}_{\text{no-em}} = 1 - \mathcal{P}$

$$
\begin{aligned}
& \mathcal{P}_{\text{no-em}}(\rho_{\max} > \rho > \rho_{\min}) \\
= \; & \mathcal{P}_{\text{no-em}}(\rho_{\max} > \rho > \rho_1) \cdot \mathcal{P}_{\text{no-em}}(\rho_1 > \rho > \rho_{\min}) \\
= \; & \lim_{n\to\infty} \prod_{i=0}^{n-1} \mathcal{P}_{\text{no-em}}(\rho_i > \rho > \rho_{i+1}) \\
= \; & \lim_{n\to\infty} \prod_{i=0}^{n-1} (1 - \mathcal{P}(\rho_i > \rho > \rho_{i+1})) \\
= \; & \lim_{n\to\infty} \exp\left(- \sum_{i=0}^{n-1} \mathcal{P}(\rho_i > \rho > \rho_{i+1}) \right) \\
= \; & \exp\left(- \int_{\rho_{\min}}^{\rho_{\max}} d\mathcal{P}(\rho) \right)
\end{aligned}
$$

Now we have emission and no-emission probability:

$$
\begin{aligned}
d\mathcal{P}(\rho) &= \frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}} P(z)dz \\
\mathcal{P}_{\text{no-em}}(\rho_1, \rho_2) &= \exp\left(-\int_{\rho_2}^{\rho_1}\frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}} P(z)dz\right)
\end{aligned}
$$

Probability of parton i to have hardest splitting follows Poission statistics:

$$
d\mathcal{P}_{\text{first}}(\rho) = d\mathcal{P}(\rho)\cdot\mathcal{P}_{\text{no-em}}(\rho_{\max}, \rho)
$$

$\Delta(\rho_1, \rho_2) = \mathcal{P}_{\text{no-em}}(\rho_1, \rho_2)$ is so-called Sudakov form factor.

Sudakov Form Factor

We can expand Sudakov form factor in orders of strong coupling:

$$\begin{aligned}\Delta(\rho_1,\rho_2) &= \exp\left(-\int_{\rho_2}^{\rho_1}\frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}}P(z)dz\right)\\ &= 1-\int_{\rho_2}^{\rho_1}\frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}}P(z)dz\\ &\quad -\frac{1}{2}\left(\int_{\rho_2}^{\rho_1}\frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}}P(z)dz\right)^2+\ldots\end{aligned}$$

No emission probability $\rightarrow$ no change in state $\rightarrow$ virtual corrections

Sudakov contains all orders of divergencies of virtual corrections

The Sudakov Veto Algorithm

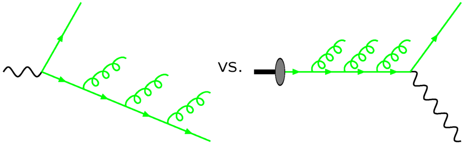
- Start with n partons at scale ρ, evolve simultaneously.
- Sudakov form factors factorize:

$$\Delta(\rho_1, \rho) = \prod_{i=1}^{n} \Delta_i(\rho_1, \rho), \ \Delta_i(\rho_1, \rho) = \prod_{j=q,g} \Delta_{i\to j}(\rho_1, \rho).$$

- Use veto algorithm to find scales of subsequent emissions
  - Propose ρ using MC based on overestimate $P_{ij}^{\max}(z)$.
  - Determine “winner” parton i and new flavour j.
  - Select splitting variable z according to overestimate $P_{ij}^{\max}(z)$.
  - Accept splitting with probability $P_{ij}(z)/P_{ij}^{\max}(z)$, else continue sampling from present scale.
- Construct full splitting kinematics and color configuration.
- Iterate until reaching cutoff $\rho_{\min} \sim 1$ GeV.

Initial State Radiation

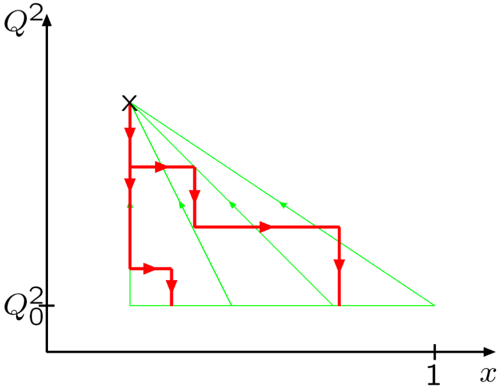
2 Theory for Parton Shower



We can modify the emission and no-emission probabilities to include PDFs: $x' = x/z$.

$$
\begin{aligned}
d\mathcal{P}(\rho) &= \frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}} P(z)\frac{x' f_i(x',\rho)}{x f_j(x,\rho)}dz \\
\mathcal{P}_{\text{no-em}}(\rho_1,\rho_2) &= \exp\left(-\int_{\rho_2}^{\rho_1}\frac{d\rho}{\rho}\frac{\alpha_S}{2\pi}\int_{z_{\min}}^{z_{\max}} P(z)\frac{x' f_i(x',\rho)}{x f_j(x,\rho)}dz\right)
\end{aligned}
$$

A possible evolution:



Green: physical process.

Red: shower generation.

Angular Radiation Pattern

Consider $e^+e^- \to q\bar{q}g$ with soft gluon. Matrix element can be written in terms of energies and angles:
[Marchesini and Webber, 1988]

$$\frac{2p_q p_{\bar{q}}}{(p_q p_g)(p_g p_{\bar{q}})} = \frac{W_{q\bar{q}}}{E_g^2}$$

The “Antenna” radiation term:

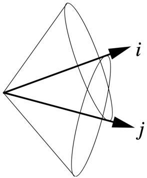
$$W_{q\bar{q}} = \frac{1 - \cos\theta_{q\bar{q}}}{(1 - \cos\theta_{qg})(1 - \cos\theta_{\bar{q}g})}$$

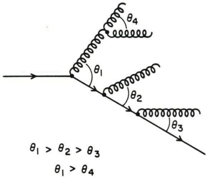
Split $W_{q\bar{q}}$ into "Dipole" terms $W_{q\bar{q}} = W_{q\bar{q}}^{(q)} + W_{q\bar{q}}^{(\bar{q})}$, where

$$W_{q\bar{q}}^{(q)} = \frac{1}{2}\left[W_{q\bar{q}} + \frac{1}{1-\cos\theta_{qg}} - \frac{1}{1-\cos\theta_{\bar{q}g}}\right]$$

Azimuthal integration gives angular ordering:

$$\int_0^{2\pi} \frac{d\phi_{qg}}{2\pi} W_{q\bar{q}}^{(q)} = \begin{cases} \frac{1}{1-\cos\theta_{qg}}, & \text{if } \theta_{qg} < \theta_{q\bar{q}} \\ 0, & \text{else} \end{cases}$$





Pseudorapidity of the third jet from [CDF, 1994]

- Very old Pythia: purely virtual ordered: too many events in the central region.
- Very old Pythia+: additional phase space constraint on initial-final dipole (angular veto). It seems OK.
- Herwig angular ordered: OK.

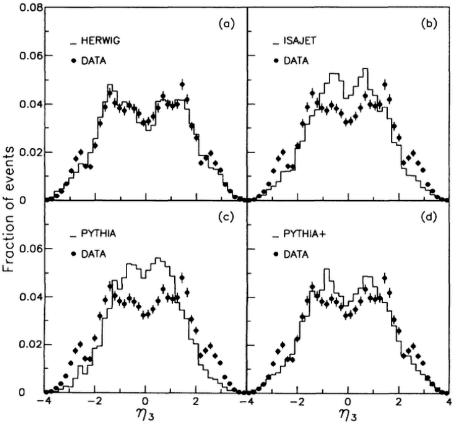


FIG. 13. Observed η_3 distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

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Fixed-Order Calculations

In general, expectation value of an observable $\mathcal{O}$ at LO is

$$\langle\mathcal{O}\rangle_{LO} = \int d\Phi_n \mathcal{B}(\Phi_n)\mathcal{O}(\Phi_n)$$

At NLO:

$$\begin{aligned}\langle\mathcal{O}\rangle_{NLO} &= \int d\Phi_n \left[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n) + \mathcal{I}(\Phi_n)\right]\mathcal{O}(\Phi_n) \\ &\quad + \int d\Phi_{n+1}\left[\mathcal{R}(\Phi_{n+1})\mathcal{O}(\Phi_{n+1}) - \mathcal{S}(\Phi_{n+1})\mathcal{O}(\Phi_n)\right],\end{aligned}$$

where $\mathcal{I}(\Phi_n) = \int d\Phi_1 \mathcal{S}(\Phi_{n+1})$.

Matching at LO

We can simply replace the $\mathcal{O}$ with parton shower:

$$\langle \mathcal{O} \rangle_{LOPS} = \int \Phi_n \mathcal{B}(\Phi_n) \mathrm{PS}_n(\rho_0, \mathcal{O})$$

where

$$\mathrm{PS}_n(\rho_0, \mathcal{O}) = \Delta_n(\rho_0, \rho_c)\mathcal{O}(\Phi_n) + \int_{\rho_c}^{\rho_0} d\Phi_1 K_n(\Phi_1)\Delta_n(\rho_0, \rho)\mathrm{PS}_{n+1}(\rho, \mathcal{O})$$

is called "shower operator".

LOPS is trivial. There is no overlap in the terms included in the parton shower and the leading order ME.

Matching at NLO

What happens if we do the same thing to NLO?

$$
\begin{aligned}
\langle\mathcal{O}\rangle_{NLOPS} \;\; = \;\; & \int d\Phi_n \left[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n) + \mathcal{I}(\Phi_n)\right] \mathrm{PS}_n(\rho_0, \mathcal{O}) \\
& + \int d\Phi_{n+1} \left[\mathcal{R}(\Phi_{n+1})\mathrm{PS}_{n+1}(\rho_R, \mathcal{O}) - \mathcal{S}(\Phi_{n+1})\mathrm{PS}_n(\rho_0, \mathcal{O})\right],
\end{aligned}
$$

Obviously, there is a problem in the second line. $\mathcal{R}$ and $\mathcal{S}$ obtain different parton shower corrections.

Matching at NLO

The solution is

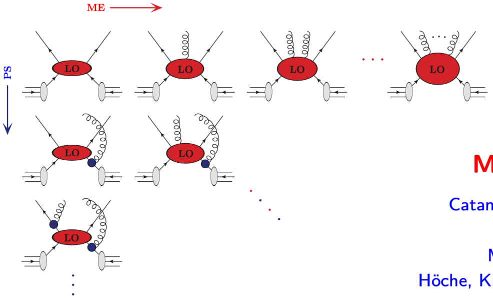
$$\begin{aligned}\langle \mathcal{O} \rangle_{NLOPS} &= \int d\Phi_n \left[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n) + \mathcal{I}_K(\Phi_n)\right] \widetilde{\mathrm{PS}}_n(\rho_0, \mathcal{O}) \\ &\quad + \int d\Phi_{n+1} \left[\mathcal{R}(\Phi_{n+1}) - \mathcal{S}_K(\Phi_{n+1})\right] \mathrm{PS}_{n+1}(\rho_R, \mathcal{O}),\end{aligned}$$

where

$$\mathcal{B}(\Phi_n)\widetilde{\mathrm{PS}}_n(\rho_0, \mathcal{O}) = \left[\mathcal{B}(\Phi_n) - \mathcal{I}_K(\Phi_n)\right] \mathcal{O}(\Phi_n) + \mathcal{S}_K(\Phi_{n+1})\mathcal{O}(\Phi_{n+1})$$

There are two famous matching scheme: MC@NLO and POWHEG. They have different choices of $\mathcal{S}_K$ and ρ_0.

Merging at LO



MePs (CKKW,MLM)

Catani, Krauss, Kuhn, Webber JHEP11(2001)063
Lönnblad JHEP05(2002)046
Mangano, Moretti, Pittau NPB632(2002)343
Höche, Krauss, Schumann, Siegert JHEP05(2009)053
Lönnblad, Prestel JHEP03(2012)019

- Matrix Element (ME) and Parton Shower (PS) are approximations in different region of phase space.
- ME handles the hard wide-angle emissions.
- PS resummates the soft and collinear limit.
- MEPS combines multiple LOPS.

Merging at LO

$$
\begin{aligned}
\langle \mathcal{O} \rangle_{MEPS} \quad = \quad & \int \Phi_n \mathcal{B}_n \mathrm{PS}_n(\mathcal{O}) \Theta(Q_{cut} - Q_{n+1}) \\
& + \int \Phi_{n+1} \mathcal{B}_{n+1} \Theta(Q_{n+1} - Q_{cut}) \Delta_n(\rho_n, \rho_{n+1}) \mathrm{PS}_{n+1}(\mathcal{O}) \Theta(Q_{cut} - Q_{n+2}) \\
& + \int \Phi_{n+2} \mathcal{B}_{n+2} \Theta(Q_{n+2} - Q_{cut}) \Delta_n(\rho_n, \rho_{n+1}) \Delta_{n+1}(\rho_{n+1}, \rho_{n+2}) \\
& \times \mathrm{PS}_{n+2}(\mathcal{O}) \Theta(Q_{cut} - Q_{n+3}) + \ldots
\end{aligned}
$$

- PS of $2 \to n$ process generate the first emission, with retriction $Q_{n+1} < Q_{cut}$.
- The $n+1$ ME and its parton shower is added. Events with $Q_{n+1} < Q_{cut}$ are generated by ME, while $Q_{n+2} > Q_{cut}$ are excluded in PS.
- The $2 \to n$ Sudakov is multiplied to restore resummation.
- Iterate

$$
\begin{aligned}
\langle\mathcal{O}\rangle_{MEPS@NLO} \quad = \quad & \int d\Phi_n \bar{\mathcal{B}}_n \widetilde{\mathrm{PS}}_n(\mathcal{O})\Theta(Q_{cut} - Q_{n+1}) \\
& + \int d\Phi_{n+1} \mathcal{H}_n \Theta(Q_{cut} - Q_{n+1}) \mathrm{PS}_{n+1}(\mathcal{O}) \\
& + \int d\Phi_{n+1} \bar{\mathcal{B}}_{n+1} \Theta(Q_{n+1} - Q_{cut}) \left(\Delta_n(\rho_n, \rho_{n+1}) - \Delta^{(1)}(\rho_n, \rho_{n+1})\right) \\
& \times \widetilde{\mathrm{PS}}_{n+1}(\mathcal{O})\Theta(Q_{cut} - Q_{n+2}) \\
& + \int d\Phi_{n+2} \mathcal{H}_{n+1} \Theta(Q_{n+1} - Q_{cut}) \mathrm{PS}_{n+1}(\mathcal{O})\Theta(Q_{cut} - Q_{n+2}) + \dots
\end{aligned}
$$

- Similar to LO, but we replace the LOPS by NLOPS.
- When multiplying the Sudakov, we have to remove overlap of Δ_n and $d\sigma_{n+1}^{NLO}$.

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Summary

Final states generated by parton shower depend on

- Choice of evolution variable: t, $p_\perp^2$, θ^2.
- Choice of phase space mapping $d\Phi_n \to d\Phi_{n+1}$.
- Choice of radiation functions, i.e. DGLAP vs. dipole/antenna.
- Choice of renormalization scale μ_R.
- Choice of starting and ending scales, i.e. phase space cuts, hadronization scale.
- Handling of azimuthal correlations and colour configuration.

Matching and Merging improve the accuracy of the parton shower.

Q&A

Thank you for listening!

Your feedback will be highly appreciated!