Book: Essential Math. Methods for the Physical Science

Quantum Computing

Lecture - 7, Part - 1

Matrix Mech.

1. **Trace** :- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ sum

Then, $Tr\, A = 1 + 4 = 5$

2. **Determinant** :- $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$

$= 4 - 6 = -2$

3. **Transpose** :- Row → Column ; Column → Row

So, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

If, $B = \begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 7 & 1 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 2 & 4 & 7 \\ 3 & 9 & 1 \end{bmatrix}$

4. ~~Conjugate~~ **Complex Conjugate**: All elements inside the matrix will be conjugate

So, $A^* = \begin{bmatrix} 1^* & 2^* \\ 3^* & 4^* \end{bmatrix}$

If, $A = \begin{bmatrix} 1-2i & 2 \\ 3 & 4+5j \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} 1+2i & 2 \\ 3 & 4-5j \end{bmatrix}$

5. Hermitian Conjugate: Transpose + Conjugate

So, $A^{\dagger} = (A^{*})^{T}$ or $(A^{T})^{*}$

So, $A = \begin{bmatrix} 1+3j & 4 \\ 2 & 5-2j \end{bmatrix} \Rightarrow A^{\dagger} = \begin{bmatrix} 1-3j & 2 \\ 4 & 5+2j \end{bmatrix}$

6. Cofactor:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Cofactor $1(45-48) = -3$

$-2(36-42) = 12$

7. Inverse:

$A^{-1} = \frac{C^T}{|A|}$

C^T = Transpose of Cofactormatrix of matrix A

prerequisite: Needs to be square matrix.

$\boxed{\text{Note}}$ In general, $AB \neq BA$.

But for Diagonal Matrix $AB = BA$

Diagonal Matrix Example = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. Real :

$A^* = A$ [Because there will be no imaginary numbers]

9. Imaginary : $A^* = -A$

10. Symmetric : Matrix is equal to it's transpose matrix. So, $A = A^T$

Say, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

11. Hermitian : The matrix is equal to it's Hermitian Conjugate.

$A = A^\dagger = (A^T)^*$

12. Anti-Hermitian : $A^\dagger = -A$

~~13. Unitary :~~

13. Orthogonal : $A^T = A^{-1}$ | The transpose is equal to the inverse.

14. Unitary : $A^\dagger = (A^T)^* = A^{-1}$ | Hermitian conjugate is equal to inverse

15. Singular: The matrix has 0 determinant.

So, $|A| = 0$

Effects of matrix operation on matrix product:

Hermitian Conjugate: $(A \cdot B \cdots G)^{\dagger} = G^{\dagger} \cdots B^{\dagger} \cdot A^{\dagger}$

Eigenvalue & Eigenvectors

Say, $\vec{B} = 5\hat{i} + 6\hat{j} + 7\hat{k}$

we can write, $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

In that case, we can write B as,

$$|B\rangle = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$ [$|\,\rangle$ is called KET]

B has to be a column matrix.

Let's say we have an eqn :-

Eigenvector

$$\text{Hermitian} \leftarrow A \cdot |B\rangle = 5\,|B\rangle$$

Eigenvalue

Here, if A is Hermitian and 5 is Real Number. Hence, $|B\rangle$ is eigenvector of A and 5 is eigenvalue.

Math Example From Wikipedia:

$$|A - \lambda I| = 0$$

eigen vector eqn :-

$$A|B\rangle = \lambda|B\rangle$$

$$\Rightarrow A|B\rangle = \lambda(I|B\rangle)$$

$$\Rightarrow A|B\rangle - \lambda(I|B\rangle) = 0$$

~~$\Rightarrow A.|B\rangle -$~~

$$\Rightarrow |B\rangle(A - \lambda I) = 0$$

Given, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $\therefore A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$\therefore |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 1$

$= \lambda^2 - 4\lambda + 3$

From, $|A - \lambda I| = 0$

we get, $\lambda^2 - 4\lambda + 3 = 0$ $[\because |A - \lambda I| = \lambda^2 - 4\lambda + 3]$

$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$

$\Rightarrow \lambda(\lambda - 3) - 1(\lambda - 3) = 0$

$\Rightarrow (\lambda - 3)(\lambda - 1) = 0$

$\Rightarrow \lambda = 3$ & $\lambda = 1$

So, Two eigen value is 3 & 1.

~~for $V_{\lambda=1}$~~ $\Rightarrow$ <u>Eigen vector eq^n:</u>

$$\vec{A}\,|V_{\lambda=1}\rangle = 1\,|V_{\lambda=1}\rangle \quad \text{------ (a)}$$

$$\&,\ \vec{A}\,|V_{\lambda=3}\rangle = 3\,|V_{\lambda=3}\rangle \quad \text{------ (b)}$$

Now let, $|V_{\lambda=1}\rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\therefore \vec{A}\cdot|V_{\lambda=1}\rangle \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

from here, we get,

$$2x_1 + x_2 = x_1 \quad \text{--- (i)}$$

$$x_1 + 2x_2 = x_2 \quad \text{--- (ii)}$$

$$\left.\begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix}\right\} \text{Same eq}^n$$

$$x_1 = -x_2$$

But, To get unique eigen vector, Let us Consider that the eigen vectors are normalized :-

$$\Rightarrow x_1^2 + x_2^2 = 1$$

$$\therefore x_1^2 + (-x_1)^2 = 1$$

$$\Rightarrow x_1^2 + x_1^2 = 1$$

$$\Rightarrow 2x_1^2 = 1$$

$$\Rightarrow x_1^2 = \frac{1}{2}$$

$$\therefore x_1 = \frac{1}{\sqrt{2}}$$

$$\therefore x_2 = -\frac{1}{\sqrt{2}}$$

So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = V_{\lambda=1}$

~~Putting this value in eq^n (a) we get,~~

Also, Let, $V_{\lambda=3} = 3\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\therefore \vec{A} \cdot |V_{\lambda=3}\rangle \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3y_1 \\ 3y_2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6y_1 + 3y_2 \\ 3y_1 + 6y_2 \end{bmatrix} = 3\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ from eq^n (b)

$\Rightarrow 6y_1 + 3y_2 = 3y_1$

$\Rightarrow 3y_1 + 3y_2 = 0$

again, $3y_1 + 6y_2 = 3y_2$

$\Rightarrow 3y_1 + 3y_2 = 0 \quad \therefore y_1 = -y_2$

But, To get the unique eigen vectors let us consider that the eigen vectors are normalized

Then, $y_1^2 + y_2^2 = 1$

$\Rightarrow (-y_2)^2 + y_2^2 = 1$

$\Rightarrow 2y_2^2 = 1 \Rightarrow y_2^2 = \frac{1}{2} \quad \therefore y_2 = \pm\frac{1}{2}$

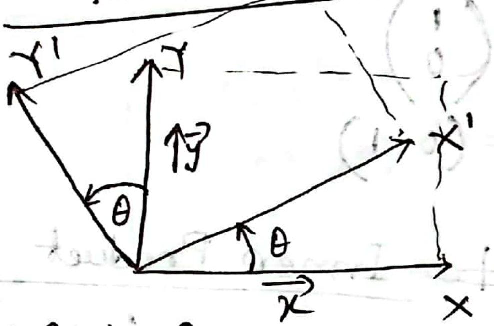
$\therefore y_1 = \pm\frac{1}{2}$

Lecture-7 Part-2

We are working on Hilbert space not vector space

That's why
$$|A\rangle = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 9 \\ \vdots \end{pmatrix}$$
is possible.

Rotational Matrix



we know,

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_r \\ E_\theta \\ E_\varphi \end{bmatrix}$$

Transformation matrix; Electro magnetic field; Resultant magnetic field

Here, $\vec{x} = 1\hat{i} + 0\hat{j}$; $\vec{y} = 0\hat{i} + 1\hat{j}$

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{R} \cdot |x\rangle = |x'\rangle \quad ; \quad \hat{R}|y\rangle = |y'\rangle$$

$$\therefore \hat{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} ; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\vec{x'} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad ; \quad \vec{y'} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$| \ \rangle \to$ Ket ; $\langle \ | \to$ Bra

In quantum computing :-

$$\underset{\text{Input/Initial state}}{|x\rangle} \longrightarrow \underset{}{\overset{\text{Gate}}{\boxed{\hat{R}}}} \longrightarrow \underset{\text{output/final state}}{|x'\rangle}$$

So, for above instance

$$\text{for, } X \to \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \boxed{\hat{R}} \longrightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\text{for, } Y \to \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \boxed{\hat{R}} \longrightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

If, $\vec{x} = |X\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(X^T)^* = \langle X| = (0 \ \ 1)$

Inner Product

$$\langle x||x\rangle = (0 \ \ 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \langle X|X\rangle = [1] \quad \because \underline{\hat{i}\cdot\hat{i} = 1}$$

This is called Inner product

Again, $\langle Y||X\rangle = (1 \ \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \langle Y|X\rangle = [0], \ \underline{\hat{j}\cdot\hat{i} = 0}$$

They are orthonormal base. Because $\left| \begin{matrix} \hat{i}\cdot\hat{i} = 1 \\ \hat{i}\cdot\hat{j} = 0 \end{matrix} \right.$

Outer Product

$$|x\rangle\langle x| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

& $$|Y\rangle\langle Y| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

Inner Product Representation

Function Rep.	Hilbert Space	Vector Space
$\langle \varphi_2 \| \varphi_1\rangle$	$\langle Y\|x\rangle$	$\hat{j}\cdot\hat{i}$
$\langle \varphi_1, \varphi_2\rangle$	$= (0^* \; 1^*)\begin{pmatrix}1\\0\end{pmatrix}$	
$\int \varphi_2^* \varphi_1 \, dv$		
$\int \varphi_1 \varphi_2^* \, dv$		

$\hat{R}|x\rangle = |x'\rangle$

$\Rightarrow \left(\hat{R}|x\rangle\right)\langle x| = |x'\rangle\langle x|$

$\Rightarrow \hat{R}\left(|x\rangle\langle x|\right) = |x'\rangle\langle x|$ --- (i)

$\hat{R}|Y\rangle = |Y'\rangle$

$\Rightarrow \left(\hat{R}|Y\rangle\right)\langle Y| = |Y'\rangle\langle Y|$

$\Rightarrow \hat{R}\left(|Y\rangle\langle Y|\right) = |Y'\rangle\langle Y|$ --- (ii)

(i)+(ii)

$\hat{R}\left(|x\rangle\langle x|\right) + \hat{R}\left(|Y\rangle\langle Y|\right) = |x'\rangle\langle x| + |Y'\rangle\langle Y|$

$\Rightarrow \hat{R}\left(|x\rangle\langle x| + |Y\rangle\langle Y|\right) = |x'\rangle\langle x| + |Y'\rangle\langle Y|$

--- (iii)

Now,

$$|X'\rangle\langle X| = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (1, 0) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix}$$

And, $|Y'\rangle\langle Y| = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} (0 \ \ 1) = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix}$

Putting all the values in eq^n (iii) we get

$$\Rightarrow \hat{R}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{bmatrix}$$

$$\Rightarrow \hat{R}\,\bar{\bar{I}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \hat{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

* Completeness theorem :-

$$|X\rangle\langle X| + |Y\rangle\langle Y| + |Z\rangle\langle Z| = \bar{\bar{I}} \ \text{[Identity matrix]}$$

also, $\hat{\text{operator}} = |\text{output}_1\rangle\langle \text{Input}_1| + |\text{output}_2\rangle\langle \text{input}_2| + |\text{output}_3\rangle\langle \text{input}_3|$.

Book: LAD → Linear Alg. Demystified

LAD → 7.8 :

Given, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and, $H_{v_1} = \dfrac{V_1 + V_2}{\sqrt{2}}$; $H_{v_2} = \dfrac{V_1 - V_2}{\sqrt{2}}$

$H = 2$

Now,

$\hat{H} \,|v_1\rangle = |v_1'\rangle = H_{v_1} = \frac{1}{\sqrt{2}} v_1 + \frac{1}{\sqrt{2}} v_2$

again, $\hat{H} \,|v_2\rangle = |v_2'\rangle = H_{v_2} = \frac{1}{\sqrt{2}} v_1 - \frac{1}{\sqrt{2}} v_2$

we know, $\hat{H} = |\text{output}_1\rangle\langle \text{Input}_1| + |\text{output}_2\rangle\langle \text{Input}_2|$

$\therefore \hat{H} = |v_1'\rangle\langle v_1| + |v_2'\rangle\langle v_2| \quad \ldots (i)$

$|v_1'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|v_2'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

putting the values in eqn. (i)

$\Rightarrow \hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\langle X|X\rangle = 1$

$\langle X|Y\rangle = 0$

Lecture 8 - Part 1

Alternative way to find Rotational Matrix :-

$\hat{R}\cdot|X\rangle = |X'\rangle$

Let, $\hat{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \hat{R}|X\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{bmatrix} a \\ c \end{bmatrix} = |X'\rangle$

$\Rightarrow \hat{R}|Y\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = |Y'\rangle$

$\therefore |X'\rangle = a|X\rangle + c|Y\rangle$

$\Rightarrow \langle X|X'\rangle = a\langle X|X\rangle + c\langle X|Y\rangle$ [mult. wrt $\langle X|$]

$= a\cdot 1 + c\cdot 0 = a$

$\Rightarrow \langle Y|X'\rangle = a\langle Y|X\rangle + c\langle Y|Y\rangle$ [mult. wrt $\langle Y|$]

$= a\cdot 0 + c\cdot 1 = c$

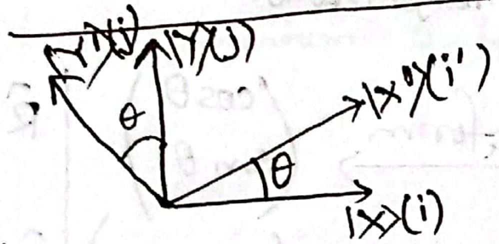
Again, $|Y'\rangle = b\,|X\rangle + d\,|Y\rangle$

$$\Rightarrow \langle X|Y'\rangle = b\langle X|X\rangle + d\langle X|Y\rangle$$
$$= b\cdot 1 + d\cdot 0 = b$$

$$\Rightarrow \langle Y|Y'\rangle = b\langle Y|X\rangle + d\langle Y|Y\rangle$$
$$= b\cdot 0 + d\cdot 1 = d$$

putting the value of a, b, c, d in matrix $\hat{R}$,

$$\Rightarrow \hat{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \langle X|X'\rangle & \langle X|Y'\rangle \\ \langle Y|X'\rangle & \langle Y|Y'\rangle \end{bmatrix}$$



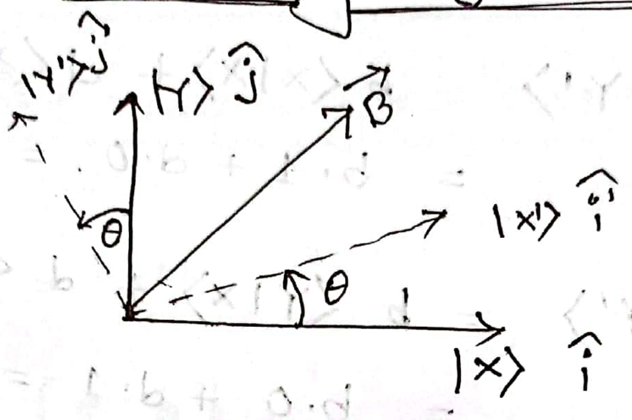
$$= \begin{bmatrix} \hat{i}\cdot\hat{i}' & \hat{i}\cdot\hat{j}' \\ \hat{j}\cdot\hat{i}' & \hat{j}\cdot\hat{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \langle X|R|X\rangle & \langle X|R|Y\rangle \\ \langle Y|R|X\rangle & \langle Y|R|Y\rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1\ 0)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} & (1\ 0)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ (0\ 1)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} & (0\ 1)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Change of basis



Previously,
$$\hat{R}|X\rangle = |X'\rangle = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \cos\theta\,|X\rangle + \sin\theta\,|Y\rangle$$

But for $\vec{B}$ vector after change of basis,

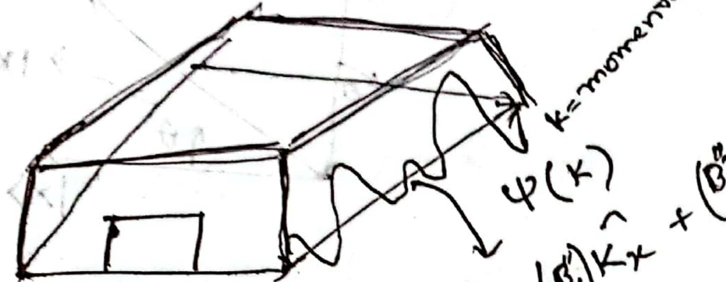
from $\vec{B} = B_x|X\rangle + B_y|Y\rangle \underset{\text{Tranformation}}{\overset{\text{to}}{=}} B_x''|X_0\rangle + B_y''|Y'\rangle$

So, Previously, $\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{\text{Transform}} \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ | $\hat{R}|X\rangle = |X'\rangle$

$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{\text{Tranform}} \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ | $\hat{R}|Y\rangle = |Y'\rangle$

Now, $\begin{pmatrix}B_x''\\ B_y\end{pmatrix} \xrightarrow{\text{Transform}} \begin{pmatrix}B_x''\\ B_y''\end{pmatrix}$ | $\hat{A}\begin{pmatrix}B_x\\ B_y\end{pmatrix} = \begin{pmatrix}B_x''\\ B_y''\end{pmatrix}$

For example : Fourier Transform



$\psi(x)$ $\psi(K)$

Suppose

$\vec{B} = 2\hat{x} + 5\hat{y}$

x = space $\vec{B} = (B_x)\hat{K_x} + (B_y)\hat{K_y}$

function space

$$\psi(K) = \int_{-\infty}^{\infty} \psi(x) \cdot \underbrace{\left[\frac{e^{-jkx}}{\sqrt{2\pi}}\right]}_{\langle K|x\rangle} dx$$

ψ এর only projection নিয়ে কাজ করা হয়।

X space -এ ψ এর projection $\langle x|\psi\rangle$

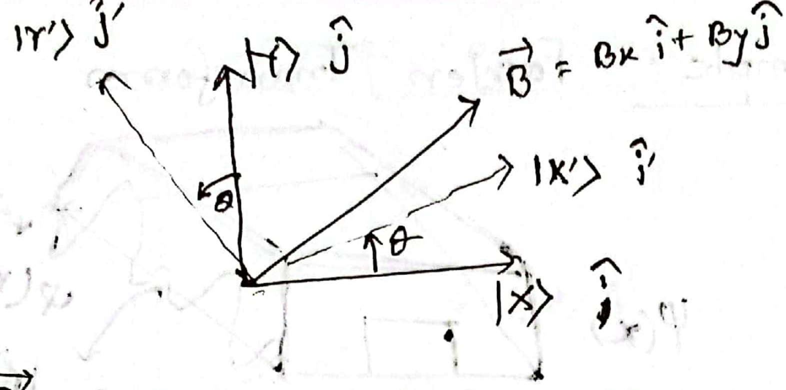
K momentum -এ ψ এর projection $\langle K|\psi\rangle$

$$\langle K|\psi\rangle = \sum \langle x|\psi\rangle\langle K|x\rangle$$ (discrete form -এ $\int \rightarrow \sum$)

Hilbert Space

$$= \sum \langle K|x\rangle\langle x|\psi\rangle$$ [completeness theory $|x\rangle\langle x| + |Y\rangle\langle Y| \ldots = \hat{I}$]

$$= \sum \langle K|\psi\rangle$$



$$\vec{B} = B_x\hat{i} + B_y\hat{j} = B_x''\hat{i}' + B_y''\hat{j}'$$

$$\hat{A}\cdot\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} B_x'' \\ B_y'' \end{pmatrix}$$

If, $\vec{B} = 2\hat{x} + 5\hat{y}$

$$\hat{A}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} B_x'' \\ B_y'' \end{pmatrix}$$

we know, $|B\rangle = B_x|X\rangle + B_Y|Y\rangle$

$$= B_x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B_Y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

Again, $|B\rangle = B_x''|X'\rangle + B_y''|Y'\rangle$ ---- (i)

$$\langle X'|X'\rangle = \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\langle Y'|X'\rangle = \begin{pmatrix} -\cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = 0$$

That's why they are in orthonormal.

From eq^n (1)

$\underline{\langle x'| \times (i)}$

$$\langle x'|B\rangle = B_x'' \cdot \langle x'|x'\rangle + B_y'' \langle x'|Y'\rangle$$

$$= B_x'' \cdot 1 + B_y'' \cdot 0$$

$$= B_x''$$

$\underline{\langle Y'| \times (i)}$

$$\langle Y'|B\rangle = B_x'' \langle Y'|x'\rangle + B_y'' \langle Y'|Y'\rangle$$

$$= B_x'' \cdot 0 + B_y'' \cdot 1$$

$$= B_y''$$

So, we can write,

$$\cancel{|B\rangle = \langle x'|B\rangle \langle x'| \ldots}$$

$$|B\rangle = \langle x'|B\rangle |x'\rangle + \langle Y'|B\rangle |Y'\rangle$$

Note:

$$\vec{A} = 2\hat{i} + 7\hat{j}$$

we can also write

$$\vec{A} = (\hat{i} \cdot \vec{A})\hat{i} + (\hat{j} \cdot \vec{A})\hat{j}$$

$$\boxed{\begin{aligned}\hat{i}\cdot\hat{i} &= 1 \\ \hat{i}\cdot\hat{j} &= 0\end{aligned}}$$

So, $$\vec{A} = (\hat{i}' \cdot \vec{A})\hat{i}' + (\hat{j}' \cdot \vec{A})\hat{j}'$$

$$\vec{A}\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} B_x'' \\ B_y'' \end{pmatrix} \qquad \text{Let, } \vec{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

L.H.S

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} aB_x + bB_y \\ cB_x + dB_y \end{pmatrix}$$

R.H.S

Again, $\begin{pmatrix} B_x'' \\ B_y'' \end{pmatrix} = \begin{pmatrix} \langle x'|B\rangle \\ \langle Y'|B\rangle \end{pmatrix}$

$$\therefore \quad aB_x + bB_y = \langle x'|B\rangle$$
$$cB_x + dB_y = \langle Y'|B\rangle$$

$|B\rangle = B_x|X\rangle + B_y|Y\rangle$

first $\rightarrow$ $\langle x'| \times |B\rangle$
then $\langle Y'| \times |B\rangle$

$$\Rightarrow \quad aB_x + bB_y = B_x\langle x'|X\rangle + B_y\langle x'|Y\rangle$$

$$cB_x + dB_y = B_x\langle Y'|X\rangle + B_y\langle Y'|Y\rangle$$

$$\therefore \quad a = \langle x'|X\rangle \; ; \; b = \langle x'|Y\rangle$$

$$c = \langle Y'|X\rangle \; ; \; d = \langle Y'|Y\rangle$$

$$\therefore \quad \vec{A} = \begin{bmatrix} \langle x'|X\rangle & \langle x'|Y\rangle \\ \langle Y'|X\rangle & \langle Y'|Y\rangle \end{bmatrix}$$

We know from Rotational Matrix :

$$\hat{R} = \begin{bmatrix} \langle X|X'\rangle & \langle X|Y'\rangle \\ \langle Y|X'\rangle & \langle Y|Y'\rangle \end{bmatrix}$$

We can write :

$$\langle X|Y\rangle = \langle Y|X\rangle^*$$

$$(\hat{R})^T = \begin{bmatrix} \langle X|X'\rangle & \langle Y|X'\rangle \\ \langle X|Y'\rangle & \langle Y|Y'\rangle \end{bmatrix}$$

$$(\hat{R}^T)^* = \begin{bmatrix} \langle X'|X\rangle & \langle X'|Y\rangle \\ \langle Y'|X\rangle & \langle Y'|Y\rangle \end{bmatrix} = \hat{A}$$

$\therefore$ Relation between $\hat{A}$ and $\hat{R}$

$$\hat{A} = (R^T)^*$$

Lecture 8 - Part : 2

1. Span (Linear dependant & Independent basis vector)
2. Diagonalization.

Sumarize :

$$\hat{R} = \begin{bmatrix} \langle X|X' \rangle & \langle X|Y' \rangle \\ \langle Y|X' \rangle & \langle Y|Y' \rangle \end{bmatrix} = \begin{bmatrix} \langle in_1|out_1 \rangle & \langle in_1|out_2 \rangle \\ \langle in_2|out_1 \rangle & \langle in_2|out_2 \rangle \end{bmatrix}$$

$$\hat{A} = \left((\hat{R})^T\right)^* = \begin{bmatrix} \langle X'|X \rangle & \langle X'|Y \rangle \\ \langle Y'|X \rangle & \langle Y'|Y \rangle \end{bmatrix} = \begin{bmatrix} \langle \text{out basis}_1 | \text{input basis}_1 \rangle & \langle \text{out basis}_1 | \text{in basis}_2 \rangle \\ \langle \text{out basis}_2 | \text{inp basis}_1 \rangle & \langle \text{out basis}_2 | \text{in basis}_2 \rangle \end{bmatrix}$$

LAD → 9.12

Given two basis,

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{\psi} = \alpha\hat{i} + \beta\hat{j} \quad ; \quad |\psi\rangle = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right.$$

$$= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and, Second basis,

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \hat{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}B_x''\\ B_y''\end{pmatrix}$$

$$\hat{A} = \begin{bmatrix}\langle u_1|v_1\rangle & \langle u_1|v_2\rangle\\ \langle u_2|v_1\rangle & \langle u_2|v_2\rangle\end{bmatrix}$$

$$\langle u_1|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1+0\end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_1|v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0+1\end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_2|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1+0\end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_2|v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0-1\end{bmatrix} = -\frac{1}{\sqrt{2}}$$

$$\therefore \hat{A} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

Applying this matrix to the arbitrary vector,

$$\hat{A}|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\end{bmatrix} = \begin{bmatrix}\frac{\alpha+\beta}{\sqrt{2}}\\ \frac{\alpha-\beta}{\sqrt{2}}\end{bmatrix}$$

So, ~~the~~ $\hat{A}|\psi\rangle = \begin{pmatrix}B_x''\\ B_y''\end{pmatrix} = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|u_1\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|u_2\rangle$

$\vec{\psi} = \alpha\hat{i} + \beta\hat{j}$; $\vec{\varphi}$ after basis change, $\vec{\varphi} = \frac{\alpha+\beta}{\sqrt{2}}\hat{i}' + \frac{\alpha-\beta}{\sqrt{2}}\hat{j}'$

~~All~~ Alternative way:

$$\vec{\psi} = B_x'' \hat{i}' + B_y'' \hat{j}'$$

$$|\psi\rangle = B_x'' |u_1\rangle + B_y'' |u_2\rangle$$

$$|\psi\rangle = \underbrace{\langle u_1|\psi\rangle}_{C_1} |u_1\rangle + \underbrace{\langle u_2|\psi\rangle}_{C_2} |u_2\rangle$$

$$\vec{\psi} = (\hat{i}' \cdot \vec{\psi})\hat{i}' + (\hat{j}' \cdot \vec{\psi}) \cdot \hat{j}'$$

$$\Rightarrow |\psi\rangle = C_1 |u_1\rangle + C_2 |u_2\rangle$$

Wave Mech,

$$\psi_{(t)} = \underset{(\omega)}{C_1} \phi_1^{(t)} + \underset{(\omega)}{C_2} \phi_2^{(t)} + \underset{(\omega)}{C_3} \phi_3^{(t)}$$

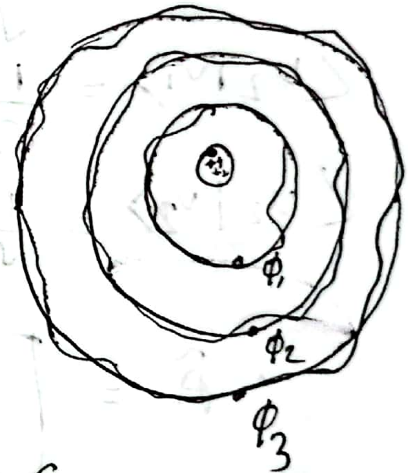
$\Rightarrow$ Vector,

$$\vec{\psi} = C_1 \hat{i} + C_2 \hat{j} + C_3 \hat{k}$$

Hilbert,

$$\xrightarrow[\text{B.M}]{} |\psi\rangle = C_1 |\phi_1\rangle + C_2 |\phi_2\rangle + C_3 |\phi_3\rangle$$

A.M $\Rightarrow$ $C_1 |\phi_1\rangle =$



C_1 = Probability Amplitude

final State   initial state

$$C_1 = \langle \phi_1 | \psi \rangle = \langle \text{final state} | \text{initial state} \rangle$$

$$C_1^* = \langle \psi | \phi_1 \rangle = \langle \text{initial state} | \text{final state} \rangle$$

$$\Psi = c_1 |x\rangle + c_2 |Y\rangle + c_3 |z\rangle$$

$$= \langle x|\psi\rangle |x\rangle + \langle Y|\psi\rangle |Y\rangle + \langle z|\psi\rangle |z\rangle$$

$$= |x\rangle\langle x|\psi\rangle + |Y\rangle\langle Y|\psi\rangle + |z\rangle\langle z|\psi\rangle$$

$$= \Big(\underbrace{|x\rangle\langle x| + |Y\rangle\langle Y| + \langle z+\psi\rangle}\Big) |\psi\rangle$$

$$= \hat{I}|\psi\rangle \quad [\because \text{Identity matrix}]$$

$$= |\psi\rangle$$

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \langle\psi| = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}$$

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

$$\Rightarrow \langle\psi|\psi\rangle = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \left[|c_1|^2 + |c_2|^2 + |c_3|^2\right] = [1]$$

$$\therefore \langle\psi|\psi\rangle = 1$$

wiki → Mathmatical formulation of Quantum Mech.

Postulates:

Description of the state of a system.

Postulate I : The state of an isolated physical system is represented, at a fixed time t, by a state vector $|\psi\rangle$ belonging to a Hilbert Space $\mathcal{H}$ called the state space.

Measurement on a system

Description of Physical Quantities :

Postulate II.a : Every measurable physical quantity $\mathcal{A}$ is described by a Hermitian Operator A acting in the state space $\mathcal{H}$. This operator is an observable, meaning that it's eigenvectors form a basis for $\mathcal{H}$. The result of measuring a physical quantity $\mathcal{A}$ must be one of the eigenvalues of the corresponding observable

A.

Measurable physical Quantity ⇒ Energy (E), Position(x), Momentum (P), Spin (S) ⇒ A (observable)

III. Results of measurement

Postulate II.b : When the physical quantity A is measured on a system in a normalized state $|\psi\rangle$, the probability of obtaining an eigenvalue (denoted by a_n for discrete spectra and α for continuous spectra) of the corresponding observable A is given by the amplitude squared of the appr. wave function (projection onto corresponding eigenvector).

IV Effect of measurement on the state :

Postulate II.C : If the measurement of the physical quantity A on the system in the state $|\psi\rangle$ gives the result a_n, then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigen-subspace associated with a_n.

$$|\psi\rangle \overset{a_n}{\Longrightarrow} \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

Now, Let, $|\psi\rangle_{B.M} = c_1|X\rangle + c_2|Y\rangle + c_3|Z\rangle$

generalised, $|\psi\rangle_{A.M} = |X\rangle$ or $|\psi\rangle_{A.M} = |Y\rangle$ or $|\psi\rangle_{A.M} = |Z\rangle$

more rigorous, $|\psi\rangle_{A.M} = \left(\frac{c_1}{|c_1|}\right)|X\rangle$

$= e^{j\theta_1}|X\rangle$

$c_1 = 2+5j$
magnitude of c_1,
$\Rightarrow \left(\sqrt{2^2+5^2}\right)\cdot e^{j\tan^{-1}5/2}$
$= |c_1|$
$\therefore \frac{c_1}{|c_1|} = e^{j\theta_1}$

Time evolution of a system:

Postulate III: The time evolution of state vector $|\psi(t)\rangle$ is governed by the Schrodinger equation, where $H(t)$ is the observable associated with the total energy of the system. (Called the Hamiltonian)

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

[Schrodinger eqn]

equivalently,

Postulate III: The time evolution of a closed system is described by a unitary trasformation on the initial state

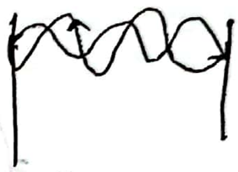
$$|\psi(t)\rangle = U(t;t_0)|\psi(t_0)\rangle$$

Two state vector formalism

Lecture 9 - Part 1

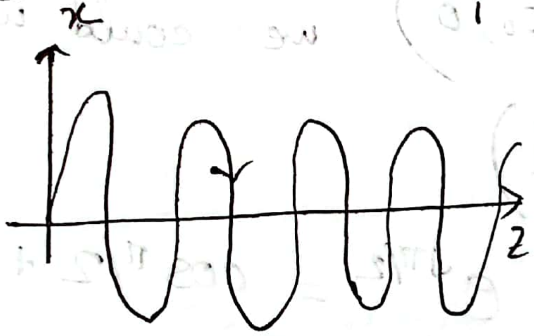
$y = E_0 \cos(\omega t) \longrightarrow$ 

$y = E_0 \cos(kz - \omega t)$ 

$y = E_0 \cos(kz) \cdot \sin(\omega t) \longrightarrow$ 

Now, $\vec{E} = \left[E_0 \cdot e^{-j(kz - \omega t)}\right] \hat{x} = E_x \hat{i}$

$\hat{x} = \hat{i}$

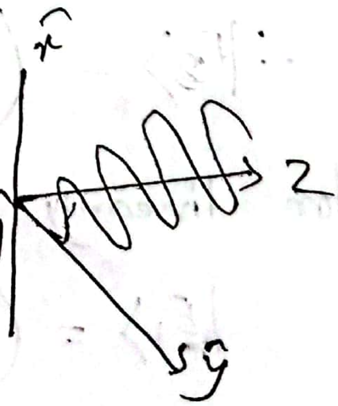


$$|E\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} E_x$$

Shorthand Notation,

$$\Rightarrow |E_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, $E_y \hat{j} = \vec{E} = \left[E_0 e^{-j(kz - \omega t)}\right] \hat{j}$



$$\Rightarrow |E_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|h\rangle = \begin{pmatrix}1\\0\end{pmatrix}$; $|v\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

horizontal $= |E_1\rangle$, vertical $= |E_2\rangle$

* $\vec{E_s} = E_x\hat{i} + E_y\hat{j} + 0\hat{k}$ [$E_x\hat{i}$ and $E_y\hat{j}$ has a $90°$ phase difference]

$= (E_x, E_y, 0)$; $A = |A| \angle 90°$

$= (E_0, E_0 e^{j\pi/2}, 0)$; $A = |A| e^{+j\pi/2}$

if $\vec{E_s} = (E_0, E_0, 0)$ we could write

$|E_3\rangle = \begin{pmatrix}1\\1\end{pmatrix}$

from euler's, $e^{j\pi/2} = \cos\pi/2 + j\sin\pi/2 = j$

$\therefore \vec{E_s} = (E_0, E_0(i))$

$\therefore |E_3\rangle = \begin{pmatrix}1\\i\end{pmatrix}$ $\boxed{\begin{pmatrix}1\\e^{j\pi/2}\end{pmatrix}}$

for linearly polarized wave,

$|E_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$; $|E_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$; $|E_3\rangle = \begin{pmatrix}1\\i\end{pmatrix}$

$= E_0\hat{i} + (E_0 i)\hat{j}$

$\boxed{i = \sqrt{-1}}$

Before Normalization,

$$|E_3\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \cdot e^{j\pi/2} \end{pmatrix}$$

magnitude of $E_3 = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$|E_3\rangle_{\text{Normalized}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$ $\boxed{\text{Right } \cancel{\text{Rotation}} \text{ hand Circular}}$ $\Rightarrow |R\rangle = |E_3\rangle_{\text{Normalized}}$

Now, $\langle E_3 | E_3 \rangle$ or $\langle R | R \rangle$ $= \frac{1}{\sqrt{2}} (1^* \; i^*) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \cancel{\frac{1}{2}}$

$$= \frac{1}{\sqrt{2}} (1 \; -i) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} (1 - i^2) = \frac{1}{2} \cdot 2$$

$$= 1$$

And Left Hand Circular $\Rightarrow |L\rangle = |E_4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Now, $\langle L | R \rangle$ / $\langle E_4 | E_3 \rangle$ $= \frac{1}{\sqrt{2}} (1^* \; -i^*) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} (1 \; i) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} (1 + i^2)$$

$$= \frac{1}{2} (1 - 1) = \frac{1}{2} \cdot 0$$

$$= 0$$

So, $|L\rangle$ and $|R\rangle$ are in orthonormal basis.

We can write,

$$\underset{\begin{pmatrix}1\\0\end{pmatrix}}{|h\rangle} = \cancel{a|L\rangle + b|R\rangle} = a\,|R\rangle + b\,|L\rangle \;\cdots\; (1)$$

$$= \frac{a}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}a+b\\ ia-ib\end{pmatrix}$$

we get two eqn.

$$\frac{a+b}{\sqrt{2}} = 1 \;\cdots\; (i)$$

$$\frac{ia-ib}{\sqrt{2}} = 0 \;\cdots\; (ii)$$

$$\Rightarrow \frac{i(a-b)}{\sqrt{2}} = 0$$

$$\Rightarrow a-b = 0 \Rightarrow a = b$$

from eqn (i)

$$\frac{b+b}{\sqrt{2}} = 1$$

$$\Rightarrow \frac{2b}{\sqrt{2}} = 1$$

$$\Rightarrow b = \frac{1}{\sqrt{2}}$$

$$\therefore a = b = \frac{1}{\sqrt{2}}$$

$$\therefore |h\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle + |L\rangle\right)$$ eqn (1) থেকে

33

Again, $|R\rangle = a|h\rangle + b|v\rangle$

$$\Rightarrow a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$\Rightarrow \begin{matrix}a\\b\end{matrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$\therefore a = \frac{1}{\sqrt{2}} \quad ; \quad b = \frac{i}{\sqrt{2}}$$

$$\therefore |R\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{i}{\sqrt{2}}|v\rangle = \frac{|h\rangle + i|v\rangle}{\sqrt{2}}$$

We saw, $\hat{R}|x\rangle = |x'\rangle$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

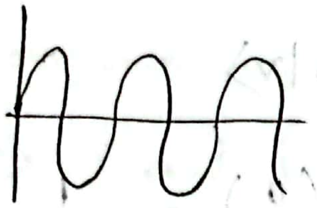
$\Rightarrow$ for $|h\rangle$,

$$\hat{R}|h\rangle = |\theta\rangle = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

If $\theta = \pi/2$, $\hat{R}|h\rangle = \begin{bmatrix}\cos\pi/2\\ \sin\pi/2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$$= 0|h\rangle + 1\cdot|v\rangle$$

$$= |v\rangle$$



$E = E_0[e^{-j(kz-\omega t)}]\hat{y}$

Intensity, $I_0 = |E_0|^2$

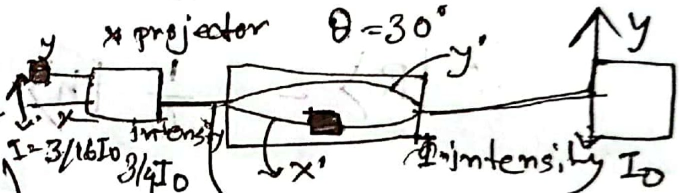
~~[illegible]~~, $\{C_1|x\rangle + C_2|y\rangle$

$|C_1|^2 + |C_2|^2 = 1$ Intensity

Book: An Introduction to Quantum Physics.
MIT Introductory physics series (French & Taylor)

Ch. 7-4: ~~Exp. [illegible]~~ see Experiment (a, b, c)

For experiment (a):



$|y\rangle = |v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Let, $|\psi\rangle_{int}$

$\boxed{\text{Stage 1}}$

Method 1:

$$|\psi_{int}\rangle = C_1|x\rangle + C_2|y\rangle$$

Now,

$$\hat{R}|v\rangle = \hat{R}|y\rangle = |y'\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= -\underbrace{\sin\theta\, |h\rangle}_{0} + \cos\theta\, |v\rangle$$

$$\therefore |\psi_{int}\rangle = C_1 \cdot 0 + \underbrace{\cos\theta}_{C_2}\, |v\rangle$$

$$\text{Intensity, } I = |C_2|^2 = (\cos\theta)^2 = (\cos 30°)^2$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

P_1

Method 2 : $|\psi_{int}\rangle = c_1 |Y\rangle + C_2 |x\rangle$

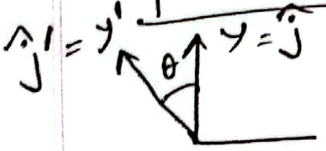
Because x' blocked

$\langle Y'|Y\rangle = \hat{j}' \cdot \hat{j} = \cos\theta$

$= \langle Y'|Y\rangle |Y\rangle + \langle Y'|x\rangle |x\rangle$

$C_* = \langle$ final state | init. state$\rangle$

$\hat{j}' = y'$, $y = \hat{j}$

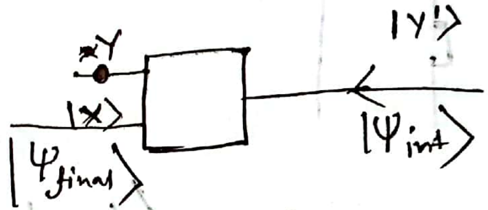


$= \cos\theta \cdot |Y\rangle + \underbrace{-\sin\theta\, |x\rangle}_{\cdot 0}$

$= \cos\theta\, |Y\rangle$

$\therefore$ Intensity, $I = \cos^2\theta = \cos^2 30^\circ = \frac{3}{4}$

Stage 2 :



Method 2 : $|\psi_{final}\rangle = c_1 |Y\rangle + c_2 |x\rangle$

$= c_1 \cdot 0 + c_2$ $\langle$ final state | init. state$\rangle$

$= \langle x | Y' \rangle$

$\therefore |C_2|^2 = |\langle x|Y'\rangle|^2 = (\sin\theta)^2 = (\sin 30^\circ)^2$

$= \frac{1}{4}$

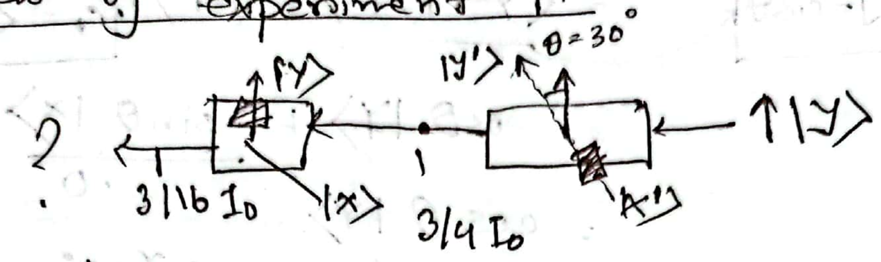
P_2

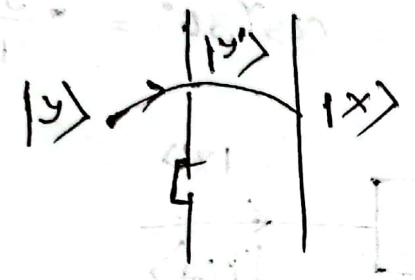
Whole system, Total Probability, $P_T = P_1 \cdot P_2$

$= \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$

Lecture-9: Part-2

Review of experiment i.





$$C = \langle \text{final state} | \text{initial state} \rangle$$

$\therefore$ intensity, $I = |C|^2 = \langle f | i \rangle^2$

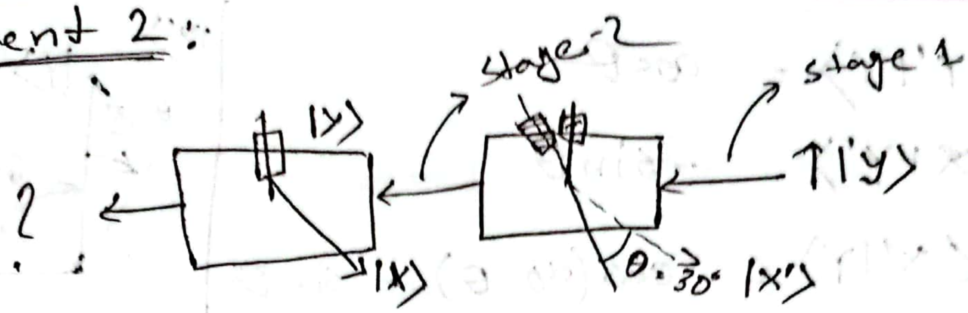
$C_1 = \langle y' | y \rangle \quad ; \quad C_2 = \langle y | y' \rangle$

$P_1 = |C_1|^2 = |\langle y' | y \rangle|^2 = -\sin^2\theta = 1/4$

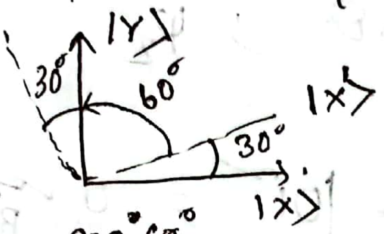
$P_2 = |C_2|^2 = |\langle x | y' \rangle|^2 = \cos^2\theta = 3/4$

$\therefore P_T = P_1 \cdot P_2 = 1/4 \cdot 3/4 = 3/16$

Experiment 2 :



Stage 1 : $|c_1|^2 = |\langle x'|Y\rangle|^2 = \cos^2 60 = \underbrace{1/4}_{P_1}$

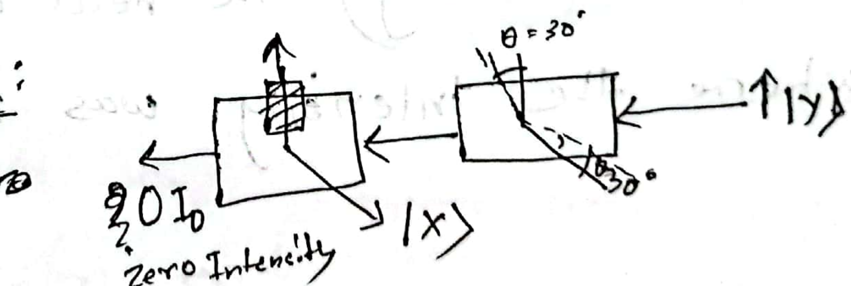


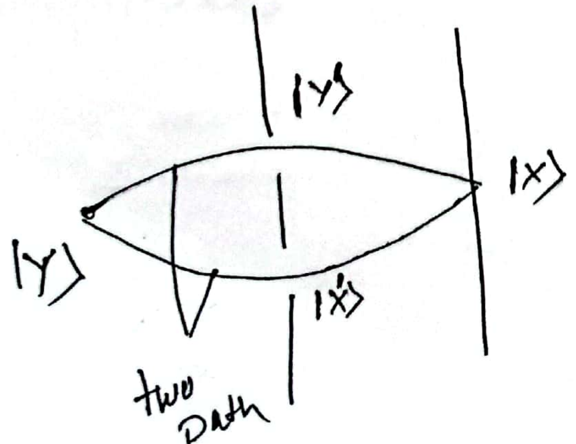
cos 60°

Stage 2 : $|c_2|^2 = |\langle x|x'\rangle|^2 = \cos^2 30 = \underbrace{3/4}_{P_2}$

$$P_T = P_1 \cdot P_2 = 1/4 \cdot 3/4 = \frac{3}{16}$$

Experiment 3 :





Path 1 $(\psi_1) = \langle x|y'\rangle\langle y'|Y\rangle$

Path 2 (ψ_2) $\langle x|x'\rangle\langle x'|Y\rangle$

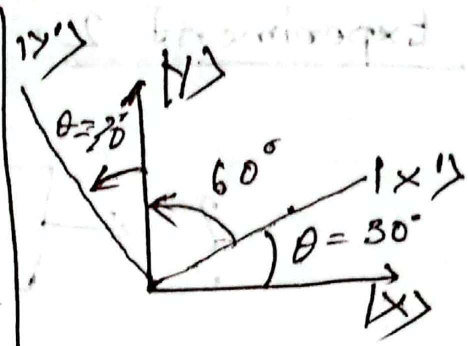
$\Rightarrow$

$\langle Y'|Y\rangle = \cos\theta$

$\langle X|Y'\rangle = -\sin\theta$

$\langle X'|Y\rangle = \cos(90-\theta) = \sin\theta$

$\langle X|X'\rangle = \cos\theta$



$\therefore \psi_1 = -\sin\theta\cdot\cos\theta \quad , \quad \psi_2 = \cos\theta\cdot\sin\theta$

for, $|\psi_1+\psi_2|^2 = |0|^2 = 0$ Because of indistinguish. paths.

and, $|\psi_1|^2 + |\psi_2|^2 = (-\sin\theta\cdot\cos\theta)^2 + (\sin\theta\cdot\cos\theta)^2$

$= 2\sin^2\theta\cdot\cos^2\theta$

This doesn't satisfy the real case scenario where the intensity was zero.

~~We~~ 3 Rules :-

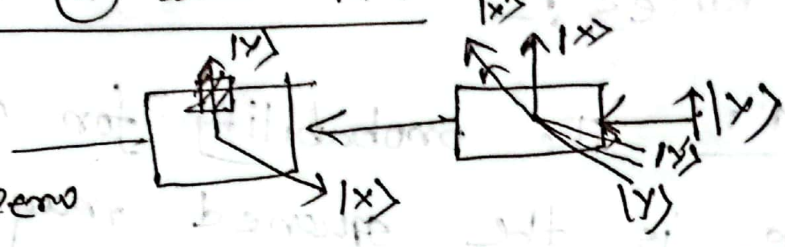
Rule 1 : The probability for any experimental outcome is the squared magnitude of a number called the quantum amplitude for that outcome.

$$P = |c_1|^2 \qquad c_1 : \text{quantum amplitude}$$

Rule 2 : For an interference experiment, the resultant quantum amplitude is the sum of the amplitudes for each alternative path.

Rule 3 : The quantum amplitude for a series of projections is the product of the quantum amplitudes for each projection in the series.

Experiment (iii) with $\hat{R}$: Theory



$$\hat{R}\,|Y\rangle = |\psi_{int}\rangle = |Y\rangle$$

Because, $|Y\rangle$, 100 intensity was not blocked

$$\langle X|\psi_{int}\rangle = \langle X|Y\rangle = 0.$$

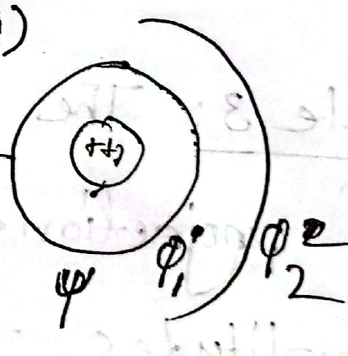
Becaus $|Y\rangle$ have no projection in $|X\rangle$.

Completeness :

we can write any vector $|2\rangle$

$$|2\rangle = c_1\,|h\rangle + c_2\,|v\rangle \cdots (1)$$

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$$



→ |initial superposition state⟩

= c_1 |Possible future State 1⟩ + c_2 |Possible future state 2⟩

$\langle f_1|i_1\rangle$ $\qquad$ $\langle f_2|i_2\rangle$

$= \langle \phi_1|\psi\rangle$ $\qquad$ $= \langle \phi_2|\psi\rangle$

So, from (i).

$$|z\rangle = \langle h|z\rangle |h\rangle + \langle v|z\rangle |v\rangle$$

$$= |h\rangle\langle h|z\rangle + |v\rangle\langle v|z\rangle$$

$$= \left(|h\rangle\langle h| + |v\rangle\langle v|\right) |z\rangle$$

$$= |z\rangle$$

Chapter 3: Postulates of Q.M (2edi.)

Postulate 3:

$$\hat{E}|\psi\rangle = \hat{E}\left(c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + \dots\right)$$

$$\hat{E}|\varphi_n\rangle = \left(\text{value of energy}\right)|\varphi_n\rangle$$

when observed or measured

$$\hat{E}|\varphi_2\rangle = (e_2)|\varphi_2\rangle$$

$$\hat{E}|\varphi_1\rangle = (e_1)|\varphi_1\rangle$$

observable

Postulate 5:

$t = 0,$

$|\psi\rangle = |\psi(x,0)\rangle$

$$|\psi(t)\rangle = \underbrace{e^{-i(E/\hbar)t}}_{\hat{U}} |\psi(x,0)\rangle$$

Math: 3.6, 3.7, 3.8, 3.11

Problem 3.6

Given, $H = \mathcal{E}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, initial state $|\psi_0\rangle = \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$

Hamoltonian

$\mathcal{E}$ = has the dimensions of energy

Eigen Energy, $E_1 = \mathcal{E}$ and $E_2 = E_3 = -\mathcal{E}$

$|\hat{H} - \lambda I| \rightarrow$ eigen values.

Charactaristic Eqn

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}; \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}; \quad |\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi_0\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix} = \underbrace{c_1/\sqrt{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}_{\phi_1} + \underbrace{c_2/\sqrt{2}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}}_{\phi_2} + \underbrace{c_3/\sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\phi_3}$$

আবার, $c_1 = \langle \phi_1 | \psi_0 \rangle$; $c_2 = \langle \phi_2 | \psi_0 \rangle$; $c_3 = \langle \phi_3 | \psi_0 \rangle$

$= \sqrt{2/5}$; $= \sqrt{2/5}$; $= \frac{1}{\sqrt{5}}$

Probability, $P_1(E_1) = \frac{|\langle \phi_1 | \psi_0 \rangle|}{|c_i|} = \left| \sqrt{\frac{2}{5}} \langle \phi_1 | \phi_1 \rangle \right|^2 = 2/5$

$P_2(E_2) = |\langle \phi_2 | \psi_0 \rangle|^2 + |\langle \phi_3 | \psi_0 \rangle|^2 = \frac{2}{5} + \frac{1}{5} = 3/5$

Because of $\underset{\text{double}}{\underline{\text{degenerate}}}$ eigenvalue

$$\langle \hat{H} \rangle = \langle \psi_0 | \hat{H} | \psi_0 \rangle = \frac{\varepsilon}{5} (1+i \quad 1+i \quad 1)\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix} = -\frac{1}{5}\varepsilon$$

Problem 3.7

given $H = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $A = a \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

a) Eigen energies $E_1 = 0, E_2 = -\varepsilon_0, E_3 = 2\varepsilon_0$

Eigen vectors

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|\phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

b) Solving positing vector A with characteristic eq.

we get eigen values, $|a_1\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 4 \\ -\sqrt{17} \\ 1 \end{pmatrix}$

$$|a_2\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$$|a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 4 \\ \sqrt{17} \\ 1 \end{pmatrix}$$

System in the state ϕ_2:

$\therefore P_1(a_1) = |\langle a_1|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{34}} (4 \; -\sqrt{17} \; 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{34}$

similarly, $P_2(a_2) = |\langle a_2|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{17}} (1 \; 0 \; -4) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{16}{17}$

$P_3(a_3) = |\langle a_3|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{2}} (4 \; \sqrt{17} \; 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{34}$

Lecture 10 – Part 1

Problem 3.8 :

Given state $|\psi(t)\rangle = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

a) Eigen values of A, $a_1 = -1$, $a_2 = 0$, $a_3 = 1$

Respective normalized eigen vectors,

$$|a_1\rangle = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \; ; \; |a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; \; |a_3\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Normalizing the state, $|\psi(t)\rangle$ we get,

$$\Rightarrow |\psi_N(t)\rangle = \frac{|\psi(t)\rangle}{\sqrt{\langle \psi(t) | \psi(t) \rangle}}$$

$\because \langle \psi(t) | \psi(t) \rangle = (-1 \; 2 \; 1) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 6$

> if, $|a\rangle = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$
> Normalized,
> $|a_N\rangle = \frac{1}{\sqrt{4^2+5^2}} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

The probability of obtaining $a_1 = -1$ is

$$P(-1) = \frac{|\langle a_1 | \psi(t) \rangle|^2}{\langle \psi(t) | \psi(t) \rangle} = \frac{1}{6} \left| \frac{1}{2} (-1 \; \sqrt{2} \; -1) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right|^2$$

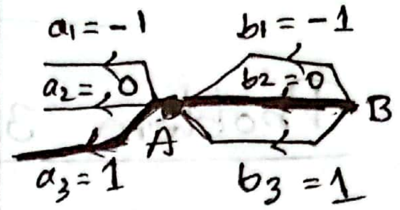
$$= 1/3$$

b) Eigenvalues of B : $b_1 = -1$, $b_2 = 0$, $b_3 = 1$

eigenvectors:

$$|b_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \quad |b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad |b_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Overall Probability



The system is in state $|\Psi(t)\rangle$, and the probability of obtaining $b_2 = 0$ for B is,

$$P(b_2) = \frac{|\langle b_2|\Psi(t)\rangle|^2}{\langle\Psi(t)|\Psi(t)\rangle} = \frac{1}{6}\left|(0\ 1\ 0)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right|^2 = 2/3$$

After measuring B the system is in ~~an~~ an intermidiate state, let's call it $|\phi\rangle$.

$$|\phi\rangle = |b_2\rangle\langle b_2|\Psi(t)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Now, the system is in state $|\phi\rangle$ and the probability of obtaining $a_3 = 1$ for A is

$$P(a_3) = \frac{|\langle a_3|\phi\rangle|^2}{\langle\phi|\phi\rangle} = \frac{1}{4}\left|\frac{1}{2}(1\ \ \sqrt{2}\ \ 1)\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right|^2$$

$$= 1/2$$

overall probability, $P(b_2, a_3) = P(b_2) \cdot P(a_3)$

$$= \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} \text{ (Answer)}$$

c) Same process but different order.

d) If we can measure $|\psi_n(t=0)\rangle = |\psi_n(0)\rangle$

Then, $|\psi_n(t=1)\rangle = |\psi_n(0)\rangle \cdot e^{-j\omega_n t}$

$E_n = \hbar \omega_n$

$\therefore \omega_n = E_n / \hbar$

Problem 3.11:

Given, $|\psi(0)\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$; $H = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}$

a) Eigen values,

$E_1 = -5$; $E_2 = 3$, $E_3 = 5$

Eigen vectors,

$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$; $|\phi_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

The probability of finding the values, E_1, E_2, E_3 :-

$$P(E_1) = |\langle\phi_1|\psi(0)\rangle|^2 = \left|\frac{1}{5\sqrt{2}}(0\ -1\ 1)\begin{pmatrix}3\\0\\4\end{pmatrix}\right|^2 = \frac{8}{25}$$

$$P(E_2) = |\langle\phi_2|\psi(0)\rangle|^2 = \left|\frac{1}{5}(1\ 0\ 0)\begin{pmatrix}3\\0\\4\end{pmatrix}\right|^2 = \frac{9}{25}$$

$$P(E_3) = |\langle\phi_3|\psi(0)\rangle|^2 = \left|\frac{1}{5\sqrt{2}}(0\ 1\ 1)\begin{pmatrix}3\\0\\4\end{pmatrix}\right|^2 = \frac{8}{25}$$

(b) we know,

$$|\psi(0)\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle$$

$$= \langle\phi_1|\psi(0)\rangle|\phi_1\rangle + \langle\phi_2|\psi(0)\rangle|\phi_2\rangle + \langle\phi_3|\psi(0)\rangle|\phi_3\rangle$$

$$= \frac{2\sqrt{2}}{5}|\phi_1\rangle + \frac{3}{5}|\phi_2\rangle + \frac{2\sqrt{2}}{5}|\phi_3\rangle$$

again, $|\psi_n(t)\rangle = |\psi_n(0)\rangle \cdot e^{-iE_n/\hbar \cdot t}$

$$= \frac{2\sqrt{2}}{5}\cdot e^{-i\cdot E_1/\hbar t}|\phi_1\rangle + \frac{3}{5}\cdot e^{-iE_2/\hbar \cdot t}|\phi_2\rangle + \frac{2\sqrt{2}}{5}\cdot e^{-iE_3/\hbar \cdot t}|\phi_3\rangle$$

~~$|\psi(t)\rangle = \frac{b}{5}($~~

Answer

(b) $\langle \Psi_{(0)} | \hat{E} | \Psi_{(0)} \rangle$ = Expectation value or Average Energy value | for that state

→ in wave mechanics

$$\underbrace{\int_{-\infty}^{\infty} \left(\frac{-\hbar}{2m} \nabla^2 \psi \right)}_{H} \psi^* \, dx$$

$\boxed{H = \hat{E}}$

H = Hamoltonian Operator

$$\langle \Psi_{(0)} | \left[\underset{\hat{E}}{H} | \Psi_{(0)} \rangle \right]$$

(c) $E_{(0)} = \langle \Psi_{(0)} | \hat{H} | \Psi_{(0)} \rangle$

$$= \cancel{cp} \ \frac{8}{25} \langle \phi_1 | \hat{H} | \phi_1 \rangle + \frac{9}{25} \langle \phi_2 | \hat{H} | \phi_2 \rangle + \frac{8}{25} \langle \phi_3 | \hat{H} | \phi_3 \rangle$$

$$\Rightarrow \frac{8}{25}(-5) + \frac{9}{25}(3) + \frac{8}{25}(5) = \frac{27}{25}$$ Answer

~~Matrix Mechanics way:~~

Matrix Algebra way:

$$E_{(0)} = \langle \Psi_{(0)} | \hat{H} | \Psi_{(0)} \rangle = \frac{1}{25} (3 \ 0 \ 4) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$= \frac{27}{25}$$

Probabilities:

$$E(0) = \sum_{n=1}^{2} \underbrace{P(E_n)}_{|C_n|^2} \overbrace{E_n}^{\text{Eigen values}} = |C_1|^2 E_1 + |C_2|^2 E_2 + |C_3|^2 E_3$$

$$= \frac{8}{25}(-5) + \frac{9}{25}(3) + \frac{8}{25}(5)$$

$$= \frac{27}{25} = E(0)$$ Answer no 1

For more math practice: Book: Demystified Matrix Mechanics

The spin observable

Book: A First Introduction to Quantum Physics — Pieter Kok

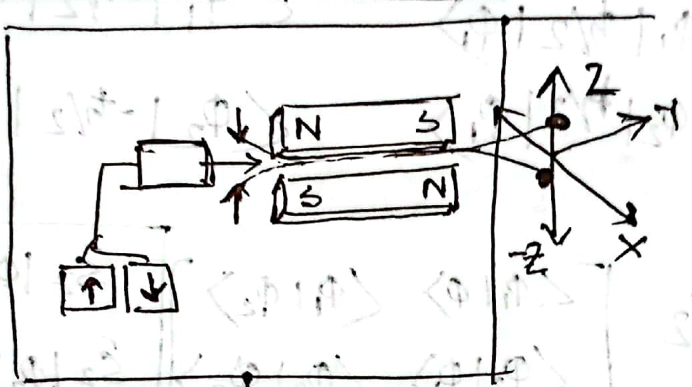
Chapter - 3

Topic : Spin

Eigen values :- $\frac{\hbar}{2}$ $\quad -\frac{\hbar}{2}$

$+1/2 \qquad -1/2$

Stern - Gerlach Experiment :



Measuring Device (M.D) = $\hat{S}_z$

States:

$$|\phi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \; ; \; |\phi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\therefore \hat{S}_z \begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \hat{S}_z |\phi_1\rangle = \frac{\hbar}{2}|\phi_1\rangle \quad \text{— (i)}$$

$$\therefore \hat{S}_z \begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \hat{S}_z |\phi_2\rangle = -\frac{\hbar}{2}|\phi_2\rangle \quad \text{— (ii)}$$

from (i) × $\langle\phi_1|$

$$\hat{S}_z |\phi_1\rangle\langle\phi_2| = \hbar/2\, |\phi_1\rangle\langle\phi_2|$$

from (ii) × $\langle\phi_2|$

$$+\; \hat{S}_z |\phi_2\rangle\langle\phi_2| = -\hbar/2\, |\phi_2\rangle\langle\phi_2|$$

$$\hat{S}_z = \hbar/2 \left(|\phi_1\rangle\langle\phi_2| - |\phi_2\rangle\langle\phi_2|\right)$$

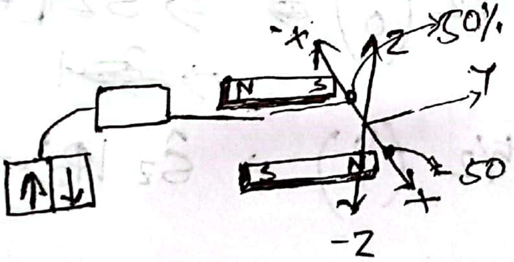
Again, $\hat{S}_z = \begin{bmatrix} \langle \varphi_1 | \hat{S}_z | \varphi_1 \rangle & \langle \varphi_1 | \hat{S}_z | \varphi_2 \rangle \\ \langle \varphi_2 | \hat{S}_z | \varphi_1 \rangle & \langle \varphi_2 | \hat{S}_z | \varphi_2 \rangle \end{bmatrix}$

$$\Rightarrow \hat{S}_z = \begin{bmatrix} \langle \varphi_1 | \hbar/2 | \varphi_1 \rangle & \langle \varphi_1 | -\hbar/2 | \varphi_2 \rangle \\ \langle \varphi_2 | \hbar/2 | \varphi_1 \rangle & \langle \varphi_2 | -\hbar/2 | \varphi_2 \rangle \end{bmatrix}$$

$$\Rightarrow \hat{S}_z = \hbar/2 \begin{bmatrix} \langle \varphi_1 | \varphi \rangle & \langle \varphi_1 | \varphi_2 \rangle \\ \langle \varphi_2 | \varphi_1 \rangle & \langle \varphi_2 | \varphi_2 \rangle \end{bmatrix} \quad \boxed{\begin{aligned} \hat{S}_z |\varphi_1\rangle &= \hbar/2 |\varphi_1\rangle \\ \hat{S}_z |\varphi_2\rangle &= -\hbar/2 |\varphi_2\rangle \end{aligned}}$$

$$\therefore \hat{S}_z = \hbar/2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, Rotating the magnets 90° anti-clockwise :—



~~Does~~ ↑↓ 50% time goes to x axis and 50% to -x axis

So, $\hat{S}_x |\varphi_n\rangle = \hbar/2 |\varphi_n\rangle$ → does not satisfy

let, $|\varphi_1\rangle = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

as they are equal probability, $|c_1|^2 + |c_2|^2 = 1$

$[c_1 = c_2] \longrightarrow 2|c|^2 = 1$

$\therefore c_1 = c_2 = 1/\sqrt{2}$

$\therefore |\phi\rangle_{N-1} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

~~we~~ we also can write,

$$|\phi_{N-2}\rangle = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

~~$|\phi_{N-2}\rangle =$~~

$\therefore \cancel{|\phi_{N-1}\rangle = c_1 |\phi_{N-1}\rangle + c_2 |\phi_{N-2}\rangle = 1/\sqrt{2}\, |\phi_1}$

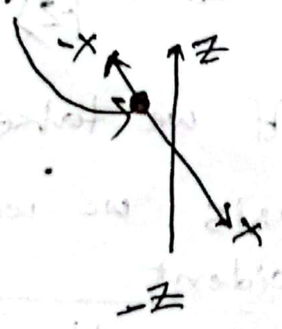
$\therefore |\phi_{N-1}\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle = 1/\sqrt{2}\, |\phi_1\rangle + 1/\sqrt{2}\, |\phi_2\rangle$

$\therefore |\phi_{N-2}\rangle = c_1 |\phi_1\rangle - c_2 |\phi_2\rangle = 1/\sqrt{2}\, |\phi_1\rangle - 1/\sqrt{2}\, |\phi\rangle_2$

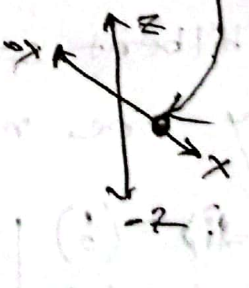
Now, Modeling,

$\hat{S}_x |\phi_{N-1}\rangle = \hbar/2\, |\phi_{N-1}\rangle$; $\hat{S}_x |\phi_{N-2}\rangle = -\hbar/2\, |\phi_{N-2}\rangle$

100% $-x$ axis



100% x axis



So, we write,

$\underset{|\phi_{N-1}\rangle}{|+\rangle} = \frac{1}{\sqrt{2}} |\uparrow\rangle + 1/\sqrt{2}\, |\downarrow\rangle$ | $\underset{|\phi_{N-2}\rangle}{|-\rangle} = \frac{1}{\sqrt{2}} |\uparrow\rangle - 1/\sqrt{2}\, |\downarrow\rangle$

Summary

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \; ; \; |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{S}_z|\uparrow\rangle = \hbar/2\,|\uparrow\rangle \; ; \; \hat{S}_z|\downarrow\rangle = -\hbar/2\,|\downarrow\rangle$$

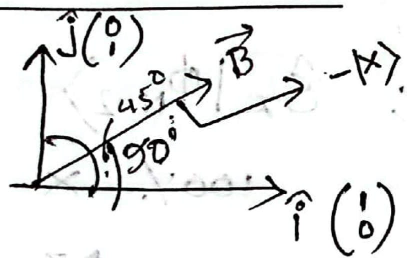
$$\hat{S}_z = \hbar/2\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

$$|\phi_{N-1}\rangle = |+\rangle = 1/\sqrt{2}\,|\uparrow\rangle + 1/\sqrt{2}\,|\downarrow\rangle$$

$$|\phi_{N-2}\rangle = |-\rangle = 1/\sqrt{2}\,|\uparrow\rangle - 1/\sqrt{2}\,|\downarrow\rangle$$

Real space & Hilbert Space: Analysis of Experiment

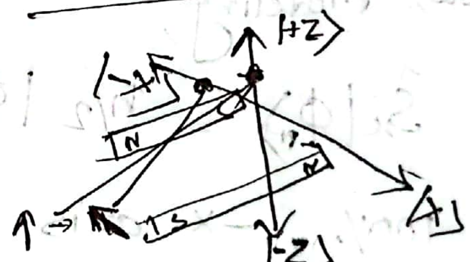
Hilbert



In Hilbert space they are represented

$|z\rangle = \hat{i} = \begin{pmatrix}1\\0\end{pmatrix}$ | And has 90° angle between them
$|-z\rangle = \hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$

Real



So, if we take $|-x\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ basis we would get that incident

But convention wise only z axis we can take $|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|-z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ ~~otho~~

~~we need to fin~~

$|+z\rangle$ and $|-z\rangle$ has 180° angle between the

So, Real space = θ

Hilbert Space $\frac{\theta}{2}$

$$\hat{i}+\hat{j} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = c\begin{pmatrix}1\\1\end{pmatrix} = 1/\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$\hat{S}_x \rightarrow$

$$\hat{S}_x |x\rangle = \hbar/2 \; |+\rangle$$

$$\hat{S}_x |-\rangle = -\hbar/2 \; |-\rangle$$

$$\therefore \hat{S}_x = \hbar/2 \left(|+\rangle\langle+| - |-\rangle\langle-| \right)$$

$$\therefore \hat{S}_x = \begin{bmatrix} \langle+|\hat{S}_x|+\rangle & \langle+|\hat{S}_x|-\rangle \\ \langle-|\hat{S}_x|+\rangle & \langle-|\hat{S}_x|-\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle+|\hbar/2|+\rangle & \langle+|-\hbar/2|-\rangle \\ \langle-|\hbar/2|+\rangle & \langle-|-\hbar/2|-\rangle \end{bmatrix}$$

$$= \hbar/2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Something Wrong

(See Lect 11 - Part 1)

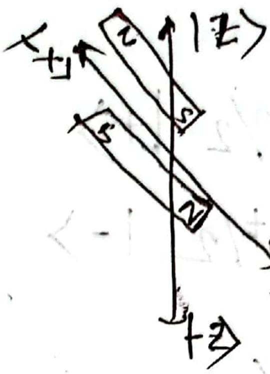
$$\hat{S}_x = \hbar/2 \left(|+\rangle\langle+| - |-\rangle\langle-| \right)$$

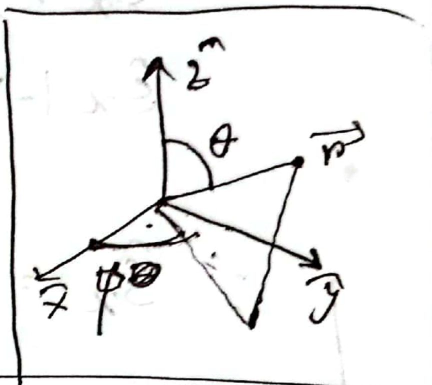
$$= \hbar/2 \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \right) - \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} \right)$$

$$= \hbar/2 \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right)$$

$$= \hbar/2 \cdot 1/2 \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now, for θ to z axis experiment:





$$\hat{S}_\theta |+_\theta\rangle = \hbar/2 \, |+_\theta\rangle$$

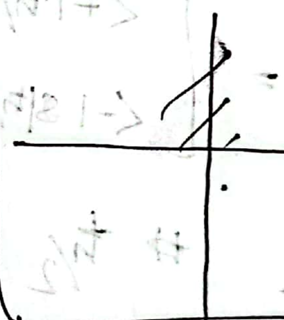
$$\hat{S}_\theta |-_\theta\rangle = -\hbar/2 \, |-_\theta\rangle$$

$$|\psi(\theta)\rangle = \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 \end{pmatrix}$$

$$\hat{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

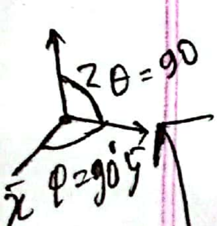
$= 45^0$



Hilbert space

Real space 90^0

$|\psi_y\rangle =$



$$|\psi(\theta,\phi)\rangle = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} . \sin \theta/2 \end{pmatrix}$$

$$|\psi_y\rangle = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} . \sin \theta/2 \end{pmatrix} \quad \left[\because \theta = 90,\ \phi = 90\right]$$

$$= \begin{pmatrix} \cos 45^0 . \\ e^{i\pi/2} . \sin 45^0 \end{pmatrix}$$

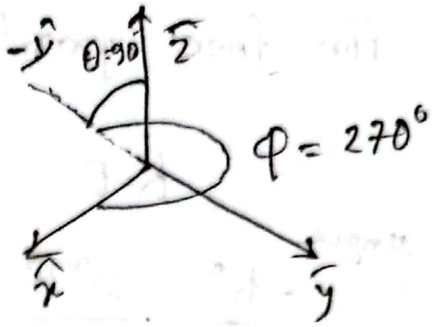
$$= \begin{pmatrix} 1/\sqrt{2} \\ i . 1/\sqrt{2} \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\hat{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$\therefore \theta = 135^0$

$$|\psi_{-y}\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\cdot\sin\theta/2 \end{pmatrix}$$



$$= \begin{matrix} \cos 45° \\ e^{i3\pi/2}\cdot\sin 45° \end{matrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -i\cdot 1/\sqrt{2} \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Multiparticle Quantum System:

from wave mech. (single part. → wave)

$$\underbrace{-\hbar/2m\,\frac{\partial^2\psi}{\partial x^2}}_{K.E} + \underbrace{V\psi}_{P.E} = i\hbar\frac{\partial\psi}{\partial t}$$

$$K.E + P.E = T.E$$

$$\Rightarrow \tfrac{1}{2}mv^2 + V = E$$

$$\Rightarrow p^2/2m + V = E$$

for multiple particle

$$(K.E_1 + P.E_1) + (K.E_2 + P.E_2) = (T.E_1 + T.E_2) = T.E$$

ψ 1 state

Schrodigen eq^n →

for. Non Interacting

$$-\hbar^2/2m\,\frac{\partial^2\psi}{\partial x_1^2} + V_1\psi + \frac{-\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x_2^2} + V_2\psi = i\hbar\frac{\partial\psi}{\partial t}$$

For free particle,

$$K.E = T.E$$

single

$$-\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + 0 = i\hbar \frac{\partial \psi}{\partial t}$$

multiple non-Int.

$$-\frac{\hbar^2}{2m_1} \cdot \frac{\partial^2 \psi}{\partial x_1^2} + \frac{(-\hbar^2)}{2m_2} \frac{\partial^2 \psi}{\partial x_2^2} = i\hbar \frac{\partial \psi}{\partial t}$$

for Int. a Interaction potential, $V_{I.P}(x_1 - x_2)$ will be added to the system.

$$-\frac{\hbar^2}{2m_1} \cdot \frac{\partial^2 \psi}{\partial x_1^2} + \frac{(-\hbar^2)}{2m_2} \cdot \frac{\partial^2 \psi}{\partial x_2^2} + \underline{V_{I.P}(x_1 - x_2)} = i\hbar \frac{\partial \psi}{\partial t}$$

A Book: Amit Goshwami : Quntum Computing
Explanation Chap. 9 : 215p

Lecture 11 - Part 1

Fix:

$$\hat{S}_x |+\rangle = \hbar/2\, |+\rangle \qquad \text{where, } |+\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle$$

$$\hat{S}_x |-\rangle = -\hbar/2\, |-\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\rangle$$

$\therefore$

$$\hat{S}_x = \begin{bmatrix} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{bmatrix}$$

$$\cancel{\hat{S}_x|+\rangle} \; \hat{S}_x\left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right) = \frac{\hbar}{2}\left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)$$

First,

$$\Rightarrow \hat{S}_x|\uparrow\rangle + \hat{S}_x|\downarrow\rangle = \hbar/2\,(|\uparrow\rangle + |\downarrow\rangle) \quad \text{— (a)}$$

$\times \langle\uparrow|$

$$\Rightarrow \langle\uparrow|\hat{S}_x|\uparrow\rangle + \langle\uparrow|\hat{S}_x|\downarrow\rangle = \hbar/2\,(\underbrace{\langle\uparrow|\uparrow\rangle}_{1} + \underbrace{\langle\uparrow|\downarrow\rangle}_{0})$$

$$\Rightarrow \langle\uparrow|\hat{S}_x|\uparrow\rangle + \langle\uparrow|\hat{S}_x|\downarrow\rangle = \hbar/2 \quad \text{— (i)}$$

$$\hat{S}_x\left(\frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}\right) = -\hbar/2\left(\frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}\right) \quad \text{— (b)}$$

$\times \langle\uparrow|$

$$\Rightarrow \langle\uparrow|\hat{S}_x|\uparrow\rangle - \langle\uparrow|\hat{S}_x|\downarrow\rangle = -\hbar/2 \quad \text{— (ii)}$$

(i)+(ii)

$$2\langle\uparrow|\hat{S}_x|\uparrow\rangle = 0$$

$$\therefore \langle\uparrow|\hat{S}_x|\uparrow\rangle = 0$$

(i) - (ii)

$$2\langle\uparrow|\hat{S}_x|\downarrow\rangle = \hbar$$

$$\therefore \langle\uparrow|\hat{S}_x|\downarrow\rangle = \hbar/2$$

Again, $\cancel{\hat{S}_x|+\rangle = \hbar/2\,|+\rangle}$

$\Rightarrow$ (a) $\times \langle\downarrow|$

$$\langle\downarrow|\hat{S}_x|\uparrow\rangle + \langle\downarrow|\hat{S}_x|\downarrow\rangle = \hbar/2 \quad \text{---(iii)}$$

(b) $\times \langle\downarrow|$

$$\langle\downarrow|\hat{S}_x|\uparrow\rangle - \langle\downarrow|\hat{S}_x|\downarrow\rangle = \hbar/2 \quad \text{---(iv)}$$

(iii) + (iv)

$$2\langle\downarrow|\hat{S}_x|\uparrow\rangle = \hbar$$

$$\therefore \langle\downarrow|\hat{S}_x|\uparrow\rangle = \hbar/2$$

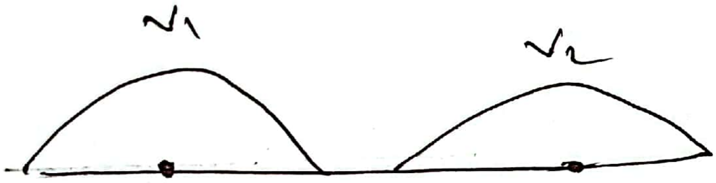
(iii) - (iv)

$$2\langle\downarrow|\hat{S}_x|\downarrow\rangle = 0$$

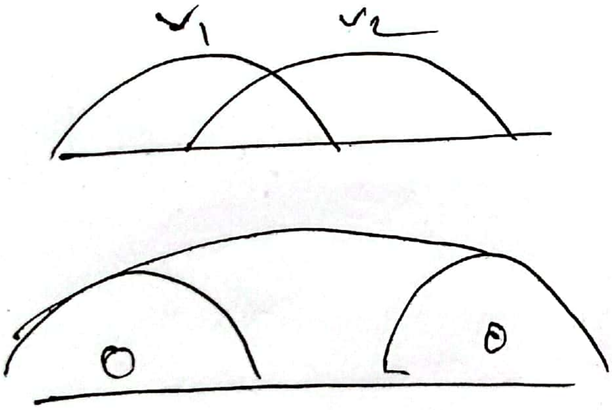
$$\therefore \langle\downarrow|\hat{S}_x|\downarrow\rangle = 0$$

$$\hat{S}_x = \begin{bmatrix} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} = \hbar/2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



→ Product state



→ Entangled state
or
Correlated Particles

Book Ref : P. Quantum Mech. for Pedestrian 2.

Chapter 20