Boundary Element Method

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1 Governing Equations

Equation of motion for elastic waves:

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + f_i = \rho \ddot{u}_i \tag{1}$$

where λ and μ are Lame constants, u_i is the displacement vector, f_i is the body force applied to the mass point and ρ is the density of the medium.

2 Boundary Integral

2.1 Elastic Medium

Betti Reciprocal Theorem

Betti reciprocal theorem (without considering the inertial force): In the elastic domain D with boundary S, let u_i and u_i^* be the displacement fields from the body force f_i and f_i^*, respectively. The two fields exhibit the relation:

$$\int_D f_i^* u_i dD + \int_S t_i^* u_i dS = \int_D f_i u_i^* dD + \int_S t_i u_i^* dS$$

yields

$$\int_D f_i^* u_i dD - \int_D f_i u_i^* dD + = \int_S t_i u_i^* dS - \int_S t_i^* u_i dS. \tag{2}$$

Introduction of Green's Function

Now we suppose that displacement field u_i^* is exicited by a unit pulse force in the j direction at point $x^{'}$ in domain D. The force can be expressed as fucntion:

$$f_i^* = \delta_{ij}\delta(x - x')$$

where δ_{ij} is the Kronecker delta ande $\delta(x - x')$ is the Dirac function.

The displacement field excited by this single source is called the Green's function in elastodynamics and is written as $G_{ij}(x; x')$. From equation (1), there is

$$(\lambda + \mu)G_{kj,ik} + \mu G_{ij,kk} - \rho \ddot{G_{ij}} = -\delta_{ij}\delta(x - x').$$

Direct BEM and Singularity on Boundary S

According to the property of the Dirac function, we have

$$\int_D u_i \delta_{ij} \delta(x - x') dD_{x'} = \delta_{ij} u_i(x). \tag{3}$$

From equation (2), replacing u^*, t^* by G_{ij}, T_{ij} respectively , we deduce

$$\int_D u_i\delta_{ij}\delta(x-x')dD_{x'} - \int_D G_{ij}f_i dD = \int_S (G_{ij}t_i - u_iT_{ij})dS,$$

and using equation (3), we arrive at

$$\delta_{ij}u_i(x) = \int_S [G_{ij}(x;x')t_i(x) - u_iT_{ij}(x,x')]dS_{x'} + \int_D G_{ij}(x;x')f_i dD \qquad (4)$$

which is derived under the condition of x' being inside D.

Expressions such as eq.(4) for the displacement within region D, that involve integrals like

$$\int_S G_{ij}\psi_i ds_x \qquad (5)$$

and

$$\int_S T_{ij}\psi_i ds_x \qquad (6)$$

The Green's function G_{ij} in 2-D static P-SV motion is given by

$$G^s_{ij}(x;x') = \frac{-1}{8\pi\mu(1-v)}[(3-4v)\delta_{ij}\log r - \hat{r}_i\hat{r}_j] + A_{ij},$$

where v is Poisson's ratio, $r = |x-x'|$, $\hat{r}_i = (x_i - x'_i)/r$, and A_{ij} is a constant tensor. There is therefore a logarithmic singularity in G_{ij} at $x = x'$, but this is integrable.

Considering eq.(6), T_{ij} is given by

$$T_{ij}(x;x') = c_{ipkl}\hat{n}_p(x)\frac{\partial}{\partial x_l}G_{kj}(x;x'),$$

where $\hat{n}$ is the outward normal to the boundary S. The static equivalent of this is

$$T^s_{ij}(x;x') = \frac{-1}{4\pi r(1-v)}\{(1-2v)(\hat{n}_j\hat{r}_i - \hat{n}_i\hat{r}_j) + [(1-2v)\delta_{ij} + 2\hat{r}_i\hat{r}_j]\hat{r}_k\hat{n}_k\},$$

and this has a $(\frac{1}{r})$ singularity, which is not integrable when $x' = x^S \in S$.

Consider the point x' within the region D as it approaches a point x^Son the boundary S.We split the curve. There we have,

$$S = S' + S_\epsilon,$$

where S_ϵ is a section of curve centred on x^S and length of ϵ on either side. Thus the main task is to evaluate

$$\lim_{x'\to x^S}\int_{S_\epsilon} T_{ij}\psi_i ds_x = \lim_{x'\to x^S}\{\psi_i(x^S)\int_{S_\epsilon} T_{ij}(x;x')ds_x + \int_{S_\epsilon} T_{ij}(x;x')[\psi_i(x) - \psi(x^S)]ds_x\}. \tag{7}$$

If ψ is Hölder continuous on S, there we have

$$|\psi_i(x_1) - \psi_i(x_2)| \leq L|x_1 - x_2|^\alpha,$$

where $0 < \alpha \leq 1$. On the assumption that this inequality holds, the second integral in eq. (7) is integrable and bounded when $x' = x^S$. Thus the main task is to evaluate the integral $\int_{S_\epsilon} T_{ij}(x;x')ds_x$.

We have assumed that S is smooth and so we may replace the curve S_ϵ

by a section of straight line, tangent to S_ϵ at x^S with error of order of ϵ:

$$\int_{S_\epsilon} T_{ij}(x; x')ds_x = \int_{-\epsilon}^{\epsilon} T_{ij}(x; x') + O(\epsilon),$$

$x' = (0, \eta), \hat{n}(x) = (0, -1)$.

If let $x' \to x^S$,the integral $\int_{S_\epsilon} T_{ij}(x; x')$ of static form for T_{ij} is given by

$$\int_{S_\epsilon} T_{ij}(x; x')ds_x = -\frac{1}{2}\delta_{ij} + O(\epsilon).$$

Indirect BEM

2.2 Fluid Medium

Consider the linear acoustic first-order wave equation:

$$\frac{1}{K}\dot{p} + \nabla \cdot v = f,$$

and Newton's Second Law:

$$\rho\dot{v} + \nabla p = 0,$$

where v is velocity field and f is the loading.

Consider the dispalcement potential function ψ that satisfy the following diferential relations:

$$u = \nabla\psi$$

$$p = -\rho\ddot{\psi}.$$

The scalar wave equation can be expressed as:

$$[\frac{1}{c^2}\dddot{\psi} - \nabla^2\psi]_t = f, \tag{8}$$

where $c = \dfrac{K}{\rho}$ is the wave velocity. Integrating eq.(8), we have second order scalar wave equation:

$$\frac{1}{c^2}\ddot{\psi} - \nabla^2\psi = F.$$

Define Fourier transform as

$$\hat{u}(\omega) = \int_{-\infty}^{+\infty} u(t)e^{i\omega t}dt,$$

and Fourier transform the wave equation, we have

$$-\frac{\omega^2}{c^2}\hat{\psi} - \nabla^2\hat{\psi} = \hat{F}. \tag{9}$$

The Green's function for this problem satisfies

$$-\frac{\omega^2}{c^2}\hat{G}(x;x') - \nabla^2\hat{G}(x;x') = \delta(x - x').$$

Multiply the Green's function onto the eq.(9), and integrate over the finite volume V, we have

$$\int_V -\frac{\omega^2}{c^2}\psi G - G\nabla^2\psi - GF dV_x = 0.$$

Then, apply the Green's second identity

$$\int_V G\nabla^2\psi - \psi\nabla^2 G dV = \int_{\partial V} \boldsymbol{n}\cdot(G\nabla\psi - \psi\nabla G)dS,$$

and substitute the term $G\nabla^2\psi$, we have

$$\int_V -\frac{\omega^2}{c^2}\psi G - GFdV_x = \int_V \psi\nabla^2 G dV + \int_{\partial V} \boldsymbol{n}\cdot(G\nabla\psi - \psi\nabla G)dS_x,$$

then rearrange this equation, we have

$$\int_V \psi[-\frac{\omega^2}{c^2}G - \nabla^2 G]dV_x = \int_V GFdV + \int_{\partial V} \boldsymbol{n}\cdot(G\nabla\psi - \psi\nabla G)dS_x,$$

Use the property of the Green's function:

$$\psi(x) = \int_V \psi(x)\delta(x-x')dV_x = \int_V GFdV_x + \int_{\partial V} \boldsymbol{n}\cdot(G\nabla\psi - \psi\nabla G)dS_x$$

The term in absence of sources,

$$\psi(x) = \int_{\partial V} \boldsymbol{n}\cdot(G\nabla\psi - \psi\nabla G)dS_x$$

3 Green's Function

3.1 Contitutive Model

$$T_{ij}(x,x') = c_{ipkl}\hat{n}_p(x)\frac{\partial}{\partial x_l}G_{kj}(x,x'),$$

where $\hat{n}(x)$ is the outward normal vector to the boundary S. And $c_{ipkl} = \lambda\delta_{ip}\delta_{kl} + \mu(\delta_{ik}\delta_{pl} + \delta_{il}\delta_{pk})$,where $\lambda = \dfrac{2v\mu}{1-2v}$.

3.2 Green's Function of Displacement and Traction

For 2D in plane (P-SV) problem,
$\dfrac{\partial H_0^{(2)}(kr)}{\partial x_j} = -k\hat{r}_j H_1^{(2)}(kr)$,
$\dfrac{\partial r}{\partial x_j} = \hat{r}_j$,
$\dfrac{\partial^2 r}{\partial x_j \partial x_l} = \hat{r}_{jl} = -\dfrac{\hat{r}_j \hat{r}_l}{r} + \delta_{jl}\dfrac{1}{r}$,
in order to avoid singularity, we use $\dfrac{\partial H_1^{(2)}(x)}{\partial x} = \dfrac{1}{2}(H_0^{(2)}(x) - H_2^{(2)}(x))$.
$\dfrac{\partial H_1^{(2)}(kr)}{\partial x_j} = k\hat{r}_j \dfrac{1}{2}(H_0^{(2)}(kr) - H_2^{(2)}(kr))$,

$$T_{ij}(x,x') = c_{ipkl}\hat{n}_p(x)\frac{\partial}{\partial x_l}G_{kj}(x,x')$$

$$= c_{ipkl}\hat{n}_p(x)\frac{i}{4\mu}\{-\delta_{kj}k_\beta\hat{r}_l H_1^{(2)}(k_\beta r) - \frac{1}{k_\beta}(\frac{\partial}{\partial x_l}\frac{1}{r}\hat{r}_k\hat{r}_j + \frac{1}{r}\hat{r}_k\hat{r}_{jl} + \frac{1}{r}\hat{r}_j\hat{r}_{kl})[H_1^{(2)}(k_\beta r) - \frac{\alpha}{\beta}H_1^{(2)}(k_\alpha r)]$$

$$-\frac{1}{2k_\beta r}\hat{r}_k\hat{r}_j\hat{r}_l[k_\beta H_0^{(2)}(k_\beta r) - k_\beta H_2^{(2)}(k_\beta r) - \frac{\beta}{\alpha}k_\alpha H_2^{(2)}(k_\alpha r) + \frac{\beta}{\alpha}k_\alpha H_0^{(2)}(k_\alpha r)]\}$$

$$-\frac{i}{4\mu}\{(\hat{r}_k\hat{r}_{jl}+\hat{r}_j\hat{r}_{kl})[H_0^{(2)}(k_\beta r)-\frac{\beta^2}{\alpha^2}H_0^{(2)}(k_\alpha r)]+\hat{r}_k\hat{r}_j\hat{r}_l[\frac{\beta^2}{\alpha^2}k_\alpha H_1^{(2)}(k_\alpha r)-k_\beta H_1^{(2)}(k_\beta r)]\}.$$

Considering T_{nk}, where suffix n stands for normal vector outwards the boundary, T_{nk} $=\hat{n}_l T_{lk}$. ($\hat{n}=(cos\theta, sin\theta)$).

$$T_{ij}(x,x') \quad = \quad c_{ipkl}\hat{n}_p(x)\frac{\partial}{\partial x_l}G_{kj}(x,x')$$

$$= c_{ipkl}\hat{n}_p(x)\frac{i}{4\mu}\{-\delta_{kj}k_\beta\hat{r}_l H_1^{(2)}(k_\beta r)-\frac{1}{k_\beta}(\frac{\partial}{\partial x_l}\frac{1}{r}\hat{r}_k\hat{r}_j+\frac{1}{r}\hat{r}_k\hat{r}_{jl}+\frac{1}{r}\hat{r}_j\hat{r}_{kl})[H_1^{(2)}(k_\beta r)-\frac{\alpha}{\beta}H_1^{(2)}(k_\alpha r)]$$

$$-\frac{1}{2k_\beta r}\hat{r}_k\hat{r}_j\hat{r}_l[k_\beta H_0^{(2)}(k_\beta r)-k_\beta H_2^{(2)}(k_\beta r)-\frac{\beta}{\alpha}k_\alpha H_2^{(2)}(k_\alpha r)+\frac{\beta}{\alpha}k_\alpha H_0^{(2)}(k_\alpha r)]\}$$

$$-\frac{i}{4\mu}\{(\hat{r}_k\hat{r}_{jl}+\hat{r}_j\hat{r}_{kl})[H_0^{(2)}(k_\beta r)-\frac{\beta^2}{\alpha^2}H_0^{(2)}(k_\alpha r)]+\hat{r}_k\hat{r}_j\hat{r}_l[\frac{\beta^2}{\alpha^2}k_\alpha H_1^{(2)}(k_\alpha r)-k_\beta H_1^{(2)}(k_\beta r)]\}.$$

Considering T_{nk}, where suffix n stands for normal vector outwards the boundary, T_{nk} $=\hat{n}_l T_{lk}$. ($\hat{n}=(cos\theta, sin\theta)$).

4 viscous fluid

4.1 Governing equation

Governing Equation:

$$\boldsymbol{u}_{,tt} - c_p^2\nabla(\nabla\cdot\boldsymbol{u}) - \frac{\mu}{\rho}\nabla^2\boldsymbol{u}_{,t} = f$$

Using identity (10), we have

$$\omega^2\boldsymbol{u} + (c_p^2 - \frac{i\omega\mu}{\rho})\nabla(\nabla\cdot\boldsymbol{u}) + \frac{i\omega\mu}{\rho}\nabla\times(\nabla\times\boldsymbol{u}) = -\boldsymbol{f}$$

$$\omega^2\boldsymbol{u} + \hat{c}_p^2\nabla^2\boldsymbol{u} - \hat{c}_s^2\nabla\times(\nabla\times\boldsymbol{u}) = -\boldsymbol{f}$$

where $c_p = \sqrt{\frac{K}{\rho}}$, $\hat{c}_p = \sqrt{\frac{-i\omega\mu}{\rho} + c_p^2}$, $\hat{c}_s = \sqrt{\frac{-i\omega\mu}{\rho}}$.

It's interesting that the wave speed in this equation is complex, which means the enery damping with time and also exactly the definition of "Viscousity".

Green's function for 2-D LNS:

$$G_{ij} = \frac{i}{4\mu}\{\delta_{ij}H_0^{(2)}(k_s r) - \frac{1}{k_s}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}[H_1^{(2)}(k_s r) - \frac{\hat{c}_s}{\hat{c}_p}H_1^{(2)}(k_p r)]\}$$
$$- \frac{i}{4\mu}\{\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}[H_0^{(2)}(k_s r) - \frac{\hat{c}_s^2}{\hat{c}_p^2}H_0^{(2)}(k_s r)]\}$$

On the boundary S,

$$u_{total} = u_{incident} + u_{diffracted} = u_{refracted}$$

Thus we have

$$u_j^E(\boldsymbol{x}) - u_j^I(\boldsymbol{x}) = -u_j^{(i)}(\boldsymbol{x})$$

$$t_j^E(\boldsymbol{x}) - t_j^I(\boldsymbol{x}) = -t_j^{(i)}(\boldsymbol{x}),\ \boldsymbol{x} \in S$$

where E and I indicates the (displacment & traction) response casused by boundary S alone.

The discrete version is:

$$\sum_{l=1}^{M} \bar{G}_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}_l)\phi_{kl}^E - \sum_{l=1}^{M} \bar{G}_{jk}^I(\boldsymbol{x}_m, \boldsymbol{\xi}_l)\phi_{kl}^I = -u_j^{(i)}(\boldsymbol{x}_m),\ m = 1...M$$

$$\sum_{l=1}^{M} \bar{T}_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}_l)\phi_{kl}^E - \sum_{l=1}^{M} \bar{T}_{jk}^I(\boldsymbol{x}_m, \boldsymbol{\xi}_l)\phi_{kl}^I = -t_j^{(i)}(\boldsymbol{x}_m),\ m = 1...M$$

4.2 Boundary Integral Representation

$$\bar{G}_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}_l) = \int_{\Delta S_l} G_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}) dS_{\boldsymbol{\xi}},$$

$$\bar{T}_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}_l) = \pm\frac{1}{2}\delta_{jk}\delta_{ml} + \int_{\Delta S_l} T_{jk}^E(\boldsymbol{x}_m, \boldsymbol{\xi}) dS_{\boldsymbol{\xi}}$$

$$\bar{G}^I_{jk}(\boldsymbol{x}_m, \boldsymbol{\xi}_l) = \int_{\Delta S_l} G^I_{jk}(\boldsymbol{x}_m, \boldsymbol{\xi}) dS_{\boldsymbol{\xi}},$$

$$\bar{T}^I_{jk}(\boldsymbol{x}_m, \boldsymbol{\xi}_l) = \pm\frac{1}{2}\delta_{jk}\delta_{ml} + \int_{\Delta S_l} T^I_{jk}(\boldsymbol{x}_m, \boldsymbol{\xi}) dS_{\boldsymbol{\xi}}$$

$$\boldsymbol{\mathcal{L}u} + \boldsymbol{f} = 0$$

$$\boldsymbol{\mathcal{L}u} = \omega^2\boldsymbol{u} + c_p^2\nabla(\nabla\cdot\boldsymbol{u}) - i\omega\frac{\mu}{\rho}\nabla^2\boldsymbol{u}$$

$$\begin{aligned}\int v_j(\mathcal{L}u_i)_j - u_j(\mathcal{L}v_i)_j dV &= c_p^2\int v_j u_{i,ij} - u_j v_{i,ij} dV - i\omega\frac{\mu}{\rho}\int v_i u_{i,jj} - u_i v_{i,jj} dV \\ &= c_p^2\int(\nabla\cdot\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{v} - (\nabla\cdot\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{u}dS - i\omega\frac{\mu}{\rho}\int\boldsymbol{v}\nabla\boldsymbol{u} - \boldsymbol{u}\nabla\boldsymbol{v}dS, \\ &(Green's\ Idenities)\end{aligned}$$

$$\begin{aligned}\alpha u(\boldsymbol{x}) = &\int_\Omega f(\boldsymbol{\xi})G(\boldsymbol{x},\boldsymbol{\xi})d\boldsymbol{\xi} + \\ &\int_S \boldsymbol{u}(\boldsymbol{\xi})[c_p^2(\nabla\cdot G(\boldsymbol{x},\boldsymbol{\xi}))\boldsymbol{n} - i\omega\frac{\mu}{\rho}\nabla G(\boldsymbol{x},\boldsymbol{\xi})] - G(\boldsymbol{x},\boldsymbol{\xi})[c_p^2(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi}))\boldsymbol{n} - i\omega\frac{\mu}{\rho}\nabla\boldsymbol{u}(\boldsymbol{\xi})]dS_\xi\end{aligned}$$

APPENDIX

Vector Denotation and Identities

consider scalar u and vector $\boldsymbol{V}$:

$$grad\ u = \nabla u = u_{,i} = \frac{\partial u}{\partial x_i}$$

$$div\ u = \nabla \cdot u = u_{i,i}$$

$$\nabla \cdot (\nabla u) = \nabla^2 u = u_{,ii}$$

$$\nabla \cdot (\nabla \boldsymbol{V}) = \nabla^2 \boldsymbol{V} = \nabla^2 v_i = v_{i,jj}$$

$$\nabla \times \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times \nabla u = 0,\ \text{curl of a gradient is zero}$$

$$\nabla \cdot (\nabla \times \boldsymbol{V}) = 0,\ \text{divergence of a curl is zero}$$

$$\nabla^2 \boldsymbol{V} = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla \times (\nabla \times \boldsymbol{V}), \;\; \nabla \times \nabla \times (\nabla \times \boldsymbol{V}) = -\nabla \times (\nabla^2 \boldsymbol{V}). \quad (10)$$

$$\text{divergency theorem:} \;\; \int_\Omega \nabla \cdot \boldsymbol{V} dA = \int_{\partial\Omega} \boldsymbol{V} \cdot \boldsymbol{n} ds$$