§泰勒公式

局部泰勒公式：$x \to x_0$时 $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2} + \cdots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + o((x-x_0)^n)$

佩亚诺余项

证明：令 $T_n(x) = f(x) - o((x-x_0)^n)$

$$\lim_{x\to x_0} \frac{f(x) - T_n(x)}{(x-x_0)^n} = \lim_{x\to x_0} \frac{f^{(n)}(x) - T_n^{(n)}(x)}{n!} = 0$$

证等式等同于证差为0或无穷小量

借分母低阶无穷小量凑洛必达的结构

证毕

麦克劳林公式：$x_0=0$ 时 $f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)x^n}{n!}$

常用麦克劳林公式：

$$e^x = 1 + x + \cdots \frac{1}{n!}x^n + o(x^n)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha!}{(\alpha-n)!\,n!}x^n + o(x^n)$$

$$\sin x = x - \frac{1}{3!}x^3 + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \begin{cases} o(x^{2n-1}) \\ 0 + o(x^{2n}) \end{cases}$$

$$\cos x = 1 - \frac{1}{2}x^2 + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!} + \begin{cases} o(x^{2n}) \\ 0 + o(x^{2n+1}) \end{cases}$$

局部泰勒展开的唯一性定理：若 $f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots A_n(x-x_0)^n + o((x-x_0)^n)$，$x\to x_0$

则有 $A_k = \frac{f^{(k)}(x_0)}{k!}$

也就是说用多项式去逼近某个函数，局部的泰勒展开公式只会有一种

证明：显然 $A_0 = f(x_0)$ 即证 $\frac{f(x)-f(x_0)}{x-x_0} = A_1 + A_2(x-x_0) + \cdots + A_n(x-x_0)^{n-1} + o((x-x_0)^{n-1})$，$x\to x_0$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} f'(x)$$
$$= f'(x_0) = A_1$$
$$= \lim_{x\to x_0}(\text{右式})$$

同理 $f^{(k)}(x_0) = k!A_k$

证毕

求 e^{-x^2} 的麦克劳林展开式

令 $t = -x^2$ $e^t = $ —— ——

$=$ —— ——

求 $e^{\cos x}$ 的麦克劳林展开式

保证 $x\to 0$时，$t\to 0$，即最后一项为无穷小量

带拉格朗日余项的泰勒展开式：$f(x) = f(x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

证明：令 $F(t) = f(t) - f(x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$G(t) = (t-x_0)^{n+1}$

显然 $F(x_0) = 0$，$G(x_0) = 0$

$$\frac{F(x)-F(x_0)}{G(x)-G(x_0)} = \frac{F'(x_1)-F'(x_0)}{G'(x_1)-G'(x_0)} = \frac{F^{(n+1)}(\xi)}{G^{(n+1)}(\xi)}$$

即 $F(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

证毕

常见函数的拉格朗日余项：$x \to 0$

$$e^x \sim \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

$$\sin x \sim (-1)^n \frac{\cos\xi}{(2n+1)!} x^{2n+1}$$

$$\ln(1+x) \sim \frac{1}{(n+1)} \cdot \frac{1}{(1+\xi)^{n+1}} x^{n+1}$$

$$\cos x \sim (-1)^{n-1} \frac{\cos\xi}{(2n+2)!} x^{2n+2}$$

$$(1+x)^{\alpha} \sim \frac{\alpha!}{(\alpha-n-1)!\,(n+1)!} x^{n+1}$$

常用求极限方法

1° 选择合适换元方法

2° 提前提出确定极限

3° 无穷小量代换

4° 泰勒展开至适当级数

5° 洛必达法则

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