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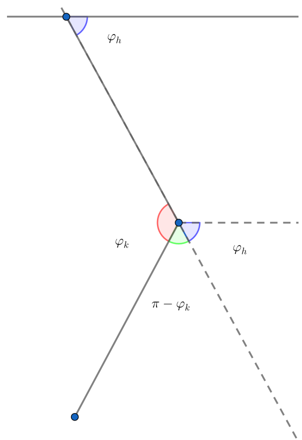
Problem Set 2: Swing Leg and Running

1. (a) **Derive for this swing leg model the equation of motion for the knee.**



$x_1 = l_1 \cos(\varphi_h)$
$\dot{x}_1 = -\dot{\varphi}_h l_1 \sin(\varphi_h)$

$y_1 = -l_1 \sin(\varphi_h)$
$\dot{y}_1 = -\dot{\varphi}_h l_1 \cos(\varphi_h)$

$x_2 = x_1 + l_2 \cos(\varphi_h + \pi - \varphi_k)$
$x_2 = x_1 + l_2 \left(\cos(\varphi_h - \varphi_k)\cos(\pi) + \sin(\varphi_h - \varphi_k)\sin(\pi)\right)$
$x_2 = l_1 \cos(\varphi_h) - l_2 \cos(\varphi_h - \varphi_k)$
$\dot{x_2} = \frac{\partial x_2}{\partial \varphi_h} + \frac{\partial x_2}{\partial \varphi_k} = -\dot{\varphi}_h l_1 \sin(\varphi_h) + \dot{\varphi}_h l_2 \sin(\varphi_h - \varphi_k) + \dot{\varphi}_k l_1 \sin(\varphi_h - \varphi_k)$
$\dot{x_2} = -\dot{\varphi}_h l_1 \sin(\varphi_h) + (\dot{\varphi}_h + \dot{\varphi}_k) l_2 \sin(\varphi_h - \varphi_k)$

$y_2 = y_1 - l_2 \sin(\varphi_h + \pi - \varphi_k)$
$y_2 = y_1 - l_2 \left(\sin(\varphi_h - \varphi_k)\cos(\pi) + \sin(\pi)\cos(\varphi_h - \varphi_k)\right)$
$y_2 = -l_1 \sin(\varphi_h) + l_2 \sin(\varphi_h - \varphi_k)$
$\dot{y_2} = \frac{\partial y_2}{\partial \varphi_h} + \frac{\partial y_2}{\partial \varphi_k} = -\dot{\varphi}_h l_1 \cos(\varphi_h) + \dot{\varphi}_h l_2 \cos(\varphi_h - \varphi_k) - \dot{\varphi}_k l_2 cos(\varphi_h - \varphi_k)$
$\dot{y_2} = -\dot{\varphi}_h (phi_h - phi_k)(l_1 \cos(\varphi_h) + l_2 \cos(\varphi_h - \varphi_k)) - \dot{\varphi}_k l_2 \cos(\varphi_h - \varphi_k)$

$T = \frac{m_1}{2}(\dot{x_1}^2 + \dot{y_1}^2) + \frac{m_2}{2}(\dot{x_2}^2 + \dot{y_2}^2)$
$T = \frac{m_1}{2}(\dot{\varphi_h}^2 {l_1}^2 \sin(\varphi_h)^2 + \dot{\varphi_h}^2 {l_1}^2 \cos(\varphi_h)^2) + \frac{m_2}{2}(\dot{x_2}^2 + \dot{y_2}^2)$
$T = \frac{\dot{\varphi_h}^2 {l_1}^2 m_1}{2} + \frac{m_2}{2}(-\dot{\varphi_h} l_1 \sin(\varphi_h) + (\dot{\varphi_h} + \dot{\varphi_k}) l_2 \sin(\varphi_h - \varphi_k))^2 + (-\dot{\varphi_h} l_1 \cos(\varphi_h) + (\dot{\varphi_h} - \dot{\varphi_k}) l_2 \cos(\varphi_h - \varphi_k))^2]$
$T = \frac{1}{2}(\dot{\varphi_h}^2 {l_1}^2 m_1 + \dot{\varphi_h}^2 {l_1}^2 m_2 + \dot{\varphi_h}^2 l_2^2 m_2 + \dot{\varphi_k}^2 l_2^2 m_2) - \dot{\varphi_h}\dot{\varphi_k} l_2^2 m_2 - \dot{\varphi_h}^2 l_1 l_2 m_2 \cos(\varphi_k) + \dot{\varphi_h}\dot{\varphi_k} l_1 l_2 m_2 \cos(\varphi_k)$
$U = -g m_2 (l_1 \sin(\varphi_h) - l_2 \sin(\varphi_h - \varphi_k)) - g l_1 m_1 \sin(\varphi_h)$
$L = g m_2 (l_1 \sin(\varphi_h) - l_2 \sin(\varphi_h - \varphi_k)) + \frac{1}{2}\left(\dot{\varphi_h}^2 l_1^2 m_1 + \dot{\varphi_h}^2 l_1^2 m_2 + \dot{\varphi_h}^2 l_2^2 m_2 + \dot{\varphi_k}^2 {l_2}^2 m_2\right) - \dot{\varphi_h}\dot{\varphi_k} {l_2}^2 m_2 + g l_1 m_1 \sin(\varphi_h) - \dot{\varphi_h}^2 l_1 l_2 m_2 \cos(\varphi_k) + \dot{\varphi_h}\dot{\varphi_k} l_1 l_2 m_2 \cos(\varphi_k)$
$\frac{\partial L}{\partial \dot{\varphi_k}} = \dot{\varphi_k} {l_2}^2 m_2 - \dot{\varphi_h} {l_2}^2 m_2 + \dot{\varphi_h} l_1 l_2 m_2 \cos(\varphi_k)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi_k}}\right) = \ddot{\varphi_k} {l_2}^2 m_2 - \ddot{\varphi_h} {l_2}^2 m_2 + \ddot{\varphi_h} l_1 l_2 m_2 \cos(\varphi_k) - \dot{\varphi_h} l_1 l_2 m_2 \sin(\varphi_k)$
$\frac{\partial L}{\partial \varphi_k} = l_1 l_2 m_2 \sin(\varphi_k) \dot{\varphi_h}^2 - \dot{\varphi_k} l_1 l_2 m_2 \sin(\varphi_k) \dot{\varphi_h} + g l_2 m_2 cos(\varphi_h - \varphi_k)$
$\tau_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi_k}}\right) - \frac{\partial L}{\partial \varphi_k}$
$\tau_k = -l_2 m_2 (\ddot{\varphi_h} l_2 - \ddot{\varphi_k} l_2 + g \cos(\varphi_h - \varphi_k) - \ddot{\varphi_h} l_1 \cos(\varphi_k) + \dot{\varphi_h} l_1 \sin(\varphi_k) + \dot{\varphi_h}^2 l_1 \sin(\varphi_k) - \dot{\varphi_h}\dot{\varphi_k} l_1 \sin(\varphi_k))$

(b) Develop an intuition about the swing leg behavior at the start of locomotion by making simplifying assumptions. Assume $m_1 = m_2 = m$ and $l_1 = l_2 = l$. Show that the straight-down leg position $\varphi_h = \pi/2$ and $\varphi_k = \pi$ which humans have at the start of locomotion leads to $\ddot{\varphi_k} - 2\ddot{\varphi_h} = \frac{\tau_k}{ml^2}$ and $3\ddot{\varphi_h} - \ddot{\varphi_k} = \frac{\tau_k + \tau_h}{ml^2}$.

$\cos(\varphi_h) = 0$
$\cos(\varphi_k) = -1$
$\sin(\varphi_k) = 0$
Simplifying the hip equation:
$3ml\ddot{\varphi_h} - ml\ddot{\varphi_k} = \frac{\tau_k + \tau_h}{l} => 3\ddot{\varphi_h} - \ddot{\varphi_k} = \frac{\tau_k + \tau_h}{ml^2}$
Simplifying the knee equation:
$\tau_k = -lm(\ddot{\varphi_h} l - \ddot{\varphi_k} l + \ddot{\varphi_h} l) => \ddot{\varphi_k} - 2\ddot{\varphi_h} = \frac{\tau_k}{ml^2}$

2. (a) **What is the constant system energy E_{sys} of the model? What are the horizontal velocity and the vertical position at apex 1?**

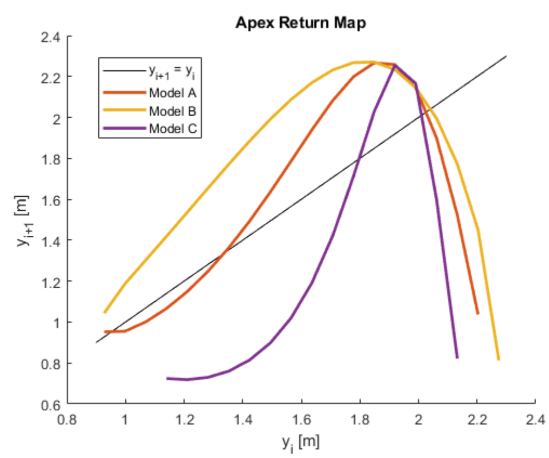
   The constant system energy is 1784.8 Joules. At apex 1, the horizontal velocity is 5.08612 m/s and the vertical position is 0.955 m

   (b) **What are the horizontal velocity and the vertical position at apex 1 for the versions B and C?**

   For model B at apex 1, the horizontal velocity is 4.613 m/s and the vertical position is 1.190

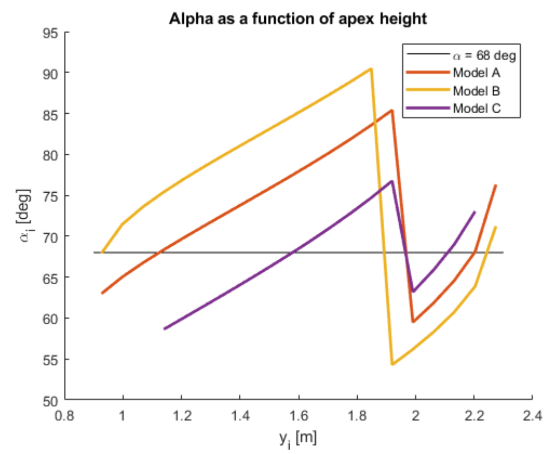
   Model C does not reach an apex because the leg never leaves the ground. When the simulation ends, the horizontal velocity is 5.404 m/s and the vertical position is 0

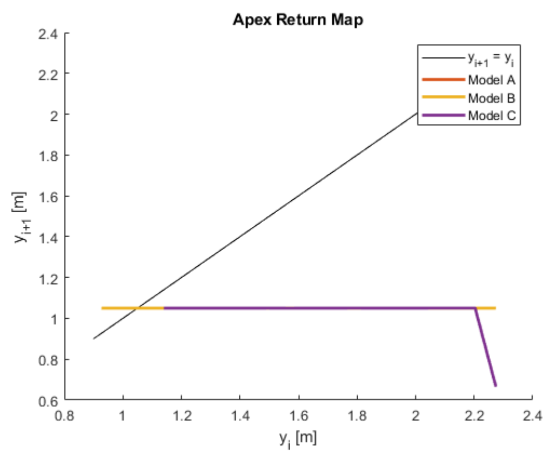
   (c) **Compute and plot in one figure the resulting apex return maps**



   The most robust is model A since it has a stable fixed point at $y_0 = 0.95$, though it is close to the minimum apex height before the leg fails to leave the ground. The rest of the models have at least one unstable fixed point so under ideal circumstances with no disturbances they could remain stable at those points. Model C is second most robust because it has two fixed points but both of them are unstable. Finally, model B is least stable because it only has a single unstable fixed point.

(d) **Repeat this search for a series of initial conditions from the minimum to the maximum possible values and extract and plot the relationship $\alpha(y_i)$ for each model version.**





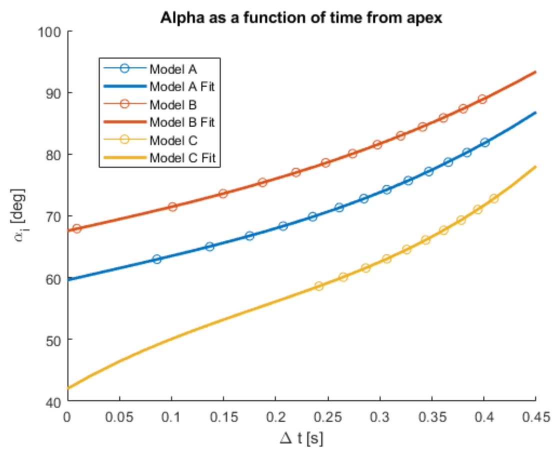
The apex return map is almost flat for each model, meaning it is able to maintain its height of $y_{Target} = 1.05$ no matter what y_i is. The basin of attraction for each function is at least $[1.2, 2.2]$ meters and only varies between models because model C fails to get enough height with its lower spring stiffness when dropped from heights below ~ 1.2 m.

(e) $\Delta y = \frac{g}{2}(t - t_{apex})^2$

$t - t_{apex} = \sqrt{\frac{2}{g}\Delta y}$

$t - t_{apex} = \sqrt{\frac{2}{g}(y_i - l_0 sin(\alpha(y_i)))}$

Limiting y_i to $[0.8, 1.8]$ so that α increases monotonically and fitting a 3rd order polynomial to $\alpha(t - t_{apex})$ produces the following figure.



(f) In order to determine model stability, the 3 models were simulated with increasing levels of noise in leg angle α. For each trial, the simulation ran for 10 seconds or until the model fell over. The total time without falling was logged for each trial. Graphed below is the mean time until fall for each model along with the confidence interval (0.975). Model C is the most stable, staying upright for the full 10 seconds in 50/50 trials until the noise surpasses ± 0.6 degrees and staying upright for an average of 5 seconds at a noise level of ± 12 degrees.