R16-868 F2024
Problem Set 2: Swing Leg and Running (due: 8 Oct)

1. **Theory.** Swing leg dynamics. Figure 1A shows a double pendulum as a model for the human swing leg. The thigh and the shank-foot complex are represented as segments whose mass is concentrated at the end points. **(6+2pts)**

   a) Derive for this swing leg model the equation of motion for the knee.

      Hints: The equation of motion for the knee can be obtained from $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_K}\right) - \frac{\partial L}{\partial \varphi_K} = \tau_K$ where the Lagrangian $L = T - U$ is the difference between the kinetic and the potential energy, T and U respectively, and τ_K is the torque applied to the knee by an actuator. In cartesian coordinates, $T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$ and $U = m_1 g y_1 + m_2 g y_2$. Represent these energies in the coordinates φ_H and φ_K and then apply the differential equation. Be careful when calculating the total derivative d/dt after you have calculated the partial derivative $\partial L/\partial \dot{\varphi}_K$. (3pts)

   b) The equation of motion for the hip is

      $$[(m_1 + m_2)l_1 - m_2 l_2 \cos\varphi_K]\,\ddot{\varphi}_H + m_2 l_2 \cos\varphi_K \ddot{\varphi}_K - m_2 l_2 (\dot{\varphi}_K - \dot{\varphi}_H)^2 \sin\varphi_K - (m_1 + m_2) g \cos\varphi_H = \frac{\tau_H + \tau_K}{l_1}.$$

      The knee and the hip equation form a set of coupled nonlinear differential equations. No closed form solution exists for this set of equations. Develop an intuition about the swing leg behavior at the start of locomotion by making simplifying assumptions.

      Hints: Assume $m_1 = m_2 = m$ and $l_1 = l_2 = l$. Show that the straight-down leg position $\varphi_H = \pi/2$ and $\varphi_K = \pi$ which humans have at the start of locomotion leads to $\ddot{\varphi}_K - 2\ddot{\varphi}_H = \frac{\tau_K}{ml^2}$ and $3\ddot{\varphi}_H - \ddot{\varphi}_K = \frac{\tau_K + \tau_H}{ml^2}$. What happens if you apply the same torque τ_0 only at the hip ($\tau_K = 0$) or only at the knee ($\tau_H = 0$)? Which is more effective for initiating gait? How does this change if you change the ratio m_1/m_2? (3pts)

      Option: Can you find at least one other characteristic position and develop an intuition about the swing leg behavior? (2pts)

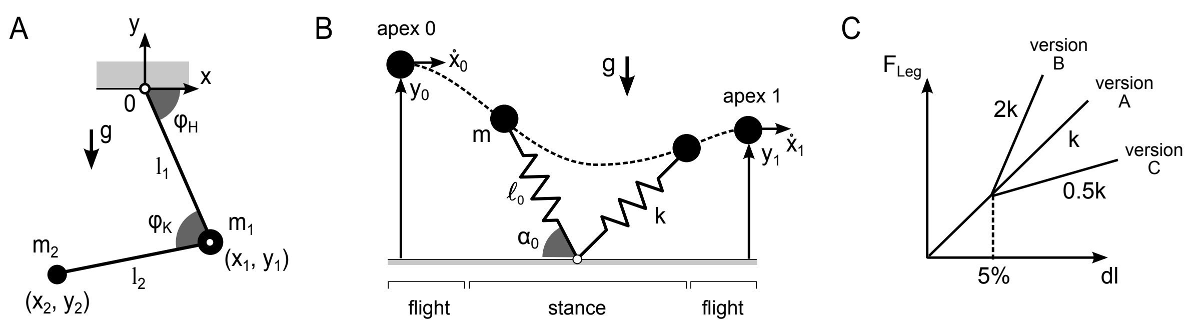

Figure 1: Swing leg dynamics (A), apex return map of spring mass model (B), and stance leg behavior (C).

2. **Simulation.** Develop three versions of the 2D spring-mass running model and investigate the influence of the stance leg behavior on gait stability and robustness of locomotion (Figs. 1B,C). **(9+2pts)**

   a) Use the Simulink model of spring-mass running that we have developed in class with a preliminary angle of attack α_0=68deg (Fig. 1B; m=80kg, k=20kN/m, l_0=1m, g=9.81ms^{-2}, landing condition y_l=$l_0 \sin\alpha_0$). Start the simulation at the apex in flight with an initial velocity $\dot{x}_0$=5ms^{-1} and an initial height y_0=1m. What is the constant system energy E_{sys} of the model? What are the horizontal velocity and the vertical position at apex 1? (1pt)

b) Develop two additional versions of the model with altered stance leg behavior (Fig. 1C). Version B has twice the original leg stiffness and version C has half the original leg stiffness once the leg compresses more than 5%. Repeat the previous experiment. What are the horizontal velocity and the vertical position at apex 1 for the versions B and C? (1pt)

c) Write a matlab script that can vary the initial condition y_0, then calls the simulation model, and gets the next apex state y_1 back from the simulation. Compute and plot in one figure the resulting apex return maps y_{i+1}=P(y_i) at α_0=68deg for each version of the model. Which of the three models is most robust to changes in ground height? Which is least robust? (2pts)

Hints: When varying the initial height, you also need to vary the initial velocity to make sure that the system energy E_{sys} stays constant. For assessing stability, the size of the basin of attraction is a good measure. It can be found in 1D-return maps following the slope of the mapping function.

d) Now extend your matlab code such that you can also vary the landing angle α_0. Embed the code in a matlab optimization which, for a given initial state y_0 (and $\dot{x}_0$ adapted via constant system energy), minimizes by varying α_0 the difference between y_1 and y_{tgt}, where y_{tgt} = 1.05m is the target apex state. Repeat this search for a series of initial conditions from the minimum to the maximum possible values and extract and plot the relationship $\alpha(y_i)$ for each model version. Show that this function improves gait stability by plotting the resulting Poincare maps and comparing the basin of attraction to the ones in the corresponding Poincare maps with constant α_0=68deg. (2pts)

Hint: You can use brute force computation. Alternatively, matlab provides optimization functions, for instance, the command `fminsearch` calls a minimization function.

e) Use the relationship between $\alpha(y_i)$ and the falling time from apex y_i to touchdown y_l=l_0 sinα_0 to transform the function $\alpha(y_i)$ into a function α(t-t$_{apex}$) that varies the angle of attack after the apex event. (1pt)

Hints: (i) The transformation only works if the relationship between α and time is monotonic. To satisfy this condition, limit the lowest initial height you consider for the time law calculation to y_i^{min} = 0.8m. (ii) In addition, use a 3rd order polynomial fit to obtain the control law $\alpha(t-t_{apex}) = c_0 + c_1(t - t_{apex}) + c_2(t - t_{apex})^2 + c_3(t - t_{apex})^3$ that you can embed in simulink.

f) Embed this function α(t-t$_{apex}$) in your model as the commanded leg angle in flight after the apex event. Add uniform noise (block in simulink library) to the commanded angle to model the uncertainty in regulating the leg angle in humans and humanoids. Systematically increase the noise level for the three model versions and find the maximum level that each can tolerate. Think of a good way to present your comparison results in one figure and use the figure to discuss the outcomes. (2pts)

g) Option: Use your previous result to identify a leg function $F(dl)$ that further improves the tolerance against noise in leg angle regulation. Demonstrate the performance improvement by repeating steps d) through f) for a model with this leg function. (+2pts)