Mariano Cantero

II. CHANNEL FLOW MODEL

A. Geometry

The channel is composed of two infinite parallel walls, spaced a distance 2 h apart. The model assumes that the flow is periodic in the plane of the walls. Thus, a finite sized section can be used to model the infinite channel. The section used in the model is shown in Figure 2.1. The x coordinate points in the streamwise direction and the y coordinate points in the wall–normal direction. The z coordinate points transversely across the channel, in the spanwise direction. The size of the box along the streamwise and spanwise directions is $4\pi h$ and $\frac{4}{3}\pi h$, respectively.

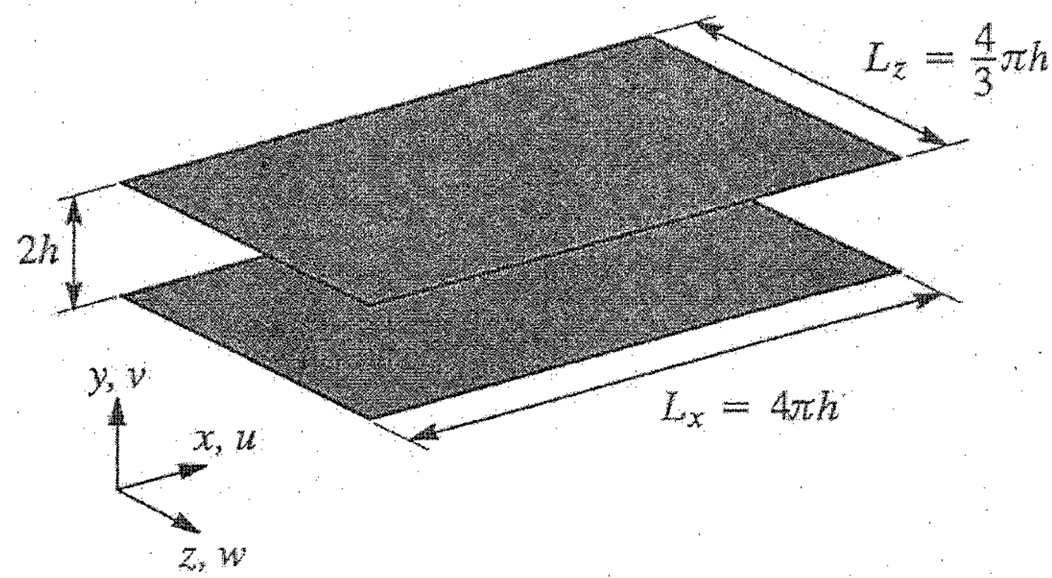

Figure 2.1. Channel geometry.

B. Governing Equations

The Navier–Stokes equations for an incompressible fluid with no body forces are

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + v\frac{\partial \tilde{u}}{\partial y} + w\frac{\partial \tilde{u}}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\left\{\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial^2 \tilde{u}}{\partial z^2}\right\} \tag{2.2}$$

$$\frac{\partial v}{\partial t} + \tilde{u}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right\} \tag{2.3}$$

$$\frac{\partial w}{\partial t} + \tilde{u}\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right\}. \tag{2.4}$$

Equation (2.1) is a statement of conservation of mass, while Equations (2.2), (2.3) and (2.4) are statements of conservation of linear momentum in the x, y, and z directions, respectively. The convective form of the momentum equations are shown. Note that the streamwise velocity is Reynolds decomposed into mean and and perturbation terms: $\tilde{u} = U + u$. The tilde denotes the total, the capital denotes the mean, and the lowercase denotes the perturbation.

These governing equations can be rescaled using the half channel height, h, for the length scale, the wall friction velocity, $u*$, for the velocity scale, and ϱu^{*2} for the pressure scale. The friction velocity is defined as $u^* = \sqrt{\frac{\tau_w}{\varrho}} = \sqrt{\nu\frac{\partial\tilde{u}}{\partial y}\bigg|_{y=\pm h}}$.

The variables are nondimensionalized as follows:

$$\begin{aligned}
\tilde{x} &= \frac{x}{h} \\
\tilde{y} &= \frac{y}{h} \\
\tilde{z} &= \frac{z}{h} \\
\tilde{u}^+ &= \frac{\tilde{u}}{u^*} \\
v^+ &= \frac{v}{u^*} \\
w^+ &= \frac{w}{u^*} \\
p^+ &= \frac{p}{\varrho u^{*2}} \\
\tilde{t} &= \frac{t\,u^*}{h}.
\end{aligned} \tag{2.5}$$

Using the nondimensionalization shown in Equation (2.5), Equations (2.1) through (2.4) are rescaled:

$$\frac{\partial\tilde{u}^+}{\partial\tilde{x}} + \frac{\partial v^+}{\partial\tilde{y}} + \frac{\partial w^+}{\partial\tilde{z}} = 0 \tag{2.6}$$

$$\frac{\partial\tilde{u}^+}{\partial\tilde{t}} + \tilde{u}^+\frac{\partial\tilde{u}^+}{\partial\tilde{x}} + v^+\frac{\partial\tilde{u}^+}{\partial\tilde{y}} + w^+\frac{\partial\tilde{u}^+}{\partial\tilde{z}} = -\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2\tilde{u}^+}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{u}^+}{\partial\tilde{y}^2} + \frac{\partial^2\tilde{u}^+}{\partial\tilde{z}^2}\right\} \tag{2.7}$$

$$\frac{\partial v^+}{\partial \tilde{t}} + \tilde{u}^+\frac{\partial v^+}{\partial \tilde{x}} + v^+\frac{\partial v^+}{\partial \tilde{y}} + w^+\frac{\partial v^+}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2 v^+}{\partial \tilde{x}^2} + \frac{\partial^2 v^+}{\partial \tilde{y}^2} + \frac{\partial^2 v^+}{\partial \tilde{z}^2}\right\} \tag{2.8}$$

$$\frac{\partial w^+}{\partial \tilde{t}} + \tilde{u}^+\frac{\partial w^+}{\partial \tilde{x}} + v^+\frac{\partial w^+}{\partial \tilde{y}} + w^+\frac{\partial w^+}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{z}} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2 w^+}{\partial \tilde{x}^2} + \frac{\partial^2 w^+}{\partial \tilde{y}^2} + \frac{\partial^2 w^+}{\partial \tilde{z}^2}\right\} \tag{2.9}$$

where $\mathrm{Re}_\tau = \dfrac{u^* h}{\nu}$ is the Reynolds number based on the wall friction velocity and the half channel height.

From here on, the superscripts will be dropped and the variables will be assumed to be nondimensionalized as above, unless otherwise noted.

The flow is driven through the channel by a constant mean pressure gradient in the streamwise direction. By performing a force balance on the channel, the value of the mean pressure gradient is determined to be –1. The gradient of the mean pressure is zero in the spanwise and wall–normal directions. The pressure can be Reynolds decomposed into mean and perturbation terms:

$$\tilde{p}(x,y,z) = P(x) + p'(x,y,z)\,. \tag{2.10}$$

Using Equation (2.10) and the value of (–1, 0, 0) for the mean pressure gradient term, Equations (2.7) through (2.9) can be expressed as:

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + v\frac{\partial \tilde{u}}{\partial y} + w\frac{\partial \tilde{u}}{\partial z} = 1 - \frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial^2 \tilde{u}}{\partial z^2}\right\} \tag{2.11}$$

$$\frac{\partial v}{\partial t} + \tilde{u}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p'}{\partial y} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right\} \tag{2.12}$$

$$\frac{\partial w}{\partial t} + \tilde{u}\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}_\tau}\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right\}. \tag{2.13}$$

C. Boundary Conditions

The model has periodic boundary conditions in the streamwise and spanwise directions. In the normal direction, the velocities satisfy no slip and no penetration at the walls.

III. NUMERICAL METHODS

A. Temporal Discretization

An operator splitting method [Zang et al. (1989) and Streett and Hussaini (1990)] is used to solve the Navier–Stokes equations along with the incompressibility condition. The flow field is advanced from time level $t^{(n)}$ to $t^{(n+1)}$ in two steps. First, an advection–diffusion equation is used to advance from time level $t^{(n)}$ to an intermediate time level, $t^{(n*)}$. The boundary conditions for this first step will be discussed in the following section. After these intermediate (*) level velocities are determined, a pressure correction step is used to advance to the level $t^{(n+1)}$.

Temporal discretization of the advection–diffusion step is accomplished using a low–storage mixed third order Runge–Kutta and Crank–Nicolson scheme [Hussaini et al. (1988)]. The scheme is carried out in three stages. The time step from level $t^{(n)}$ to level $t^{(n+1)}$, Δt, is split into three smaller steps, with pressure correction at the end of each stage. A backward Euler scheme is used to discretize the pressure correction step. The pressure correction is carried out in two parts. First, a pressure Poisson equation is solved and once the pressure is determined, the velocities can then be updated.

In order to simplify the notation, advection and diffusion operators are defined. The diffusion operator is defined as

$$D(\,) = \frac{1}{\mathrm{Re}_\tau}\nabla^2(\,)\,. \tag{3.1}$$

Two different advection operators are used:

$$A_c(\,) = -\,\vec{u}\cdot\nabla(\,) \tag{3.2}$$

$$A_d(\,) = -\,\nabla\cdot[\,\vec{u}\,(\,)]\,. \tag{3.3}$$

These correspond to the convective and divergence forms of the advection operator. By alternating between these two advection operators for successive time steps, the results are very similar to results obtained using a skew–symmetric form of the governing equations, but the method has the advantage that the number of derivatives which need to be evaluated to complete one time step is cut by a factor of two [Zang (1991)]. Henceforth, the advection operator will be referred to as $A(\,)$, without specifying which form is to be used. Also note that $\vec{u} = (\tilde{u}, v, w)$.

One complete time step through the entire three stage scheme is shown below. The superscripts 0, 1 ,2, and 3 denote the stage level. The zeroeth stage level field is equal to the velocities at time level $t^{(n)}$, i. e.

$$\vec{u}^{(0)} = \vec{u}\left(t^{(n)}\right). \tag{3.4}$$

Next, the stage (1*) velocities are calculated using the advection diffusion equation. Then the corresponding pressure is evaluated by solving a Poisson equation. Next, the pressure gradients are used to correct the velocity to a divergence free field:

$$\vec{H}^{(1)} = \Delta t \;\; A\left(\vec{u}^{(0)}\right) \tag{3.5}$$

$$\vec{u}^{(1*)} = \vec{u}^{(0)} + \frac{1}{3}\vec{H}^{(1)} + \frac{\Delta t}{6}\left[D\left(\vec{u}^{(0)}\right) + D\left(\vec{u}^{(1*)}\right)\right] \tag{3.6}$$

$$\nabla^2 p^{(1)} = \frac{3}{\Delta t}\; \nabla \cdot \vec{u}^{(1*)} \tag{3.7}$$

$$\vec{u}^{(1)} = \vec{u}^{(1*)} - \frac{\Delta t}{3}\; \nabla p^{(1)}. \tag{3.8}$$

Once the first stage is complete, the second stage of the scheme is performed:

$$\vec{H}^{(2)} = \Delta t \;\; A\left(\vec{u}^{(1)}\right) - \frac{5}{9}\; \vec{H}^{(1)} \tag{3.9}$$

$$\vec{u}^{(2*)} = \vec{u}^{(1)} + \frac{15}{16}\vec{H}^{(2)} + \frac{5}{24}\Delta t\left[D\left(\vec{u}^{(1)}\right) + D\left(\vec{u}^{(2*)}\right)\right] \tag{3.10}$$

$$\nabla^2 p^{(2)} = \frac{12}{5\Delta t}\; \nabla \cdot \vec{u}^{(2*)} \tag{3.11}$$

$$\vec{u}^{(2)} = \vec{u}^{(2*)} - \frac{5\Delta t}{12}\; \nabla p^{(2)}. \tag{3.12}$$

The third stage is performed next:

$$\vec{H}^{(3)} = \Delta t \;\; A\left(\vec{u}^{(2)}\right) - \frac{153}{128}\; \vec{H}^{(2)} \tag{3.13}$$

$$\vec{u}^{(3*)} = \vec{u}^{(2)} + \frac{8}{15}\vec{H}^{(3)} + \frac{1}{8}\Delta t\left[D\left(\vec{u}^{(2)}\right) + D\left(\vec{u}^{(3*)}\right)\right] \tag{3.14}$$

$$\nabla^2 p^{(3)} = \frac{4}{\Delta t}\ \nabla \cdot \vec{u}^{(3*)} \tag{3.15}$$

$$\vec{u}^{(3)} = \vec{u}^{(3*)} - \frac{\Delta t}{4}\ \nabla p^{(3)}\,. \tag{3.16}$$

Once the last stage is complete, the velocity field is updated, and the process can be restarted, until the desired number of time steps have been performed:

$$\vec{u}\left(t^{(n+1)}\right) = \vec{u}^{(3)}\,. \tag{3.17}$$

These equations can be simplified if we define a stage number, m, a coefficient for the first nonlinear term, $cnl1$, a coefficient for the second nonlinear term, $cnl2$, and the coefficient of the diffusion term, cd. These quantities are defined as:

$$m = (1, 2, 3) \tag{3.18}$$

$$cnl1(m) = \left(0, -\frac{5}{9}, -\frac{153}{128}\right) \tag{3.19}$$

$$cnl2(m) = \left(\frac{1}{3}, \frac{15}{16}, \frac{8}{15}\right) \tag{3.20}$$

$$cd(m) = \left(\frac{\Delta t}{6}, \frac{5\Delta t}{24}, \frac{\Delta t}{8}\right)\,. \tag{3.21}$$

The complete three stage scheme is shown below, using the notation introduced in Equations (3.18) through (3.21):

$$\vec{u}^{(0)} = \vec{u}\left(t^{(n)}\right) \tag{3.22}$$

$$\vec{H}^{(m)} = \Delta t\ A\left(\vec{u}^{(m-1)}\right) + cnl1(m)\ \vec{H}^{(m-1)} \tag{3.23}$$

$$\vec{u}^{(m*)} = \vec{u}^{(m-1)} + cnl2(m)\ \vec{H}^{(m)} + cd(m)\left[D\left(\vec{u}^{(m-1)}\right) + D\left(\vec{u}^{(m*)}\right)\right] \tag{3.24}$$

$$\nabla^2 p^{(m)} = \frac{1}{2\ cd(m)}\ \nabla \cdot \vec{u}^{(m*)} \tag{3.25}$$

$$\vec{u}^{(m)} = \vec{u}^{(m*)} - 2\ cd(m)\ \nabla p^{(m)} \tag{3.26}$$

$$\vec{u}\left(t^{(n+1)}\right) = \vec{u}^{(3)}\,. \tag{3.27}$$

Thus this step must be solved ... space.

B. Spatial Discretization

The periodic boundary conditions in the streamwise and spanwise directions suggest the use of Fourier expansions. In the streamwise and spanwise directions, the collocation points are uniformly spaced. In contrast, the nonperiodicity of the wall normal direction, coupled with the need to resolve the high gradients close to the walls, suggests that an expansion with nonuniformly spaced collocation points should be employed in this direction. Chebyshev polynomials are used as the basis functions for the wall–normal expansion. The expansion used to discretize a typical variable, u, is

$$u\left(x_j, y_k, z_l, t\right) = \sum_{k_x=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \sum_{k_y=0}^{N_y-1} \sum_{k_z=-\frac{N_z}{2}}^{\frac{N_z}{2}-1} \hat{u}\left(k_x, k_y, k_z, t\right) e^{\left(ik_x x_j\right)} e^{\left(ik_z z_l\right)} T_{k_y}\left(y_k\right) . \tag{3.28}$$

Equation (3.28) shows the Note that T_{k_y} denotes the k_yth Chebyshev polynomial. In the wall–normal direction, Chebyshev–Gauss Lobatto points are used. These points provide a higher resolution near the walls, as well as a point on both of the walls, which allow for straightforward treatment both Dirichlet and Neumann boundary conditions. The locations of the collocation points in the streamwise, wall–normal, and spanwise directions, respectively are

$$x_j = \frac{jL_x}{hN_x}, \qquad 0 \le j \le N_x - 1 \tag{3.29}$$

$$y_k = \cos\left[\frac{(k-1)\pi}{N_y - 1}\right], \qquad 1 \le k \le N_y \tag{3.30}$$

$$z_l = \frac{lL_z}{hN_z}, \qquad 0 \le l \le N_z - 1 . \tag{3.31}$$

Spectrally accurate derivatives are obtained by using fast Fourier transform (FFT) techniques in the streamwise and spanwise directions and by matrix multiplication techniques in the wall–normal direction. Vectorized routines to perform these techniques are available in the Cray Science Library.

In order to complete one time step, Equations (3.22) through (3.27) are used. Equations (3.23) through (3.26) must be solved for each of the three stages.

Rearrangement of Equations (3.24) and (3.25) yield Helmholtz/Poisson equations for the three components of velocity and for the pressure. The diffusion term involving $\vec{u}^{(m*)}$ can be moved to the left hand side. After performing this operation, the x–component of the momentum equation can be rewritten as

$$\tilde{u}^{(m*)} - cd(m)\, D\left(\tilde{u}^{(m*)}\right) = \tilde{u}^{(m-1)} + cnl2(m)\, H_x^{(m)} + cd(m)\, D\left(\tilde{u}^{(m-1)}\right). \tag{3.32}$$

Next, $D(\,)$ is replaced by its definition in scalar form:

$$\begin{aligned}\left(\frac{\partial^2\tilde{u}^{(m*)}}{\partial x^2} + \frac{\partial^2\tilde{u}^{(m*)}}{\partial y^2} + \frac{\partial^2\tilde{u}^{(m*)}}{\partial z^2}\right) - \frac{\mathrm{Re}_\tau}{cd(m)}\tilde{u}^{(m*)} \\ = -\frac{\mathrm{Re}_\tau}{cd(m)}\left[\tilde{u}^{(m-1)} + cnl2(m)\; H_x^{(m)}\right] - \left(\frac{\partial^2\tilde{u}^{(m-1)}}{\partial x^2} + \frac{\partial^2\tilde{u}^{(m-1)}}{\partial y^2} + \frac{\partial^2\tilde{u}^{(m-1)}}{\partial z^2}\right).\end{aligned} \tag{3.33}$$

At this point, the right hand side can be computed from known information and stored. In order to simplify the analysis, the right hand side of Equation (3.33) will be denoted by rhs_x^{m*}.

Next, the equation is Fourier transformed in the x and z directions. Note that the hat represents transformed coefficients in Fourier space:

$$\frac{\partial^2\hat{\tilde{u}}^{(m*)}}{\partial y^2} - \left(\frac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2\right)\hat{\tilde{u}}^{(m*)} = \widehat{rhs}_x^{(m*)}. \tag{3.34}$$

Discretization in the y direction can be expressed in terms of a matrix multiplication operation [Zang, Streett, Hussaini, 1989]. By combining a discrete forward Chebyshev transform, a differentiation by recursion, and an inverse chebyshev transform, a single matrix multiply operation can be defined to differentiate a dicretized variable. The matrix for the first derivative with respect to y is symbolically denoted as $\frac{\partial(\,)}{\partial y} = \vec{\vec{D}}_{C1}\{\,\}$. $\vec{\vec{D}}_{C1}$ is an asymmetric matrix with N_y rows and columns [Zang, Streett, and Hussaini (1989)]. An operator for performing a second partial derivative with respect to y can be formulated by simply multiplying two first derivative matrix operators:

$$\frac{\partial^2(\,)}{\partial y^2} = \frac{\partial\left(\frac{\partial(\,)}{\partial y}\right)}{\partial y} = \vec{\vec{D}}_{C1}\left\{\vec{\vec{D}}_{C1}\{\,\}\right\} = \vec{\vec{D}}_{C2}\{\,\}. \tag{3.35}$$

The notation of Equation (3.35) can be used to replace the partial derivatives in Equation (3.34):

$$\vec{\vec{D}}_{C2}\hat{\tilde{u}}^{(m*)} - \left(\frac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2\right)\hat{\tilde{u}}^{(m*)} = \hat{rhs}_x^{(m*)} \tag{3.36}$$

Similarly, the wall–normal and spanwise components of the advection–diffusion equations can be discretized:

$$\vec{\vec{D}}_{C2}\hat{v}^{(m*)} - \left(\frac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2\right)\hat{v}^{(m*)} = \hat{rhs}_y^{(m*)} \tag{3.37}$$

$$\vec{\vec{D}}_{C2}\hat{w}^{(m*)} - \left(\frac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2\right)\hat{w}^{(m*)} = \hat{rhs}_z^{(m*)} . \tag{3.38}$$

C. Boundary Conditions

The periodic boundary conditions in the streamwise and spanwise directions are automatically satisfied by the use of the Fourier expansions. Because the time splitting scheme separates the momentum equation into two parts, the no slip and incompressibility conditions cannot be satisfied simultaneously. A proper choice for the intermediate level velocity boundary conditions must be made in order to minimize the slip at the final level. Also, a boundary condition for the pressure must be specified during the pressure–Poisson step. Although there is no natural boundary condition for pressure, it can be shown that a self–consistent, pure Neumann condition will allow the slip velocity to be minimized. An analysis of the pressure part of the time splitting routine yields all of the necessary boundary conditions.

Define $\vec{\tau}$ and $\vec{\eta}$ to be unit vectors tangential and normal to the wall, respectively. The tangential and normal components of the pressure correction step are

$$\vec{\tau} \cdot \vec{u}^{(m)}\Big|_{y=\pm 1} = \vec{\tau} \cdot \left[\vec{u}^{(m*)} - 2\ cd(m)\ \nabla p^{(m)}\right]\Big|_{y=\pm 1} \tag{3.39}$$

$$\vec{\eta} \cdot \vec{u}^{(m)}\Big|_{y=\pm 1} = \vec{\eta} \cdot \left[\vec{u}^{(m*)} - 2\ cd(m)\ \nabla p^{(m)}\right]\Big|_{y=\pm 1} . \tag{3.40}$$

Ideally, the no slip and no penetration conditions require the left hand sides of Equations (3.39) and (3.40) to be zero. This condition can be used to obtain the boundary conditions for the intermediate level velocity field:

$$\vec{\tau}\cdot\vec{u}^{(m^*)}\Big|_{y=\pm 1} = 2\ cd(m)\ \vec{\tau}\cdot\nabla p^{(m)}\Big|_{y=\pm 1} \tag{3.41}$$

$$\vec{\eta}\cdot\vec{u}^{(m^*)}\Big|_{y=\pm 1} = 2\ cd(m)\ \vec{\eta}\cdot\nabla p^{(m)}\Big|_{y=\pm 1}\ . \tag{3.42}$$

Because $\nabla p^{(m)}$ has not yet been computed when the intermediate level velocity boundary conditions are needed, an approximation to $\nabla p^{(m)}$ must be used instead. A Taylor series expansion about $t = t^{(m-1)}$ can be used to estimate $\nabla p^{(m)}$:

$$\nabla p^{(m)} = \nabla p^{(m-1)} + cd(m)\frac{\partial\left(\nabla p^{(m-1)}\right)}{\partial t} + O\left(\Delta t^2\right). \tag{3.43}$$

The second term can be approximated by a backward finite difference:

$$\frac{\partial\left(\nabla p^{(m-1)}\right)}{\partial t} = \frac{\nabla p^{(m-1)} - \nabla p^{(m-2)}}{2\ cd(m-1)} + O(\Delta t)\ . \tag{3.44}$$

Substituting Equation (3.44) into (3.43), the second order accurate approximation to the pressure gradient is obtained:

$$\nabla p^{(m)} = \nabla p^{(m-1)}\left[1 + \frac{cd(m)}{cd(m-1)}\right] - \nabla p^{(m-2)}\left[\frac{cd(m)}{cd(m-1)}\right] + O\left(\Delta t^2\right). \tag{3.45}$$

The estimate for ∇p^m from Equation (3.45) is substituted into Equation (3.41) to obtain the third order accurate boundary conditions for the intermediate level streamwise and spanwise velocity components:

$$\vec{u}^{(m^*)}\Big|_{y=\pm 1} = 2\ cd(m)\left\{\frac{\partial p^{(m-1)}}{\partial x}\left[1 + \frac{cd(m)}{cd(m-1)}\right] - \frac{\partial p^{(m-2)}}{\partial x}\left[\frac{cd(m)}{cd(m-1)}\right]\right\}\Bigg|_{y=-\pm 1} \tag{3.46}$$

$$w^{(m^*)}\Big|_{y=\pm 1} = 2\ cd(m)\left\{\frac{\partial p^{(m-1)}}{\partial z}\left[1 + \frac{cd(m)}{cd(m-1)}\right] - \frac{\partial p^{(m-2)}}{\partial z}\left[\frac{cd(m)}{cd(m-1)}\right]\right\}\Bigg|_{y=\pm 1} \tag{3.47}$$

In order to satisfy global mass conservation, the final penetration velocity must be identically zero and no penetration errors can be tolerated. A consistent approach will be to apply a pure Neumann boundary condition in the computation of the pressure field:

$$\frac{\partial p^{(m)}}{\partial y}\Bigg|_{y=\pm 1} = 0\ . \tag{3.48}$$

The boundary condition for the intermediate level penetration velocity is simply zero:

$$v^{(m*)}\Big|_{y=\pm 1} = 0\,. \tag{3.49}$$

D. Solution Procedure

In order to solve fc :: entire flow field at the intermediate time level,the Helmholtz equations for the three components of velocity (Equations (3.36) through (3.38)) must be solved for each combination of horizontal wave numbers. In addition, the pressure Poisson equation must be solved. Thus, for a 60 x 61 x 60 grid, 1-D Helmholtz or Poisson equations must be solved 43,200 (three components of velocity, one pressure, for three fractional steps, at 60 x 60 horizontal wave numbers) times for each complete time step. Clearly, a technique to quickly solve these 1-D Helmholtz and Poisson equations is a necessity. In their present form, the equations require a matrix inversion for each combination of wave numbers. By diagonalizing the matrix, it is possible eliminate the ~~the~~ dependence on wave numbers from the inversion, thus, it is only necessary to invert the modal matrix once and store the inverse. Define $\vec{\vec{T}}$ as the modal matrix of $\vec{\vec{D}}_{C2}$, $\vec{\vec{T}}^{-1}$, as the inverse of $\vec{\vec{T}}$, $\vec{\lambda}$ as a vector made up of the eigenvalues of $\vec{\vec{D}}_{C2}$, and $\vec{\vec{\Lambda}}$ as an N_y by N_y diagonal matrix made up of the vector $\vec{\lambda}$ on the diagonal and zeros elsewhere. Because $\vec{\vec{D}}_{C2}$ is diagonalizable, it can be represented as $\vec{\vec{D}}_{C2} = \vec{\vec{T}}\vec{\vec{\Lambda}}\vec{\vec{T}}^{-1}$. Using this representation, Equation (3.36) can be rewritten as

$$\vec{\vec{T}}\vec{\vec{\Lambda}}\vec{\vec{T}}^{-1}\hat{\vec{u}}^{(m*)} - \left(\frac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2\right)\hat{\vec{u}}^{(m*)} = \hat{\vec{rhs}}_x^{(m*)}\,. \tag{3.50}$$

The coefficient of $\hat{\vec{u}}^{m*}$ involving the Reynolds number and the wave numbers can be combined with $\vec{\vec{\Lambda}}$. Let $\vec{\vec{I}}$ be the N_y by N_y identity matrix, and let $\alpha = \dfrac{\mathrm{Re}_\tau}{cd(m)} + k_x^2 + k_z^2$. Using this notation, Equation (3.50) can be rearranged to yield an expression for $\hat{\vec{u}}^{(m*)}$:

$$\hat{\vec{u}}^{(m*)} = \vec{\vec{T}}\frac{1}{\left(\vec{\vec{\Lambda}} - \alpha\,\vec{\vec{I}}\right)}\vec{\vec{T}}^{-1}\hat{\vec{rhs}}_x^{(m*)}\,. \tag{3.51}$$

This is a diagonal matrix

UUVat = Real one