Theorem 1. *Euler's Theorem. For a connected multi-graph G, G is Eulerian if and only if every vertex has even degree.*

Proof: If G is Eulerian then there is an Euler circuit, P, in G. Every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So every edge is accounted for and there are no repeats. Thus every degree must be even.
Suppose every degree is even. We will show that there is an Euler circuit by induction on the number of edges in the graph. The base case is for a graph G with two vertices with two edges between them. This graph is obviously Eulerian.
Now suppose we have a graph G on $m > 2$ edges. We start at an arbitrary vertex v and follow edges, arbitrarily selecting one after another until we return to v. Call this trail W. We know that we will return to v eventually because every time we encounter a vertex other than v we are listing one edge adjacent to it. There are an even number of edges adjacent to every vertex, so there will always be a suitable unused edge to list next. So this process will always lead us back to v.
Let E be the edges of W. The graph $G - E$ has components $C_1, C_2, \ldots, C_k$. These each satisfy the induction hypothesis: connected, less than m edges, and every degree is even. We know that every degree is even in $G - E$, because when we removed W, we removed an even number of edges from those vertices listed in the circuit. By induction, each circuit has an Eulerian circuit, call them $E_1, E_2, \ldots, E_k$.
Since G is connected, there is a vertex a_i in each component C_i on both W and E_i. Without loss of generality, assume that as

we follow W, the vertices $a_1, a_2, \ldots, a_k$ are encountered in that order.

We describe an Euler circuit in G by starting at v follow W until reaching a_1, follow the entire E_1 ending back at a_1, follow W until reaching a_2, follow the entire E_2, ending back at a_2 and so on. End by following W until reaching a_k, follow the entire E_k, ending back at a_k, then finish off W, ending at v. $\square$

Corollary 1.1. *A connected multi-graph G is semi-Eulerian if and only if there are exactly 2 vertices of odd degree.*

Proof:

($\Rightarrow$)

If G is semi-Eulerian then there is an open Euler trail, P, in G. Suppose the trail begins at u_1 and ends at u_n. Except for the first listing of u_1 and the last listing of u_n, every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So every edge is accounted for and there are no repeats. Thus every degree must be even, except for u_1 and u_n which must be odd.

($\Leftarrow$)

Suppose u and v are the vertices of odd degree. Consider $G+uv$. This graph has all even degrees. By Theorem 1, G has an Eulerian circuit. This circuit uses the edge uv. Thus we have an Euler path in G when we omit the edge uv.$\square$