Homework 01-25-2021

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Problem 1

Develop two Monte Carlo methods for the estimation of $\theta = \int_0^1 e^{x^2} dx$ and implement in **R** .

Answer: your answer starts here...

Method 1

Introduce Let $Y \sim U(0, 1)$

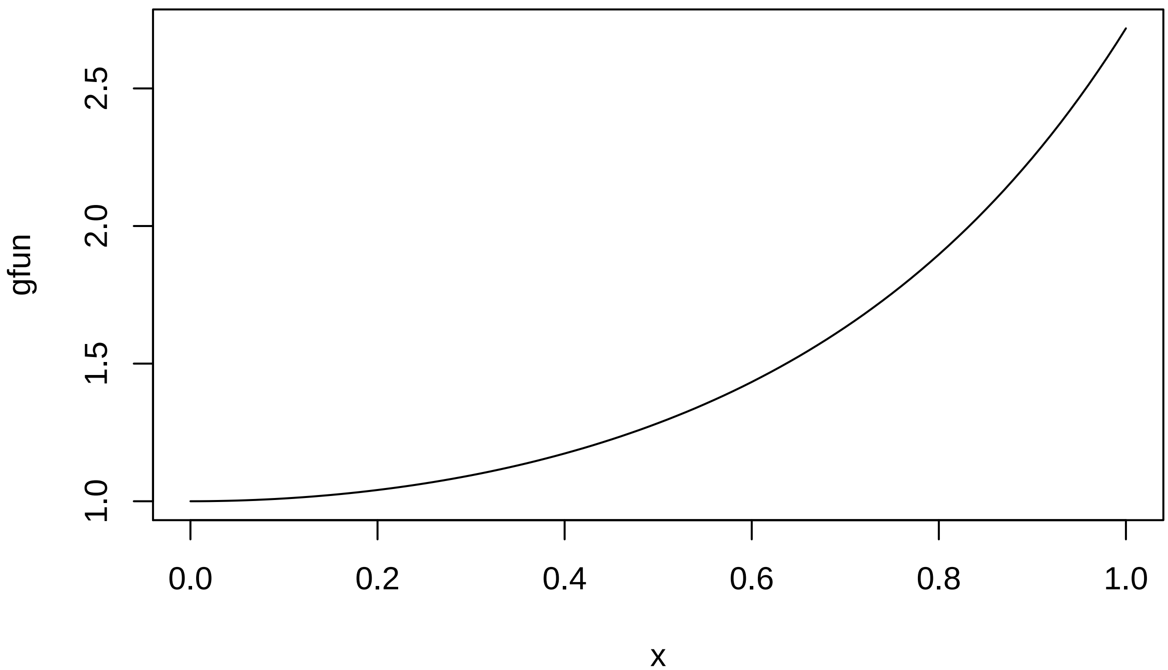
$$\theta = \int_0^1 e^{y^2} * 1 dx = E[e^{y^2}]$$

```
set.seed(123123)
y = runif(1000, 0, 1)

gfun = function(y) {
  return((exp(y ^ 2)))
}

gx =
  gfun(y)

theta_estimate =
  list(theta = mean(gx),
       var_theta = var(gx))

integrate(function(x)
  exp(x ^ 2), 0, 1)
```

```
## 1.462652 with absolute error < 1.6e-14
```

Method 2

```
plot.function(gfun,0,1)
```



looking at the function, which is similar to a exponetial funciton, thus introducing $f(x) = e^x$ as control variate, we have:

$$\theta = \int_0^1 \beta * e^x + (e^{x^2} - \beta * e^x)dx = \beta * e^x|_0^1 + E[(e^{x^2} - \beta * e^x)]$$

```
ffun = function(y){
  return(exp(y))
}

fx = ffun(y)

b = lm(gx~fx)$coef[2]

theta_estimate_2 =
  list(theta = mean(b*(exp(1)-1)+(gx-b*fx)),
       theta2 = sum(lm(gx~fx)$coef*c(1,mean(fx))),
       var_theta =var((b*(exp(1)-1)+(gx-b*fx)))
       )

theta_estimate_2
```

```
## $theta
## [1] 1.462708
##
```

```
## $theta2
## [1] 1.453559
##
## $var_theta
## [1] 0.01238334
```

Problem 2

Show that in estimating $\theta = E\{\sqrt{1-U^2}\}$ it is better to use U^2 rather than U as the control variate, where $U \sim U(0,1)$. To do this, use simulation to approximate the necessary covariances. In addition, implement your algorithms in **R**.

Answer: your answer starts here...

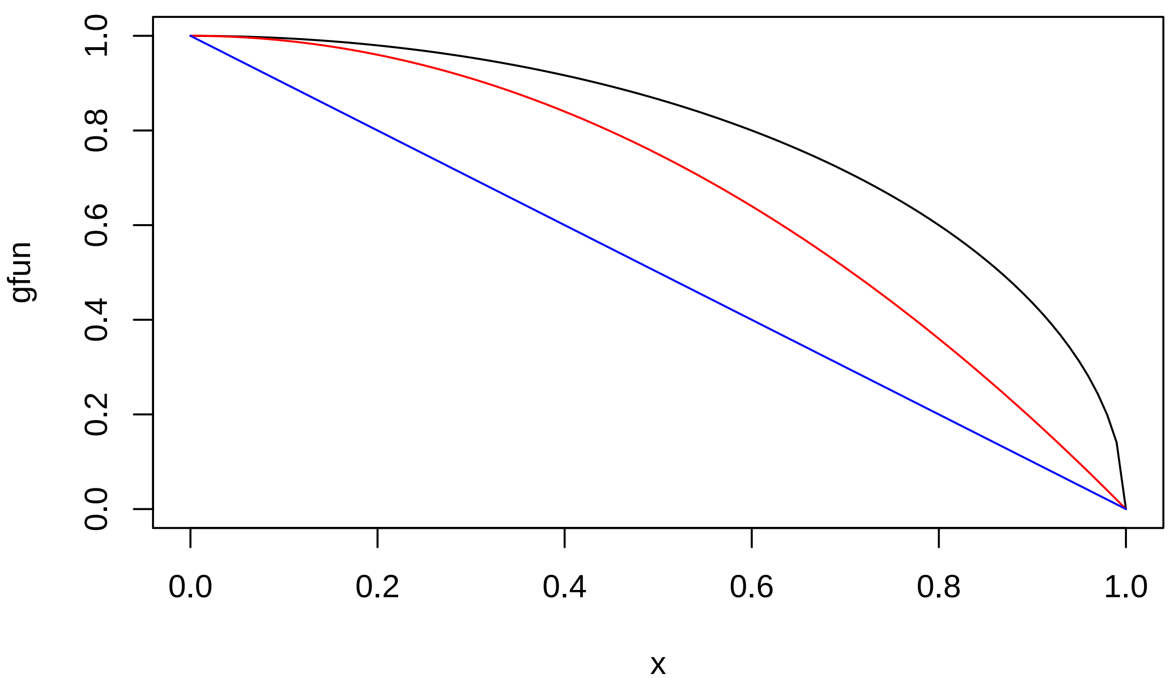
```r
set.seed(123123)

y = runif(1000, 0, 1)

gfun = function(x)
  return(sqrt(1 - x ^ 2))

f1fun = function(x)
  return(x ^ 2)

f2fun = function(x)
  return(x)

gx = gfun(y)

f1x = f1fun(y)

b1 = lm(gx ~ f1x)$coef[2]

f2x = f2fun(y)

b2 = lm(gx ~ f2x)$coef[2]

theta_1 =
  b1 * (1 / 3) + gx - b1 * f1x

theta_2 =
  b2 * (1 / 2) + gx - b2 * f2x

result =
  tibble(
    model = c("gx", "U^2", "U"),
    theta = c(mean(gx), mean(theta_1), mean(theta_2)),
    effic = c(0, (var(gx) - var(theta_1)), (var(gx) - var(theta_2))) / var(gx)
  )
```

```
knitr::kable(result)
```

model	theta	effic
gx	0.7901179	0.0000000
U^2	0.7859641	0.9696349
U	0.7859847	0.8537123

This result shows that U^2 has higher efficiency than U, the plot shows that, $1 - U^2$ is more resemble to the original function. Corelation between simulated data shows that same conclusion(-0.9847004, -0.9239655)

```
plot.function(gfun,0,1)
plot.function(function(x) 1-f1fun(x),0,1,add=T,col = "red")
plot.function(function(x) 1-f2fun(x),0,1,add=T,col = "blue")
```



Problem 3

Obtain a Monte Carlo estimate of

$$\int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

by importance sampling and evaluate its variance. Write a **R** *function to implement your procedure.*

Answer: your answer starts here…

Method 1

$$\theta = \int \frac{\frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} * (I > 1) + 0 * (I <= 0)}{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{1}$$

```
set.seed(123123)

y = rnorm(10000)

gfun = function(x) return(x^2/sqrt(2*pi)*exp(-x^2/2)*(x>1))

gx = gfun(y)

theta = mean(gx/y)
```

Method 2

We first us change of variables, where we take $x = tan(y)$, s.t

$$\begin{aligned} \theta &= \int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_{\pi/4}^{\pi/2} \frac{tan(y)^2}{\sqrt{2\pi}} * e^{-\frac{tan(y)^2}{2}} \frac{2}{cos(2y) + 1} dy \end{aligned}$$