Homework 01-25-2021

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Problem 1

Develop two Monte Carlo methods for the estimation of $\theta = \int_0^1 e^{x^2} dx$ and implement in **R** .

Answer: your answer starts here...

Method 1

Introduce Let $Y \sim U(0, 1)$

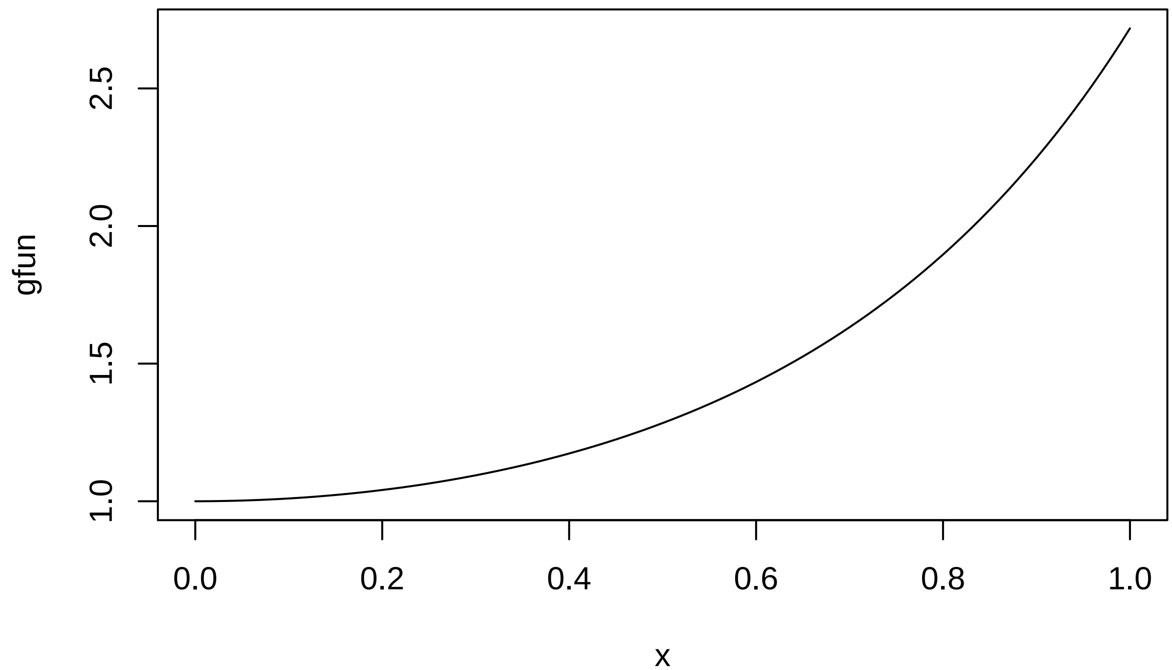
$$\theta = \int_0^1 e^{y^2} * 1dx = E[e^{y^2}]$$

```
set.seed(123123)
y = runif(1000, 0, 1)

gfun = function(y) {
  return((exp(y ^ 2)))
}

gx =
  gfun(y)

theta_estimate =
  list(theta = mean(gx),
       var_theta = var(gx))

integrate(function(x)
  exp(x ^ 2), 0, 1)
```

```
## 1.462652 with absolute error < 1.6e-14
```

Method 2

```
plot.function(gfun,0,1)
```



looking at the function, which is similar to a exponetial funciton, thus introducing $f(x) = e^x$ as control variate, we have:

$$\theta = \int_0^1 \beta * e^x + (e^{x^2} - \beta * e^x)dx = \beta * e^x|_0^1 + E[(e^{x^2} - \beta * e^x)]$$

```r
ffun = function(y){
  return(exp(y))
}

fx = ffun(y)

b = lm(gx~fx)$coef[2]

theta_estimate_2 =
  list(theta = mean(b*(exp(1)-1)+(gx-b*fx)),
       theta2 = sum(lm(gx~fx)$coef*c(1,mean(fx))),
       var_theta =var((b*(exp(1)-1)+(gx-b*fx)))
       )

theta_estimate_2
```

```
## $theta
## [1] 1.462708
##
```

```
## $theta2
## [1] 1.453559
##
## $var_theta
## [1] 0.01238334
```

Problem 2

Show that in estimating $\theta = E\{\sqrt{1 - U^2}\}$ it is better to use U^2 rather than U as the control variate, where $U \sim U(0, 1)$. To do this, use simulation to approximate the necessary covariances. In addition, implement your algorithms in **R**.

Answer: your answer starts here...

```
set.seed(123123)

y = runif(1000, 0, 1)

gfun = function(x)
  return(sqrt(1 - x ^ 2))

f1fun = function(x)
  return(x ^ 2)

f2fun = function(x)
  return(x)

gx = gfun(y)

f1x = f1fun(y)

b1 = lm(gx ~ f1x)$coef[2]

c(cor(gx,f1x),b1)
```

```
##                   f1x
## -0.9847004 -0.7285452
```

```
f2x = f2fun(y)

b2 = lm(gx ~ f2x)$coef[2]

c(cor(gx,f2x),b2)
```

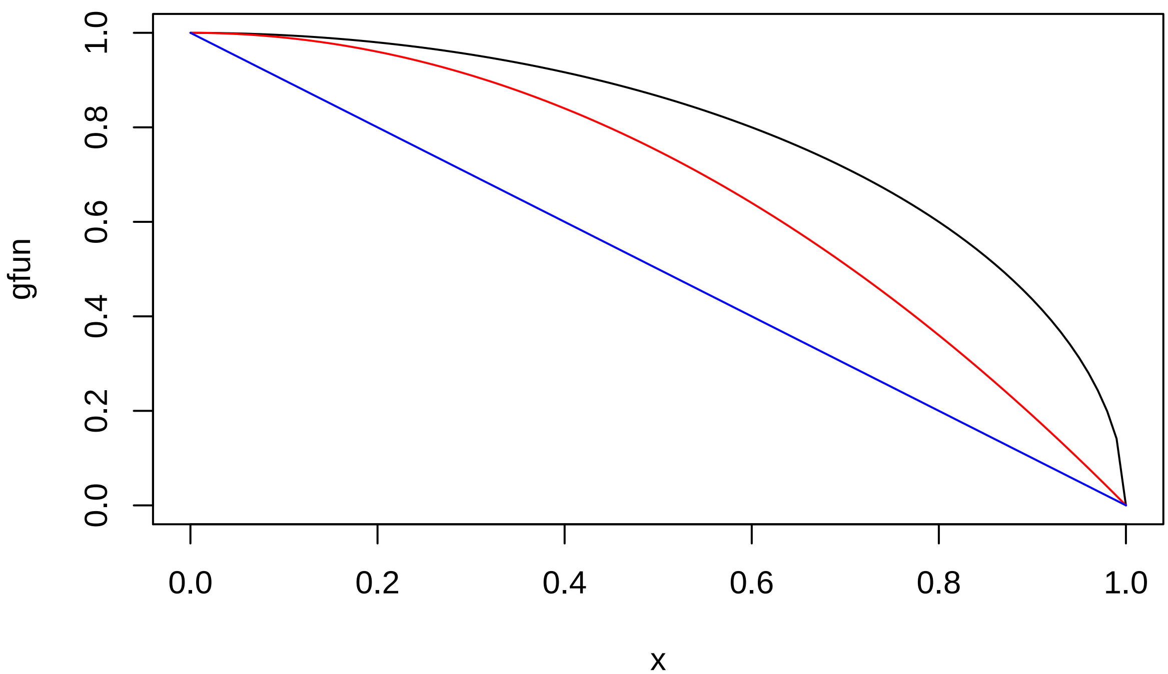
```
##                   f2x
## -0.9239655 -0.7011728
```

```r
theta_1 =
  b1 * (1 / 3) + gx - b1 * f1x

theta_2 =
  b2 * (1 / 2) + gx - b2 * f2x

result =
  tibble(
    model = c("gx", "U^2", "U"),
    theta = c(mean(gx), mean(theta_1), mean(theta_2)),
    effic = c(0, (var(gx) - var(theta_1)), (var(gx) - var(theta_2))) / var(gx)
  )

knitr::kable(result)
```

model	theta	effic
gx	0.7901179	0.0000000
U^2	0.7859641	0.9696349
U	0.7859847	0.8537123

This result shows that U^2 has higher efficiency than U, the plot shows that, $1 - U^2$ is more resemble to the original function. Corelation between simulated data shows that same conclusion(-0.9847004, -0.9239655)

```r
plot.function(gfun,0,1)
plot.function(function(x) 1-f1fun(x),0,1,add=T,col = "red")
plot.function(function(x) 1-f2fun(x),0,1,add=T,col = "blue")
```



Problem 3

Obtain a Monte Carlo estimate of

$$\int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

by importance sampling and evaluate its variance. Write a **R** *function to implement your procedure.*

Answer: your answer starts here...

From wolfarm/alpha we know that the integral integrate to 0.40.

Method 1

$$\theta = \int \frac{\frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} * (I > 1) + 0 * (I <= 0)}{Gamma(1, 1/2)} * Gamma(1, 1/2) dx \tag{1}$$

```
set.seed(123123)

y = rgamma(1000,1,1/2)
```

```
gfun = function(x)
  return(x ^ 2 / sqrt(2 * pi) * exp(-x ^ 2 / 2) * (x > 1))

gx = gfun(y)

y = dgamma(y,1,1/2)

theta = mean(gx/y)

theta
```

```
## [1] 0.4113953
```

Method 2

We first us change of variables, where we take $x = tan(y)$, s.t

$$
\begin{aligned}
\theta &= \int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{\pi/4}^{\pi/2} \frac{tan(y)^2}{\sqrt{2\pi}} * e^{-\frac{tan(y)^2}{2}} \frac{1}{cos(y)^2} dy \\
&= \int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{\frac{\pi/4^{\pi/2} \frac{tan(y)^2}{\sqrt{2\pi}} * e^{-\frac{tan(y)^2}{2}} \frac{1}{cos(y)^2}}{\frac{1}{(\pi/2-\pi/4)}} * \frac{1}{(\pi/2-\pi/4)}} dy
\end{aligned}
$$

```
set.seed(123123)

y = runif(1000,pi/4,pi/2)

yfun =
  function(x) {
    return((tan(x)^2*exp(-tan(x)^2/2))/(sqrt(2*pi)*cos(x)^2))
  }

gx = yfun(y)

yx = 1/(pi/2 - pi/4)

theta = mean(gx/yx)

theta
```

```
## [1] 0.4033183
```

Method 3

$$\int_1^\infty x^2 * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

,

if we use $g(x) = x^2 * I(x > 1) + 0 * I(x \leq 0)$ *and introduce standard normal as* $p(x)$*, the function becomes:*

$$\int_1^{\infty} \frac{x^2 * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = E[x^2 * I(x > 1)]$$

,

```r
set.seed(123123)

y = rnorm(1000)

gfun = function(x)
  return(x ^ 2* (x > 1))

gx = gfun(y)

theta = mean(gx)

theta
```

```
## [1] 0.3624538
```