Homework 2

Problem 1

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$\nabla f = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 4x_2 = 0$$
$$x_1 = x_2$$

$$-4x_1 + 3x_2 + 1 = 0$$
$$-4x_1 + 3x_1 = -1$$
$$x_1 = 1 = x_2$$

zero gradient at $(1,1)$ $f(1,1) = 0.5$

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \quad \det(H) = 4(3) - (-4)(-4)$$
$$12 - 16 = -4 \quad \therefore H \text{ is indefinite}$$

$$f(x_1, x_2) \approx f(1,1) + \nabla f^T \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^T H \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$\approx 0.5 + [4x_1 - 4x_2, -4x_1 + 3x_2 + 1]_{(1,1)} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + (x_1 - 1, x_2 - 1)\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$\approx 0.5 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}(\partial x_1, \partial x_2)\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$\approx f(1,1) + \frac{1}{2}(4\partial x_1 - 4\partial x_2, -4\partial x_1 + 3\partial x_2)\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(1,1) + \frac{1}{2}(4\partial x_1^2 - 4\partial x_1\partial x_2 - 4\partial x_1\partial x_2 + 3\partial x_2^2)$$

$$f(1,1) + \frac{1}{2}(4\partial x_1^2 - 8\partial x_1\partial x_2 + 3\partial x_2^2)$$

$$f(x_1, x_2) \approx f(1,1) + \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 + 1.5\partial x_2)$$

$$f(x_1, x_2) \approx f(1,1) + (\partial x_1 - .5\partial x_2)(\partial x_1 + .75\partial x_2)$$

Problem 2

a) $d = \text{distance} = (x_1+1)^2 + x_2^2 + (x_3-1)^2$

point $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\min_{x_1, x_2, x_3} d$$

subject to: $x_1 + 2x_2 + 3x_3 = 1$

unconstrain:

$x_1 = 1 - 2x_2 - 3x_3$

$d = (1-2x_2-3x_3+1)^2 + x_2^2 + (x_3-1)^2$

$(2-2x_2-3x_3)^2 = 4x_2^2 + 12x_2x_3 + 9x_3^2 - 8x_2 - 12x_3 + 4$

$g = \begin{bmatrix} 10x_2 + 12x_3 - 8 \\ 20x_3 + 12x_2 - 14 \end{bmatrix} = 0$ for minimum

$H = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$

$(\lambda-28)(\lambda-2)$

$\boxed{\lambda_1 = 28, \lambda_2 = 2, \therefore \text{pd and convex}}$

$10x_2 + 12x_3 - 8 = 0$

$x_2 = -1.2x_3 + 0.8$

$20x_3 + 12(-1.2x_3 + .8) - 14 = 0$

$5.6x_3 = 4.4$

$x_3 = \frac{11}{14}$

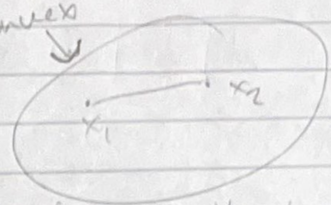
$x_2 = -\frac{1}{7}$

$x_1 = -\frac{15}{14}$

$\boxed{\text{point} = \begin{bmatrix} -\frac{15}{14} \\ -\frac{1}{7} \\ \frac{11}{14} \end{bmatrix}}$

Problem 3

kinda proves hyperplane is convex



$a^T x = c$ for $x \in \mathbb{R}^n$

S = hyperplane , x_1 and x_2 are any point in the hyperplane
then

$$\lambda x_1 + (1-\lambda) x_2 \in S$$

$a^T x_1 = c = a^T x_2$ $\qquad$ $\lambda a^T x_1 + (1-\lambda) a^T x_2 = c$

so b/c $\lambda x_1 + (1-\lambda) x_2 \in S$, it is convex

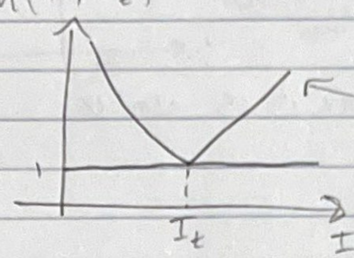
Problem 4

$$\min_{p} \ \max_{k} \{h(a_k^T p, I_t)\}$$

$$\text{st.} \quad 0 \le p_i \le p_{max} \qquad k=1,\dots m \quad i=1,\dots n$$

$$h(I, I_t) = \begin{cases} I_t/I & \text{if } I \le I_t \\ I/I_t & \text{if } I \ge I_t \end{cases}$$

a) $h(I, I_t)$



$$\frac{dh}{dp} = \frac{dh}{dI} \cdot \frac{da^T p}{dp} = h' a$$

b) $p_i \le p^*$ for $i = 1, 2, \dots 10$

set $p_i = p^*$ $\quad \max_k \{h(a_k^T p, I_t)\}$ $\quad \frac{dh}{dp} = \frac{\partial h}{\partial I} \frac{\partial a_k^T p}{\partial p} = h' a$

$$\frac{d^2 h}{dp^2} = \frac{\partial h}{\partial I} \frac{\partial a^T}{\partial p} \cdot a^T = h'' \underbrace{a a^T}_{\text{symmetric matrix}}$$

h is convex, therefore $h'' > 0$, and is pd, therefore there is a unique solution.

c) $p_i \ge 0$ for $i = 1, 2, \dots 10$, $i \le 10$

When $m < 10$, there is no unique solution because the function is psd. When $m \ge 10$, there is a unique solution because the function is pd

Problem 5

$c(x)$ = cost for x amount of product A

y = unit price

$$c^*(y) = \max_x (xy - c(x))$$

$$\nabla c^*(y) = x$$

$H = 0$ - psd, therefore it is convex

The function is linear, therfore it is convex.