Self-sustained turbulence in magnetized shear flow

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$$\mathrm{D}_{\overline{\tau}}^{\text{shear}}\mathbf{v}_{\mathbf{Q}}(\tau) = -v_{x,\mathbf{Q}}\mathbf{e}_y + 2n_y\mathbf{n}v_{x,\mathbf{Q}} - \nu'_{\mathrm{k}}Q^2\mathbf{v}_{\mathbf{Q}} + \Pi^{\perp\mathbf{Q}}\cdot\sum_{\mathbf{Q}'}\left[\mathbf{v}_{\mathbf{Q}'}\otimes\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} + \mathbf{b}_{\mathbf{Q}'}\otimes\mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}\right]\cdot\mathbf{Q}, \tag{1}$$

$$\mathrm{D}_{\overline{\tau}}^{\text{shear}}\mathbf{b}_{\mathbf{Q}}(\tau) = b_{x,\mathbf{Q}}\mathbf{e}_y - (\mathbf{Q}\cdot\alpha)\,\mathbf{v}_{\mathbf{Q}} - \nu'_{\mathrm{m}}Q^2\mathbf{b}_{\mathbf{Q}} + \Pi^{\perp\mathbf{Q}}\cdot\sum_{\mathbf{Q}'}\left[\mathbf{b}_{\mathbf{Q}'}\otimes\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} - \mathbf{v}_{\mathbf{Q}'}\otimes\mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}\right]\cdot\mathbf{Q}, \tag{2}$$

$$\mathbf{Q}\cdot\mathbf{N}_v = 0, \quad \mathbf{v}_{\mathbf{Q}}(\overline{\tau}_0) = \Pi^{\perp\mathbf{Q}}\mathbf{v}_{\mathbf{Q}}(\overline{\tau}_0), \quad \mathbf{Q}\cdot\mathbf{N}_b = 0, \quad \mathbf{b}_{\mathbf{Q}}(\overline{\tau}_0) = \Pi^{\perp\mathbf{Q}}\mathbf{b}_{\mathbf{Q}}(\overline{\tau}_0), \tag{3}$$

$$\mathrm{D}_{\tau}^{\text{shear}} \equiv \partial_\tau + \mathbf{U}_{\text{shear}}(\mathbf{Q})\cdot\partial_{\mathbf{Q}} = \partial_\tau - Q_y\partial_{Q_x} = \partial_\tau + \partial_{\overline{\tau}}, \quad \mathbf{U}_{\text{shear}}(\mathbf{Q}) \equiv -Q_y\mathbf{e}_x, \tag{4}$$

$$\mathrm{d}_{\overline{\tau}}\, v_y = -v_x + 2n_y^2 v_x + Q_y b_y - \nu'_{\mathrm{k}}Q^2 v_y + N_v^y, \tag{5}$$

$$\mathrm{d}_{\overline{\tau}}\, b_y = b_x - Q_y v_y - \nu'_{\mathrm{m}}Q^2 b_y + N_b^y. \tag{6}$$

$$\mathrm{d}_{\overline{\tau}}^2 b_y + \nu'_{\text{tot}}Q^2\mathrm{d}_{\overline{\tau}}b_y + [Q_\alpha^2 + 2\nu'_{\mathrm{m}}\overline{\tau}Q_y^2 + \nu'_{\mathrm{m}}\nu'_{\mathrm{k}}Q^4]b_y = \mathrm{d}_{\overline{\tau}}N_b^y - Q_\alpha N_v^y - 2Q_\alpha n_y^2 v_x + \nu'_{\mathrm{k}}QN_b^y + (\nu'_{\mathrm{k}} - \nu'_{\mathrm{m}})b_x. \tag{7}$$

$$\lim_{\nu'_{\mathrm{k}}\to 0}\left(\frac{\int_0^\infty \nu'_{\mathrm{k}}\overline{\tau}^2 e^{-\nu'_{\text{tot}}Q_y^2\overline{\tau}^2/6}\mathrm{d}\overline{\tau}}{\int_0^\infty e^{-\nu'_{\text{tot}}Q_y^2\overline{\tau}^2/6}\mathrm{d}\overline{\tau}} = \frac{6^{2/3}}{3\Gamma(\frac{4}{3})}\frac{\nu_{\mathrm{k}}'^{1/3}}{Q_y^{4/3}(1+\frac{1}{\mathrm{P_m}})^{2/3}}\right) = 0. \tag{8}$$

$$\left[\mathrm{d}_{\overline{\tau}}^2 + \nu'_{\text{tot}}Q^2\mathrm{d}_{\overline{\tau}} + Q_\alpha^2\right]b_y = \mathrm{d}_{\overline{\tau}}N_b^y - Q_\alpha N_v^y. \tag{9}$$

$$
\mathbf{N}_v = \Pi^{\perp\mathbf{Q}} \cdot \sum_{\mathbf{Q}'} \left[\begin{pmatrix} v^x_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & v^x_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & v^x_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \\ v^y_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \\ v^z_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & v^z_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & v^z_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \end{pmatrix} + \begin{pmatrix} b^x_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & b^x_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & b^x_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \\ b^y_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \\ b^z_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & b^z_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & b^z_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \end{pmatrix} \right] \cdot \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix}, \tag{10}
$$

$$
\mathbf{N}_b = \Pi^{\perp\mathbf{Q}} \cdot \sum_{\mathbf{Q}'} \left[\begin{pmatrix} b^x_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & b^x_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & b^x_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \\ b^y_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & b^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & b^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \\ b^z_{\mathbf{Q}'} v^x_{\mathbf{Q}-\mathbf{Q}'} & b^z_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} & b^z_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} \end{pmatrix} - \begin{pmatrix} v^x_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & v^x_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & v^x_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \\ v^y_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & v^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & v^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \\ v^z_{\mathbf{Q}'} b^x_{\mathbf{Q}-\mathbf{Q}'} & v^z_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} & v^z_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} \end{pmatrix} \right] \cdot \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix}, \tag{11}
$$

$\Pi^{\perp\mathbf{Q}}$ is the projection operator and have

$$
\lim_{\tau\to\infty} \Pi^{\perp\mathbf{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}
$$

$$
\lim_{\tau\to\infty} \mathbf{N}_b = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (b^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} - v^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} - v^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \\ \sum_{\mathbf{Q}'} (b^z_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} - v^z_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^z_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} - v^z_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \end{pmatrix} \tag{13}
$$

$$
\lim_{\tau\to\infty} \mathbf{N}_v = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} + b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} + b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \\ \sum_{\mathbf{Q}'} (v^z_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} + b^z_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (v^z_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} + b^z_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \end{pmatrix} \tag{14}
$$

$$
\begin{aligned}
\mathrm{d}_{\overline{\tau}} N^y_b &= \mathrm{d}_{\overline{\tau}} \left[\sum_{\mathbf{Q}'} (b^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'} - v^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'} - v^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \right] \\
&= \sum_{\mathbf{Q}'} [(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z](Q_y - Q'_y), \\
\mathrm{d}_{\overline{\tau}} v^y_{\mathbf{Q}-\mathbf{Q}'} &= (Q_y - Q'_y) b^y_{\mathbf{Q}-\mathbf{Q}'}, \quad \mathrm{d}_{\overline{\tau}} b^y_{\mathbf{Q}-\mathbf{Q}'} = -(Q_y - Q'_y) v^y_{\mathbf{Q}-\mathbf{Q}'}
\end{aligned}
$$

For the resulting external force acting on the out oscillator we have

$$
\begin{aligned}
&F_{\mathrm{ext}} = \mathrm{d}_{\overline{\tau}} N_b^y - Q_y N_v^y = \\
&\sum_{\mathbf{Q}'} \left\{ \left[(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \right] (Q_y - Q'_y) \right\} - && (15)\\
&\sum_{\mathbf{Q}'} \left\{ \left[(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \right] (-Q_y) \right\} = && (16)\\
&-\sum_{\mathbf{Q}'} \left\{ \left[(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z \right] Q'_y \right\} && (17)
\end{aligned}
$$

$$
\begin{aligned}
b_y(\overline{\tau}, Q_y, Q_z) &= \int_{-\infty}^{\overline{\tau}} \mathrm{d}\overline{\tau_0} \frac{\sin[Q_y(\overline{\tau} - \overline{\tau}_0)]}{Q_y} \exp\left(-\frac{1}{2} \int_{\overline{\tau}_0}^{\overline{\tau}} \nu'_{\mathrm{tot}} Q^2(\tau) \mathrm{d}\tau \right) \theta(\overline{\tau} - \overline{\tau}_0) F_{\mathrm{ext}}(\overline{\tau}_0) = \qquad (18)\\
&= \sin(Q_y \overline{\tau}) \int_{-\infty}^{\overline{\tau}} \mathrm{d}\overline{\tau_0} \frac{1}{Q_y} \cos(Q_y \overline{\tau_0}) \exp\left[-\frac{\nu'_{\mathrm{tot}}}{6} (\overline{\tau}^3 - \overline{\tau_0}^3) Q_y^2 \right] F_{\mathrm{ext}}(\overline{\tau}_0)\\
&- \cos(Q_y \overline{\tau}) \int_{-\infty}^{\overline{\tau}} \mathrm{d}\overline{\tau_0} \frac{1}{Q_y} \sin(Q_y \overline{\tau_0}) \exp\left[-\frac{\nu'_{\mathrm{tot}}}{6} (\overline{\tau}^3 - \overline{\tau_0}^3) Q_y^2 \right] F_{\mathrm{ext}}(\overline{\tau}_0)
\end{aligned}
$$

$$
\begin{aligned}
X(\overline{\tau}, Q_y, Q_z) &= \int_{-\infty}^{\overline{\tau}} \mathrm{d}\overline{\tau_0} \frac{1}{Q_y} \cos(Q_y \overline{\tau_0}) \exp\left[-\frac{\nu'_{\mathrm{tot}}}{6} (\overline{\tau}^3 - \overline{\tau_0}^3) Q_y^2 \right] F_{\mathrm{ext}}(\overline{\tau}_0)\\
Y(\overline{\tau}, Q_y, Q_z) &= -\int_{-\infty}^{\overline{\tau}} \mathrm{d}\overline{\tau_0} \frac{1}{Q_y} \sin(Q_y \overline{\tau_0}) \exp\left[-\frac{\nu'_{\mathrm{tot}}}{6} (\overline{\tau}^3 - \overline{\tau_0}^3) Q_y^2 \right] F_{\mathrm{ext}}(\overline{\tau}_0)
\end{aligned}
$$

$$
b_y(\overline{\tau}, Q_y, Q_z) = \sin(Q_y \overline{\tau}) X(\overline{\tau}) + \cos(Q_y \overline{\tau}) Y(\overline{\tau}) \approx X_0 \cos(Q_y \overline{\tau}) + Y_0 \sin(Q_y \overline{\tau}) \qquad (19)
$$

$$
\begin{aligned}
b_z(\xi) &= \frac{2K_z}{K_\perp}\int_{-\infty}^{\xi}\frac{\sin\left[K_\perp(\xi-\xi')\right]}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\mathrm{d}_{\xi'}\psi(\xi')-\xi'\psi(\xi')\right]\mathrm{d}\xi' = \\
&\frac{2K_z\sin(K_\perp\xi)}{K_\perp}\int_{-\infty}^{\xi}\frac{\cos\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\left(C_\mathrm{g}\mathrm{d}_{\xi'}\psi_\mathrm{g}(\xi')+C_\mathrm{u}\mathrm{d}_{\xi'}\psi_\mathrm{u}(\xi')\right)-\xi'\left(C_\mathrm{g}\psi_\mathrm{g}(\xi')+C_\mathrm{u}\psi_\mathrm{u}(\xi')\right)\right]\mathrm{d}\xi' - \\
&\frac{2K_z\cos(K_\perp\xi)}{K_\perp}\int_{-\infty}^{\xi}\frac{\sin\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\left(C_\mathrm{g}\mathrm{d}_{\xi'}\psi_\mathrm{g}(\xi')+C_\mathrm{u}\mathrm{d}_{\xi'}\psi_\mathrm{u}(\xi')\right)-\xi'\left(C_\mathrm{g}\psi_\mathrm{g}(\xi')+C_\mathrm{u}\psi_\mathrm{u}(\xi')\right)\right]\mathrm{d}\xi' = \\
&\frac{2K_z}{K_\perp}\left[J_{c,\mathrm{u}}C_\mathrm{u}\sin(K_\perp\xi)-J_{s,\mathrm{g}}C_\mathrm{g}\cos(K_\perp\xi)\right]\approx\frac{4K_z}{K_\perp^2}C_\mathrm{u}\sin(K_\perp\xi)+\frac{4K_z}{3K_\perp}C_\mathrm{g}\cos(K_\perp\xi), \\
&J_{c,\mathrm{u}} = \int_{-\infty}^{\infty}\frac{\cos\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\left(C_\mathrm{u}\mathrm{d}_{\xi'}\psi_\mathrm{u}(\xi')\right)-\xi'C_\mathrm{u}\psi_\mathrm{u}(\xi')\right],\ \lim_{K_\perp\to 0}J_{c,\mathrm{u}}=\frac{2}{K_\perp} \\
&J_{s,\mathrm{g}} = \int_{-\infty}^{\infty}\frac{\sin\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\left(C_\mathrm{g}\mathrm{d}_{\xi'}\psi_\mathrm{g}(\xi')\right)-\xi'C_\mathrm{g}\psi_\mathrm{g}(\xi')\right],\ \lim_{K_\perp\to 0}J_{c,\mathrm{u}}=-\frac{2}{3} \\
v_z(\xi) &= \frac{2K_z}{K_\perp}\int_{-\infty}^{\xi}\frac{\cos\left[K_\perp(\xi-\xi')\right]}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)\mathrm{d}_{\xi'}\psi(\xi')-\xi'\psi(\xi')\right]\mathrm{d}\xi' = \\
&\frac{2K_z\cos(K_\perp\xi)}{K_\perp}\int_{-\infty}^{\xi}\frac{\cos\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)C_\mathrm{u}\mathrm{d}_{\xi'}\psi_\mathrm{u}(\xi')-\xi'C_\mathrm{u}\psi_\mathrm{u}(\xi')\right]\mathrm{d}\xi' + \\
&\frac{2K_z\sin(K_\perp\xi)}{K_\perp}\int_{-\infty}^{\xi}\frac{\sin\left(K_\perp\xi'\right)}{K_\perp(1+\xi')^{5/2}}\left[(1+\xi'^2)C_\mathrm{g}\mathrm{d}_{\xi'}\psi_\mathrm{g}(\xi')-\xi'C_\mathrm{g}\psi_\mathrm{g}(\xi')\right]\mathrm{d}\xi' = \\
&\frac{2K_z}{K_\perp}\left[J_{c,\mathrm{u}}C_\mathrm{u}\cos(K_\perp\xi)-J_{s,\mathrm{g}}C_\mathrm{g}\sin(K_\perp\xi)\right]\approx\frac{4K_z}{K_\perp^2}C_\mathrm{u}\cos(K_\perp\xi)+\frac{4K_z}{3K_\perp}C_\mathrm{g}\sin(K_\perp\xi) \\
b_y &= -\frac{2K_z^2}{K_\perp K_y}[\sin(K_\perp\xi)I_c(\xi)-\cos(K_\perp\xi)I_s(\xi)]+\frac{K_\perp}{K_y}\frac{\xi\psi(\xi)}{\sqrt{1+\xi^2}}\approx-\frac{4K_z^2}{K_\perp^2K_y}C_\mathrm{u}\sin(K_\perp\xi)-\frac{4K_z^2}{K_\perp K_y}C_\mathrm{g}\cos(K_\perp\xi)+\frac{K_\perp}{K_y}\psi(\xi) \\
v_y &= -\frac{2K_z^2}{K_\perp K_y}[\cos(K_\perp\xi)I_c(\xi)-\sin(K_\perp\xi)I_s(\xi)]+\frac{\mathrm{d}_\xi\psi(\xi)}{K_y}\approx-\frac{4K_z^2}{K_\perp^2K_y}C_\mathrm{u}\cos(K_\perp\xi)-\frac{4K_z^2}{K_\perp K_y}C_\mathrm{g}\sin(K_\perp\xi)+\frac{\mathrm{d}_\xi\psi(\xi)}{K_y}
\end{aligned}
\tag{20}
$$

In calculating the wave–wave interaction we will use the following scheme:
The amplified wave with wave-vector $\mathbf{Q}_1$ interacts with another amplified wave with wave-vector $\mathbf{Q}_2$ and as a result we have a wave with wave-vector $\mathbf{Q}=\mathbf{Q}_1+\mathbf{Q}_2$ (see Fig. **??**).
The domain of amplification is

$$Q_1^x<0,\qquad Q_2^x>0,\qquad Q_1^y>0,\qquad Q_2^y<0,$$

$$Q_1^x = \frac{Q_x}{2} + P_x, \qquad Q_2^x = \frac{Q_x}{2} - P_x, \qquad Q_1^y = \frac{Q_y}{2} + P_y, \qquad Q_2^y = \frac{Q_y}{2} - P_y,$$

$$\mathbf{P} \to \mathbf{K}, (P_x, P_y) = (-K_x, K_y),$$

$$Q_1^x = -\left(K_x - \frac{Q_x}{2}\right) < 0, \qquad Q_2^x = +\left(K_x + \frac{Q_x}{2}\right) > 0, \tag{21}$$

$$Q_1^y = +\left(K_y + \frac{Q_y}{2}\right) > 0, \qquad Q_2^y = -\left(K_y - \frac{Q_y}{2}\right) < 0, \tag{22}$$

$$K_x \subset \left(\frac{Q_x}{2}, \infty\right), \qquad K_y \subset \left(\frac{Q_y}{2}, \infty\right). \tag{23}$$

After a change of variables we obtain

$$F^{(1)} = -\int_{Q_x/2}^{\infty} \frac{\mathrm{d}K_x}{2\pi} \int_{Q_y/2}^{\infty} \frac{\mathrm{d}K_y}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}K_z}{2\pi}\, Q_1^y \left[(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) Q_z\right]. \tag{24}$$

$$\cos\left(Q_1^y \overline{\tau_1}\right) \cos\left(Q_2^y \overline{\tau_2}\right) + \sin\left(Q_1^y \overline{\tau_1}\right) \sin\left(Q_2^y \overline{\tau_2}\right) = \cos(2K_x).$$

$$(b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'}) = \frac{-16 K_z^4 C_{\mathrm{u}}^{\mathbf{Q}'} C_{\mathrm{u}}^{\mathbf{Q}-\mathbf{Q}'} \cos(2K_x)}{\left[K_z^2 + \left(K_y + \frac{Q_y}{2}\right)^2\right]\left[K_z^2 + \left(K_y - \frac{Q_y}{2}\right)^2\right]\left(K_y + \frac{Q_y}{2}\right)\left(K_y - \frac{Q_y}{2}\right)} \tag{25}$$

$$(b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'}) = \frac{-16 K_z^3 C_{\mathrm{u}}^{\mathbf{Q}'} C_{\mathrm{u}}^{\mathbf{Q}-\mathbf{Q}'} \cos(2K_x)}{\left[K_z^2 + \left(K_y + \frac{Q_y}{2}\right)^2\right]\left[K_z^2 + \left(K_y - \frac{Q_y}{2}\right)^2\right]\left(K_y + \frac{Q_y}{2}\right)} \tag{26}$$

$$\tag{27}$$

The argument of the exponential function in both relations above is strictly negative

$$\begin{aligned}(Q_1^y)^2\overline{\tau_1}^3 &= -\frac{(Q_1^x)^3}{Q_1^y} = \left(\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2}\right) > 0,\\ (Q_2^y)^2\overline{\tau_2}^3 &= -\frac{(Q_2^x)^3}{Q_2^y} = \left(\frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right) > 0.\end{aligned}$$

The resulting exponent function in the expression $b^y_{Q_1} b^y_{Q_2} + v^y_{Q_1} v^y_{Q_2}$ is

$$\exp[-\frac{\nu'_{\text{tot}}}{6}(Q_1^y)^2\overline{\tau_1}^3]\cdot\exp[-\frac{\nu'_{\text{tot}}}{6}(Q_2^y)^2\overline{\tau_2}^3] = \exp\left\{-\frac{\nu'_{\text{tot}}}{6}\left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right]\right\}.$$

$$\begin{aligned}F^{(1)} &= \int_{Q_x/2}^{\infty}\frac{\mathrm{d}K_x}{2\pi}\exp\left\{-\frac{\nu'_{\text{tot}}}{6}\left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right]\right\}\cos(2K_x)\cdot\\ &\int_{Q_y/2}^{\infty}\frac{\mathrm{d}K_y}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}K_z}{2\pi}\,Q_1^y\left\{\frac{16K_z^3 C_{\mathrm{u}}^{\mathbf{Q}'}C_{\mathrm{u}}^{\mathbf{Q}-\mathbf{Q}'}\left(\frac{K_zQ_y}{K_y-Q_y/2}+Q_z\right)}{\left[K_z^2+\left(K_y+\frac{Q_y}{2}\right)^2\right]\left[K_z^2+\left(K_y-\frac{Q_y}{2}\right)^2\right]\left(K_y+\frac{Q_y}{2}\right)}\right\}\end{aligned} \tag{28}$$

To extract the viscosity outside the integral, we introduce the following variables:

$$K_x \equiv \left(\frac{6}{\nu'_{\text{tot}}}\right)^{1/3} x, \qquad Q_x \equiv \left(\frac{6}{\nu'_{\text{tot}}}\right)^{1/3} q_x, \quad \omega = 2\left(\frac{6}{\nu'_{\text{tot}}}\right)^{1/3}.$$

The integral over $\mathrm{d}K$ can be represented as a sum of a highly oscillatory integral from 0 to infinity and a simple integral of trigonometrical function in finite interval. In the second integral the exponential function does not affect the result because the viscosity ν'_{tot} tends to zero:

$$\begin{aligned}&\int_{Q_x/2}^{\infty}\exp\left\{-\frac{\nu'_{\text{tot}}}{6}\left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right]\right\}\cos(2K_x)\frac{\mathrm{d}K_x}{2\pi}\\ &= \frac{1}{2\pi}\left(\frac{6}{\nu'_{\text{tot}}}\right)^{1/3}\int_0^{\infty}\exp\left\{-\left[\frac{(x - q_x/2)^3}{K_y + Q_y/2} + \frac{(x + q_x/2)^3}{K_y - Q_y/2}\right]\right\}\cos(2\omega x)\mathrm{d}x\\ &-\int_0^{Q_x/2}\exp\left\{-\frac{\nu'_{\text{tot}}}{6}\left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right]\right\}\cos(2K_x)\frac{\mathrm{d}K_x}{2\pi},\end{aligned} \tag{29}$$

$$\begin{aligned}&\lim_{\omega\to\infty}\left\{\frac{1}{2\pi}\left(\frac{6}{\nu'_{\rm tot}}\right)^{1/3}\int_0^\infty \exp\left\{-\left[\frac{(x-q_x/2)^3}{K_y+Q_y/2}+\frac{(x+q_x/2)^3}{K_y-Q_y/2}\right]\right\}\cos(\omega x)\mathrm{d}x\right\}\\&=\frac{1}{8\pi}\left(\frac{\nu'_{\rm tot}}{6}\right)^{1/3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left\{-\left[\frac{(x-q_x/2)^3}{K_y+Q_y/2}+\frac{(x+q_x/2)^3}{K_y-Q_y/2}\right]\right\}\right)\Bigg|_0=0,\end{aligned}\tag{30}$$

$$\begin{aligned}&\lim_{\nu'_{\rm tot}\to 0}\left\{\int_0^{Q_x/2}\exp\left\{-\frac{\nu'_{\rm tot}}{6}\left[\frac{(K_x-Q_x/2)^3}{K_y+Q_y/2}+\frac{(K_x+Q_x/2)^3}{K_y-Q_y/2}\right]\right\}\cos(2K_x)\frac{\mathrm{d}K_x}{2\pi}\right\}\\&=\int_0^{Q_x/2}\cos(2K_x)\frac{\mathrm{d}K_x}{2\pi}=\frac{1}{4\pi}\sin(Q_x),\end{aligned}\tag{31}$$

Finally we can assemble previously calculated results for $F_{\rm ext}$ to obtain

$$F^{(1)}=\frac{1}{4\pi}\sin(Q_x)\int_{Q_y/2}^\infty\frac{\mathrm{d}K_y}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}K_z}{2\pi}\,Q_1^y\left\{\frac{16K_z^3C_{\rm u}^{\mathbf{Q}'}C_{\rm u}^{\mathbf{Q}-\mathbf{Q}'}\left(\frac{K_zQ_y}{K_y-Q_y/2}+Q_z\right)}{\left[K_z^2+\left(K_y+\frac{Q_y}{2}\right)^2\right]\left[K_z^2+\left(K_y-\frac{Q_y}{2}\right)^2\right]\left(K_y+\frac{Q_y}{2}\right)}\right\}$$

$$\begin{aligned}X&=0,\\Y&=-\frac{1}{4\pi Q_y}\left(\frac{6}{\nu'_{\rm tot}}\right)^{1/3}\int_0^\infty\frac{1}{2}\exp\left\{-\frac{q_x^3}{Q_y}\right\}\mathrm{d}q_x\cdot\\&\quad\int_{Q_y/2}^\infty\frac{\mathrm{d}K_y}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}K_z}{2\pi}\,Q_1^y\left\{\frac{16K_z^3C_{\rm u}^{\mathbf{Q}'}C_{\rm u}^{\mathbf{Q}-\mathbf{Q}'}\left(\frac{K_zQ_y}{K_y-Q_y/2}+Q_z\right)}{\left[K_z^2+\left(K_y+\frac{Q_y}{2}\right)^2\right]\left[K_z^2+\left(K_y-\frac{Q_y}{2}\right)^2\right]\left(K_y+\frac{Q_y}{2}\right)}\right\}.\end{aligned}$$

After integration over $\mathrm{d}q_x$ we have

$$Y=\frac{\Gamma(\frac{4}{3})}{16\pi Q_y^{2/3}}\left(\frac{6}{\nu'_{\rm tot}}\right)^{1/3}\int_{Q_y/2}^\infty\frac{\mathrm{d}K_y}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}K_z}{2\pi}\,Q_1^y\left\{\frac{16K_z^3C_{\rm u}^{\mathbf{Q}'}C_{\rm u}^{\mathbf{Q}-\mathbf{Q}'}\left(\frac{K_zQ_y}{K_y-Q_y/2}+Q_z\right)}{\left[K_z^2+\left(K_y+\frac{Q_y}{2}\right)^2\right]\left[K_z^2+\left(K_y-\frac{Q_y}{2}\right)^2\right]\left(K_y+\frac{Q_y}{2}\right)}\right\}.$$

$$R \equiv \frac{\Gamma(\frac{4}{3})}{16\pi}\left(\frac{6}{\nu'_{\rm tot}}\right)^{1/3}, \quad \tilde{Y} \equiv YR, \quad \tilde{C} \equiv CR.$$

$$\nu_{\rm wave} \approx \frac{R}{(2\pi)^3}\int_0^\infty Q_y \tilde{D}_{\rm f}^2(Q_y, Q_z)\, {\rm d}Q_y {\rm d}Q_z. \tag{32}$$

$$
\begin{aligned}
& v^y_{Q_1} v^y_{Q_2} = A^{yy} \\
& \left(\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2}\cos(-Q^x_1)\cos(-Q^x_2) + C^g_1 C^g_2 \sin(-Q^x_1)\sin(-Q^x_2) + \frac{C^u_1 C^g_2}{K^\perp_1}\cos(-Q^x_1)\sin(-Q^x_2) + \frac{C^g_1 C^u_2}{K^\perp_2}\sin(-Q^x_1)\cos(-Q^x_2)\right) \\
& b^y_{Q_1} b^y_{Q_2} = A^{yy} \\
& \left(\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2}\sin(-Q^x_1)\sin(-Q^x_2) + C^g_1 C^g_2 \cos(-Q^x_1)\cos(-Q^x_2) + \frac{C^u_1 C^g_2}{K^\perp_1}\sin(-Q^x_1)\cos(-Q^x_2) + \frac{C^g_1 C^u_2}{K^\perp_2}\cos(-Q^x_1)\sin(-Q^x_2)\right) \\
& v^y_{Q_1} v^z_{Q_2} = A^{yz} \\
& \left(\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2}\cos(-Q^x_1)\cos(-Q^x_2) + \frac{C^g_1 C^g_2}{3}\sin(-Q^x_1)\sin(-Q^x_2) + \frac{C^u_1 C^g_2}{3K^\perp_1}\cos(-Q^x_1)\sin(-Q^x_2) + \frac{C^g_1 C^u_2}{K^\perp_2}\sin(-Q^x_1)\cos(-Q^x_2)\right) \\
& b^y_{Q_1} b^z_{Q_2} = A^{yz} \\
& \left(\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2}\sin(-Q^x_1)\sin(-Q^x_2) + \frac{C^g_1 C^g_2}{3}\cos(-Q^x_1)\cos(-Q^x_2) + \frac{C^u_1 C^g_2}{3K^\perp_1}\sin(-Q^x_1)\cos(-Q^x_2) + \frac{C^g_1 C^u_2}{K^\perp_2}\cos(-Q^x_1)\sin(-Q^x_2)\right)
\end{aligned}
$$

$$K^\perp_1 \equiv \sqrt{K_z^2 + (K_y + Q_y/2)^2}, \qquad K^\perp_2 \equiv \sqrt{K_z^2 + (K_y - Q_y/2)^2}, \tag{33}$$

$$\tilde{A}^y \equiv A^{yy} Q_y, \quad \tilde{A}^z \equiv A^{yz} Q_z, \quad A^{yy} = \frac{16K_z^4}{K^y_1 K^\perp_1 K^y_2 K^\perp_2}, \quad A^{yz} = \frac{-16K_z^3}{K^y_1 K^\perp_1 K^\perp_2} \tag{34}$$

$$
\begin{aligned}
& b^y_{\mathbf{Q}'} b^y_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^y_{\mathbf{Q}-\mathbf{Q}'})Q_y + (b^y_{\mathbf{Q}'} b^z_{\mathbf{Q}-\mathbf{Q}'} + v^y_{\mathbf{Q}'} v^z_{\mathbf{Q}-\mathbf{Q}'})Q_z = \\
& [\cos(-Q^x_1)\cos(-Q^x_2) + \sin(-Q^x_1)\sin(-Q^x_2)] \left[\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + C^g_1 C^g_2 (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right] + \qquad (35) \\
& [\cos(-Q^x_1)\sin(-Q^x_2) + \sin(-Q^x_1)\cos(-Q^x_2)] \left[\frac{C^g_1 C^u_2}{K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + \frac{C^u_1 C^g_2}{K^\perp_1} (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right] = \qquad (36) \\
& \cos(2K_x) \left[\frac{C^u_1 C^u_2}{K^\perp_1 K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + C^g_1 C^g_2 (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right] - \sin(Q_x) \left[\frac{C^g_1 C^u_2}{K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + \frac{C^u_1 C^g_2}{K^\perp_1} (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right] \qquad (37)
\end{aligned}
$$

$$
\int_{Q_x/2}^{\infty} \frac{\mathrm{d}K_x}{2\pi} \exp\left\{ -\frac{\nu'_{\mathrm{tot}}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \approx \frac{1}{2\pi} \left(\frac{6}{\nu'_{tot}} \right)^{1/3} \frac{\Gamma(4/3)}{3\left(\frac{1}{K_y+Q_y/2} + \frac{1}{K_y-Q_y/2} \right)^{1/3}} \qquad (38)
$$

$$
\begin{aligned}
X &= 0, \\
Y &= \frac{\Gamma(4/3) Q_z}{4\pi Q_y^{2/3}} \left(\frac{6}{\nu'_{\mathrm{tot}}} \right)^{2/3} \int_0^\infty \frac{1}{2} \exp\left\{ -\frac{q_x^3}{Q_y} \right\} \mathrm{d}q_x \cdot \\
& \int_{Q_y/2}^{\infty} \frac{\mathrm{d}K_y}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}K_z}{2\pi} \frac{Q^y_1}{3\left(\frac{1}{K_y+Q_y/2} + \frac{1}{K_y-Q_y/2} \right)^{1/3}} \left[\frac{C^g_1 C^u_2}{K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + \frac{C^u_1 C^g_2}{K^\perp_1} (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right].
\end{aligned}
$$

After integration over $\mathrm{d}q_x$ we have

$$
Y = \frac{\Gamma^2(\frac{4}{3}) Q_z}{16\pi Q_y^{2/3}} \left(\frac{6}{\nu'_{\mathrm{tot}}} \right)^{2/3} \int_{Q_y/2}^{\infty} \frac{\mathrm{d}K_y}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}K_z}{2\pi} Q^y_1 \frac{Q^y_1}{3\left(\frac{1}{K_y+Q_y/2} + \frac{1}{K_y-Q_y/2} \right)^{1/3}} \left[\frac{C^g_1 C^u_2}{K^\perp_2} (\tilde{A}^y + \tilde{A}^z) + \frac{C^u_1 C^g_2}{K^\perp_1} (\tilde{A}^y + \frac{\tilde{A}^z}{3}) \right].
$$

$$
C^g = D_f D_u \sin(\phi_f - \delta_u), \qquad C^u = D_f D_g \sin(\phi_f - \delta_g) \qquad (39)
$$

$$R \equiv \frac{\Gamma^2(\frac{4}{3})}{16\pi}\left(\frac{6}{\nu'_{\mathrm{tot}}}\right)^{2/3}, \quad \tilde{Y} \equiv YR, \quad \tilde{C} \equiv CR.$$

$$\nu_{\mathrm{wave}} \approx \frac{R}{(2\pi)^3}\int_0^\infty Q_y \tilde{D}_{\mathrm{f}}^2(Q_y, Q_z)\,\mathrm{d}Q_y \mathrm{d}Q_z. \tag{40}$$