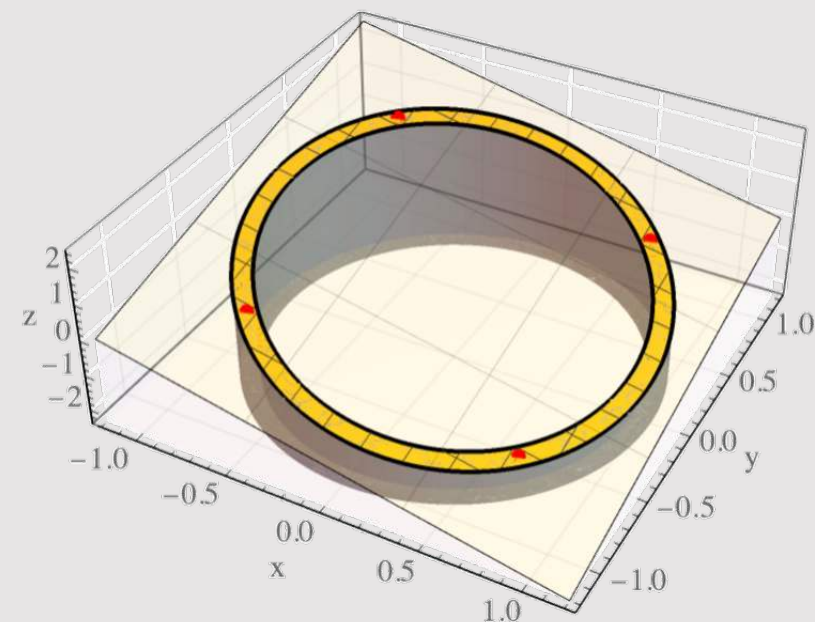
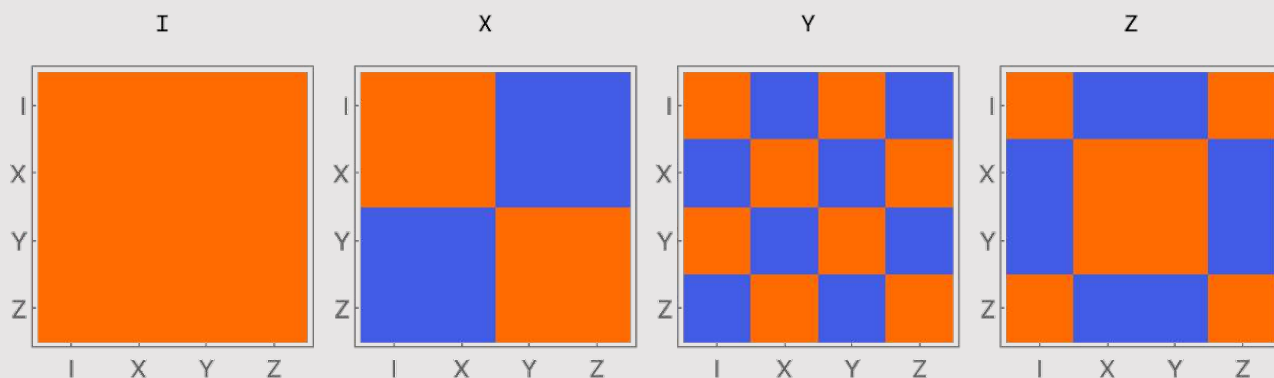
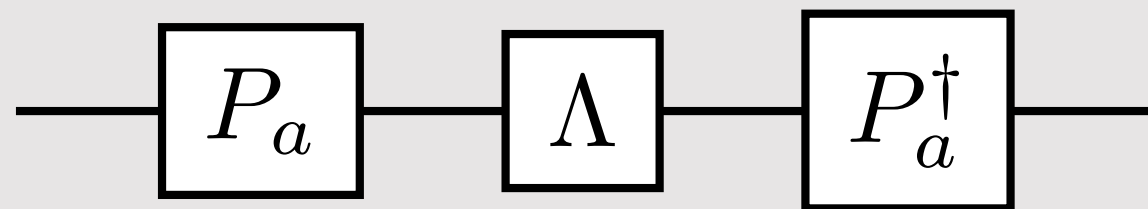


Primer on Pauli Twirling



Zlatko Minev

2022-04-20, 07-11

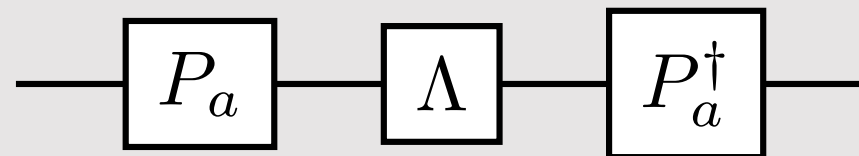
Twirling 101: Overview

Twirl operationally

Simple example

General application

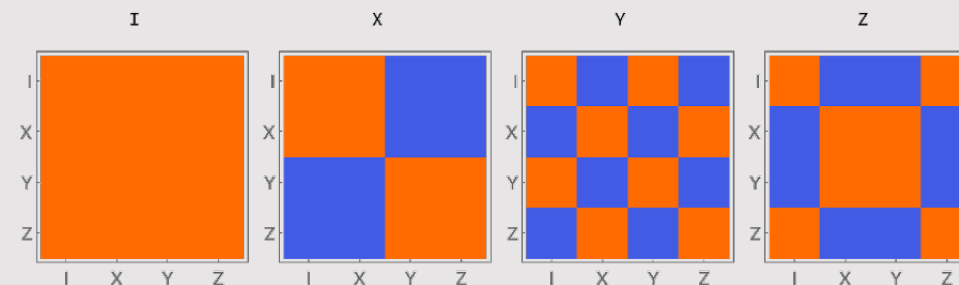
Summary



Theory of twirling

Why does twirling work?

Masking channels

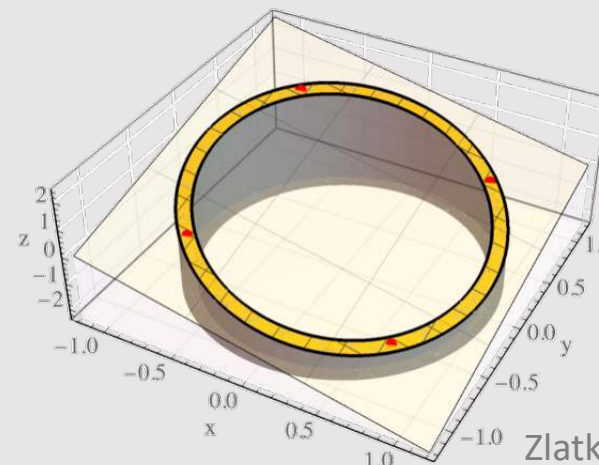


Optional: Advanced

Why is the Pauli group special for twirling?

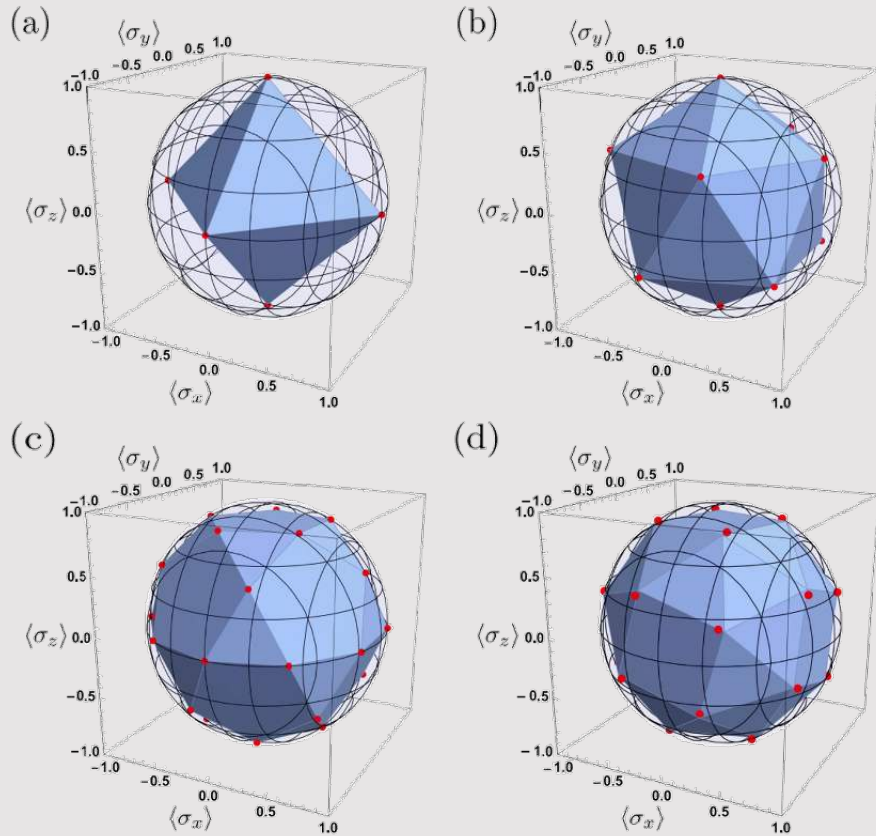
Other twirl groups

Designs



Designs

Unitary t -design



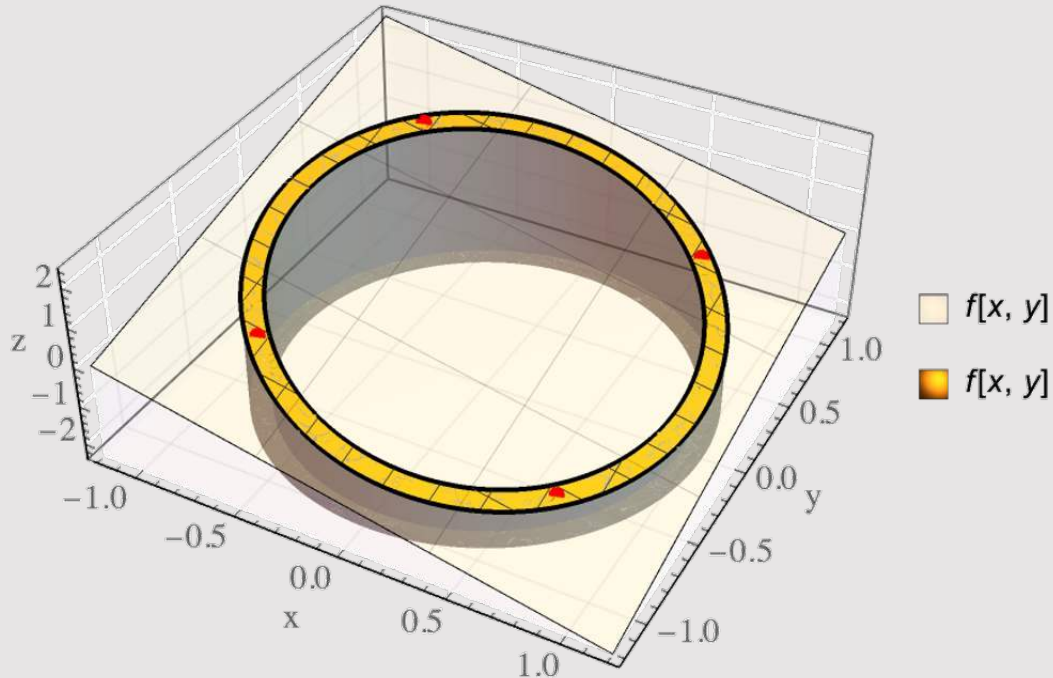
Where does this get all used?

- Average channel fidelity metrics
- RB fidelity
- Random unitaries / matrices
- Random twirling

image arxiv: 2004.08402

Classical Design

$$f(x, y) = y - x$$



Suppose we have some polynomial $f(x, y)$ in $d=2$ variables, and we would like to compute its average over the unit sphere S_d .

A spherical design is a set of representative points (i.e. d -dimensional unit vectors) on the surface of the sphere such that computing the average of the function only over these points is identical to taking the average over the entire unit sphere.

A set $X = \{x : x \in \mathcal{S}(\mathbb{R}^a)\}$ is a *spherical t -design* if

$$\frac{1}{|X|} \sum_{x \in X} p_t(x) = \int_{\mathcal{S}(\mathbb{R}^d)} f_t(u) d\mu(u)$$

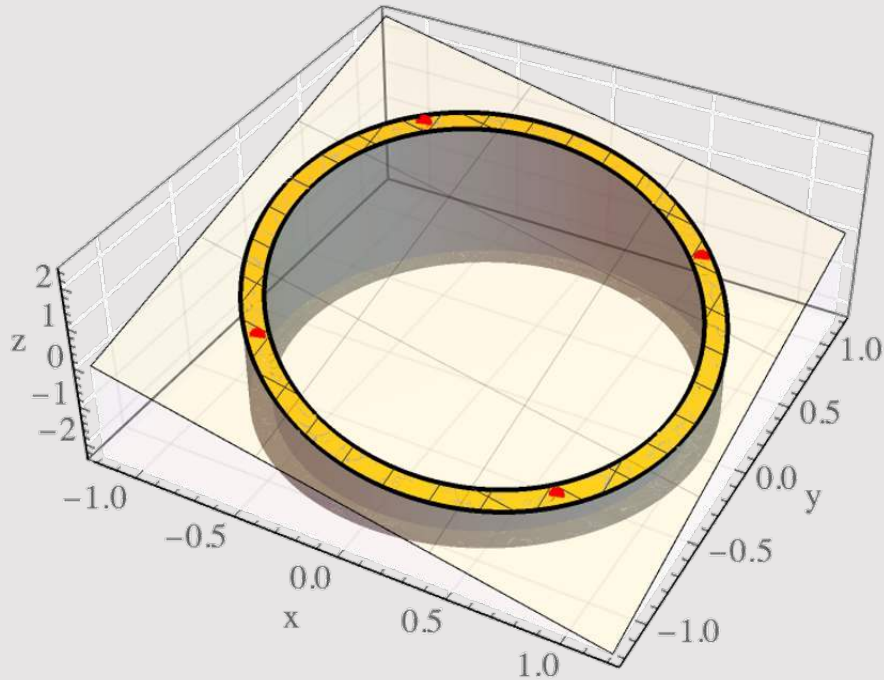
where the polynomial terms are at most degree t .

Quantum? We will have polynomials of unitaries.

Some statement of symmetries

Example of a classical design on functions over 2D plane

$$f(x, y) = y - x$$



$f[x, y]$
 $f[x, y]$

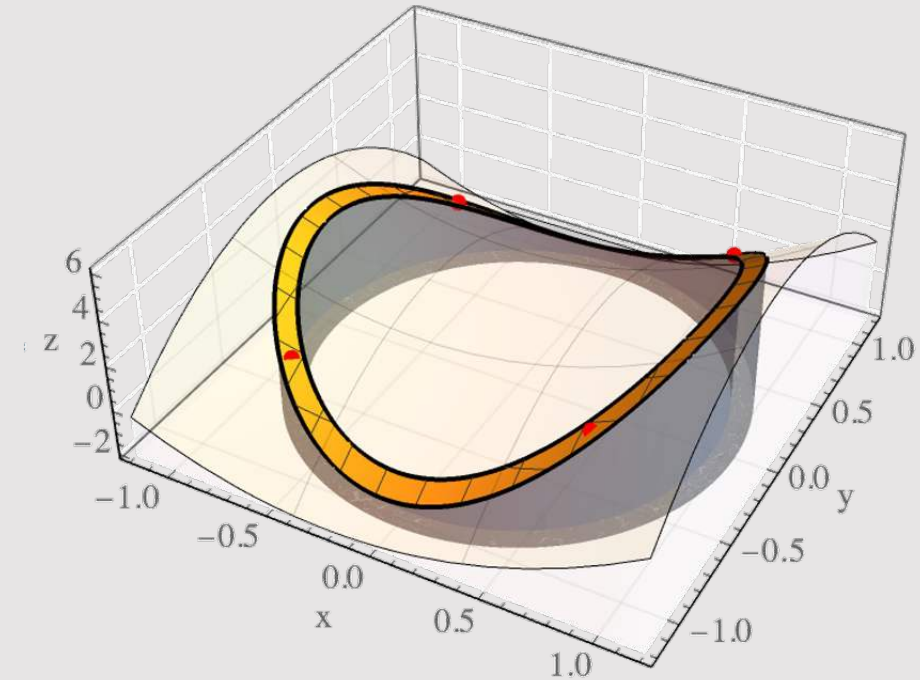
```

N = Integrate[1, {x, y} ∈ Circle[]];
Print["∫ -> ", Integrate[f[x, y], {x, y} ∈ Circle[]] / N]

N = Length[ptsf];
Print["Σ -> ", Sum[x[[2]], {x, ptsf}] / N]
    
```

$\int \rightarrow 0$
 $\Sigma \rightarrow 0$

$$f(x, y) = 2x^2 + x - 3y^2 - 0.2y + 2$$



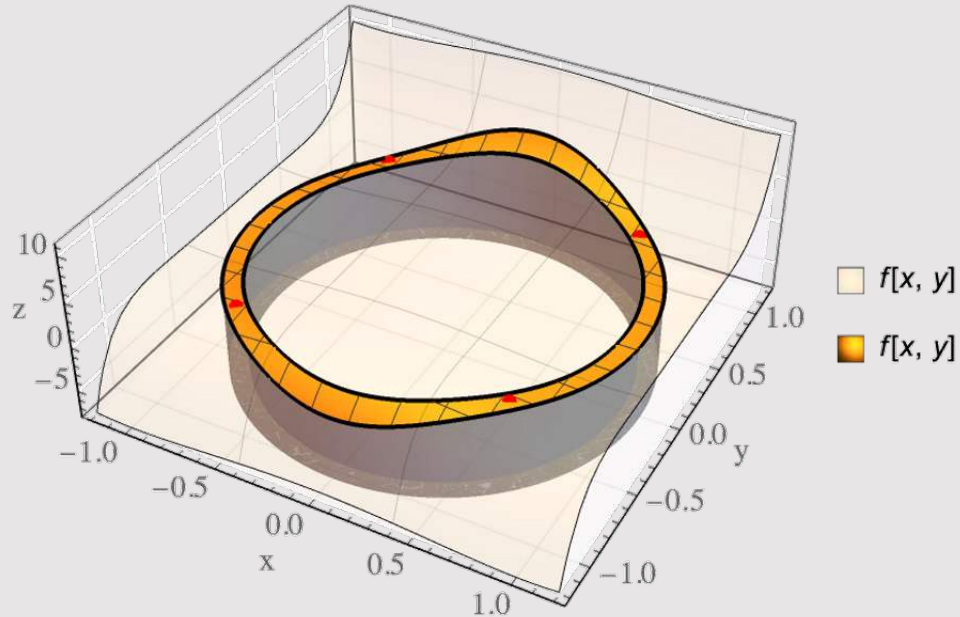
$f[x, y]$
 $f[x, y]$

$$\frac{1}{|X|} \sum_{x \in X} p_t(x) = \int_{\mathcal{S}(\mathbb{R}^d)} f_t(u) d\mu(u)$$

$\int \rightarrow 1.5$
 $\Sigma \rightarrow 1.5$

When does it fail? Higher designs?

$$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$$

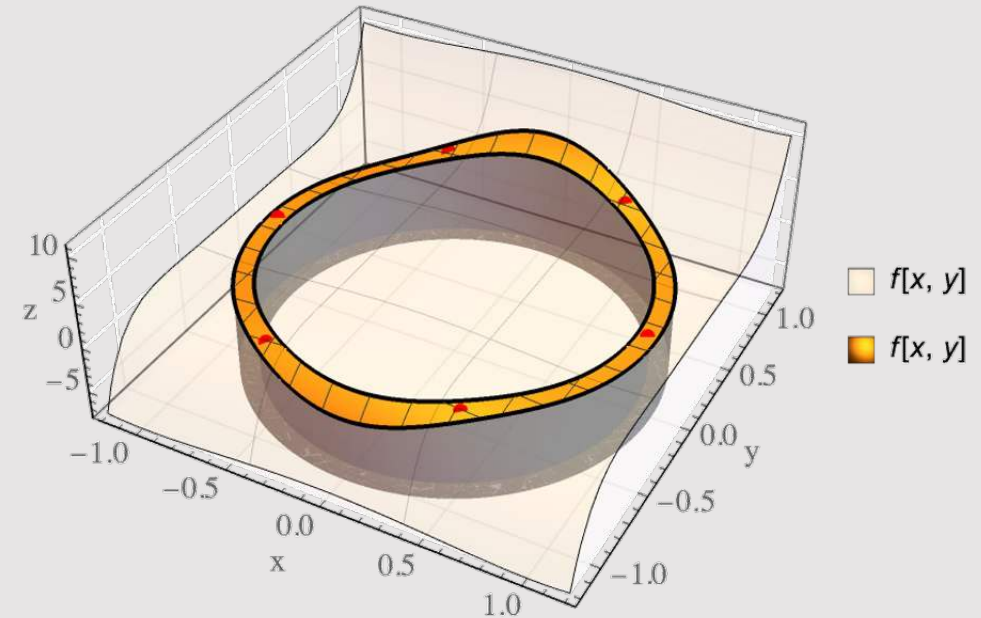


$$\int \rightarrow \frac{11}{8}$$

$$\Sigma \rightarrow \frac{5}{4}$$

Fails with only 4 points in the design set
not $t > 2$ design

$$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$$



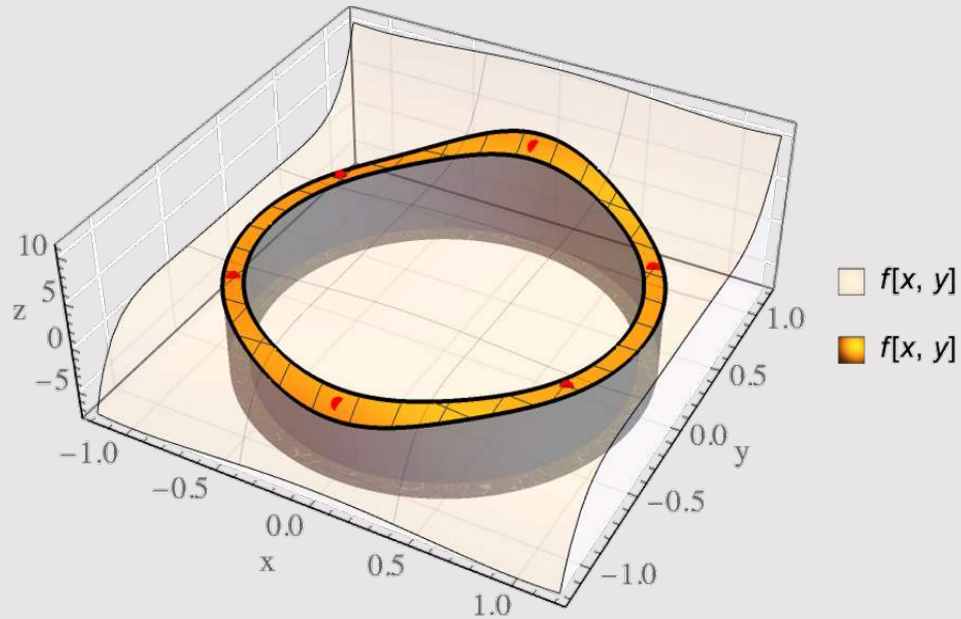
$$\int \rightarrow \frac{11}{8}$$

$$\Sigma \rightarrow \frac{11}{8}$$

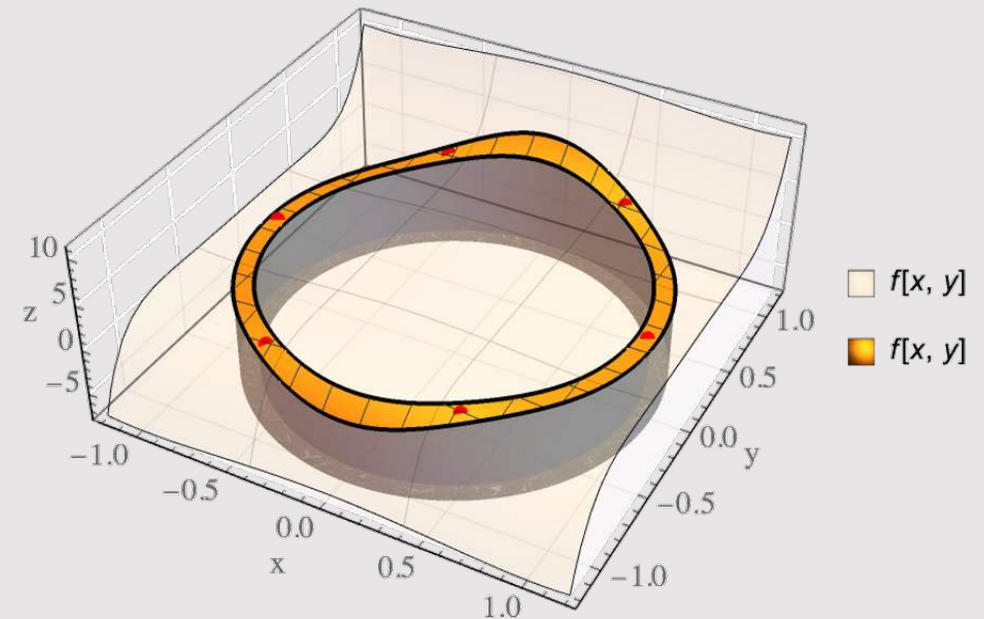
Succeed when we expand design set
to include more

Rotations?

$$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$$



$$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$$



Can also rotate points in set so long as they preserve certain properties

$$N = 6; \Delta\theta = \frac{2\pi}{N};$$

$$v0 = \text{RotationMatrix}\left[(\sqrt{2} + 0.1) * \Delta\theta\right] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$vs = \text{Table}[\text{Flatten}[\text{RotationMatrix}[\Delta\theta * n] \cdot v0], \{n, 0, N - 1\}];$$

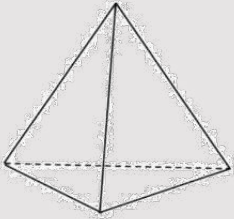
$$\int \rightarrow \frac{11}{8}$$

$$\Sigma \rightarrow \frac{11}{8}$$

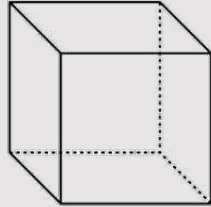
Succeed when we expand design set to include more

Higher dimensions?

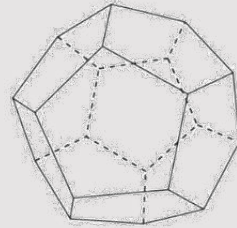
Classical 3- space



2 design



3 design



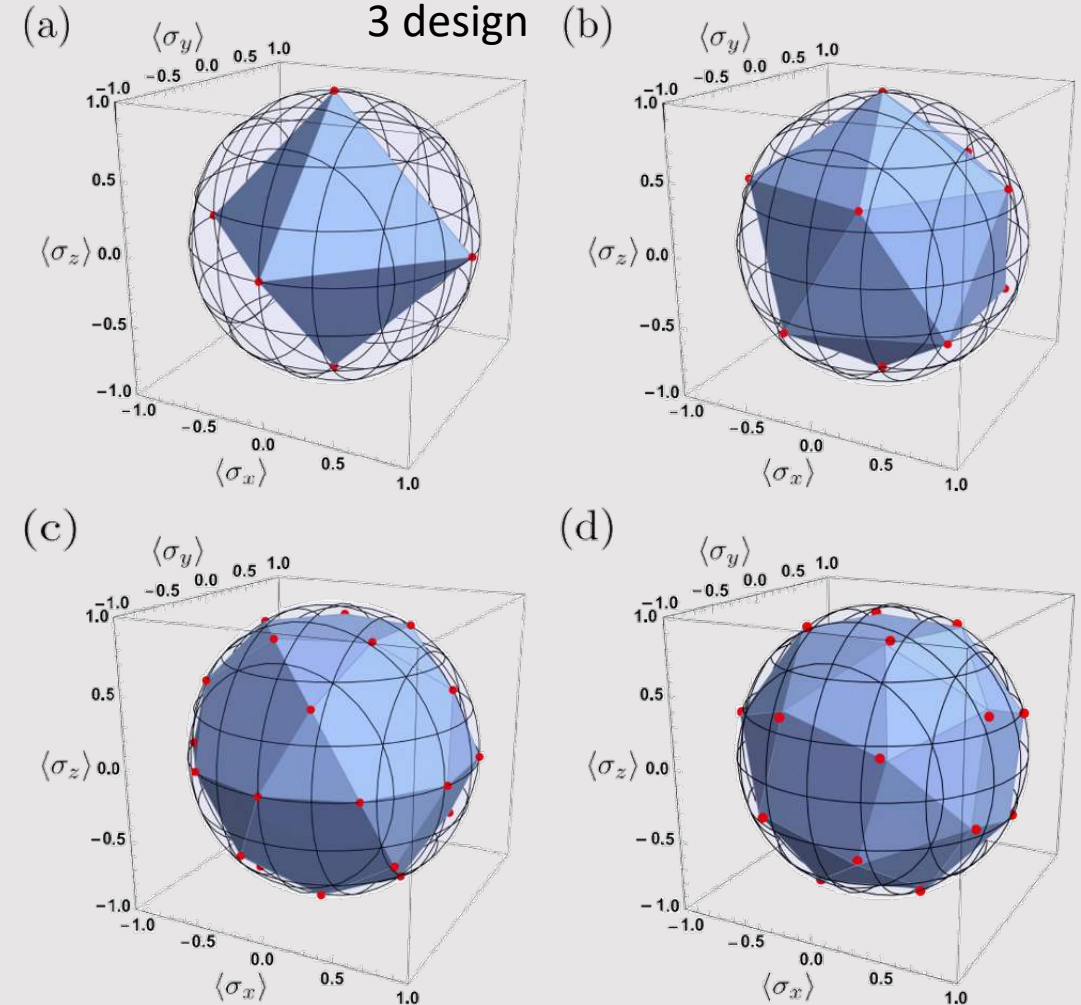
5 design

Where does this get all used?

- Average channel fidelity metrics
- RB fidelity
- Random unitaries / matrices
- Random twirling

Quantum Bloch sphere

3 design



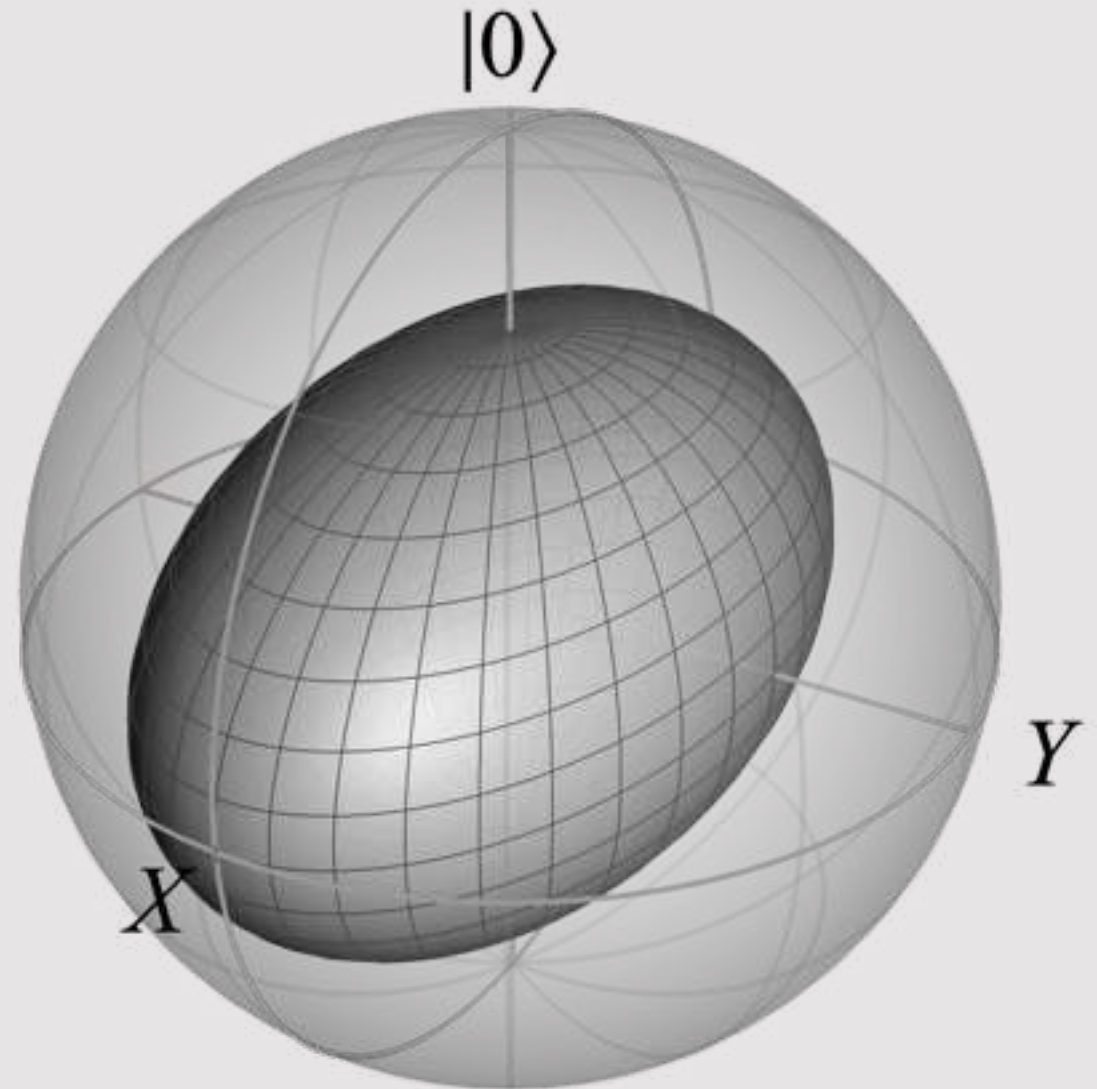
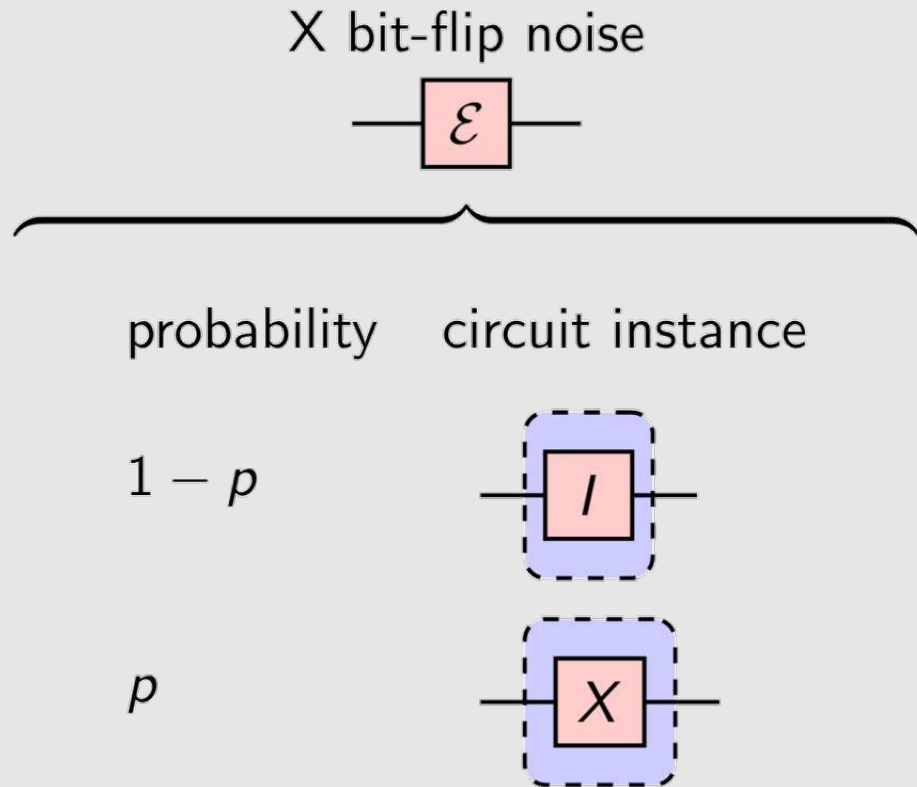


* pikisuperstar

Refresher

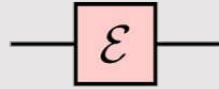
Modeling noise 101

How do we model noise?



How do we model noise?

Stochastic Pauli noise

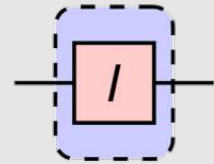


* momentarily i will use calligraphic capital E instead of Greek lambda.

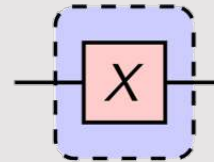
probability

circuit instance

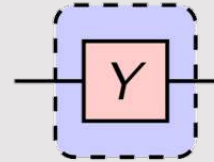
$$1 - p_X - p_Y - p_Z$$



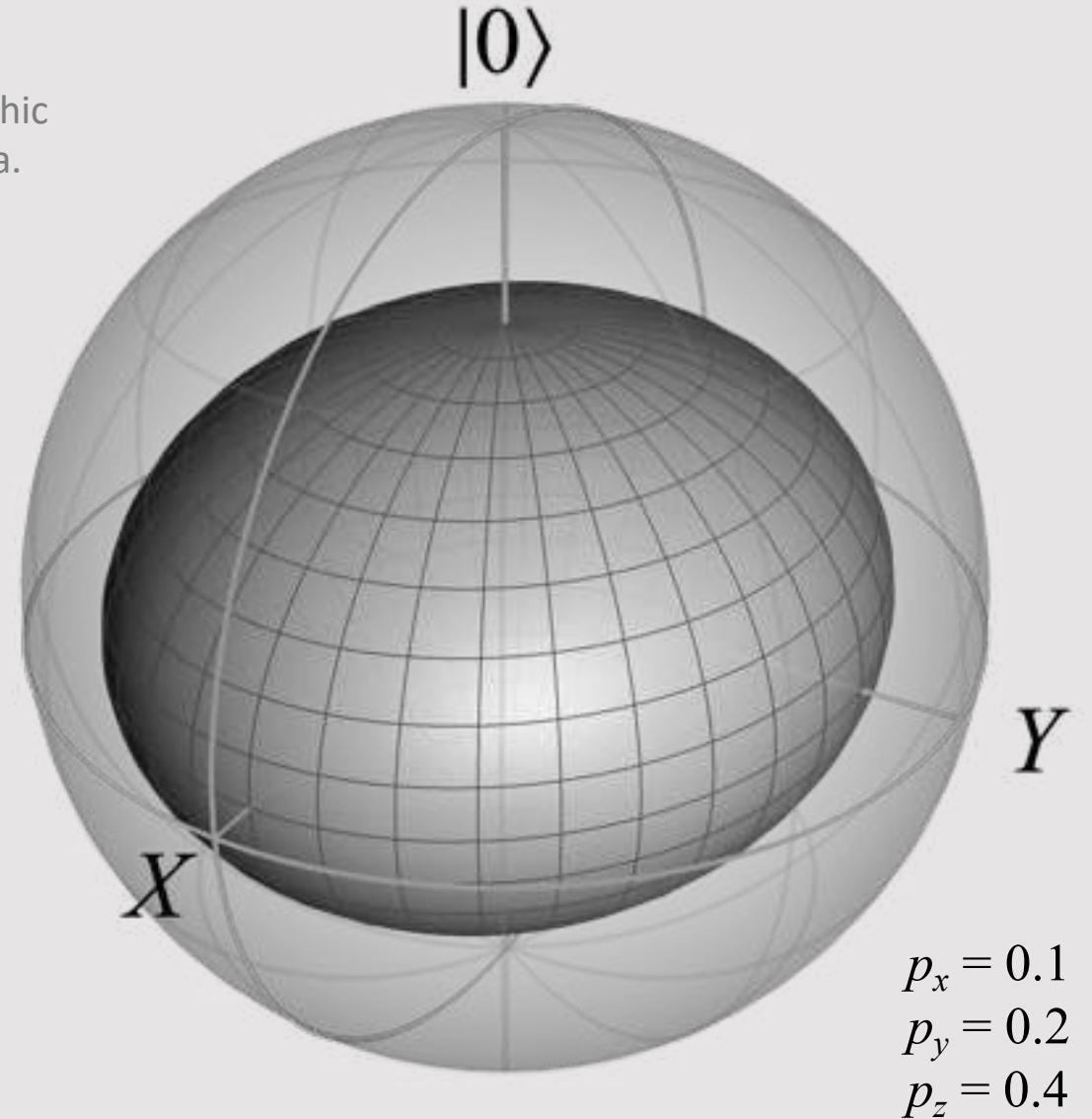
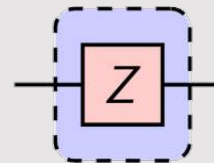
$$p_X$$



$$p_Y$$

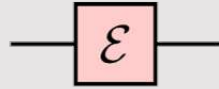


$$p_Z$$



How do we model noise?

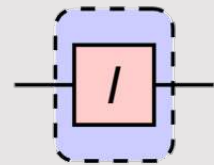
Stochastic Pauli noise



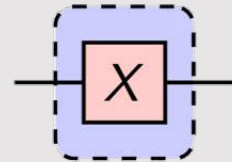
probability

circuit instance

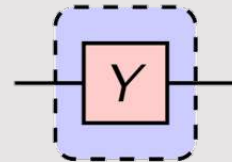
$$1 - p_X - p_Y - p_Z$$



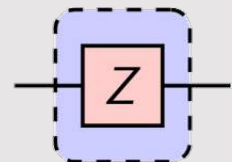
$$p_X$$



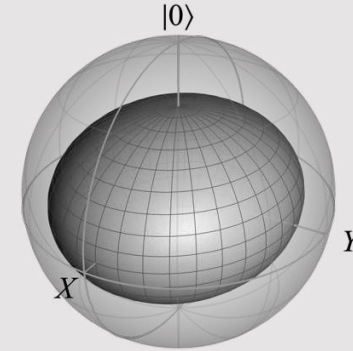
$$p_Y$$



$$p_Z$$



Picture



Math

$$\begin{aligned} \mathcal{E}(\rho) = & \chi_I \hat{I} \rho \hat{I} + \chi_X \hat{X} \rho \hat{X} \\ & + \chi_X \hat{Y} \rho \hat{Y} + \chi_X \hat{Z} \rho \hat{Z} \end{aligned}$$

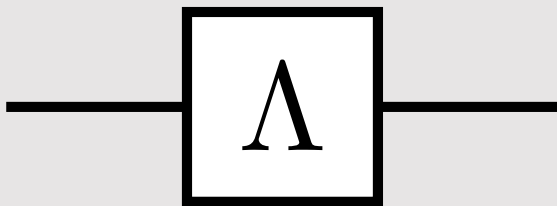
or

$$\mathcal{E}(\rho) = \sum_{a \in \{I, X, Y, Z\}} p_a \hat{P}_a \rho \hat{P}_a$$

General noise

Process matrix representation

aka. chi matrix (related to Kraus representation)



$$\Lambda(\rho) = \sum_{a,b \in \{I, X, Y, Z\}^{\otimes n}} \chi_{ab} \hat{P}_a \rho \hat{P}_b$$

Matrix elements of the Pauli transfer matrix (PTM)

$$\text{PTM}(\Lambda)_{ab} := \frac{1}{2^n} \text{Tr} [P_a^\dagger \Lambda(P_b)] = \frac{1}{2^n} \langle\langle P_a | \Lambda | P_b \rangle\rangle$$

* I dropped the hats

Amplitude damping

4D-2. Amplitude damping

Photon loss channel. The probability of losing a photon $0 \leq p \leq 1$ is given by the noise map

$$\mathcal{A}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger, \quad \text{where } A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \quad (4.5)$$

The no-loss and photon loss channels in terms of Paulis are $A_0 = (1 + \sqrt{1-p}) \hat{I} + (1 - \sqrt{1-p}) \hat{Z}$ and $A_1 = \sqrt{p} (\hat{X} + i\hat{Y})$, resp.

Kraus K_i	Pauli transfer matrix (PTM)	Process χ -matrix
$K_0 = \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix}$ $K_1 = \begin{pmatrix} 1 & \\ 0 & \sqrt{p} \end{pmatrix}$	$\begin{matrix} & \begin{matrix} I & X & Y & Z \end{matrix} \\ \begin{matrix} I \\ X \\ Y \\ Z \end{matrix} & \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{matrix}$	$\begin{matrix} & \begin{matrix} I & X & Y & Z \end{matrix} \\ \begin{matrix} I \\ X \\ Y \\ Z \end{matrix} & \begin{pmatrix} \frac{1}{4}(\sqrt{1-p}+1)^2 & & & \frac{p}{4} \\ & \frac{p}{4} & -i\frac{p}{4} & \\ & \text{H.c.} & \frac{p}{4} & \\ & & & \frac{1}{4}(\sqrt{1-p}-1)^2 \end{pmatrix} \end{matrix}$

Amplitude damping

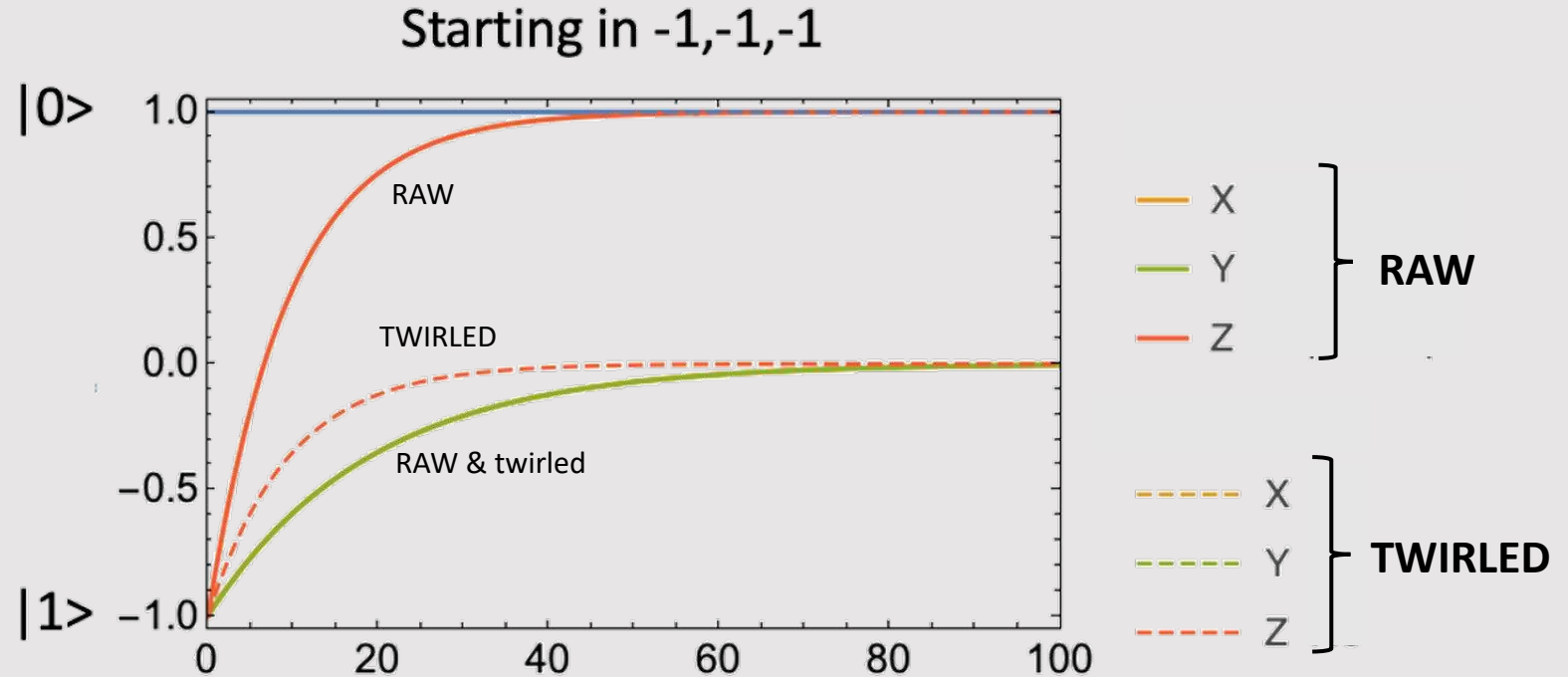
Applied m times

Pauli transfer matrix (PTM)

$$\begin{matrix} & I & X & Y & Z \\ \begin{matrix} I \\ X \\ Y \\ Z \end{matrix} & \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{matrix}$$

Twirled Pauli transfer matrix (PTM)

$$\begin{matrix} & I & X & Y & Z \\ \begin{matrix} I \\ X \\ Y \\ Z \end{matrix} & \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{matrix}$$



AD: $\langle | I \rightarrow 1, X \rightarrow X (1-\gamma)^{m/2}, Y \rightarrow Y (1-\gamma)^{m/2}, Z \rightarrow 1 + (-1+Z) (1-\gamma)^m | \rangle$

RC AD: $\langle | I \rightarrow 1, X \rightarrow X (1-\gamma)^{m/2}, Y \rightarrow Y (1-\gamma)^{m/2}, Z \rightarrow Z (1-\gamma)^m | \rangle$