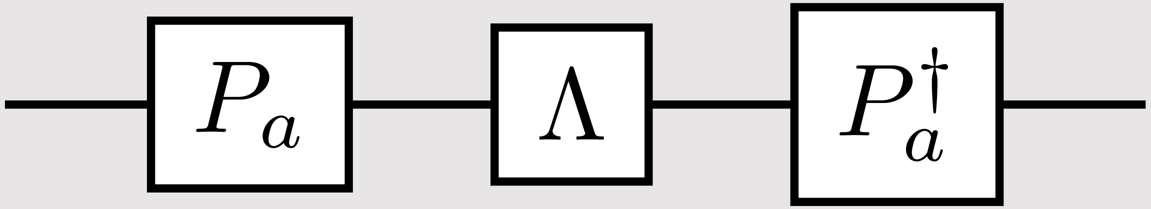
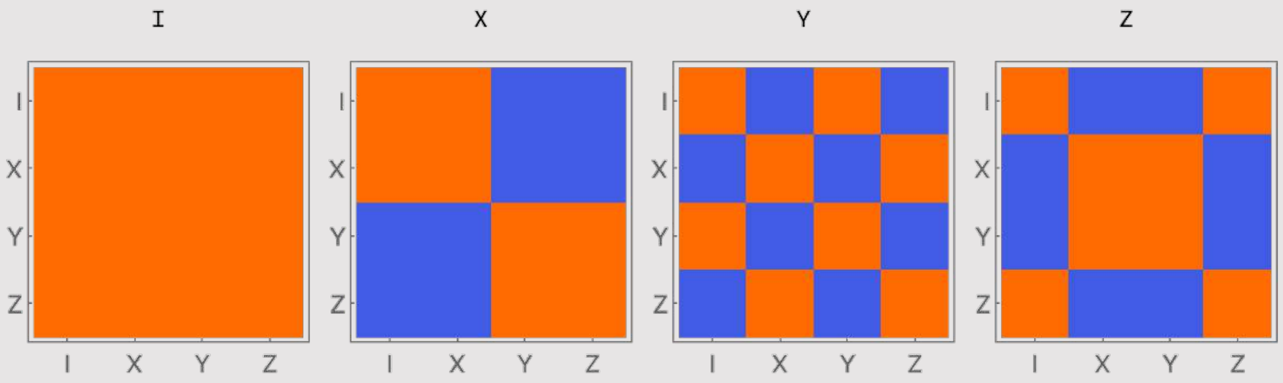
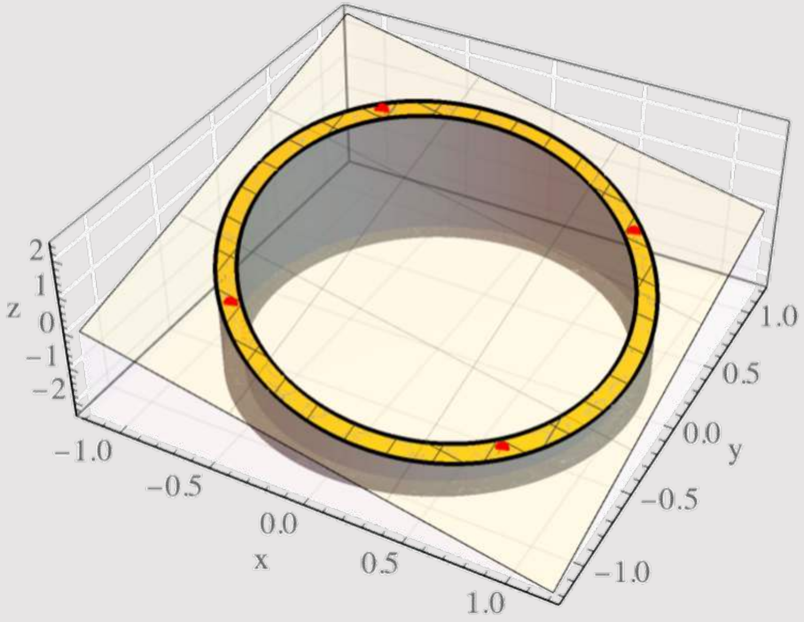
Primer on Pauli Twirling







Zlatko Minev

2022-04-20, 07-11

Twirling 101: Overview

Twirl operationally

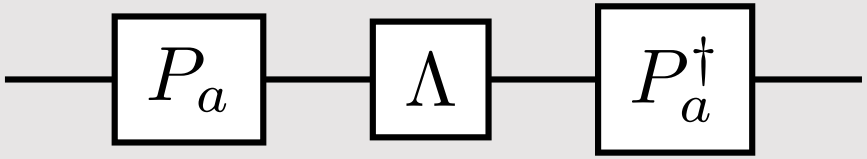
- Simple example
- General application
- Summary

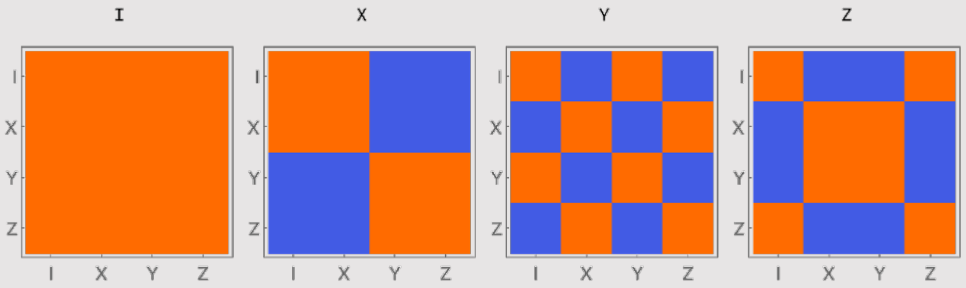
Theory of twirling

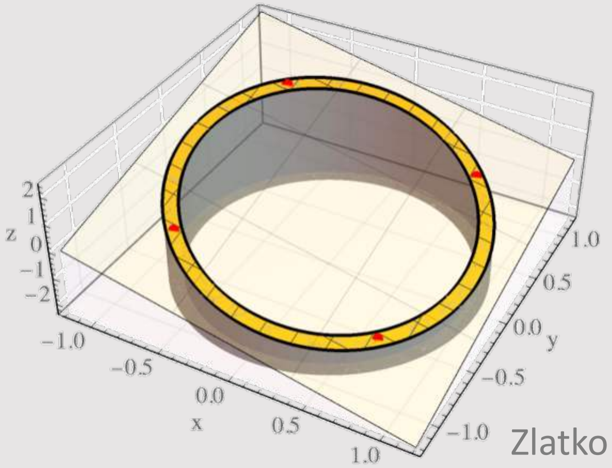
- Why does twirling work?
- Masking channels

Optional: Advanced

- Why is the Pauli group special for twirling?
- Other twirl groups
- Designs



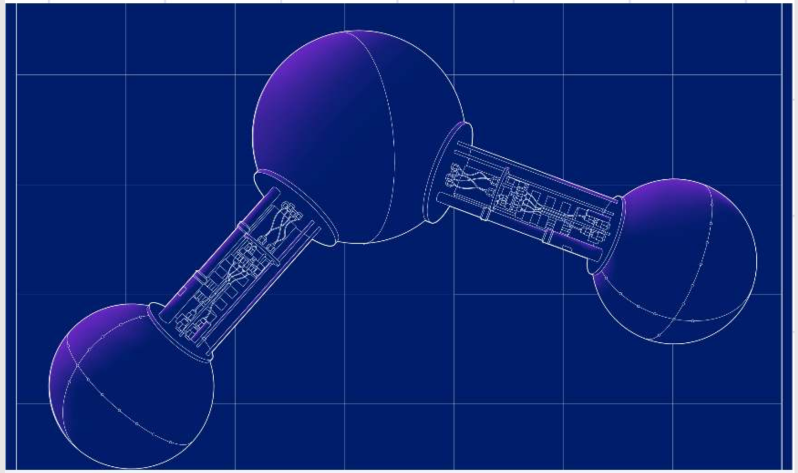




Why does twirling actually work?

Theory and my take on it

Noise basics 101



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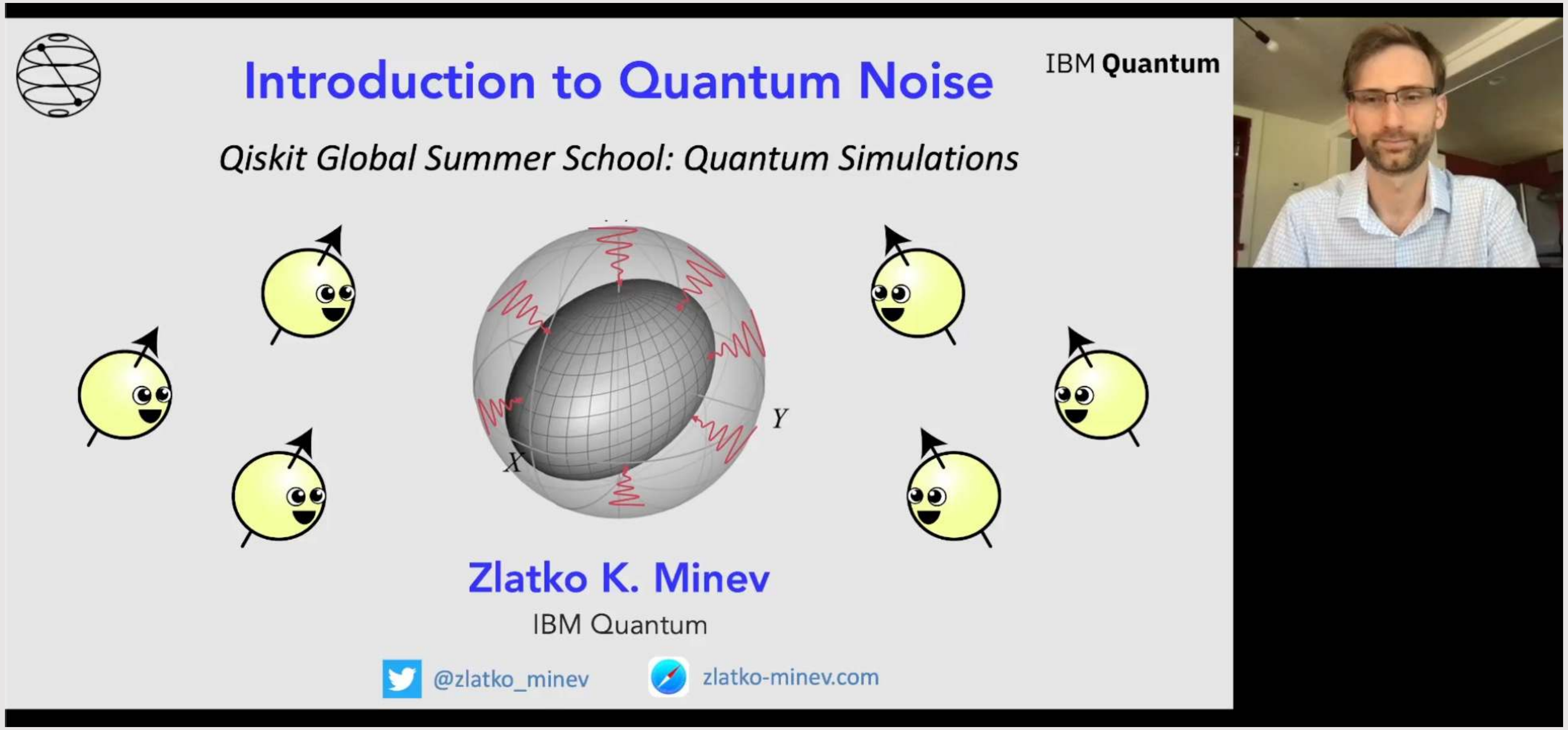
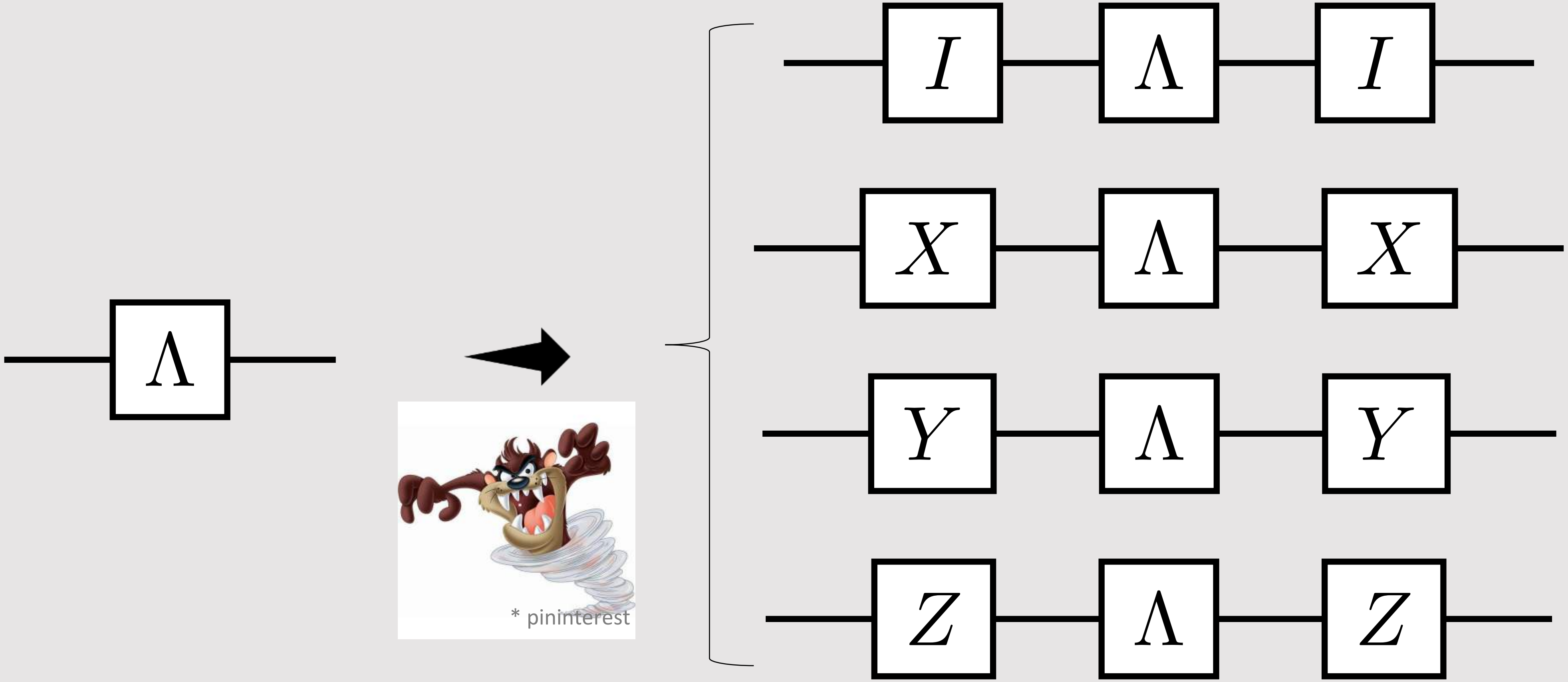


[Also: https://qiskit.org/textbook-beta/summer-school/quantum-computing-and-quantum-learning-2021](https://qiskit.org/textbook-beta/summer-school/quantum-computing-and-quantum-learning-2021)

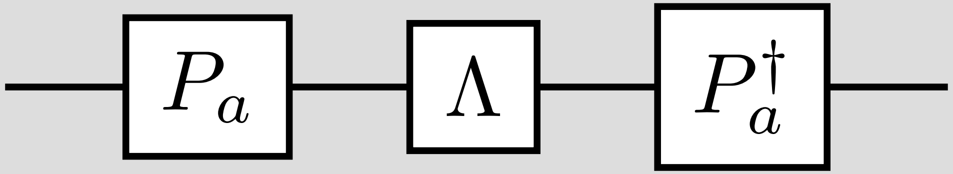
Example

Pauli twirling on a qubit

Twirl: average over instances



Twirl example on a bit flip channel



Algebraic expression of channel sequence:

$$\mathcal{P}_a \Lambda \mathcal{P}_a^\dagger = P_a^\dagger \Lambda (P_a \cdot P_a^\dagger) P_a$$

Example: Bit-flip channel

$$\begin{aligned}\Lambda(\cdot) &= (1-p)\, I \cdot I + pX \cdot X \\ &= (1-p)\,\mathcal{I} + p\mathcal{X}\end{aligned}$$

Notation

$$\mathcal{I}(\cdot) = I \cdot I \qquad \mathcal{Y}(\cdot) = Y \cdot Y$$

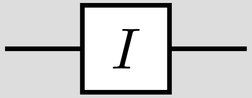
$$\mathcal{X}(\cdot) = X \cdot X \qquad \mathcal{Z}(\cdot) = Z \cdot Z$$

Recall $X^2 = Y^2 = Z^2 = I$

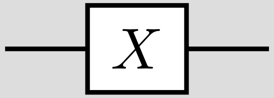
$$\mathcal{I}(X) = X \qquad \mathcal{Y}(X) = YXY = -X$$

$$\mathcal{X}(X) = X \qquad \mathcal{Z}(X) = ZXZ = -X$$

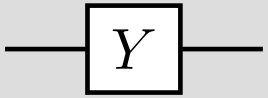
Twirl gate



$$\mathcal{I}\Lambda\mathcal{I} = \Lambda$$



$$\begin{aligned}\mathcal{X}\Lambda\mathcal{X} &= X\Lambda(X \cdot X)\,X \\ &= (1-p)\, XIX \cdot XIX + pXXX \cdot XXX \\ &= (1-p)\,\mathcal{X}(I) \cdot \mathcal{X}(I) + p\mathcal{X}(X) \cdot \mathcal{X}(X) \\ &= \Lambda\end{aligned}$$



$$\mathcal{Y}\Lambda\mathcal{Y} = Y\Lambda(Y \cdot Y)\,Y$$

Twirl example on a bit flip channel

$$-\boxed{P_a}-\boxed{\Lambda}-\boxed{P_a^\dagger}-$$

Example: Bit-flip channel

$$\begin{aligned}\Lambda(\cdot) &= (1-p)\, I \cdot I + pX \cdot X \\ &= (1-p)\,\mathcal{I} + p\mathcal{X}\end{aligned}$$

Notation

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Twirl gate

$-\boxed{I}-$

$$\mathcal{I}\Lambda\mathcal{I} = \Lambda$$

$-\boxed{X}-$

$$\begin{aligned}\mathcal{X}\Lambda\mathcal{X} &= X\Lambda(X \cdot X)\, X \\ &= (1-p)\, XIX \cdot XIX + pXXX \cdot XXX \\ &= (1-p)\,\mathcal{X}(I) \cdot \mathcal{X}(I) + p\mathcal{X}(X) \cdot \mathcal{X}(X) \\ &= \Lambda\end{aligned}$$

$-\boxed{Y}-$

$$\begin{aligned}\mathcal{Y}\Lambda\mathcal{Y} &= Y\Lambda(Y \cdot Y)\, Y \\ &= (1-p)\, YIY \cdot YIY + pYXY \cdot YXY \\ &= (1-p)\,\mathcal{Y}(I) \cdot \mathcal{Y}(I) + p\mathcal{Y}(X) \cdot \mathcal{Y}(X) \\ &= (1-p)\, I \cdot I + p(-X) \cdot (-X) \\ &= \Lambda\end{aligned}$$

$-\boxed{Z}-$

$$\begin{aligned}\mathcal{Z}\Lambda\mathcal{Z} &= Z\Lambda(Z \cdot Z)\, Z \\ &= (1-p)\,\mathcal{Z}(I) \cdot \mathcal{Z}(I) + p\mathcal{Z}(X) \cdot \mathcal{Z}(X) \\ &= \Lambda\end{aligned}$$

Twirl example on a bit flip channel

$$-\boxed{P_a}-\boxed{\Lambda}-\boxed{P_a^\dagger}-$$

Example: Bit-flip channel

$$\Lambda(\cdot) = (1-p)\, I \cdot I + pX \cdot X$$
$$= (1-p)\,\mathcal{I} + p\mathcal{X}$$

Notation

$$\mathcal{I}(\cdot) = I \cdot I \qquad \mathcal{Y}(\cdot) = Y \cdot Y$$
$$\mathcal{X}(\cdot) = X \cdot X \qquad \mathcal{Z}(\cdot) = Z \cdot Z$$

Recall $X^2 = Y^2 = Z^2 = I$

$$\mathcal{I}(X) = X \qquad \mathcal{Y}(X) = YXY = -X$$
$$\mathcal{X}(X) = X \qquad \mathcal{Z}(X) = ZXZ = -X$$

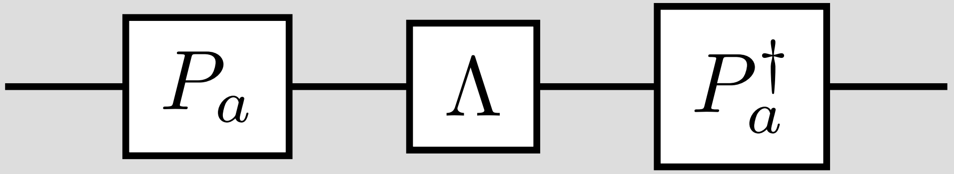
Twirl gate

$$-\boxed{I}- \;= \Lambda$$
$$-\boxed{X}- \;= \Lambda$$
$$-\boxed{Y}- \;= \Lambda$$
$$-\boxed{Z}- \;= \Lambda$$

Average

$$\Lambda \mapsto \frac{1}{4}\left(\mathcal{I}\Lambda\mathcal{I} + \mathcal{X}\Lambda\mathcal{X} + \mathcal{Y}\Lambda\mathcal{Y} + \mathcal{Z}\Lambda\mathcal{Z}\right) = \Lambda$$

Example coherent rotation channel



Example: Coherent rotation

$$U = \exp\left(-i\frac{\theta}{2}X\right) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X$$

$$\begin{aligned}\Lambda(\cdot) &= U \cdot U^\dagger \\ &= \left[\cos\left(\frac{\theta}{2}\right)\right]^2 I \cdot I + \left[\sin\left(\frac{\theta}{2}\right)\right]^2 X \cdot X \\ &\quad + \frac{i}{2}\left(\sin(\theta)\, I \cdot X - \sin(\theta)\, X \cdot I\right)\end{aligned}$$

$$\begin{aligned}&= |I\rangle\rangle\langle\langle I| + |X\rangle\rangle\langle\langle X| \\ &\quad + \cos(\theta)\left(|Y\rangle\rangle\langle\langle Y| + |Z\rangle\rangle\langle\langle Z|\right) \\ &\quad + \sin(\theta)\left(|Z\rangle\rangle\langle\langle Y| - |Y\rangle\rangle\langle\langle Z|\right)\end{aligned}$$

Chi matrix

	I	X	Y	Z
I	$\mathrm{Cos}\left[\frac{\theta}{2}\right]^2$	$\frac{1}{2}\, i\, \mathrm{Sin}[\theta]$	0	0
X	$-\frac{1}{2}\, i\, \mathrm{Sin}[\theta]$	$\mathrm{Sin}\left[\frac{\theta}{2}\right]^2$	0	0
Y	0	0	0	0
Z	0	0	0	0

Pauli transfer matrix

	I	X	Y	Z
I	1	0	0	0
X	0	1	0	0
Y	0	0	$\mathrm{Cos}[\theta]$	$-\mathrm{Sin}[\theta]$
Z	0	0	$\mathrm{Sin}[\theta]$	$\mathrm{Cos}[\theta]$

Example coherent rotation channel

$$—\boxed{P_a}—\boxed{\Lambda}—\boxed{P_a^\dagger}—$$

Example: Coherent rotation

$$\Lambda(\cdot) = U \cdot U^\dagger$$

Chi matrix

	I	X	Y	Z
I	$\mathrm{Cos}\left[\frac{\theta}{2}\right]^2$	$\frac{1}{2}\, i\, \mathrm{Sin}[\theta]$	0	0
X	$-\frac{1}{2}\, i\, \mathrm{Sin}[\theta]$	$\mathrm{Sin}\left[\frac{\theta}{2}\right]^2$	0	0
Y	0	0	0	0
Z	0	0	0	0

Pauli transfer matrix

	I	X	Y	Z
I	1	0	0	0
X	0	1	0	0
Y	0	0	Cos[θ]	-Sin[θ]
Z	0	0	Sin[θ]	Cos[θ]

Twirl gate

$$—\boxed{I}— \quad \mathcal{I}\Lambda\mathcal{I} = \Lambda$$

$$—\boxed{X}— \quad \mathcal{X}\Lambda\mathcal{X} = \cos^2\left(\frac{\theta}{2}\right)\mathcal{X}(I)\cdot\mathcal{X}(I) + \sin^2\left(\frac{\theta}{2}\right)\mathcal{X}(X)\cdot\mathcal{X}(X)$$
$$+\frac{i}{2}\left(\sin(\theta)\,\mathcal{X}(I)\cdot\mathcal{X}(X) - \sin(\theta)\,\mathcal{X}(X)\cdot\mathcal{X}(I)\right)$$
$$= \Lambda$$

$$\boxed{\begin{aligned}\mathcal{X}(I) &= I\\ \mathcal{X}(X) &= X\end{aligned}}$$

$$—\boxed{Y}— \quad \mathcal{Y}\Lambda\mathcal{Y} = \cos^2\left(\frac{\theta}{2}\right)\mathcal{Y}(I)\cdot\mathcal{Y}(I) + \sin^2\left(\frac{\theta}{2}\right)\mathcal{Y}(X)\cdot\mathcal{Y}(X)$$
$$+\frac{i}{2}\left(\sin(\theta)\,\mathcal{Y}(I)\cdot\mathcal{Y}(X) - \sin(\theta)\,\mathcal{Y}(X)\cdot\mathcal{Y}(I)\right)$$

$$\boxed{\begin{aligned}\mathcal{Y}(I) &= I\\ \mathcal{Y}(X) &= -X\end{aligned}}$$

$$= \boxed{\begin{aligned}&\cos^2\left(\frac{\theta}{2}\right) I\cdot I + \sin^2\left(\frac{\theta}{2}\right) X\cdot X\\ &+\frac{i}{2}\left(-\sin(\theta)\, I\cdot X + \sin(\theta)\, X\cdot I\right)\end{aligned}}$$

$$—\boxed{Z}— \quad \mathcal{Z}\Lambda\mathcal{Z} =$$ same

$$\boxed{\begin{aligned}\mathcal{Z}(I) &= I\\ \mathcal{Z}(X) &= -X\end{aligned}}$$

Example coherent rotation channel

$$-\boxed{P_a}-\boxed{\Lambda}-\boxed{P_a^\dagger}-$$

Example: Coherent rotation

$$\Lambda(\cdot) = U \cdot U^\dagger$$

Chi matrix

	I	X	Y	Z
I	Cos$[\frac{\theta}{2}]^2$	$\frac{1}{2}$ i Sin$[\theta]$	0	0
X	$-\frac{1}{2}$ i Sin$[\theta]$	Sin$[\frac{\theta}{2}]^2$	0	0
Y	0	0	0	0
Z	0	0	0	0

Pauli transfer matrix

	I	X	Y	Z
I	1	0	0	0
X	0	1	0	0
Y	0	0	Cos$[\theta]$	-Sin$[\theta]$
Z	0	0	Sin$[\theta]$	Cos$[\theta]$

Twirl average

$$-\boxed{I}- \quad -\boxed{X}- \quad -\boxed{Y}- \quad -\boxed{Z}-$$

$$\Lambda \mapsto \frac{1}{4}\left(\mathcal{I}\Lambda\mathcal{I} + \mathcal{X}\Lambda\mathcal{X} + \mathcal{Y}\Lambda\mathcal{Y} + \mathcal{Z}\Lambda\mathcal{Z}\right)$$

Twirl

χ =

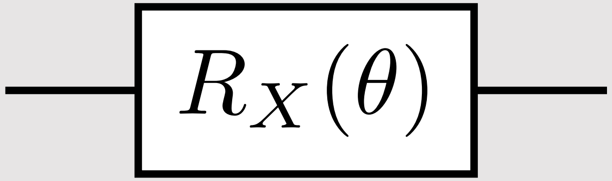
	I	X	Y	Z
I	Cos$[\frac{\theta}{2}]^2$	0	0	0
X	0	Sin$[\frac{\theta}{2}]^2$	0	0
Y	0	0	0	0
Z	0	0	0	0

Diagonal

PTM =

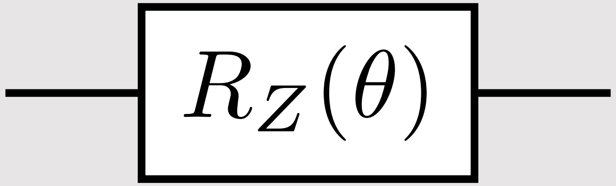
	I	X	Y	Z
I	1	0	0	0
X	0	1	0	0
Y	0	0	Cos$[\theta]$	0
Z	0	0	0	Cos$[\theta]$

Example PTM for coherent noise



$$\mathrm{PTM}\left[R_X\left(\theta\right)\right] = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \end{array}$$

Note, for other gates, permute indices
Same story

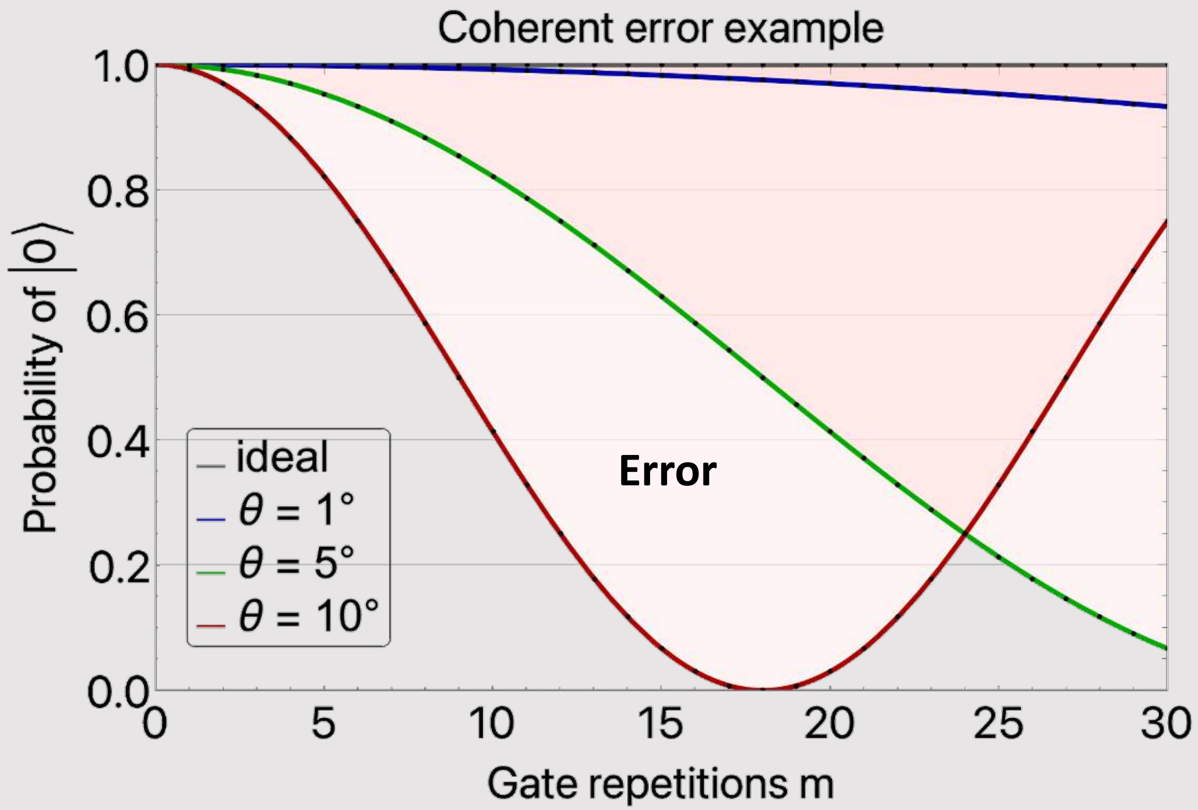


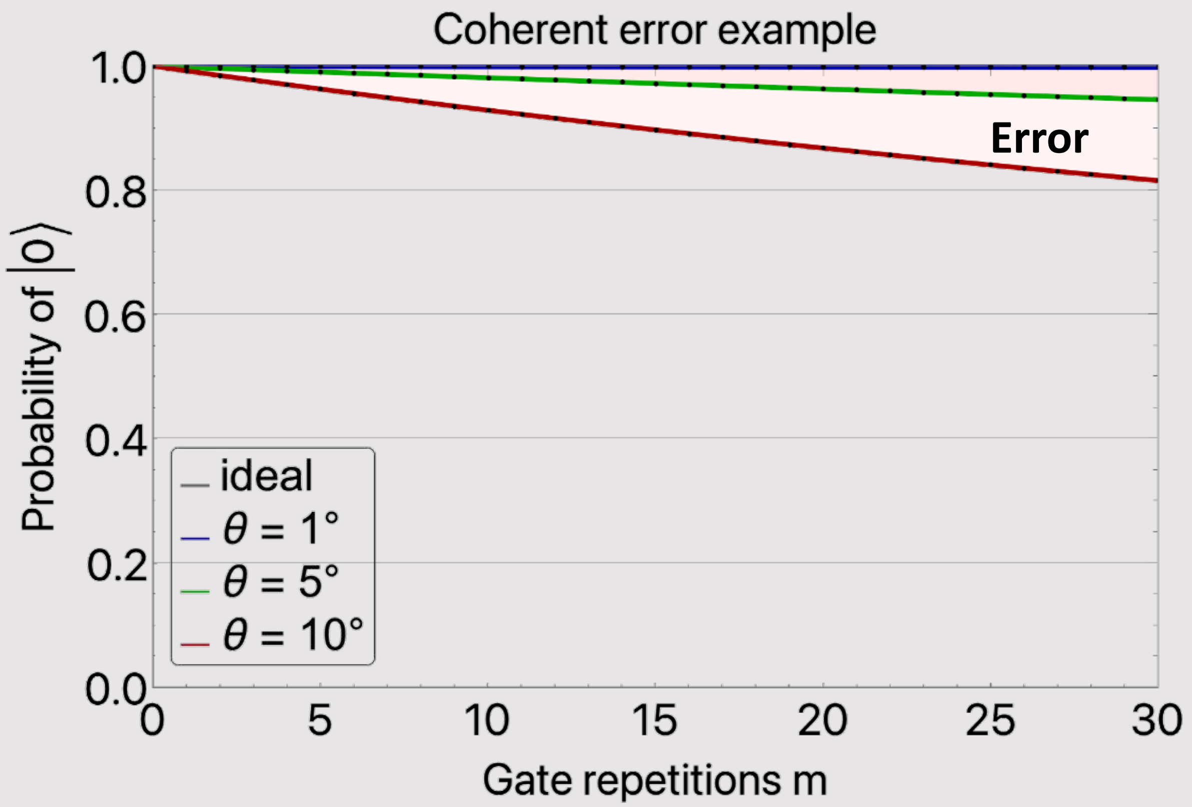
$$\mathrm{PTM}\left[R_Z\left(\theta\right)\right] = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example: Coherent over rotation

Suppose we meant to do an identity gate, but instead had a small X over rotation of angle theta

$$\left(-\boxed{R_X(\theta)}- \right)^m \qquad \left(-\boxed{P_{ai}}-\boxed{R_X(\theta)}-\boxed{P_{ai}^c}- \right)^m$$



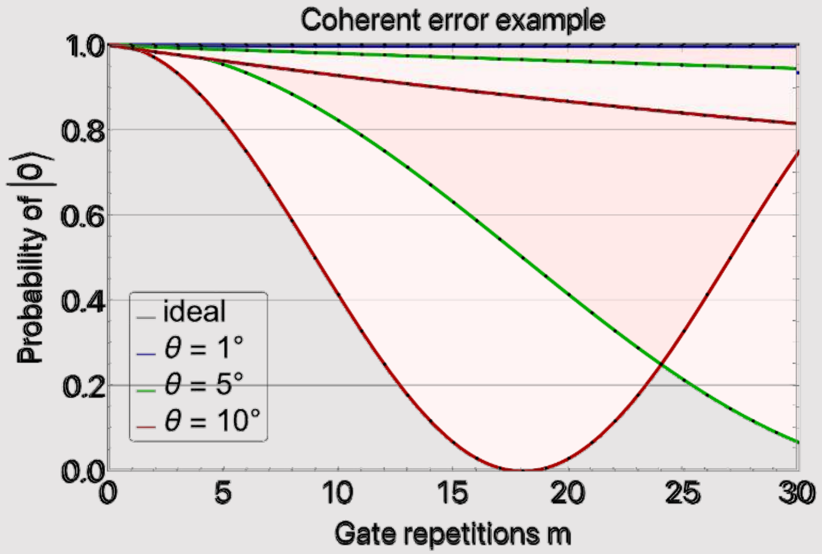


Example: Coherent over rotation

Suppose we meant to do an identity gate, but instead had a small X over rotation of angle theta

$$-\boxed{R_X(\theta)}-$$

$$\left(-\boxed{R_X(\theta)}-\right)^m$$



PTM

$$\left(\hat{R}\right) = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos(\theta) & -\sin(\theta) \\ & & \sin(\theta) & \cos(\theta) \end{pmatrix} \end{array}$$

$$\left(\hat{R}^m\right) = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos(m\theta) & -\sin(m\theta) \\ & & \sin(m\theta) & \cos(m\theta) \end{pmatrix} \end{array}$$

$$-\boxed{P_a}-\boxed{R_X(\theta)}-\boxed{P_a^c}-$$

$$\left(-\boxed{P_{ai}}-\boxed{R_X(\theta)}-\boxed{P_{ai}^c}-\right)^m$$

$$\left(\mathcal{T}\hat{R}\right) = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos(\theta) & \\ & & & \cos(\theta) \end{pmatrix} \end{array}$$

$$\left(\left[\mathcal{T}\hat{R}\right]^m\right) = \begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & [\cos(\theta)]^m & \\ & & & [\cos(\theta)]^m \end{pmatrix} \end{array}$$

$$\langle Z\rangle_{\text{noisy}} - \langle Z\rangle_{\text{ideal}} =$$

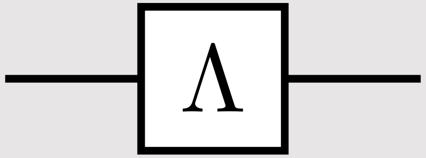
Coherent error - quadratic

$$\cos(n\theta) - 1 \approx \boxed{-\frac{n^2\theta^2}{2}} + \mathcal{O}\left(\theta^4\right)$$

Twirl error - linear

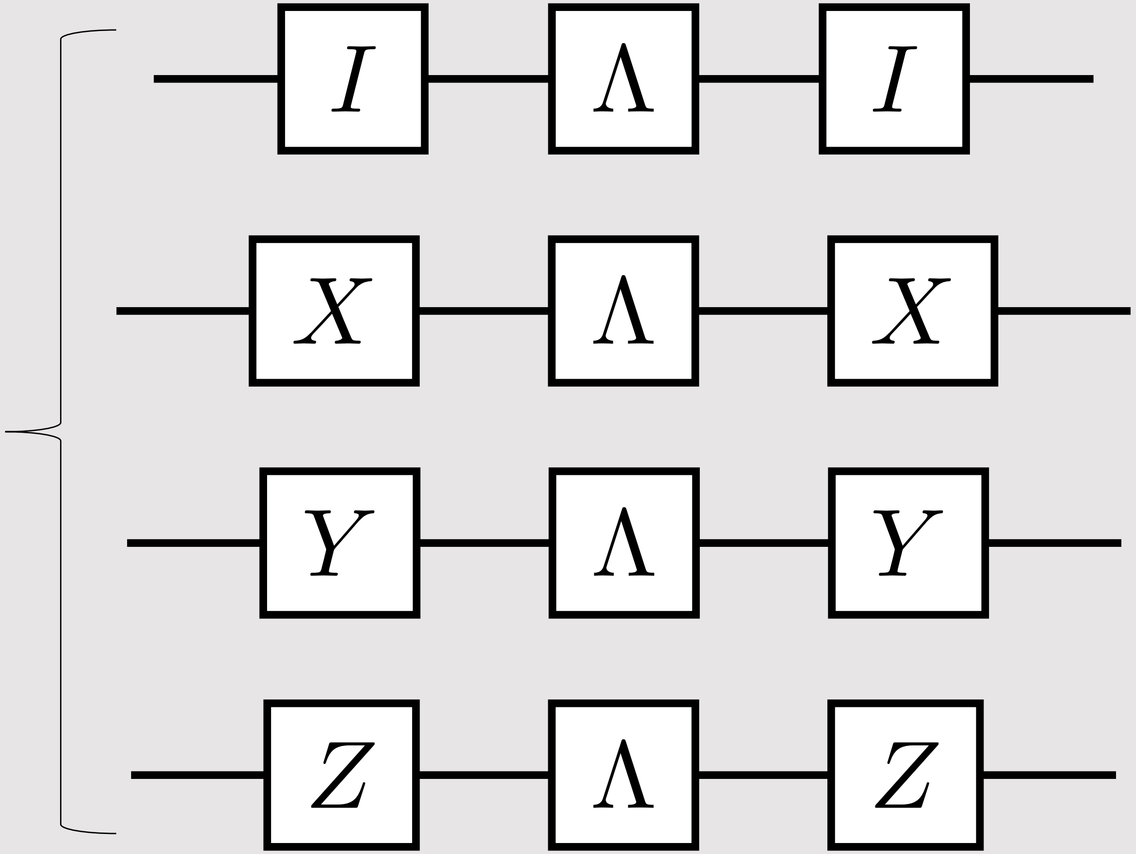
$$[\cos(\theta)]^m - 1 \approx \boxed{-\frac{n\theta^2}{2}} + \mathcal{O}\left(\theta^4\right)$$

Twirl general single qubit channel

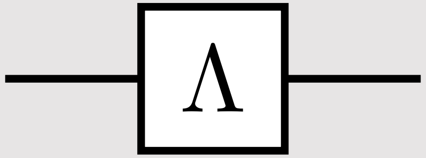


In terms of PTM

	I	X	Y	Z
I	f_{II}	f_{IX}	f_{IY}	f_{IZ}
X	f_{XI}	f_{XX}	f_{XY}	f_{XZ}
Y	f_{YI}	f_{YX}	f_{YY}	f_{YZ}
Z	f_{ZI}	f_{ZX}	f_{ZY}	f_{ZZ}

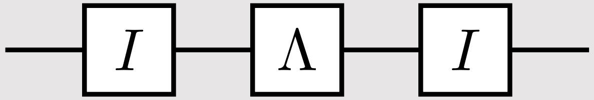


Twirl general single qubit channel



In terms of PTM

	I	X	Y	Z
I	f_{II}	f_{IX}	f_{IY}	f_{IZ}
X	f_{XI}	f_{XX}	f_{XY}	f_{XZ}
Y	f_{YI}	f_{YX}	f_{YY}	f_{YZ}
Z	f_{ZI}	f_{ZX}	f_{ZY}	f_{ZZ}



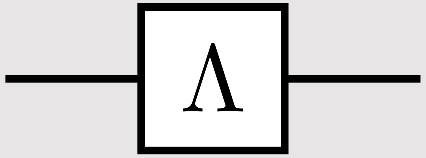
	I	X	Y	Z
I	f_{II}	f_{IX}	f_{IY}	f_{IZ}
X	f_{XI}	f_{XX}	f_{XY}	f_{XZ}
Y	f_{YI}	f_{YX}	f_{YY}	f_{YZ}
Z	f_{ZI}	f_{ZX}	f_{ZY}	f_{ZZ}

	I	X	Y	Z
I	f_{II}	f_{IX}	$-f_{IY}$	$-f_{IZ}$
X	f_{XI}	f_{XX}	$-f_{XY}$	$-f_{XZ}$
Y	$-f_{YI}$	$-f_{YX}$	f_{YY}	f_{YZ}
Z	$-f_{ZI}$	$-f_{ZX}$	f_{ZY}	f_{ZZ}

	I	X	Y	Z
I	f_{II}	$-f_{IX}$	f_{IY}	$-f_{IZ}$
X	$-f_{XI}$	f_{XX}	$-f_{XY}$	f_{XZ}
Y	f_{YI}	$-f_{YX}$	f_{YY}	$-f_{YZ}$
Z	$-f_{ZI}$	f_{ZX}	$-f_{ZY}$	f_{ZZ}

	I	X	Y	Z
I	f_{II}	$-f_{IX}$	$-f_{IY}$	f_{IZ}
X	$-f_{XI}$	f_{XX}	f_{XY}	$-f_{XZ}$
Y	$-f_{YI}$	f_{YX}	f_{YY}	$-f_{YZ}$
Z	f_{ZI}	$-f_{ZX}$	$-f_{ZY}$	f_{ZZ}

Twirl general single qubit channel



In terms of PTM

	I	X	Y	Z
I	f_{II}	f_{IX}	f_{IY}	f_{IZ}
X	f_{XI}	f_{XX}	f_{XY}	f_{XZ}
Y	f_{YI}	f_{YX}	f_{YY}	f_{YZ}
Z	f_{ZI}	f_{ZX}	f_{ZY}	f_{ZZ}

Average

	I	X	Y	Z
I	f_{II}	0	0	0
X	0	f_{XX}	0	0
Y	0	0	f_{YY}	0
Z	0	0	0	f_{ZZ}

* pikisuperstar

Refresher

More general

Pauli gates & mixed states