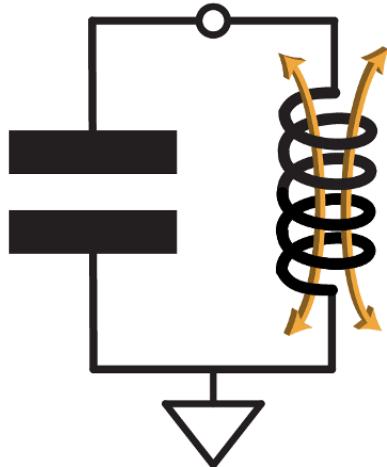


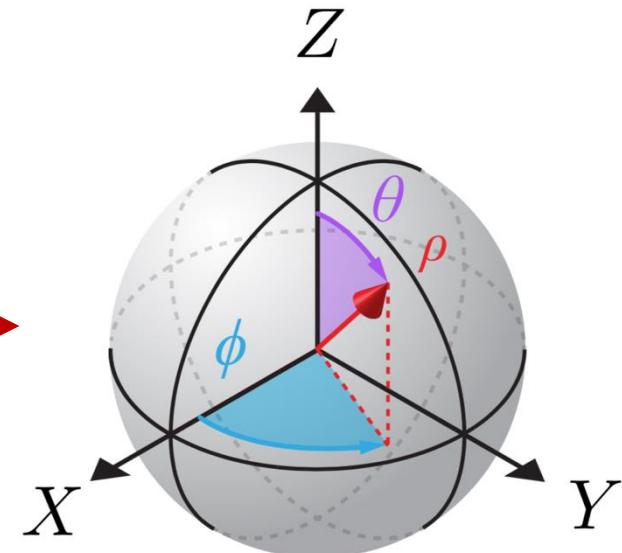
Superconducting Qubits I:

Making Your First Qubit
From an Oscillator



Introduction to Circuit
Quantum Electrodynamics (cQED)

Zlatko K. Minev



IBM Quantum
IBM T.J. Watson Research Center, Yorktown Heights, NY



@zlatko_minev



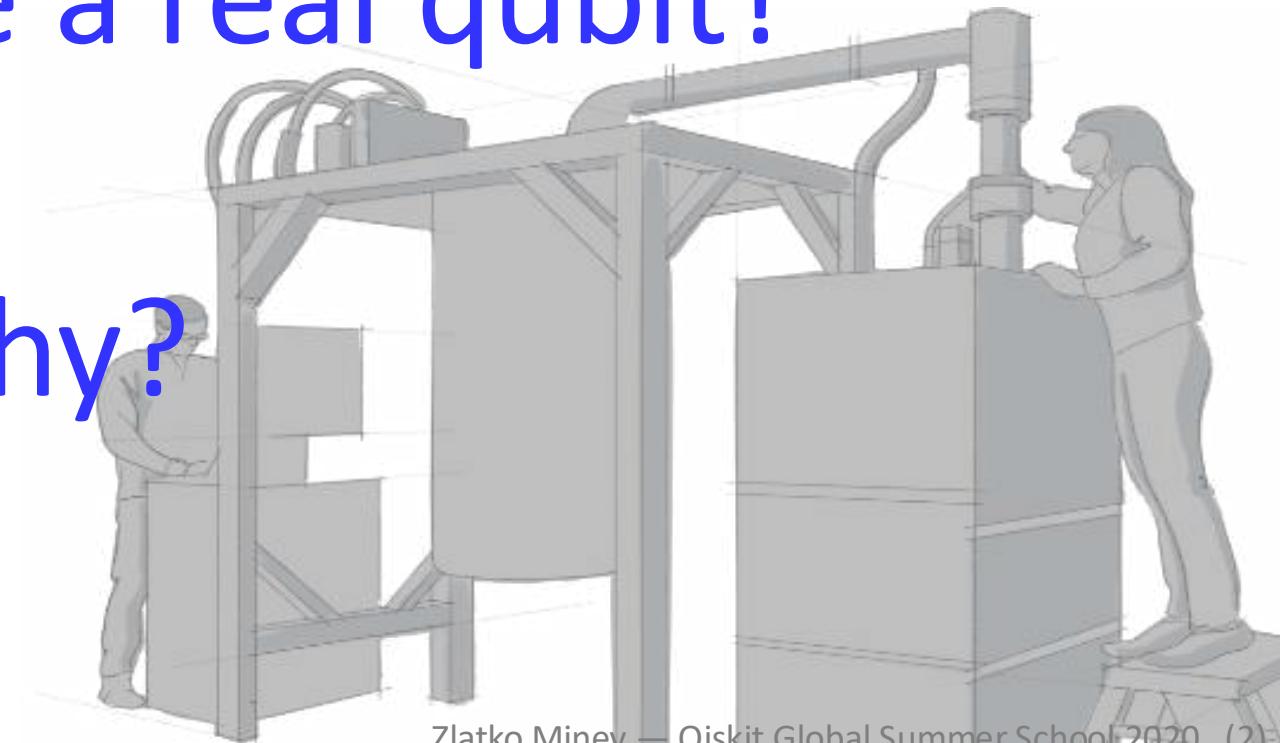
zlatko-minev.com

Image copyright:
ZKM unless otherwise noted

What is a real qubit?

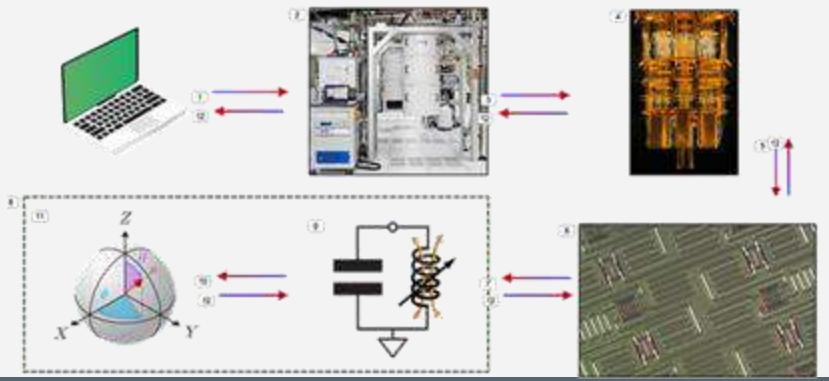
How can you design, control,
and measure a real qubit?

Why?

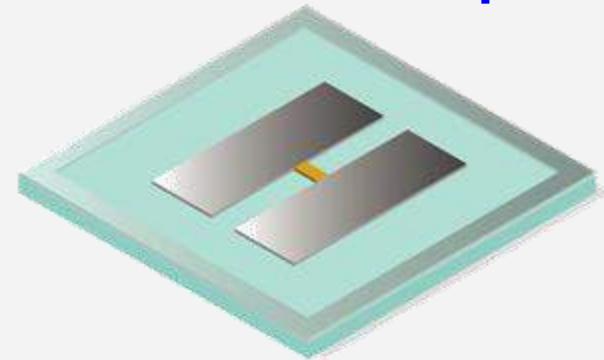


On the road ahead

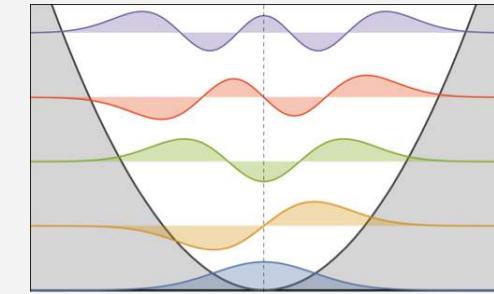
Qubit in the cloud



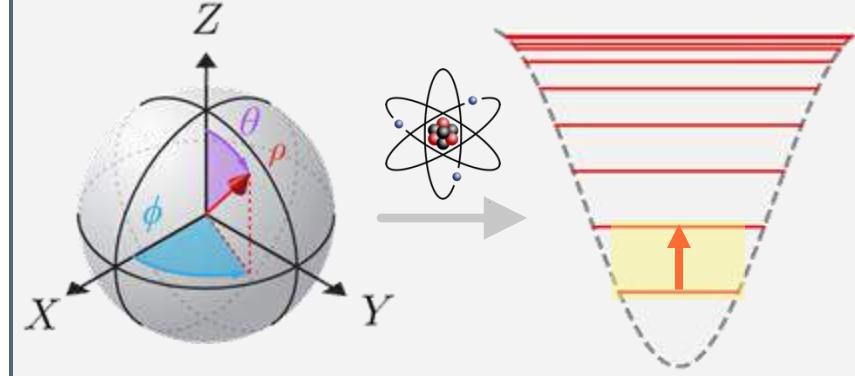
cQED: Transmon qubit



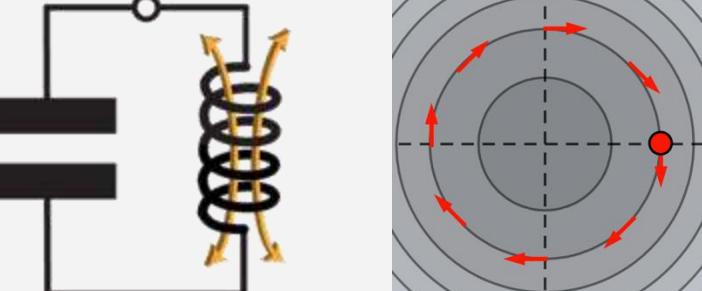
Unveiling the quantum oscillator



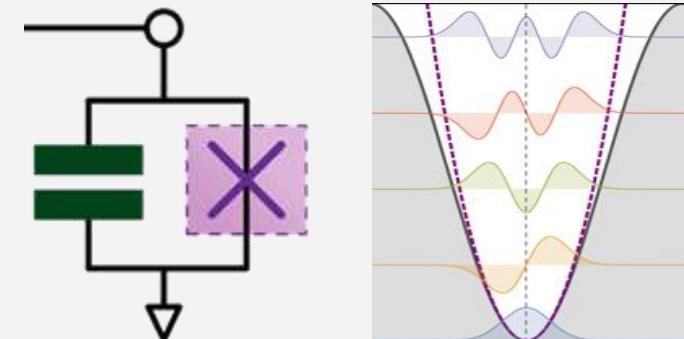
Qubit from atom / oscillator



Classical circuits & the LC



Transmon qubit



This Lecture



Introductory and skill reaffirming

Don't need to know much going in, but we will go far



Advanced material

Examples: simplest, most practical examples

Step by step

Ask questions!



Tightly integrated lab work by Dr. Nick Bronn and Co.! A circular photograph of a man with a beard and glasses, wearing a dark shirt. He is holding a clear glass beaker and looking down at it. In the background, there is a complex piece of laboratory equipment with many glass tubes and containers.

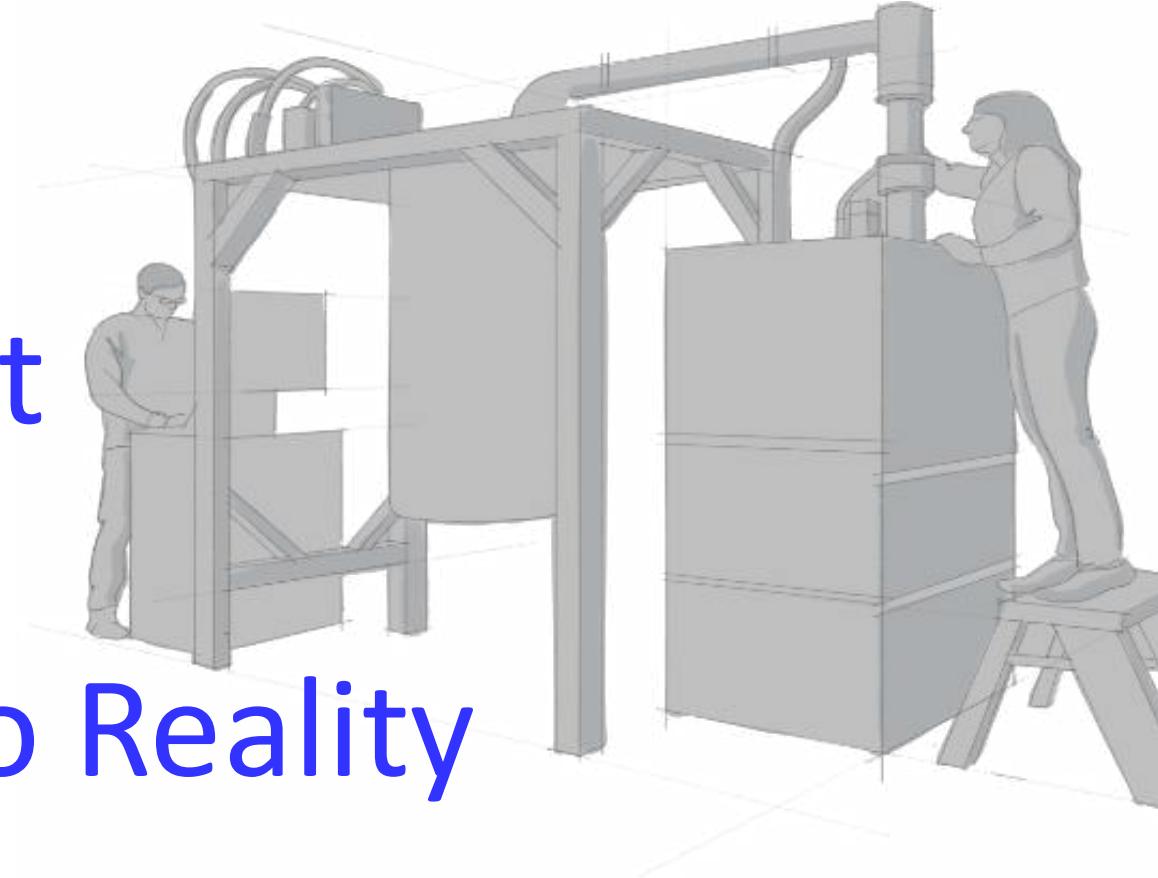
Avoid firehose of information



Thanks to Fred Moxley for gif reference.

Qubit

From Idea to Reality



THE BIG PICTURE
before calculations

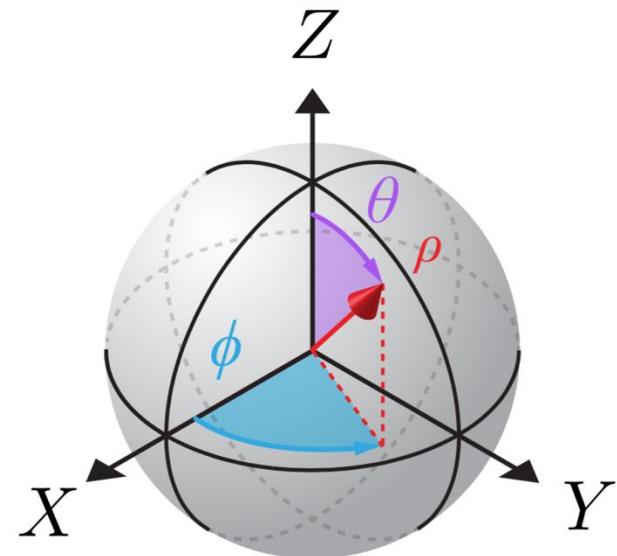
Qubit: idea

Energy levels

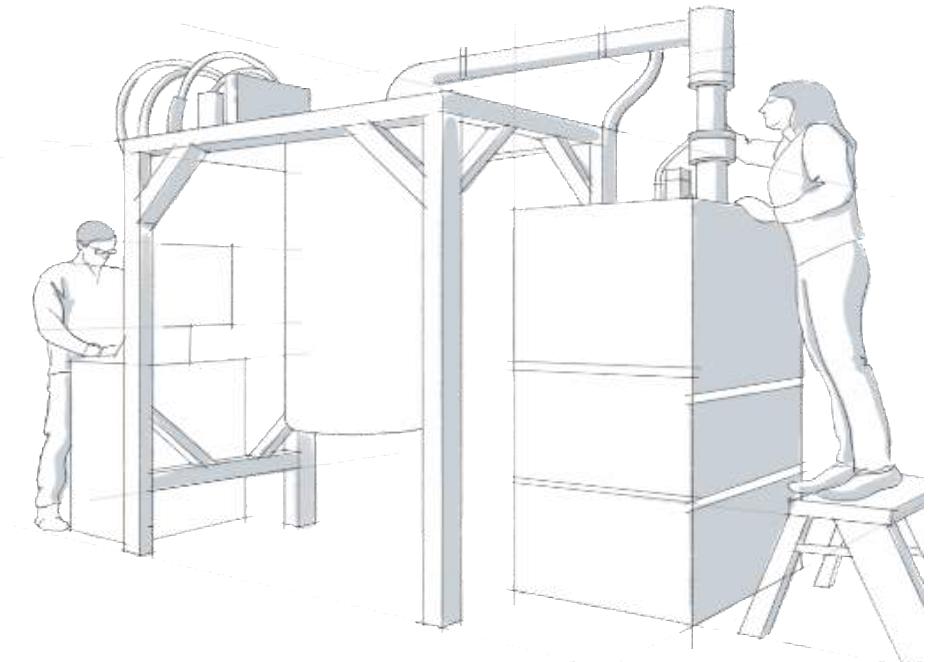
 $|1\rangle$

 $|0\rangle$

Hilbert space



Quantum cloud?



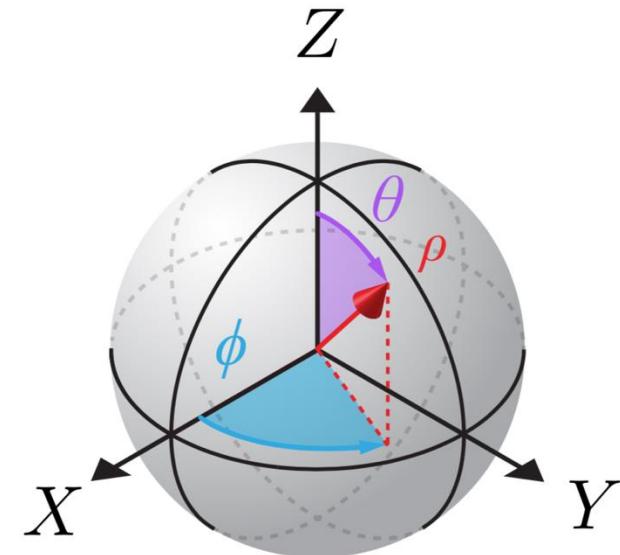
Qubit: idea and reality

Energy levels

 $|1\rangle$

 $|0\rangle$

Hilbert space



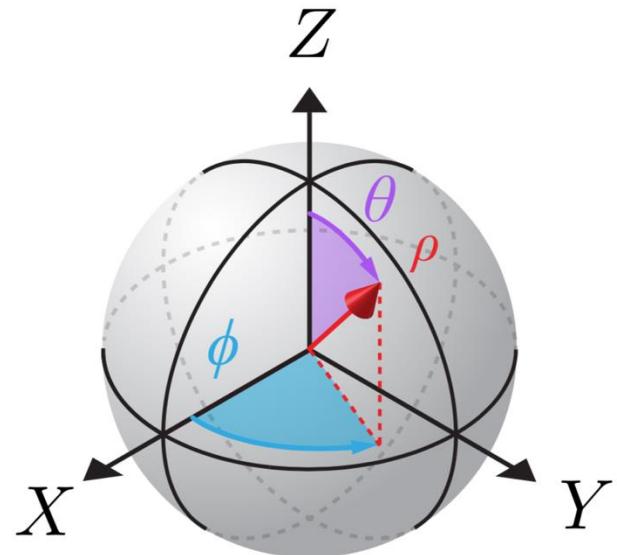
Qubit: idea and reality

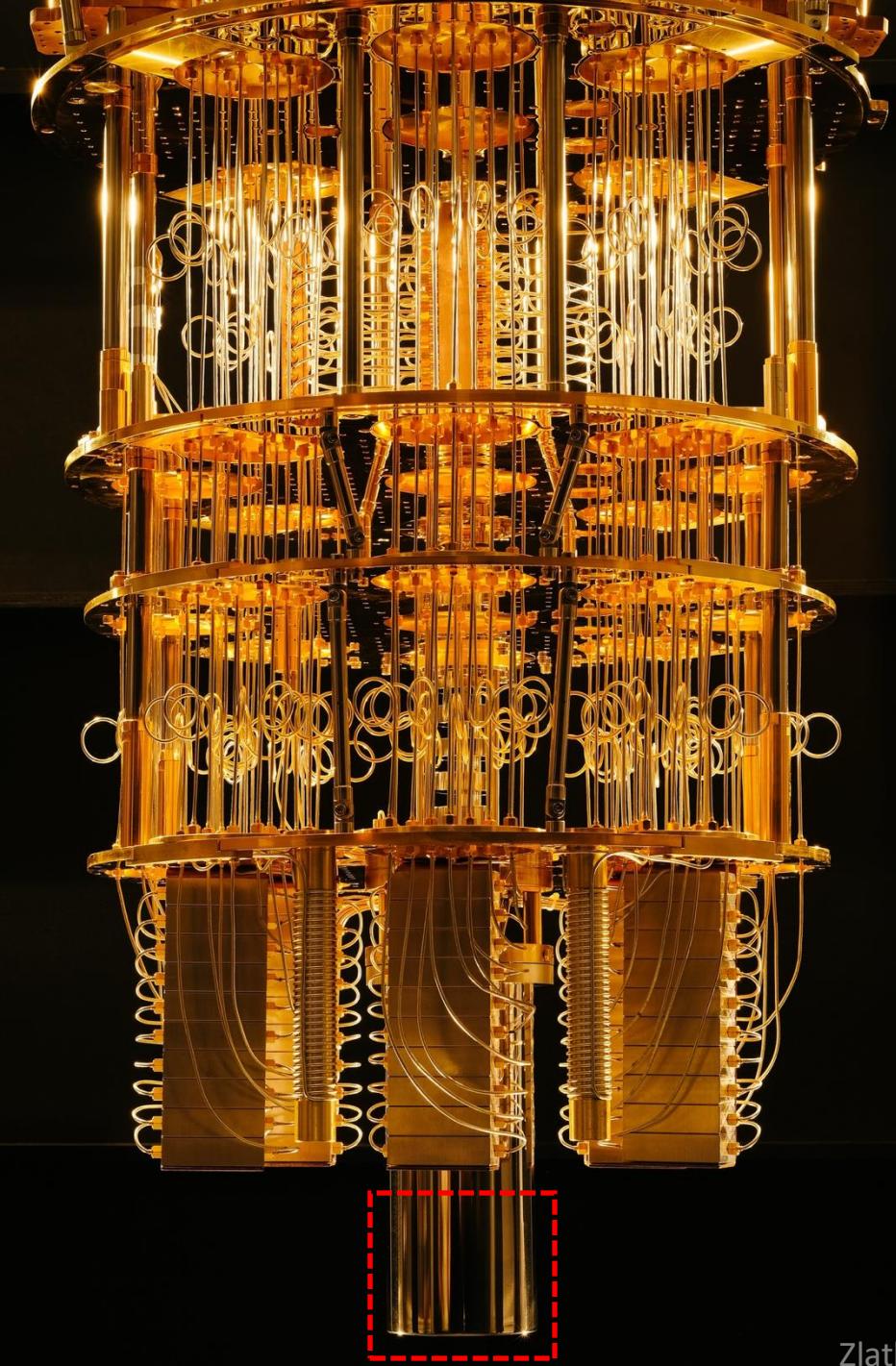
Energy levels

 $|1\rangle$

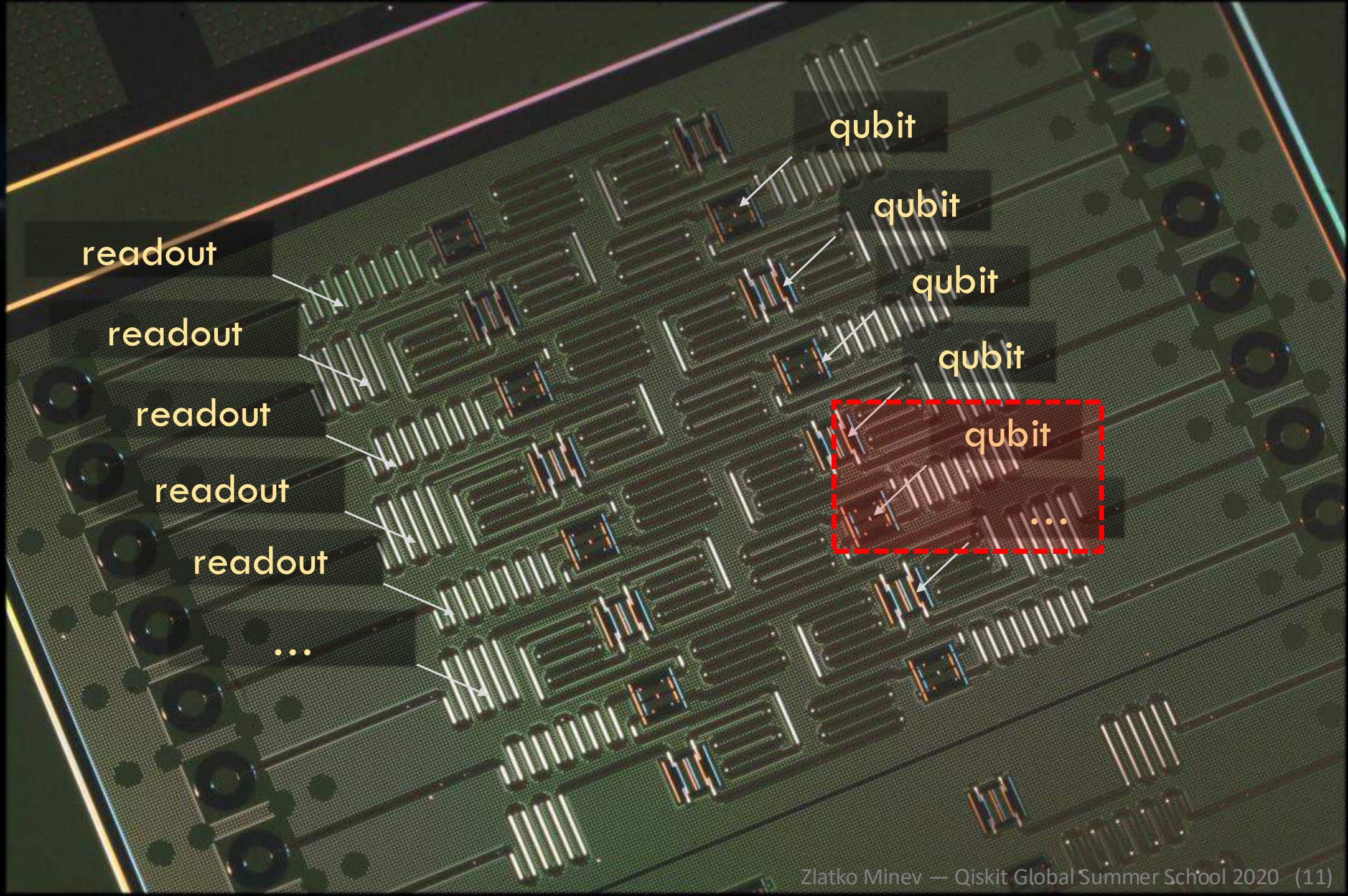
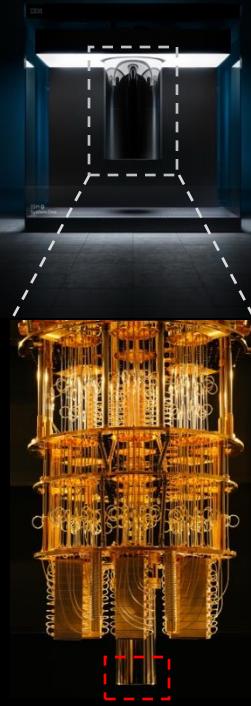
 $|0\rangle$

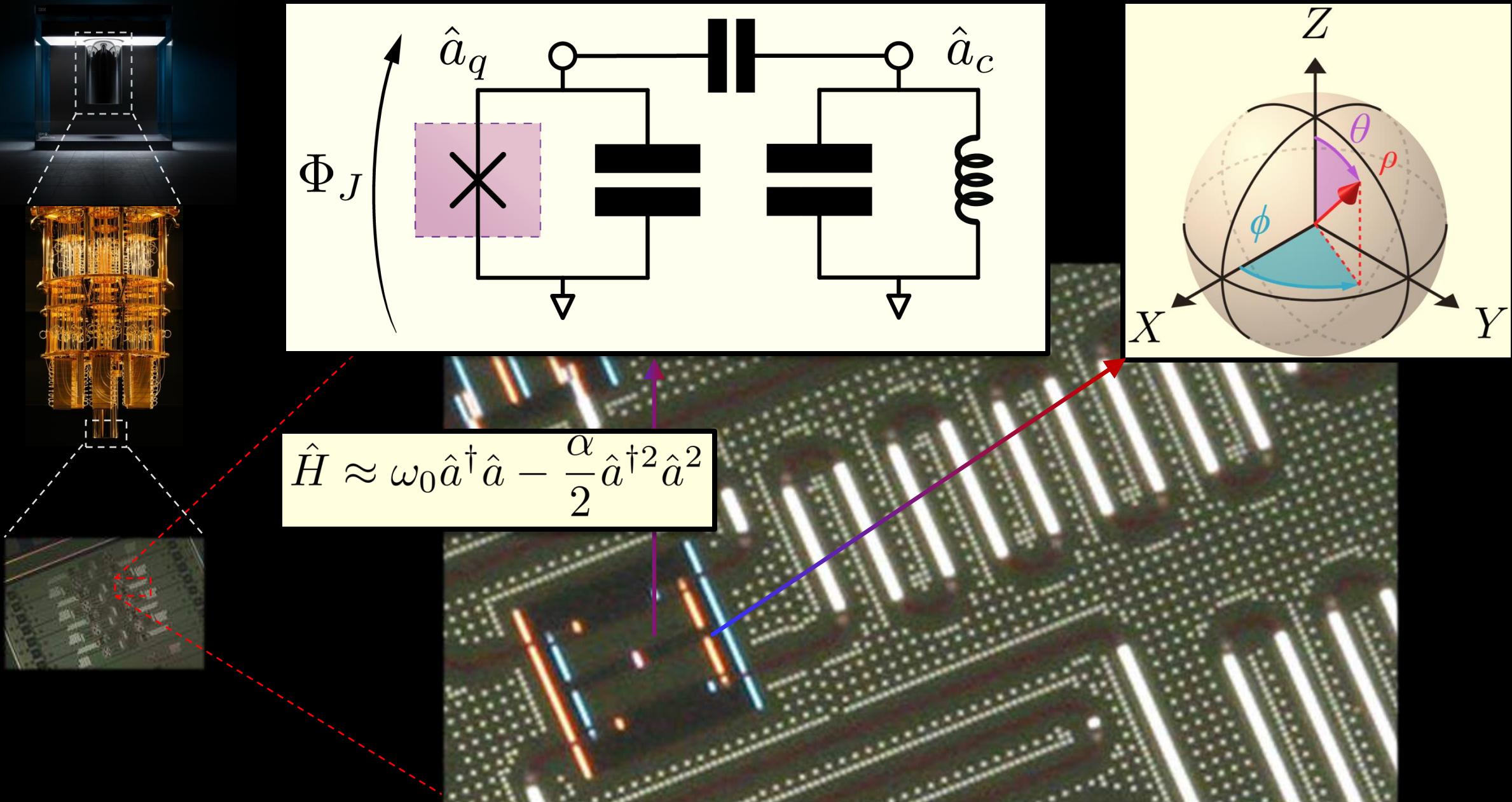
Hilbert space



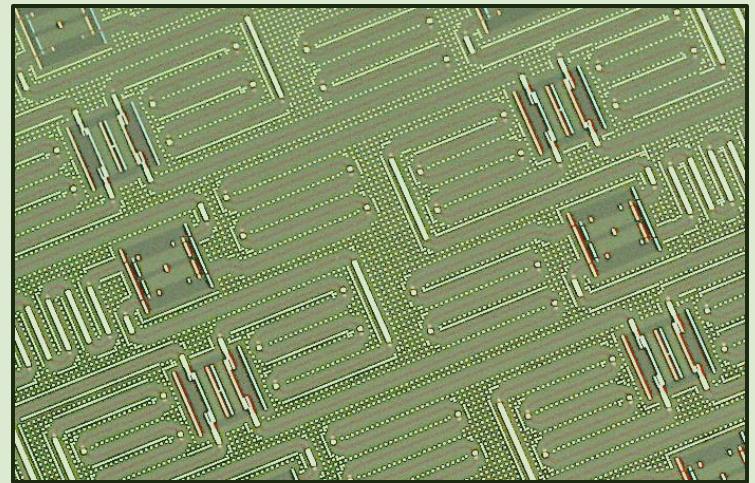
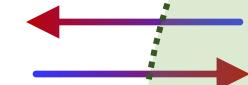
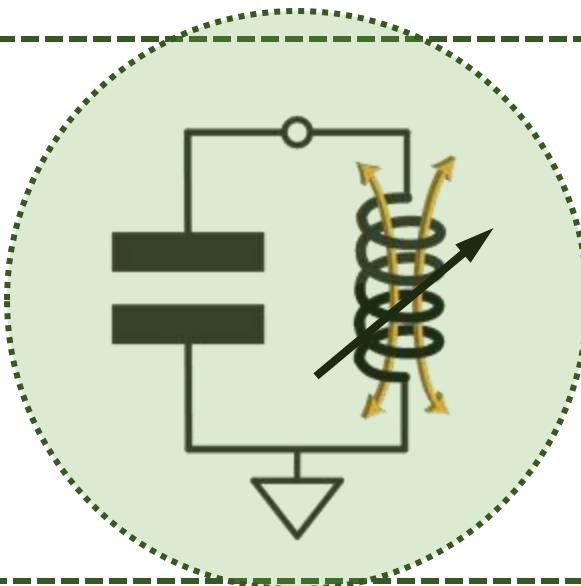
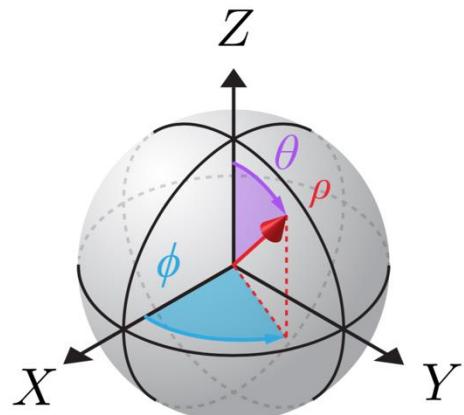
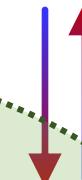
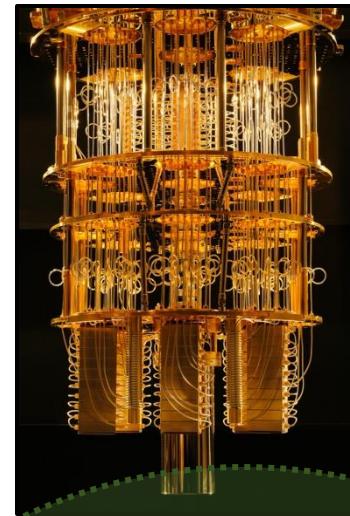


Operation at
15 mK (-273.13 °C)





cQED qubit in the cloud: Summary of flow



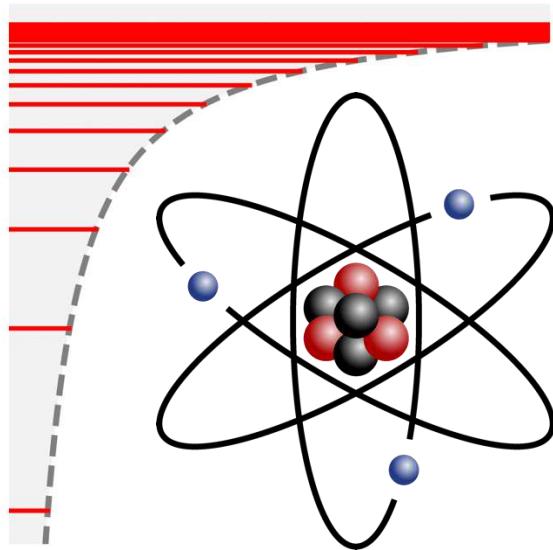
Qubit

From Idea to Reality

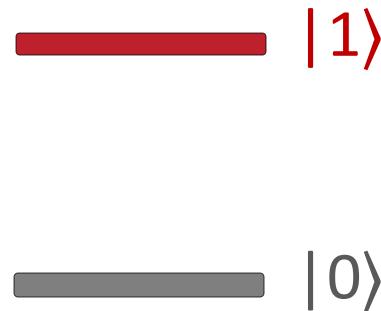
Concepts

From qubit representation to reality

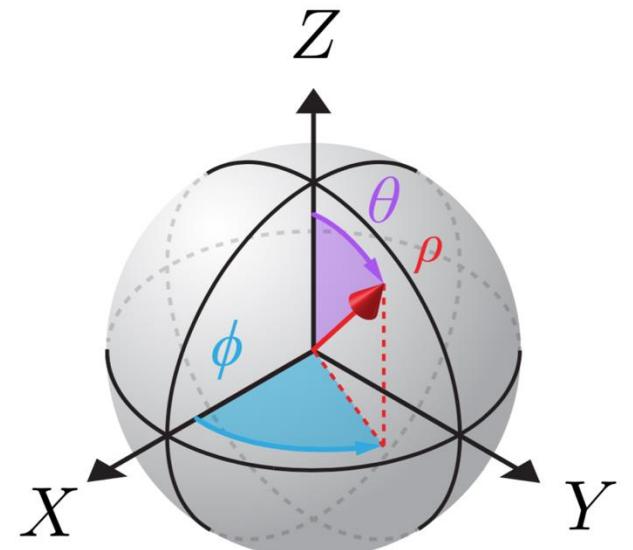
Realization



Energy levels



Hilbert space*



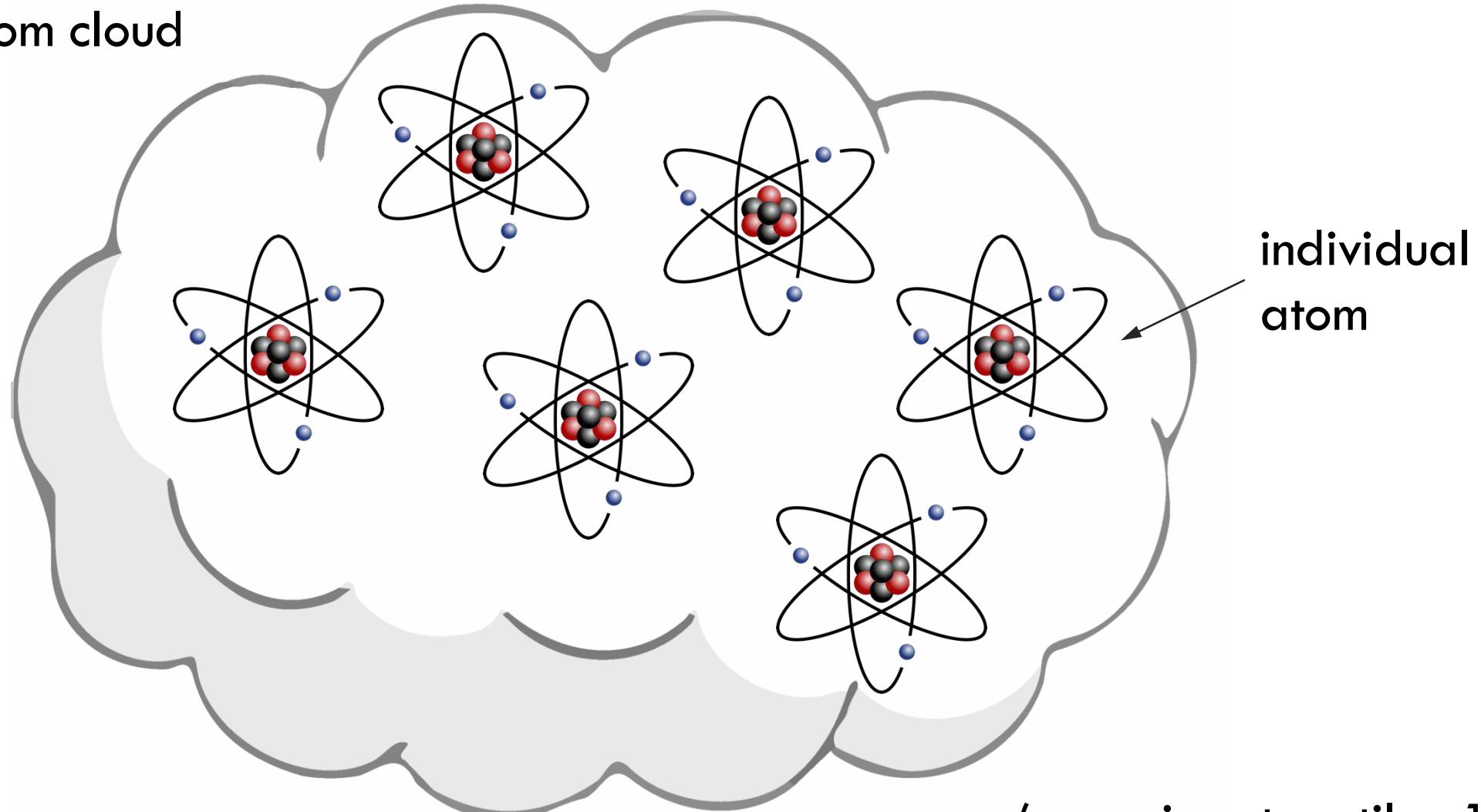
* Bloch sphere is a mere geometrical representation of $SO(3)$, but the density matrix ρ is in $SU(2)$, a double cover of $SO(3)$.

* A density matrix operator lives not in the Hilbert space H but in the Liouville space $H \otimes H$.

Images: Minev, arXiv:1902.10355; atom art: Indoleces.

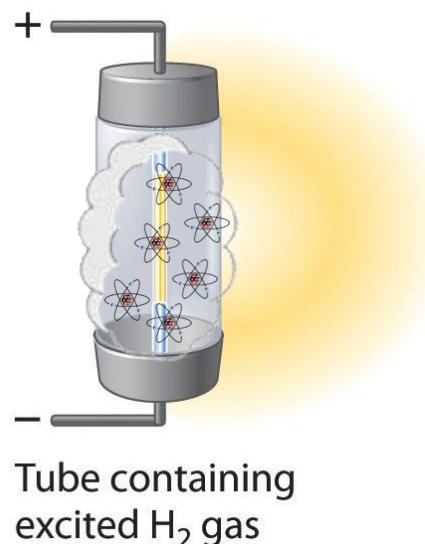
Origins of quantum

e.g., atom cloud

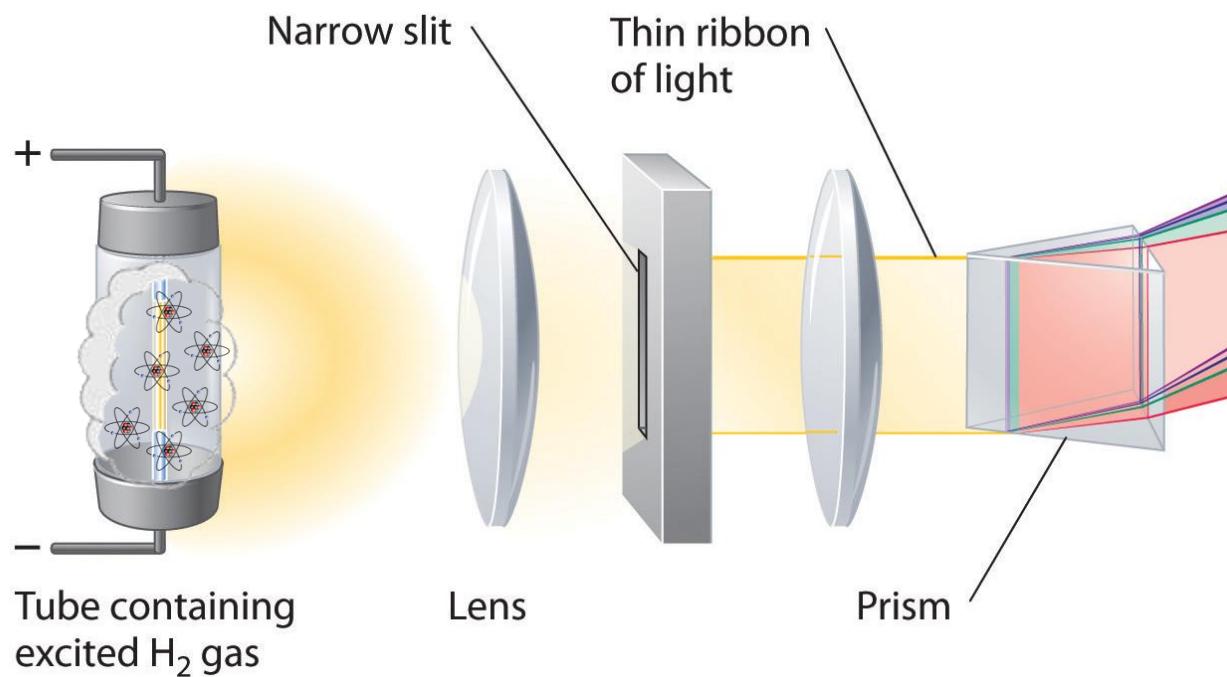


(experiments until c. 1980)

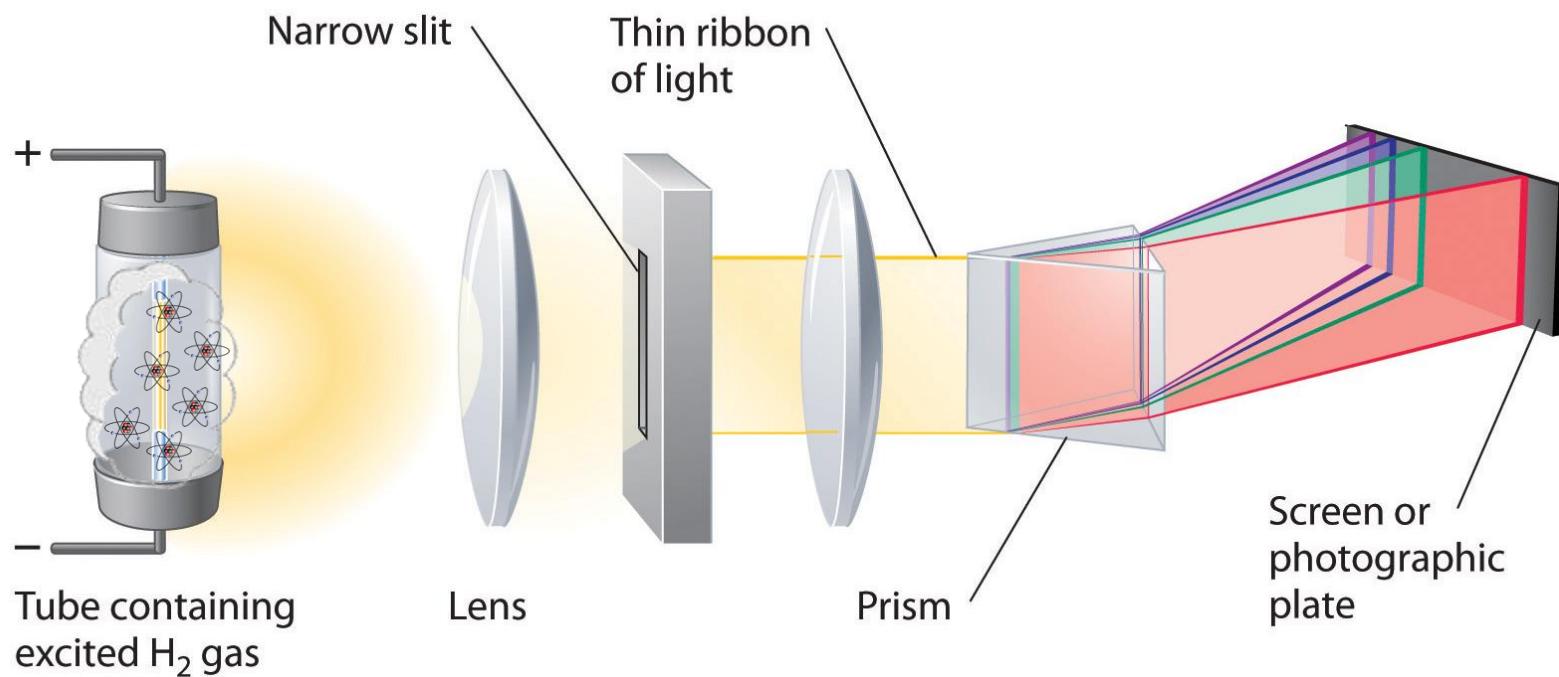
Atomic emission of light



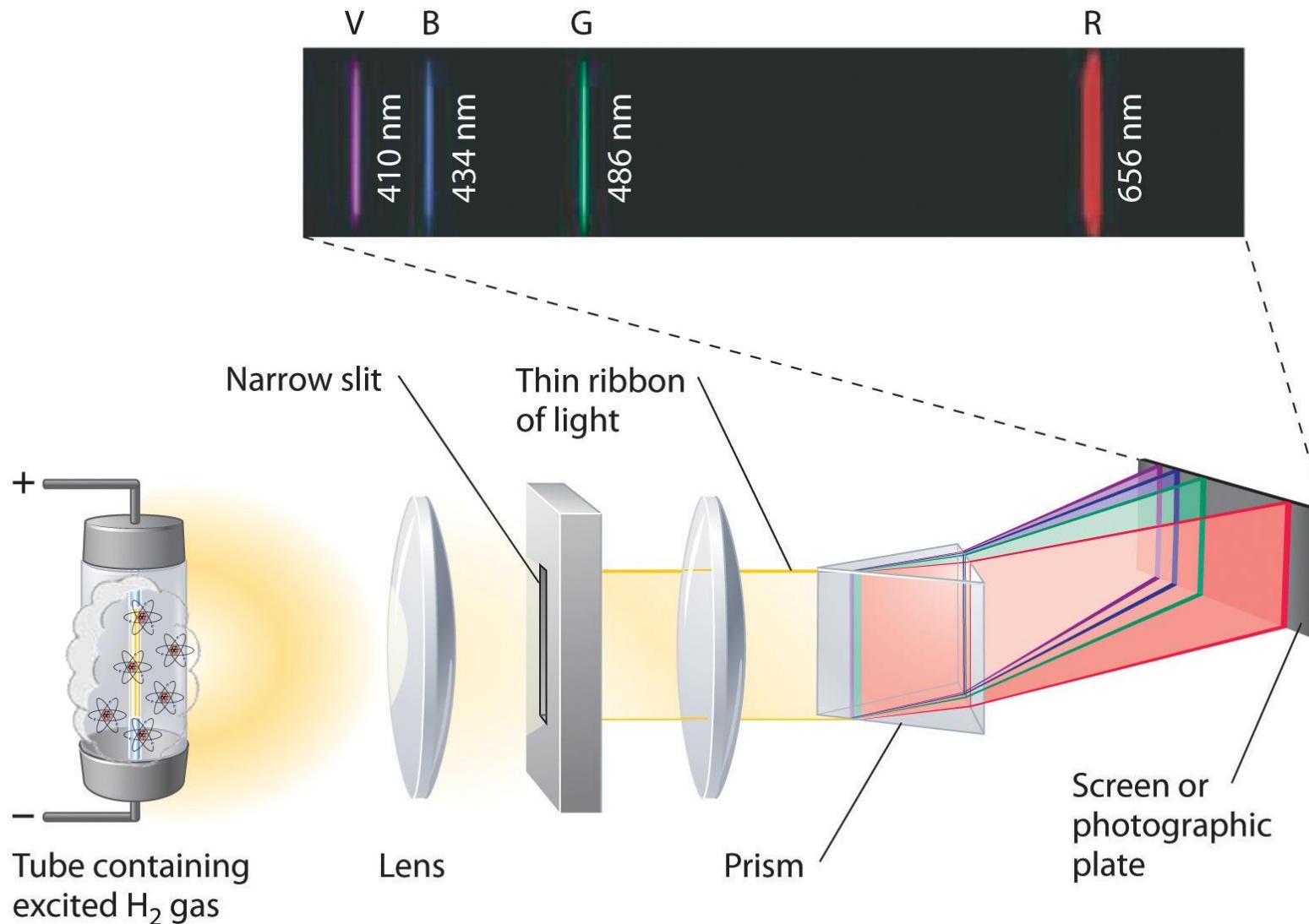
Atomic emission of light



Atomic emission of light



Atomic emission of light



Quantized levels

Atomic emission spectra

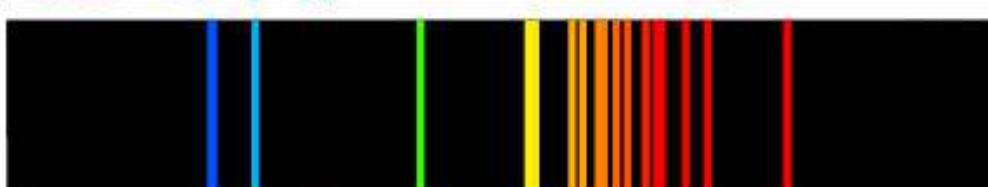
Hydrogen



Helium



Neon



Sodium

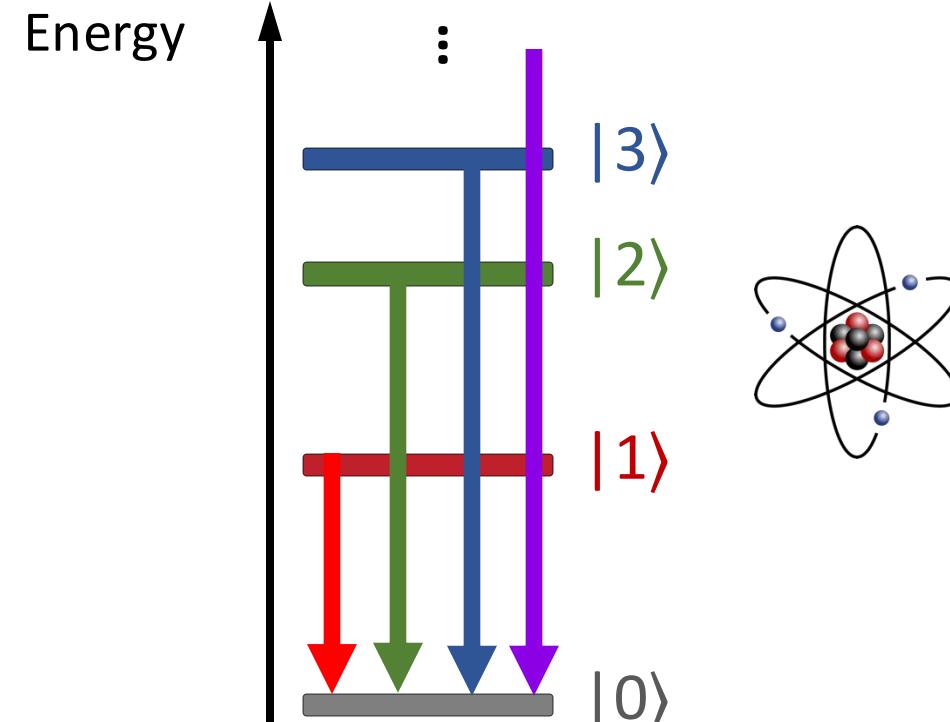


Mercury



Image credit: NMSU, N. Vogt

**Atoms are quantum:
discrete intrinsic energy levels***



Quantization of energy

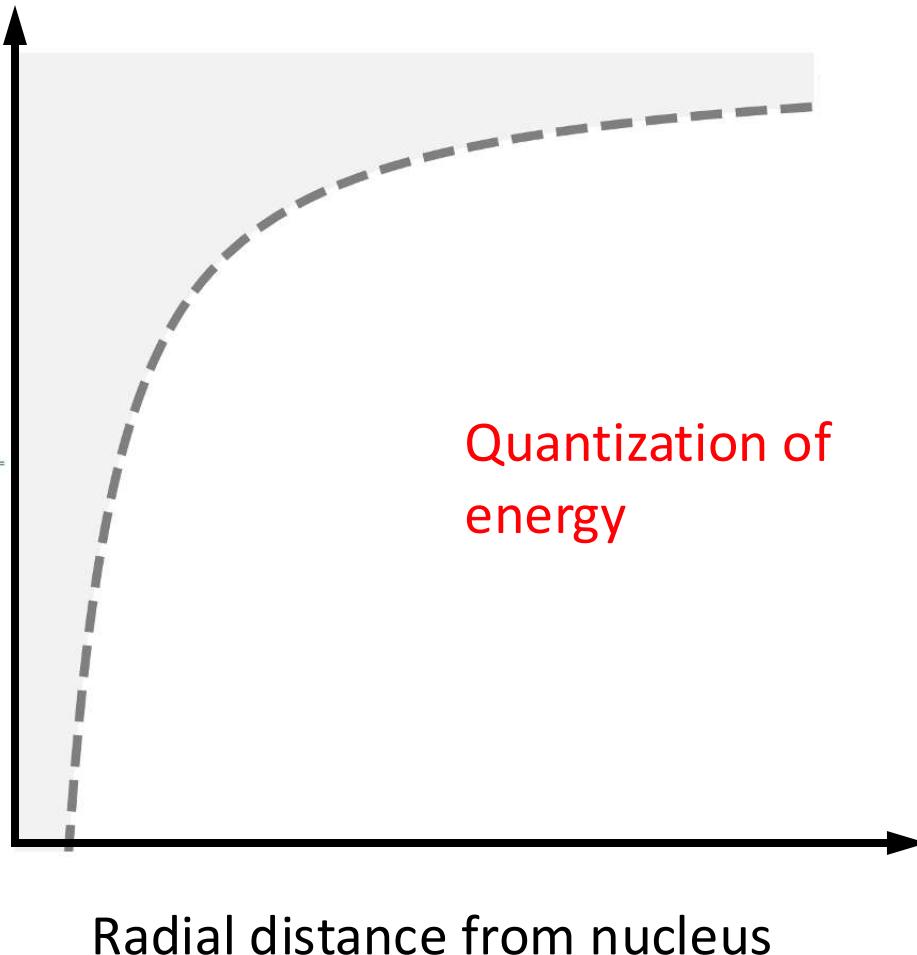
* The notion of an energy level was proposed by Bohr in 1913.
Zlatko Minev — Qiskit Global Summer School 2020 (21)

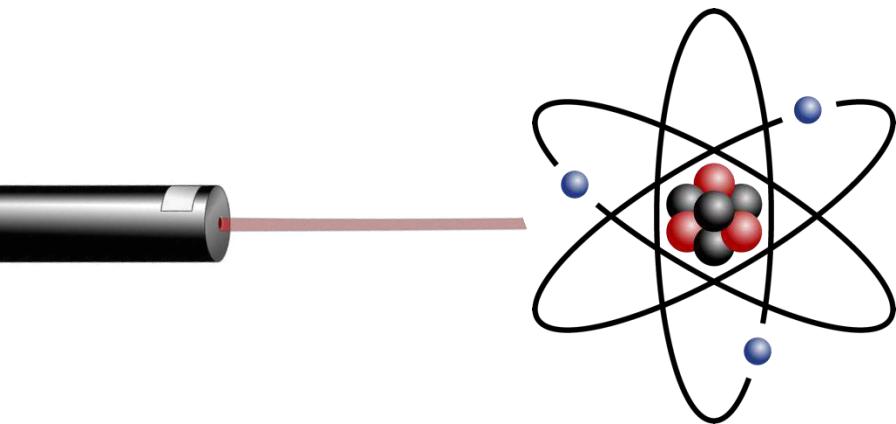
The light of atoms



Atomic energy levels and transitions

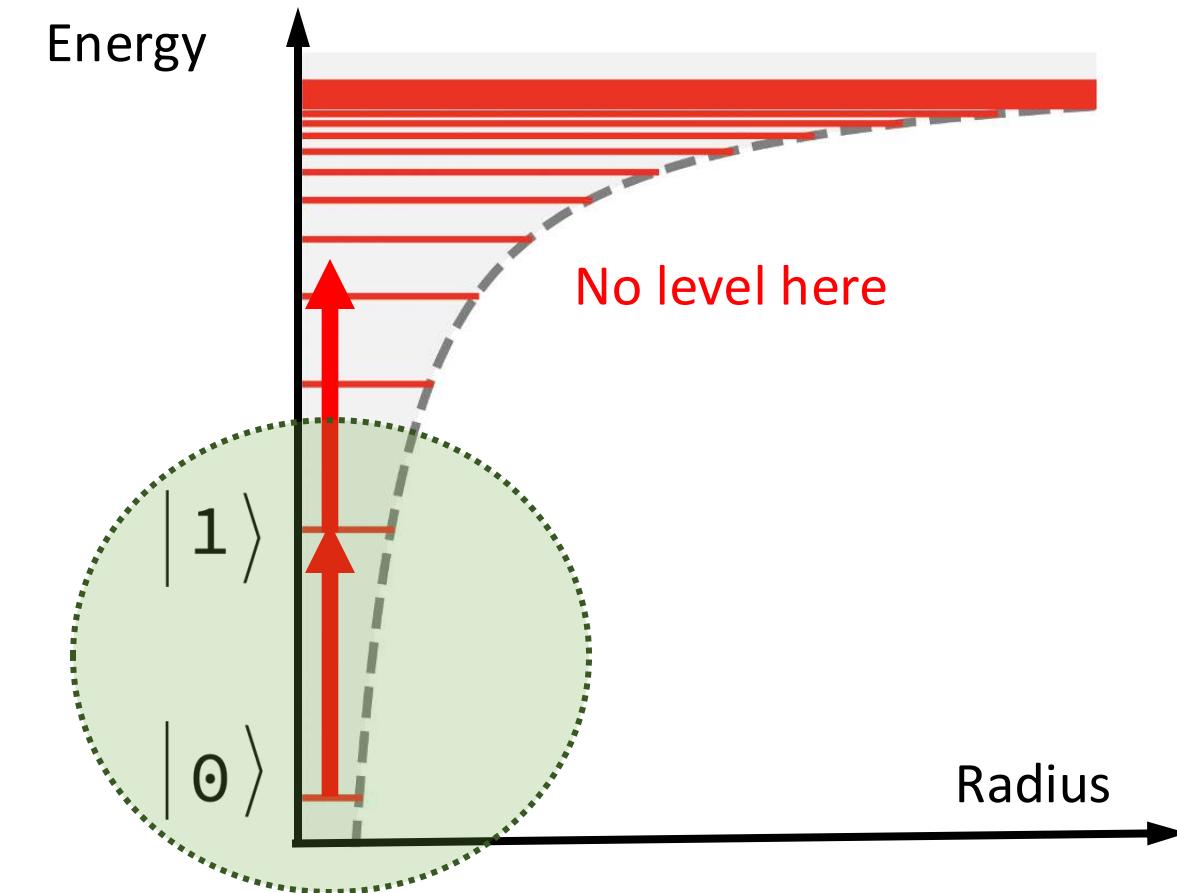
Electron potential-energy landscape





Qubit from Atom

Anharmonic degree of freedom and spin



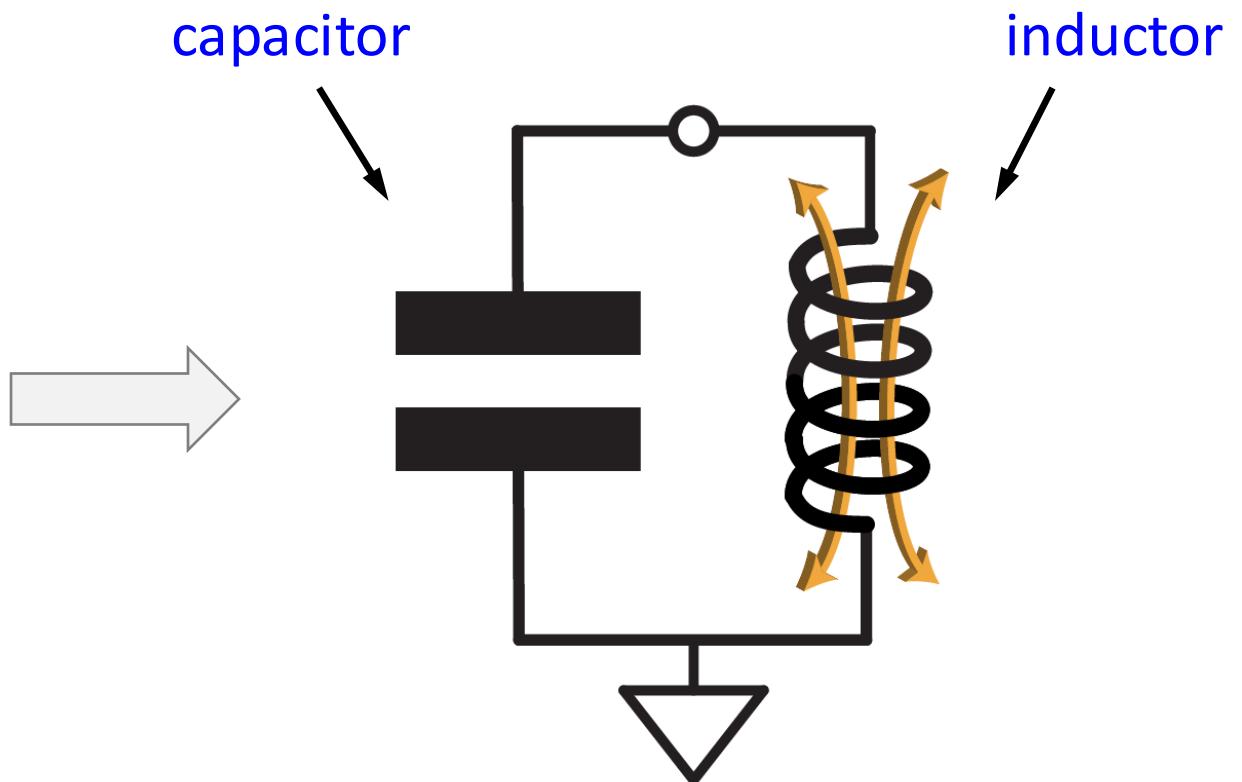
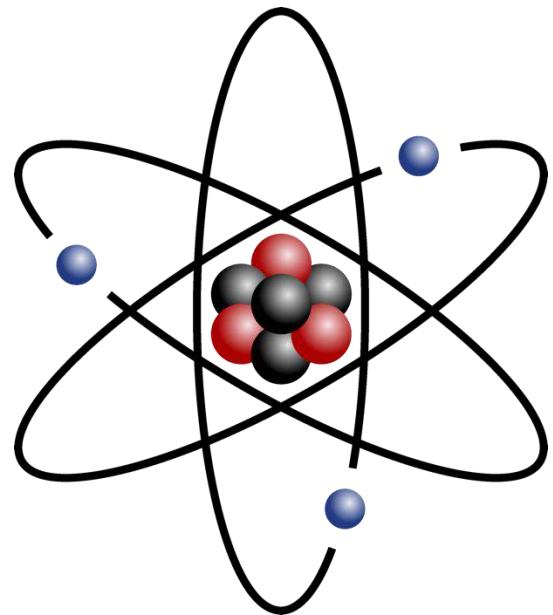
Isolated from environment and thermal bath

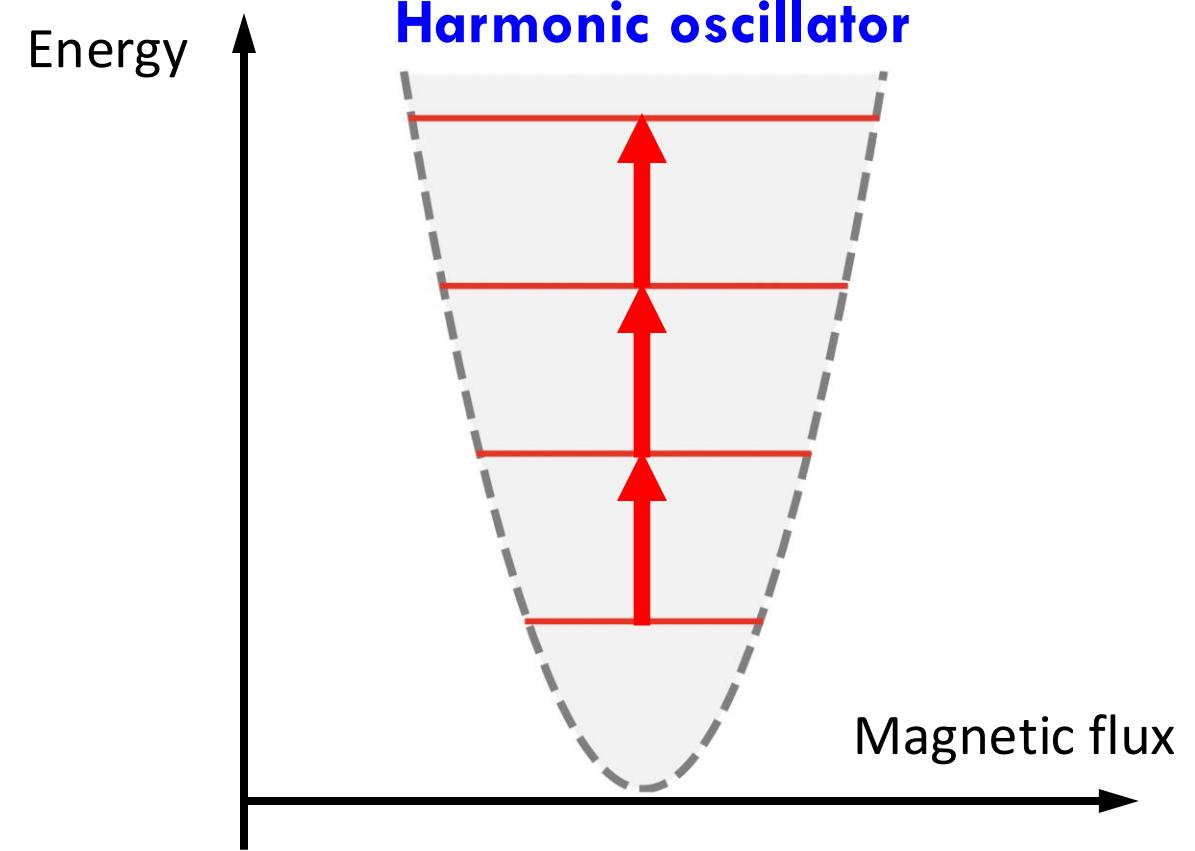
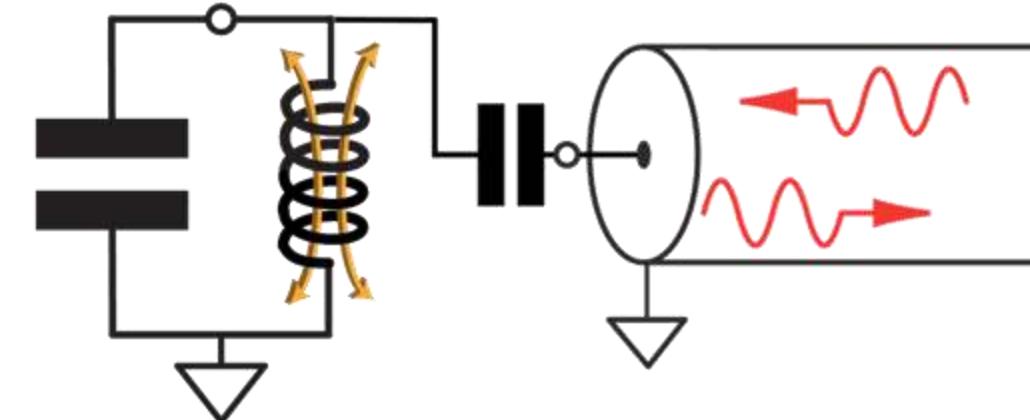
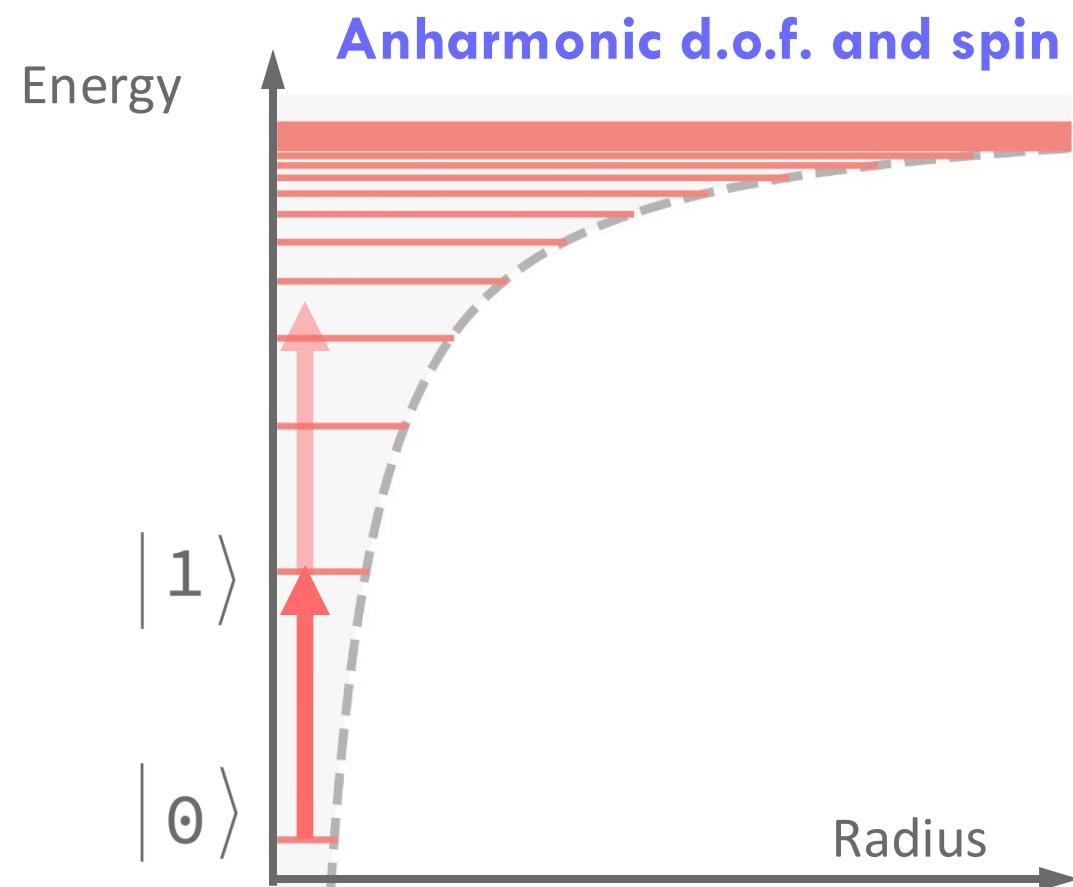
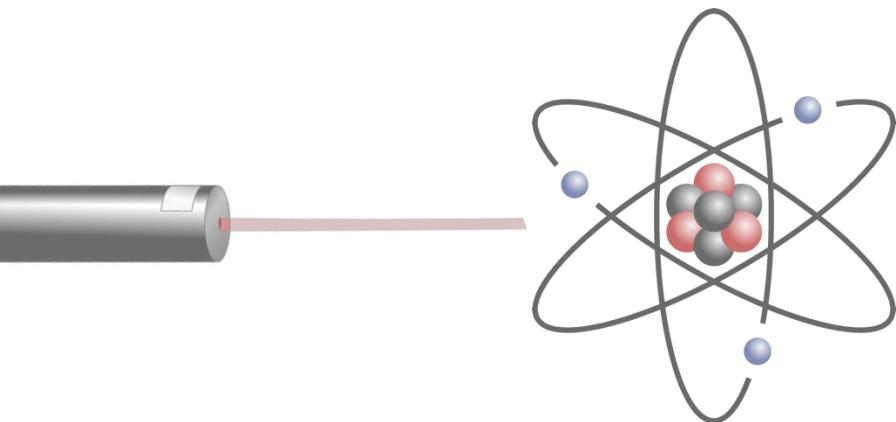
Low-loss

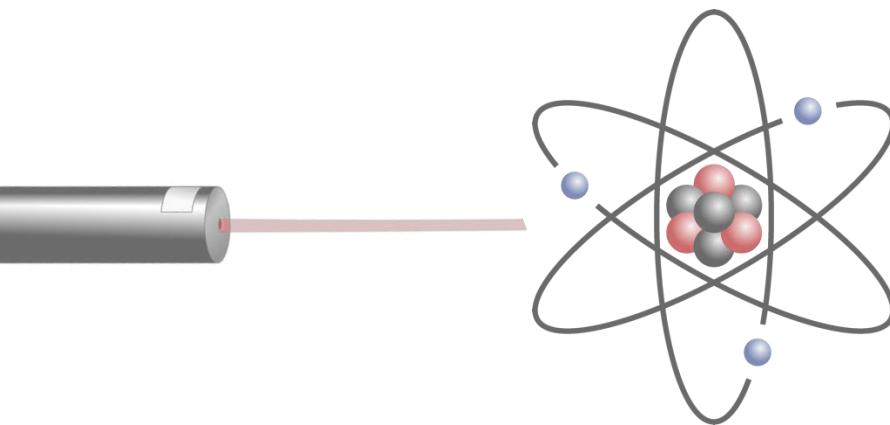
Level diagram allows for qubit-specific control and readout

There are always more than two levels

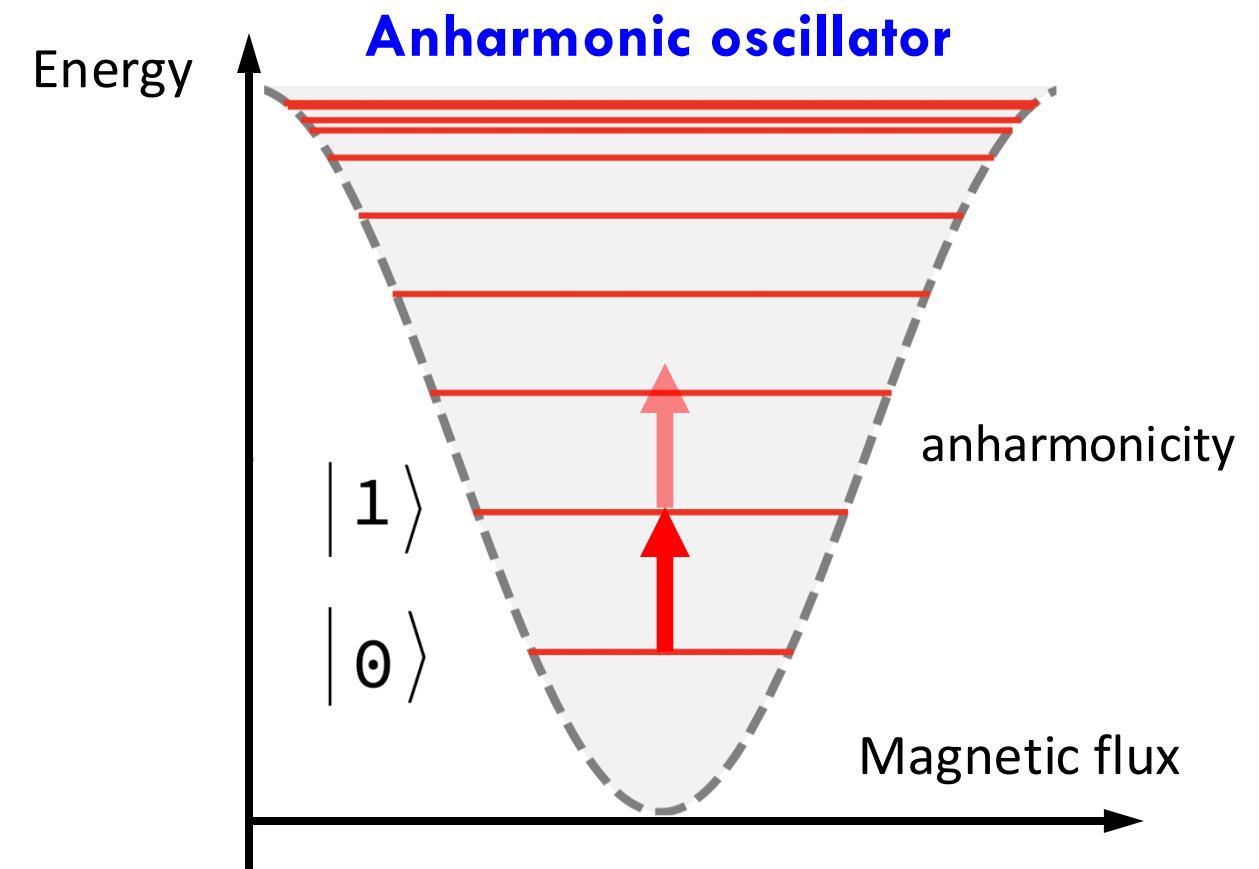
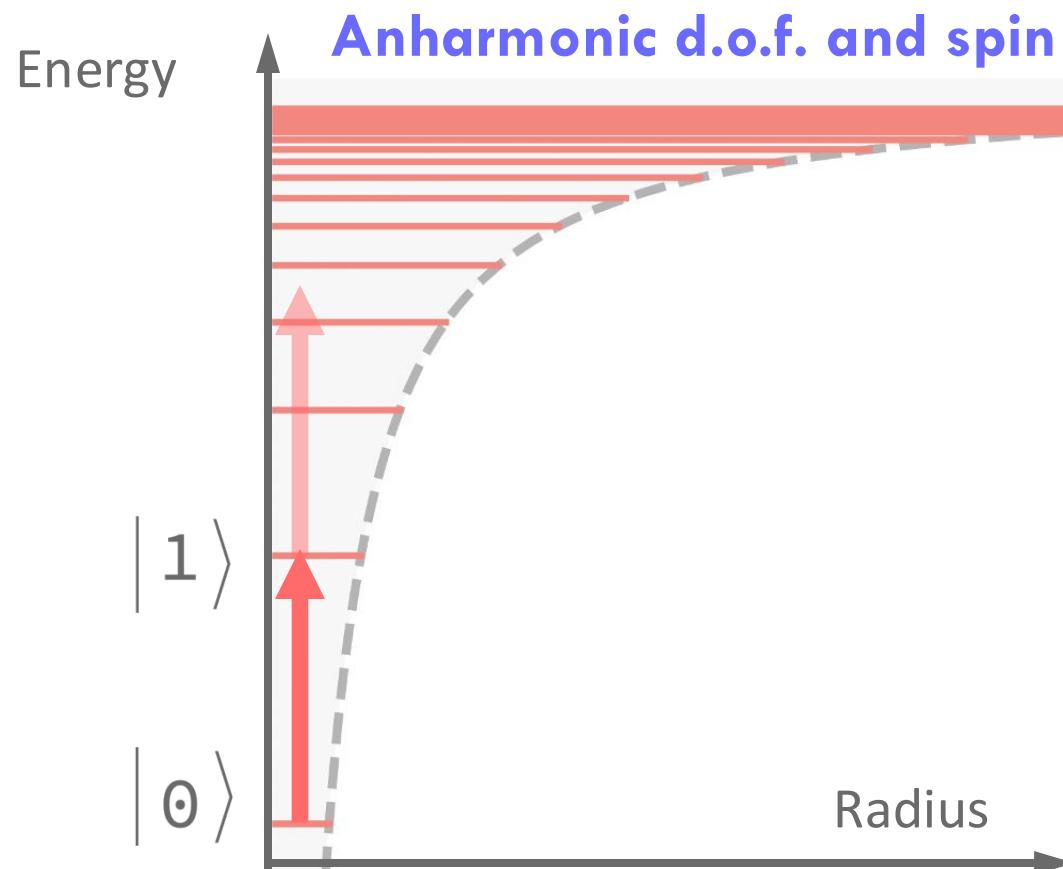
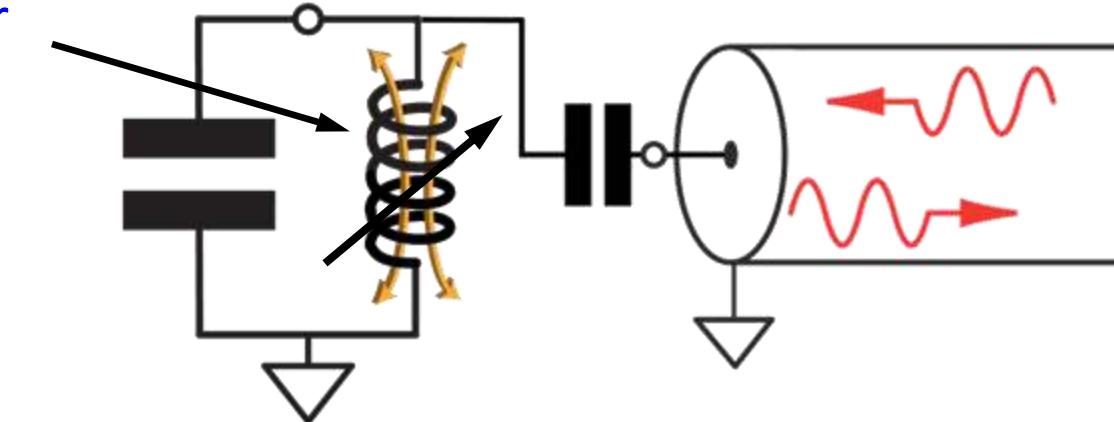
Artificial atoms





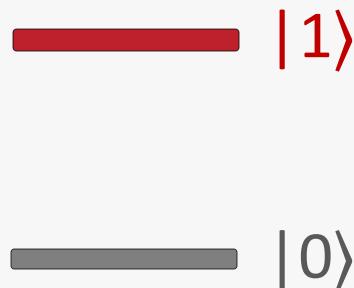


non-linear
inductor

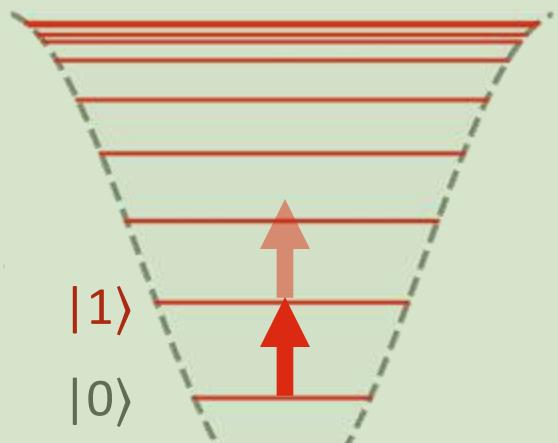


Big-picture connections

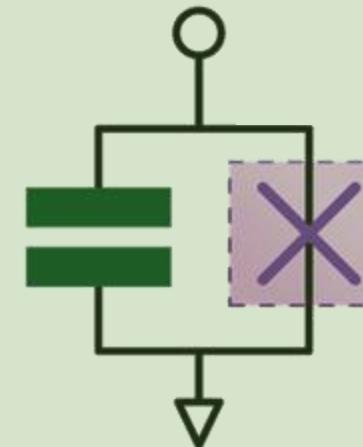
Idealization of qubit



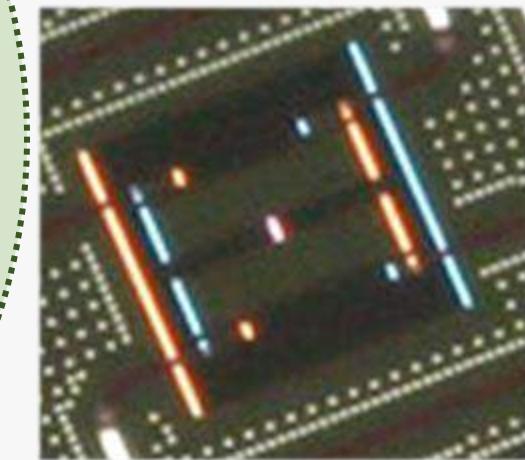
Anharmonic oscillator



Physical circuit model



Physical layout



Idealization

→

Physical reality

Circuit Quantum Electrodynamics (cQED)

Macro
overview

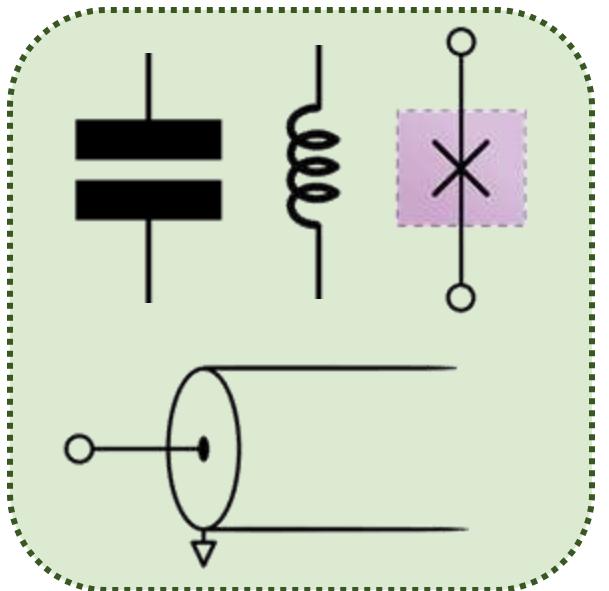
There are two kinds of physicists:

Those who believe all of physics is *spins*.

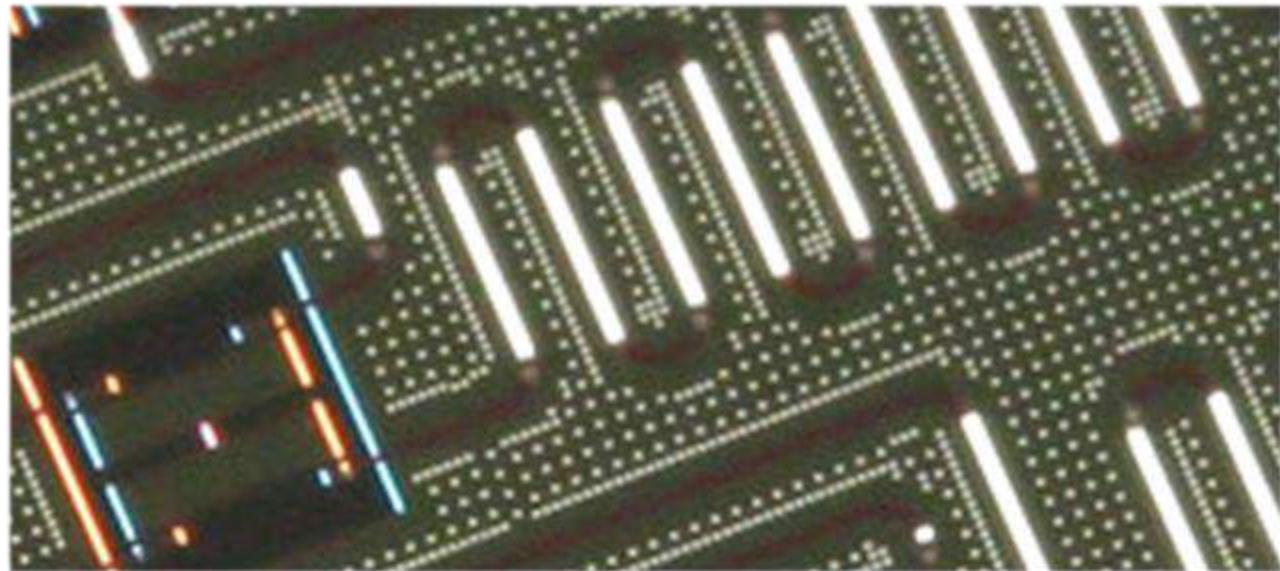
Those who believe all of physics is *oscillators*.

cQED Ingredients

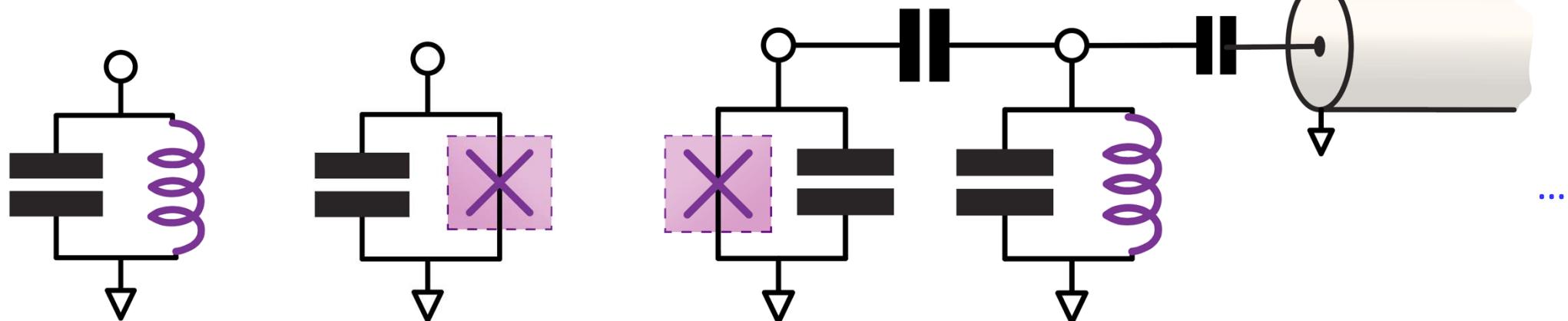
Circuit elements



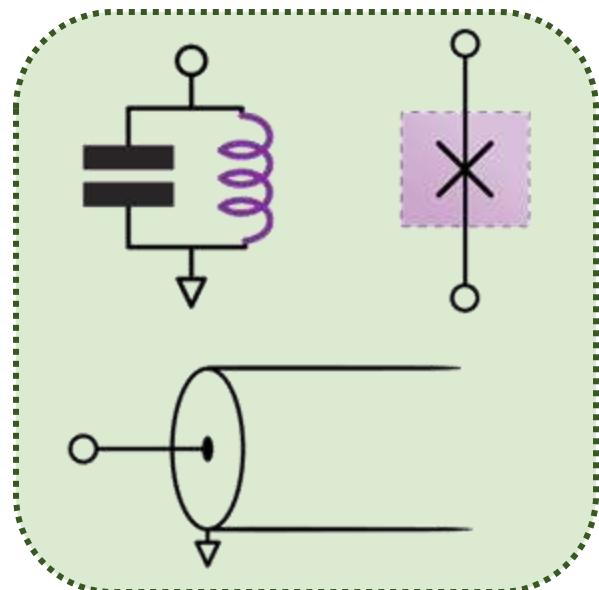
Microwave oscillators



Combine



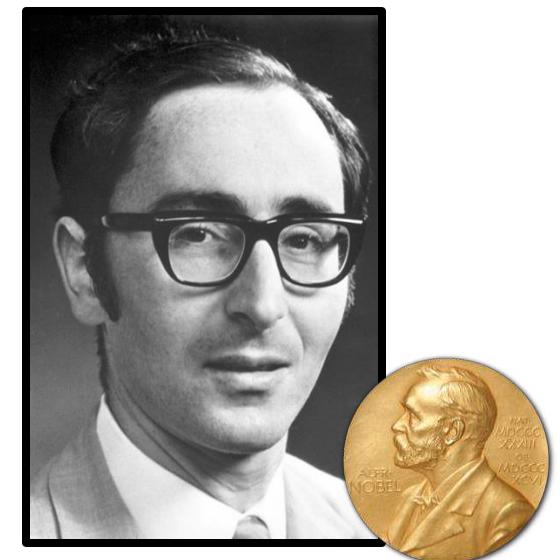
cQED Ingredients

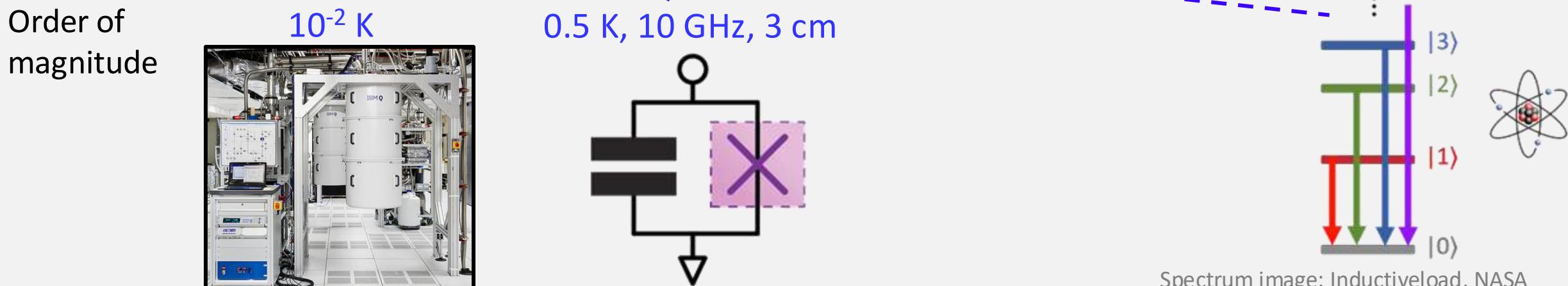
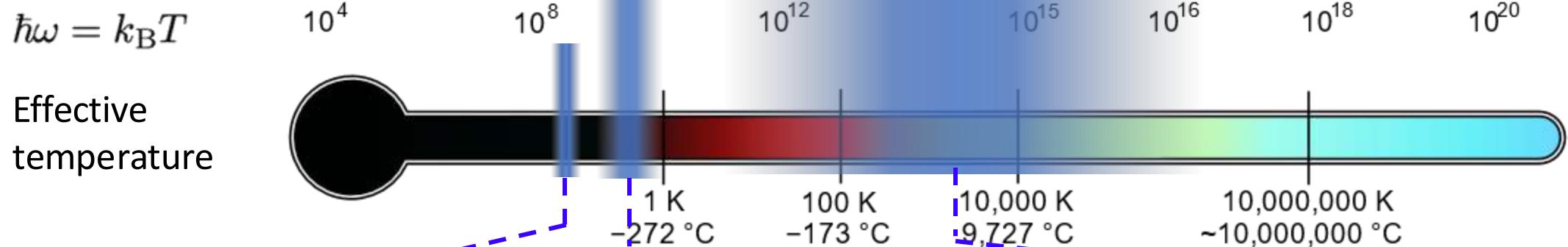
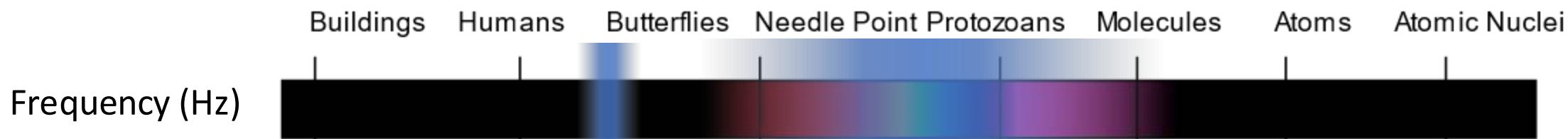
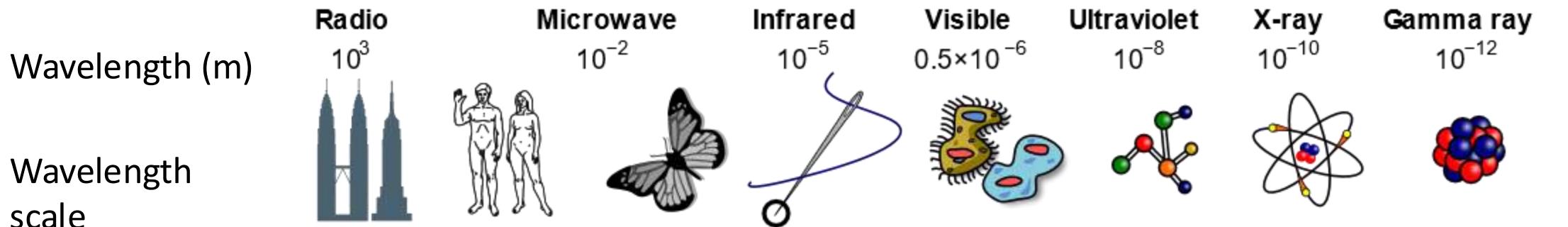


- Small dissipation
- Isolation from environment
- Low temperature
- Nonlinearity
- Large vacuum fluctuations

Superconductivity

- Nominally zero intrinsic dissipation and heat
- Nominal temperature far below energy level splitting
- Non-linear, robust Josephson tunnel junction effect





A few introductory reviews

And many more... check online or ask us for specific topic

Qiskit Textbook (2020; more chapters coming)

Blais, A., Grimsmo, A. L., Girvin, S. M., & Wallraff, A. (2020)
Circuit Quantum Electrodynamics (*arXiv:2005.12667*)

Kjaergaard, M., Schwartz, ... Oliver, W. D. (2020)
Superconducting Qubits: Current State of Play
Annual Reviews of Condensed Matter Physics 11, 369-395

Krantz, P., Kjaergaard, M., Yan, F., ... & Oliver, W. D. (2019)
A quantum engineer's guide to superconducting qubits
Applied Physics Reviews, 6(2), 021318

Corcoles, A. D., Kandala, A., ... Gambetta, J. M. (2019)
Challenges and Opportunities of Near-Term Quantum Computing Systems. *Proceedings of the IEEE*, 1–15.

Wendin, G. (2017)
Quantum information processing with superconducting circuits. *Reports on Progress in Physics*, 80(10), 106001

Gambetta, J. M., Chow, J. M., & Steffen, M. (2017)

Building logical qubits in a superconducting quantum computing system. *Npj Quantum Information*, 3(1), 2

Girvin, S. M. (2011) Circuit QED: superconducting qubits coupled to microwave photons. *Quantum machines: measurement and control of engineered quantum systems*, 113, 2.

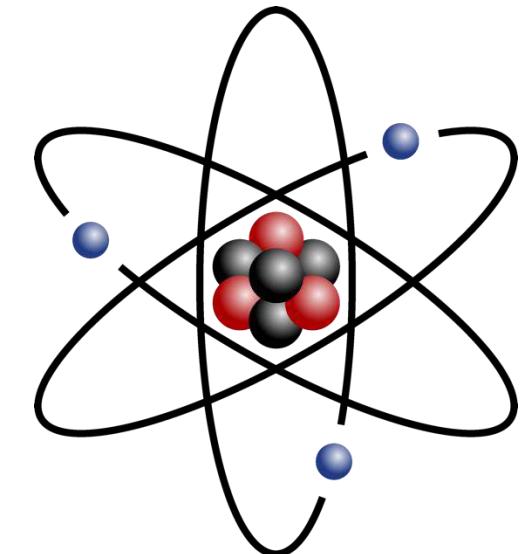
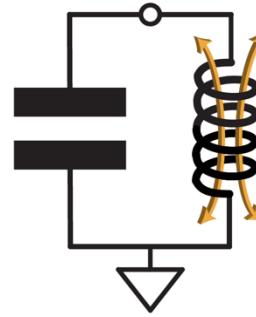
Clerk, A. A., Girvin, S. M., Marquardt, F., & Schoelkopf, R. J. (2010)
Introduction to quantum noise, measurement, and amplification
Reviews of Modern Physics, 82(2), 1155–1208

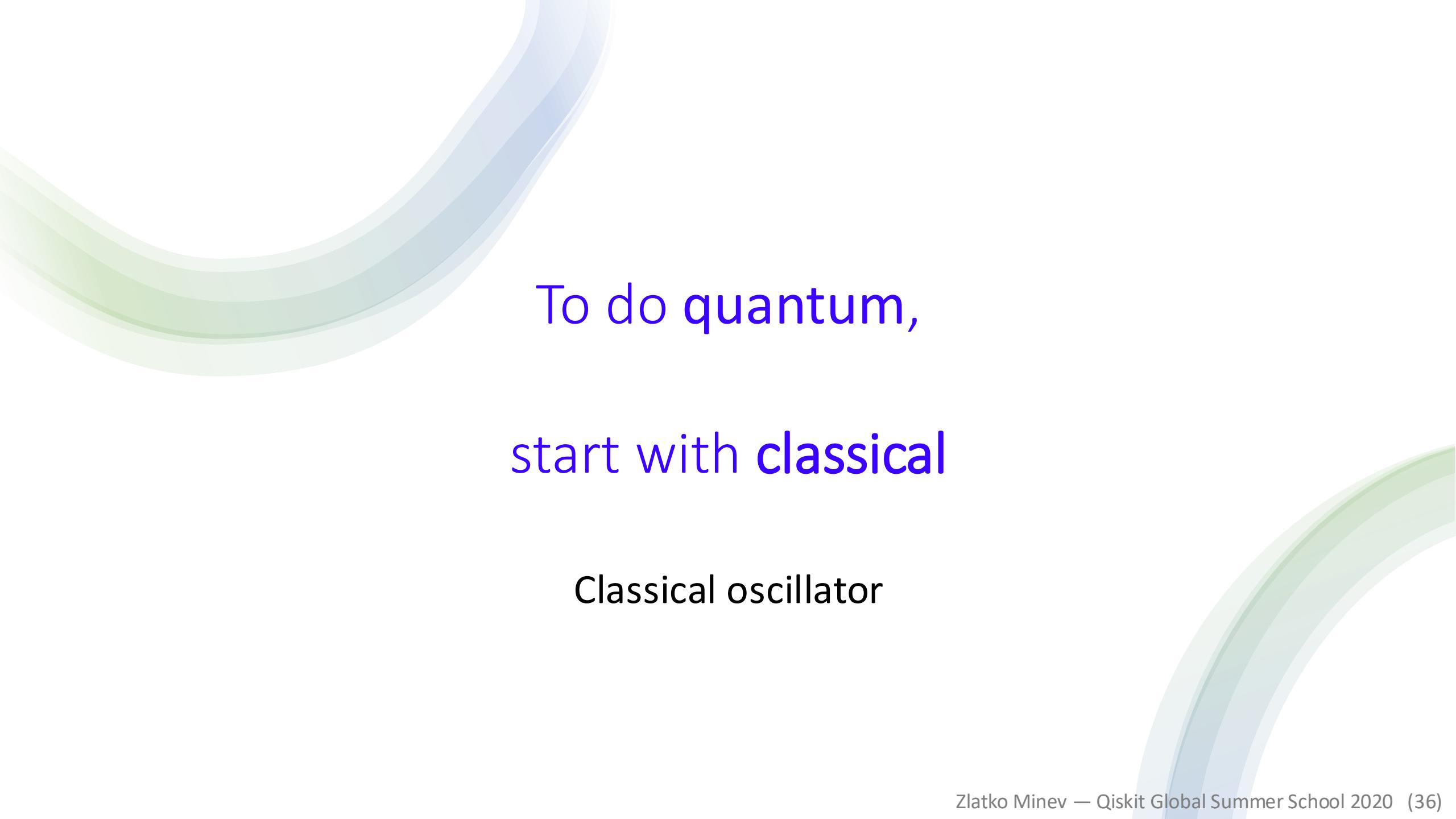
Clarke, J., & Wilhelm, F. K. (2008)
Superconducting quantum bits. *Nature*, 453(7198), 1031–1042

Devoret, M. H. (1997)
Quantum Fluctuations in Electrical Circuits.
In *Fluctuations Quantiques/Quantum Fluctuations* (p. 351)

...

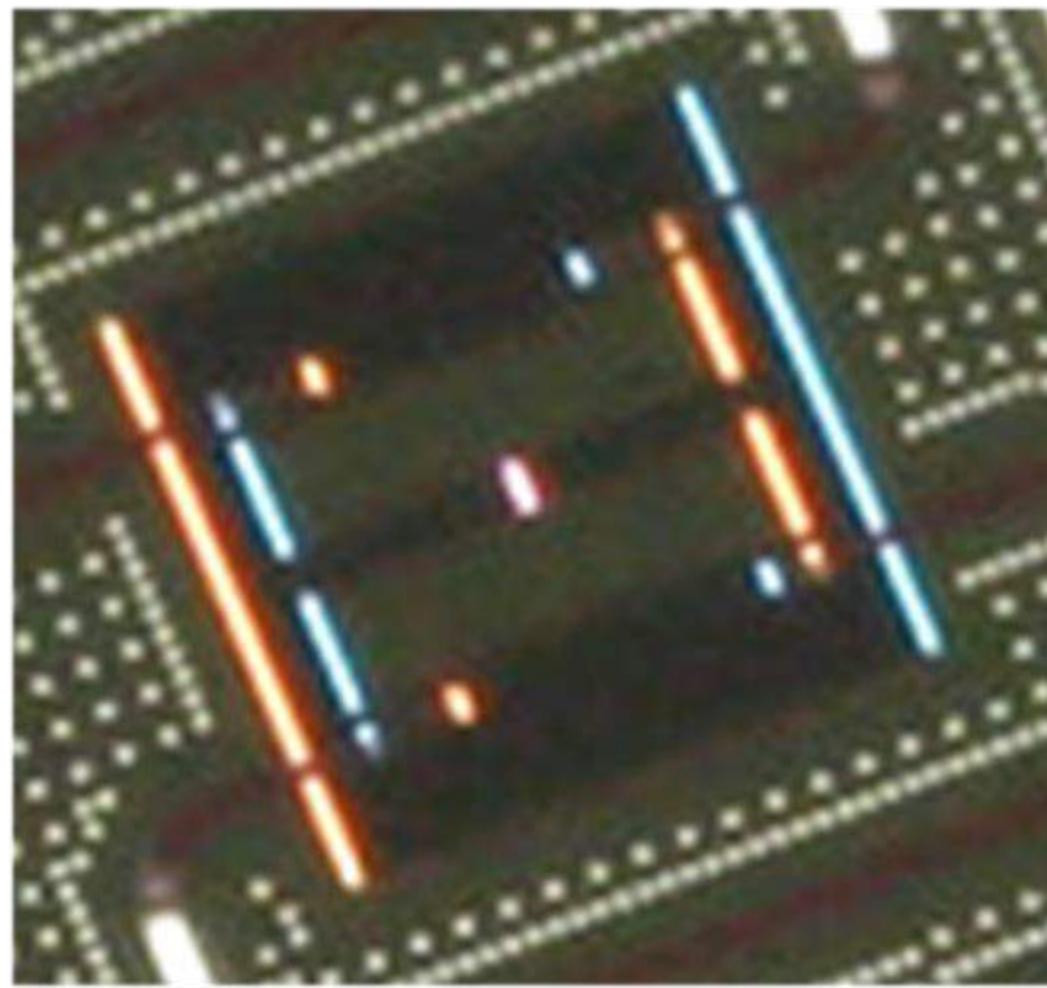
Circuit Quantum Electrodynamics



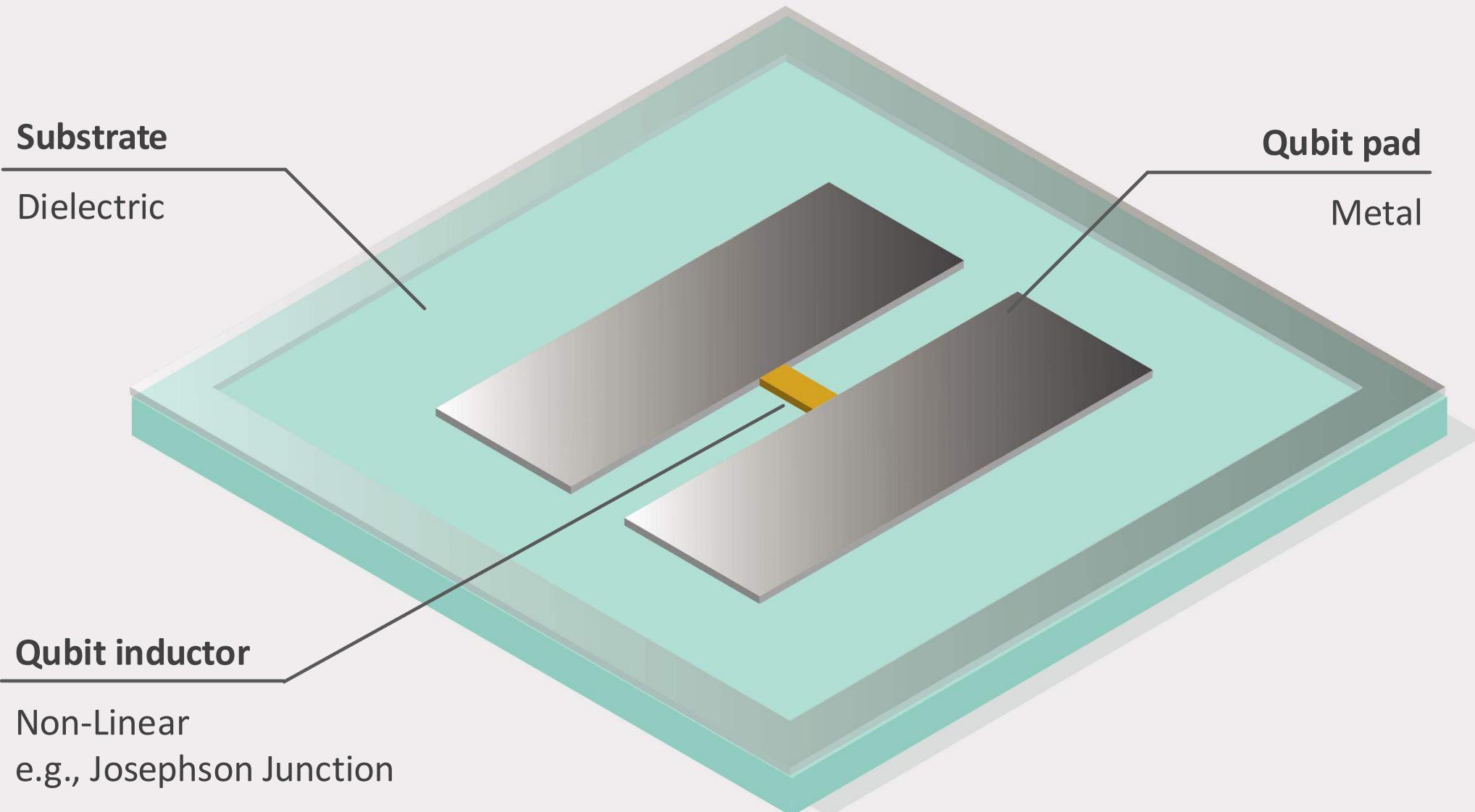


To do quantum,
start with classical
Classical oscillator

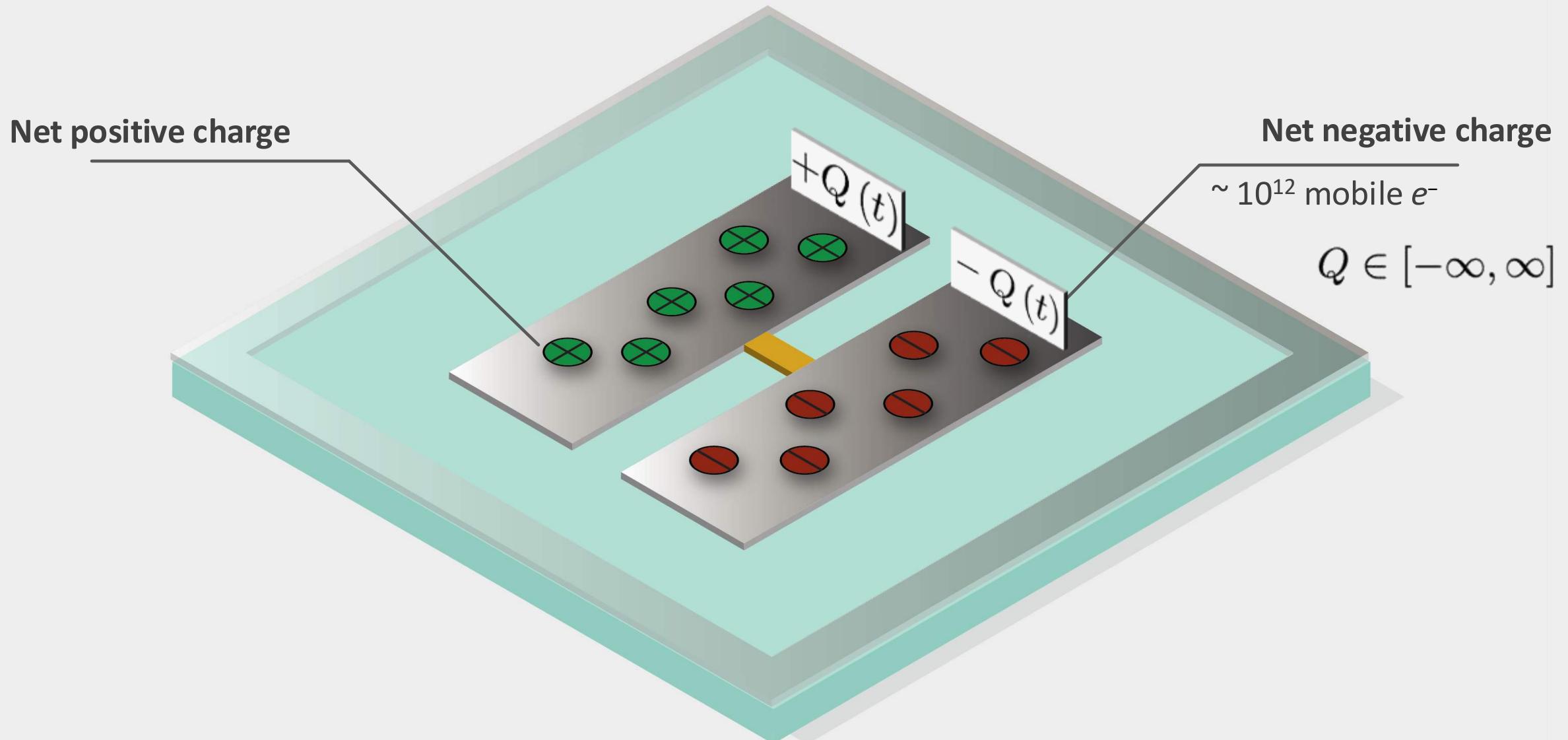
Transmon qubit



Transmon qubit

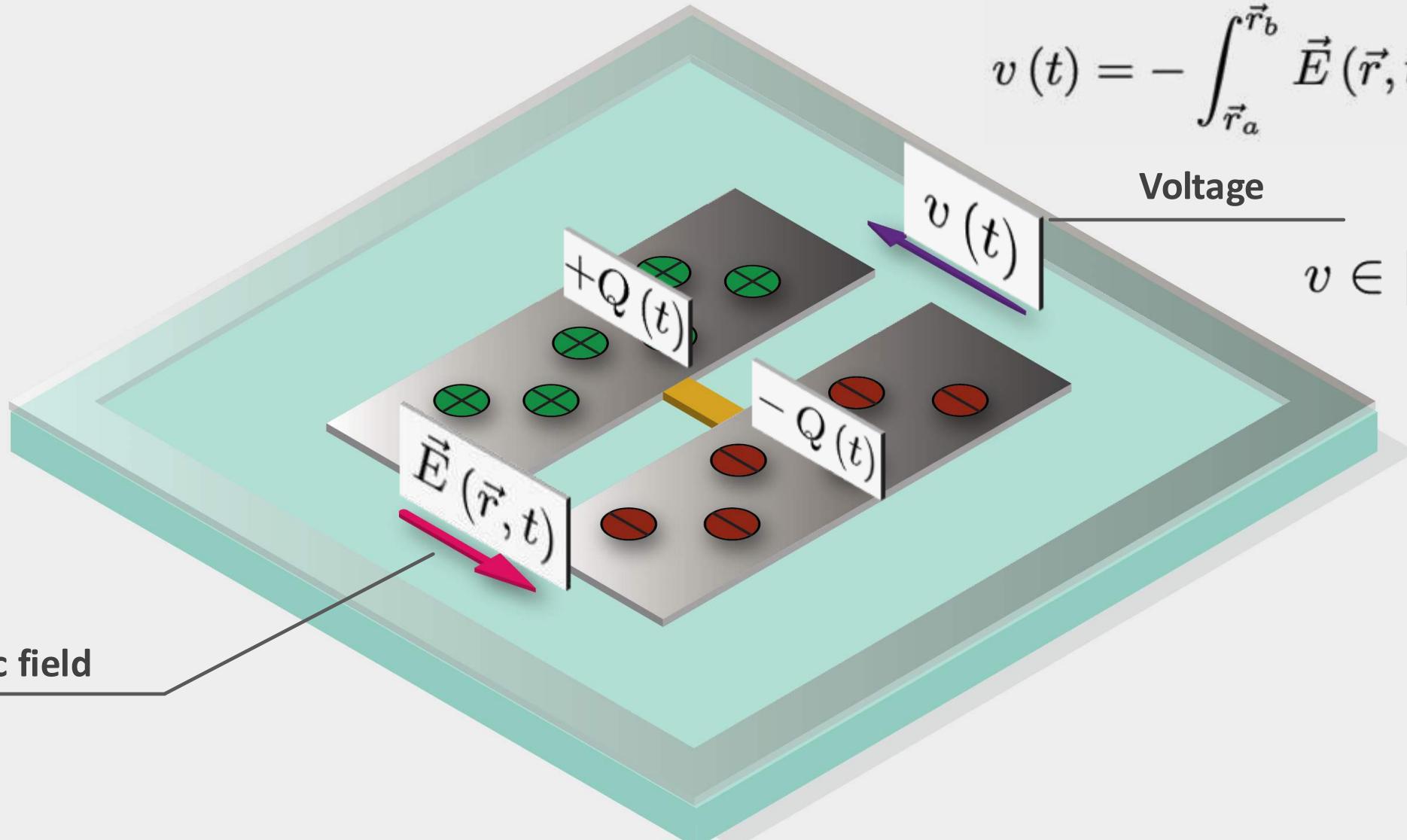


Transmon qubit: charge

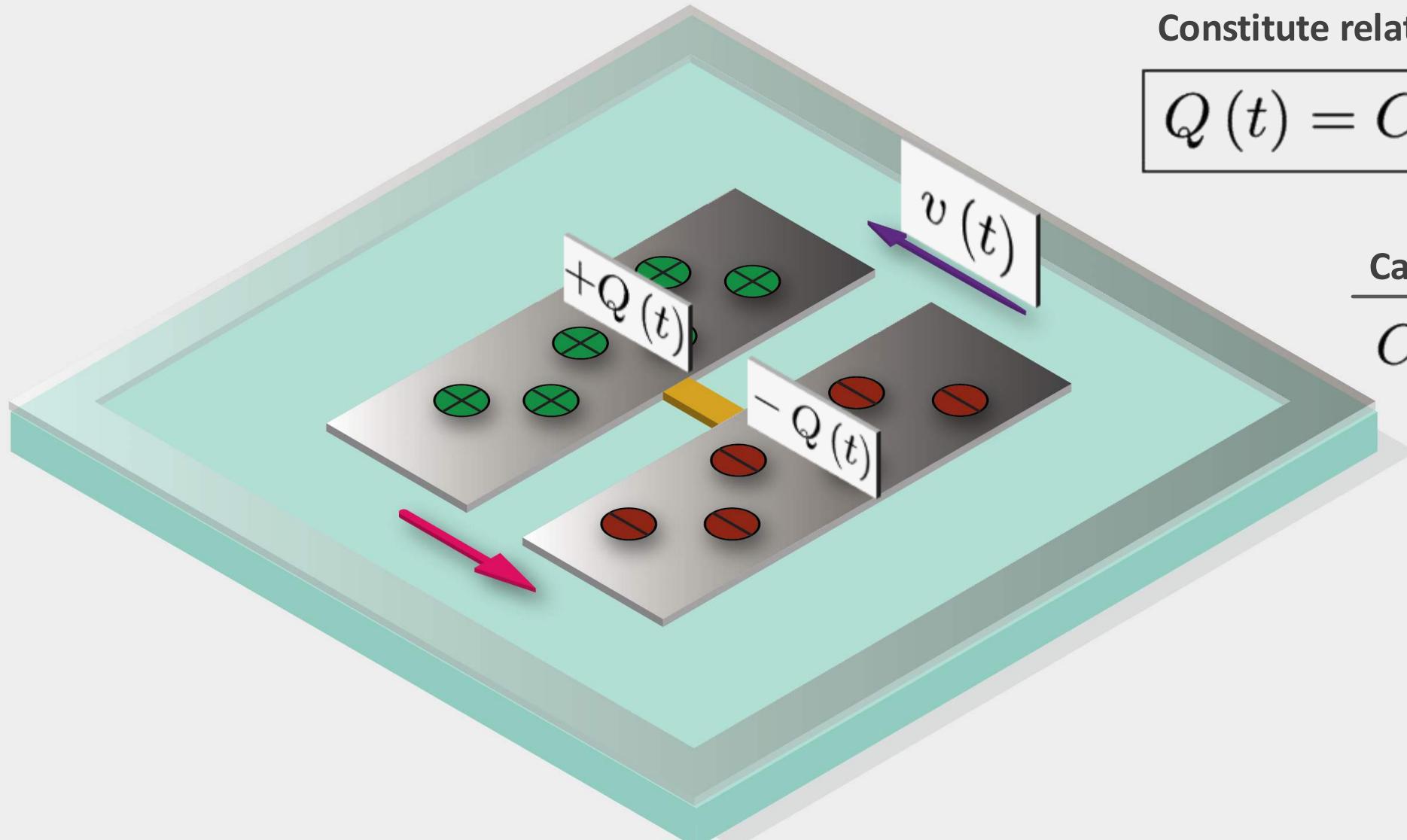


Electric field and voltage

$$v(t) = - \int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r}, t) \cdot d\vec{l}(\vec{r})$$



Charge and capacitance



Constitute relationship

$$Q(t) = Cv(t)$$

Capacitance
 $C \in (0, \infty)$



For a good discussion, see "The Feynman Lectures on Physics Vol. II
Ch. 22: AC Circuits." Caltech.

Conservation of charge

Universal relationship

$$\frac{d}{dt}Q(t) = i(t)$$

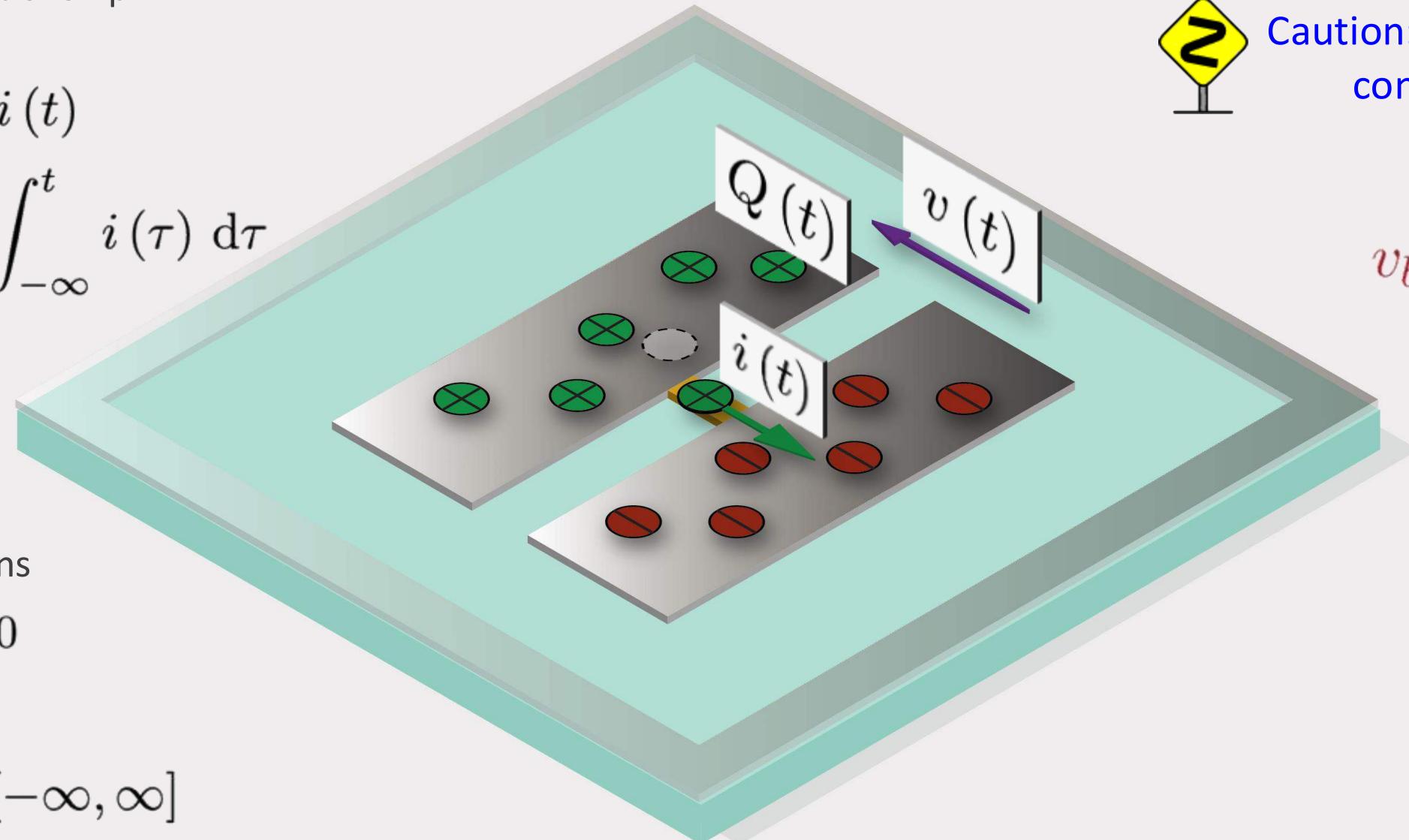
$$Q(t) = \int_{-\infty}^t i(\tau) d\tau$$

Initial conditions

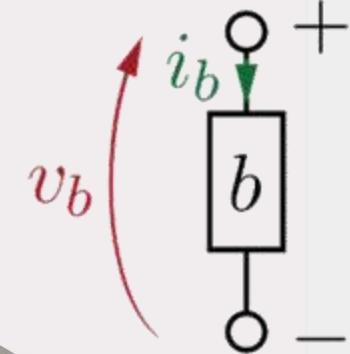
$$Q(-\infty) = 0$$

$$i \in [-\infty, \infty]$$

Charge and current



Caution: Passive sign convention



Magnetic flux and inductance

Faraday's law of induction

Universal relationship

$$\Phi(t) = \int_{-\infty}^t v(\tau) d\tau$$

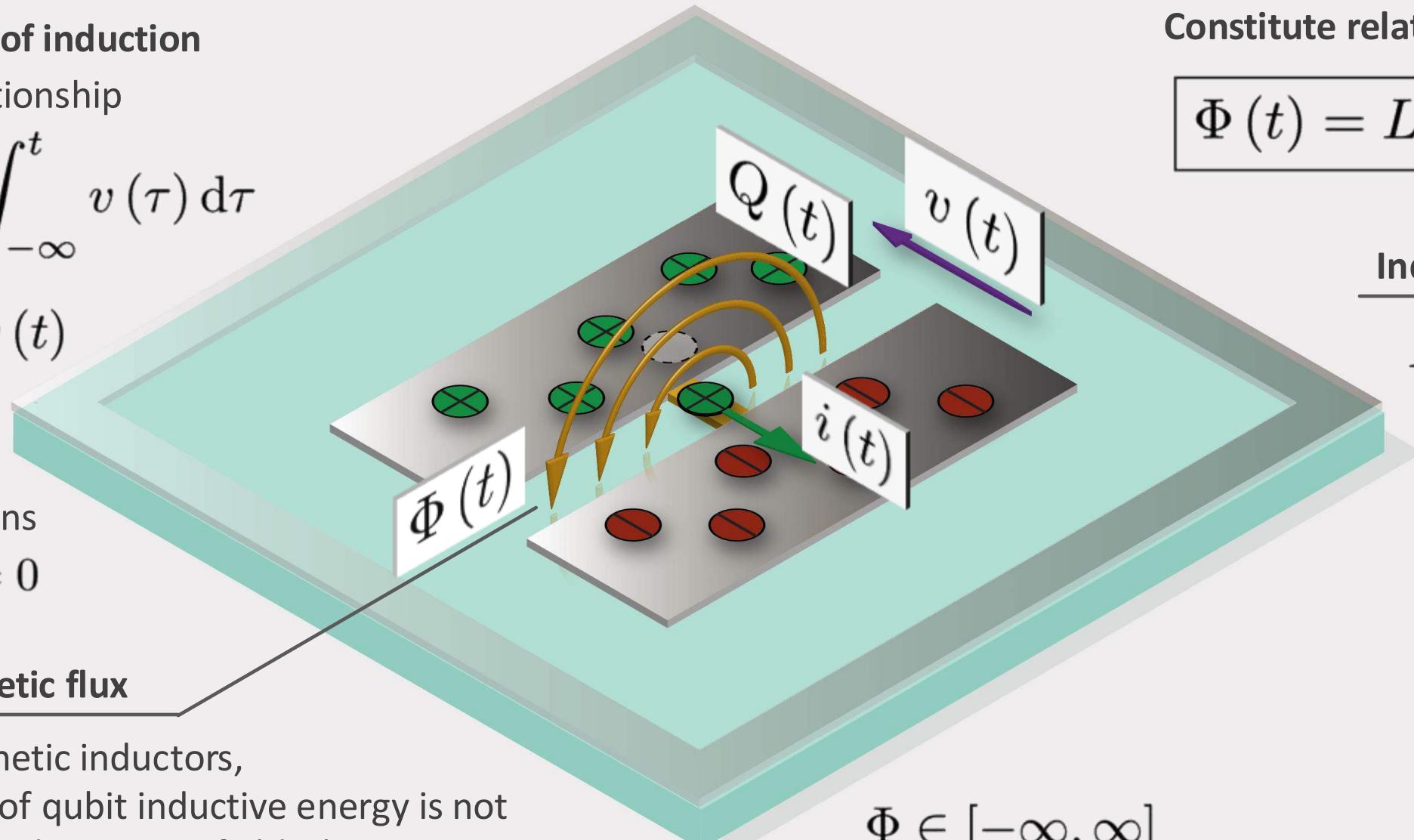
$$\frac{d}{dt}\Phi(t) = v(t)$$

Initial conditions

$$\Phi(-\infty) = 0$$

Magnetic flux

For kinetic inductors,
~98% of qubit inductive energy is not
in stored magnetic fields, but in
kinetic inductance



$$\Phi \in [-\infty, \infty]$$

Constitute relationship

$$\Phi(t) = L i(t)$$

Inductance

$$L \in (0, \infty)$$

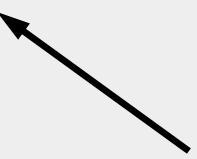
Power and energy

Universal

$$\frac{d}{dt}\mathcal{E}(t) = p(t) \equiv v(t)i(t)$$



Energy stored *in*
(delivered *to*)
component



Instantaneous
power flowing *to*
component

$$\mathcal{E}(t) = \int_{-\infty}^t p(\tau) d\tau$$

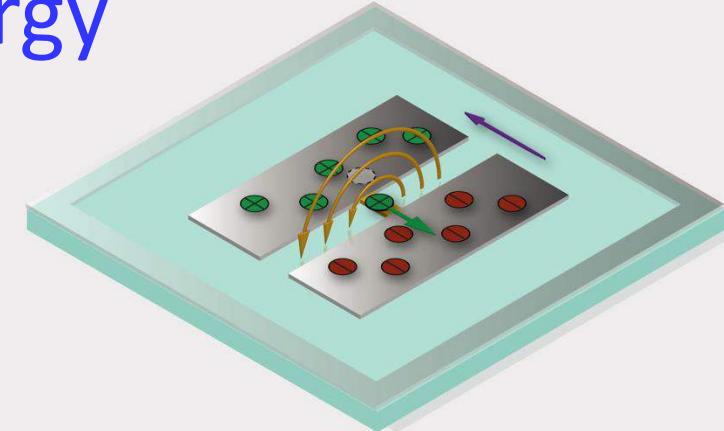


$$\mathcal{E}_{\text{cap}}(\dot{\Phi}) = \frac{1}{2}C\dot{\Phi}^2$$

$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

$$\mathcal{E}_{\text{cap}}(Q) = \frac{Q^2}{2C}$$

$$\mathcal{E}_{\text{ind}}(\dot{Q}) = \frac{1}{2}L\dot{Q}^2$$

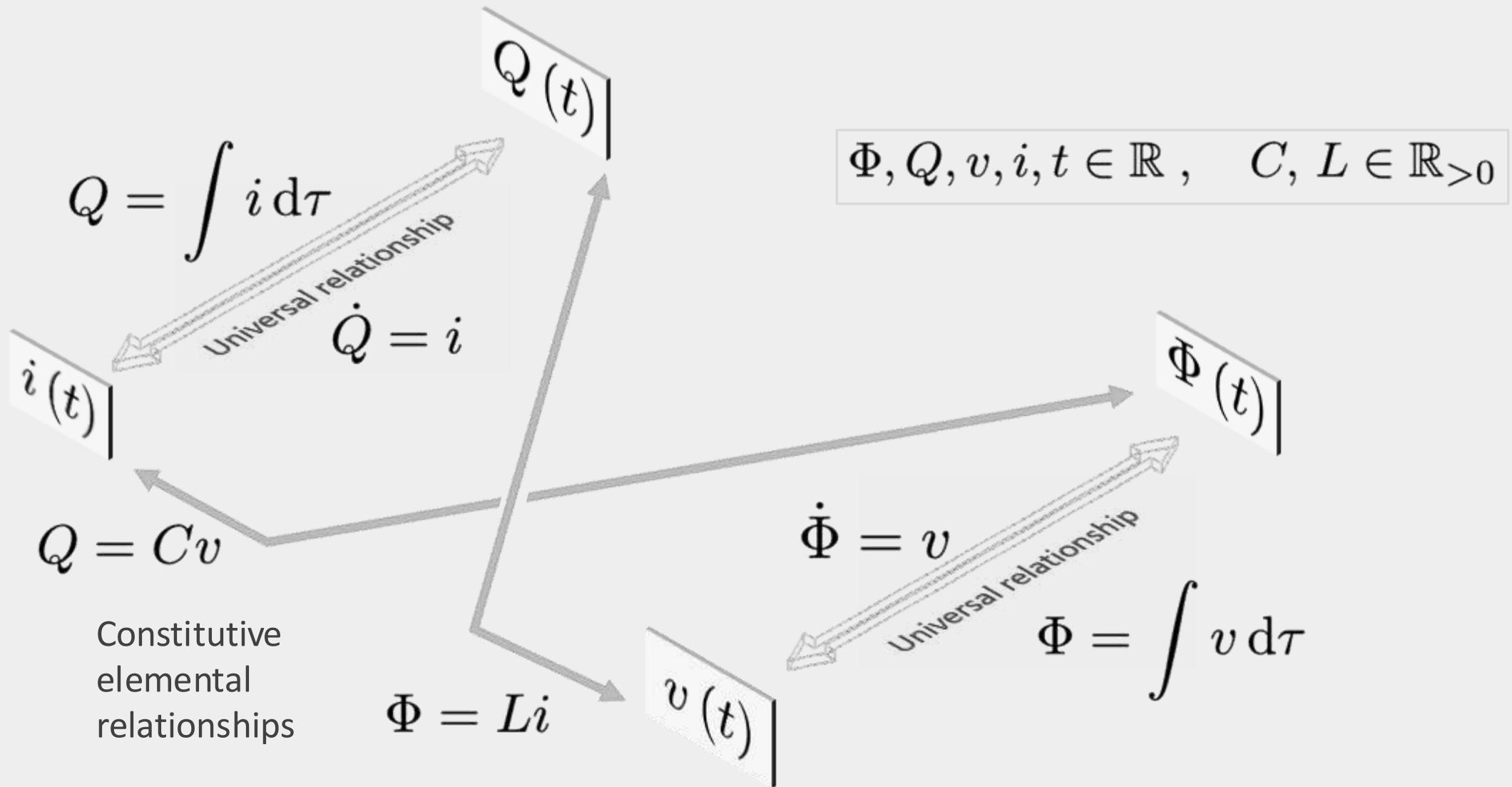


Four fundamental manifestations of electricity

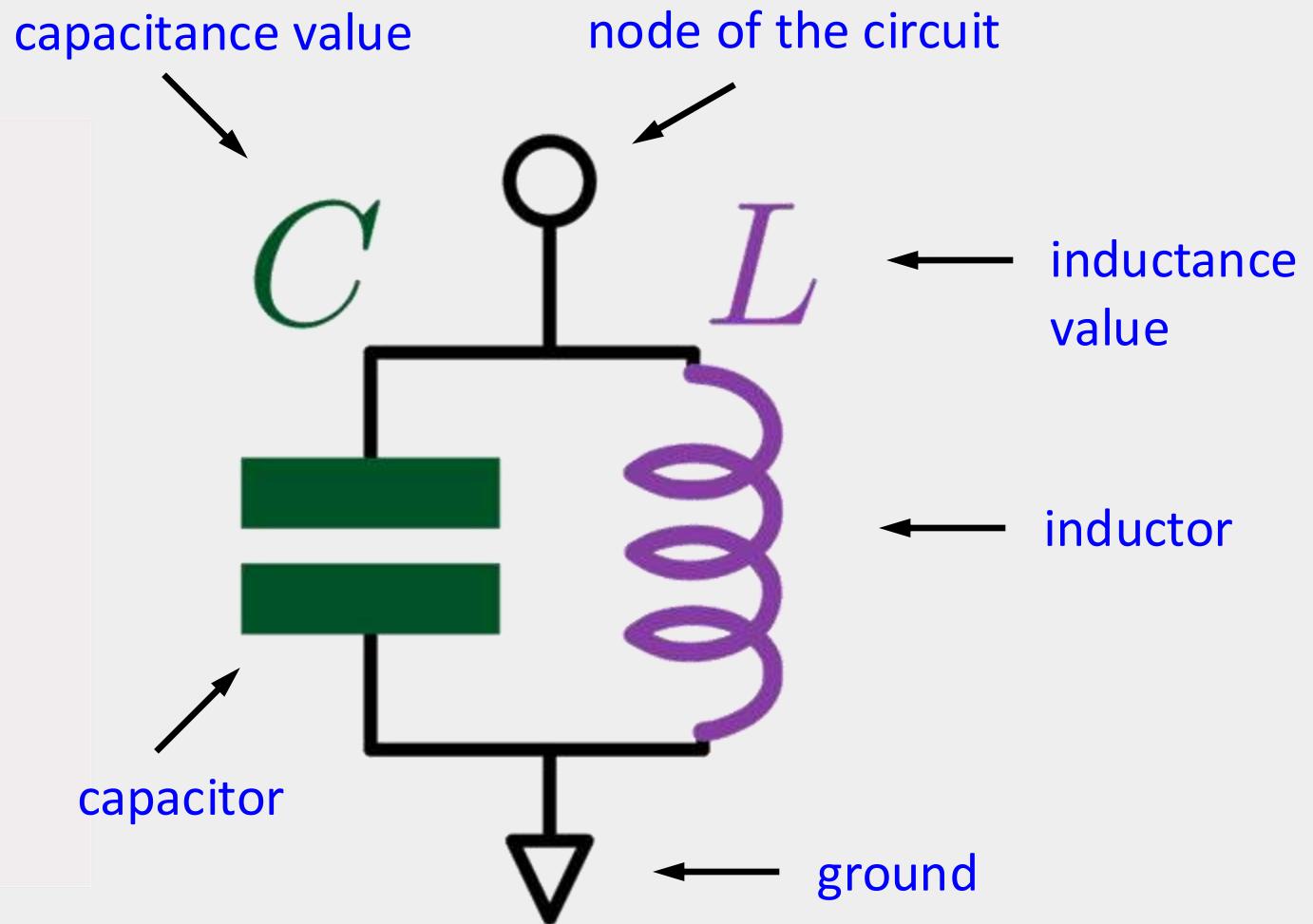
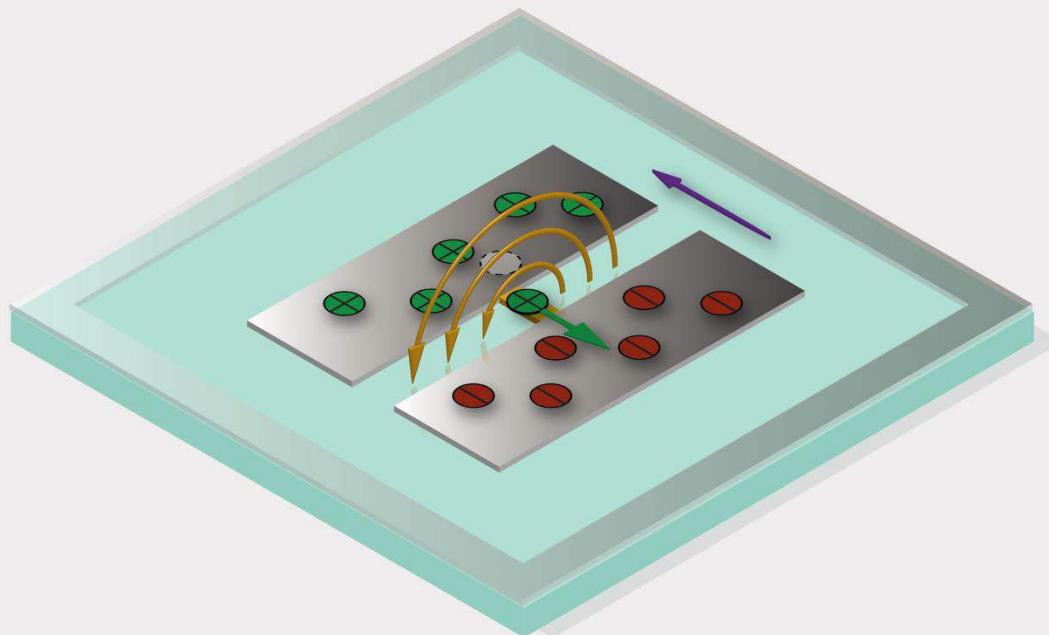
The diagram illustrates four fundamental manifestations of electricity, each represented by a 3D vector relationship:

- Charge:** A vector pointing from the origin to a point labeled $i(t)$ on the vertical axis, representing current.
- Universal relationship:** A diagonal vector labeled $Q = \int i d\tau$, representing the total charge as the integral of current over time.
- Current:** A vector pointing from the origin to a point labeled $\dot{Q} = i$ on the vertical axis, representing the rate of change of charge.
- Universal relationship:** A diagonal vector labeled $\Phi = \int v d\tau$, representing the total flux as the integral of voltage over time.
- Flux:** A vector pointing from the origin to a point labeled $v(t)$ on the vertical axis, representing voltage.
- Universal relationship:** A diagonal vector labeled $\dot{\Phi} = v$, representing the rate of change of flux.
- Electric Charge:** A vector pointing from the origin to a point labeled $Q(t)$ on the vertical axis, representing charge.

Four fundamental manifestations of electricity



Electromagnetic oscillator

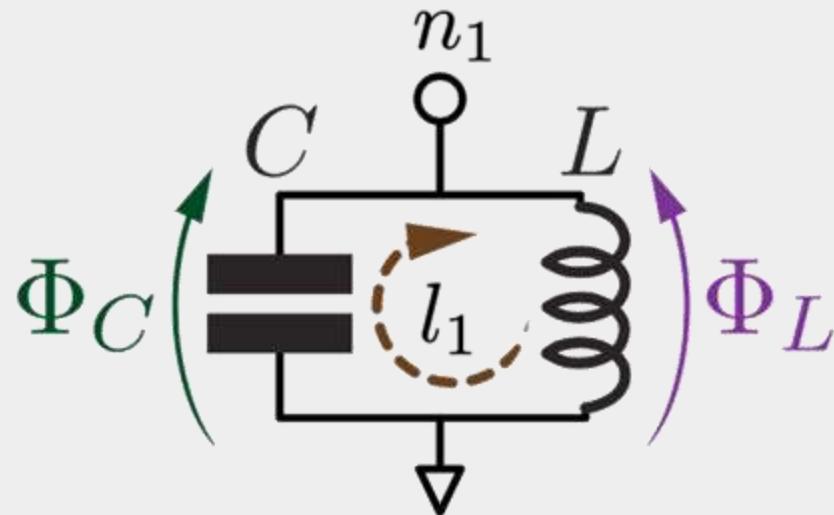


Kirchhoff's network laws*

Conservation of charge

Kirchhoff's current law

$$\sum_{b \in \text{node}} \pm \dot{Q}_b(t) = 0$$



$$n_1 : \quad \dot{Q}_C + \dot{Q}_L = 0$$

$$C\ddot{\Phi} + L^{-1}\Phi = 0$$

Faraday's law of induction

Kirchhoff's voltage law

$$\sum_{b \in \text{loop}} \pm \dot{\Phi}_b(t) = 0$$

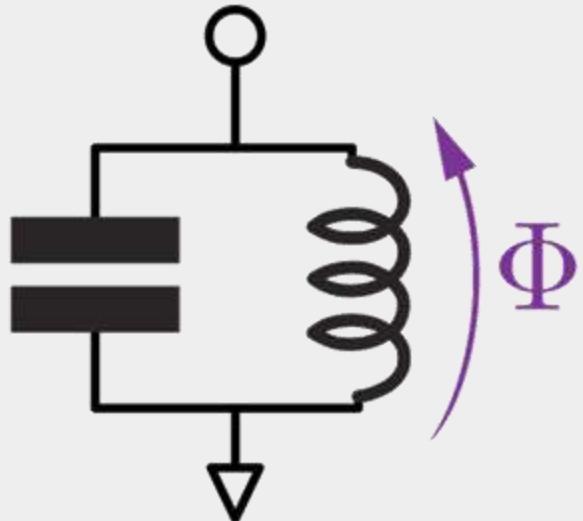
$$l_1 : \quad \dot{\Phi}_C - \dot{\Phi}_L = 0$$

$$\Phi_C = \Phi_L$$



As we will see later, for the Lagrangian description in flux basis, KVL acts as a set of holonomic constraints and KCL as the equations of motion

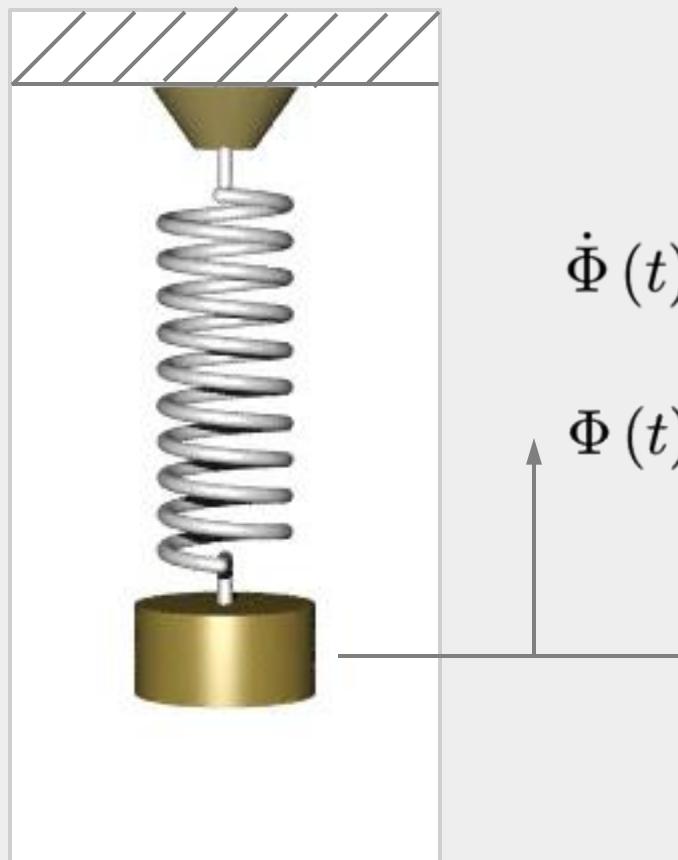
Oscillator analogy



$$C\ddot{\Phi} + L^{-1}\Phi = 0$$

$$\ddot{\Phi} = -\omega_0^2\Phi , \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Resonance
frequency

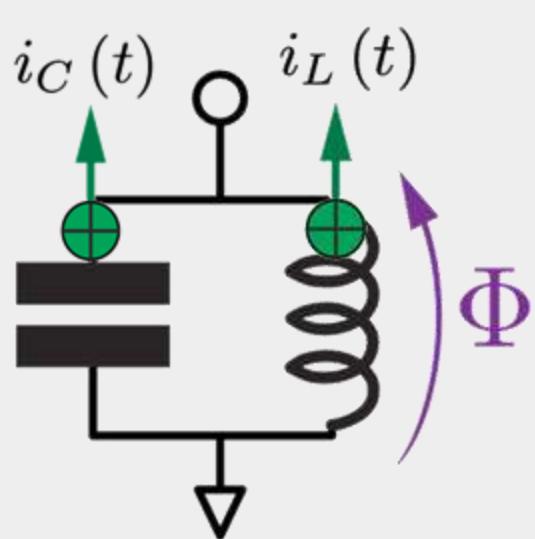


Equilibrium position
 $x = 0 , \quad \Phi = 0$

$\dot{\Phi}(t)$ Velocity
 $\Phi(t)$ Deviation from equilibrium



Lagrangian and Hamiltonian*



$$C\ddot{\Phi} + L^{-1}\Phi = 0$$

$$i_C(t) \quad i_L(t)$$

$$\mathcal{L}(\Phi, \dot{\Phi}) = \mathcal{E}_{\text{cap}}(\dot{\Phi}) - \mathcal{E}_{\text{ind}}(\Phi)$$

$$= \frac{1}{2}C\dot{\Phi}^2 - \frac{\Phi^2}{2L}$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{\partial \mathcal{L}}{\partial \Phi}$$

$$\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

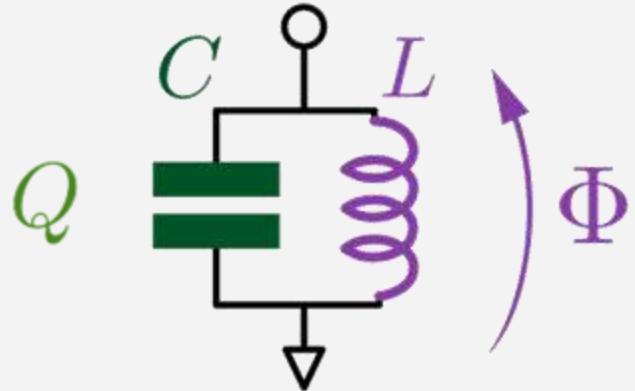
Kinetic energy – Potential energy

Canonically conjugate variable
(momentum; charge)

Euler-Lagrange equations, $F = ma$

Kinetic energy + Potential energy

* Temporarily going to assume some minimal knowledge of classical mechanics.
Since we eliminated the KVL constraints, this is a Lagrangian of the second kind.



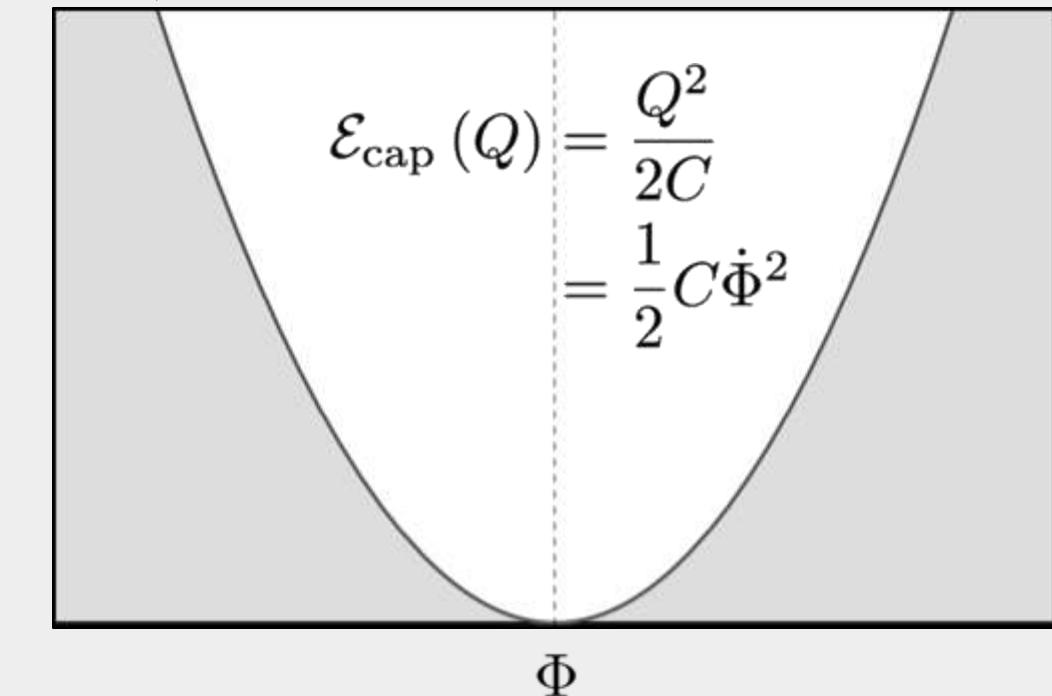
The LC classical harmonic oscillator



Position: $\Phi \mapsto x$
Mass: $C \mapsto m$

Momentum: $Q \mapsto p$
Spring constant: $L^{-1} \mapsto k$

$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$



$$\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$\omega_0^2 = \frac{1}{LC} \quad Z_0 = \frac{L}{C}$$

“It is by *logic* that we prove,
but by *intuition* that we discover.”

Henri Poincaré

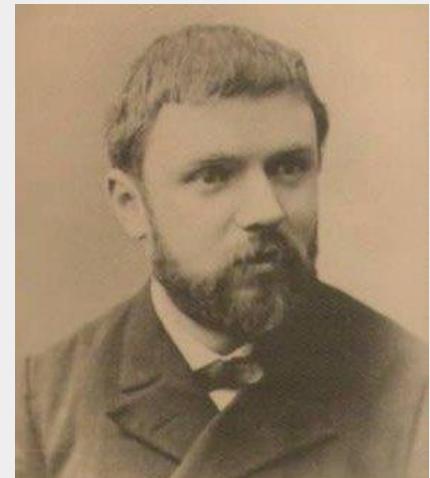


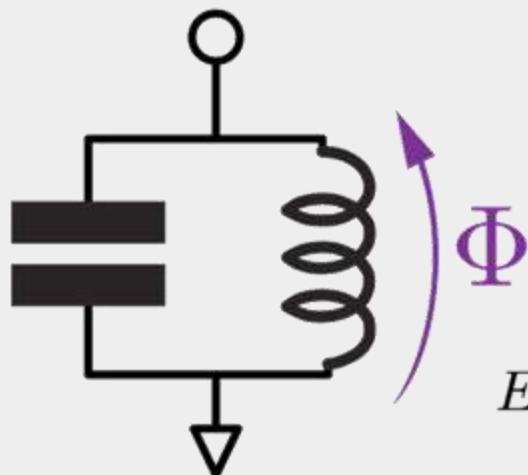
Photo by Eugène Pirou

Hamiltonian dynamics and phase space

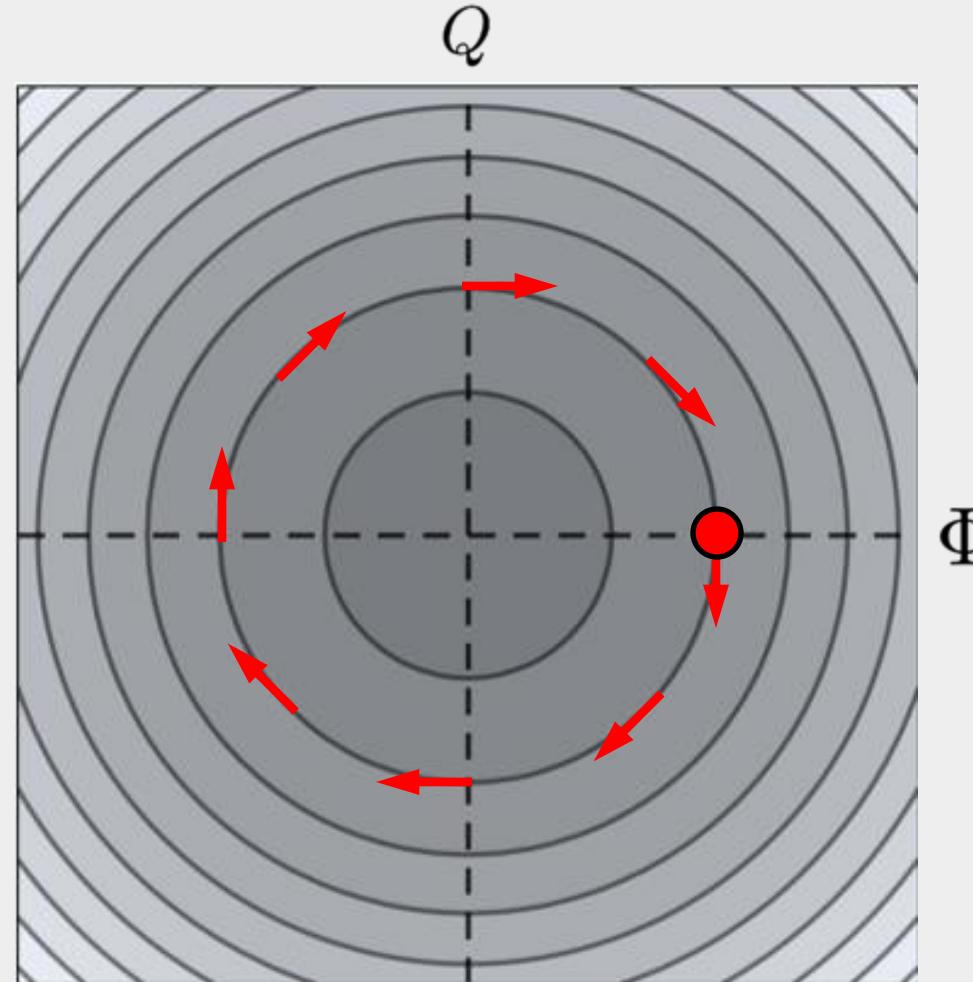
$$\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \hbar\omega_0 \left(n + \frac{1}{2} \right) = \frac{1}{2} (Q^2 + \Phi^2)$$

with $L = C = 1$



$$E = \frac{3\hbar\omega_0}{2}$$
$$E = \frac{\hbar\omega_0}{2}$$



$$\dot{\Phi} = +\frac{\partial \mathcal{H}}{\partial Q} = +Q$$
$$\dot{Q} = -\frac{\partial \mathcal{H}}{\partial \Phi} = -\Phi$$

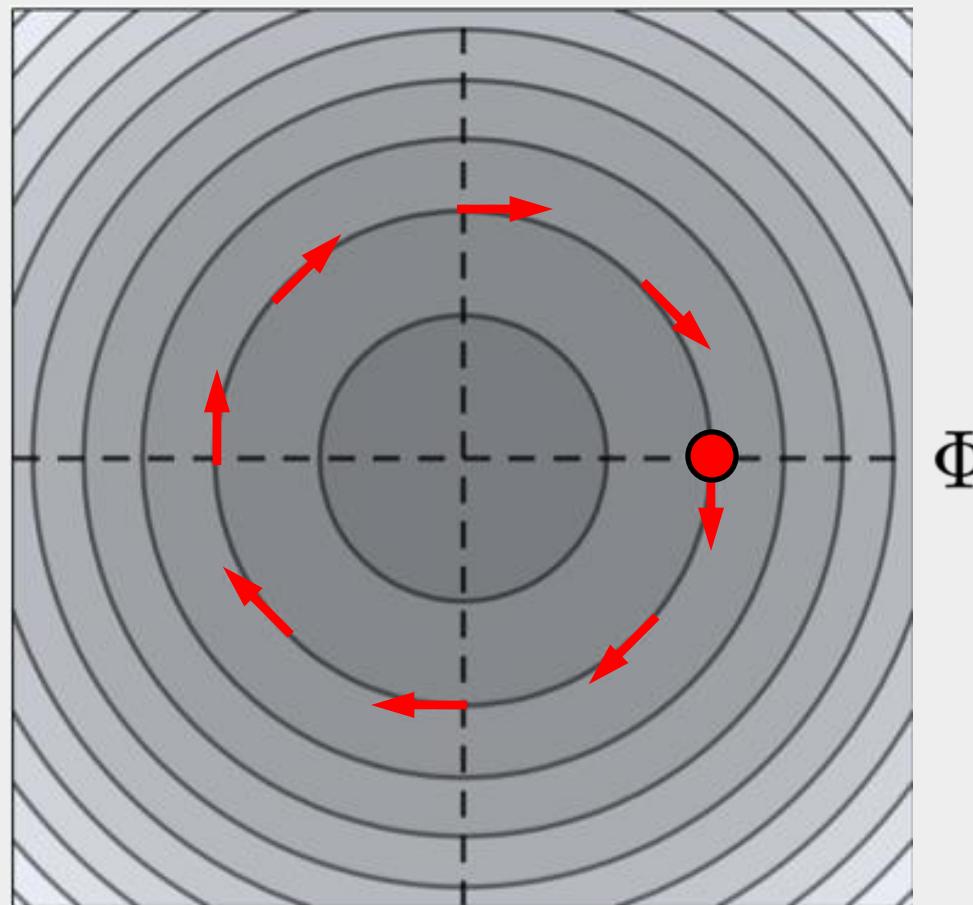
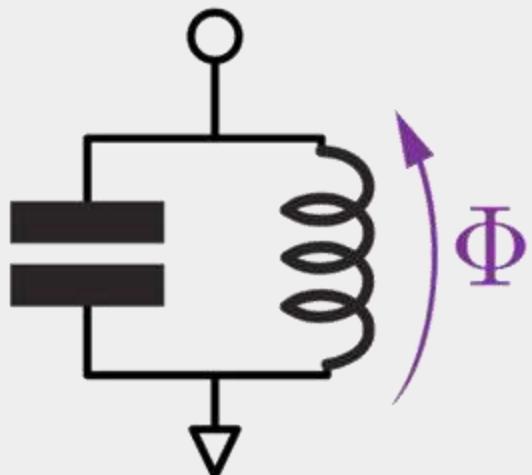
Complex action-angle variable

$$\mathcal{H}(\Phi, Q^*) = \frac{Q^2}{2C} \hbar \omega_0 \frac{\Phi^2}{2L} (\alpha^* \alpha + \alpha \alpha^*)$$

$$E = \hbar \omega_0 \left(n + \frac{1}{2} \right)$$

$$\alpha(t) = \sqrt{\frac{1}{2\hbar Z}} [\Phi(t) + iZQ(t)]$$

Q



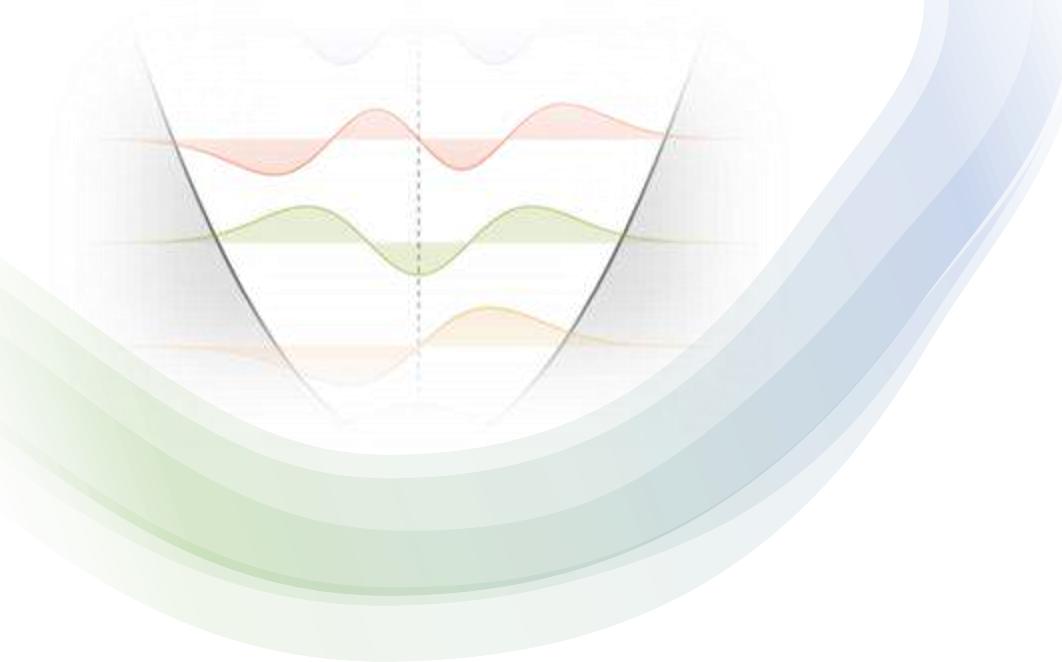
Classical analog of the
bosonic ladder operator

$$\alpha(t) = \alpha(0) e^{-i\omega_0 t}$$

Φ

Unveiling the quantum

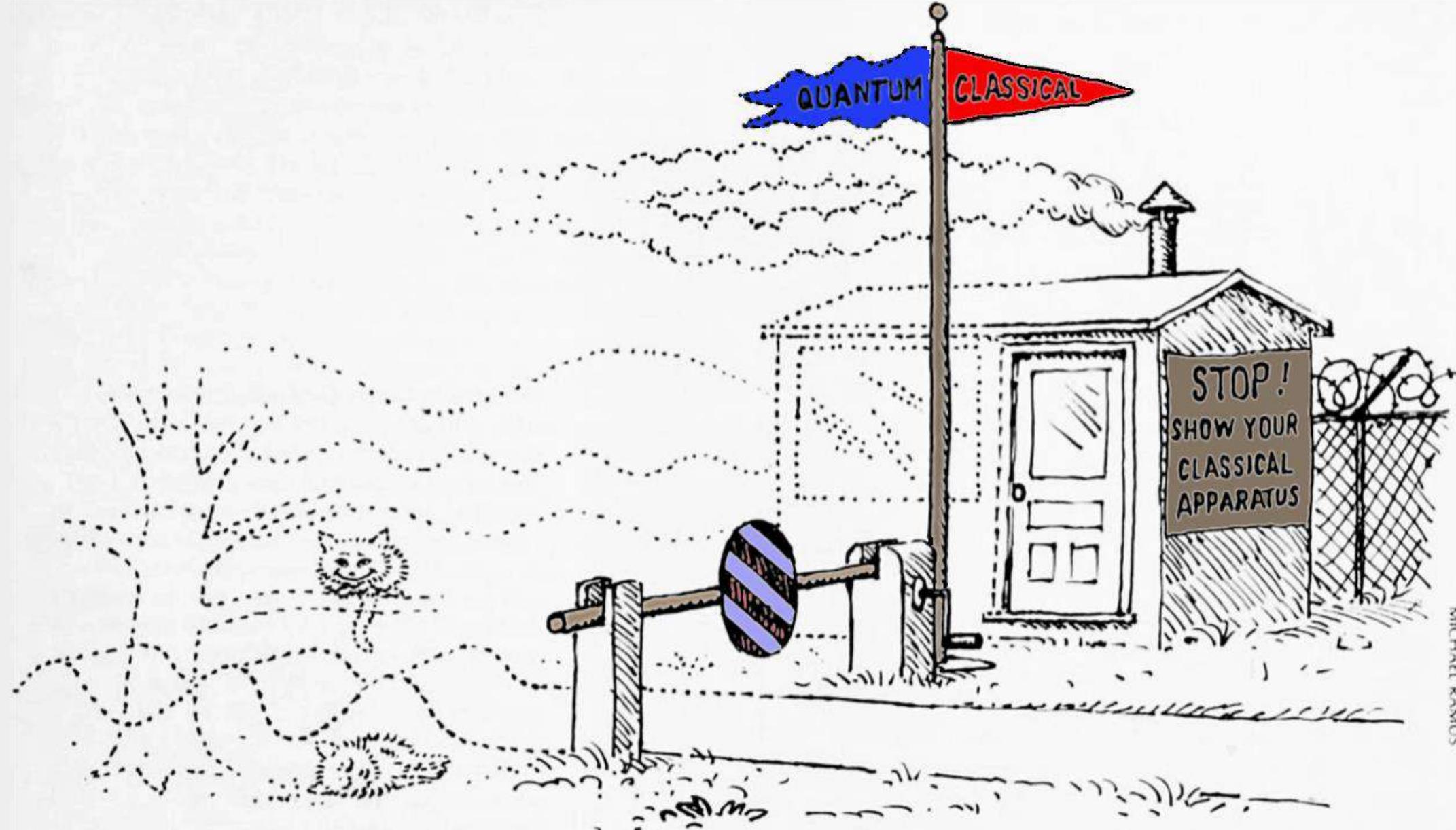
Quantum harmonic oscillator



Unveiling the quantum

Dangerous bend ahead
on quantization





Drawing: Zurek, Physics Today (1991)



Dirac's canonical quantization



Source: Cambridge University,
Cavendish Laboratory / Wikimedia
Commons

Quantum (commutator)

$$-\frac{i}{\hbar} [\hat{x}, \hat{p}] = \hat{1}$$

$$\frac{d}{dt} \hat{O} = -\frac{i}{\hbar} [\hat{O}, \hat{H}]$$

where $\hat{O}(\hat{x}, \hat{p}; t), \hat{H}(\hat{x}, \hat{p}; t)$

These look
a lot alike!

Classical (Poisson bracket)

$$\{x, p\}_P = 1$$

$$\frac{d}{dt} O = \{O, \mathcal{H}\}_P$$

where $O(x, p; t), \mathcal{H}(x, p; t)$

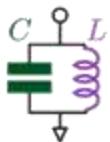
Procedure: Supplant the Poisson brackets by commutators

The Principles
of
Quantum Mechanics
FOURTH EDITION
P. A. M. DIRAC

$$\{x, p\} \mapsto \frac{1}{i\hbar} [\hat{x}, \hat{p}]$$

$$\{O, O'\} \mapsto \frac{1}{i\hbar} [\hat{O}, \hat{O}']$$

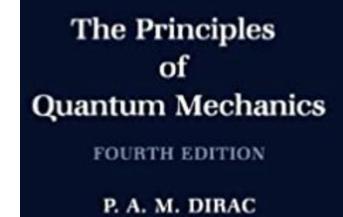
“There is, however, a fairly general method of obtaining quantum conditions, applicable to a very large class of dynamical systems. This is the **method of classical analogy**”
P.A.M. Dirac



Dirac's canonical quantization: Quick exposure

Supplant classical Poisson bracket
and all quantum algebra follows...

$$\{O, O'\} \mapsto \frac{1}{i\hbar} [\hat{O}, \hat{O}']$$



Derivation: Dirac derives the quantum form of the Poisson bracket — the commutator — from mere assuming that

1. Classical Poisson bracket rules hold (by analogy, the new theory must be consistent with the old!)
2. The dynamical variables do not commute; i.e., $xp \neq px$
3. The Poisson bracket has a single result and single unique meaning



Problems:

- Operator ordering ambiguities.
Consider A and B are polynomials in x and p ; e.g., x^2xp or pxx^2
- Curvilinear coordinate systems (potentially transmon if $\cos(\Phi/\varphi_0)$ considered wrapped)
- Quantum gravity ...



If I knew what I was doing, it
wouldn't be called research.

Albert Einstein
See Hawken *et al.* (2010)

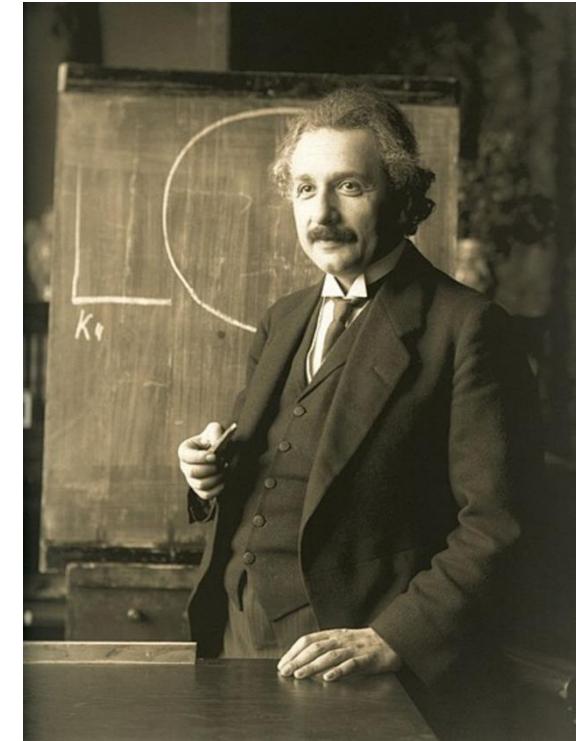
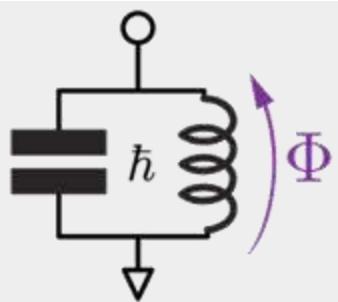


Photo: F. Schmutz



The classical and quantum oscillator

Classical

Quantum

$$\Phi(t) \mapsto \hat{\Phi}$$

$$Q(t) \mapsto \hat{Q}$$

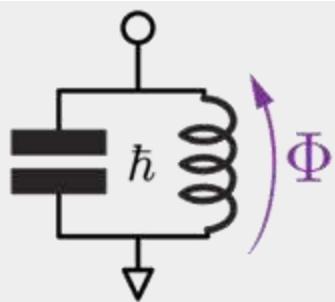
$$\mathcal{H}(\Phi, Q) \mapsto \hat{H}\left(\hat{\Phi}, \hat{Q}\right)$$

$$\{\Phi, Q\} = 1 \mapsto [\hat{\Phi}, \hat{Q}] = i\hbar\hat{1}$$

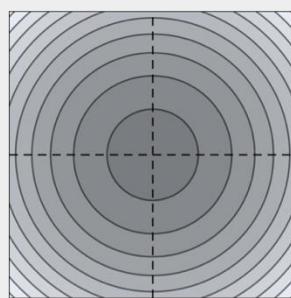
$$\{\alpha, \alpha^*\} = 1/(i\hbar) \mapsto [\hat{a}, \hat{a}^\dagger] = \hat{1}$$

$$\{A, B\} = \frac{\partial A}{\partial \Phi} \frac{\partial B}{\partial Q} - \frac{\partial B}{\partial \Phi} \frac{\partial A}{\partial Q}$$

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$



The classical and quantum oscillator



Hamiltonian

Classical

Quantum

$$\begin{aligned}\mathcal{H} &= \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \\ &= \frac{1}{2}\hbar\omega_0 (\alpha^*\alpha + \alpha\alpha^*)\end{aligned}$$

$$\begin{aligned}\hat{H} &= \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} \\ &= \hbar\omega_0 (\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\end{aligned}$$

Phase space

$$\alpha(t) = \sqrt{\frac{1}{2\hbar Z}} [\Phi(t) + iZQ(t)]$$

$$\alpha(t) = \alpha(0) e^{-i\omega_0 t}$$

$$\Phi(t) = \sqrt{\frac{\hbar Z}{2}} (\alpha^*(t) + \alpha(t))$$

$$Q(t) = i\sqrt{\frac{\hbar}{2Z}} (\alpha^*(t) - \alpha(t))$$

$$\hat{a} = \sqrt{\frac{1}{2\hbar Z}} (\hat{\Phi} + iZ\hat{Q})$$

$$\hat{a}(t) = \hat{a}(0) e^{-i\omega_0 t} \quad (\text{Heisenberg picture})$$

$$\hat{\Phi} = \Phi_{\text{ZPF}} (\hat{a}^\dagger + \hat{a})$$

$$\hat{Q} = iQ_{\text{ZPF}} (\hat{a}^\dagger - \hat{a})$$

Ladder operators and matrix representation

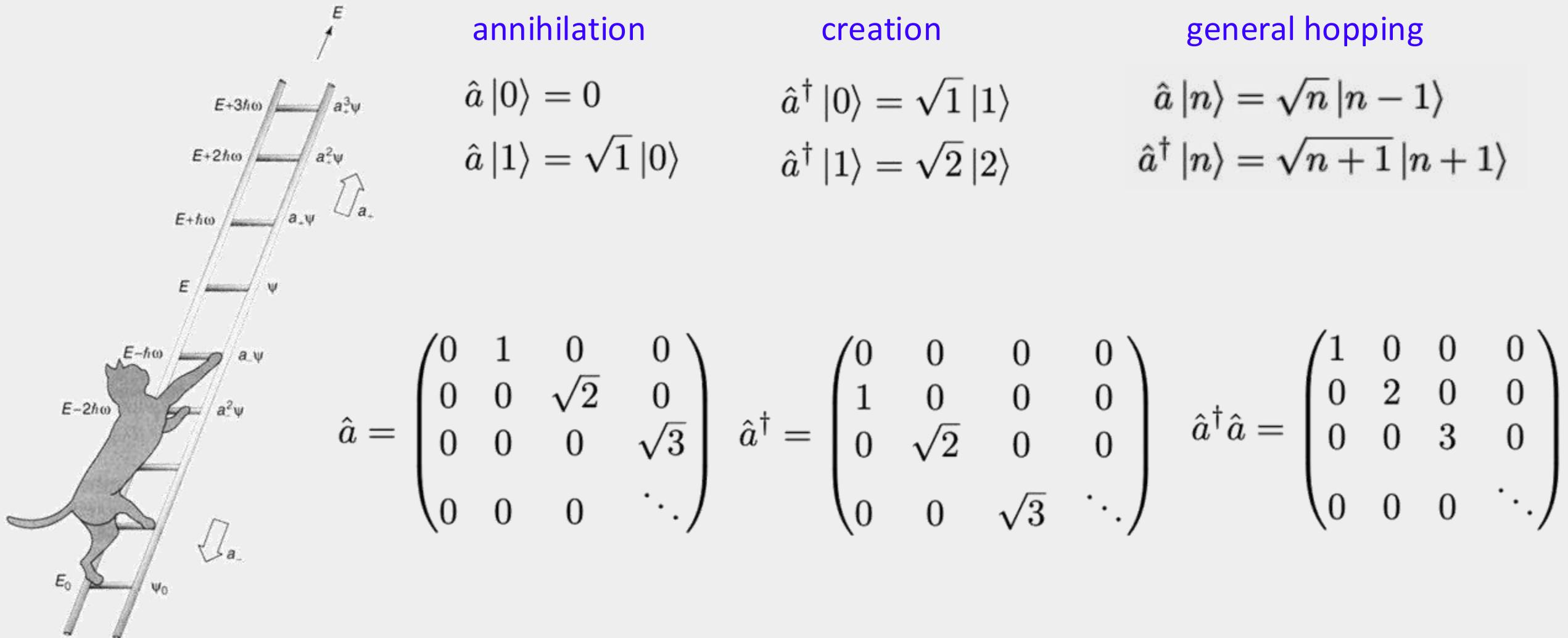


Image: Griffiths, D.J.

Hand-written notes

Hamiltonian and energy

$$\begin{aligned}\hat{H} &= \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} \\ &= \hbar\omega_0 (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)\end{aligned}$$

Mean, variance, and RMS fluctuations

$$\langle 0 | \hat{\Phi}^2 | 0 \rangle = \Phi_{\text{ZPF}}^2$$

$$\langle 0 | \hat{Q}^2 | 0 \rangle = Q_{\text{ZPF}}^2$$

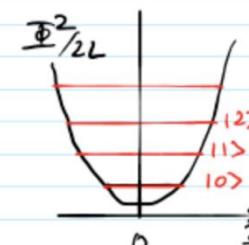
Calculations of the energy

For notational simplicity, I will drop the hats on the operators temporarily and the 0 from ω_0 .

$$\hat{H} = \frac{\hbar\omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$$
$$\hat{a}^\dagger |n\rangle = n |n\rangle \quad \text{Fock}$$
$$\hat{a}^\dagger (|n\rangle) = \hat{a}^\dagger \sqrt{n} |n-1\rangle$$
$$\downarrow \sqrt{n+1} |n+1\rangle \quad \text{using } \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$
$$\downarrow$$
$$= n |n\rangle \quad \hat{a}^\dagger a = \text{Photon number } n$$
$$[\hat{a}, \hat{a}^\dagger] = 1$$
$$a a^\dagger - a^\dagger a = 1$$
$$a a^\dagger = 1 + a^\dagger a$$
$$\hat{H} = \frac{\hbar\omega}{2} (a^\dagger a + a a^\dagger)$$
$$\approx \frac{\hbar\omega}{2} (a^\dagger a + a^\dagger a + 1)$$
$$= \frac{\hbar\omega}{2} (2a^\dagger a + 1)$$
$$\hat{H} = \hbar\omega_0 (a^\dagger a + 1/2)$$

n zero-point energy

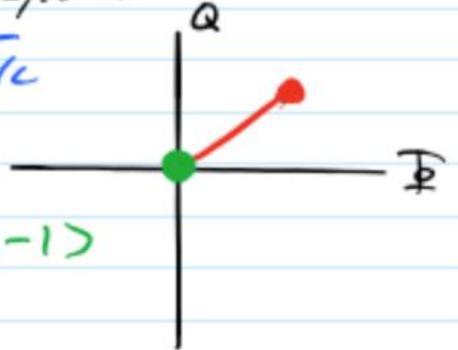
$$\hat{H}|n\rangle \approx E_n |n\rangle$$
$$= \hbar\omega_0 (n + 1/2) \quad n = 0, 1, 2, \dots$$



Expectation value of magnetic flux and charge

$$\langle \hat{\Phi} \rangle = \langle n | \hat{\Phi}_{\text{BPF}} (\hat{a} + \hat{a}^\dagger) | n \rangle$$

$\downarrow \text{some} \# = \sqrt{\frac{b^2}{2}}$ $Z = \sqrt{Lc}$

$$\langle n | a | n \rangle = \sqrt{n} \langle n | n-1 \rangle = 0$$
$$\langle n | a^\dagger | n \rangle = \sqrt{n+1} \langle n | n+1 \rangle = 0$$
$$\langle \hat{\Phi} \rangle = \hat{\Phi}_{\text{BPF}} (\langle \hat{a} \rangle + \langle a^\dagger \rangle) = 0$$
$$\begin{aligned} \langle \hat{Q} \rangle &= \langle n | -i \hat{Q}_{\text{BPF}} (a^\dagger - a) | n \rangle \\ &= -i \hat{Q}_{\text{BPF}} (\langle a^\dagger \rangle - \langle a \rangle) = 0 \end{aligned}$$
$$Q_{\text{BPF}} = \sqrt{\frac{\pi}{2Z}}$$


Fluctuations of flux

$$\begin{aligned}\text{Var}(\hat{\Phi}) &= \langle \hat{\Phi}^2 \rangle - \langle \hat{\Phi} \rangle^2 \\ &= \langle \hat{\Phi}^2 \rangle \\ &= \langle n | (a^\dagger + a)^2 | n \rangle \times \bar{\Phi}_{\text{ZPF}}^2 \\ &= \langle n | a^{+2} + a^2 + a^\dagger a + a a^\dagger | n \rangle + \dots \\ &= \cancel{\langle n | a^{+2} | n \rangle}^{\text{c.c.}} + \cancel{\langle a^2 \rangle}^{\text{c.c.}} + \langle a^\dagger a + a a^\dagger \rangle \times \bar{\Phi}_{\text{ZPF}}^2 \\ &= \bar{\Phi}_{\text{ZPF}}^2 \langle a^\dagger a + a a^\dagger \rangle \\ &= \bar{\Phi}_{\text{ZPF}}^2 \langle a^\dagger a + a^\dagger a + 1 \rangle\end{aligned}$$

Fluctuations of flux and charge

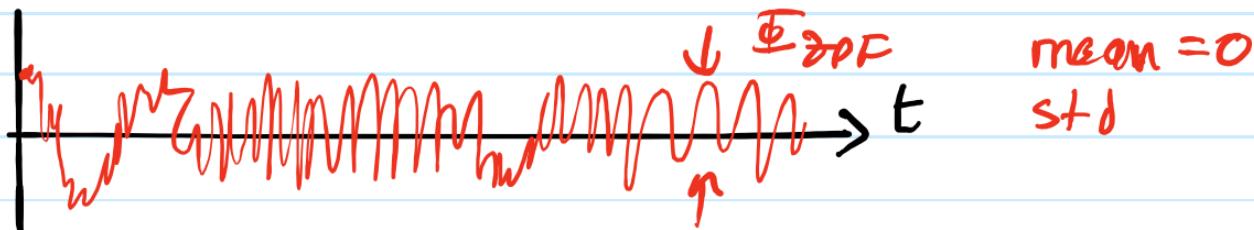
$$\text{Var}(\hat{\Phi}) = \langle \hat{\Phi}^2 \rangle = (2n+1) \Phi_{\text{ZPF}}^2$$

For $\langle n \rangle = 0$: $\langle \hat{\Phi}^2 \rangle = \Phi_{\text{ZPF}}^2$

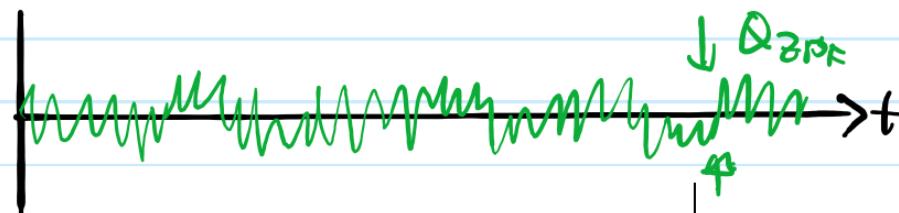
i.e., RMS fluctuation of $\hat{\Phi}$: in ground | 0 >

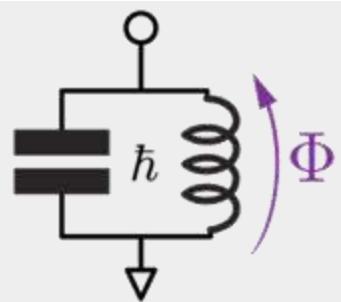
$$\sqrt{\hat{\Phi}} = \Phi_{\text{ZPF}}$$

measured $\hat{\Phi}(t)$



$Q(t)$

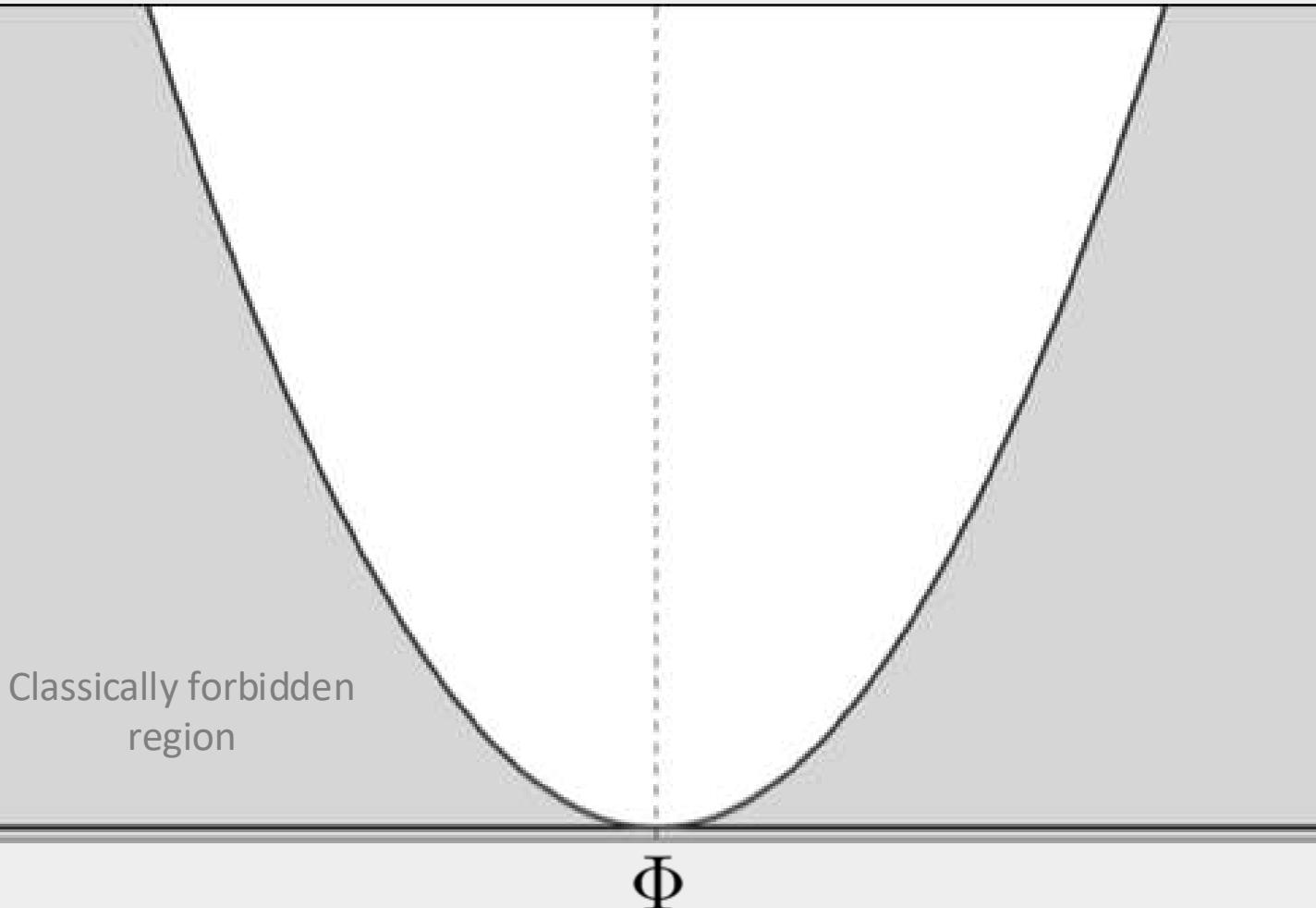


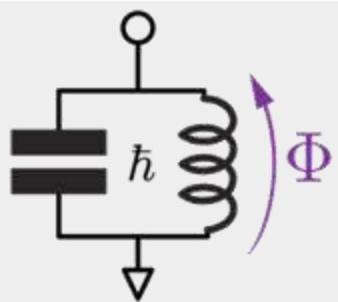


Wavefunctions of the quantum oscillator

$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

Energy



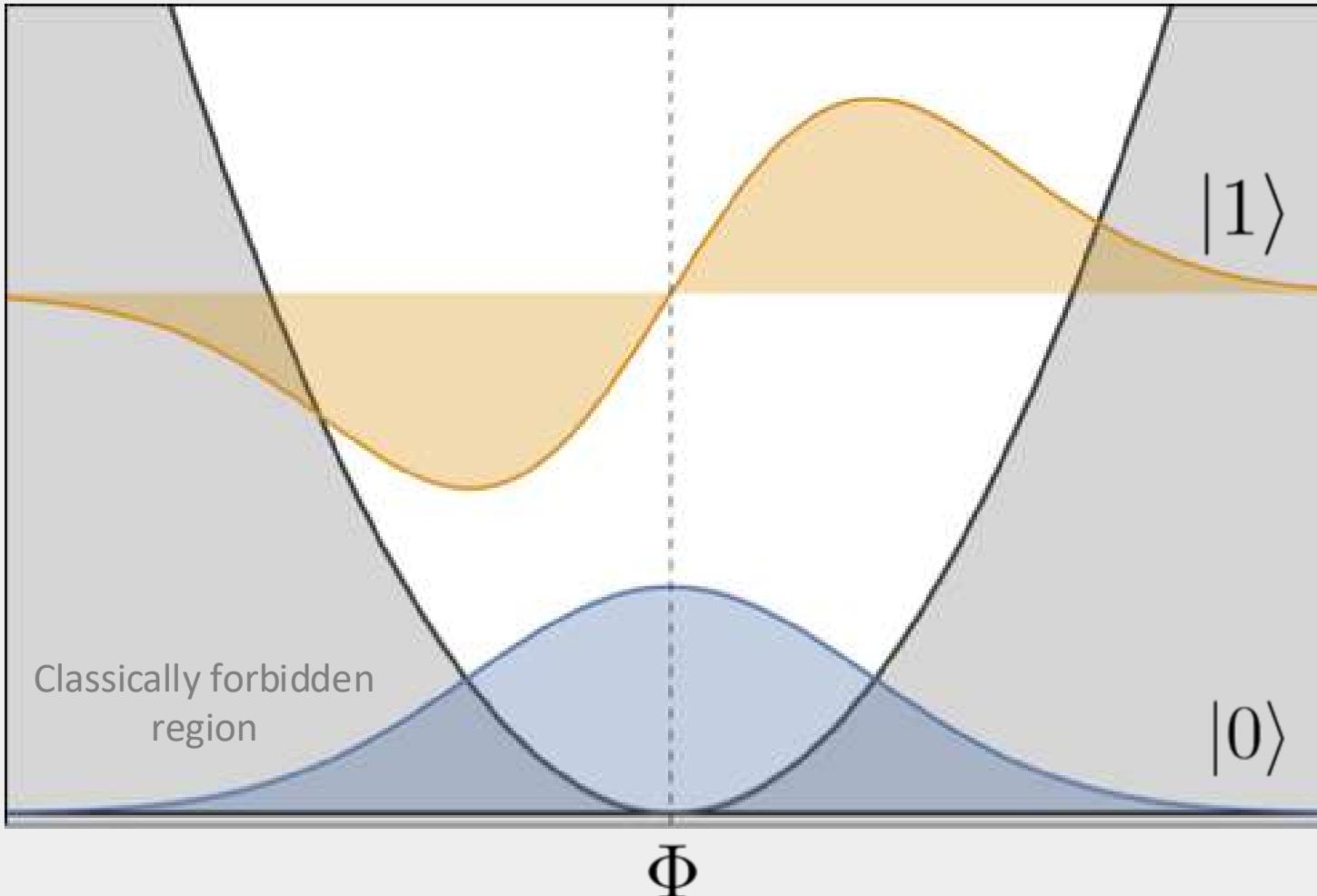


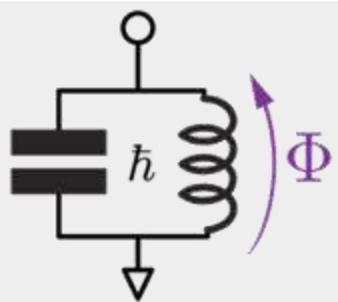
Wavefunctions of the quantum oscillator

$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

$$\psi_n(\Phi) \equiv \langle \Phi | n \rangle$$

Energy /
Scaled
wavefunction
amplitude



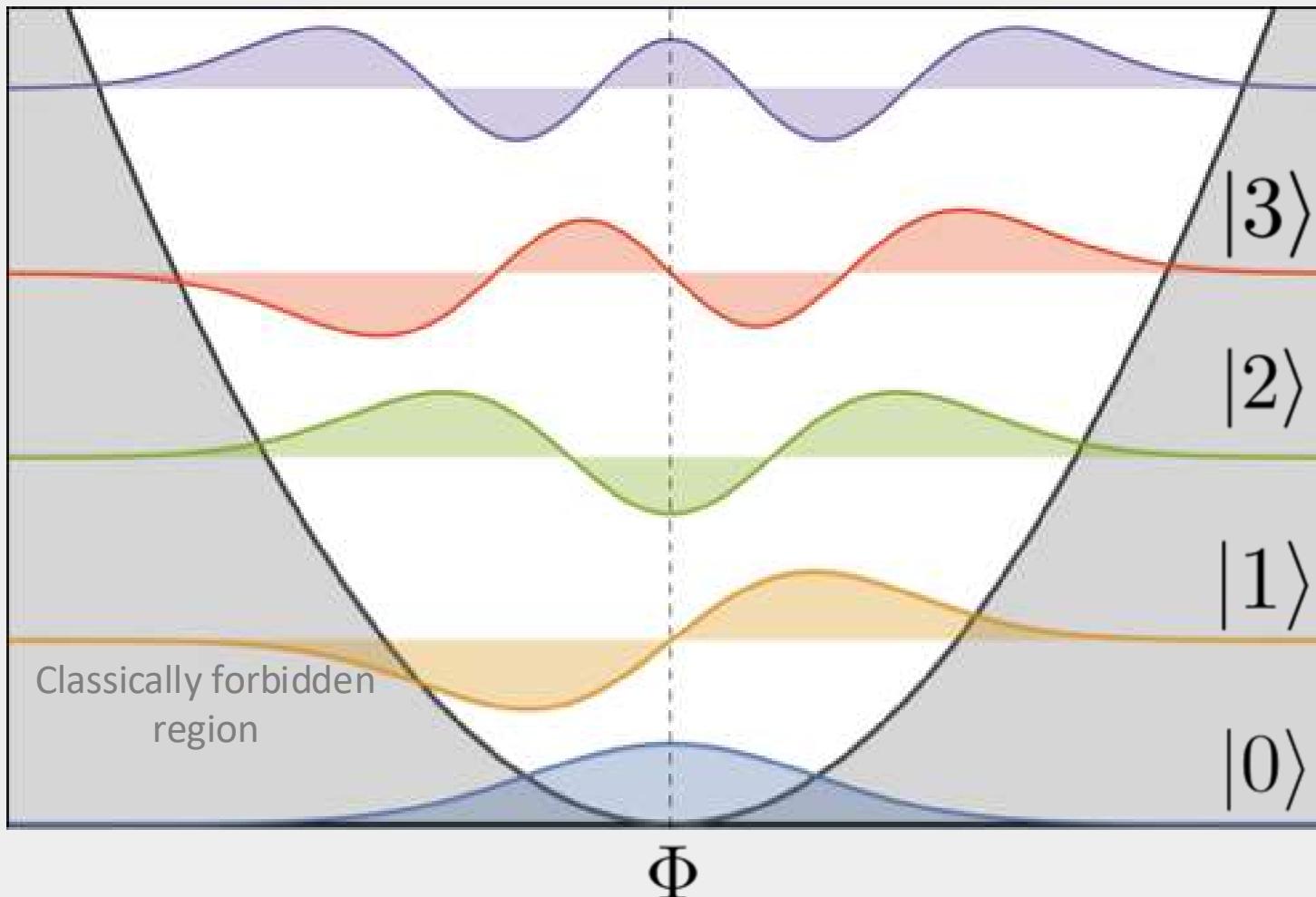


Wavefunctions of the quantum oscillator

$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

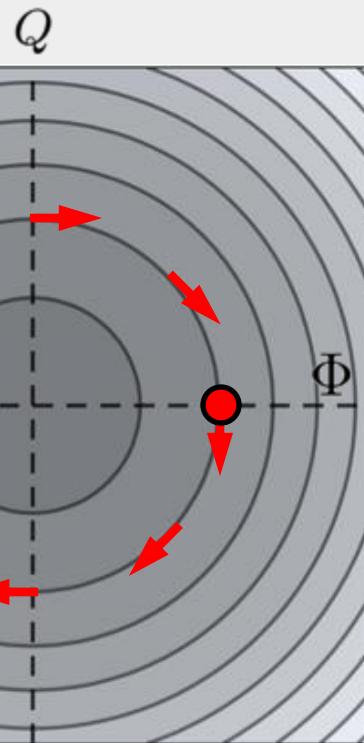
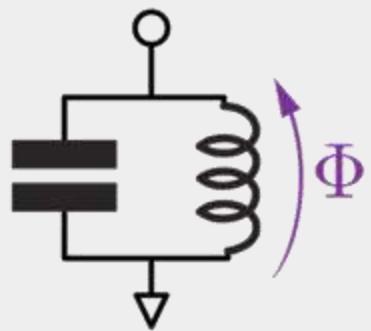
$$\psi_n(\Phi) \equiv \langle \Phi | n \rangle$$

Energy /
Scaled
wavefunction
amplitude

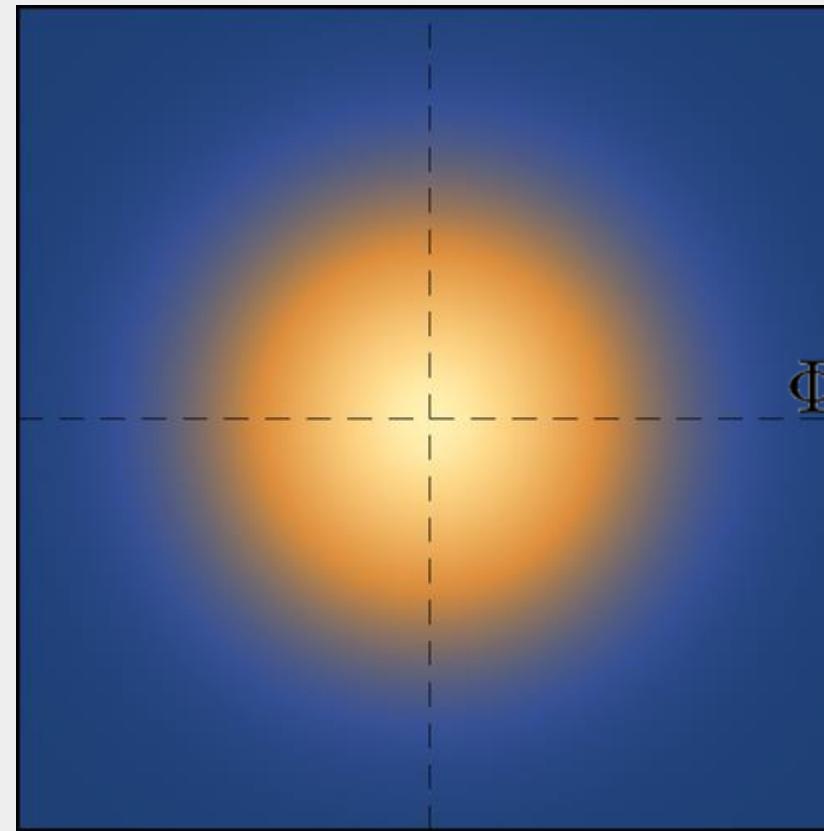




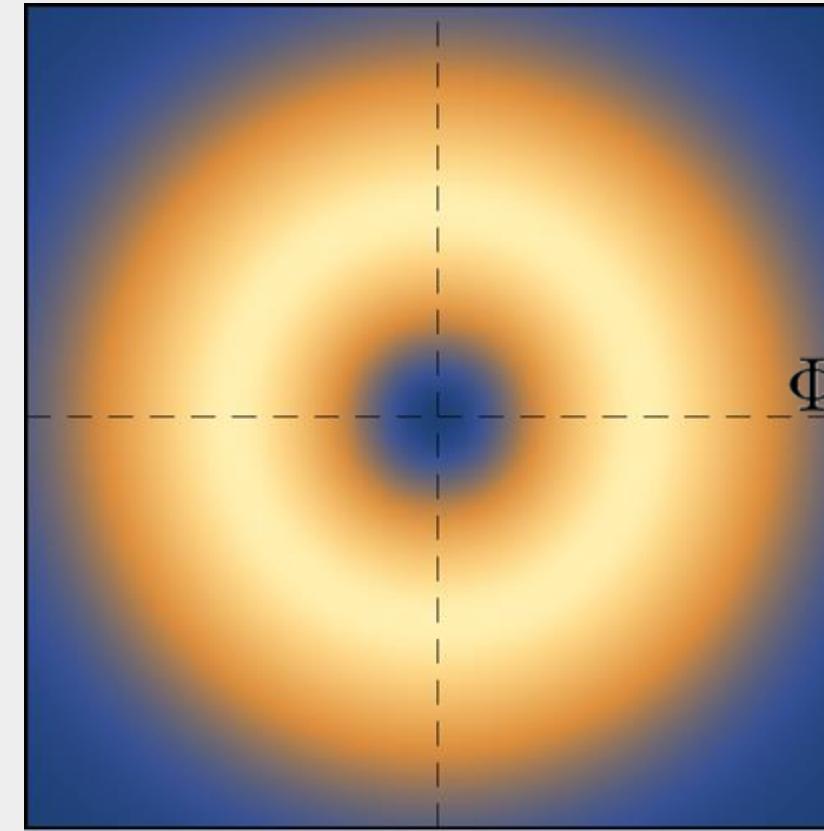
Phase-space (Husimi Q function)



$|0\rangle$
 Q

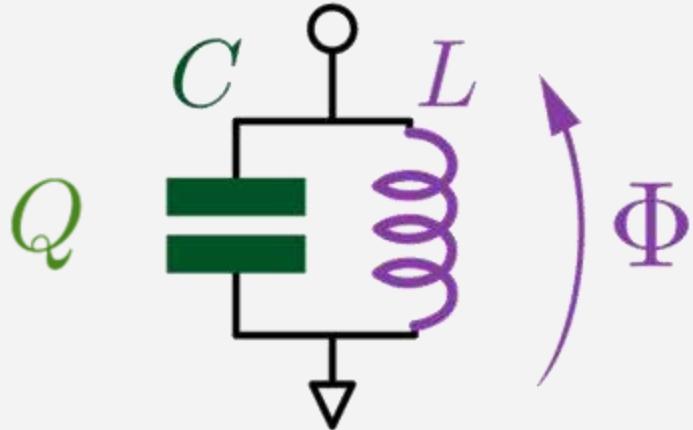


$|1\rangle$
 Q



$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle$$

Pop-up question



The flux and charge operators are Hermitian observables.

How can some expectations, such as

$$\langle 0 | \hat{\Phi} \hat{Q} | 0 \rangle = \frac{1}{4} i ,$$

be imaginary?

Or, others be negative...?

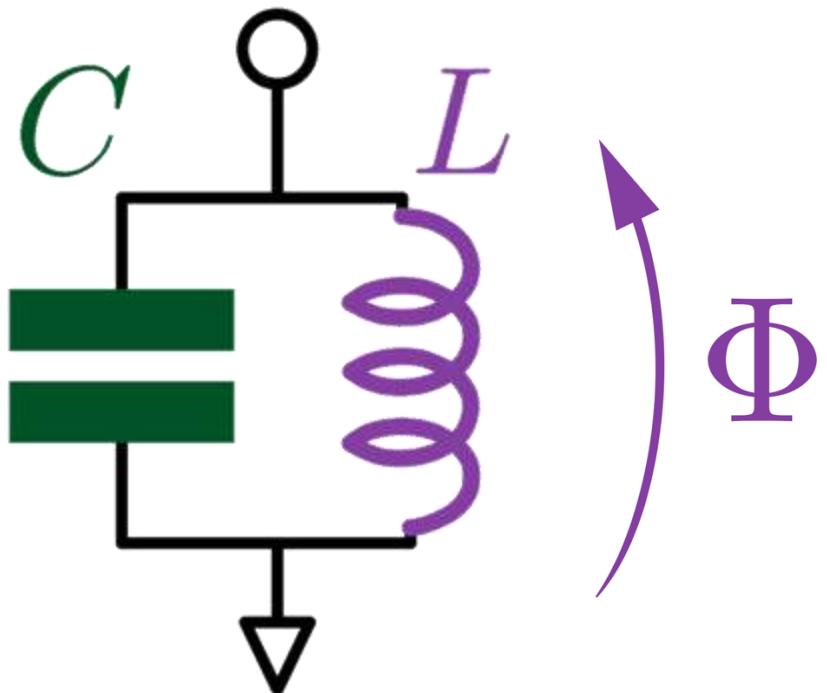
Advanced questions

1. What is $\hat{\Phi}(t)$ in terms of \hat{a} in the Heisenberg picture?
2. **Autocorrelation.** Find the autocorrelation operator $\hat{C}_{\Phi\Phi}(t) = \hat{\Phi}(t)\hat{\Phi}(0)$. What frequency components does it have?
 - (a) What is the flux autocorrelation expectation value $C_{\Phi\Phi}(t) = \langle n | \hat{\Phi}(t)\hat{\Phi}(0) | n \rangle$ for the Fock state $|n\rangle$? Is it real? Why not? What is the frequency spectrum?
 - (b) Repeat for a coherent state $|\alpha\rangle$.
3. **Thermal state.** The thermal state is $\hat{\rho}_{\text{th}} = \exp[-\beta\hat{a}^\dagger\hat{a}] / \text{Tr}[\exp(-\beta\hat{a}^\dagger\hat{a})]$, where $\beta = \hbar\omega_0/k_B T$, and T is the temperature of the oscillator.
 - (a) What is the mean and variance of the flux $\hat{\Phi}$ and charge \hat{Q} operators?
 - (b) How does the frequency spectrum of the autocorrelation $\langle \hat{C}_{\Phi\Phi}(t) \rangle = \text{Tr}[\hat{\rho}_{\text{th}}\hat{C}_{\Phi\Phi}(t)]$ change when with that for the state $|n\rangle$ and $|\alpha\rangle$?
 - (c) The spectrum is not symmetric in frequency. How can you interpret that positive and negative frequencies have different weights? How is this related to absorption and emission of the oscillator? (Consider Fermi's golden rule).
 - (d) What happens to the spectrum in the limit of high temperature, $k_B T \gg \hbar\omega_0$? How about low temperature, $k_B T \ll \hbar\omega_0$?
 - (e) How do the above conclusions change for charge; i.e., for $\langle \hat{C}_{QQ}(t) \rangle = \text{Tr}[\hat{\rho}_{\text{th}}\hat{C}_{QQ}(t)]$?

Will discuss
answers on my
blog sometime
soon



Quantum Harmonic Oscillator Applets

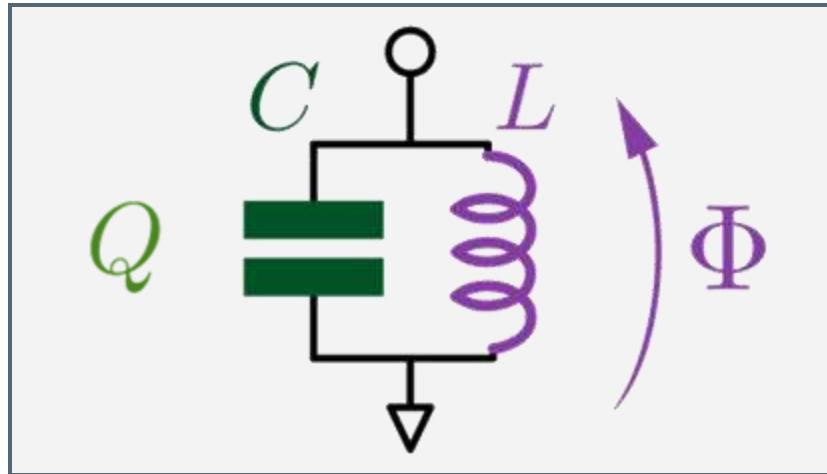


Energy levels of SHO applet from
<https://www.st-andrews.ac.uk/physics/quvis>

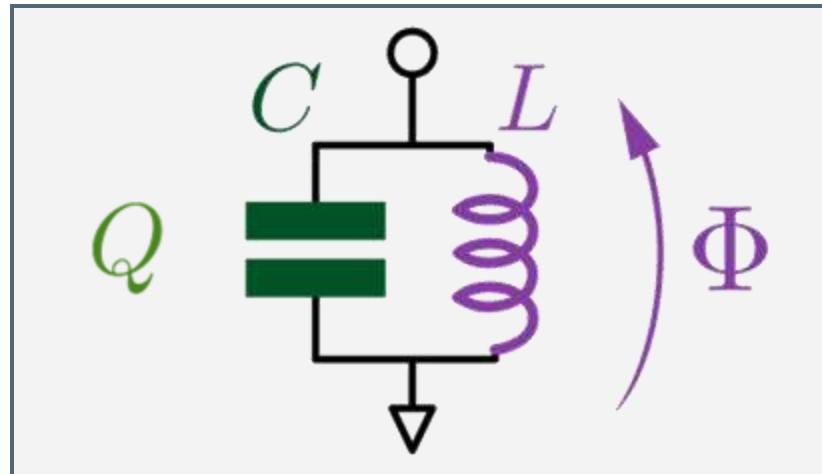
Wigner function
<https://demonstrations.wolfram.com/WignerFunctionOfHarmonicOscillator/>

Coherent states
<https://demonstrations.wolfram.com/CoherentStatesOfTheHarmonicOscillator/>

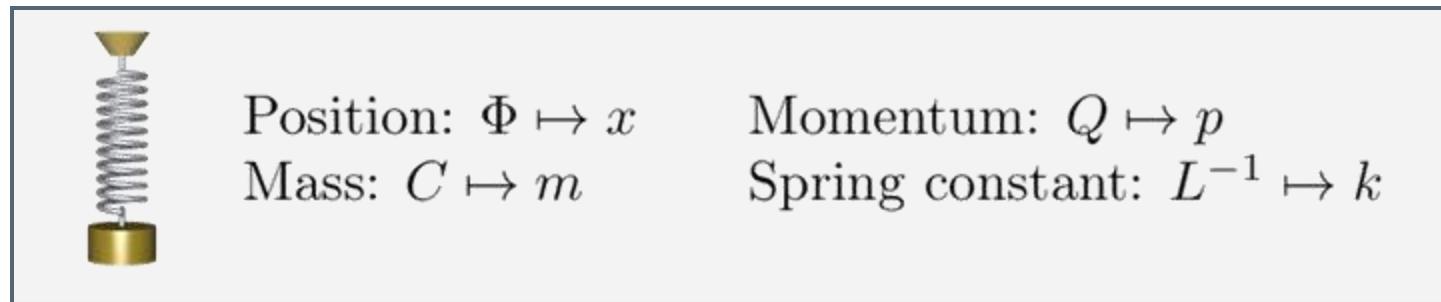
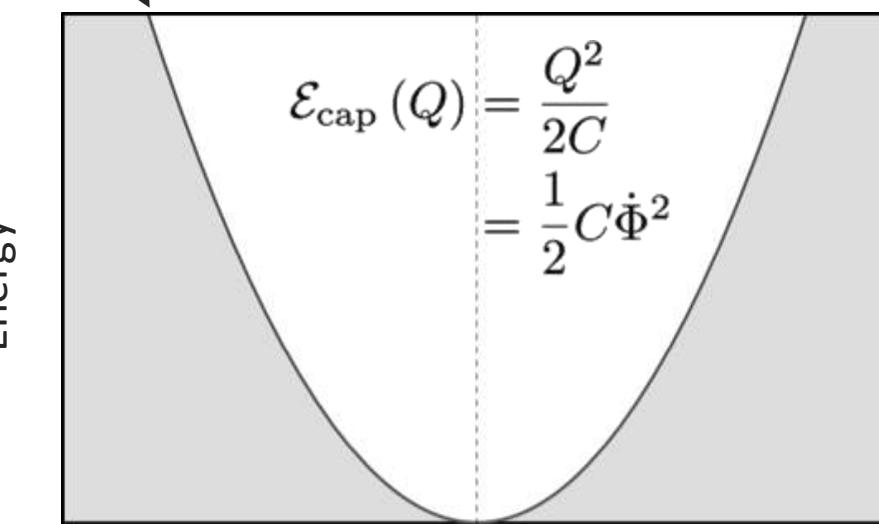
Linear harmonic oscillator summary



The LC quantum harmonic oscillator



$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$



$$\hat{H}(\hat{\Phi}, \hat{Q}) = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\hat{H}|n\rangle = \hbar\omega_0 \left(n + \frac{1}{2} \right) |n\rangle$$

$$Z_0 = \frac{L}{C}$$

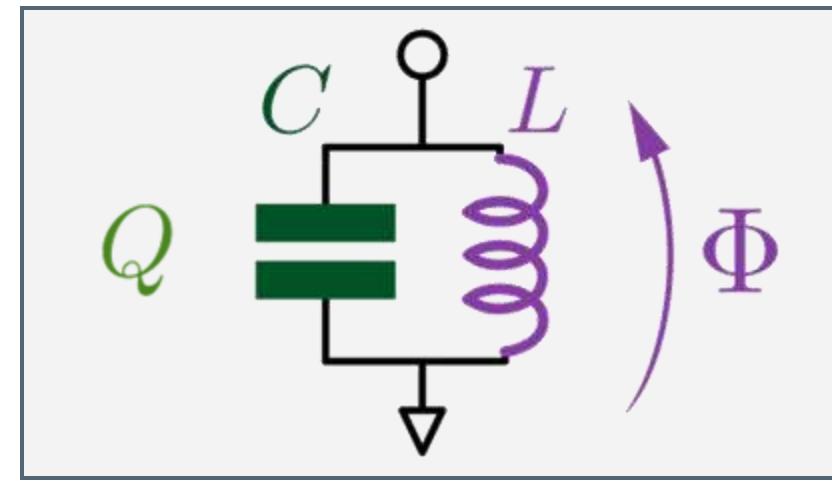
$$\hat{\Phi} = \Phi_{\text{ZPF}} (\hat{a}^\dagger + \hat{a})$$

$$\Phi_{\text{ZPF}} = \sqrt{\frac{\hbar}{2}} Z_0 = \Phi_0 \sqrt{\frac{z_0}{2\pi}},$$

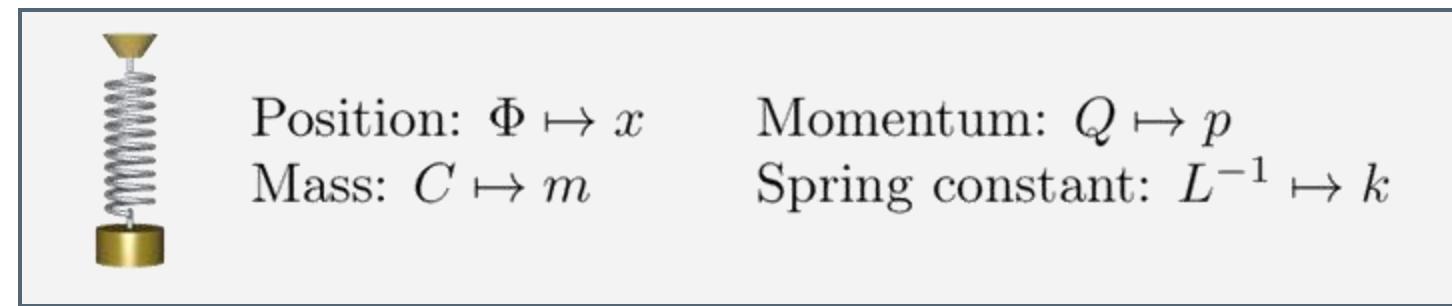
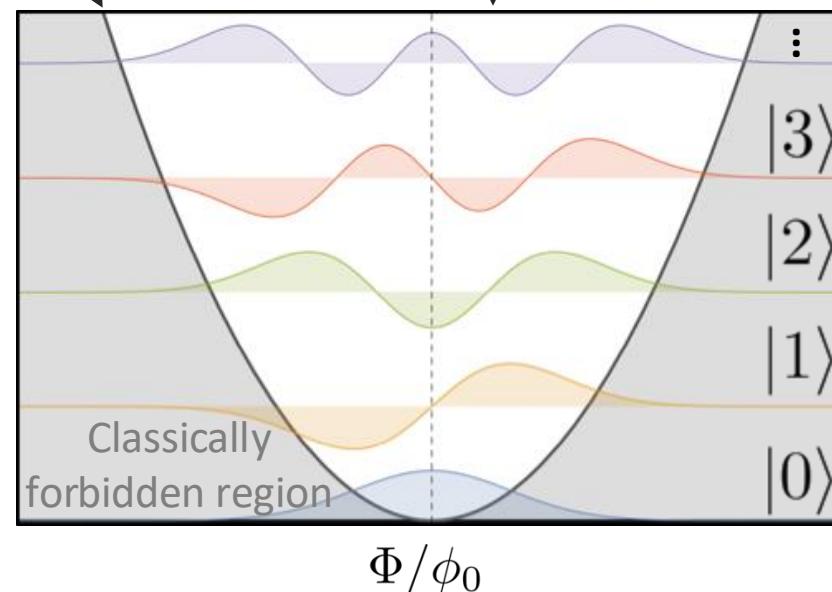
$$\hat{Q} = iQ_{\text{ZPF}} (\hat{a}^\dagger - \hat{a})$$

$$Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2}} Z_0^{-1} = (2e) \sqrt{\frac{1}{2\pi z_0}},$$

The LC quantum harmonic oscillator



$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L} \quad \psi_n(\Phi) \equiv \langle \Phi | n \rangle$$



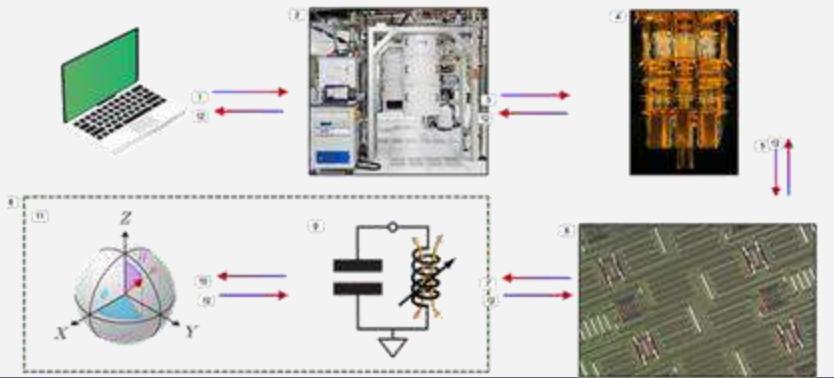
$$\hat{H}(\hat{\Phi}, \hat{Q}) = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad \omega_0^2 = \frac{1}{LC}$$

$$\hat{H}|n\rangle = \hbar\omega_0 \left(n + \frac{1}{2} \right) |n\rangle \quad \Phi_{\text{ZPF}} Q_{\text{ZPF}} = \frac{\hbar}{2} \quad Z_0 = \frac{L}{C}$$

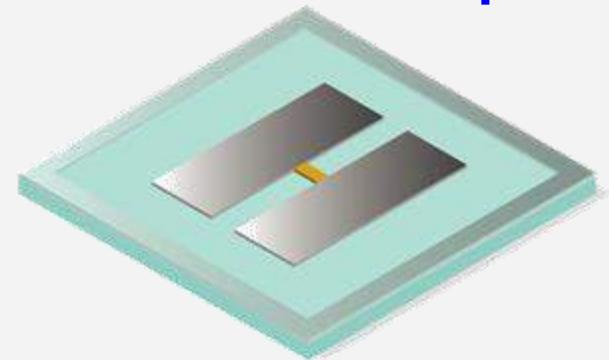
$$\begin{aligned} \hat{\Phi} &= \Phi_{\text{ZPF}} (\hat{a}^\dagger + \hat{a}) & \Phi_{\text{ZPF}} &= \sqrt{\frac{\hbar}{2} Z_0} & \langle 0 | \hat{\Phi}^2 | 0 \rangle &= \Phi_{\text{ZPF}}^2 \\ \hat{Q} &= i Q_{\text{ZPF}} (\hat{a}^\dagger - \hat{a}) & Q_{\text{ZPF}} &= \sqrt{\frac{\hbar}{2} Z_0^{-1}} & \langle 0 | \hat{Q}^2 | 0 \rangle &= Q_{\text{ZPF}}^2 \end{aligned}$$

The road behind and ahead

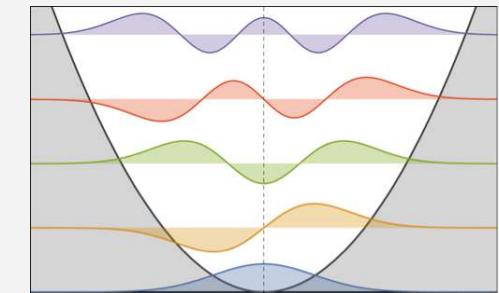
Qubit in the cloud



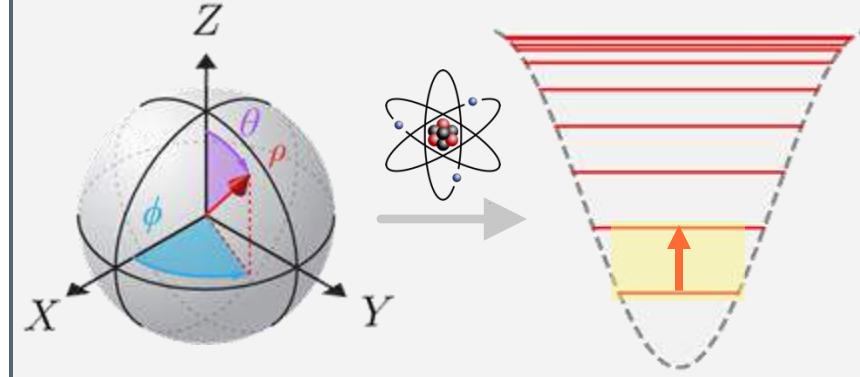
cQED: Transmon qubit



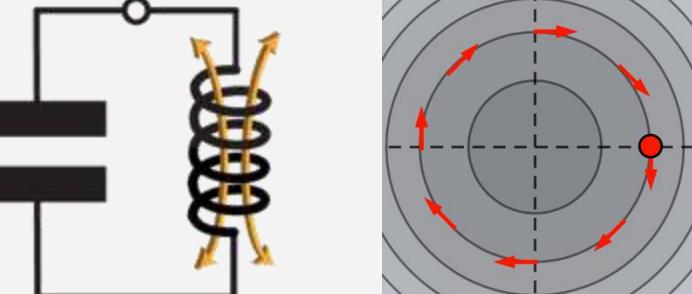
Unveiling the quantum oscillator



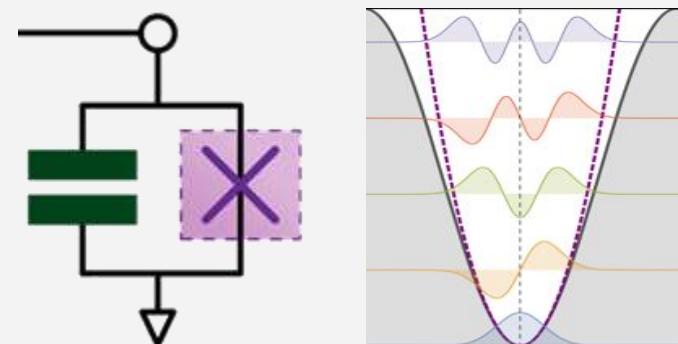
Qubit from atom / oscillator



Classical circuits & the LC



Transmon qubit



Next steps

Tightly integrated lab work with Dr. Nick Bronn and Co.!

More depth on qubit control

Run experiments on real devices



Check out references, problems given in the lecture,
dangerous bends

Break away from the rules of today



Thank you!

Zlatko K. Minev



@zlatko_minev



zlatko-minev.com

IBM Quantum