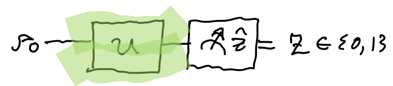
(C65B) PEC - 1Q clean - quantum derivation with ansatz

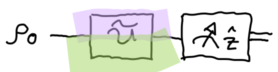
Sunday, July 23, 2023 12:18 PM

Setup

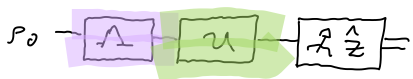


Details on notation:

Quantum register alphabet $\Sigma = \{0,1\}$
Hilbert space $\mathcal{H} = \mathbb{C}^{\Sigma}$
Initial state $\rho_0 \in D(\mathcal{H}) \subset L(\mathcal{H})$
Ideal unitary $U \in U(\mathcal{H}) \subset L(\mathcal{H})$
Ideal u-channel $\mathcal{U}(\rho) = U \rho U^{\dagger}$
$\mathcal{U} \in C(\mathcal{H}) \subset L(L(\mathcal{H}))$



Noisy gate/circuit $\tilde{\mathcal{U}} \in L(L(\mathcal{H}))$



Decompose noisy gate $\tilde{\mathcal{U}} = \mathcal{U}\Lambda$

Simple Example

Keeping it simple and illustrative, let's do a simple case

Let $U = I$
$\mathcal{U} := I \cdot I$

For the noise, let's play with the simplest bit-flip channel

$$\Lambda(\rho) = \underbrace{(1-p)}_{\text{prob of no error}} I \rho I + \underbrace{p}_{\text{prob of a bit flip error}} X \rho X$$

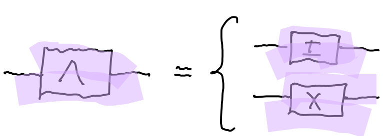
$$\Lambda_\rho = (1-p)\mathcal{I}_\rho + p\,\mathcal{X}_\rho$$

Equivalent superoperator channel representation

$\mathcal{X}_\rho = X \rho X$

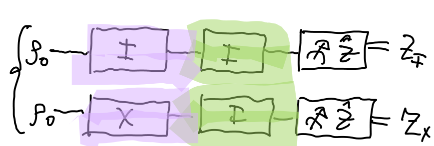
$\mathcal{I}_\rho = I \rho I = \rho$

Equivalent trajectory unraveling



: Prob $1-p$
: Prob p

Our circuit then is equivalent to either



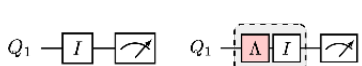
with prob $1-p$
with prob p

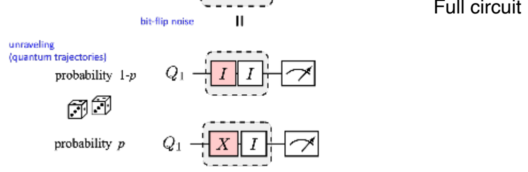
The ideal expectation value is

$$z_{ideal} = \langle \hat{z} \rangle = \mathrm{Tr}(Z \mathcal{I} \rho_0) = \mathrm{Tr}(Z \rho_0) = \rho_z$$

When the channel introduces an error however,

$$\mathbb{E}[z_X] = \mathrm{Tr}(Z X \rho X) = \mathrm{Tr}(X Z X \rho) = \mathrm{Tr}(-Z\rho) = -\rho_z$$

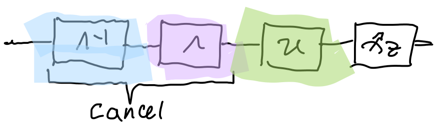




$$\Lambda(\rho) = (1-p) I \rho I + p X \rho X$$

Noise Inverse

to undo the noise, we'd like to introduce the inverse noise



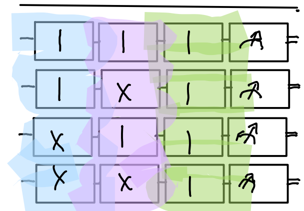
$$\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = I$$

Taking the ansatz $\Lambda^{-1}(\rho) = (1-r) I \cdot I + r(X \cdot X)$

we see 4 cases of unraveling

inverse	noise	no error	prob
I	I	✓	$(1-r)(1-p)$
I	X	X	$(1-r)p$
X	I	X	$r(1-p)$
X	X	✓	rp

Circuit



ideally, we what to interfere trajectories so that the no-error on will coherently add to unity probably, and the ones with an error will cancel.

$$\therefore \text{(A)} \quad (1-r)(1-p) + r \cdot p = 1 \qquad \wedge \qquad \text{(B)} \quad (1-r)p + r(1-p) = 0$$

$$1 - r - p + 2rp = 1 \qquad\qquad p + r - 2rp = 0$$

$$r + p - 2rp = 0 \qquad\qquad \text{same condition}$$

$$\Rightarrow \quad r(1-2p) = -p$$

$$\therefore \quad \boxed{r = \frac{-p}{1-2p}}$$

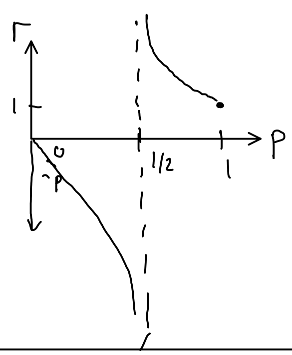
Recall p is a probability $0 \le p \le 1$,

$p = 0 \Rightarrow r = 0$ — no noise, no need to do anything

$p = 1 \Rightarrow r = 1$ — deterministic bit-flip, requires deterministic bit flip as well for Λ^{-1}

$p = 1/2 \Rightarrow r = \infty$ — singular value, since at $p = 1/2$, we fully scramble the state

$p \ll 1 \Rightarrow r \simeq -p$



Note that we could equivalently have used the algebraic condition and solved for r

$\Lambda(\Lambda^{-1}(\rho)) = I(\rho) = \rho$ Solve for r

$$= \Lambda\left[(1-r)\rho + r X \rho X \right)$$

$$= \underbrace{(1-p)(1-r)\rho + pr \cancel{XX\rho XX}^{I}}_{\text{no error}} + \underbrace{(1-p) r X \rho X + (1-r) p X \rho X}_{\text{error}}$$

$$= \overbrace{\left[(1-p)(1-r) + pr\right]}^{\text{no error}} \rho + \overbrace{\left[(1-p) r + (1-r) p\right]}^{\text{error}} X \rho X$$

same conditions as above & solution $r = \frac{-p}{1-2p}$

How to implement? Quasi-Probability

$$\Lambda^{-1} = (1-r) I \rho I + r X \rho X \qquad r = \frac{-p}{1-2p}$$

$$= \left[\frac{|1-r|}{|1-r| + |r|} \operatorname{sgn}(1-r) I \rho I + \frac{|r|}{|1-r|+|r|} \operatorname{sgn}(r) X \rho X\right] (|1-r| + |r|)$$

$$= \gamma \left[s_I p_I I \rho I + s_X p_X X \rho X \right]$$

with

$$\gamma = |1-r| + |r|$$

$$p_I = \frac{|1-r|}{\gamma} \qquad s_I = \operatorname{sgn}(1-r)$$

$$p_X = \frac{|r|}{\gamma} \qquad s_X = \operatorname{sgn}(r)$$

valid prob distribution

$$0 \le p_I, p_X \le 1 \quad \text{and} \quad |p_I| + |p_X| = 1$$

How to sample?

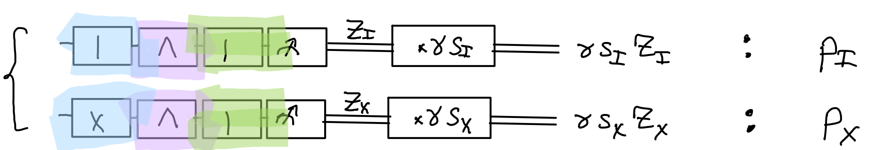
$$\langle Z \rangle = \operatorname{Tr}(Z \tilde{I} \Lambda \Lambda^{-1} \rho_0)$$

$$= \operatorname{Tr}(Z \tilde{I} [\gamma s_I p_I \rho_0 + \gamma s_X p_X X \rho_0 X])$$

$$= \gamma s_I p_I \operatorname{Tr}(Z \tilde{I} \rho_0) + \gamma s_X p_X \operatorname{Tr}(Z \tilde{I} X \rho_0 X)$$

$$= \gamma \left[s_I p_I \underbrace{\langle Z \rangle_I}_{\text{quantum circuit exp. val}} + s_X p_X \underbrace{\langle Z \rangle_X}_{\text{we can find}} \right]$$

Equivallent interportation: sample prob



Estimator $\quad E_{mitg} = \gamma s_I Z_I + \gamma s_X Z_X$

$$\mathbb{E}[E_{mitg}] = \langle \hat{Z} \rangle_{ideal}$$

$$\mathbb{V}[E_{mitg}] = \mathbb{V}[\gamma s_I Z_I] + \mathbb{V}[\gamma s_X Z_X]$$

$$= \gamma^2 \mathbb{V}[Z_I] + \gamma^2 \mathbb{V}[Z_X]$$

$$= \gamma^2 (2 \sigma_{ideal}^2)$$

$$\sigma_{ideal}^2 = \mathbb{V}[Z_I] = 4q(1-q)$$

$$q = 1 - 2 p_Z = \langle \tfrac{1-Z}{2} \rangle$$

Since the X just flip $Z \to -Z$ of ρ, it follows that the variance is the same, since symmetric

Detailed calculation: [SKIP IN LECTURE]

$Z_X, Z_I \in \{-1, +1\}$ rand Bernoulli variables

$$Z_I \sim \text{Bernoulli}\left(q;\ |0\rangle \to +1,\ |1\rangle \to -1\right)$$

$$q = \mathrm{Tr}\left(\tfrac{I-\hat{Z}}{2}\, I \Lambda I \rho_0\right) \qquad \text{prob of } |1\rangle$$

$$= \tfrac{1}{2}\left(1 - \mathrm{Tr}(\hat{Z} \Lambda \rho_0)\right)$$

$$= \tfrac{1}{2}\left(1 - \left[(1-p)\,\mathrm{Tr}(\hat{Z} I \rho_0 I) + p\, \mathrm{Tr}(\hat{Z} X \rho_0 X)\right]\right)$$

$$= \tfrac{1}{2}\left(1 - (1-2p)\, \mathrm{Tr}[Z \rho_0]\right)$$

$$= \tfrac{1}{2}\left(1 - \underbrace{(1-2p)}_{\text{fidelity of channel}} \langle \hat{Z} \rangle_{\text{ideal}}\right)$$

fidelity of channel, Bloch

For the other circuit

$$Z_X \sim \text{Bernoulli}\left(q_X;\ |0\rangle \to +1,\ |1\rangle \to -1\right)$$

$$q_X = \mathrm{Tr}\left(\tfrac{\hat{1}-\hat{Z}}{2}\, I \Lambda X \rho_0\right)$$

$$= \tfrac{1}{2}\left(1 + \mathrm{Tr}(Z \Lambda X \rho X)\right)$$

$$= \tfrac{1}{2}\left(1 + \mathrm{Tr}(X Z X \Lambda \rho)\right) \qquad XZX = -Z$$

$$= \tfrac{1}{2}\left(1 \underset{\text{for son}}{-} \underbrace{\mathrm{Tr}(Z \Lambda \rho)}_{\text{found above}}\right)$$

$$= \tfrac{1}{2}\left(1 + (1-2p) \langle \hat{Z} \rangle_{\text{ideal}}\right) \qquad \text{just flipped}$$

$$\therefore \quad q_I = \tfrac{1}{2}\left(1 - f \langle Z_{\text{ideal}} \rangle\right) \qquad f := 1 - 2p$$

$$q_X = \tfrac{1}{2}\left(1 + f \langle Z_{\text{ideal}} \rangle\right)$$

$$\mathbb{E}[E_{\text{mgt}}] = \mathbb{E}\{\gamma s_I Z_I + \gamma s_X Z_X\}$$

$$= \gamma\left(\cdots \quad \right)$$