Tech Note T22: Bounding sum of operators

Zlatko K. Minev (2022-03)

22.1 Positive semidefinite operators

Here, we prove Lemma 3.3 of Watrous (2018) in some detail and point out the neat technique used to cast a sum of operators as a product of operators in a larger, augmenting space, which mixes quantum states and amplitudes. All references in this section refer to Watrous (2018).

Lemma 3.3 of Ref. Watrous (2018). Let $\mathcal{X}$ be a complex Euclidean space, let Σ be an alphabet, let $u \in \mathbb{C}^{\Sigma}$ be a complex vector, and let $\{P_a : a \in \Sigma\} \subset \mathrm{Pos}(\mathcal{X})$ be a collection of *positive semidefinite operators*. It holds that

$$\left\| \sum_{a\in\Sigma} u(a)\, P_a \right\| \leq \|u\|_\infty \left\| \sum_{a\in\Sigma} P_a \right\| , \tag{1}$$

where $u(a) \in \mathbb{C}$ denotes the a-th element of the vector u and the spectral operator norm is equally denoted $\|P\| = \|P\|_\infty$.

In words, the spectral (infinity) norm of the sum of the operators weighted by the complex vector is bounded by the maximum (infinity) norm of u times the the spectral norm of the unweighted total sum. The largest absolute value of the elements of vector u is the ∞-norm $\|u\|_\infty = \max(|u(0)|, |u(1)|, \ldots)$. So, bound is set by the largest absolute value in the weights times the total norm of the unweighted sum. In a sense, this bounding value could represent a worst-case error.

For a looser bound, we can use the triangle inequality property of the norm, $\left\|\sum_{a\in\Sigma} P_a\right\| \leq \sum_{a\in\Sigma} \|P_a\|$.

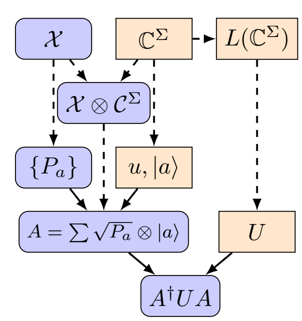
Proof.

First, a few observations:

1. $P_a = P_a^\dagger$ because P_a is positive semidefinite (see bottom of page 17).

Recasting sum of operators as a matrix product. We would like to reframe the sum as an operator product equation to make use of the Schatten norm properties more directly. To achieve this, we first combine the spaces of u and P_a by introducing the auxiliary operator $A \in L\left(\mathcal{X}, \mathcal{X} \otimes \mathbb{C}^{\Sigma}\right)$, where $\mathbb{C}^{\Sigma}$ is the space the vector u lives. The map A takes a quantum state vector and maps it to another one in a space that additionally keeps track of which P_a mapped it.

$$A \coloneqq \sum_{a\in\Sigma} \sqrt{P_a} \otimes e_a = \sum_{a\in\Sigma} \sqrt{P_a} \otimes |a\rangle , \tag{2}$$



where we can think of the unit basis vector $e_a = |a\rangle$ (and define the matrix basis element $E_{a,b} = |a\rangle\langle b|$). It follows,[1] $A^\dagger A = \sum_{a\in\Sigma} P_a$. In other words, A is a factorization of the sum. To factor the weighted sum, we just need a kind of metric, which will be u in matrix form as a diagonal

[1]First, $A^\dagger = \sum_{a\in\Sigma} \sqrt{P_a} \otimes \langle a|$, then $A^\dagger A = \sum_{a\in\Sigma}\sum_{b\in\Sigma} \left(\sqrt{P_a} \otimes \langle a|\right) \cdot \left(\sqrt{P_b} \otimes |b\rangle\right) = \sum_{a,b} \sqrt{P_a}\sqrt{P_b} \otimes \langle a|b\rangle$, using $\langle a|b\rangle = \delta_{ab}$, it follows that $A^\dagger A = \sum_{a\in\Sigma} P_a$.

matrix $U = \mathrm{Diag}(u) \in L(\mathbb{C}^{\Sigma})$. In matrix form then, $U \coloneqq \sum_{a\in\Sigma} u(a) E_{a,a}$. With this 'metric' in place, the sum of Eq. (1) is more succinctly expressed as the 'larger' matrix equation

$$\sum_{a\in\Sigma} u(a) P_a = A^{\dagger} U A , \quad \text{with } A \coloneqq \sum_{a\in\Sigma} \sqrt{P_a} \otimes |a\rangle \text{ and } U \coloneqq \mathrm{Diag}(u) . \tag{3}$$

This construction is quite general and useful. Now, all we have to do is bound $\left\|A^{\dagger} U A\right\|$.

Norms. Let's collect a few quick preparatory observations.

1. Schatten p-norm is submultiplicative: $\|ABC\|_p \leq \|A\|_p \|B\|_p \|C\|_p$ (Eq. 1.176)
2. The Schatten p-norm of the diagonal matrix is just given by the corresponding vector norm $\|U\|_p = \|u\|_p$; i.e., $\|U\| = \|u\|_{\infty}$ (follows from Eq. 1.167).
3. Schatten p-norm is invariant under adjoint $\left\|A^{\dagger}\right\|_p = \|A\|_p$ (Eq. 1.177)
4. Schatten ∞-norm has $\|A\|_{\infty}^2 = \left\|A^{\dagger} A\right\|_{\infty}$ (Eq. 1.178)
5. Schatten p-norm is multiplicative wrt tensor product $\|A \otimes B\| = \|A\| \|B\|$ (true for $p = \infty$ and $p = 2$ have to check for others, should follow from Eq. 1.167)

Bounding norm. For sum, now simply $A^{\dagger} U A$,

$$\begin{aligned}
\left\|\sum_{a\in\Sigma} u(a) P_a\right\| &= \left\|A^{\dagger} U A\right\| && \text{eq above} \\
&\leq \left\|A^{\dagger}\right\| \|U\| \|A\| && \text{remark 1} \\
&= \left\|A^{\dagger}\right\| \|u\|_{\infty} \|A\| && \text{remark 2} \\
&= \|A\| \|u\|_{\infty} \|A\| && \text{remark 3} \\
&= \|u\|_{\infty} \|A\|^2 && \\
&= \|u\|_{\infty} \left\|A^{\dagger} A\right\| && \text{remark 4 (spectral norm only)} \\
&= \|u\|_{\infty} \left\|\sum_{a\in\Sigma} P_a\right\| && \text{eq above}
\end{aligned}$$

This proves Eq. 1; QED.

Further questions

1. Generalize to sum of Hermitian operators.
2. Does it hold for $p = 1$ or $p = 2$ norm?

Bibliography

J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, 2018), ISBN 9781316848142, URL `https://books.google.com/books?id=GRNSDwAAQBAJhttps://www.cambridge.org/core/product/identifier/9781316848142/type/book`.

Nature Physics **16**, 233 (2020), ISSN 1745-2473, URL `https://doi.org/10.1038/s41567-020-0847-3http://www.nature.com/articles/s41567-020-0847-3`.