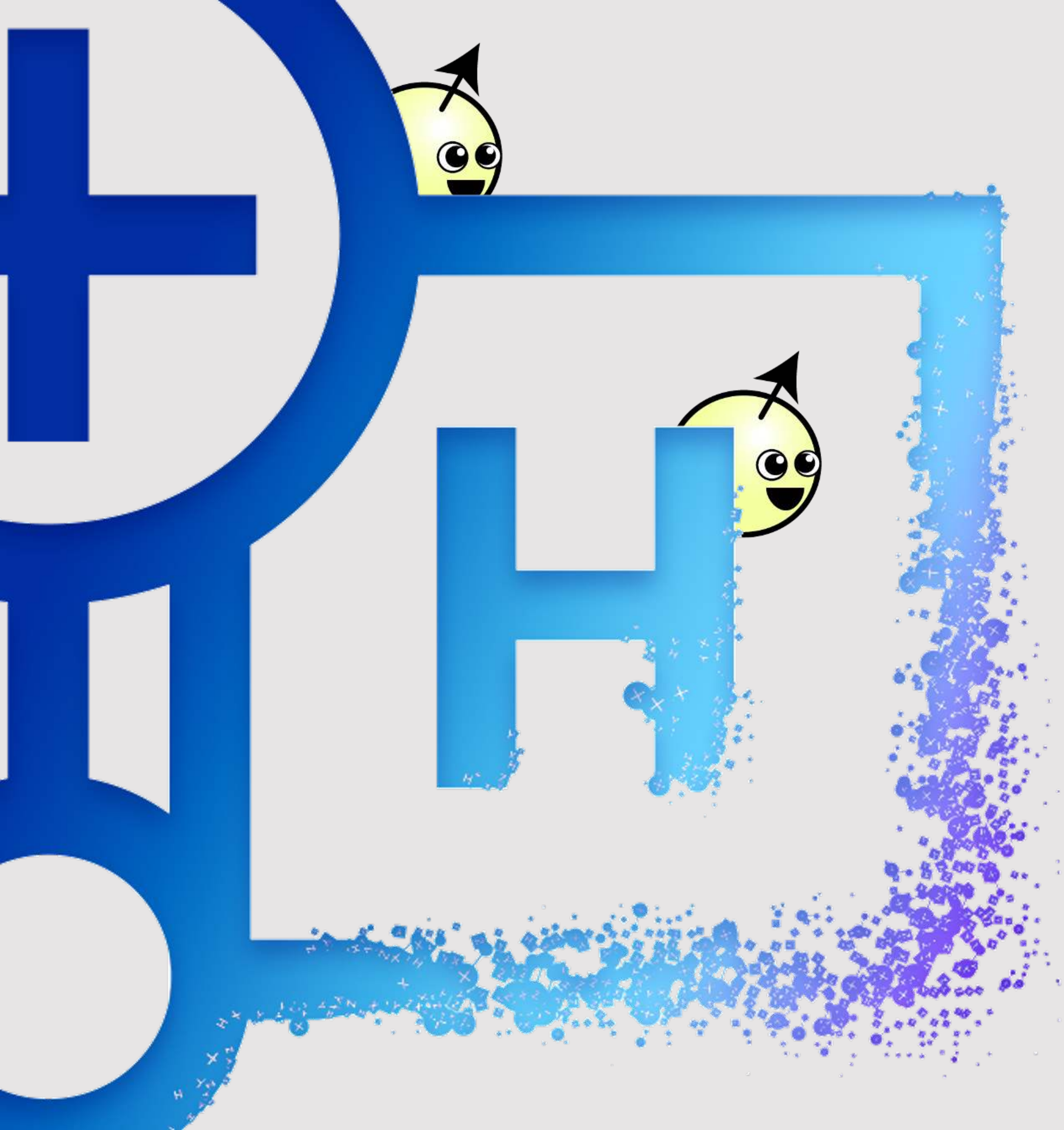
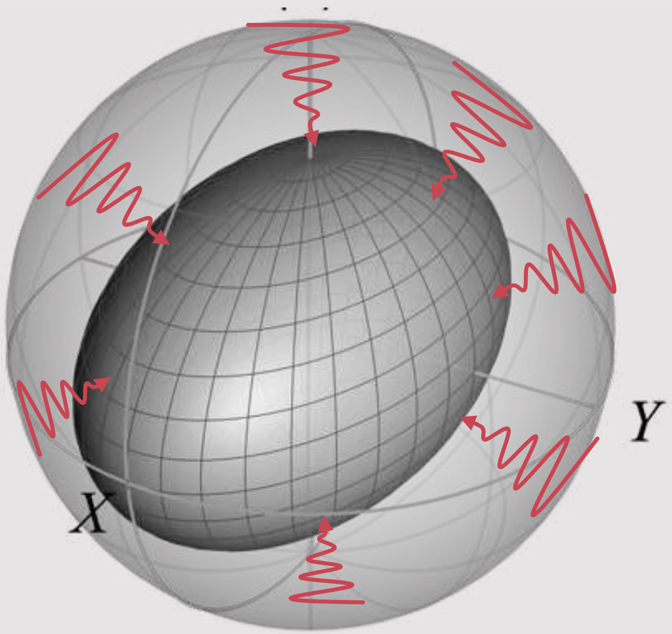


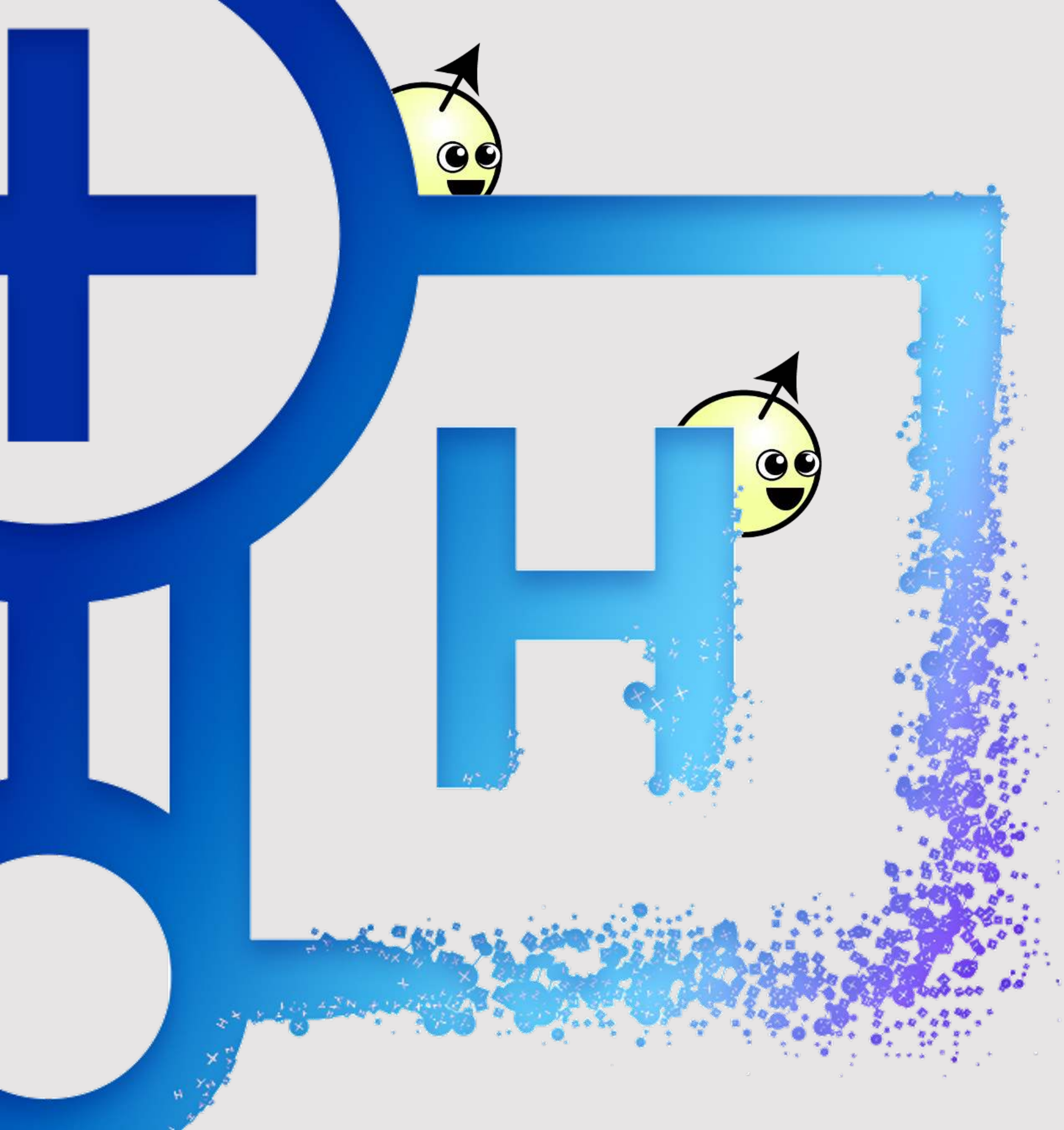
Introduction to Quantum Noise

by Zlatko K. Minev



Qiskit | Global Summer School 2023

zlatko-minev.com @zlatko_minev



Introduction to noise in quantum computers

the state $|0\rangle$ as a vector in the Bloch sphere that points from the origin of Bloch sphere all the way to the surface (radius equal to 1) in the Z direction.

Fig. 2-3 also shows a black vector along the X axis. The black vector shows the axis around which the operator X, and its corresponding unitary $R_X(\theta)$, will rotate a state vector on the Bloch sphere.

If we now look at the way that the X gate operates on the initial state $|\psi_0\rangle = |0\rangle$, we can see that the ideal X gate operation will flip $|0\rangle$ to $|1\rangle$ and $|0\rangle$ to $|1\rangle$. The physical implementation of an X gate ($R_X(\theta)$) can behave as the ideal X gate (up to an ignorable global phase). As θ increases, the $R_X(\theta)$ operator evolves the ground state $|0\rangle$ around the X axis by the right-hand rule as illustrated in Fig. 2-4 with the color changing path from $|0\rangle$ to $|1\rangle$.

Supporting this illustration with some math, $R_X(\theta)$'s action on $|\psi_0\rangle$ from $\theta = 0$ to $\theta = \pi$, evolves the state as

$$
\begin{aligned}
R_X(\theta = \pi)|\psi_0\rangle &= [\cos(\pi/2)I - i\sin(\pi/2)X]\,|0\rangle && \text{(2.6a)}\\
|\psi_\theta(\theta = \pi)\rangle &= \cos(\pi/2)I|0\rangle - i\sin(\pi/2)X|0\rangle && \text{(2.6b)}\\
&= \cos(\pi/2)|0\rangle - i\sin(\pi/2)|1\rangle && \text{(2.6c)}\\
&= -i|1\rangle && \text{(2.6d)}\\
&= |1\rangle\,. && \text{(2.6e)}
\end{aligned}
$$

In Eq. (2.6d), we see that the rotation has introduced a global phase factor of $-i$ to the state $|1\rangle$, which we will ignore from now on.

Let's look at the expectation value of the X, Y, and Z quadratures for the state as it evolves as for any angle θ

$$
\begin{aligned}
\langle X\rangle &= \langle\psi_\theta(\theta)|\,X\,|\psi_\theta(\theta)\rangle = 0\,, && \text{(2.7a)}\\
\langle Y\rangle &= \langle\psi_\theta(\theta)|\,Y\,|\psi_\theta(\theta)\rangle = -\sin(\theta)\,, && \text{(2.7b)}\\
\langle Z\rangle &= \langle\psi_\theta(\theta)|\,Z\,|\psi_\theta(\theta)\rangle = \cos(\theta)\,. && \text{(2.7c)}
\end{aligned}
$$

The three expectation values match the view of the Bloch sphere. The X expectation value is zero the entire evolution, and there are oscillations $-\sin(\theta)$ and $\cos(\theta)$ along Y and Z axes respectively as θ increases. Now we can see the reason as to why the factor of $1/2$ in the definition of the $R_X(\theta)$. The $1/2$ allows the angle by which a vector on the Bloch sphere will rotate to be the same as the angle θ in $R_X(\theta)$.

Of course, you can continue this angle θ under the $R_X(\theta)$ operation. You'll notice that for increasing the angle of θ, it will trace a full circle all the way around the Bloch sphere, back into the ground state when $\theta = 2\pi$. In other words, $R_X(2\pi) = I$.

To summarize, the initial state $|0\rangle$ starts off in the ground state, which is pointing in the plus Z direction (Fig. 2-4). The first rotation, an X gate, takes the state all the way around to the south pole $|1\rangle$. Applying a second gate X evolves the state from $|1\rangle$ all the way around to $|0\rangle$. Applying more and more X gates, the state will continue to rotate around. This visualization helps us keep in mind what's happening, and it's a particularly useful one for understanding coherent errors.

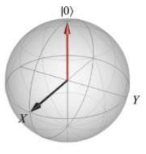


Fig. 2-3. Bloch sphere with the ground state $|0\rangle$ visualized as a red vector pointing form the origin to the surface. The red vector overlaps the Z-axis. The operator $R_X(\theta)$ is represented as the black vector pointing along the X axis.

Right-hand rule

Open your palm and point the thumb of your right hand along the X axis. The direction your fingers curl as you close your palm is the direction of θ. Your fingers will track out the trajectory illustrated by the colored path in Fig. 2-4. (TODO: add pic of right hand rule.)

Practice

On your own, draw the path $|0\rangle$ takes on the surface of the Bloch sphere for an angle θ not equal to π.

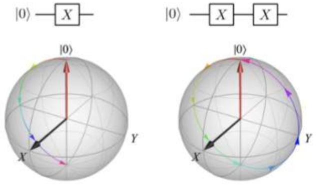
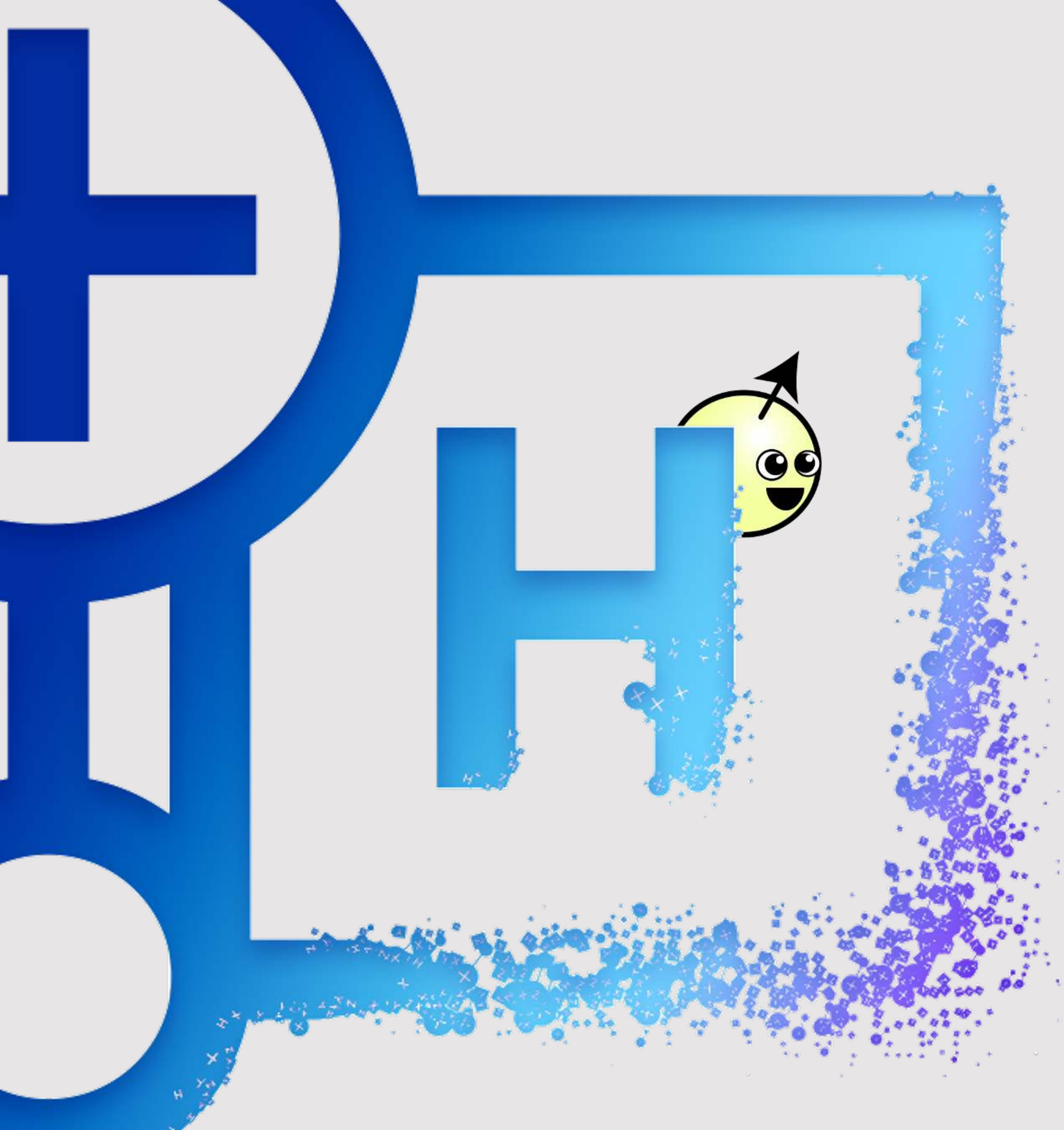


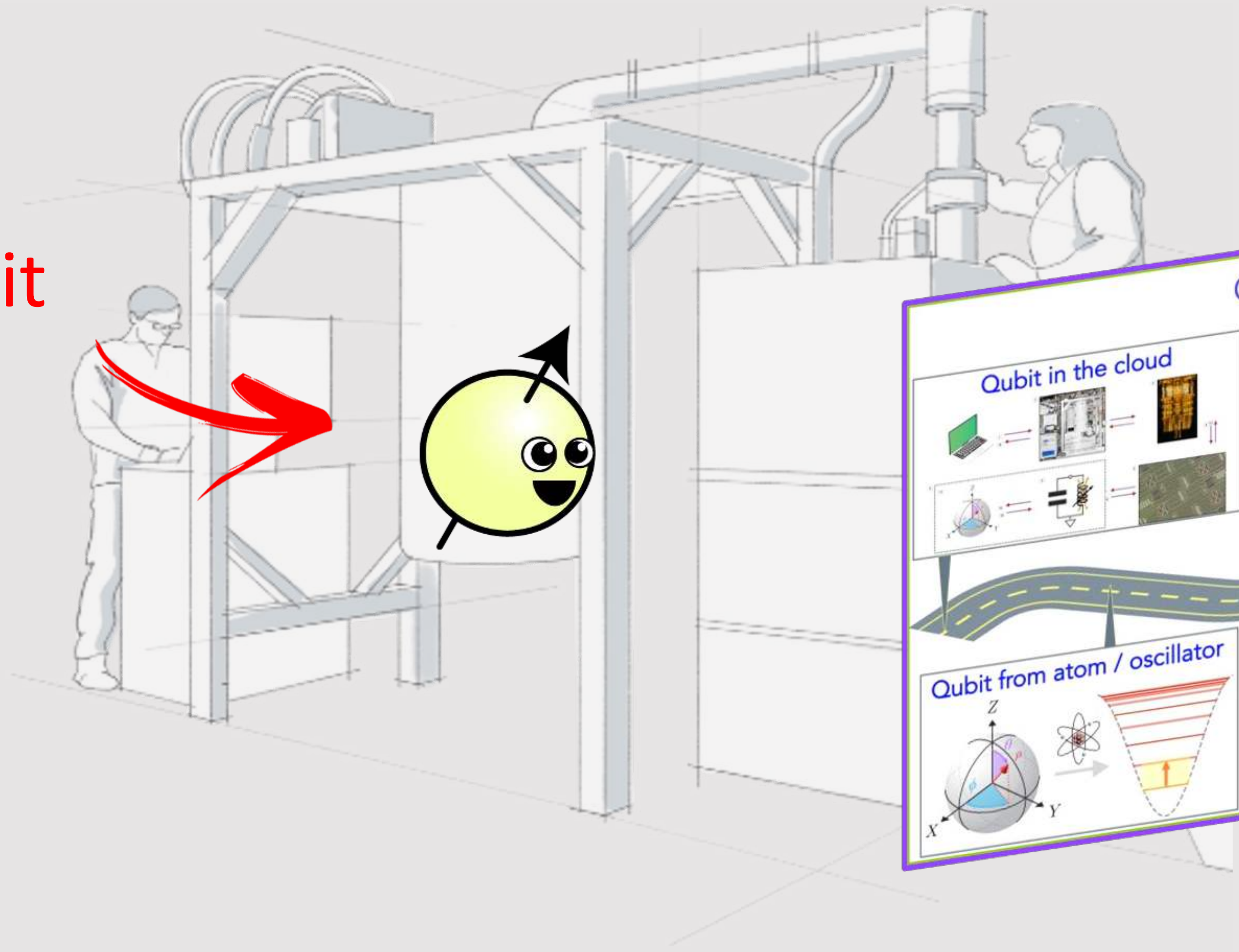
Fig. 2-4. Bloch spheres showing the initial state $|0\rangle$ as a red vector. The color changing path marks out the state's evolution during the corresponding circuit diagram above the spheres. On the left, one X gate is applied leaving the state in $|1\rangle$. On the [illegible], two X gates are applied returning the state to where it began.

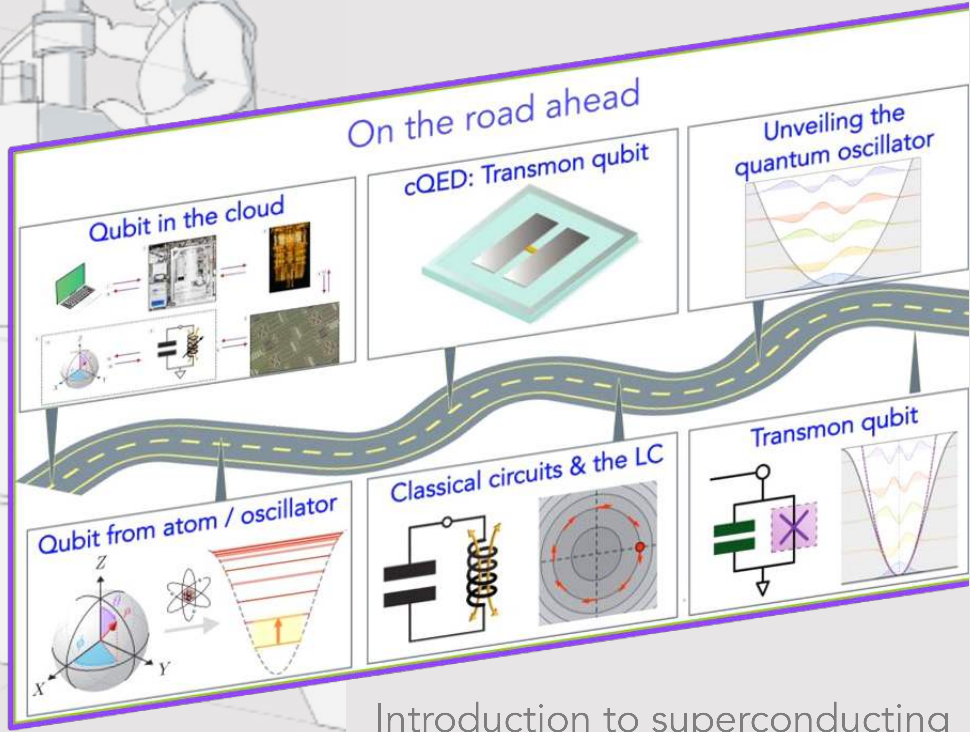


What do you need to know before putting theory to practice on a real, noisy quantum processor?

Chapter 1: Hello World with a real experiment!

A qubit

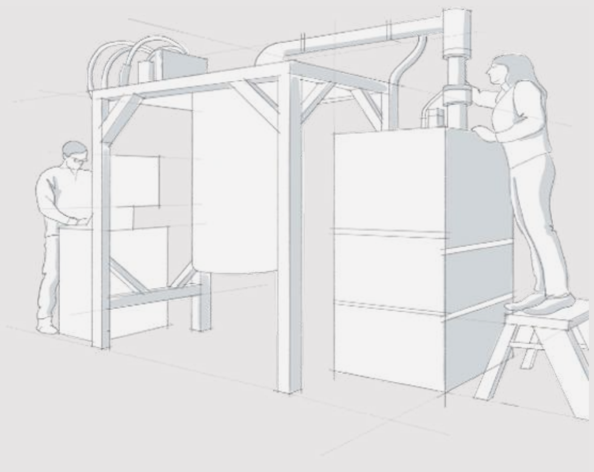




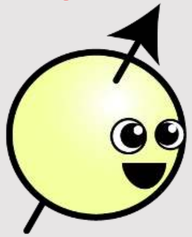
Introduction to superconducting qubits (cQED)
Lecs. 16-21 Minev
QGSS 2020 at qiskit.org/learn

image: qiskit

Hello World! building blocks



A qubit



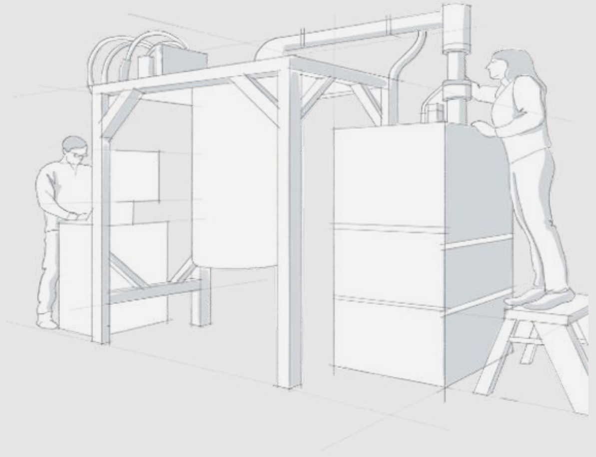
$$|1\rangle$$

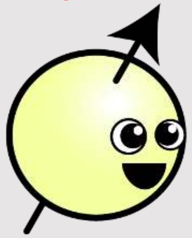
$$|0\rangle$$

Computational basis states

image: qiskit

Hello World! building blocks





A qubit

$$|1\rangle$$

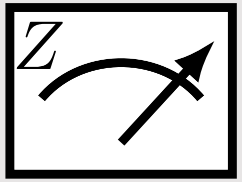
$$|0\rangle$$

Computational basis states

Operations: qubit gate

$$\boxed{X}$$

Measurements: qubit observable

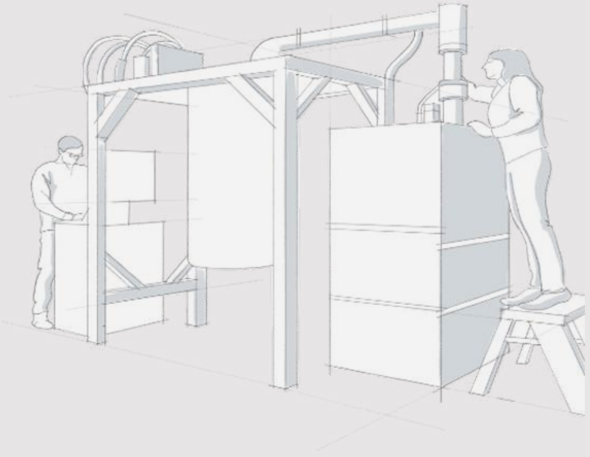


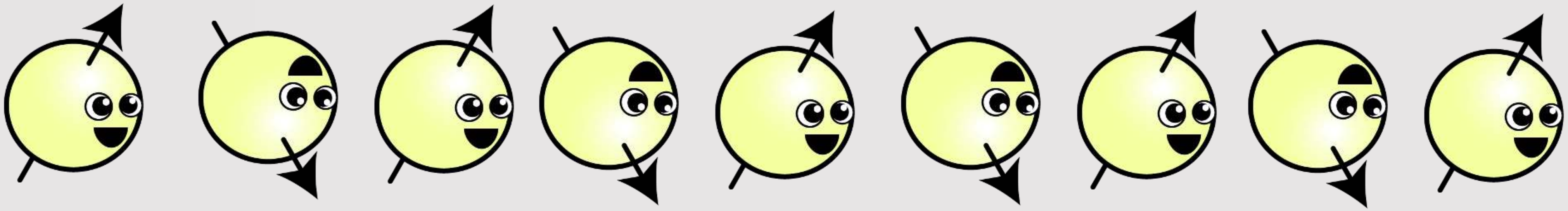
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! Even-odd algo: qubit flipper





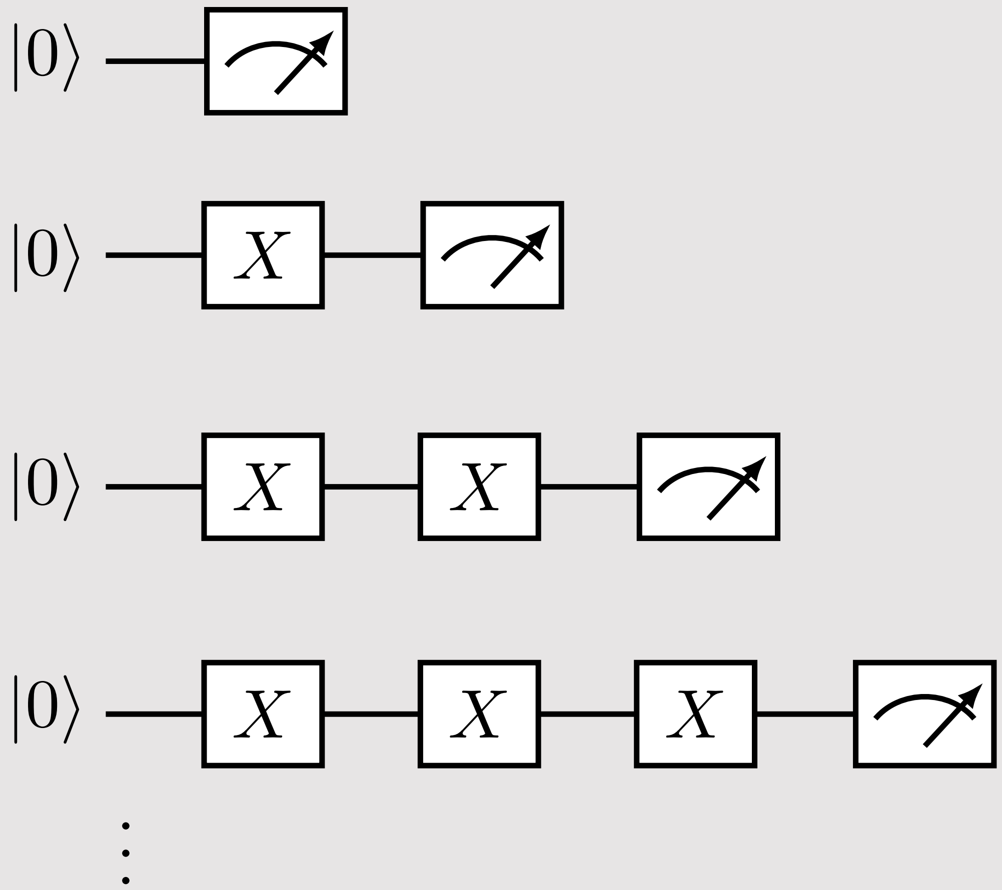
refresher:

$$X|0\rangle = |1\rangle \qquad Z|0\rangle = +1|0\rangle$$

$$X|1\rangle = |0\rangle \qquad Z|1\rangle = -1|1\rangle$$

Hello World! qubit flipper quantum circuits

depth



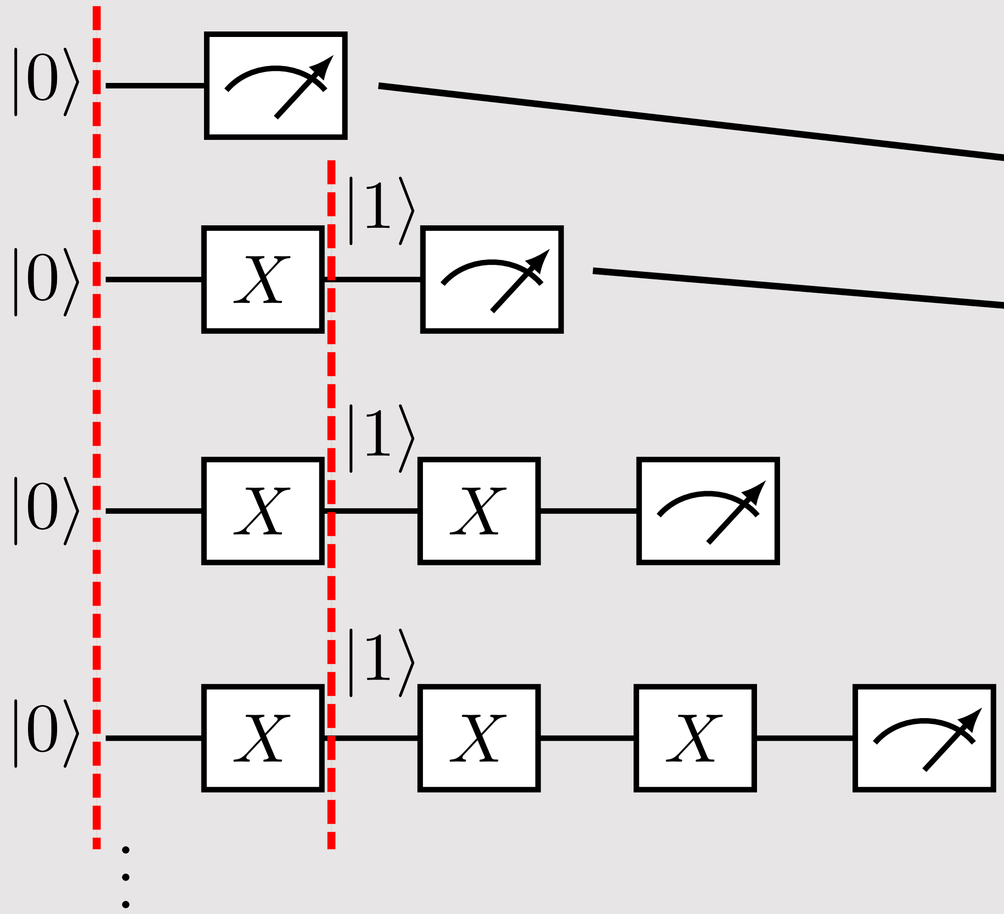
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

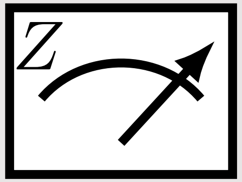
$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! "debugger" step through

depth



Measurement



$$\langle 0|Z|0\rangle = +1$$

$$\langle 1|Z|1\rangle = -1$$

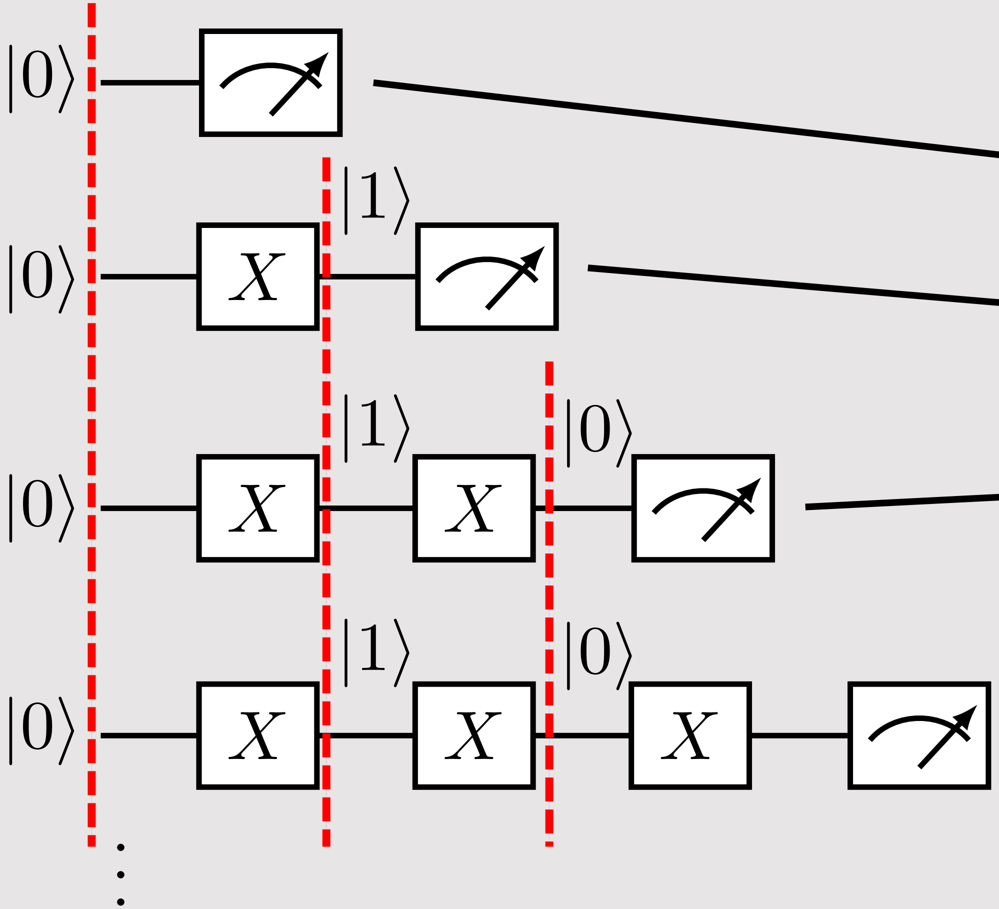
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

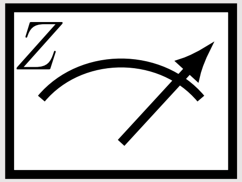
$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! “debugger” step through

depth



Measurement



$$\langle 0|Z|0\rangle = +1$$

$$\langle 1|Z|1\rangle = -1$$

$$\langle 0|Z|0\rangle = +1$$

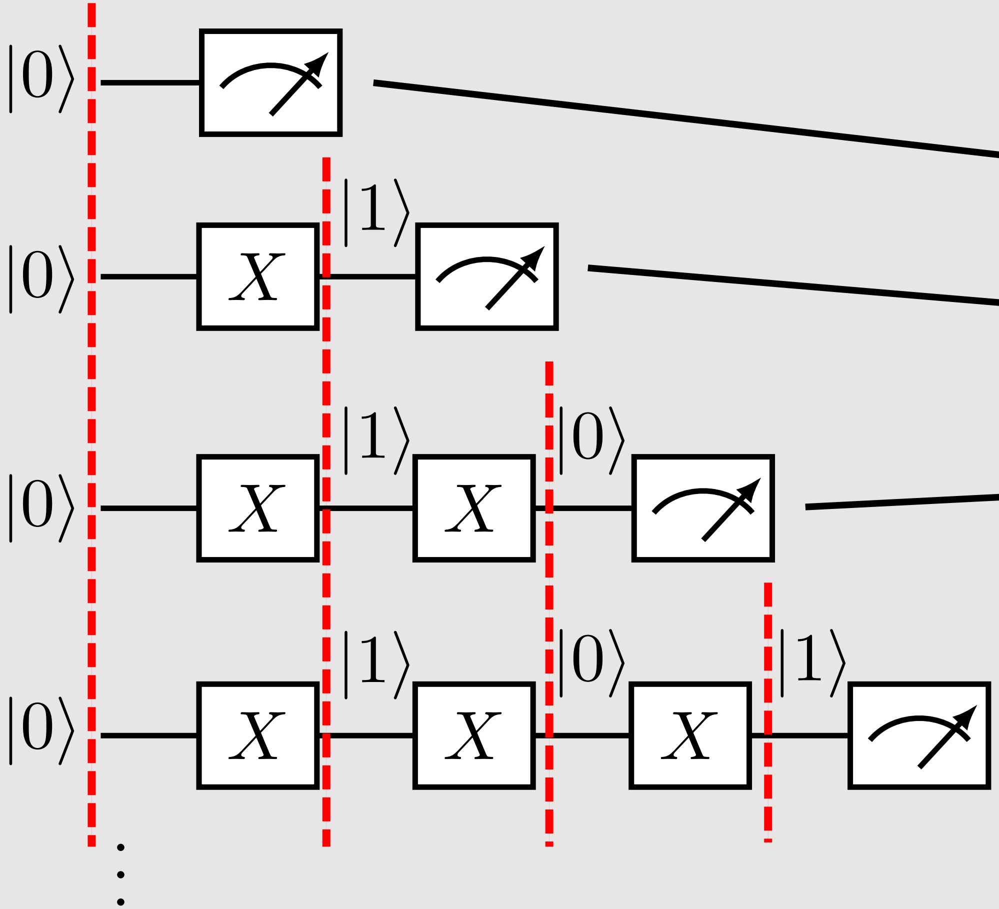
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

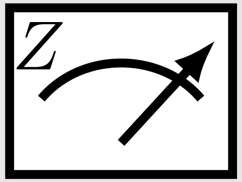
$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! "debugger" step through

depth



Measurement



$$\langle 0|Z|0\rangle = +1$$

$$\langle 1|Z|1\rangle = -1$$

$$\langle 0|Z|0\rangle = +1$$

$$\langle Z\rangle = (-1)^d$$

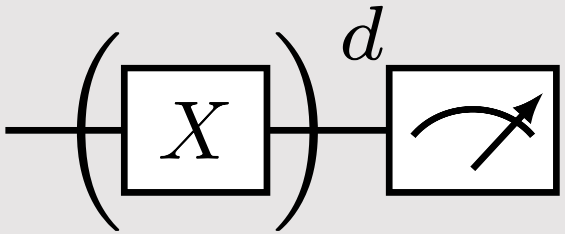
where d is the circuit depth

refresher:

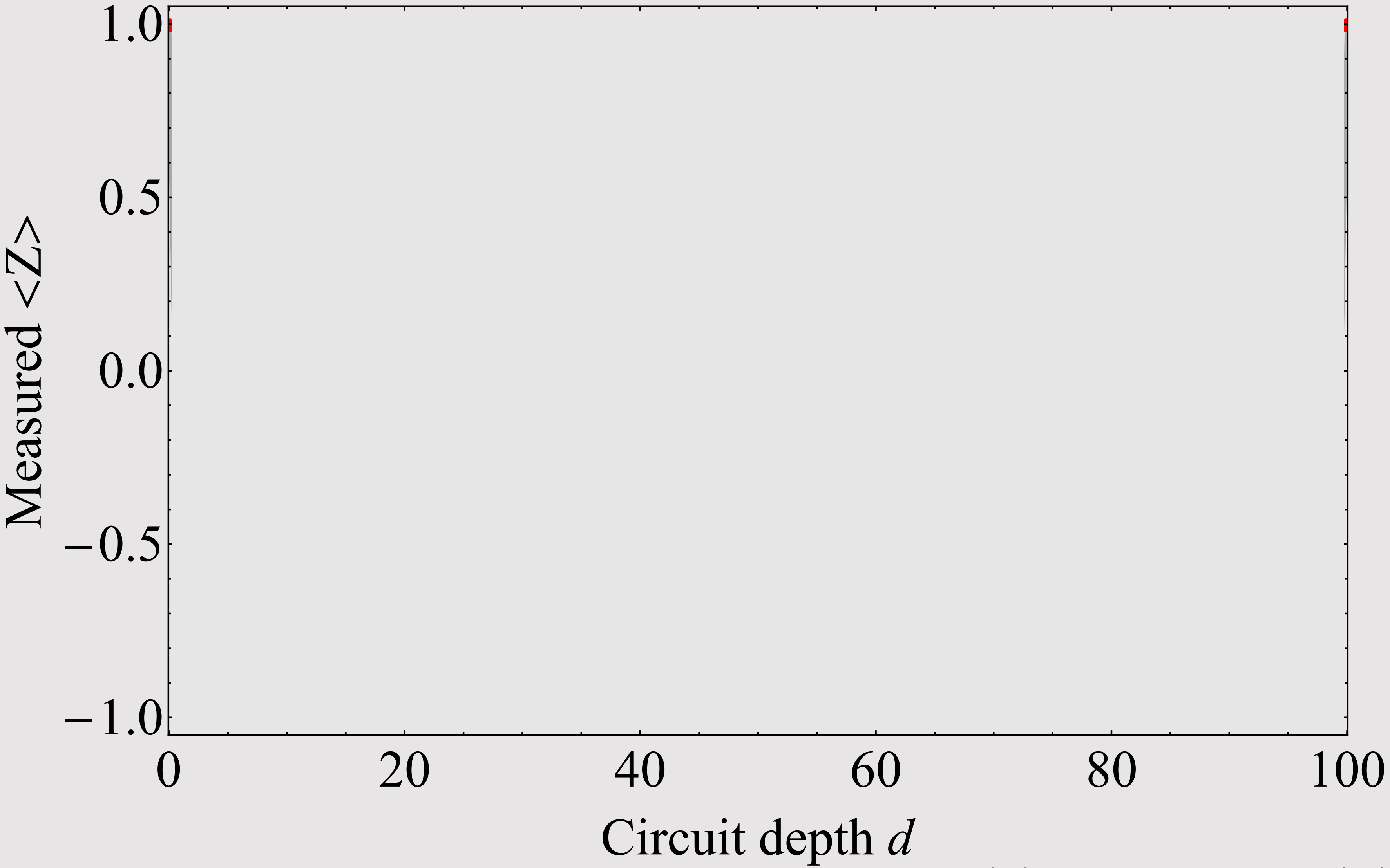
$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

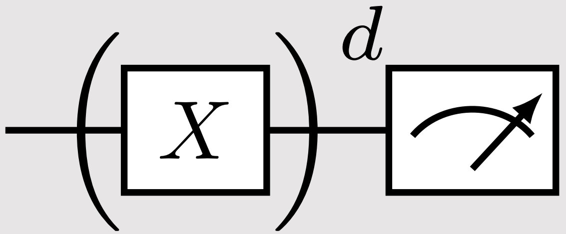
Hello World! Ideal expectation results



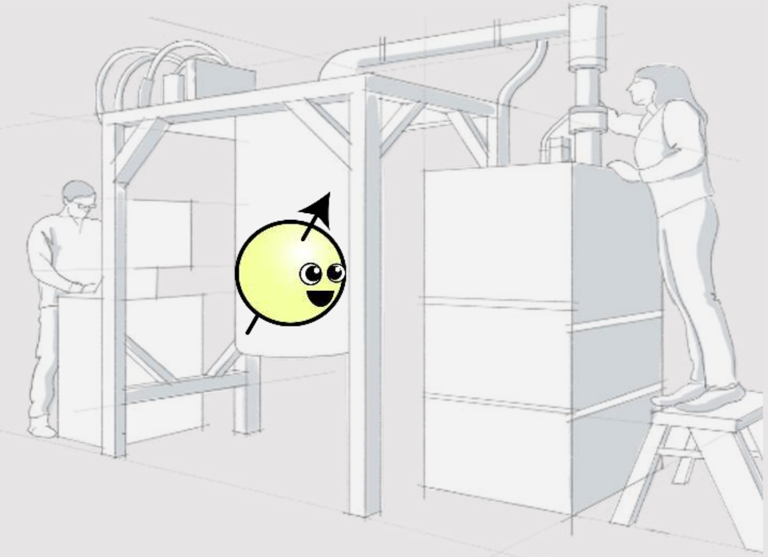
$\langle Z \rangle = (-1)^d$



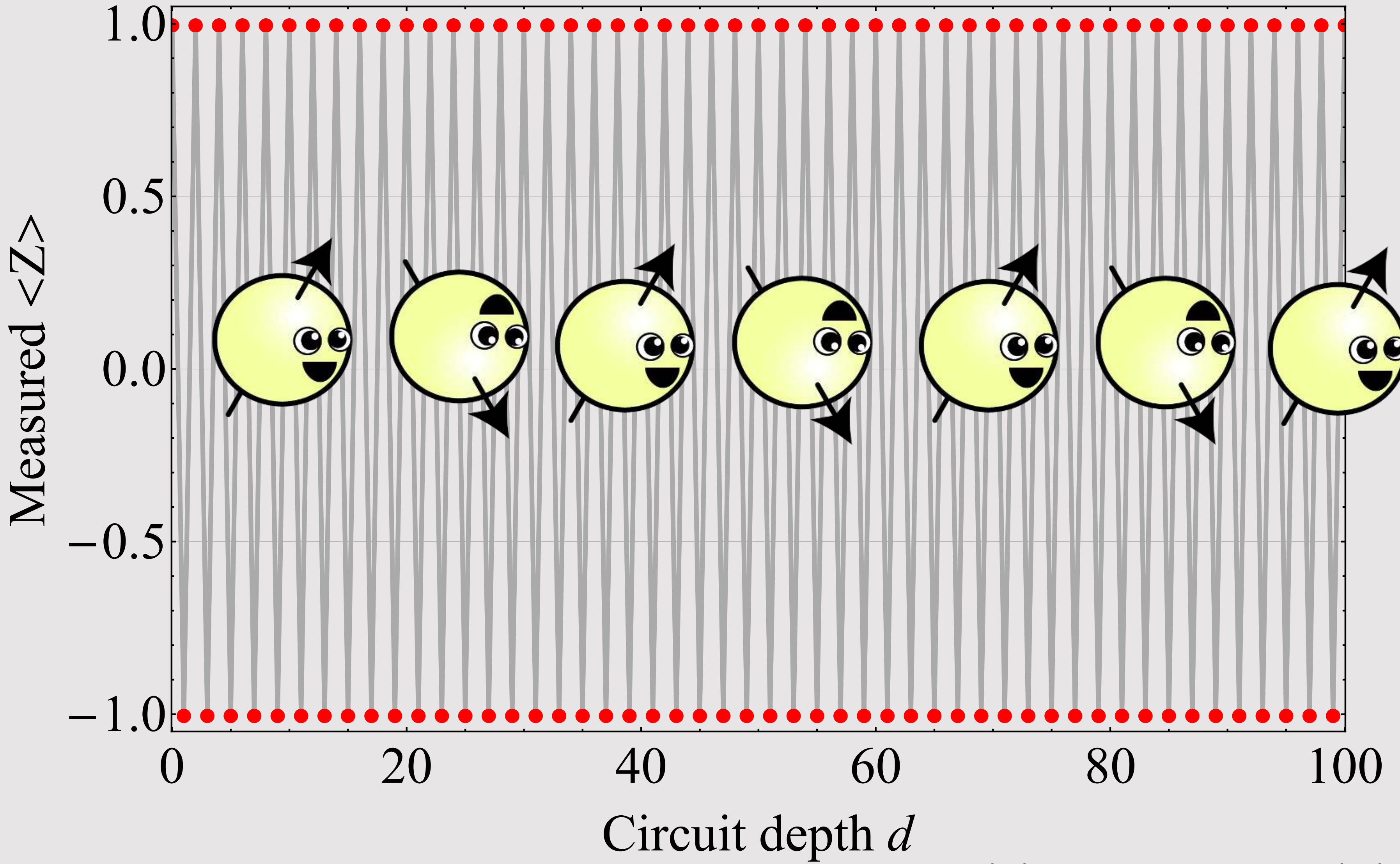
Hello World! Ideal expectation results



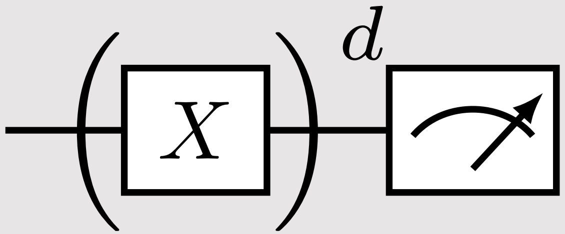
Let's run on a real device!



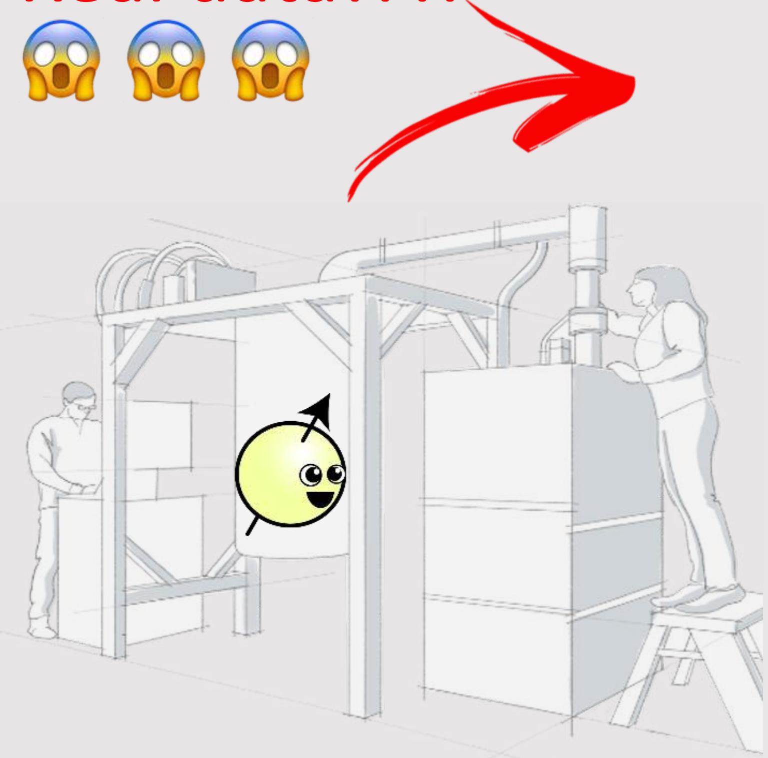
$$\langle Z \rangle = (-1)^d$$

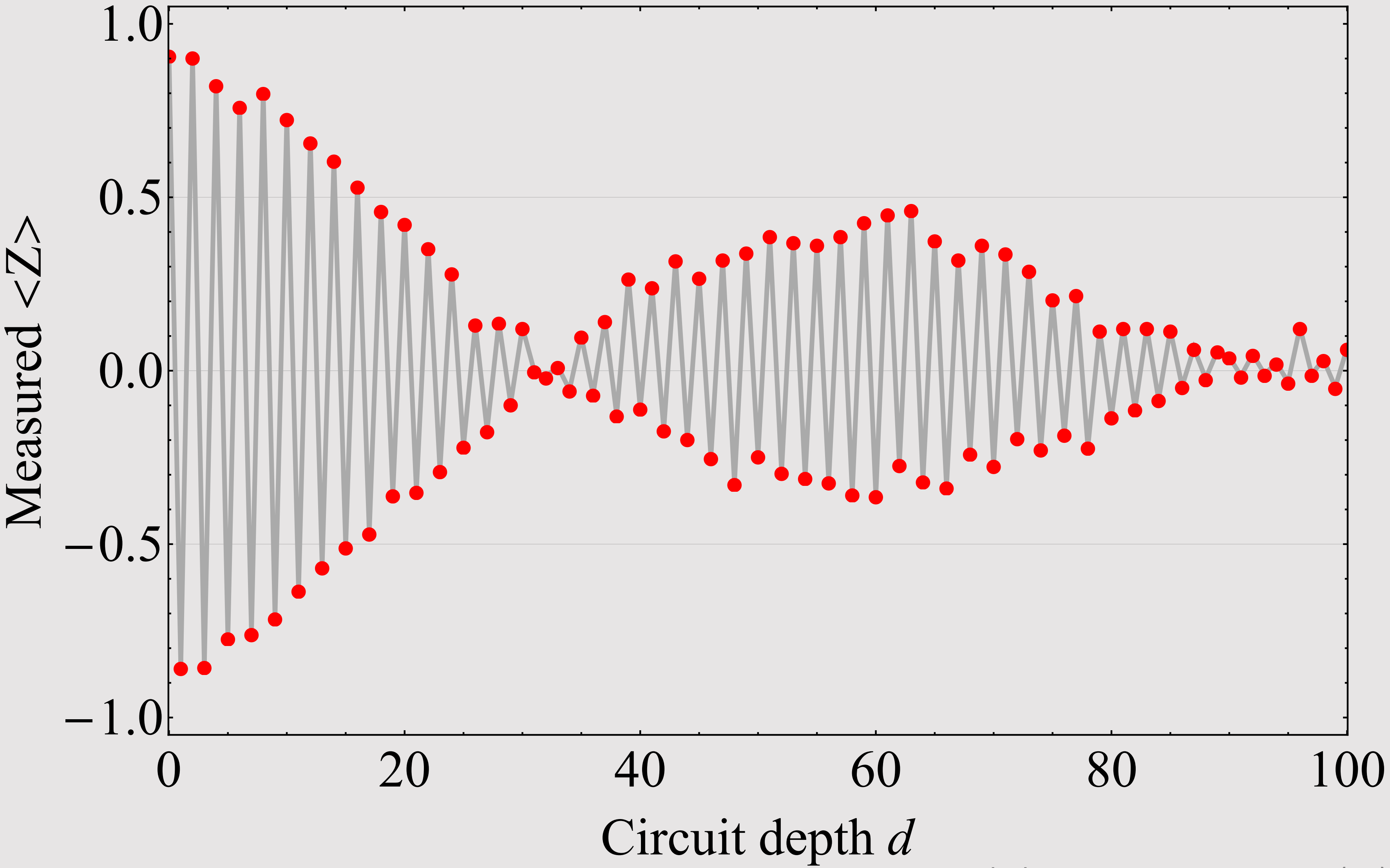


Hello World! Real expectation results

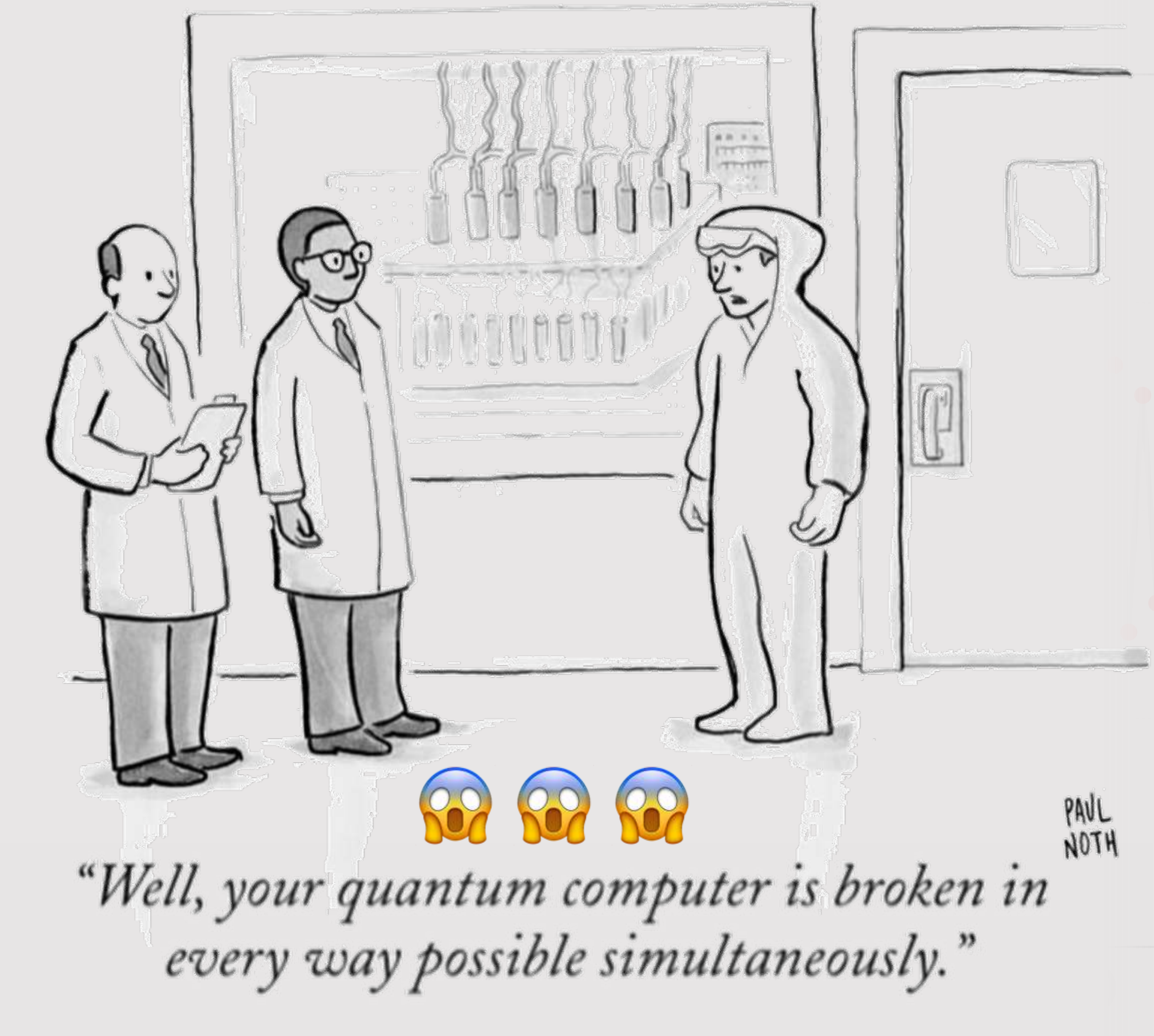


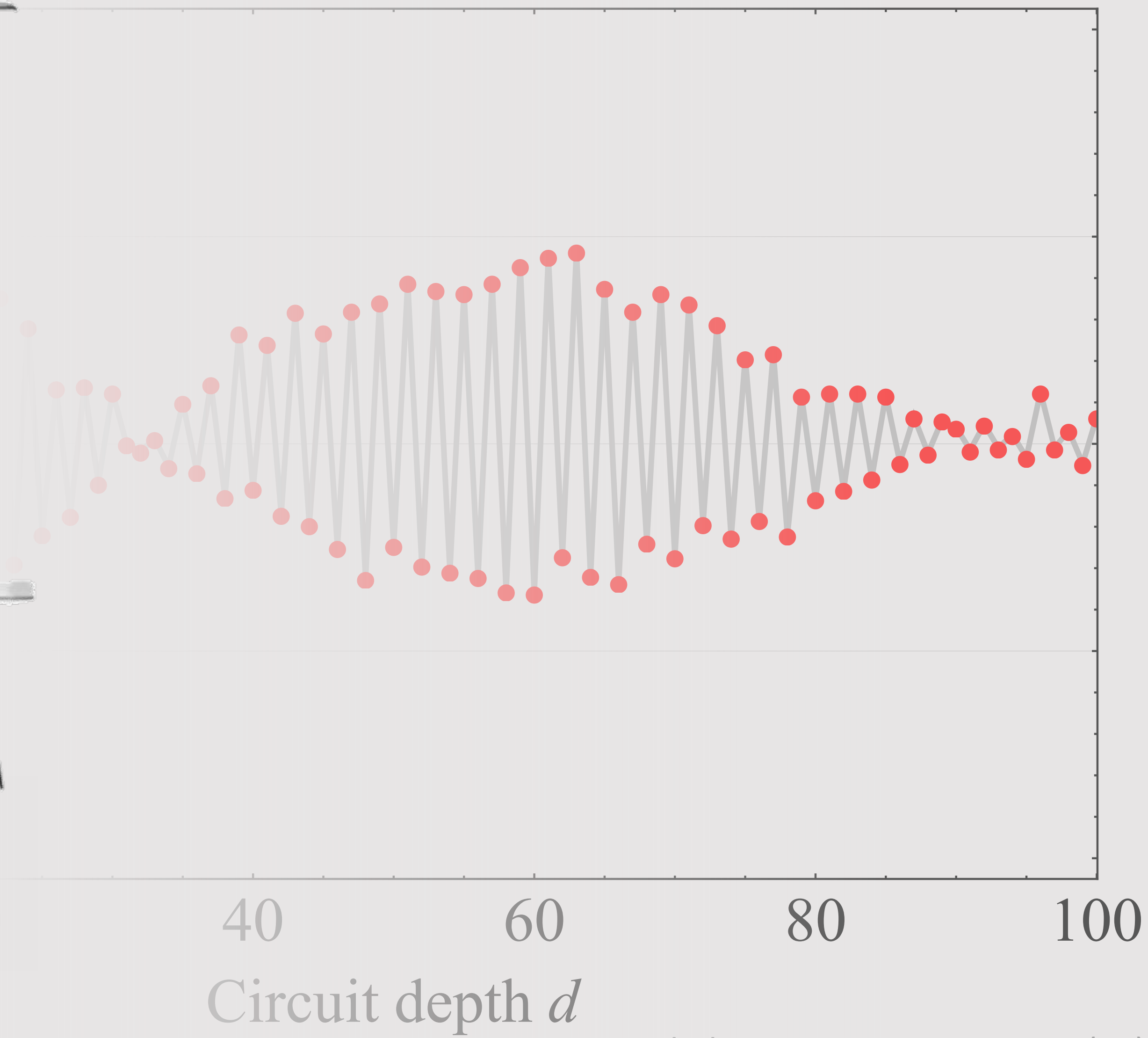
Real data!?!!



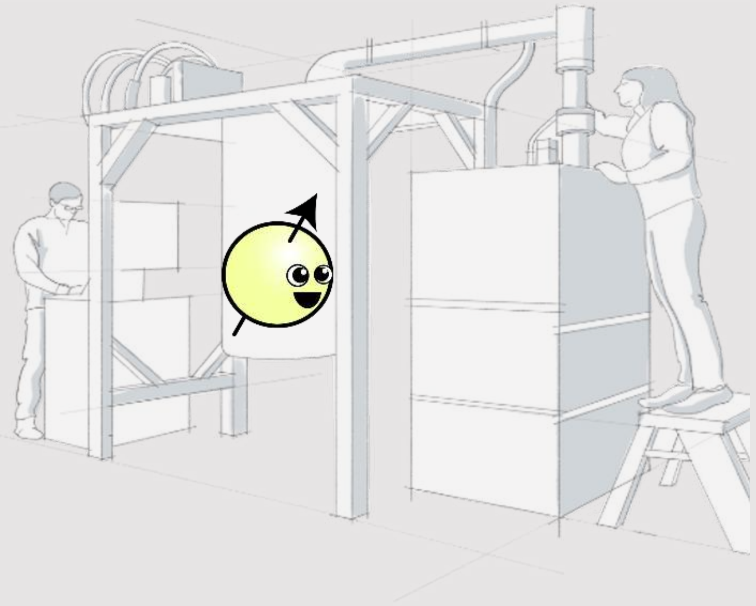


Real & noisy quantum processors: Why study noise?





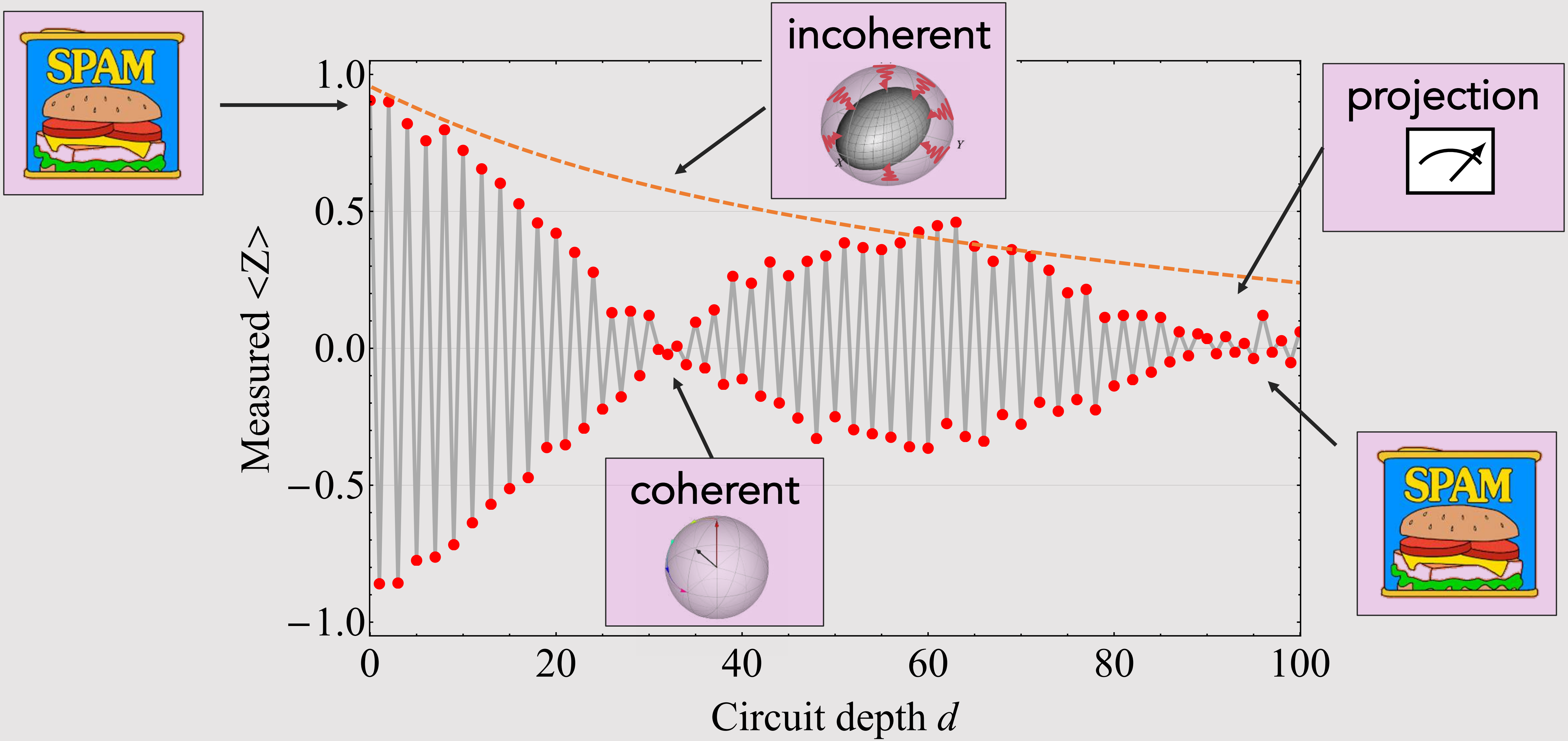
Cartoon: Paul Noth



"Quantum phenomena
do *not* occur in a Hilbert space,
they occur in a laboratory."

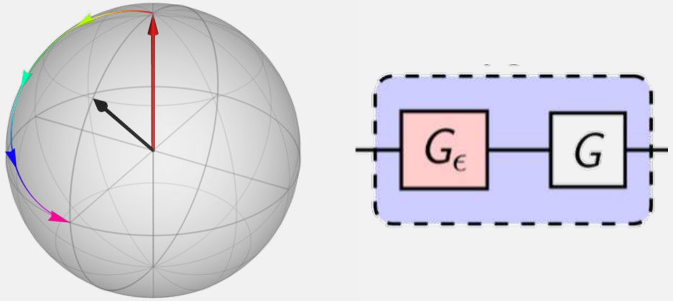
Asher Peres

Elements of 😱 noise



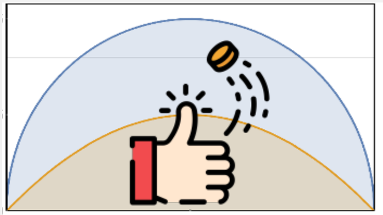
The road ahead

Coherent noise

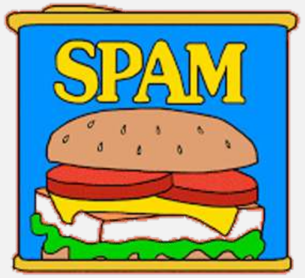


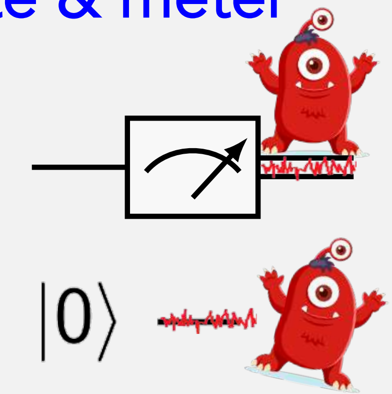
Measurements in quantum

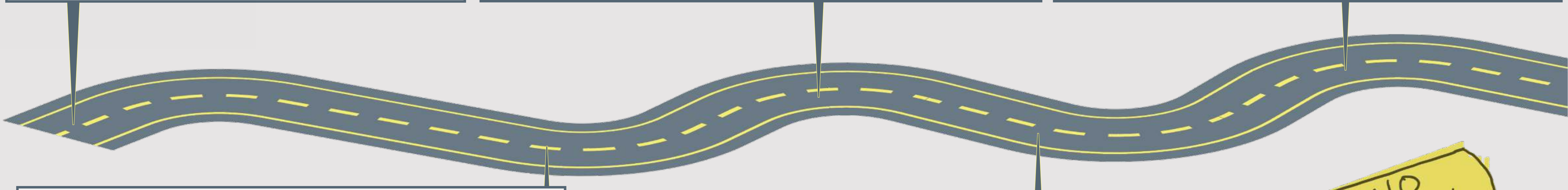
Projection noise



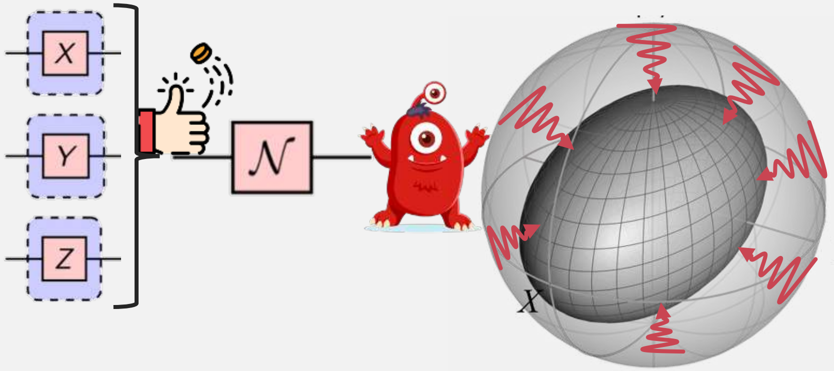
SPAM: Sate & meter







Incoherent noise



Bonus content

Coherent ZZ noise

...

coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

Guideposts

Evil monster

Eve the Evil monster is here to terrorize the quantum world

Random Sorcery

Randomness and unpredictability abounds

Green box

Key result of section

Purple box

Key challenge highlighted

Try it yourself

Pause the video and try to work it out, before I share the answer

Dangerous bend sign

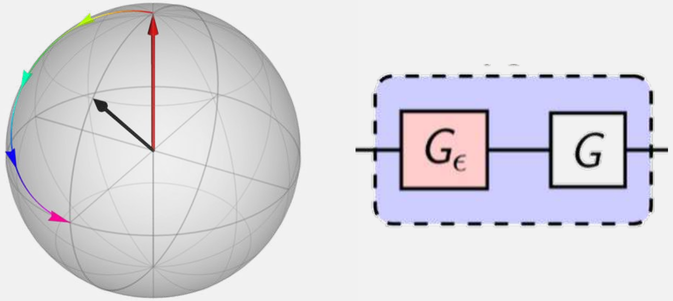
Denotes extra information,
with appendix-like depth or digression.
You can skip this material on a first reading

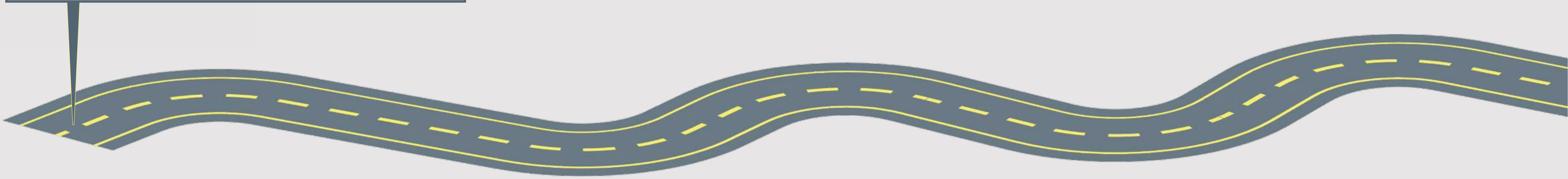
Caution

Caution or common pitfall

Monster image by jcomp on Freepik; coin-toss icon: Good Ware; pen: bclogo latex

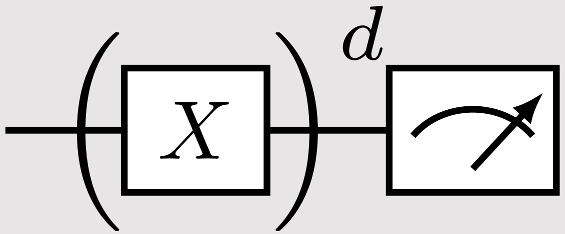
Coherent

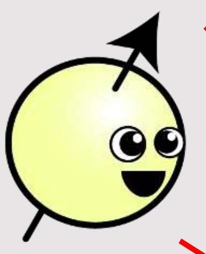


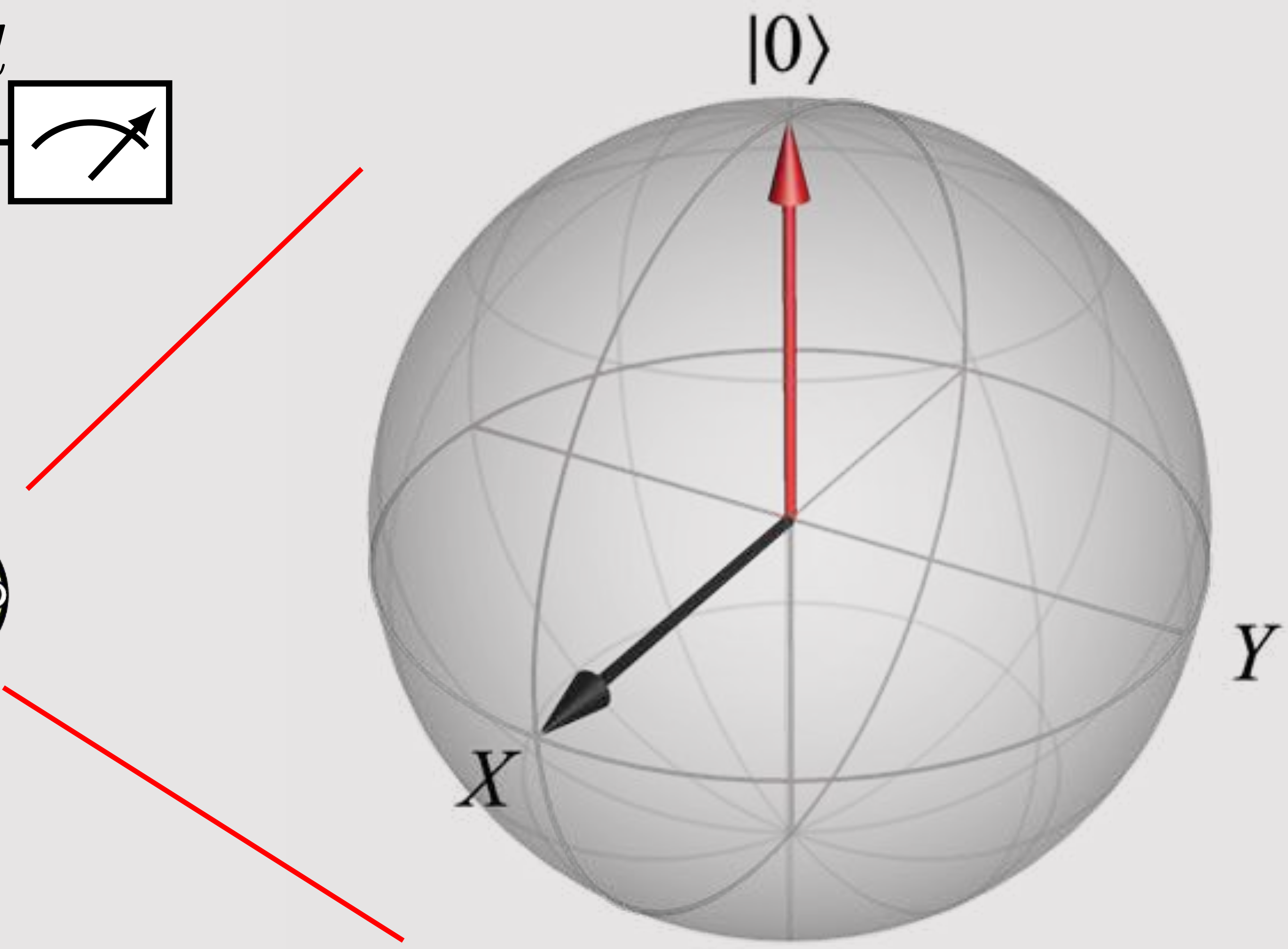


road based on: freepik

Return to the Hello World example







Try it yourself

Show that for |0>:

$$\left\langle \hat{Z} \right\rangle = 1$$
$$\left\langle \hat{X} \right\rangle = 0$$
$$\left\langle \hat{Y} \right\rangle = 0$$

Show that for |1>:

$$\left\langle \hat{Z} \right\rangle = -1$$
$$\left\langle \hat{X} \right\rangle = 0$$
$$\left\langle \hat{Y} \right\rangle = 0$$

Bloch sphere cardinal points

Solution For $|\psi\rangle = |0\rangle$, and recall from earlier lectures the spin Pauli matrices and states:

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\langle 0| = |0\rangle^\dagger = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \langle 1| = |1\rangle^\dagger = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

then

$$\begin{aligned} \left\langle \hat{Z} \right\rangle &= \langle\psi| \hat{Z} |\psi\rangle \\ &= \langle 0| \hat{Z} |0\rangle \\ &= \langle 0| \left(\hat{Z} |0\rangle\right) \\ &\quad (\text{recall } \hat{Z} |0\rangle = (+1) |0\rangle) \\ &= \langle 0| (+1 |0\rangle) \\ &= (+1) \langle 0| \cdot |0\rangle \\ &\quad (\text{recall } \langle 0| \cdot |0\rangle = \langle 0|0\rangle = 1) \\ &= 1 \end{aligned}$$

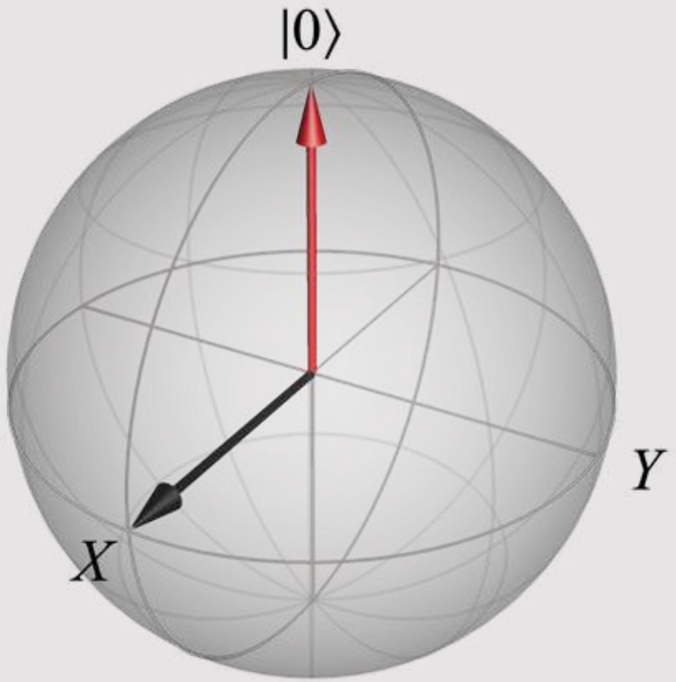
Recall that bra-ket is a dot product between the two vectors.

$$\begin{aligned} \left\langle \hat{X} \right\rangle &= \langle\psi| \hat{X} |\psi\rangle \\ &= \langle 0| \hat{X} |0\rangle \\ &(\text{recall } \hat{X} |0\rangle = |1\rangle) \\ &= \langle 0|1\rangle \\ &= 0 \end{aligned}$$

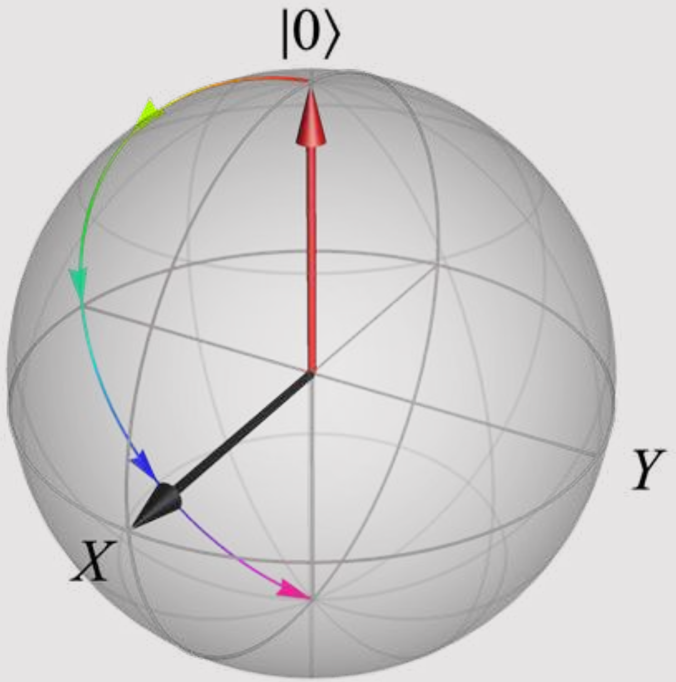
$$\begin{aligned} \left\langle \hat{Y} \right\rangle &= \langle\psi| \hat{Y} |\psi\rangle \\ &= \langle 0| \hat{Y} |0\rangle \\ &(\text{recall } \hat{Y} |0\rangle = i |1\rangle) \\ &= i \langle 0|1\rangle \\ &= 0 \end{aligned}$$

(Ideal) Evolution on the Bloch sphere

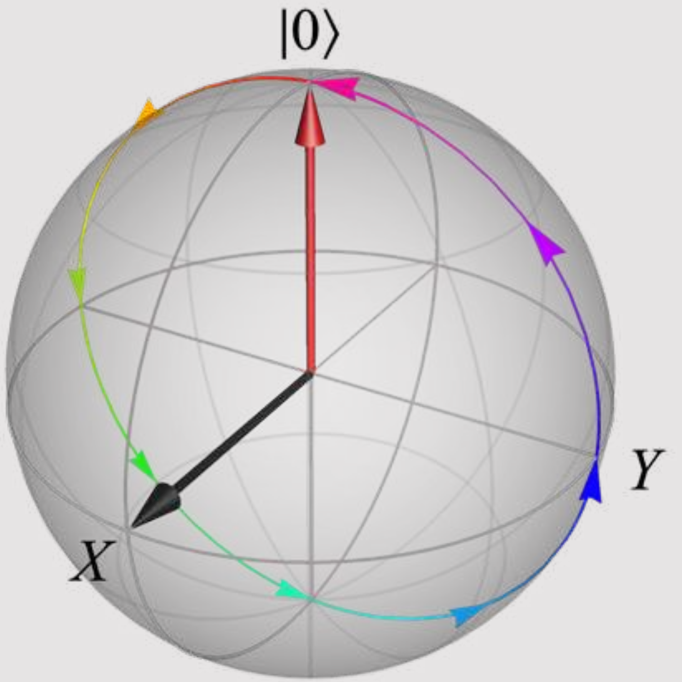
$|0\rangle$



$|0\rangle$ —[X]—



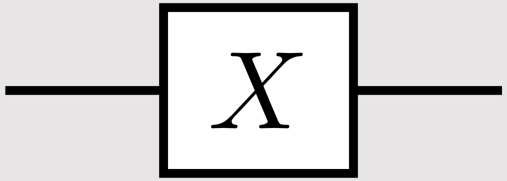
$|0\rangle$ —[X]—[X]—



Origin of our X gate: time evolution

$$X = R_X(\pi)$$

(up to global phase)



Refresher:

$$\hat{H} = \frac{\hbar\omega}{2} X$$

$$= \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U(t) = \exp\left(-it\hat{H}/\hbar\right)$$

$$\theta := \omega t$$

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2} X\right)$$

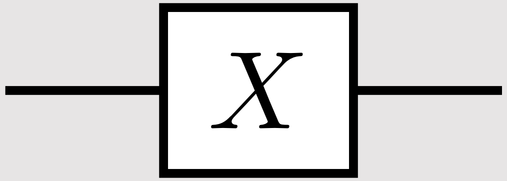
$$= \cos\left(\theta/2\right) I - i \sin\left(\theta/2\right) X$$

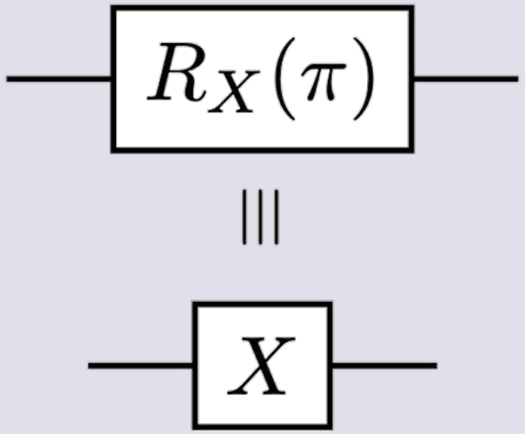
* We will often drop hats on Paulis *I*, *X*, *Y*, *Z*

Minor footnote

$$X = R_X(\pi)$$

(up to global phase)

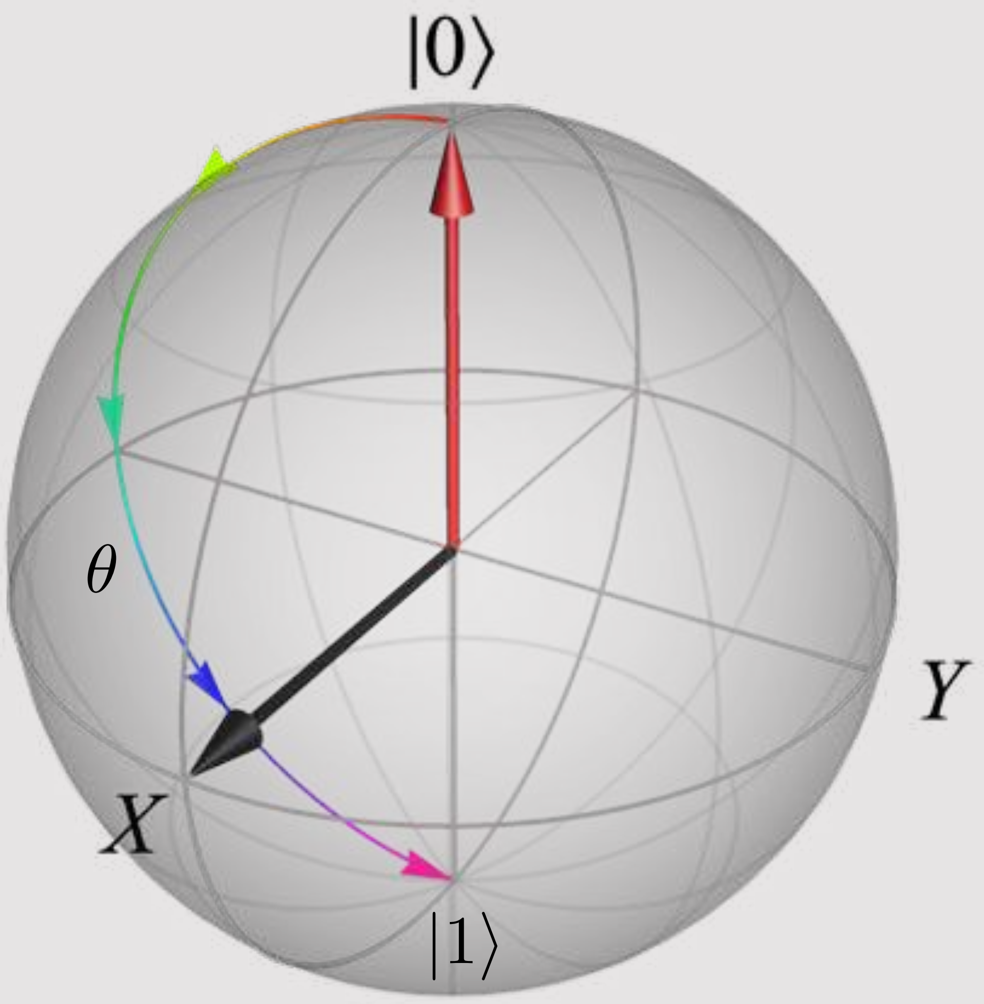




Minor footnote. There is a global phase factor between the way X and $R_X(\pi)$ are defined; i.e., $R_X(\pi) = -iX$.

Visualize: Evolution on the Bloch sphere

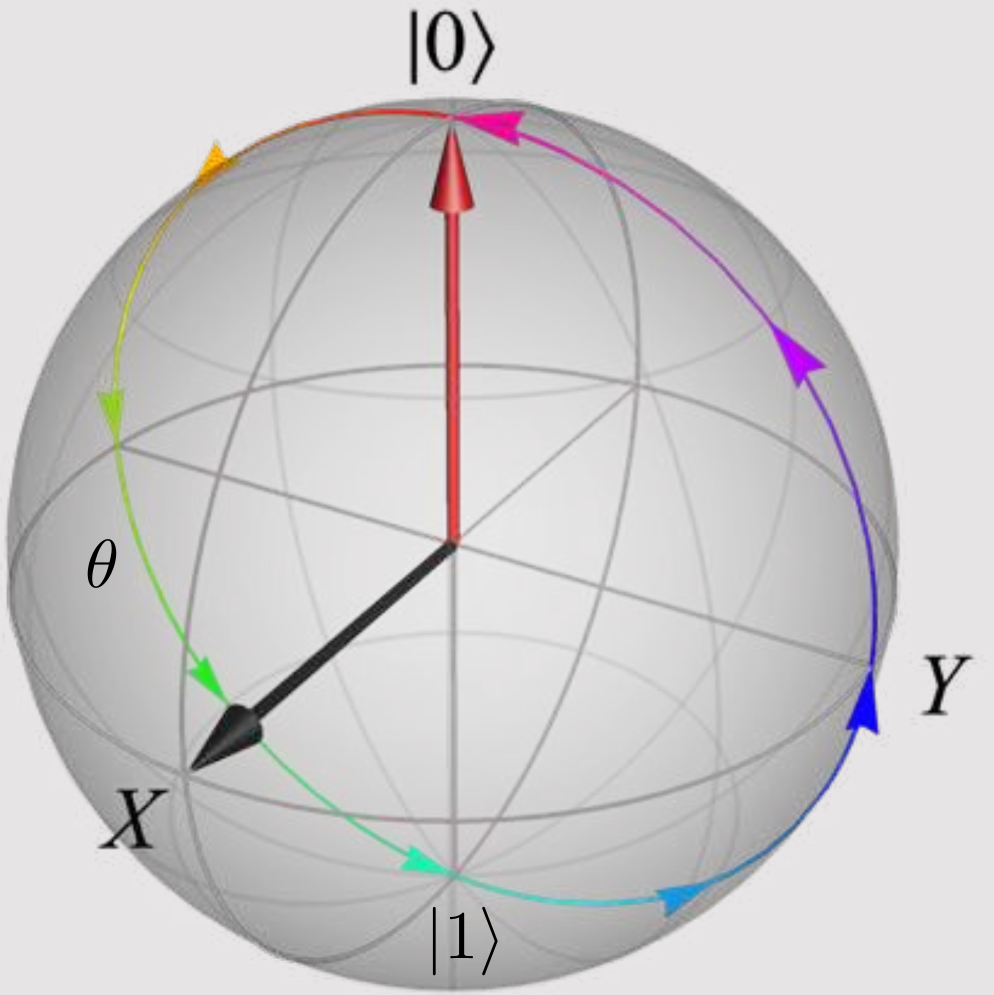
$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$



Visualize: Evolution on the Bloch sphere

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$

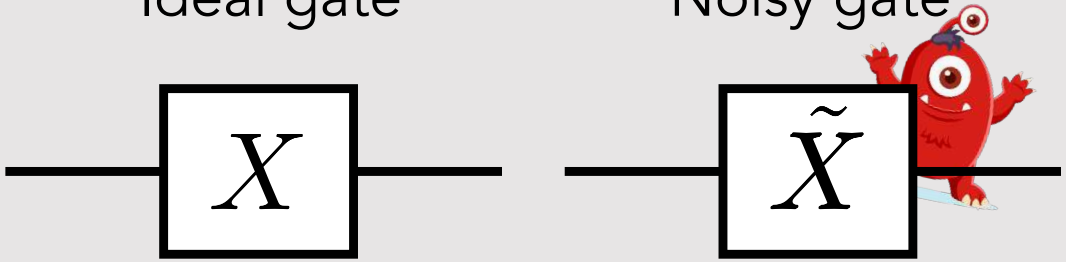
$$X\left|0\right\rangle = \left|1\right\rangle$$
$$X\left|1\right\rangle = \left|0\right\rangle$$

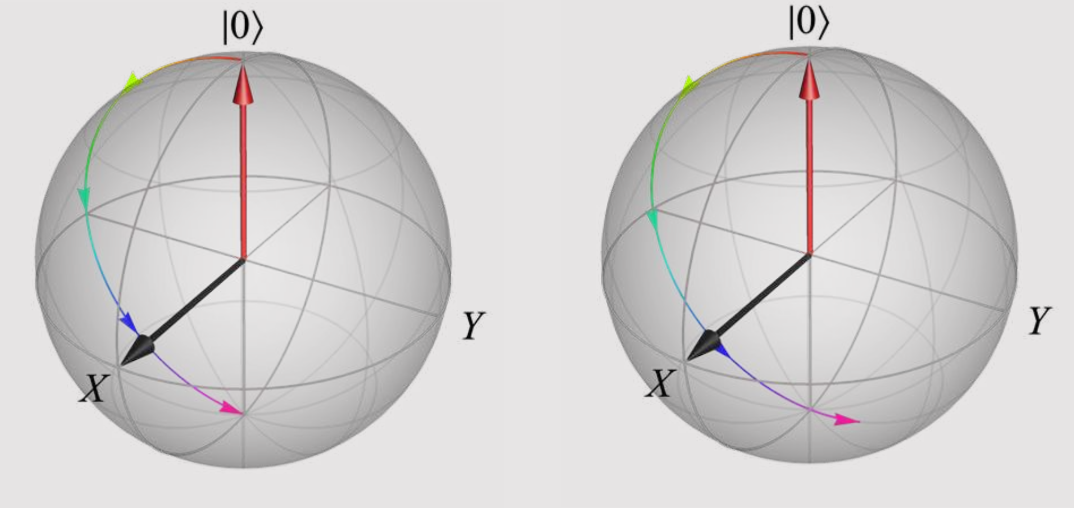


Miscalibrated gate

Ideal gate

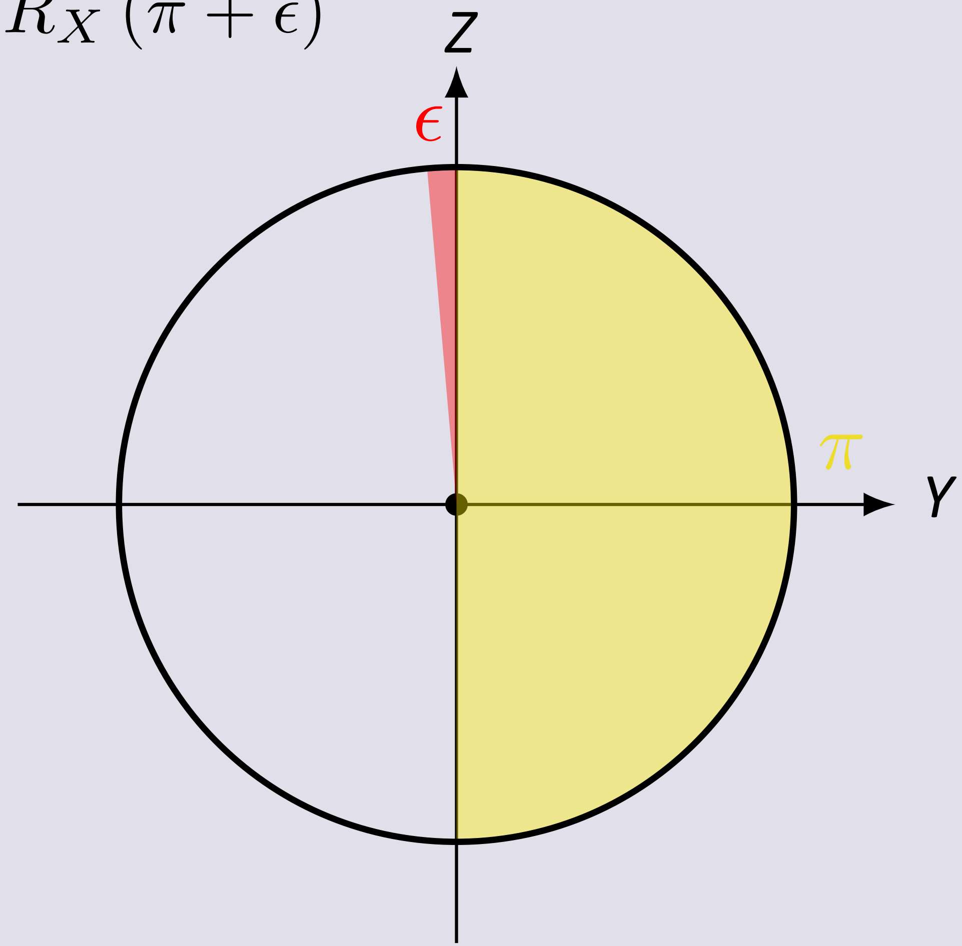
Noisy gate





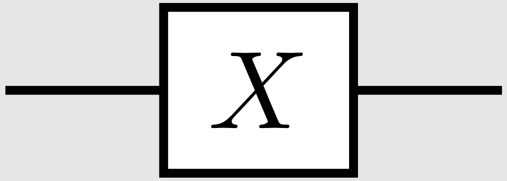
$$X = R_X(\pi)$$

$$\tilde{X} := R_X(\pi + \epsilon)$$

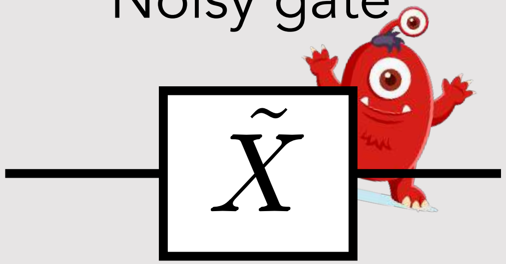


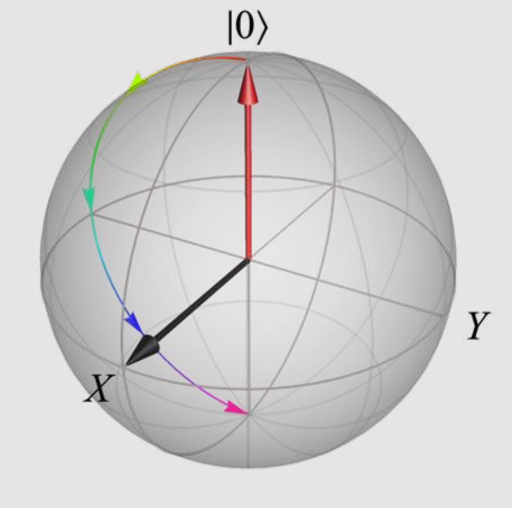
Miscalibrated gate

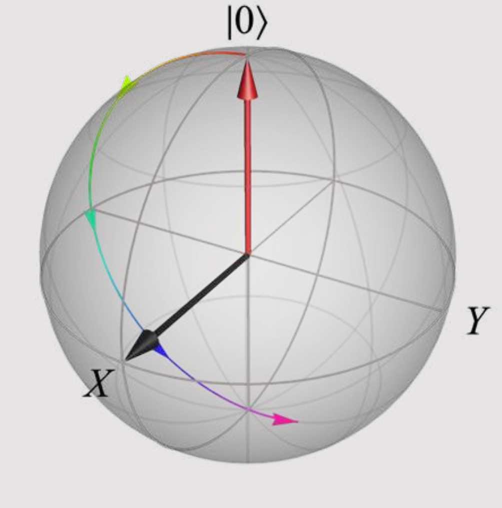
Ideal gate



Noisy gate







Shown

$$R_X(\theta+\phi) = R_X(\theta)\,R_X(\phi)$$

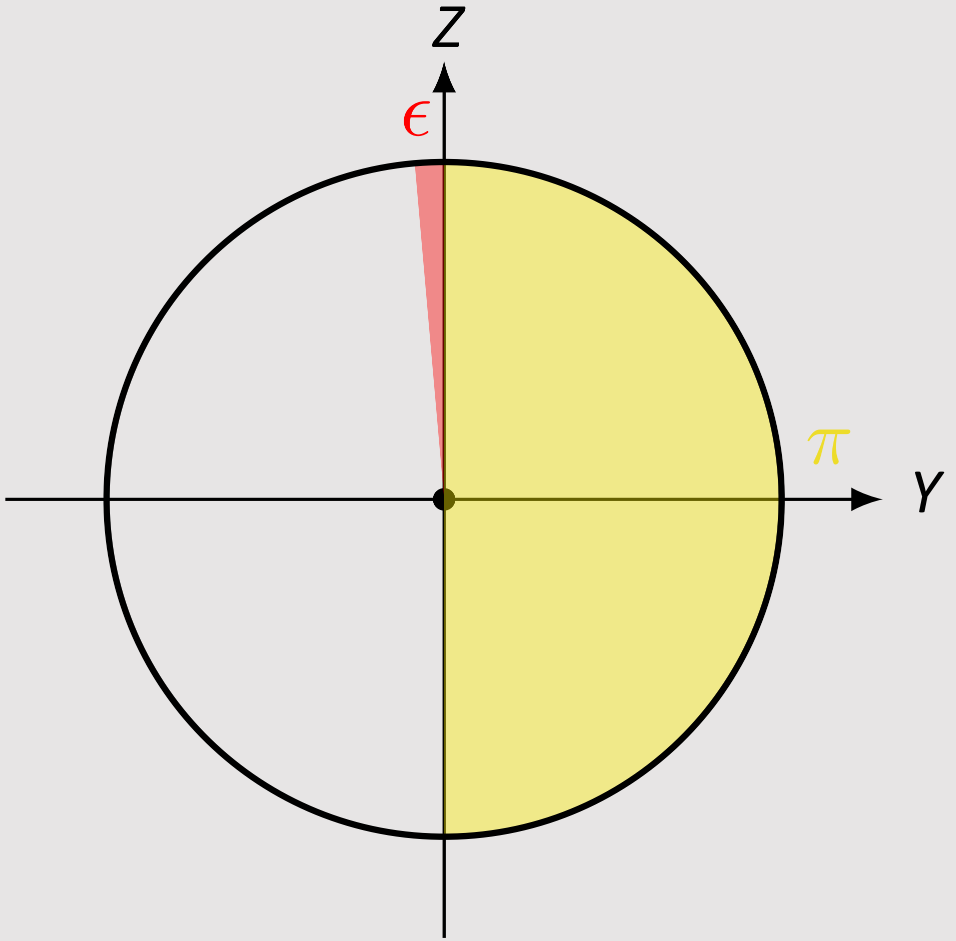
$$X = R_X(\pi)$$

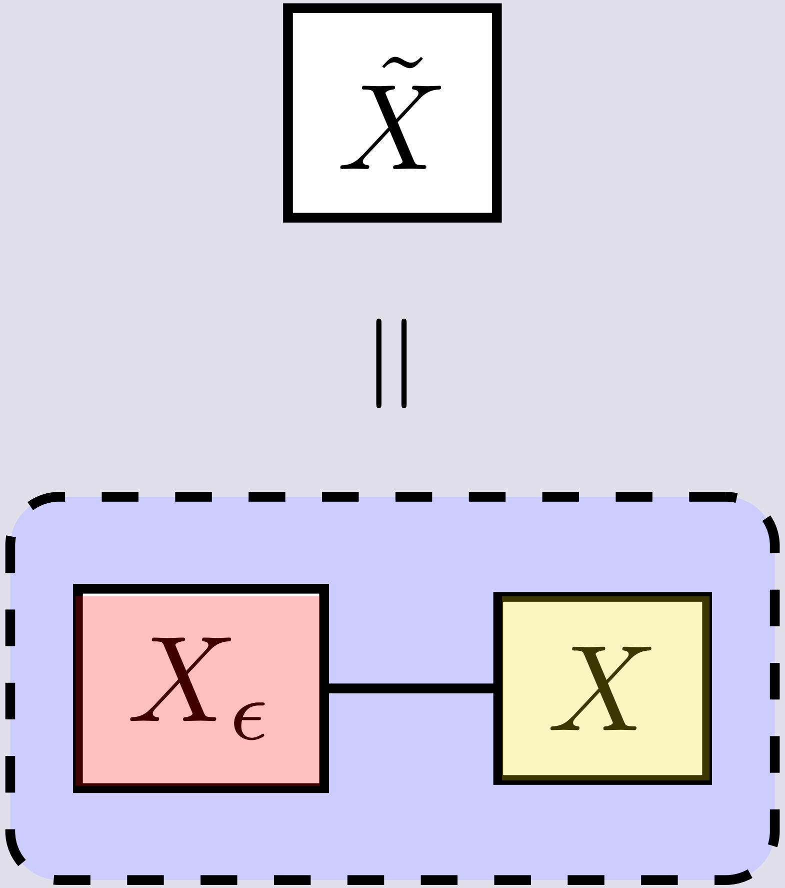
$$\tilde{X} := R_X(\pi+\epsilon)$$

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$
$$= \cos(\theta/2)I - i\sin(\theta/2)X$$

$$= \exp\left(-i\frac{\pi+\epsilon}{2}X\right)$$
$$= \exp\left(-i\frac{\pi}{2}X - i\frac{\epsilon}{2}X\right)$$
$$= \exp\left(-i\frac{\pi}{2}X\right)\exp\left(-i\frac{\pi}{2}X\right)$$
$$= R_X(\epsilon)\,R_X(\pi)$$
$$= X_\epsilon X$$

Noisy gate decomposition

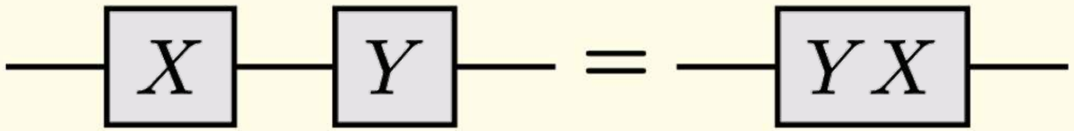


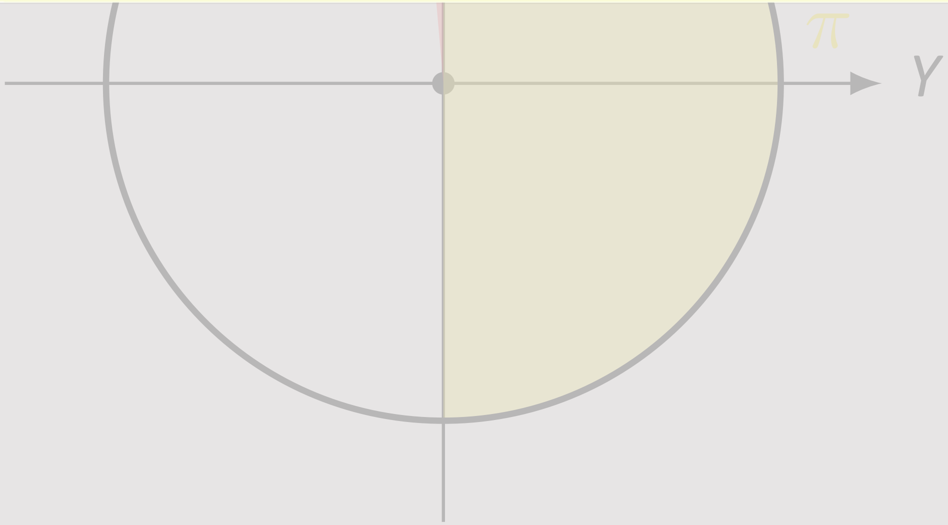


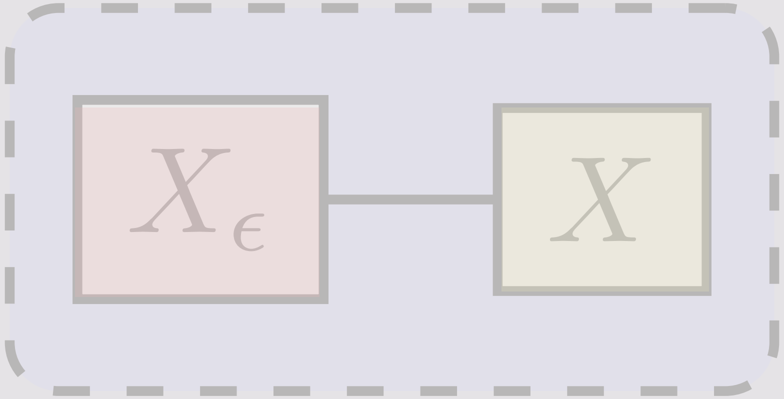
Noisy gate decomposition

Common pitfall

The order in which gates appear in a schematic is the reverse of how they appear in the algebra.

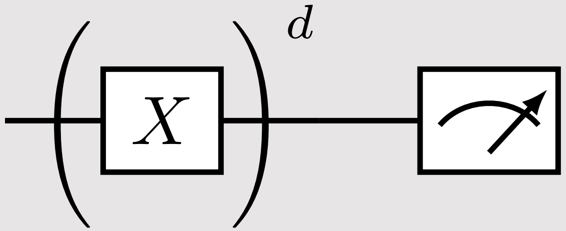






Using a noisy gate in a quantum circuit

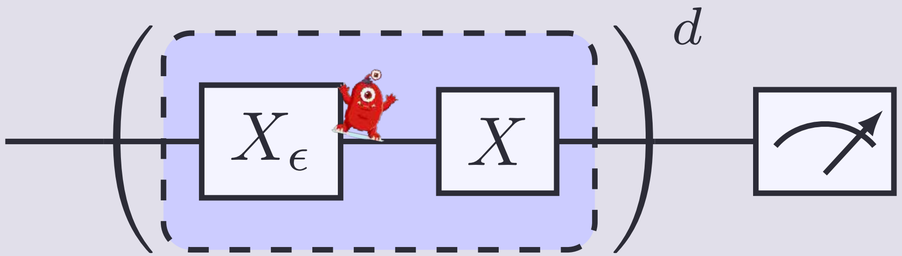
Ideal



$$X = R_X(\pi)$$

$$U_{\text{total}} = X^d$$
$$= \left[R_X(\pi)\right]^d$$
$$= R_X(d\pi)$$

Noisy

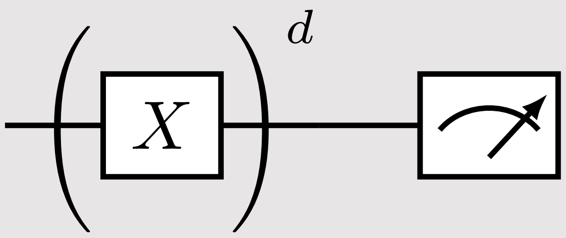


$$\tilde{X} := R_X(\pi + \epsilon) = X_\epsilon X$$

$$\tilde{U}_{\text{total}} = \tilde{X}^d$$
$$= \left[R_X(\epsilon)\, R_X(\pi)\right]^d$$
$$= R_X(d\epsilon)\, R_X(d\pi)$$
$$\tilde{U}_{\text{total}} = R_X(d\epsilon)\, U_{\text{total}}$$

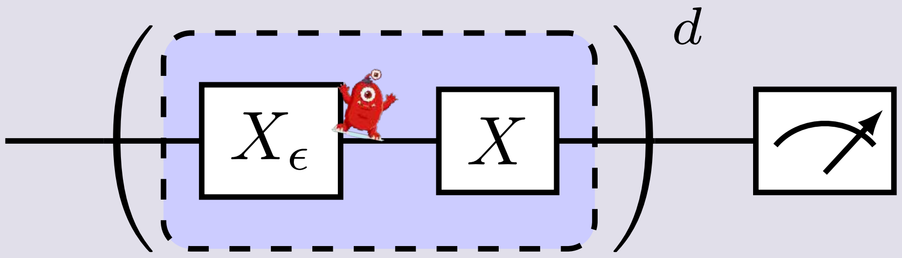
Using a noisy gate in a quantum circuit: final state

Ideal



$$U_{\text{total}} = X^d = R_X\left(d\pi\right)$$

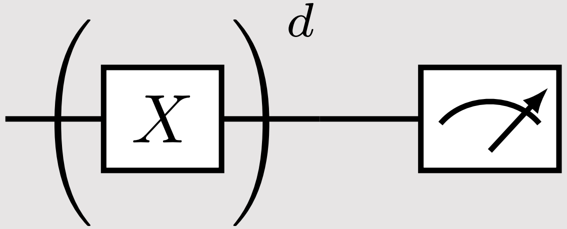
Noisy



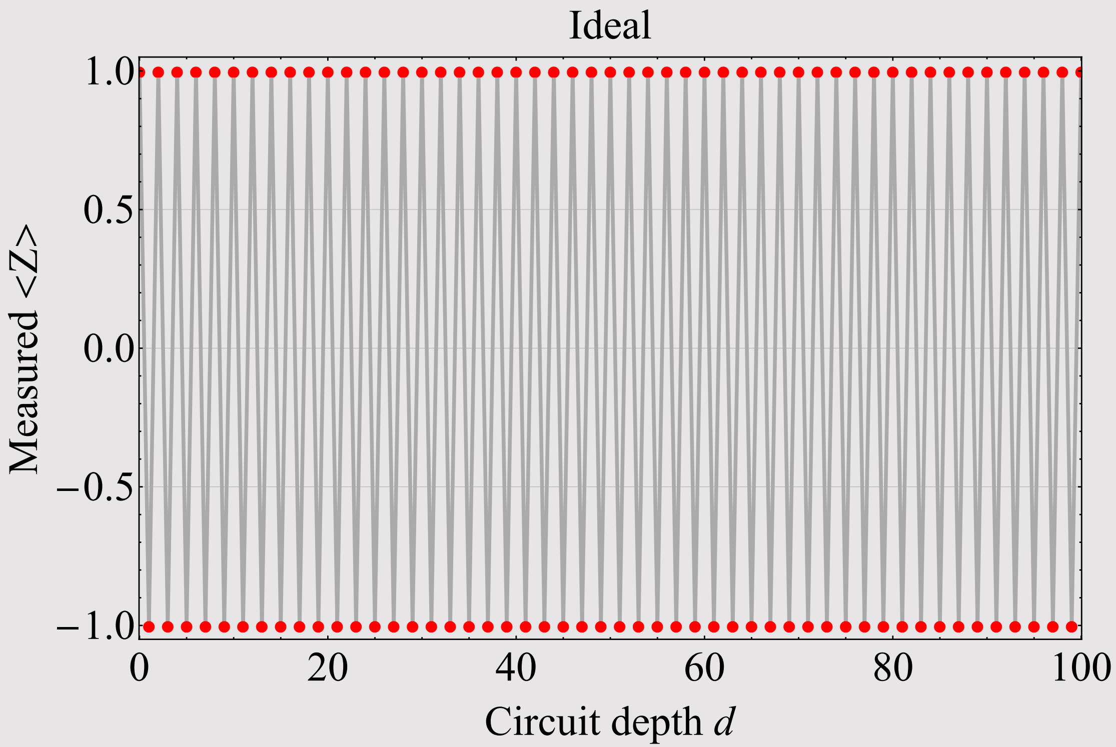
$$\tilde{U}_{\text{total}} = R_X\left(d\epsilon\right) U_{\text{total}}$$

Ideal vs. noisy observable

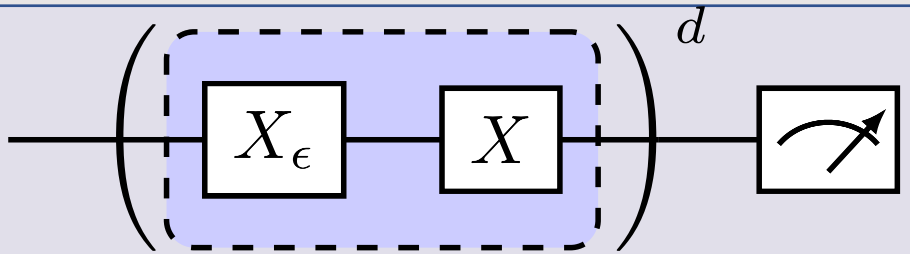
Ideal



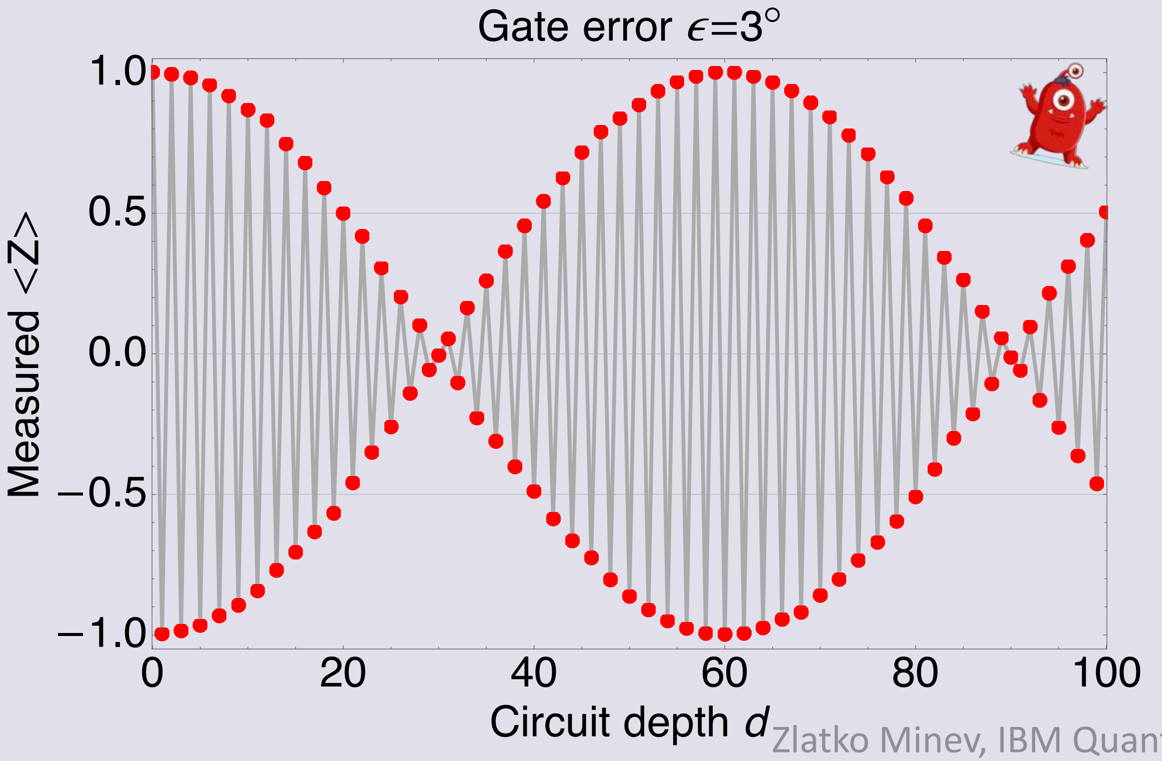
$$\langle \psi_f | Z | \psi_f \rangle = \cos(d\pi) = (-1)^d$$



Noisy

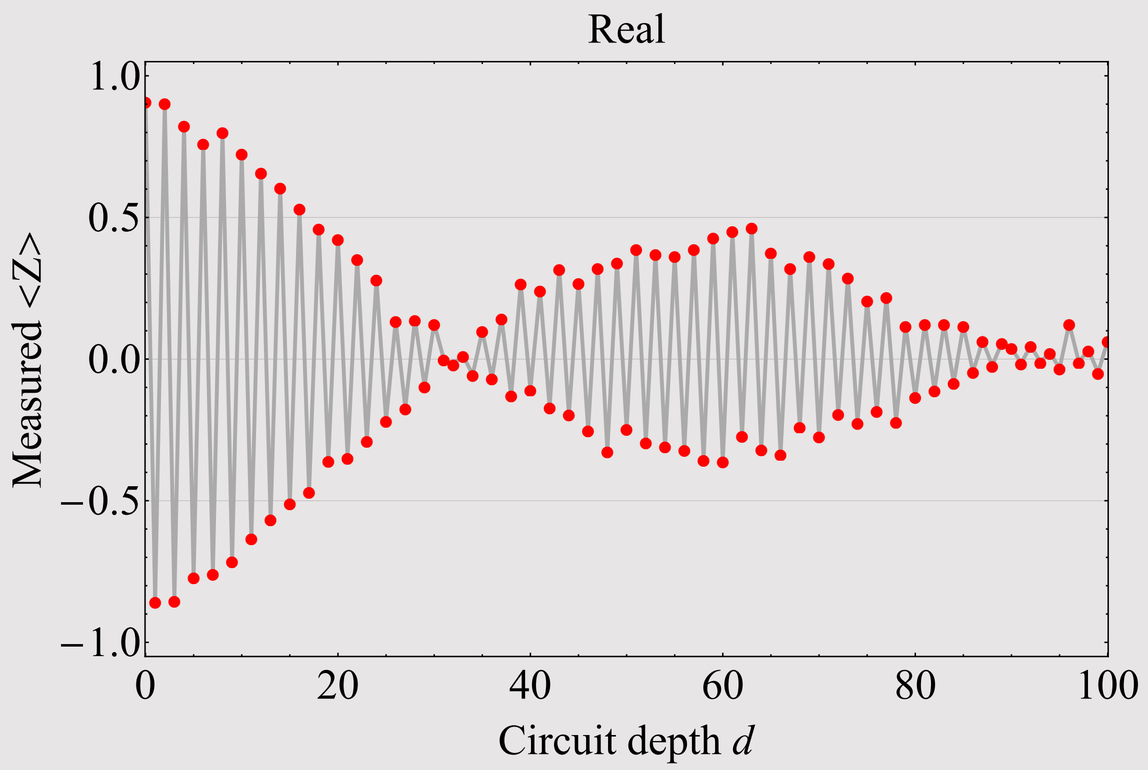


$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos(d\pi + d\epsilon)$$

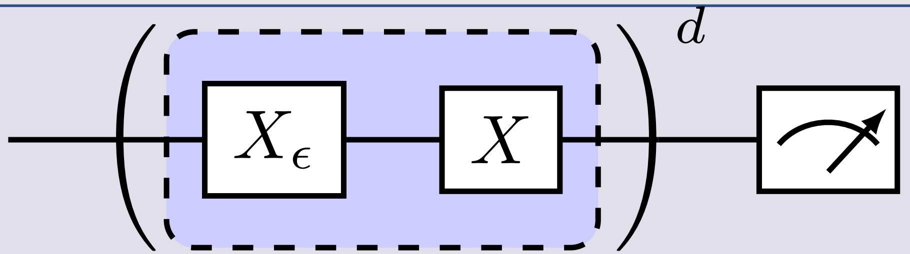


Compare to full experiment

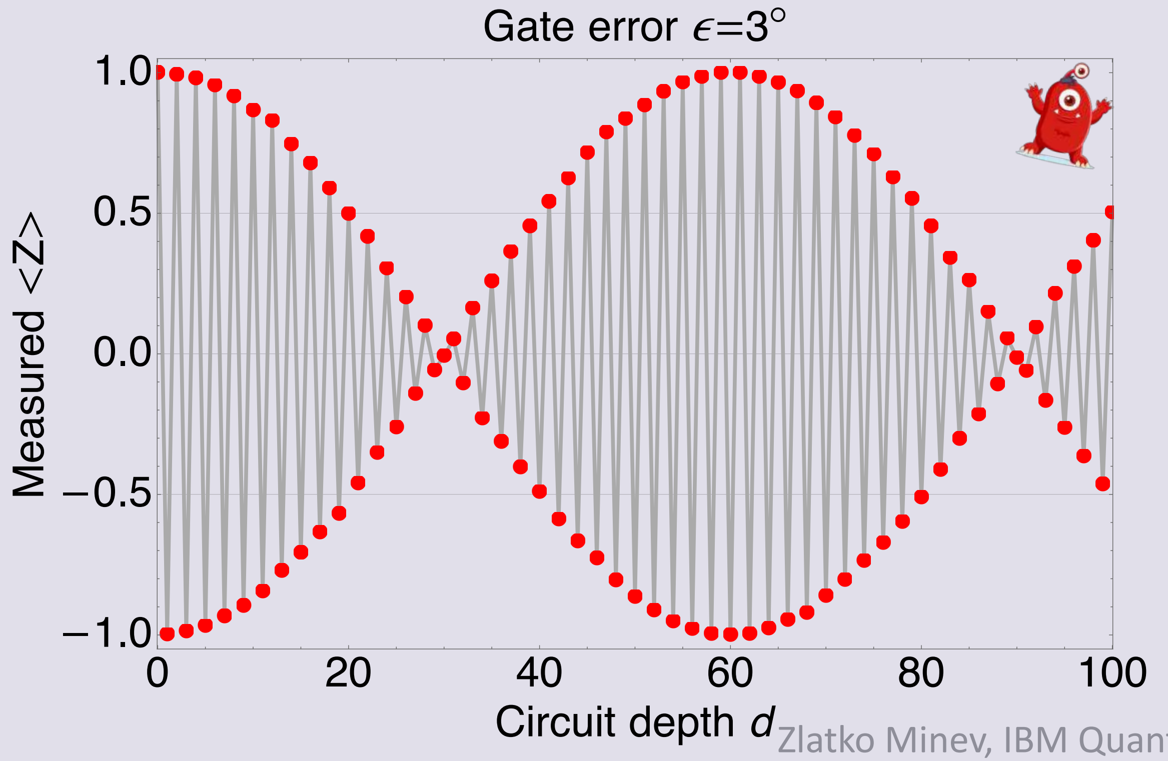
Full experiment



Noisy



$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



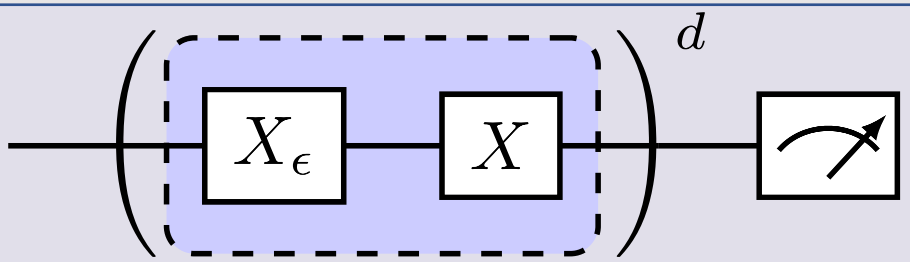
Coherent error is bad, quadratically so

Coherent errors have a *quadratic* impact on algorithmic accuracy (worst-case error)

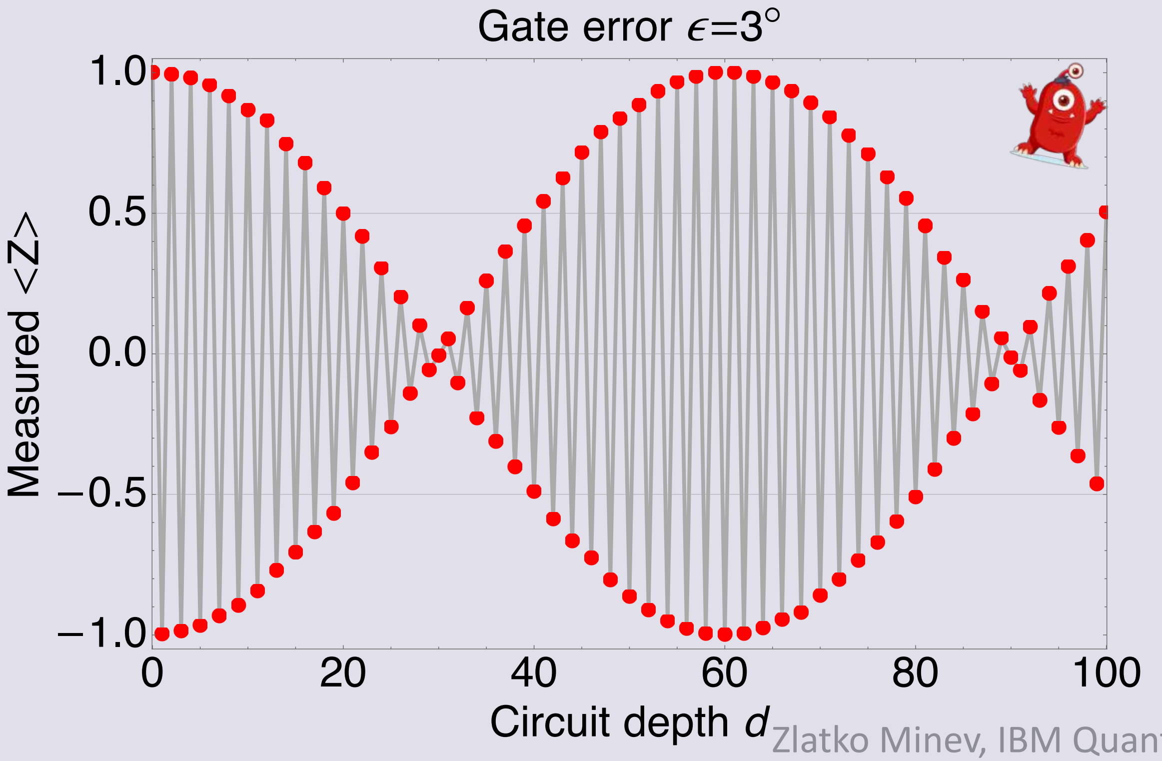
$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle - \langle \psi_f | Z | \psi_f \rangle$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}\left(x^3\right)$$

Noisy

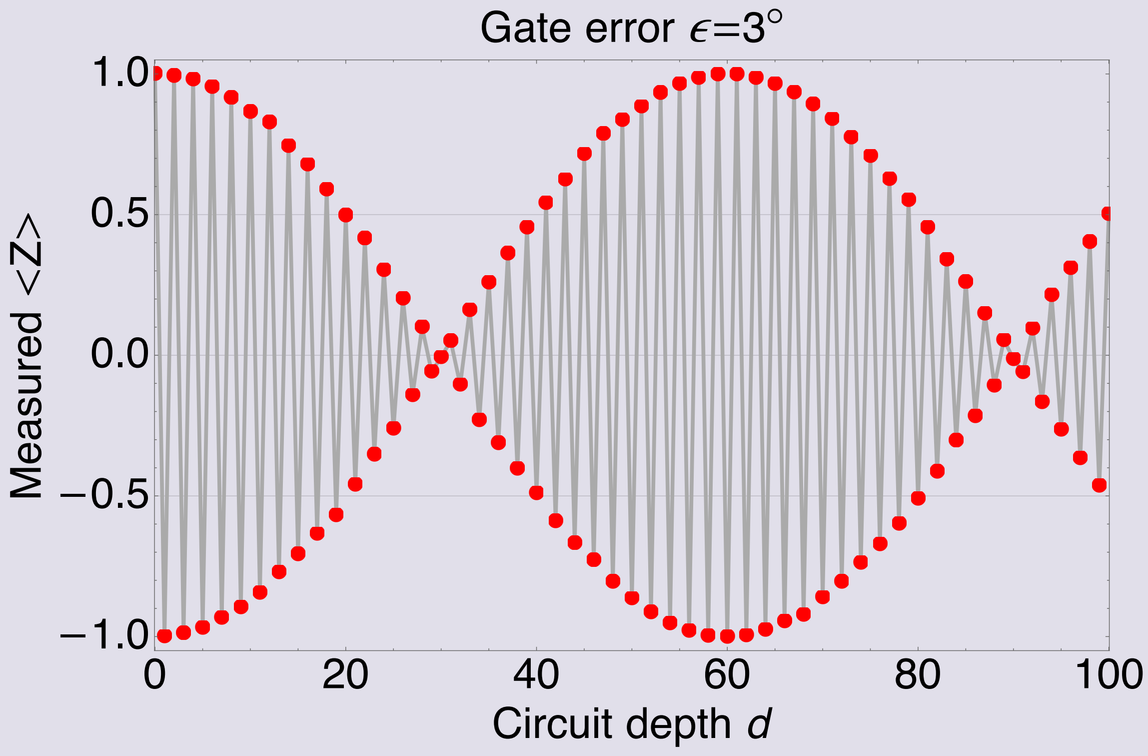


$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos(d\pi + d\epsilon)$$

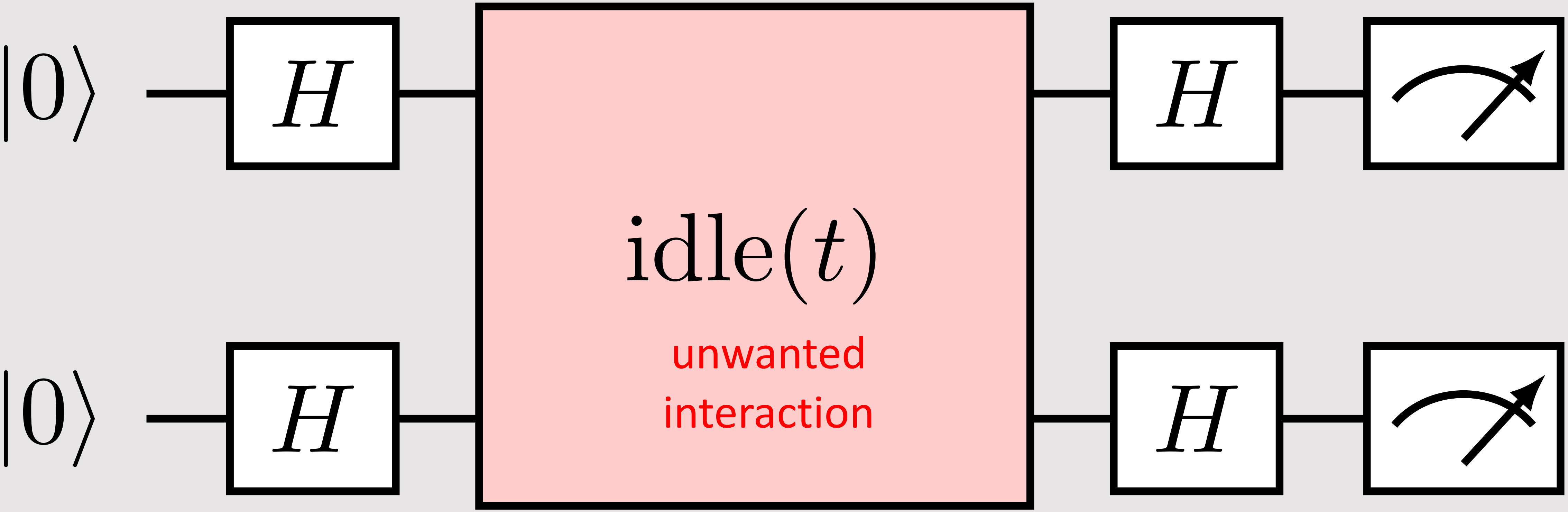


Coherent errors: brief summary

- are ubiquitous
- can be described by unitary operations
- do not loose quantum information
- data: can create oscillations in the data
- data: do not yield exponential decays
- have a *quadratic* impact on algorithmic accuracy (worst-case error)



Bonus content: two-qubit coherent ZZ error



Questions

Answer these multiple-choice questions in the chat; for example, type “1a 2b.”

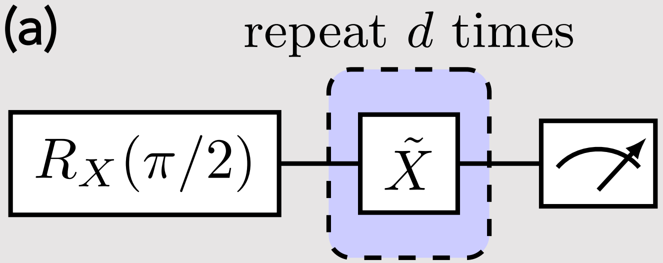
1. Coherent noise can be caused by
   a) loss of energy of the qubit
   b) miscalibration, such as over-rotation
   c) wanted coupling to neighboring qubit

2. Coherent noise can be really bad because
   a) it results in loss of information
   b) you cannot undo it
   c) the worst-case error often grows quadratically

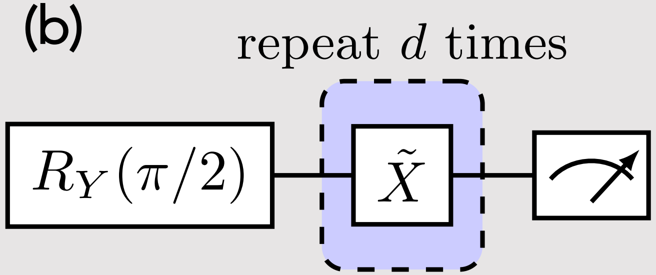
Advanced questions to dive deeper

1. Amplitude-error amplification sequences.
   (A) Calculate the expectation value <*Z* (*d*)> as a function of depth *d* of the sequence for the following sequences. Assume the noise model $\tilde{X} := R_X(\pi + \epsilon) = X_\epsilon X$
   (B) How could you use this result to fine-tune your gates?
   (C) Can you come up with an alternative or more clever error-amplification sequence?

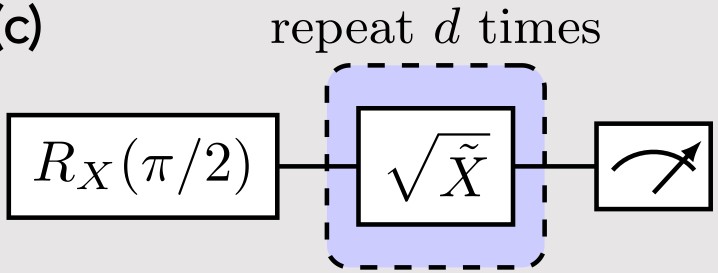
(a)



(b)



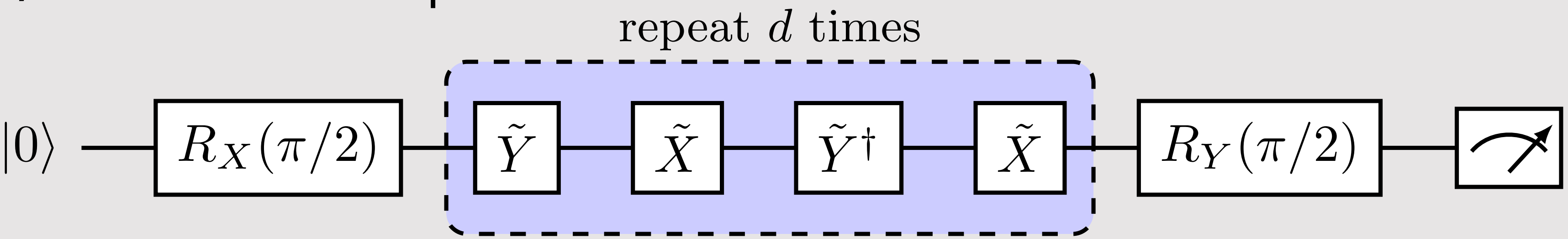
(c)



2. Phase-error amplification sequences. Use the noise model: Repeat (A), (B), and (C) for Exercise (1) for the following sequences and assuming a phase error between the X and Y gates, rather than an amplitude error.

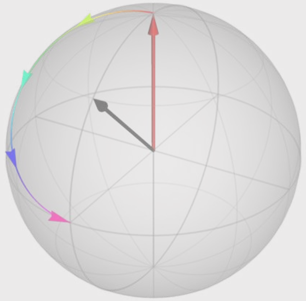
$$\tilde{X} = X\,,$$

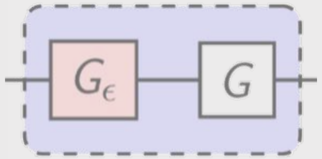
$$\tilde{Y} = \exp\left[-i\frac{\pi}{2}\left(\cos\epsilon\, Y + \sin\epsilon\, X\right)\right]\,.$$



Chapter 3

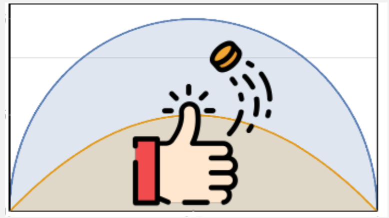
Coherent noise





Measurements in quantum

Projection noise

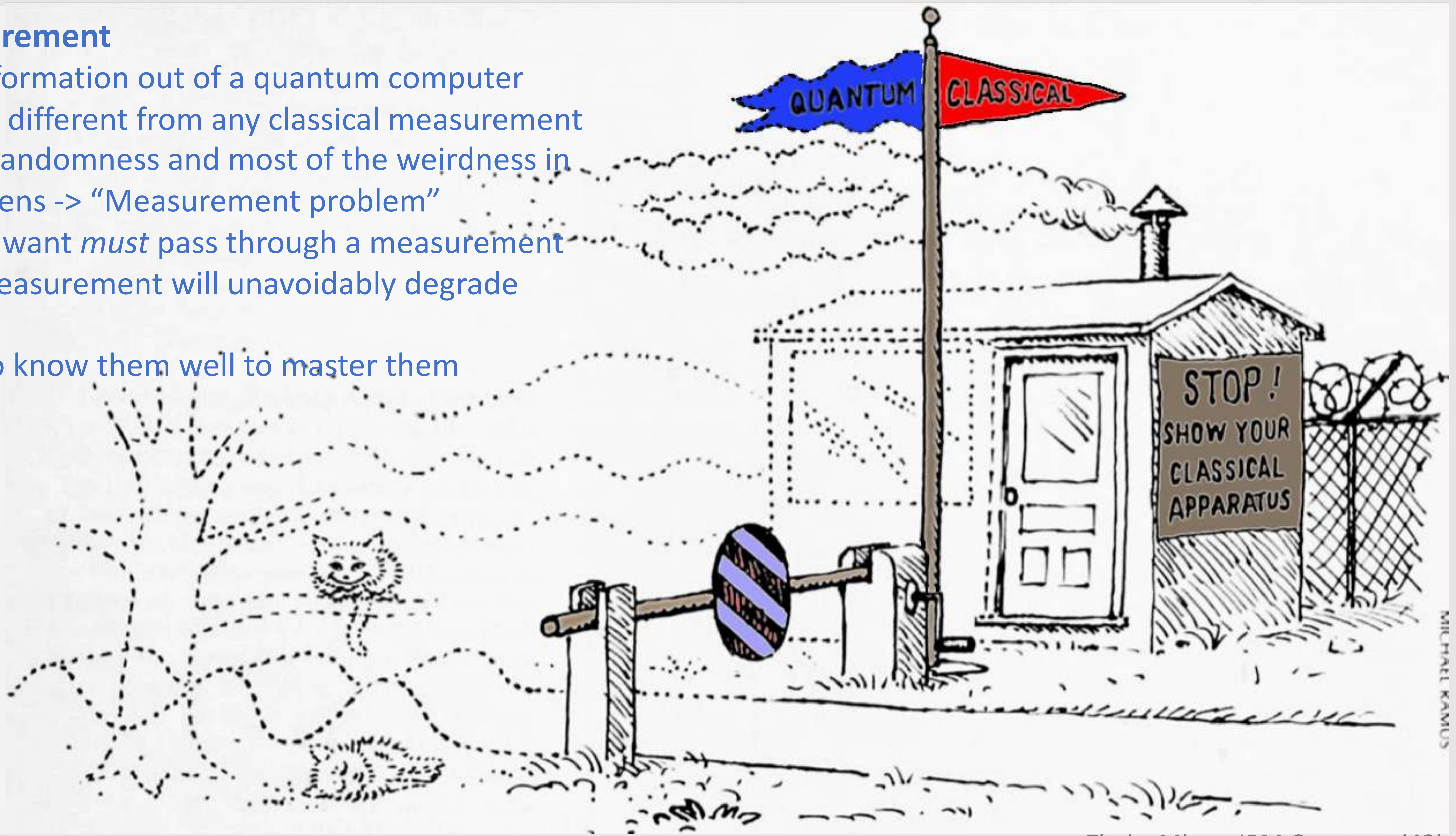


coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

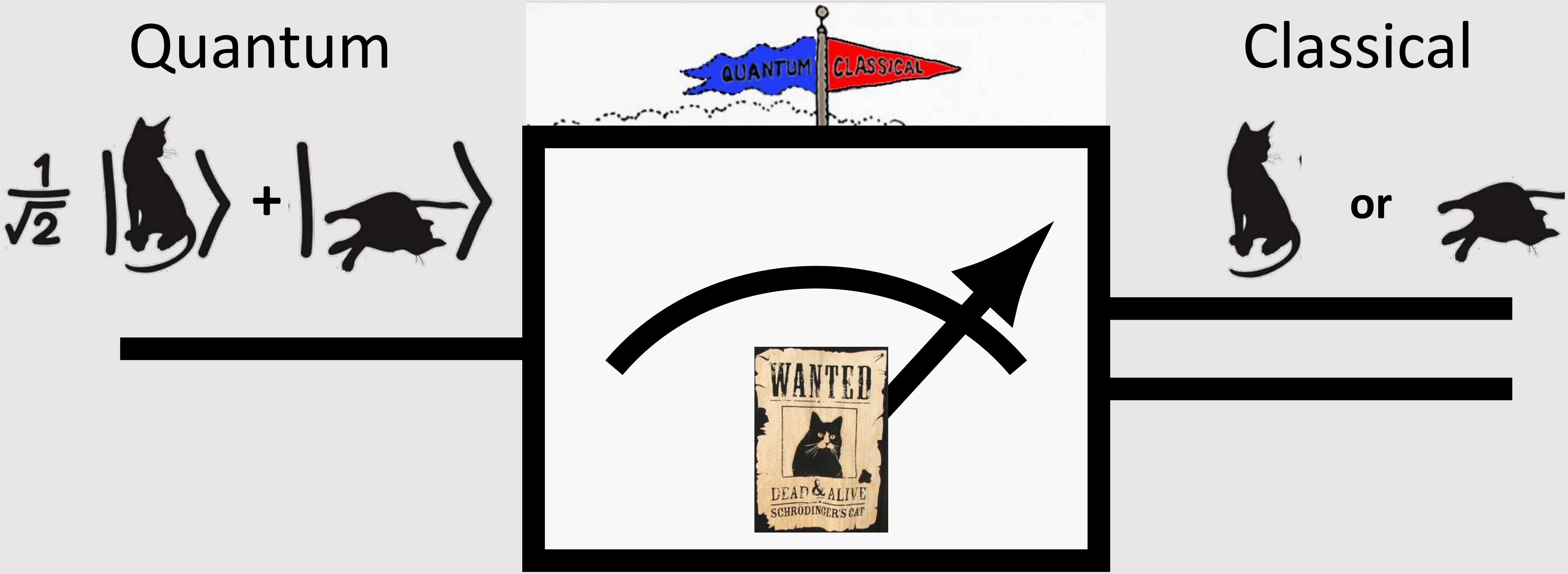
Quantum-to-classical interface

Quantum measurement

- How we get information out of a quantum computer
- Fundamentally different from any classical measurement
- Where all the randomness and most of the weirdness in quantum happens -> “Measurement problem”
- Any results we want *must* pass through a measurement
- Any noise in measurement will unavoidably degrade our results
- We must get to know them well to master them



Quantum-to-classical interface

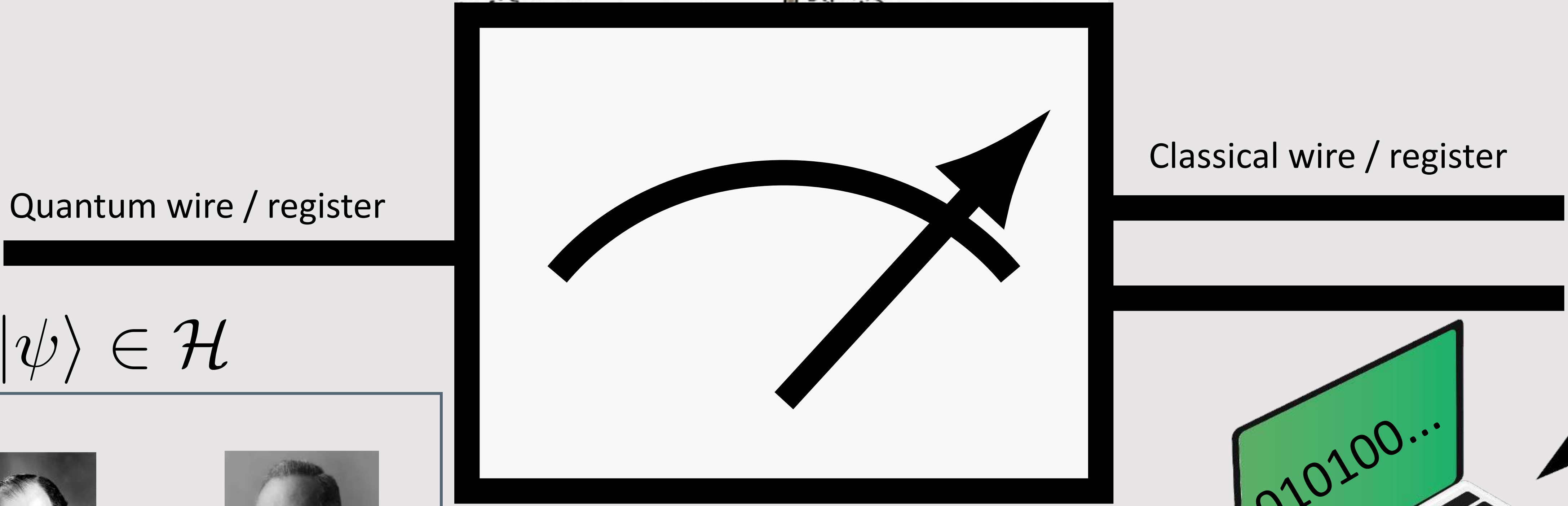


cat image: docencia

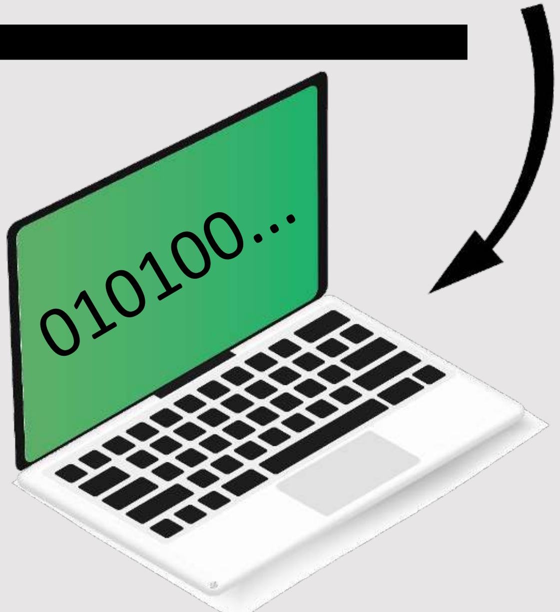
Meter in quantum computing

Quantum

Classical



$$|\psi\rangle \in \mathcal{H}$$



John von Neumann

Image source: LANL

Max Born

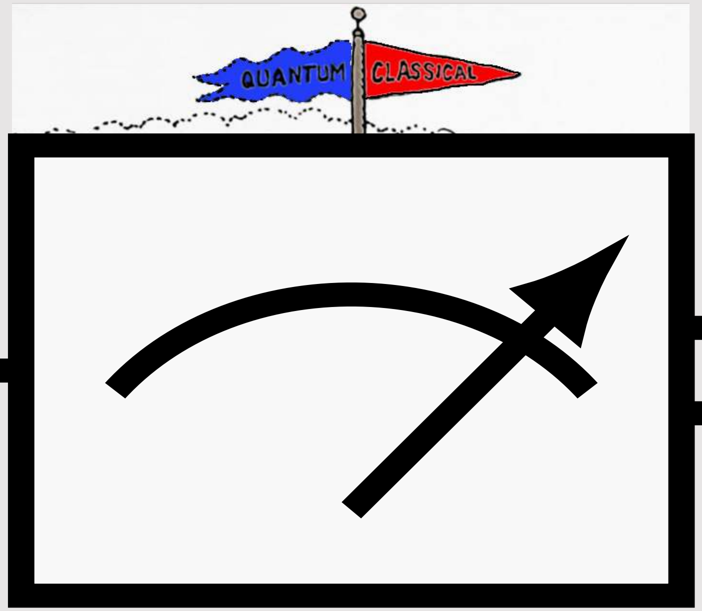
Image source: Public domain

Meter in quantum computing

Quantum

Quantum wire / register

$|\psi\rangle \in \mathcal{H}$



Classical

Classical wire / register

X

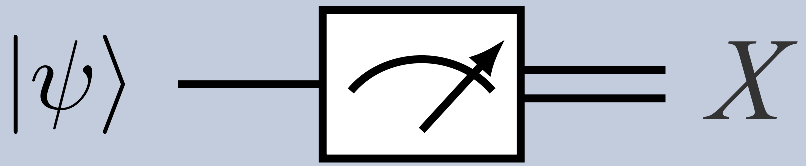
Measure in computational basis

$$|0\rangle\,,|1\rangle$$

outcome — outcome probability

$$\begin{cases} X=0 & : \quad p\,(X=0) = |\langle 0|\psi\rangle|^2 \\ X=1 & : \quad p\,(X=1) = |\langle 1|\psi\rangle|^2 \end{cases}$$

Try it yourself! Resolution of the identity



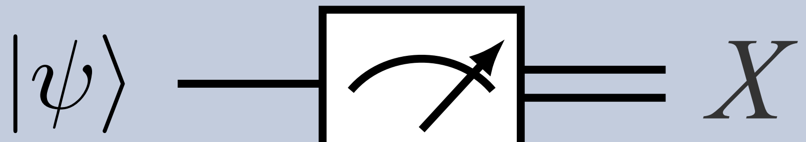
outcome — outcome probability

$$\begin{cases} X=0 \;:\; p\left(X=0\right) = \left|\langle 0|\psi\rangle\right|^2 \\ X=1 \;:\; p\left(X=1\right) = \left|\langle 1|\psi\rangle\right|^2 \end{cases}$$

Try it yourself! (Pause the video) Show that

$$p\left(X=0\right) + p\left(X=1\right) = 1$$

Solution: Resolution of the identity



outcome outcome probability

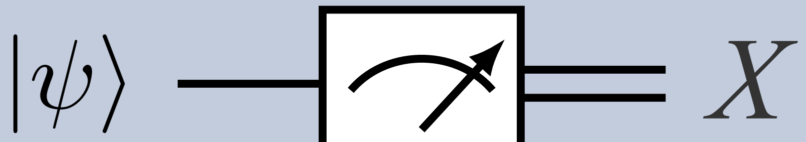
$$\begin{cases} X=0 \ : \ p\left(X=0\right)=\left|\langle 0|\psi\rangle\right|^2 \\ X=1 \ : \ p\left(X=1\right)=\left|\langle 1|\psi\rangle\right|^2 \end{cases}$$

Try it yourself! (Pause the video) Show that

$$p\left(X=0\right)+p\left(X=1\right)=1$$

$$\begin{aligned} &= \left|\langle 0|\psi\rangle\right|^2+\left|\langle 1|\psi\rangle\right|^2 \\ &= \langle\psi|0\rangle\langle 0|\psi\rangle+\langle\psi|1\rangle\langle 1|\psi\rangle \\ &= \langle\psi|\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)|\psi\rangle \\ &= \langle\psi|\hat{I}|\psi\rangle \\ &= \left||\psi\rangle\right|^2 \\ &= 1 \end{aligned}$$

Outcome probability distribution

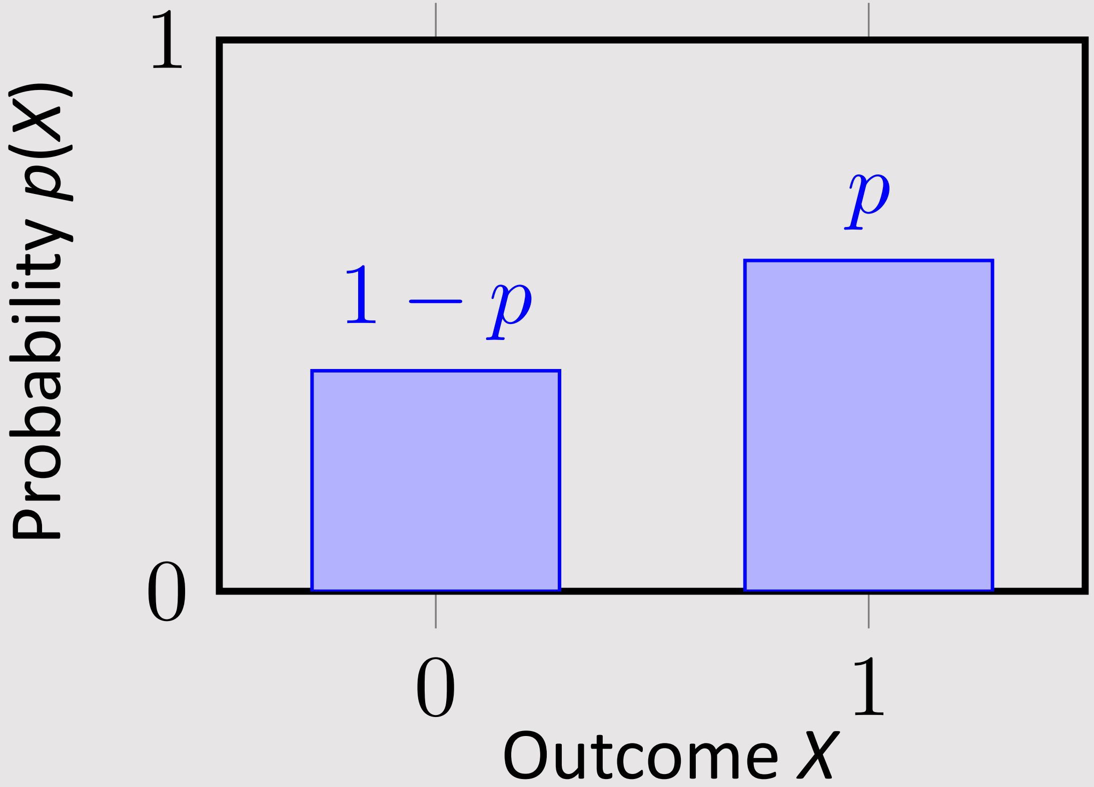


outcome — outcome probability

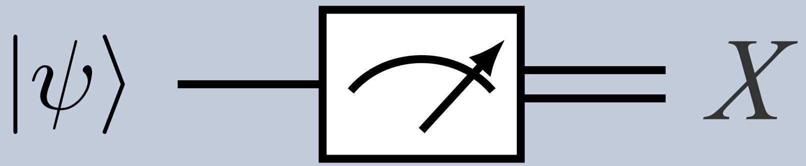
$$\begin{cases} X=0 \ : \ p\left(X=0\right) = \left|\langle 0|\psi\rangle\right|^2 \\ X=1 \ : \ p\left(X=1\right) = \left|\langle 1|\psi\rangle\right|^2 \end{cases}$$

The entire classical distribution is dictated by a single parameter ***p*** determined from the quantum state according to

$$\begin{aligned} p &:= \Pr\left[X=1\right] \\ &= \langle|1\rangle\langle 1|\rangle \\ &= \left|\langle 1|\psi\rangle\right|^2 \end{aligned}$$



Outcome distribution in terms of projectors and overlaps



outcome — outcome probability

$$
\begin{cases}
X=0 \;:\; p\left(X=0\right)=\left|\langle 0|\psi\rangle\right|^2 \\
X=1 \;:\; p\left(X=1\right)=\left|\langle 1|\psi\rangle\right|^2
\end{cases}
$$

The resolution of the total probability

$$p\left(X=0\right)+p\left(X=1\right)=1$$

$$=\langle\psi|\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)|\psi\rangle$$

In terms of projectors

$$\hat{\mu}\left(0\right)=|0\rangle\langle 0| \qquad \hat{\mu}\left(1\right)=|1\rangle\langle 1|$$

General formulation (overlap between states)

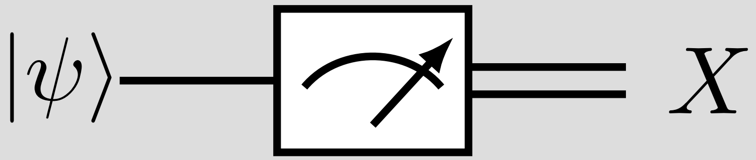
$$p\left(X=x\right)=\langle\psi|\hat{\mu}\left(x\right)|\psi\rangle$$

$$=\langle\hat{\mu}\left(x\right),|\psi\rangle\langle\psi|\rangle$$ as an inner product in operator space

$$=\mathrm{Tr}\left(\hat{\mu}\left(x\right)|\psi\rangle\langle\psi|\right)$$ as trace, to be introduced later

Classical expectation values

$$\{0,1\}$$



Set of possible outcomes

$$\Sigma = \{0,1\} \qquad X \in \Sigma$$

Measurement operators

$$\begin{array}{ll} x & \hat{\mu}(x) \\ 0: & \hat{\mu}(0) = |0\rangle\langle 0| \\ 1: & \hat{\mu}(0) = |1\rangle\langle 1| \end{array}$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Classical expectation value of classical random variable

$$\begin{aligned} \mathrm{E}[X] &= \sum_{x\in\{0,1\}} x \Pr[X = x] \\ &= 0 \Pr[X = 0] + 1 \Pr[X = 1] \\ &= \Pr[X = 1] \\ &\boxed{= p\ .} \end{aligned}$$

In terms of quantum operators

$$\begin{aligned} \mathrm{E}[X] &= \sum_{x\in\{0,1\}} x \Pr[X = x] \\ &= \sum_{x\in\{0,1\}} x \langle \hat{\mu}(x) \rangle \\ &= \sum_{x\in\{0,1\}} \langle x \hat{\mu}(x) \rangle \\ &= \left\langle \sum_{x\in\{0,1\}} x \hat{\mu}(x) \right\rangle \\ &\boxed{= \langle \hat{M} \rangle} \end{aligned}$$

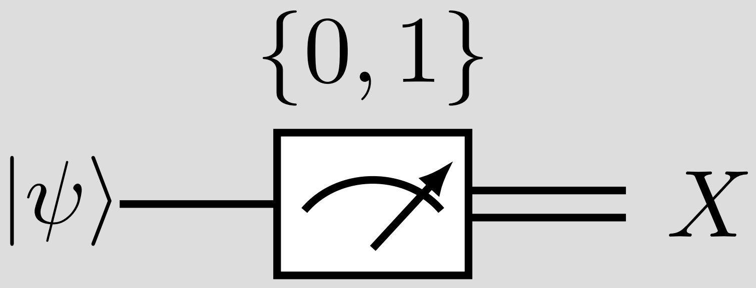
The classical expectation value gives us the expectation value of the quantum operator

$$\hat{M} := \sum_{x\in\{0,1\}} x \hat{\mu}(x)$$

$$= 0\,|0\rangle\langle 0| + 1\,|1\rangle\langle 1|$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Classical variance



Set of possible outcomes

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Measurement operators

$$\begin{array}{ll} x & \hat{\mu}(x) \\ 0: & \hat{\mu}(0) = |0\rangle\langle 0| \\ 1: & \hat{\mu}(0) = |1\rangle\langle 1| \end{array}$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Calculate the variance of the classical random variable

$$\mathbb{V}[X] = \mathrm{E}\left[\left(X - \mathrm{E}\left[X^2\right]\right)^2\right]$$
$$= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2$$

Calculate the expectation of the square

"Classical path"

$$\mathrm{E}\left[X^2\right] = \sum_{x\in\{0,1\}} x^2 \Pr[X = x]$$
$$= 0 \Pr[X = 0] + 1 \Pr[X = 1]$$
$$= p$$

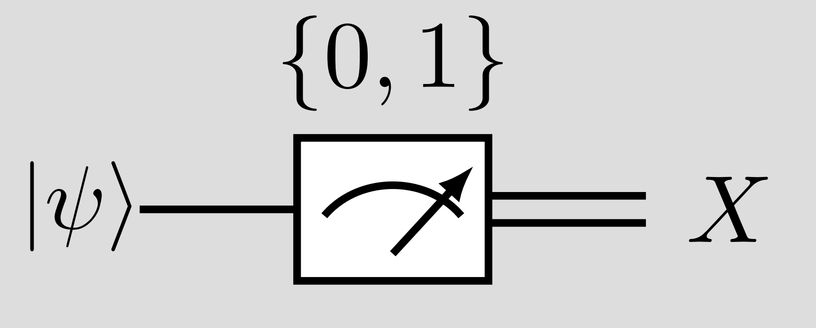
"Quantum path"

$$\mathrm{E}\left[X^2\right] = \sum_{x\in\{0,1\}} x^2 \Pr[X = x]$$
$$= \sum_{x\in\{0,1\}} x^2 \langle \hat{\mu}(x) \rangle$$
$$= \left\langle \sum_{x\in\{0,1\}} x^2 \hat{\mu}(x) \right\rangle$$
$$= \left\langle \hat{M}^2 \right\rangle$$

indimart

Coin-toss icon created by Good Ware - Flaticon

Classical variance



Set of possible outcomes

$$\Sigma = \{0,1\} \qquad X \in \Sigma$$

Measurement operators

$$\begin{array}{ll} x & \hat{\mu}(x) \\ 0: & \hat{\mu}(0) = |0\rangle\langle 0| \\ 1: & \hat{\mu}(0) = |1\rangle\langle 1| \end{array}$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Calculate the variance of the classical random variable

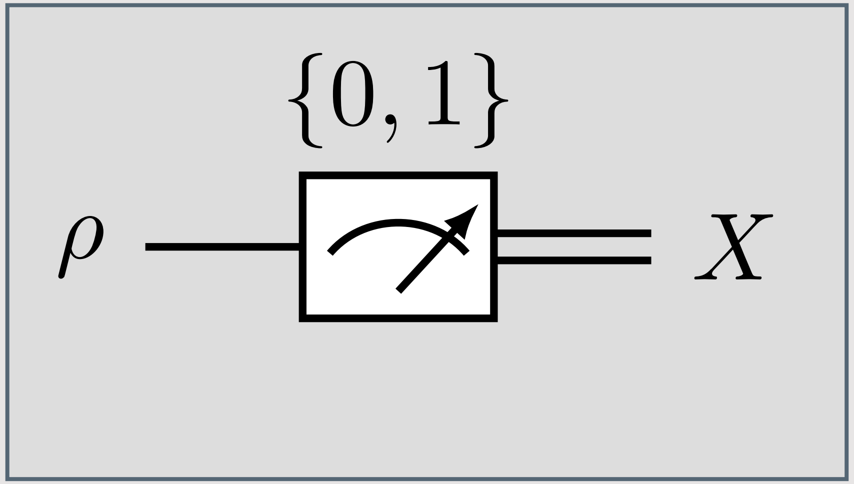
$$\begin{aligned} \mathbb{V}[X] &= \mathrm{E}\left[\left(X - \mathrm{E}\left[X^2\right]\right)^2\right] \\ &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \end{aligned}$$

Putting it together

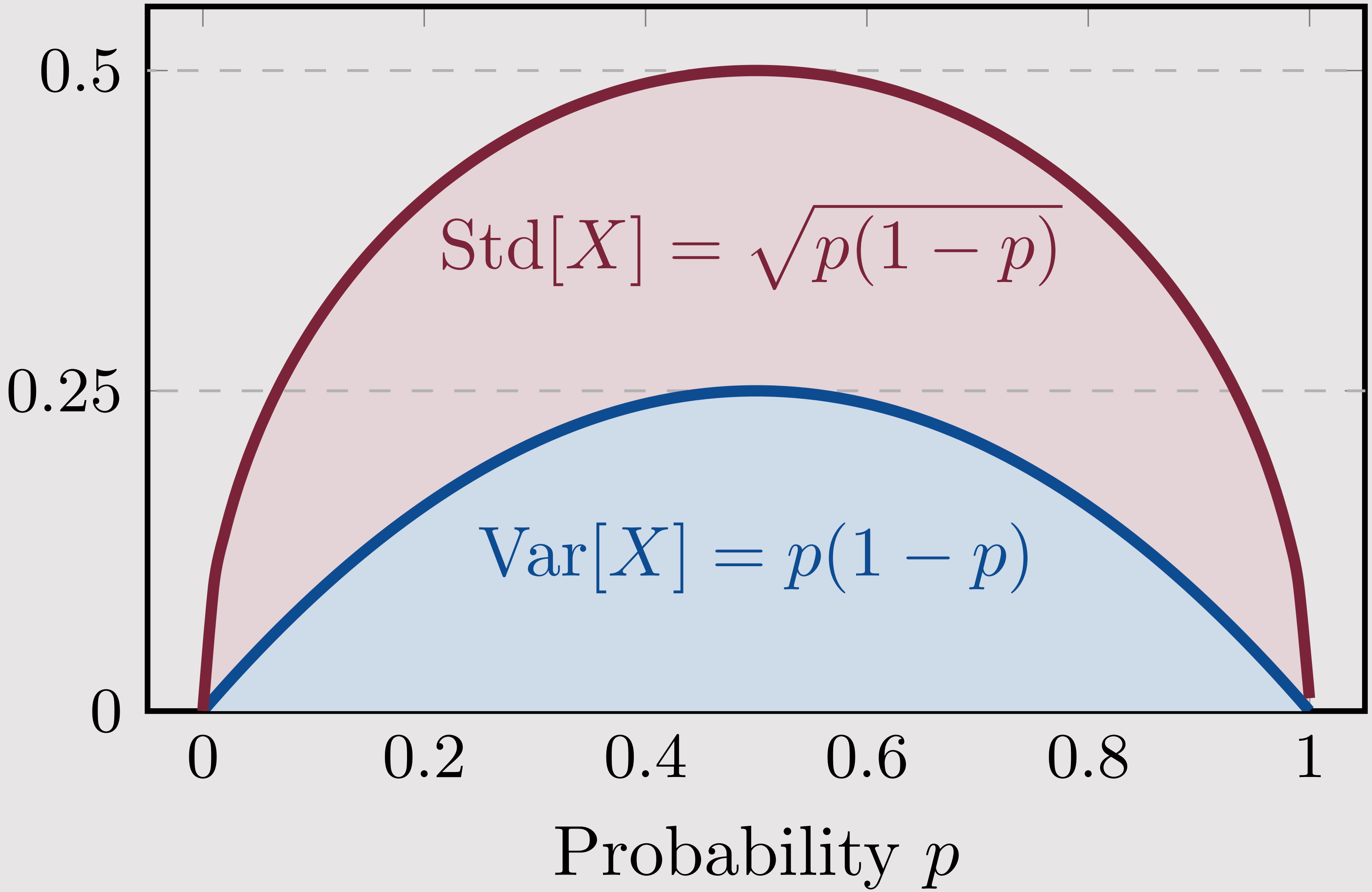
$$\mathbb{V}[X] = \left\langle \hat{M}^2 \right\rangle - \left\langle \hat{M} \right\rangle^2$$

$$\begin{aligned} \mathbb{V}[X] &= p - p^2 \\ &= p(1-p) \end{aligned}$$

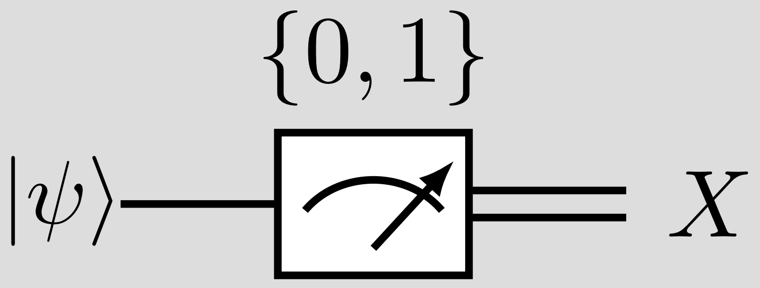
Quantum projection noise



Variance of the random classical variable vs. probability to obtain 1



Summary: Qubit measured in the computational basis



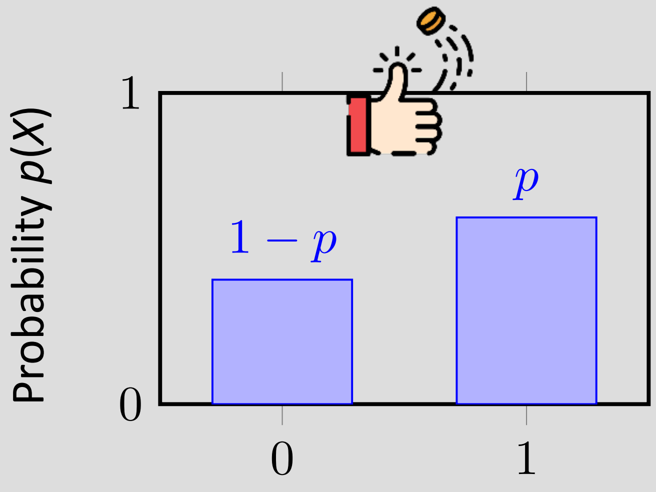
Set of possible outcomes

$$\Sigma = \{0, 1\}$$

$$X \in \Sigma$$

Measurement operators

$$\begin{array}{ll} x & \hat{\mu}\,(x) \\ 0: & \hat{\mu}(0) = |0\rangle\,\langle 0| \\ 1: & \hat{\mu}(0) = |1\rangle\,\langle 1| \end{array}$$



Probability to measure outcome

$\rho = |\psi\rangle\,\langle\psi|$ to be introduced later in detail

$$\begin{cases} X = 0: \quad p\,(X = 0) = \mathrm{Tr}\left(\hat{\mu}(0)^\dagger \rho\right) = \mathrm{Tr}\left(|0\rangle\,\langle 0|\,\rho\right) = \frac{1}{2}\left(1 + \langle Z\rangle\right) \\ X = 1: \quad p\,(X = 1) = \mathrm{Tr}\left(\hat{\mu}(1)^\dagger \rho\right) = \mathrm{Tr}\left(|1\rangle\,\langle 1|\,\rho\right) = \frac{1}{2}\left(1 - \langle Z\rangle\right) \end{cases}$$

Bernoulli distribution. Single shot outcome follows a Bernoulli distribution:

$$X \sim \mathrm{Bernoulli}\,(p) \qquad p := \mathrm{Tr}\left(|1\rangle\,\langle 1|\,\rho\right) \in [0, 1]$$

$$\mathrm{E}\,[X] = p \qquad \mathrm{Var}[X] = p(1 - p)$$

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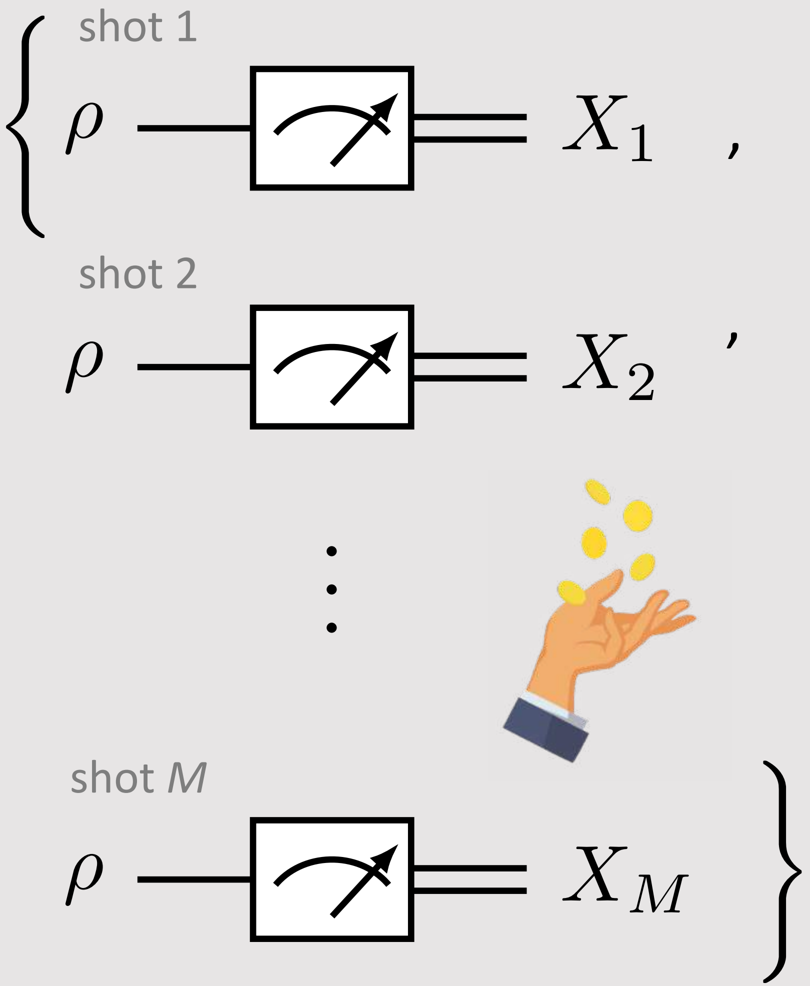
Ideal single qubit measurement with *M* shots



M shots with IID distribution

M outcomes: independent and identically distributed (iid) random variables

$$X_1, X_2, \ldots, X_M \in \Sigma$$

$$X_1, X_2, \ldots, X_M \sim \Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Empirical mean random variable

$$S = \frac{1}{M} \sum_{m=1}^{M} X_m$$

Find the expectation value and variance of the empirical mean

$$\mathrm{E}[X_m] = \mathrm{E}[X] = p \quad \forall m \in \{1, \ldots, M\}$$

$$\mathbb{V}[X_m] = \mathbb{V}[X] = p \quad \forall m \in \{1, \ldots, M\}$$

$$\mathrm{E}[aX_m + bX_n] = a\mathrm{E}[X_m] + b\mathrm{E}[X_n]$$

$$\forall m, n, \quad a, b \in \mathbb{C}$$

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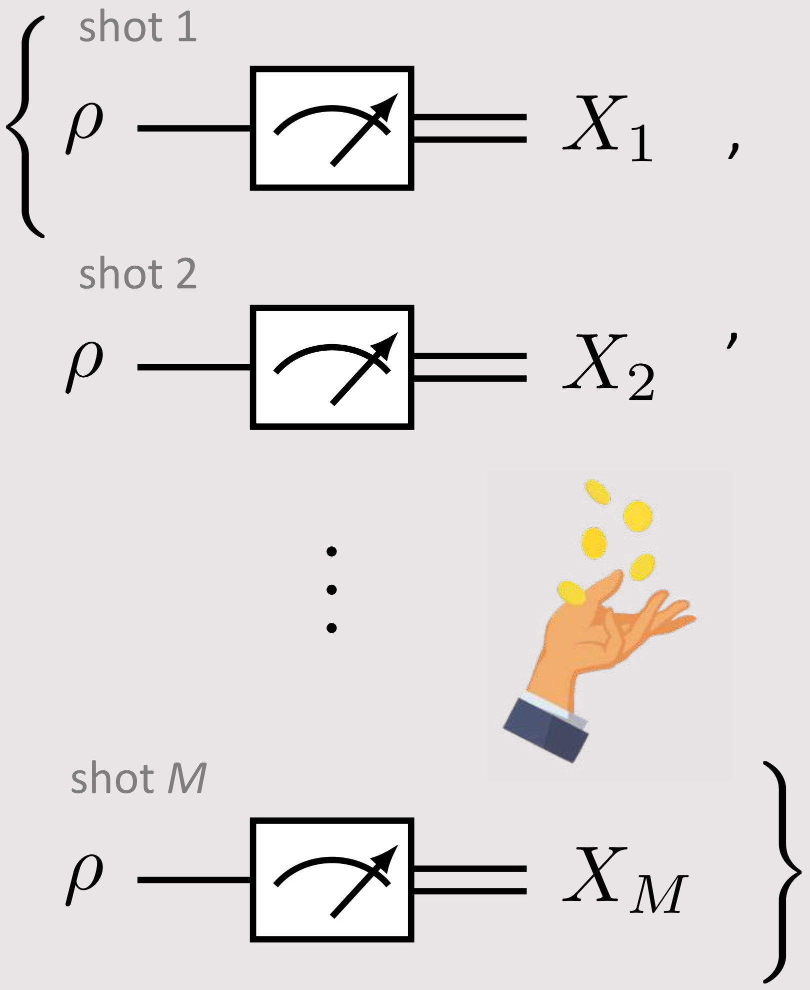
Empirical mean: an unbiased estimator



$$S = \frac{1}{M}\sum_{m=1}^{M} X_m$$

Find the expectation value and variance of the empirical mean

$$\mathrm{E}\left[S\right] = \mathrm{E}\left[\frac{1}{M}\sum_{m=1}^{M} X_m\right]$$

$$= \frac{1}{M}\sum_{m=1}^{M} \mathrm{E}\left[X\right]$$ linear functional

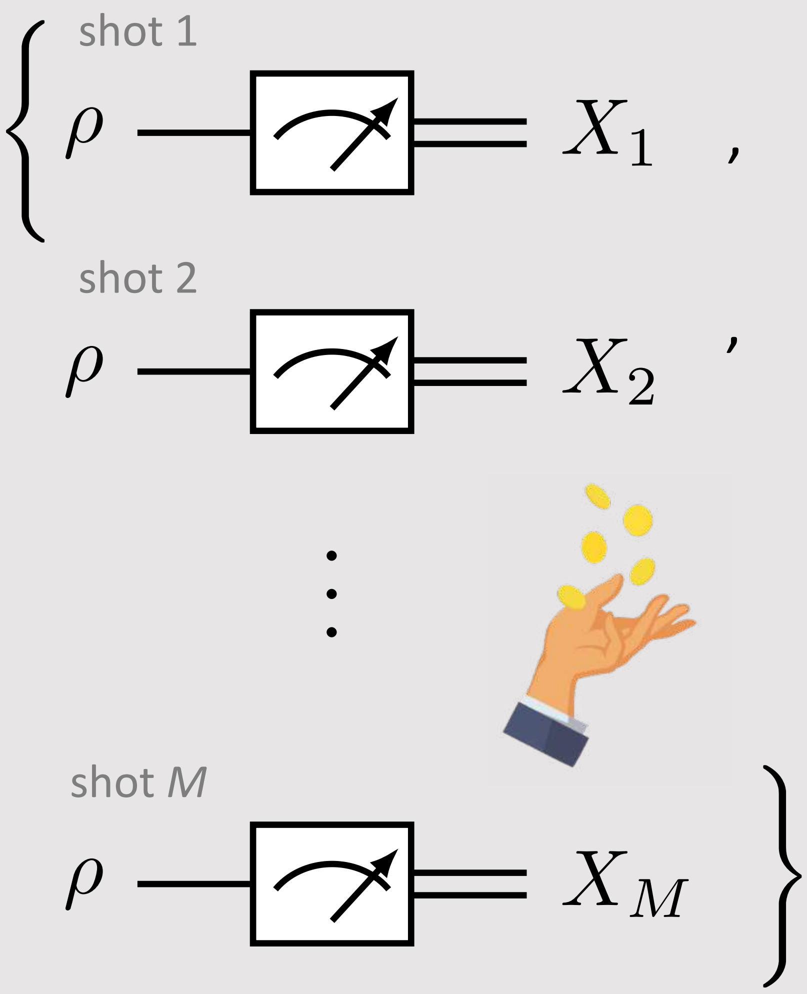
$$= \mathrm{E}\left[X\right]$$ expectation value of a single shot

$$\boxed{\begin{aligned} &= \left\langle \hat{M} \right\rangle \\ &= p \end{aligned}}$$

relation to quantum operator we derived earlier (unbiased estimator)

relation to probability to measure 1

How noisy is our estimate of the empirical mean?



$$S = \frac{1}{M}\sum_{m=1}^{M} X_m$$

$$\mathbb{V}\left[S\right] = \mathbb{V}\left[\frac{1}{M}\sum_{m=1}^{M} X_m\right]$$

$$= \frac{1}{M^2}\sum_{m=1}^{M}\mathbb{V}\left[X\right]$$

$$\boxed{\begin{aligned} &= \frac{1}{M}\mathbb{V}\left[X\right] \\ &= \frac{p\left(1-p\right)}{M}\end{aligned}}$$

$$\boxed{\begin{aligned}\sigma_S &= \sqrt{\mathbb{V}\left[S\right]} \\ &= \sqrt{\frac{\mathbb{V}\left[X\right]}{M}} \\ &= \sqrt{\frac{p\left(1-p\right)}{M}}\end{aligned}}$$

Find the expectation value and variance of the empirical mean

Use key identity for variance

$$\mathbb{V}\left[aX_m + bX_n\right] = a^2\mathbb{V}\left[X_m\right] + b^2\mathbb{V}\left[X_n\right]$$

(you can derive this from the definition)

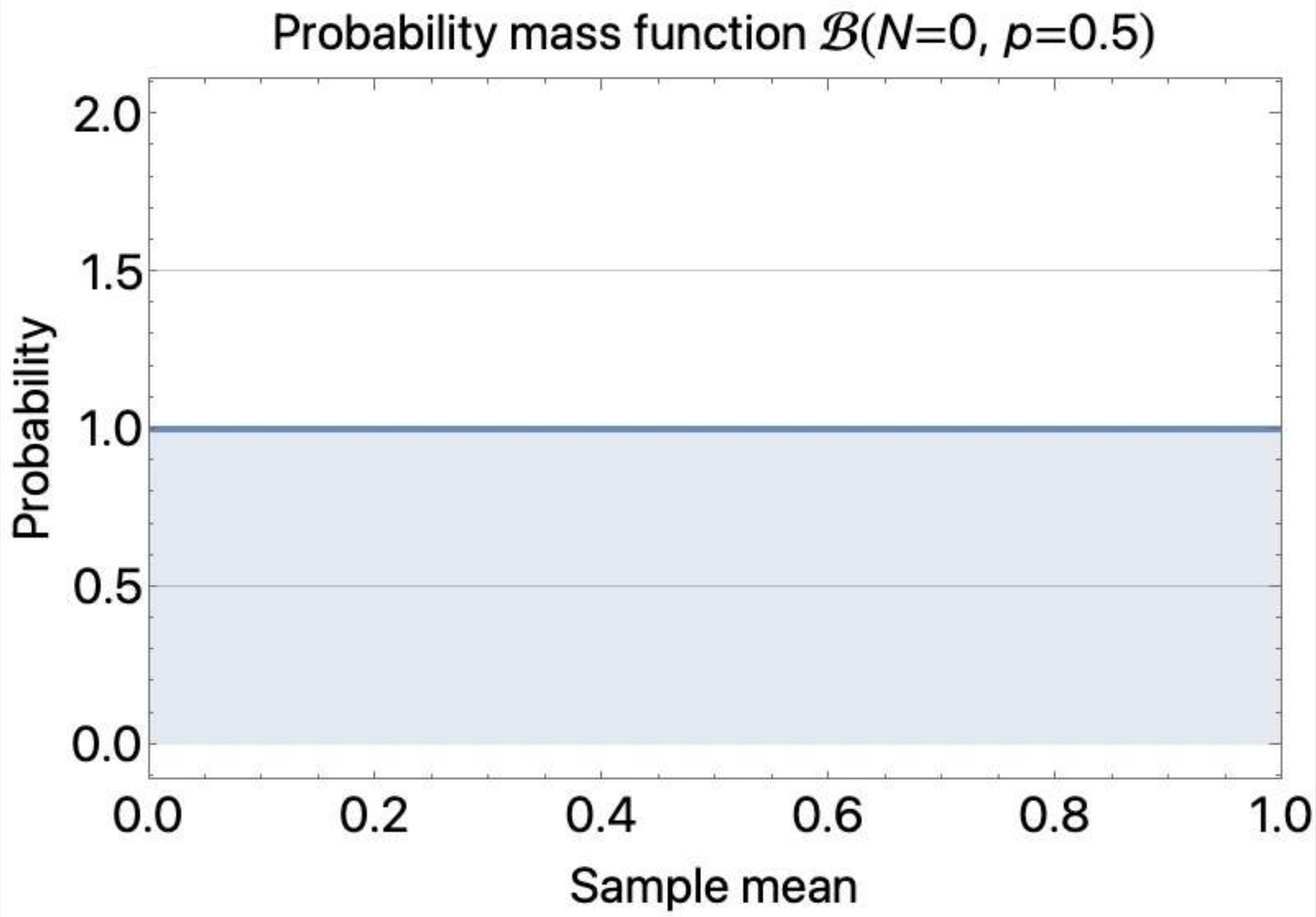
The variance is reduced by the number of sampler we take!

Thus we can suppress the quantum projection noise with enough shots.

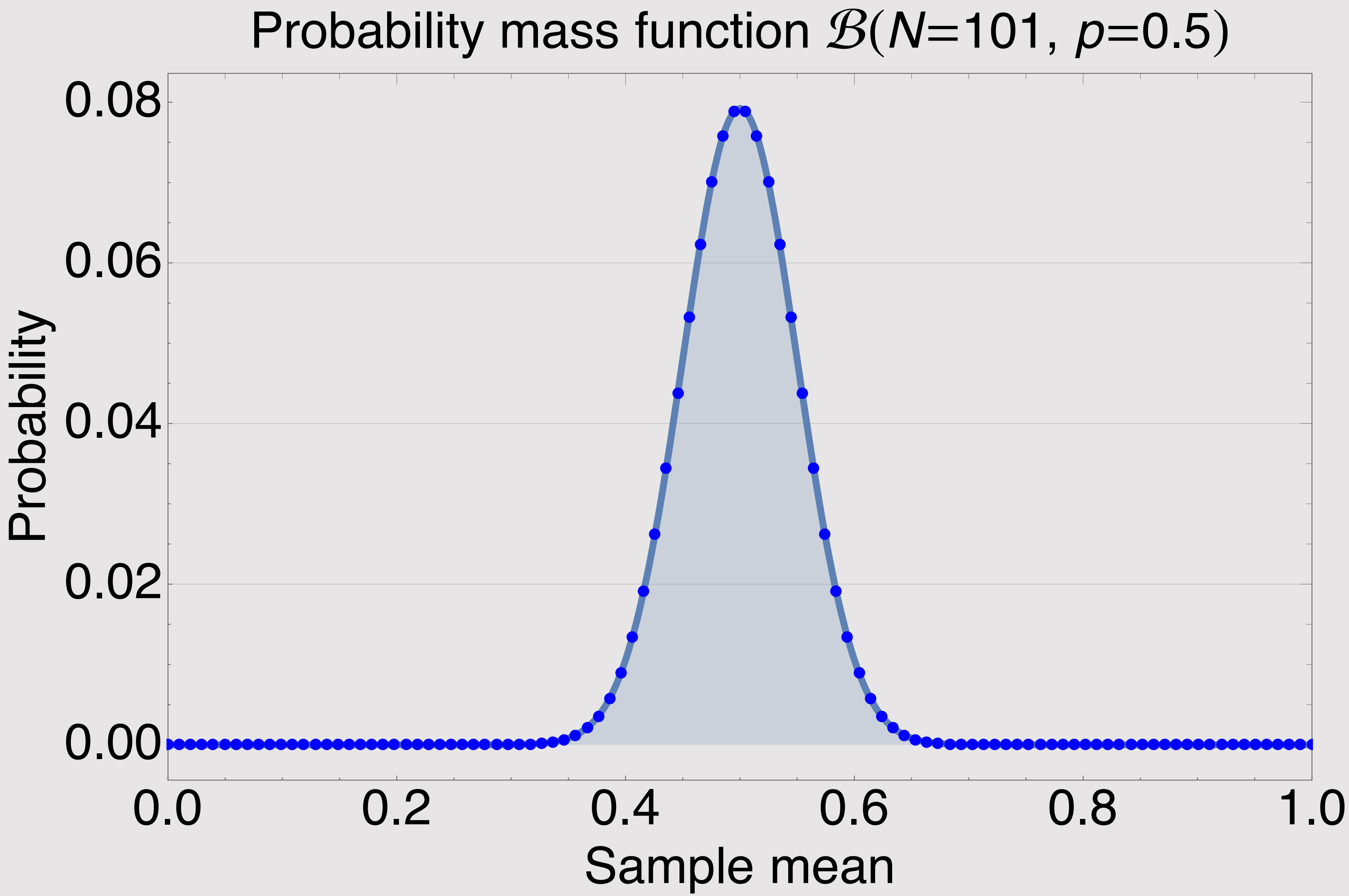
The standard deviation drop as one over square root of the number of shots

Hand Image by iconicbestiary on Freepik

Animation of convergence of shots expectation value and mean



Sampled output distribution



Concentration inequalities and tail bounds

Making a list,
checking it twice,
going to see
which inequality
is nice!

Markov? Hoeffding?
Jensen? Chebyshev?
Chernoff?

https://www.zlatko-minev.com/blog/inequalities

1. Probability (Technical note 11.9 v0.6)

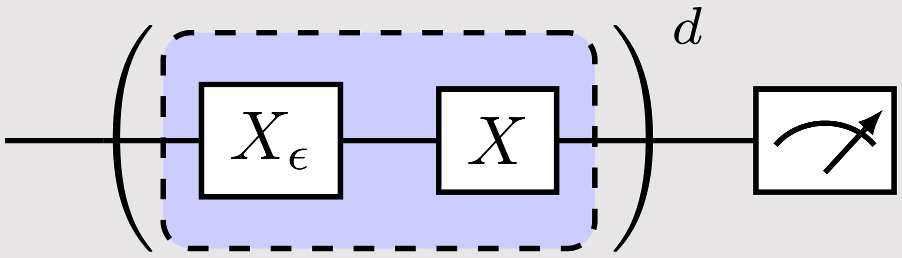
1A. Concentration inequalities and tail bounds

Unless otherwise specified, all variables are real $\mathbb{R}$. Inequalities come as one-sided $\Pr(\cdots \leq \cdots)$ and two-sided $\Pr(|\cdots| \leq \cdots)$. Notation: X is a random variable, $\mu := \mathbb{E}[X]$, $\sigma^2 := \mathrm{Var}[X]$, $S_n := X_1 + \cdots + X_n$.

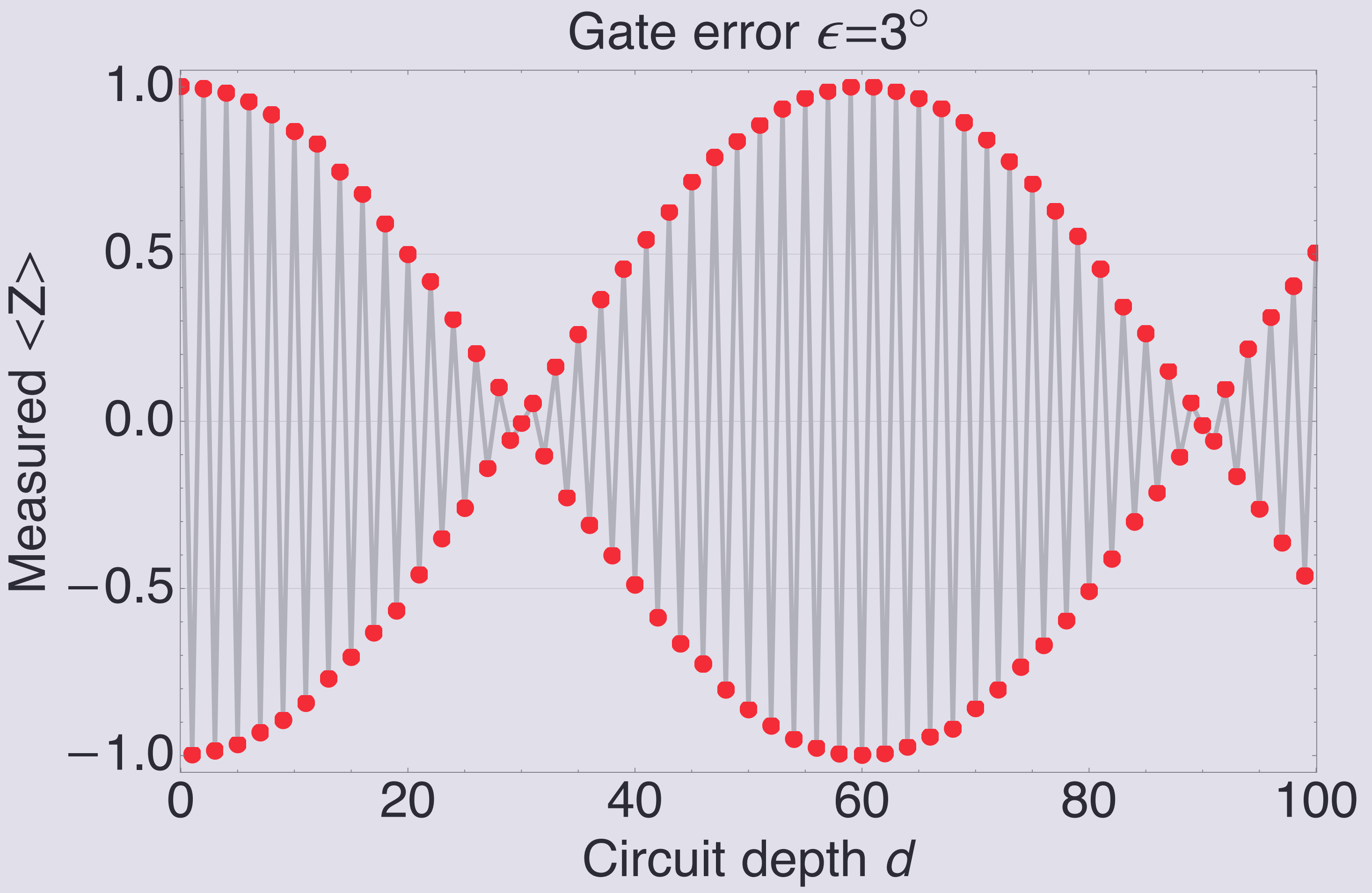
Inequality	Conditions		Common form	Notes / Alternate form	
			Single random variable		
Markov[1]	Non-negative	$X \geq 0$	$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$ $\forall a > 0$	$\Pr[X \geq kE[X]] \leq \frac{1}{k}$ $k > 1$	[3, Sec. 5.1][6, Thm 1.13]
extension	+ non-negative, strictly increasing func Φ	$X \geq 0$ $\Phi(X) \geq 0$ increasing	$\Pr[X \geq a] = \Pr[\Phi(X) \geq \Phi(a)] \leq \dfrac{\mathrm{E}(\Phi(X))}{\Phi(a)}$	$\forall a > 0$	Wiki
Reverse Markov	upper-bounded by U (can be positive)	$\max X = U$	$\Pr[X \leq a] \leq \frac{U - \mathbb{E}[X]}{U - a}$	$\forall a > 0$	[1, Sec. 3.1]
Chebyshev[2]	Finite mean and variance	$\mathbb{E}[X], \mathrm{Var}[X]$ finite	$\Pr[\lvert X - \mathbb{E}[X]\rvert \geq a] \leq \frac{\sigma^2}{a^2}$	$\Pr[\lvert X - \mathbb{E}[X]\rvert \geq a \cdot \sigma] \leq \frac{1}{a^2}$ $\forall a > 0,\ \sigma^2 = \mathrm{Var}[X]$	[1, Sec. 3.2] [3, Sec. 5.1][2, Thm 18.11]
Cantelli	Improved Chebyshev	(same; but one-sided)	$\Pr[X - \mathbb{E}[X] \geq a]) \leq \dfrac{\sigma^2}{\sigma^2 + a^2}$	$\forall a > 0,\ \sigma^2 = \mathrm{Var}[X]$	Wiki
Chernoff[3]	Generic		$\Pr[X \geq a] = \Pr\left[e^{tX} \geq e^{ta}\right]$	$\forall t > 0, \quad a \in \mathbb{R}$	[1, Sec. 3.3]
Jensen		$f: \mathbb{R} \to \mathbb{R}$; f convex	$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$		[3, Prob. 5.3][6, Thm 1.14]
Hoeffding's lemma		$\mathbb{E}[X] = \mu$ $a \leq X \leq b$	$\mathbb{E}\left[e^{\lambda X}\right] \leq e^{\lambda\mu} e^{\frac{\lambda^2 (b-a)^2}{8}}$	$\lambda \in \mathbb{R}$	[1, Sec. 3.4]
			Sum of random variables		
Chernoff-Hoeffding (one-sided)	n independent random vars	$X_1, \ldots, X_n$ indep $S_n = X_1 + \cdots + X_n$ $X_i \in [a_i, b_i]$ $\forall i$	$\Pr[S_n - \mathbb{E}[S_n] \geq t] \leq \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$		[1, Sec. 3.5]
(two-sided)[4]	(same as above)		$\Pr[\lvert S_n - \mathbb{E}[S_n]\rvert > t] \leq 2\exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$	$\forall t \in (0, \frac{1}{2})$	[5, Thm.1.1]
(two-sided iid)	same plus iid, range, mean μ for each	$X_1, \ldots, X_n \in [0, 1]$ $\mathbb{E}[X_i] = \mu$ iid	$\Pr\left[\left\lvert \frac{S_n}{n} - \mu \right\rvert \geq \epsilon\right] \leq 2\exp(-2n\epsilon^2)$	$\forall \epsilon > 0$	[6, Thm 1.16]
Thm 1.3	n independent random vars	$X_1, \ldots, X_n$ indep $S_n = X_1 + \cdots + X_n$	$\Pr[S_n - \mathbb{E}[S_n] > \epsilon] \leq 2\exp\left(\frac{-\epsilon^2}{4\sum_{i=1}^n \mathrm{Var}[X_i]}\right)$	$\epsilon \in (0, 2\,\mathrm{Var}[S_n] / (\max_i \lvert X_i - \mathbb{E}[X_i]\rvert))$	[5, Thm. 1.3]
Azuma					
Weak law of large numbers	n independent iid random vars	$X_1, \ldots, X_n$ indep $\mathbb{E}[X_i] = \mu$ iid	$\lim_{n\to\infty} \Pr\left[\lvert \frac{1}{n} S_n - \mu \rvert \geq \epsilon\right] = 0$	$\forall \epsilon > 0$	[3, Sec. 5.2][6, Thm 1.15]
Strong law of large numbers	(same)	(same)	$\Pr\left[\lim_{n\to\infty} \frac{1}{n} S_n = \mu\right] = 1$		[3, Sec. 5.5]
	Advanced				
Bennett	n independent zero-mean	$X_1, \ldots, X_n$ indep $\mathbb{E}[X_i] = 0$ iid	$\Pr[S_n > \epsilon] \leq \exp\left(-n\sigma^2 h\left(\frac{\epsilon}{n\sigma^2}\right)\right)$	$\sigma^2 := \frac{1}{n}\sum_{i=1}^n \mathrm{Var}[X_i]$, $\forall \epsilon > 0$, $h(a) := (1+a)\log(1+a) - a$ for $a \geq 0$	[1, 4.1]
Bernstein	(same)	(same)	$\Pr[S_n > \epsilon] \leq \exp\left(\frac{-n\epsilon^2}{2(\sigma^2 + \epsilon/3)}\right)$	(same)	[1, 4.2]
Efron-Stein	scalar func of vars $f: \chi^n \to \mathbb{R}$	$X_1, \ldots, X_n$ indep w/ values in set χ	$\mathrm{Var}[Z] \leq \sum_{i=1}^n \mathbb{E}\left[(Z - \mathbb{E}_i[Z])^2\right]$	$Z := g(X_1, \ldots, X_n)$ $\mathbb{E}_i[Z] := \mathbb{E}[Z \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_n]$	[1, 4.3]
McDiarmid's	scalar func of vars $f: \chi^n \to \mathbb{R}$	$X_1, \ldots, X_n$ indep w/ values in set χ	$\Pr[f(X_1, \ldots, X_n) - \mathbb{E}[f(X_1, \ldots, X_n)] \geq \epsilon] \leq \exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^n c_i^2)}\right)$	condition: c-bounded difference property $\forall \epsilon > 0$ $\lvert f(X_1, \ldots, X_i, \ldots, X_n) - f(X_1, \ldots, X_i', \ldots, X_n)\rvert \leq c_i$	[1, 4.4]

[1] Markov's inequality bounds the first moment of random variable. Use it when a constant probability bound is sufficient [1, Sec. 3.3].
[2] Chebyshev is derived from Markov. It bounds the second moment. It is the appropriate one when the variance σ is known. If σ is unknown, we can use the bounds of $X \in [a, b]$.
[3] Chernoff bound is used to bound the tails of the distribution for a sum of independent random variables. By far the most useful tool in randomized algorithms [1, Sec. 3.3].
[4] This probability can be interpreted as the level of significance ϵ (probability of making an error) for a confidence interval around the mean of size 2ϵ. Therefore, we require at least $\log(2\alpha)/2t^2$ samples to acquire $1 - \alpha$ confidence interval $\mathbb{E}[X] \pm t$.

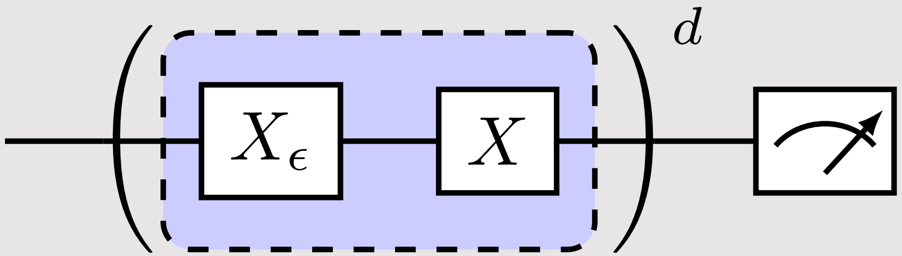
Recall gate error result



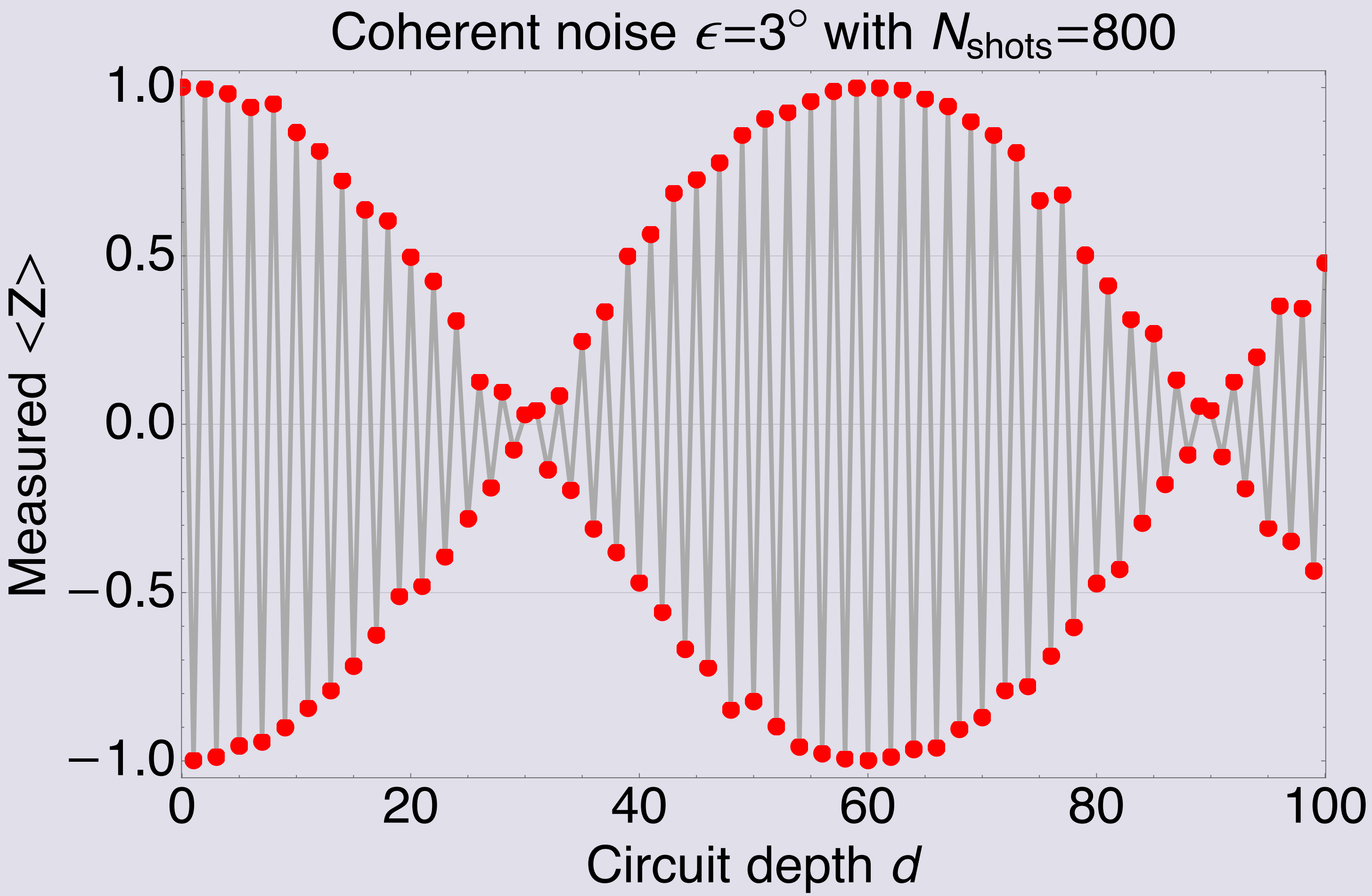
$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



Projection & sampling noise



$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



Questions

Answer these multiple-choice questions in the chat; for example, type “1a 2b.”

1. Projection noise is due to
   a) measurement apparatus that could be made more efficient
   b) classical limitations
   c) core nature of quantum physics

2. To reduce projection noise
   a) increase the number of sample
   b) you cannot undo it
   c) apply readout error mitigation

Dive deeper? Try the following

1. Calculate the following for a qubit
    1. The expectation value of the sample variance for N shots of the observable |1><1|.

       where the sample mean is defined as $$S = \frac{1}{N}\sum_{n=1}^{N} M_n$$

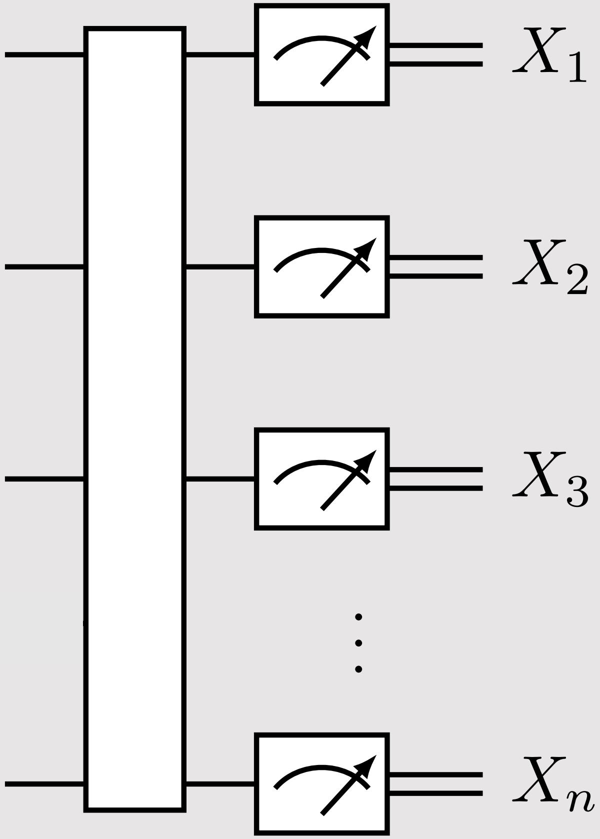
       and the sample variance is defined as $$V = \frac{1}{N}\sum_{n=1}^{N} (M_n - S)^2$$
    2. Is the estimate biased?
    3. What is the variance of V?
    4. Can you find an expression for an unbiased estimate of the variance of *S*?

2. What about two qubits?
    1. Can you find the projection operators for the observable ZZ?
    2. Find the probability distribution for the observables ZI, IZ, and ZZ for a general state.

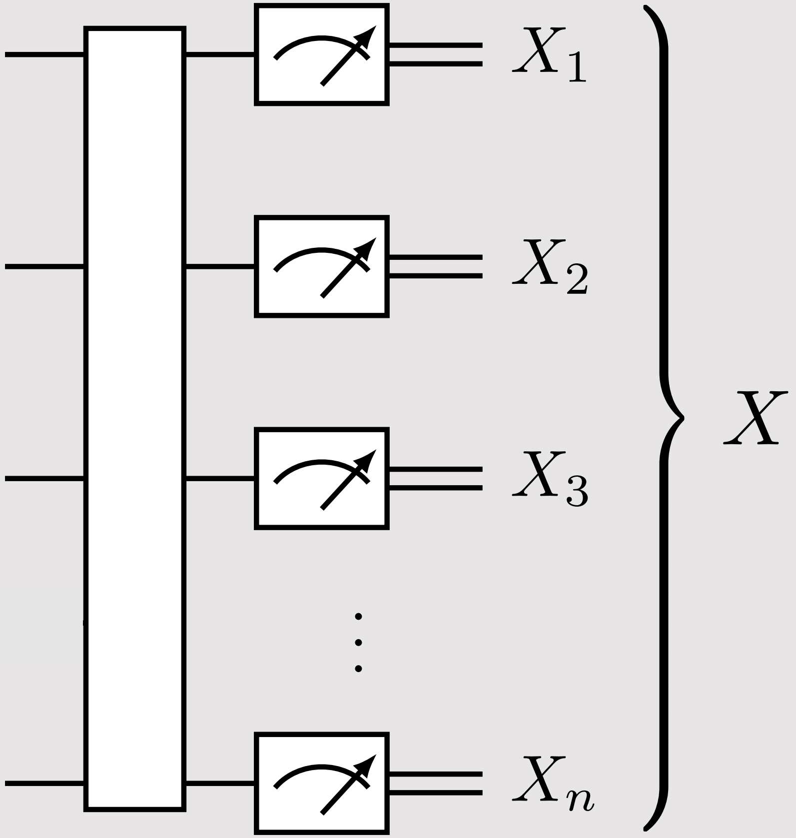
3. If you take 10 shots and find all 10 outcomes to be 1, what is the probability the qubit is in the |0> state? (hint: it's not zero!)

n qubits: preliminaries

meter



n qubits: binary strings



binary string | bits of string

$$X = X_1 X_2 \cdots X_n$$ (one convention)

$$X = X_n X_{n-1} \cdots X_1$$ (Qiskit convention)

Example:

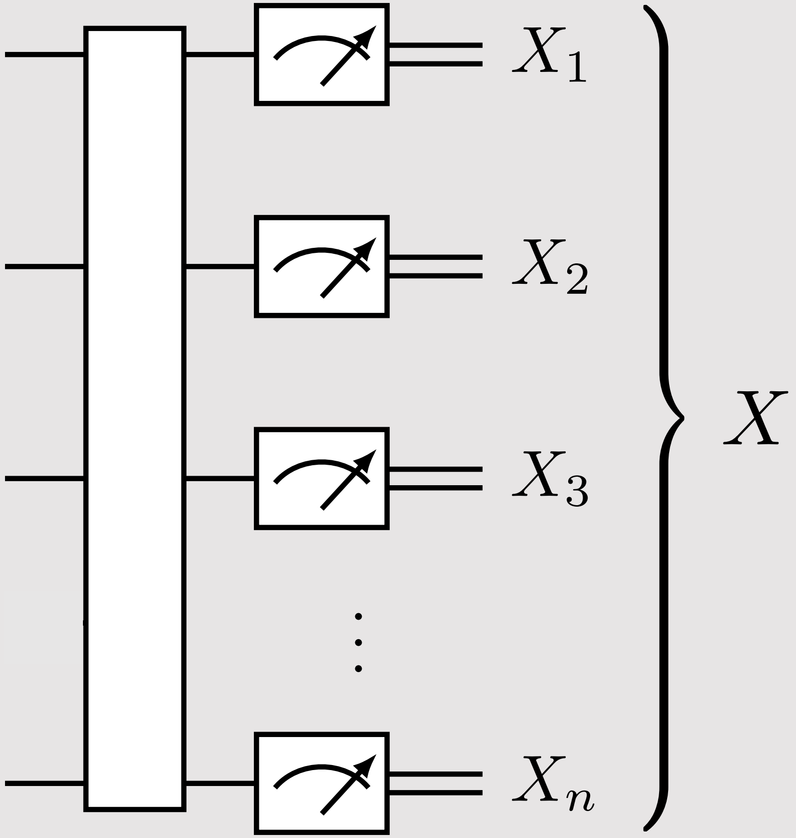
$$00 \cdots 0$$
$$00 \cdots 1$$
$$11 \cdots 0$$
$$11 \cdots 1$$

What do the variables belong to?

$$X_i \in \{0, 1\} \qquad \text{for } i = 1, \ldots, n$$

$$X \in \Sigma = \{0, 1\}^n$$

Binary strings: integer representation



binary string — bits

$$X = X_1 X_2 \cdots X_n$$

integer representation

$$X_{\text{int}} = 0, 1, \ldots, 2^n - 1$$

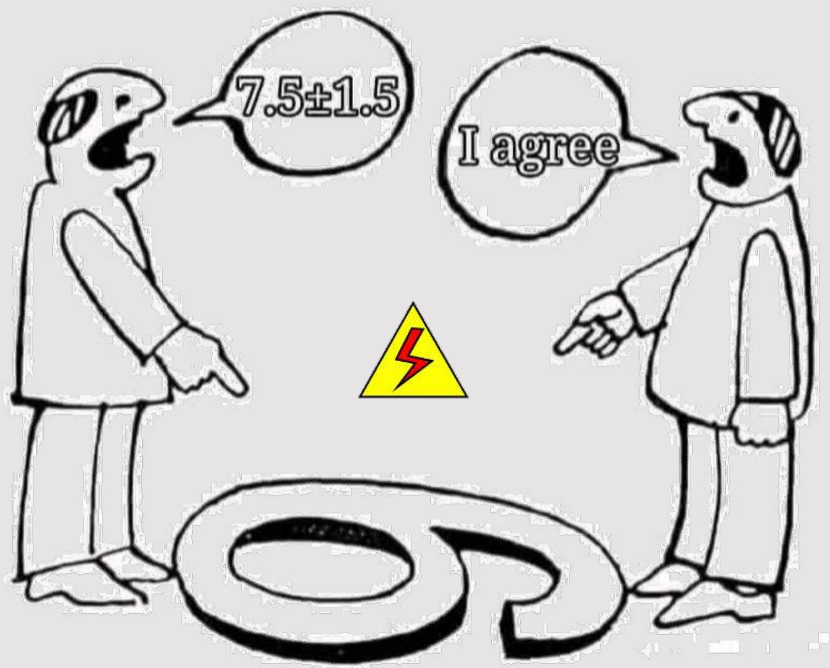
little-endian *binary expansion*

$$X_{\text{int}} = \sum_{j=1}^{n} X_j 2^{j-1}$$

note, this is one convention

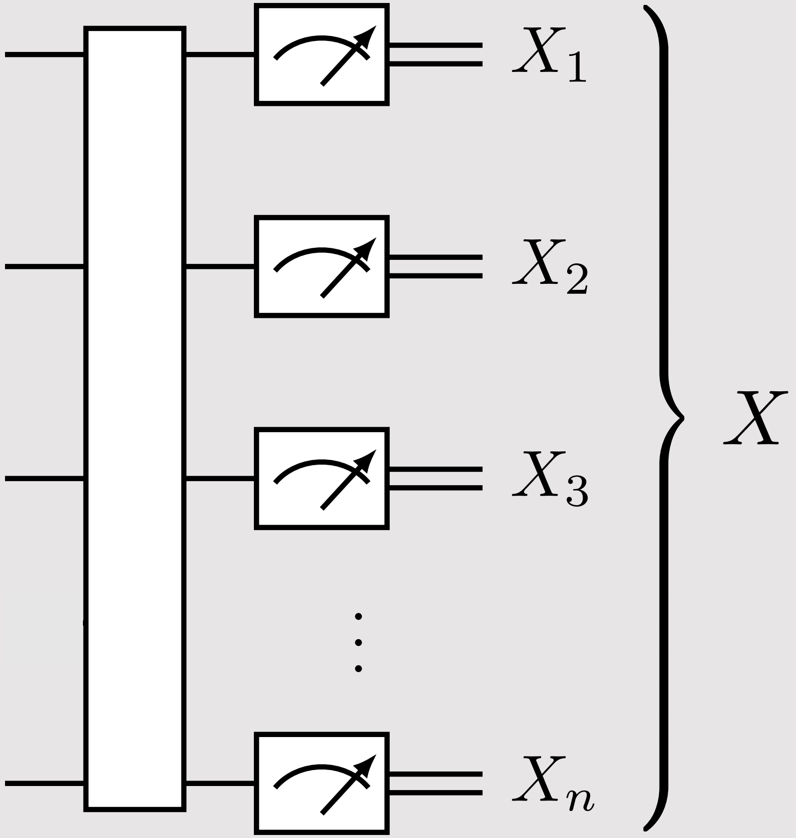
Increasing numeric significance with increasing memory addresses (index number j) is known as little-endian.
See also: bijective numeration.

Beware tool's conventions



Internet meme

Binary and Pauli strings



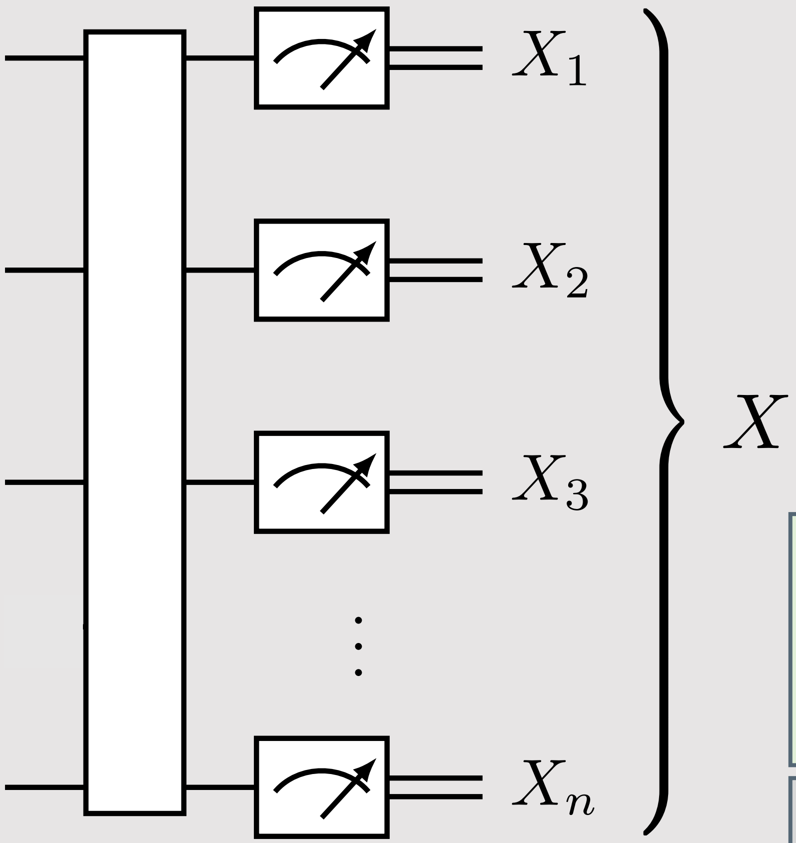
For 1 qubit

$$\langle I \rangle = p(0) + p(1)$$
$$\langle Z \rangle = p(0) - p(1)$$

For 2 qubits

$$\langle II \rangle = p(00) + p(01) + p(10) + p(11)$$
$$\langle IZ \rangle = p(00) - p(01) + p(10) - p(11)$$
$$\langle ZI \rangle = p(00) + p(01) - p(10) - p(11)$$
$$\langle ZZ \rangle = p(00) - p(01) - p(10) + p(11)$$

How to go between bitstrings and Pauli expectation values?



For 2 qubits

$$\langle II\rangle = p(00) + p(01) + p(10) + p(11)$$
$$\langle IZ\rangle = p(00) - p(01) + p(10) - p(11)$$
$$\langle ZI\rangle = p(00) + p(01) - p(10) - p(11)$$
$$\langle ZZ\rangle = p(00) - p(01) - p(10) + p(11)$$

In vector form

$$\mathbf{z} = \begin{pmatrix} \langle II\rangle \\ \langle IZ\rangle \\ \langle ZI\rangle \\ \langle ZZ\rangle \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} p(00) \\ p(01) \\ p(10) \\ p(11) \end{pmatrix}$$

Pauli expectation values from binary string probabilities

$$\mathbf{z} = \mathbf{W}\mathbf{p} \qquad \mathbf{p} = \frac{1}{2^n}\mathbf{W}\mathbf{z}$$

Walsh-Hadamard transform

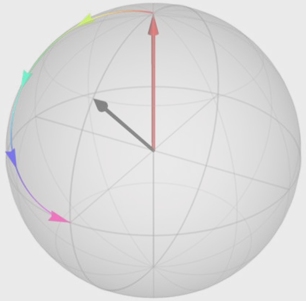
inner product over Z_2^n

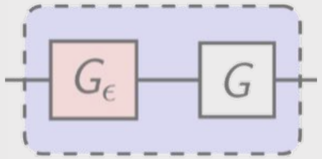
$$W_{ax} = (-1)^{\langle x,a\rangle} \qquad x \cdot a = \langle x, a\rangle = \sum_{i=1}^{n} x_i a_i$$

For deeper dive, see zlatko-minev.com/blog/t21-bit-strings

Chapter 4

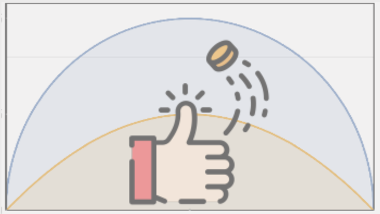
Coherent noise



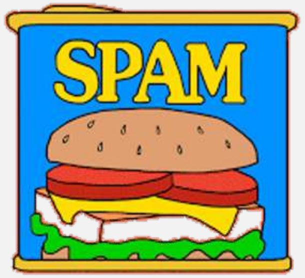


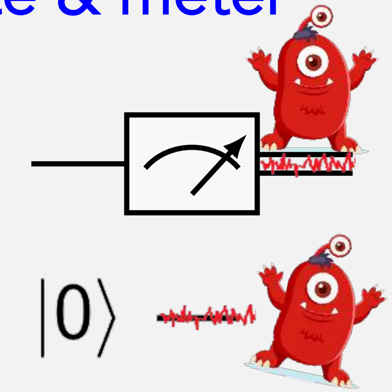
Measurements in quantum

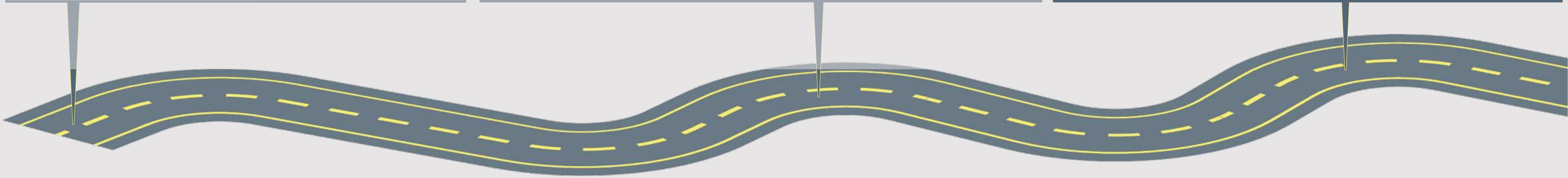
Projection noise



SPAM: Sate & meter

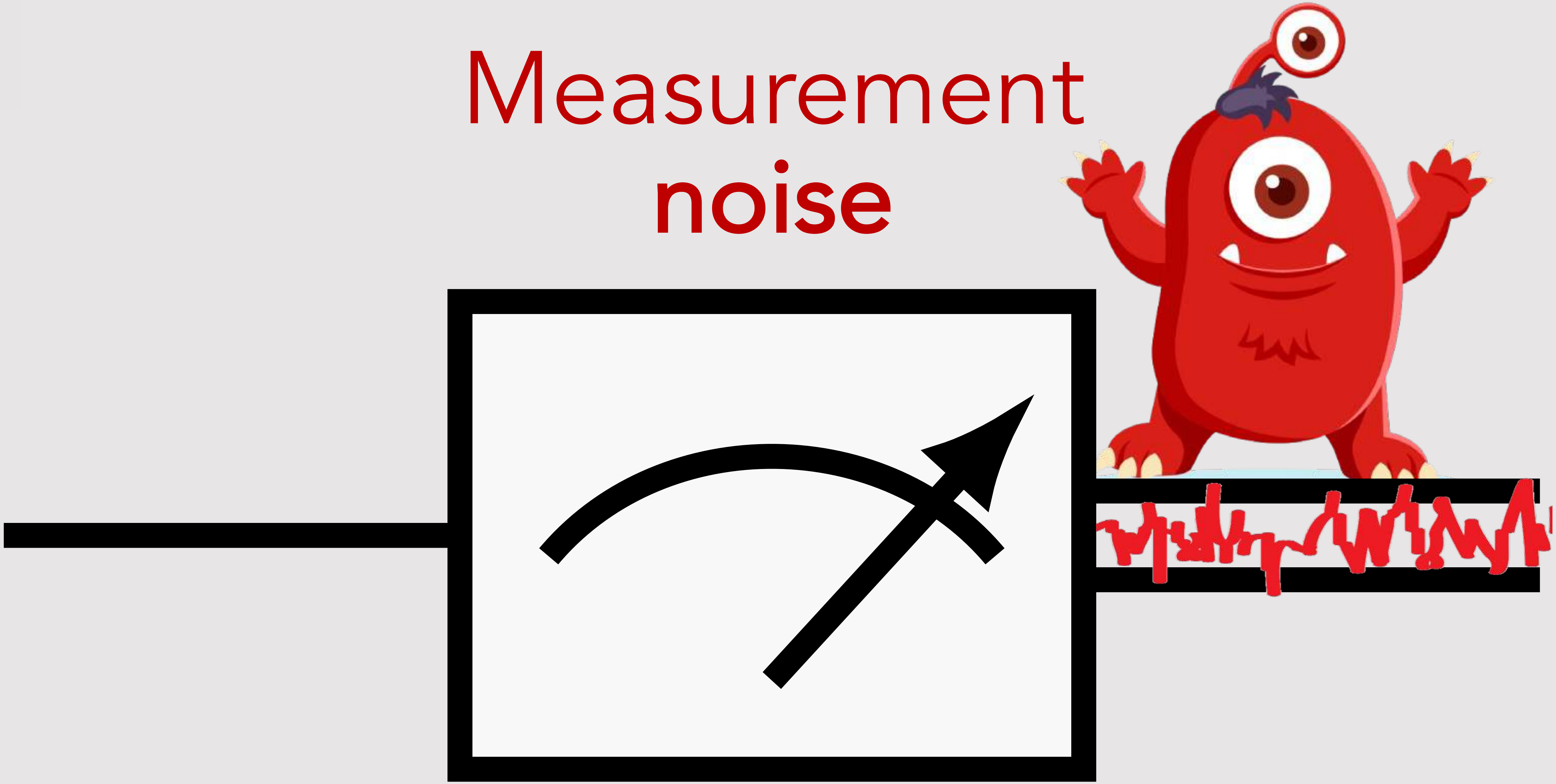






coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

Measurement **noise**



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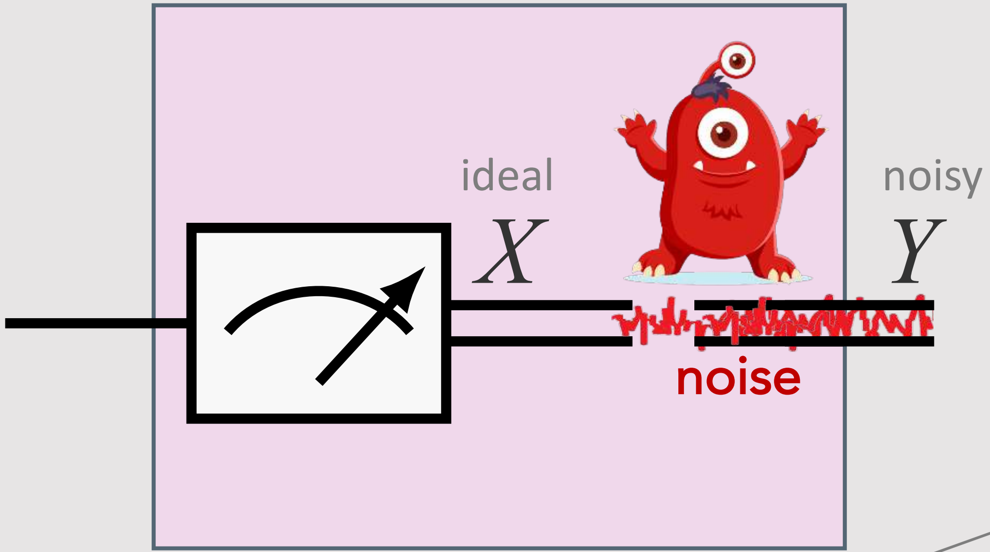
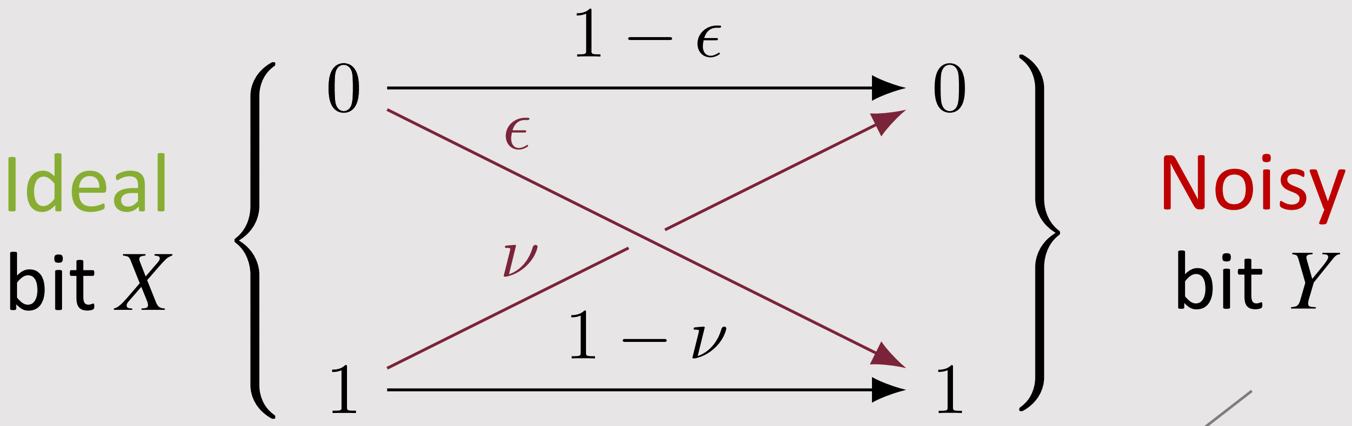
Toy model of measurement noise





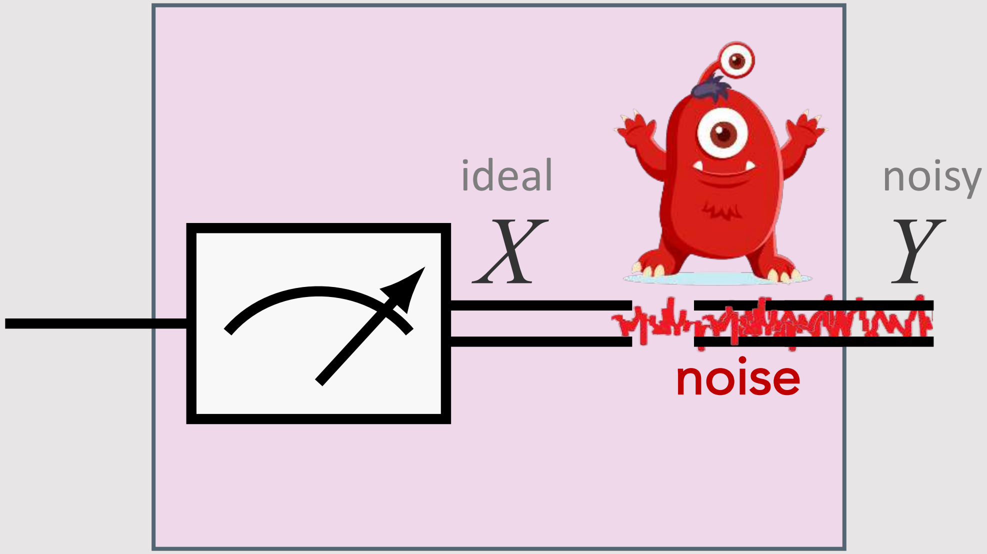
$$\mathbf{p}_{\mathrm{ideal}} = \begin{pmatrix} \mathbb{P}(X=0) \\ \mathbb{P}(X=1) \end{pmatrix}$$

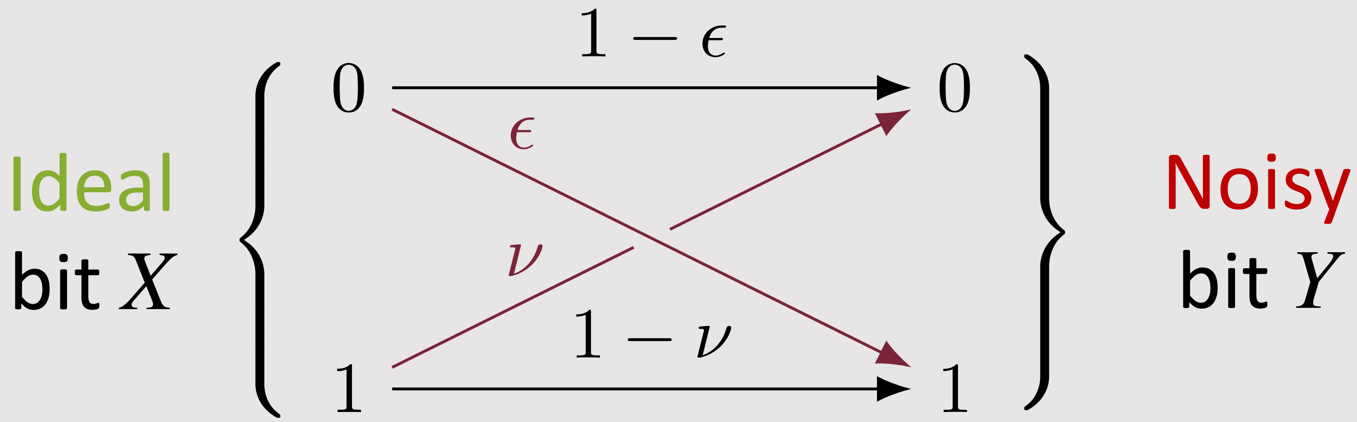
$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$

$$\mathbf{p}_{\mathrm{noisy}} = \begin{pmatrix} \mathbb{P}(Y=0) \\ \mathbb{P}(Y=1) \end{pmatrix}$$

* Monster image by jcomp on Freepik

The *A matrix* of the noisy meter



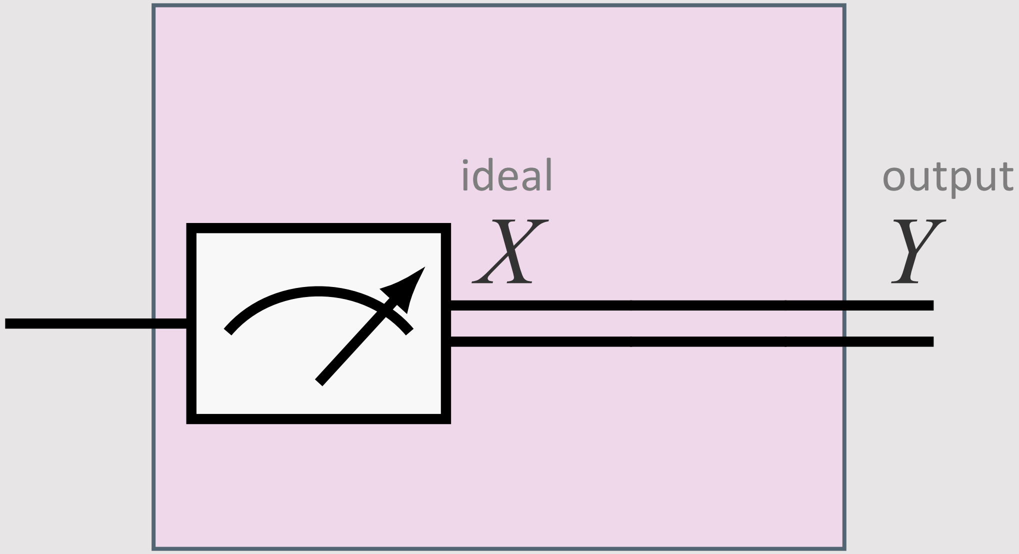


$$\mathbf{p}_{\text{noisy}} = \mathbf{A}\mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{array}{c} \\ Y=0 \\ Y=1 \end{array} \begin{array}{c} \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$

* Monster image by jcomp on Freepik

Ideal case



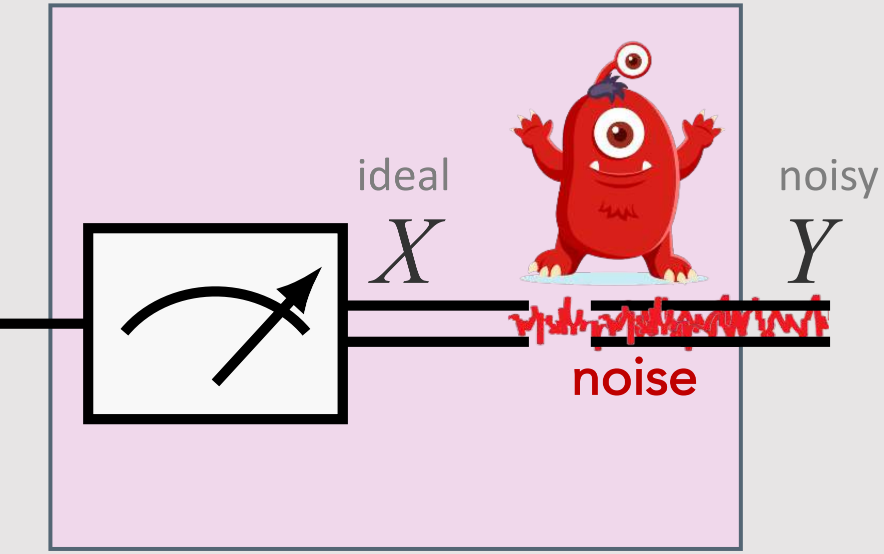
$$\mathbf{A}_{\text{ideal}} = \begin{array}{c c} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{p}_{\text{noisy}} = \mathbf{A}\mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$

* Monster image by jcomp on Freepik

The *A matrix* of the noisy meter



$$\mathbf{p}_{\text{noisy}} = \mathbf{A}\mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$

What are the properties of A you can observe?

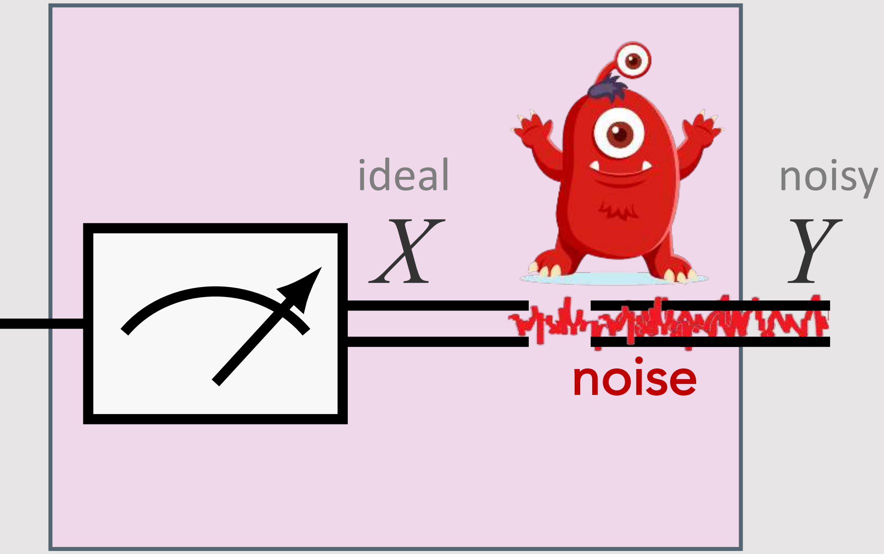
Properties

- The element $A_{y,x} \in [0,1]$ with indices $x, y \in \{0,1\}$ is a probability vector element. (i.e., A is a stochastic matrix)
- $\sum_{y\in\Sigma} A_{y,x} = 1 \ \forall x \in \{0,1\}$ is the unit normalization of the probability vector (columns of the A matrix).

What is the probability of Y?

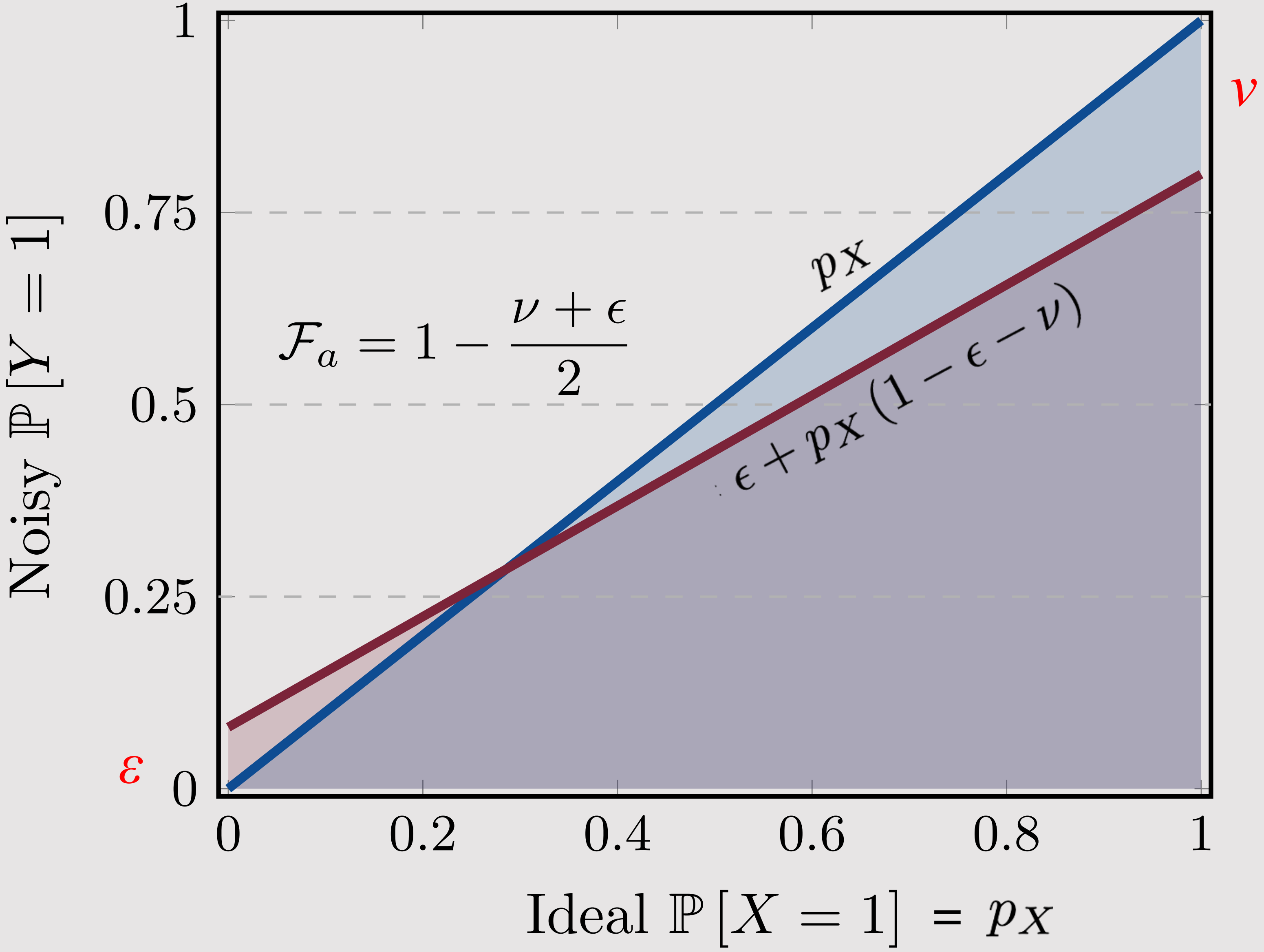
$$\begin{aligned} p_X &:= \mathbb{P}\left[X=1\right] \ , \\ p_Y &:= \mathbb{P}\left[Y=1\right] \ , \\ \mathbb{P}\left[Y=1\right] &= \mathbb{P}\left[Y=1|X=0\right]\mathbb{P}\left[X=0\right] + \mathbb{P}\left[Y=1|X=1\right]\mathbb{P}\left[X=1\right] \\ &= \epsilon\left(1-p_X\right) + \left(1-\nu\right)p_X \\ &= \epsilon + p_X\left(1-\epsilon-\nu\right) \end{aligned}$$

The effect of readout error on the output probability



$$\mathbf{p}_{\text{noisy}} = \mathbf{A}\mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$



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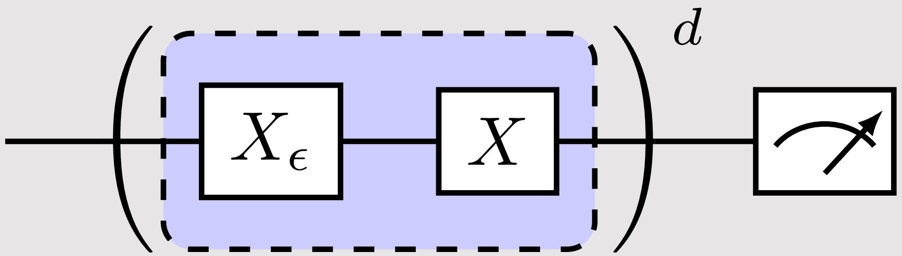
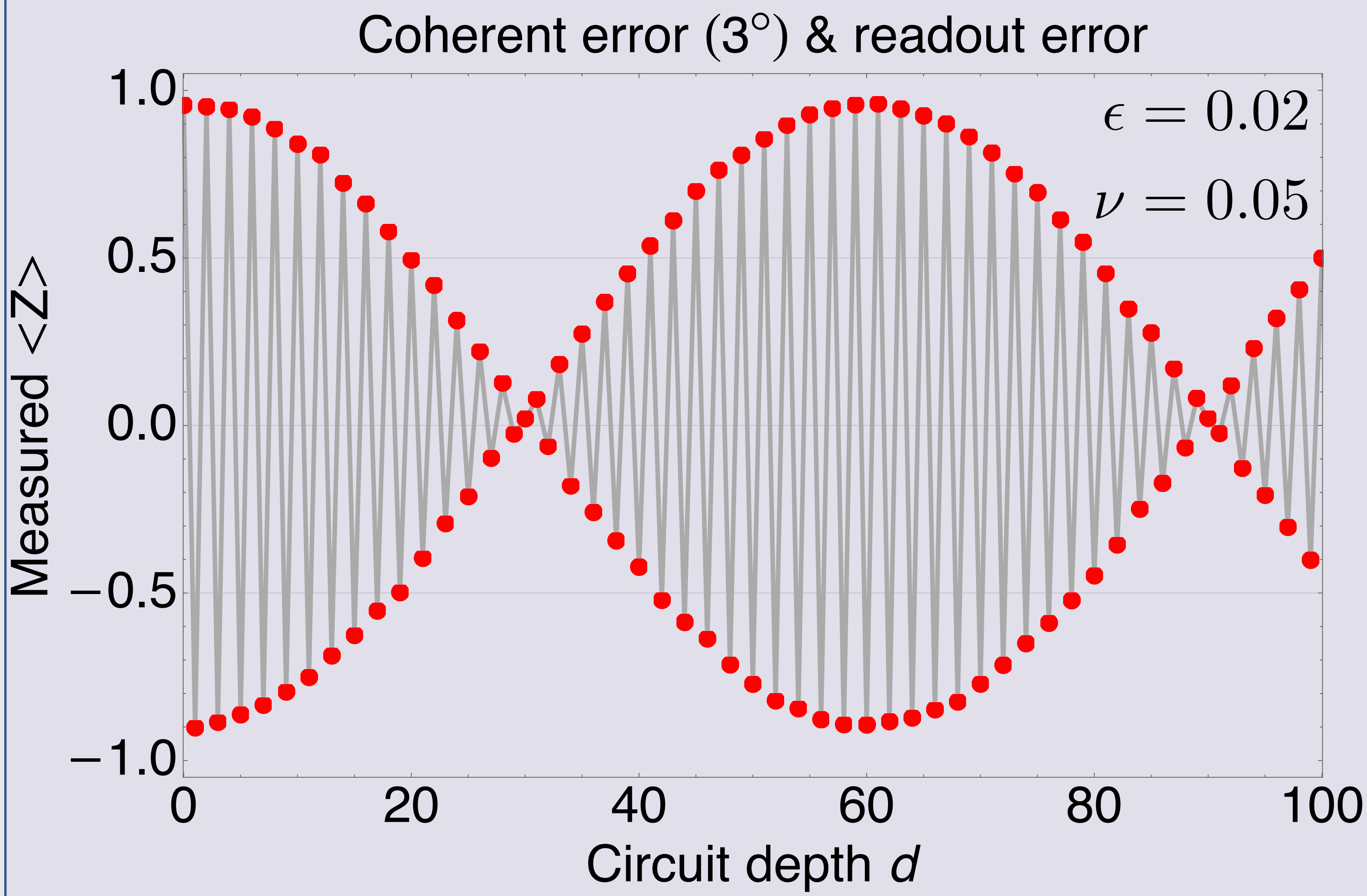
Projection & sampling noise



$$\mathbf{p}_{\text{noisy}} = \mathbf{A}\mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} X=0 & X=1 \end{array} \\ \begin{array}{c} Y=0 \\ Y=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$



Questions

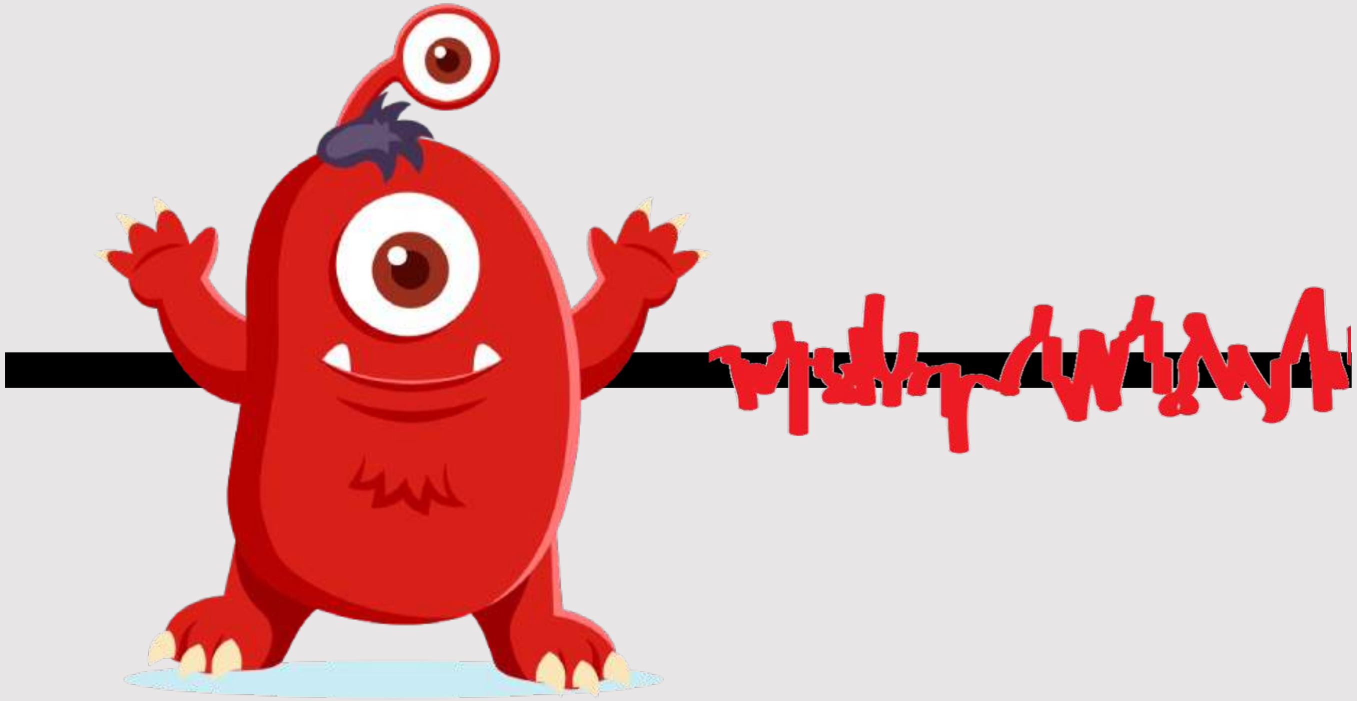
Answer these multiple-choice questions in the chat; for example, type “1a 2b.”

1. Readout error is due to
   a) measurement apparatus that could be made more efficient
   b) classical limitations
   c) core nature of quantum physics

2. To reduce readout error bias
   a) increase the number of sample
   b) you cannot undo it
   c) apply readout error mitigation

State preparation **noise**

$|0\rangle$



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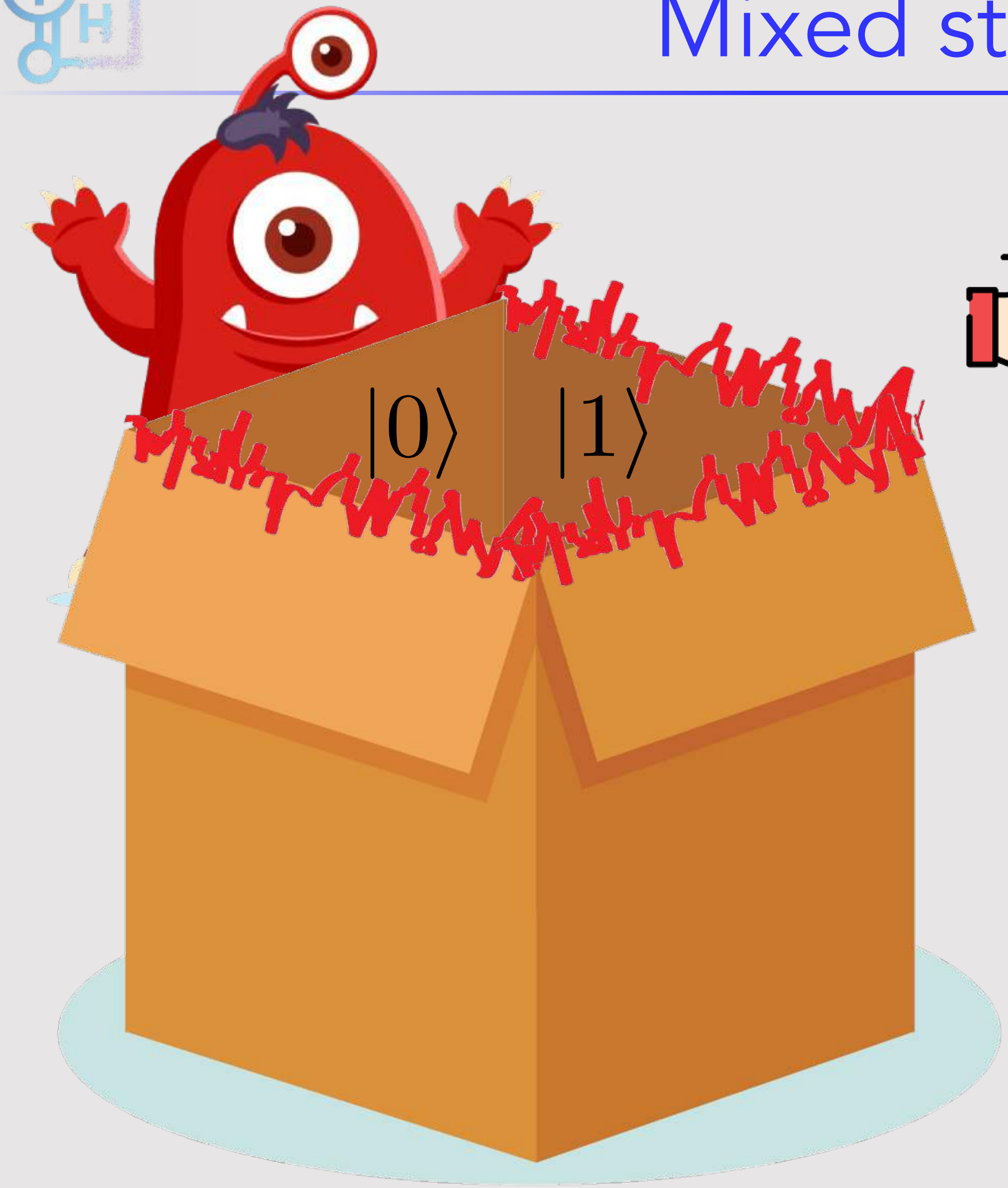
Mixed state (density matrix)



Probability 1-p $\quad|0\rangle$

Probability p $\quad|1\rangle$

$$\rho = \mathbb{P}\left[|0\rangle\right]|0\rangle\langle 0| + \mathbb{P}\left[|1\rangle\right]|1\rangle\langle 1|$$
$$= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$

Mixed state (density matrix)

Recall

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$ zero state ket

$$\langle 0| = |0\rangle^\dagger = \begin{pmatrix} 1 & 0 \end{pmatrix},$$ zero state bra

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$ one state ket

$$\langle 1| = |1\rangle^\dagger = \begin{pmatrix} 0 & 1 \end{pmatrix},$$ one state bra

Dyad

$$|a\rangle\langle b| = |a\rangle \otimes \langle b|$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Probability 1-p $\quad |0\rangle$

Probability p $\quad |1\rangle$

Equivalent to randomly applying X gate

$$\rho = \mathbb{P}\left[|0\rangle\right]|0\rangle\langle 0| + \mathbb{P}\left[|1\rangle\right]|1\rangle\langle 1|$$
$$= (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$$

$$= \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & 1-p & 0 \\ |1\rangle & 0 & p \end{array}$$

box; freepik. person: facebook

Mixed state expectation values

Expectation values

$$\left\langle \hat{A} \right\rangle_{\text{pure}} = \left\langle \psi \middle| \hat{A} \middle| \psi \right\rangle$$

$$\left\langle \hat{A} \right\rangle_{\text{mixed}} = \sum_{\psi} \mathbb{P}\left[|\psi\rangle\right] \left\langle \psi \middle| \hat{A} \middle| \psi \right\rangle$$

$$= (1-p)\left\langle 0 \middle| \hat{A} \middle| 0 \right\rangle + p \left\langle 1 \middle| \hat{A} \middle| 1 \right\rangle$$

$$= \mathrm{Tr}\left[\rho \hat{A}\right]$$

Trace of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathrm{Tr}\left[A\right] = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

Probability 1-p $\quad |0\rangle$

Probability p $\quad |1\rangle$

$$\rho = \mathbb{P}\left[|0\rangle\right] |0\rangle\langle 0| + \mathbb{P}\left[|1\rangle\right] |1\rangle\langle 1|$$

$$= (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$$

$$= \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \end{array}$$

Mixed state Bloch vector

Expectation value of Z

$$\left\langle \hat{Z} \right\rangle = \mathrm{Tr}\left[\hat{Z}\rho\right]$$
$$= (1-p)\left\langle 0\middle|\hat{Z}\middle|0\right\rangle + p\left\langle 1\middle|\hat{Z}\middle|1\right\rangle$$
$$= (1-p)(1) + (p)(-1)$$
$$= 1 - 2p$$

Trace of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathrm{Tr}\left[A\right] = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

Probability 1-*p* $|0\rangle$

Probability *p* $|1\rangle$

$$\rho = \mathbb{P}\left[|0\rangle\right]|0\rangle\langle 0| + \mathbb{P}\left[|1\rangle\right]|1\rangle\langle 1|$$
$$= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$

$$= \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \end{array}$$

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Mixed state Bloch vector

Expectation value of Z

$$\left\langle \hat{Z} \right\rangle = \mathrm{Tr}\left[\hat{Z}\rho\right]$$
$$= (1-p)\left\langle 0 \middle| \hat{Z} \middle| 0 \right\rangle + p\left\langle 1 \middle| \hat{Z} \middle| 1 \right\rangle$$
$$= (1-p)(1) + (p)(-1)$$
$$= 1 - 2p$$

Show that

$$\left\langle \hat{X} \right\rangle = 0$$
$$\left\langle \hat{Y} \right\rangle = 0$$
$$\left\langle \hat{I} \right\rangle = 1$$

and that generally for a qubit

$$\mathrm{Tr}\left[\rho\right] = 1$$

$$1/2^n \leq \mathrm{Tr}\left[\rho^2\right] \leq 1$$

Probability 1-p $\quad |0\rangle$

Probability p $\quad |1\rangle$

$$\rho = \mathbb{P}\left[|0\rangle\right]|0\rangle\langle 0| + \mathbb{P}\left[|1\rangle\right]|1\rangle\langle 1|$$
$$= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$

$$= \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \end{array}$$

Mixed state Bloch vector

Expectation value of Z

$$\left\langle \hat{Z} \right\rangle = \mathrm{Tr}\left[\hat{Z}\rho\right]$$
$$= (1-p)\left\langle 0\middle|\hat{Z}\middle|0\right\rangle + p\left\langle 1\middle|\hat{Z}\middle|1\right\rangle$$
$$= (1-p)(1) + (p)(-1)$$
$$= 1 - 2p$$

Show that

$$\left\langle \hat{X} \right\rangle = 0$$
$$\left\langle \hat{Y} \right\rangle = 0$$
$$\left\langle \hat{I} \right\rangle = 1$$

and that generally for a qubit

$$\mathrm{Tr}\left[\rho\right] = 1$$
$$1/2^n \leq \mathrm{Tr}\left[\rho^2\right] \leq 1$$

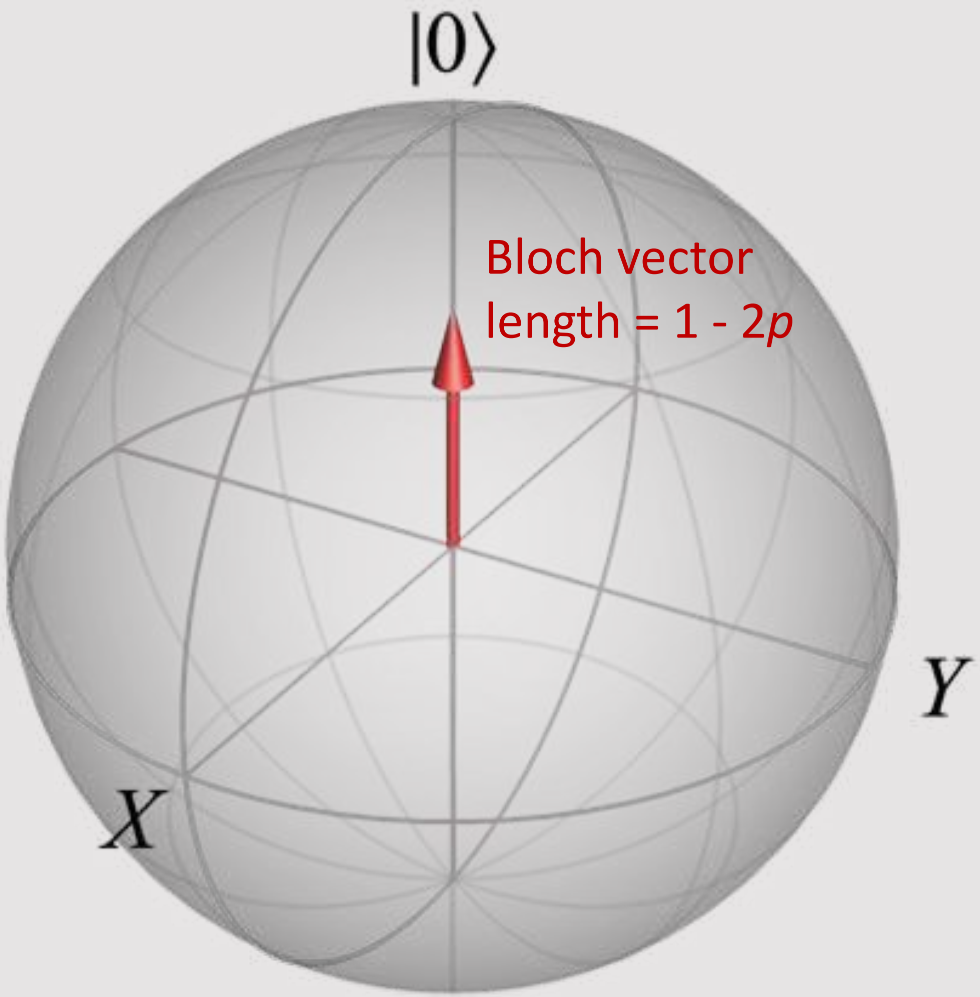
Show that

$$\rho = \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \end{array} = \frac{1}{2}\left(\hat{I} + 0\hat{X} + 0\hat{Y} + (1-2p)\,\hat{Z}\right)$$

And more generally for any qubit mixed state

$$\rho = \frac{1}{2}\left(\hat{I} + \left\langle \hat{X} \right\rangle \hat{X} + \left\langle \hat{Y} \right\rangle \hat{Y} + \left\langle \hat{Z} \right\rangle \hat{Z}\right)$$

Visualizing the mixed-state Bloch vector



For our state prep noisy monster, the state is

$$\rho = \begin{array}{c} \\ |0\rangle \\ |1\rangle \end{array} \begin{array}{c} \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \end{array} = \frac{1}{2}\left(\hat{I} + 0\hat{X} + 0\hat{Y} + (1-2p)\,\hat{Z}\right)$$

Keep in mind that a qubit mixed state in general is

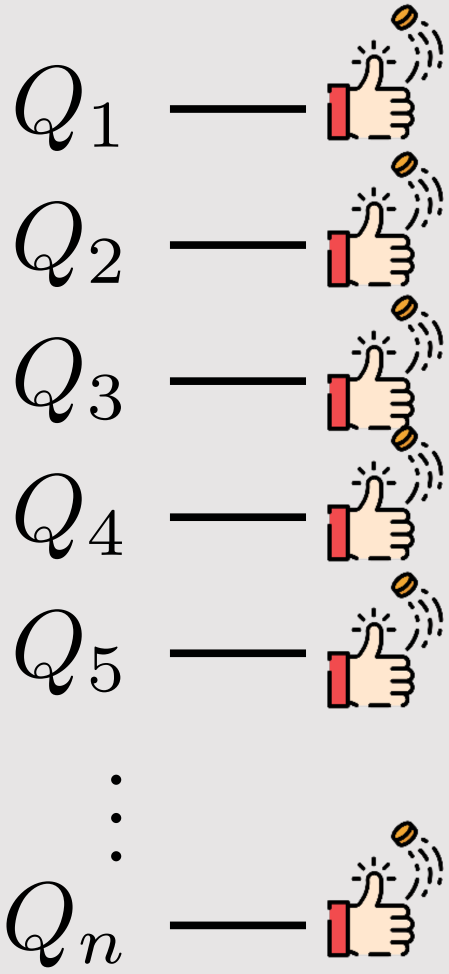
$$\rho = \frac{1}{2}\left(\hat{I} + \left\langle \hat{X} \right\rangle \hat{X} + \left\langle \hat{Y} \right\rangle \hat{Y} + \left\langle \hat{Z} \right\rangle \hat{Z}\right)$$

Purity

$$\operatorname{Tr}\left[\rho^2\right] = \frac{1}{2}\left(1 + \left\langle \hat{X} \right\rangle^2 + \left\langle \hat{Y} \right\rangle^2 + \left\langle \hat{Z} \right\rangle^2\right)$$
$$= \frac{1}{2}\left(1 + (1-2p)^2\right)$$

box; freepik. person: facebook

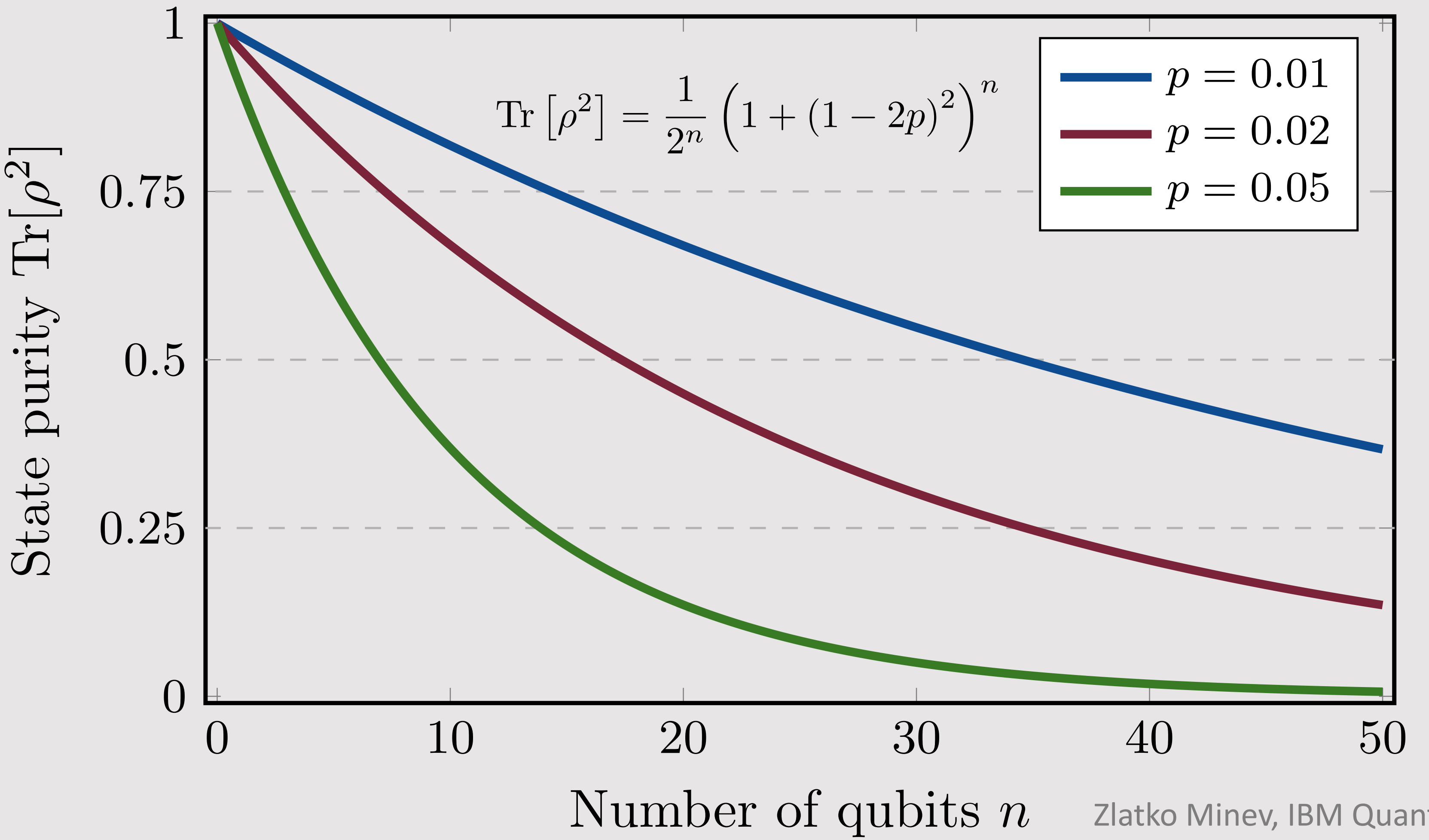
Noisy state preparation for multiple qubits



Harder, but see if you can try the derivation

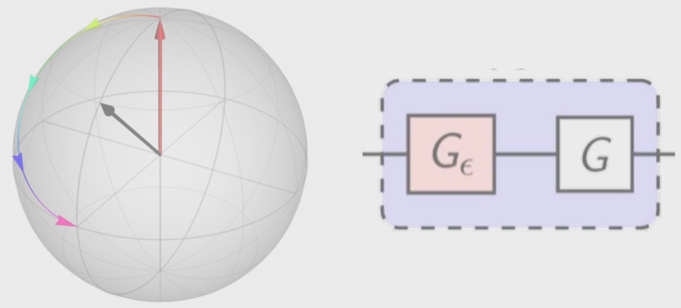
$$\rho = \left[(1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|\right]_{\mathrm{Q}_1} \otimes \cdots \otimes \left[(1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|\right]_{\mathrm{Q}_N}$$

where p is individual qubit thermal probability to be in the 1 state.



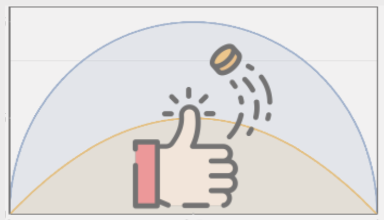
Chapter 5

Coherent noise

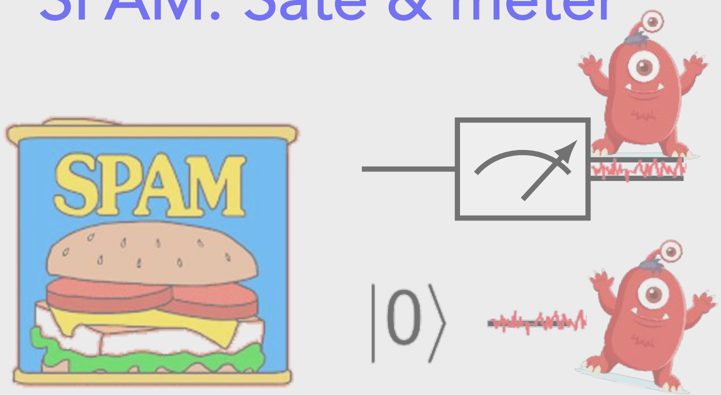


Measurements in quantum

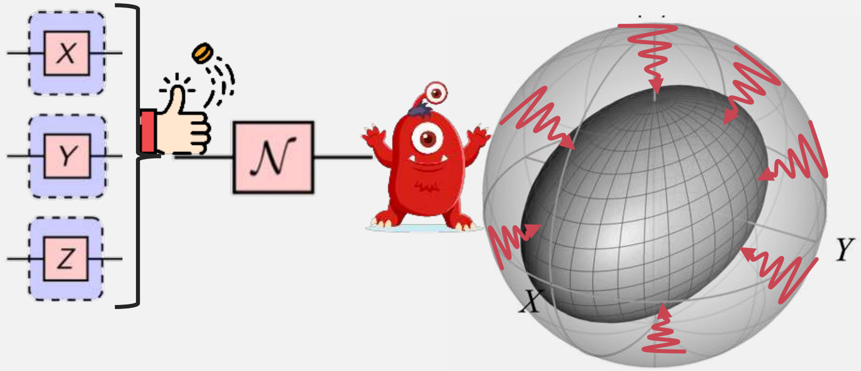
Projection noise



SPAM: Sate & meter

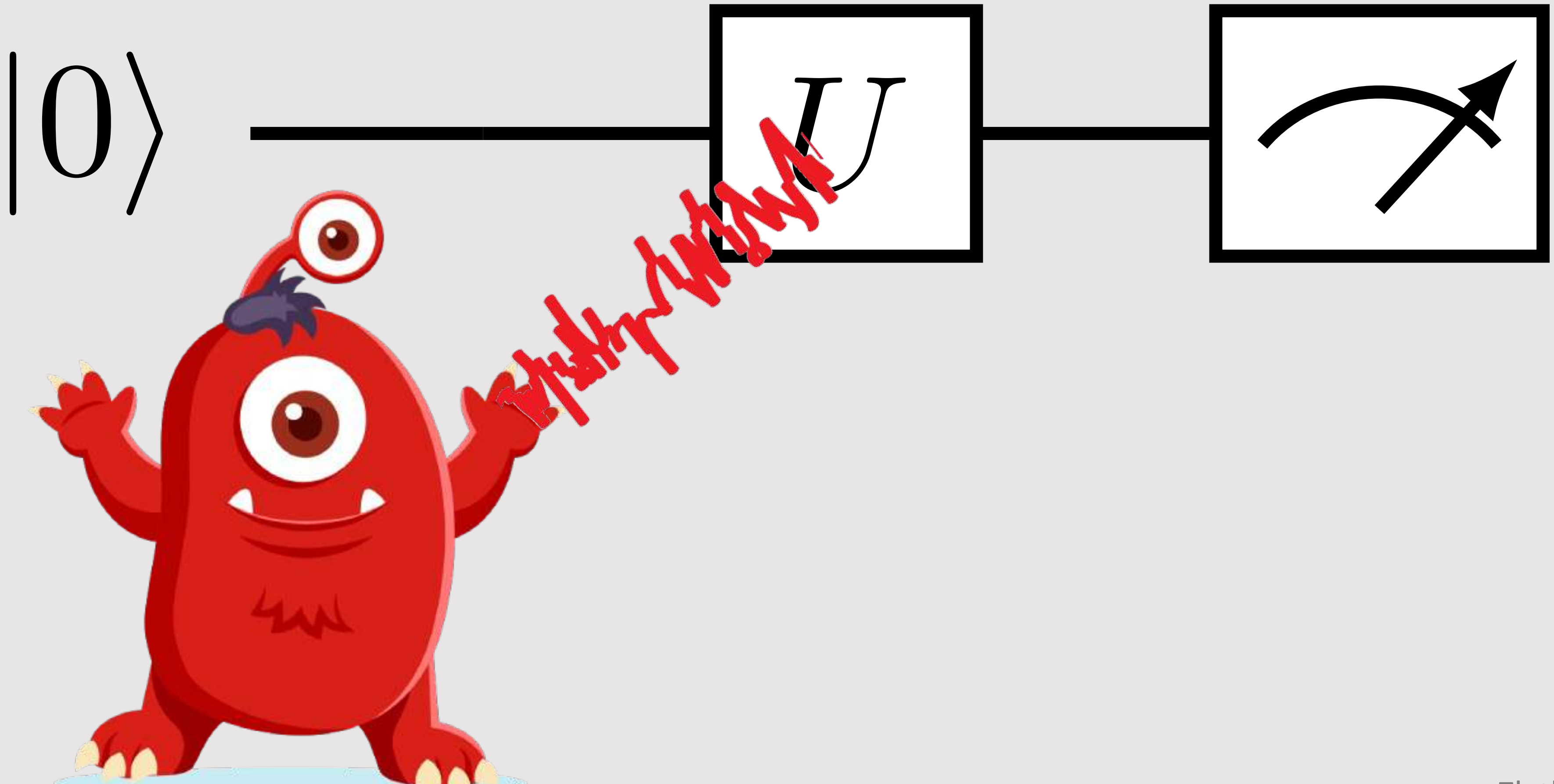


Incoherent noise



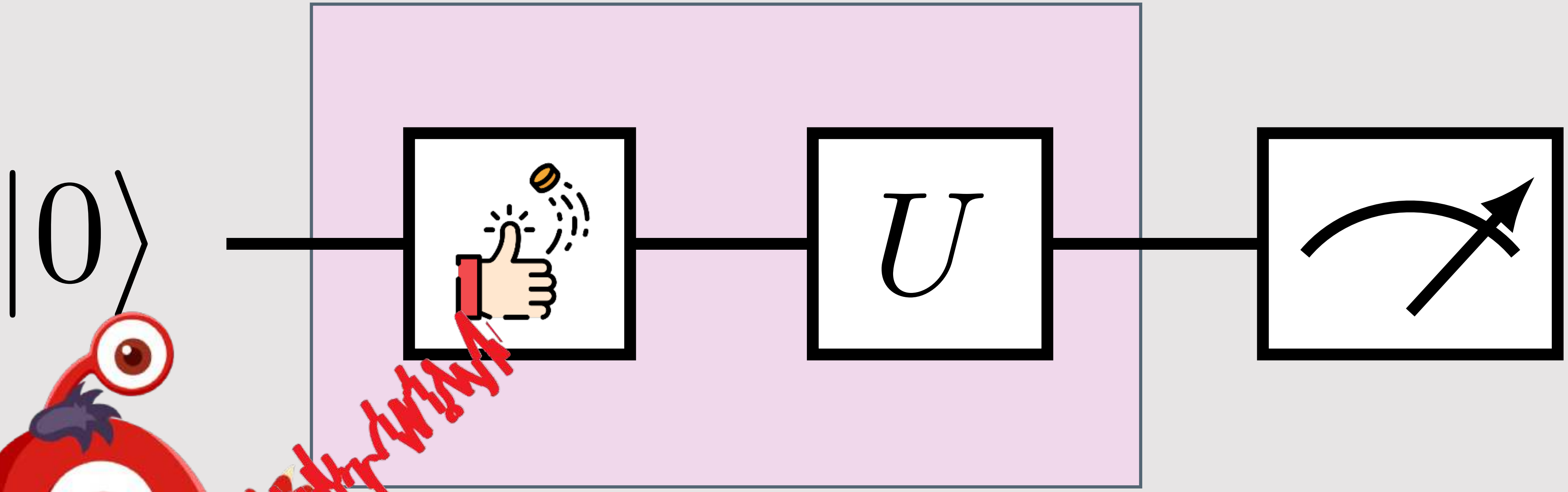
In this chapter, we will show more results, and leave more of the calculations for you to try

Quantum noise
Toy model example



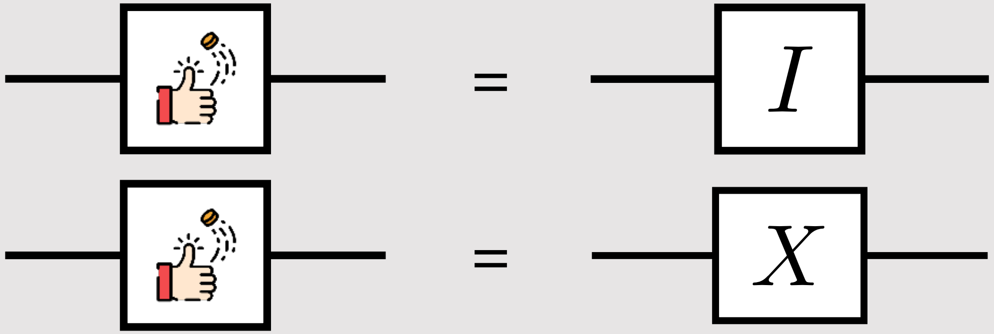
* Monster image by jcomp on Freepik

Quantum noise
Toy model example



probability $1\text{-}p$

probability p



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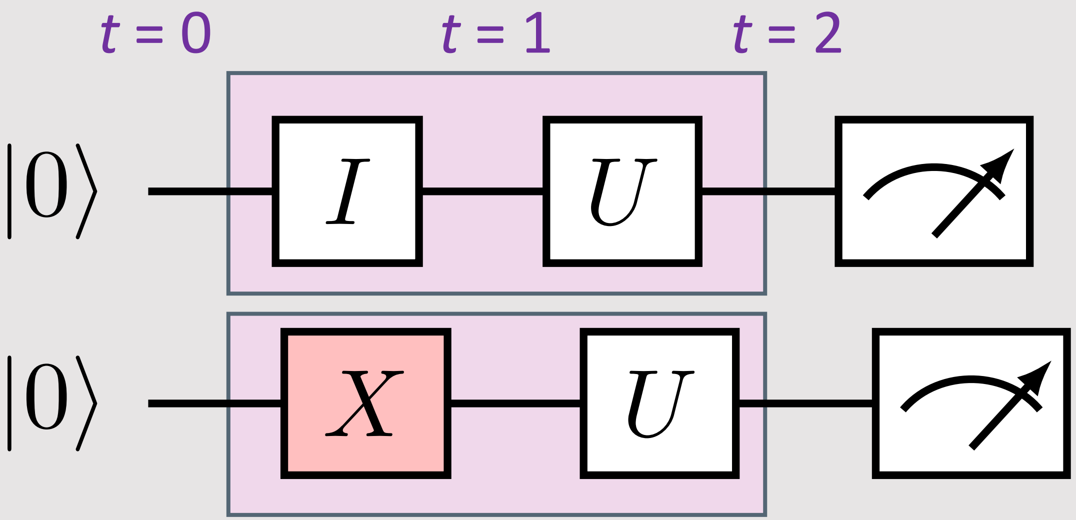
Noise raveling of quantum trajectories

Trajectory A:

probability $1\text{-}p$

Trajectory B:

probability p



Results in mixed state just like in our discussion of state preparation error and mixed states

How does this noise act on the state?

For trajectory A:

$$\rho_0^A = |0\rangle\langle 0|$$
$$\rho_1^A = I|0\rangle\langle 0| I^\dagger$$
$$\rho_2^A = UI|0\rangle\langle 0| I^\dagger U^\dagger$$

For trajectory B:

$$\rho_0^B = |0\rangle\langle 0|$$
$$\rho_1^B = X|0\rangle\langle 0| X^\dagger$$
$$\rho_2^B = UX|0\rangle\langle 0| X^\dagger U^\dagger$$

Average evolution

$$\rho_t = (1-p)\,\rho_t^A + p\rho_t^B$$

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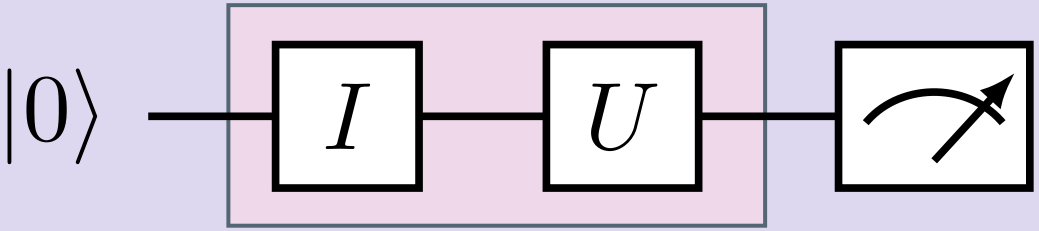
Average evolution: Quantum noise channel

Trajectory A:

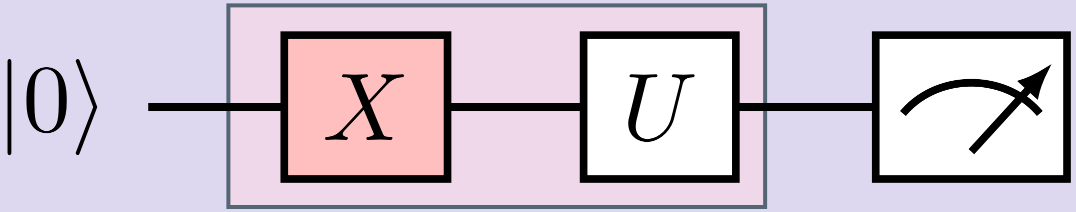
probability 1-p



+

Trajectory B:

probability p



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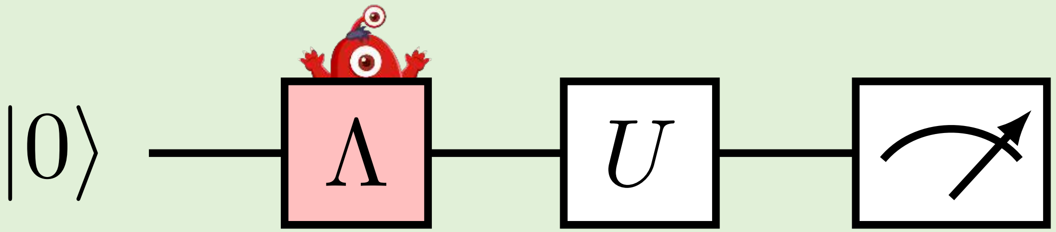
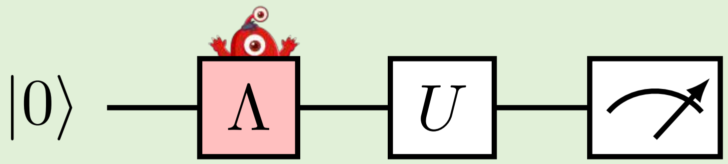


$$\Lambda(\rho) = (1-p)\, I\rho I + pX\rho X$$

* Monster image by jcomp on Freepik

Bit flip channel effects on the Bloch vector



$$\Lambda(\rho) = (1-p)\, I\rho I + pX\rho X$$

Show the effect of the noise channel on the Bloch vector of the state $\rho_0 = \frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)$

Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle\hat{I}\rangle, \langle\hat{X}\rangle, \langle\hat{Y}\rangle, \langle\hat{Z}\rangle$ for ρ_0, ρ_1

For the initial state ρ_0

$$\left\langle \hat{I} \right\rangle =$$

$$\left\langle \hat{X} \right\rangle =$$

$$\left\langle \hat{Y} \right\rangle =$$

$$\left\langle \hat{Z} \right\rangle =$$

For the final state ρ_1

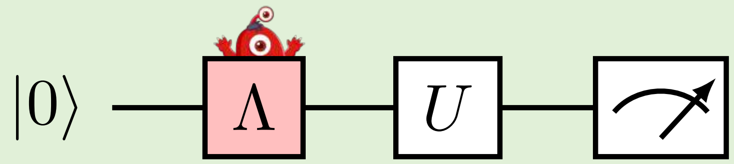
$$\left\langle \hat{I} \right\rangle =$$

$$\left\langle \hat{X} \right\rangle =$$

$$\left\langle \hat{Y} \right\rangle =$$

$$\left\langle \hat{Z} \right\rangle =$$

Bit flip channel effects on the Bloch vector



$$\Lambda(\rho) = (1-p)\, I\rho I + pX\rho X$$

Tool bag tricks:

For $\hat{A}, \hat{B} \in \left\{\hat{I}, \hat{X}, \hat{Y}, \hat{Z}\right\}$ Paulis,

$$\hat{A}^n = \begin{cases} \hat{A}\,, & n \text{ odd}\,, \\ \hat{I}\,, & n \text{ even}\,, \end{cases}$$

$$\operatorname{Tr}\left[\hat{A}\right] = 0$$

$$\operatorname{Tr}\left[\hat{A}\hat{B}\right] = 2\delta_{AB}$$

The trace is a linear functional

For $a, b \in \mathbb{C}$ and $\hat{C}, \hat{D} \in L(\mathcal{H})$,

$$\operatorname{Tr}\left[a\hat{C} + b\hat{D}\right] = a \operatorname{Tr}\left[\hat{C}\right] + b \operatorname{Tr}\left[\hat{D}\right]$$

Show the effect of the noise channel on the Bloch vector of the state $\rho_0 = \frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)$

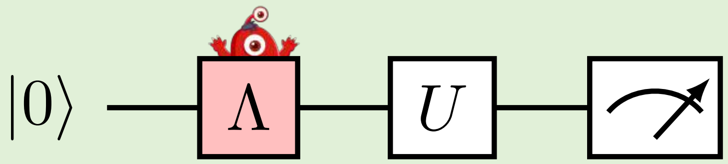
Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle\hat{I}\rangle, \langle\hat{X}\rangle, \langle\hat{Y}\rangle, \langle\hat{Z}\rangle$ for ρ_0, ρ_1

For the initial state ρ_0

$$\begin{aligned}
\langle\hat{I}\rangle &= \operatorname{Tr}\left[\hat{I}\rho\right] \\
&= \operatorname{Tr}\left[\hat{I} \cdot \frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)\right] \\
&= \operatorname{Tr}\left[\frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)\right] && \text{Identity times anything is itself} \\
&= \frac{1}{2}\operatorname{Tr}\left[\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right] && \text{trace is linear functional, move constant outside} \\
&= \frac{1}{2}\left(\operatorname{Tr}\left[\hat{I}\right] + x\operatorname{Tr}\left[\hat{X}\right] + y\operatorname{Tr}\left[\hat{Y}\right] + z\operatorname{Tr}\left[\hat{Z}\right]\right) && \text{trace is linear functional, expand sum and distribute} \\
&= \frac{1}{2}(2+0+0+0) && \text{all Pauli operators are traceless, except for the identity} \\
&= 1 && \text{we have shown the trace of the mixed state is 1 as expected}
\end{aligned}$$

* Monster image by jcomp on Freepik

Bit flip channel effects on the Bloch vector



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$$\mathrm{Tr}\left[a\hat{C} + b\hat{D}\right] = a\,\mathrm{Tr}\left[\hat{C}\right] + b\,\mathrm{Tr}\left[\hat{D}\right]$$

For the initial state ρ_0

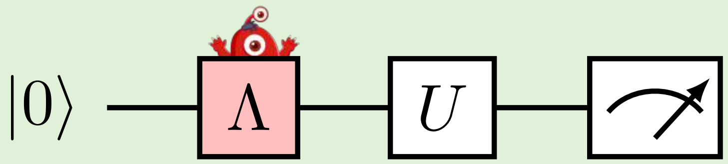
$$\begin{aligned}
\left\langle \hat{X} \right\rangle &= \mathrm{Tr}\left[\hat{X}\rho\right] \\
&= \mathrm{Tr}\left[\hat{X} \cdot \frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\hat{X}\hat{I}\right] + \frac{1}{2}x\,\mathrm{Tr}\left[\hat{X}\hat{X}\right] + \frac{1}{2}y\,\mathrm{Tr}\left[\hat{X}\hat{Y}\right] + \frac{1}{2}z\,\mathrm{Tr}\left[\hat{X}\hat{Z}\right] \\
&= \cancel{\frac{1}{2}\mathrm{Tr}\left[\hat{X}\right]} + \frac{1}{2}x\,\mathrm{Tr}\left[\hat{I}\right] + \cancel{\frac{1}{2}y\mathrm{const}\mathrm{Tr}\left[\hat{Z}\right]} + \cancel{\frac{1}{2}z\mathrm{const}\mathrm{Tr}\left[\hat{Y}\right]} \\
&= \frac{1}{2}x\,\mathrm{Tr}\left[\hat{I}\right] \\
&= x
\end{aligned}$$

trace is linear functional, expand sum and distribute

* Some constants we don't need to calculate since the terms are zero

* Monster image by jcomp on Freepik

Bit flip channel effects on the Bloch vector



$$\Lambda(\rho) = (1-p)\, I\rho I + pX\rho X$$

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For the initial state ρ_0

$$\left\langle\hat{I}\right\rangle = 1$$
$$\left\langle\hat{X}\right\rangle = x$$
$$\left\langle\hat{Y}\right\rangle = y$$
$$\left\langle\hat{Z}\right\rangle = z$$

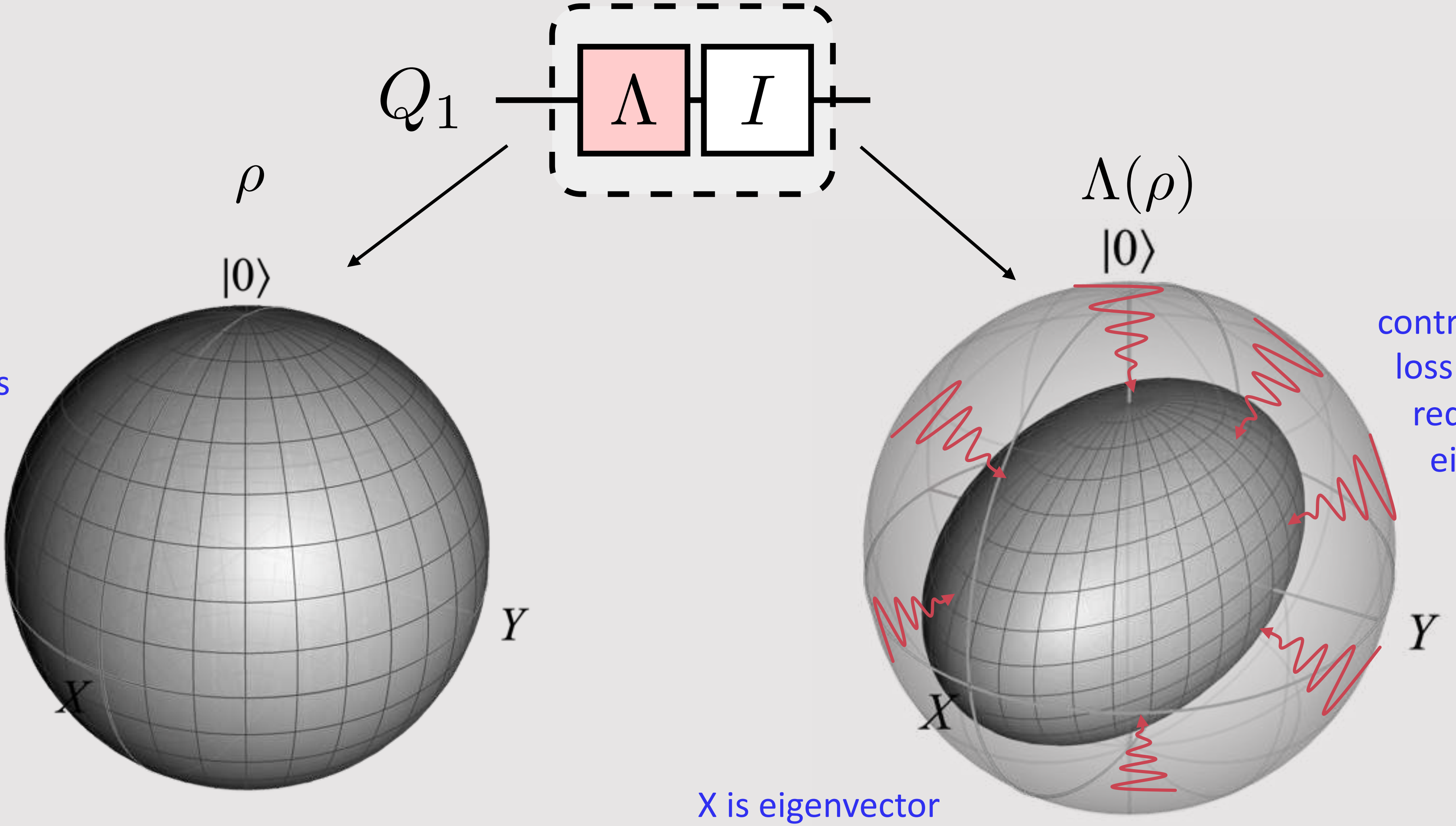
For the final state ρ_1

$$\left\langle\hat{I}\right\rangle = 1$$
$$\left\langle\hat{X}\right\rangle = x$$
$$\left\langle\hat{Y}\right\rangle = (1-2p)\, y$$
$$\left\langle\hat{Z}\right\rangle = (1-2p)\, z$$

The noisy damping factor for each Pauli is often called the *Pauli fidelity* for the relevant Pauli component

* Monster image by jcomp on Freepik

Visual explanation

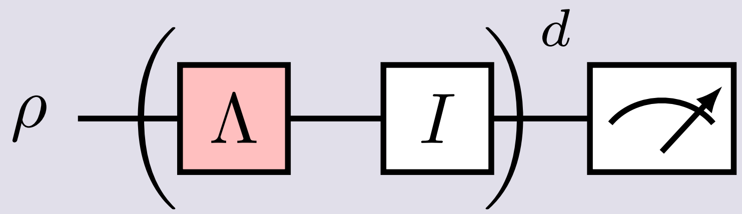


Bloch sphere
of pure states

contraction
loss of information
reduced purity
eigenvalues > 0

X is eigenvector
of map

Effect of repeated noise



For the bit flip channel, find the effect on the Bloch vector of repeating the noise *d* times with an identity gate as the unitary

$$\Lambda\left(\rho\right) = \left(1-p\right) I\rho I + pX\rho X$$

Initial state

$$\rho_0 = \frac{1}{2}\left(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}\right)$$

$$\rho_1 = \Lambda\left(\rho_0\right)$$

$$\vdots$$

$$\rho_d = \Lambda\left(\rho_{d-1}\right)$$

Use the result 2 slides back for the effect of the channel on the Bloch components with the recursive relation on the left:

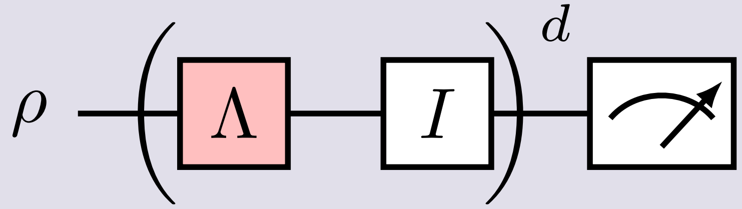
$$\left\langle \hat{I} \right\rangle = 1$$

$$\left\langle \hat{X} \right\rangle = x$$

$$\left\langle \hat{Y} \right\rangle = \left(1-2p\right) y$$

$$\left\langle \hat{Z} \right\rangle = \left(1-2p\right) z$$

Effect of repeated noise



For the bit flip channel, find the effect on the Bloch vector of repeating the noise *d* times with an identity gate as the unitary

$$\Lambda(\rho) = (1-p)\, I\rho I + pX\rho X$$

It follows that the noise causes the Bloch vector components to shift exponentially a rate given by the Pauli fidelities of the noise channel, where the fidelity is

$$f = 1 - 2p$$

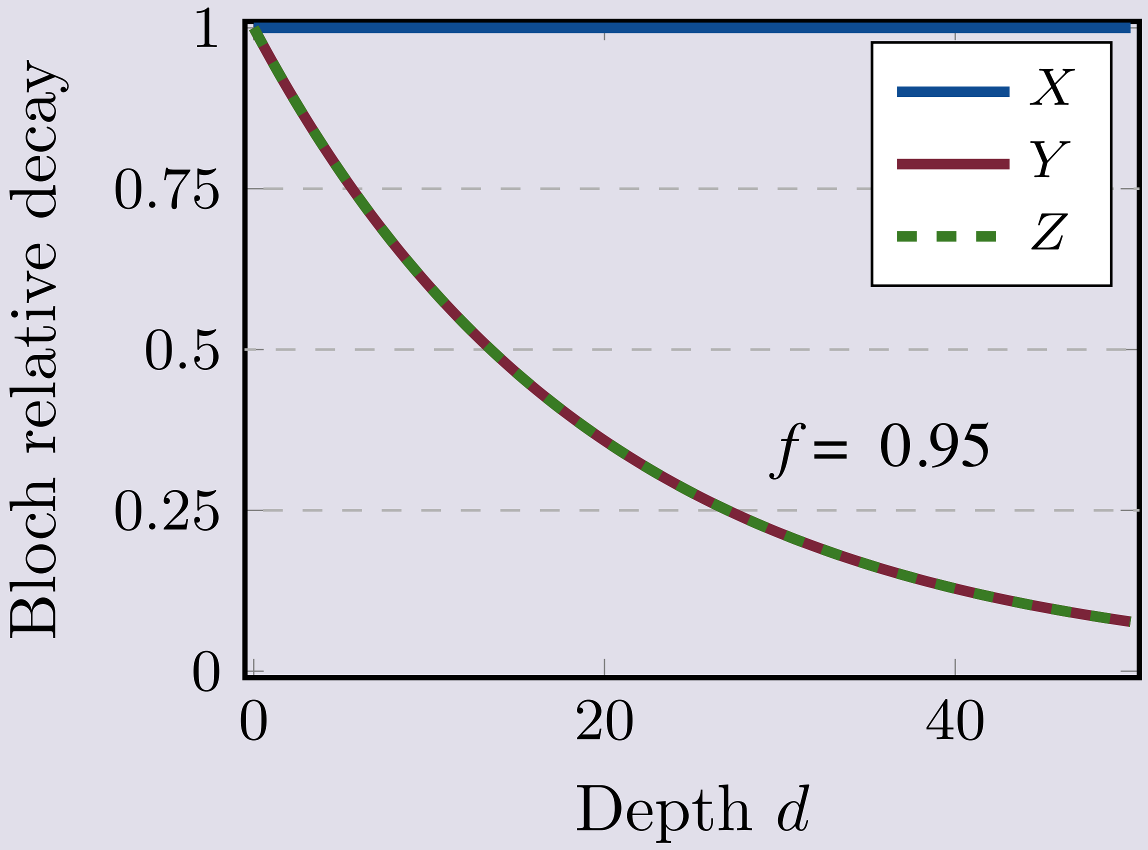
and the Bloch components are

$$\left\langle \hat{I} \right\rangle(d) = 1$$

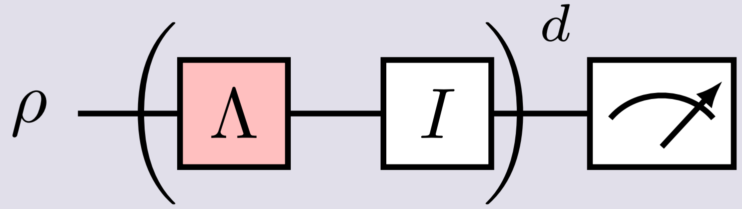
$$\left\langle \hat{X} \right\rangle(d) = x$$

$$\left\langle \hat{Y} \right\rangle(d) = f^d y$$

$$\left\langle \hat{Z} \right\rangle(d) = f^d z$$



Effect of repeated noise



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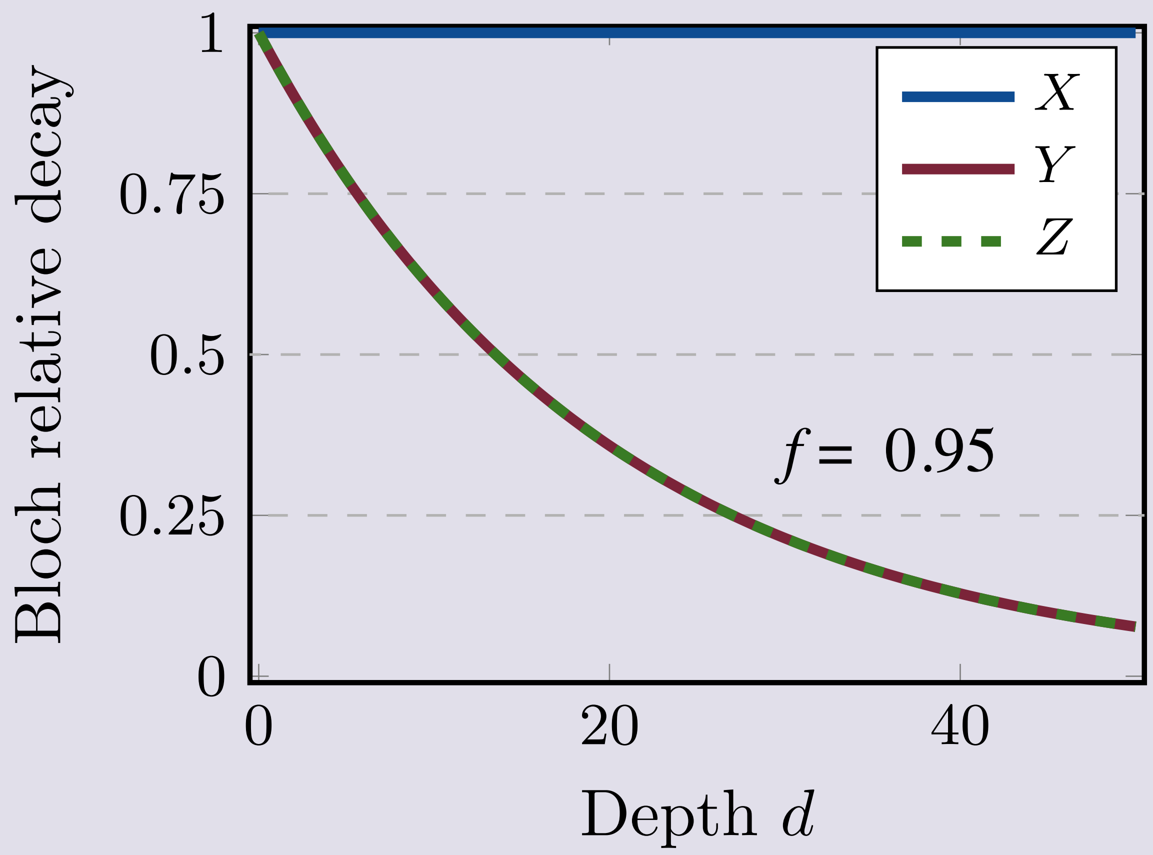
It follows that the noise causes the Bloch vector components to shift exponentially a rate given by the Pauli fidelities of the noise channel, where the fidelity is

$$f = 1 - 2p$$

Note, to first order the errors add up linearly rather than quadratically as for coherent noise, in other words, slower:

$$\left\langle \hat{Z} \right\rangle (d) = \left(1-2p\right)^d z \approx \left(1 - 2pd\right) z + \mathcal{O}\left(d^2\right)$$



Hello World example: all the pieces explained

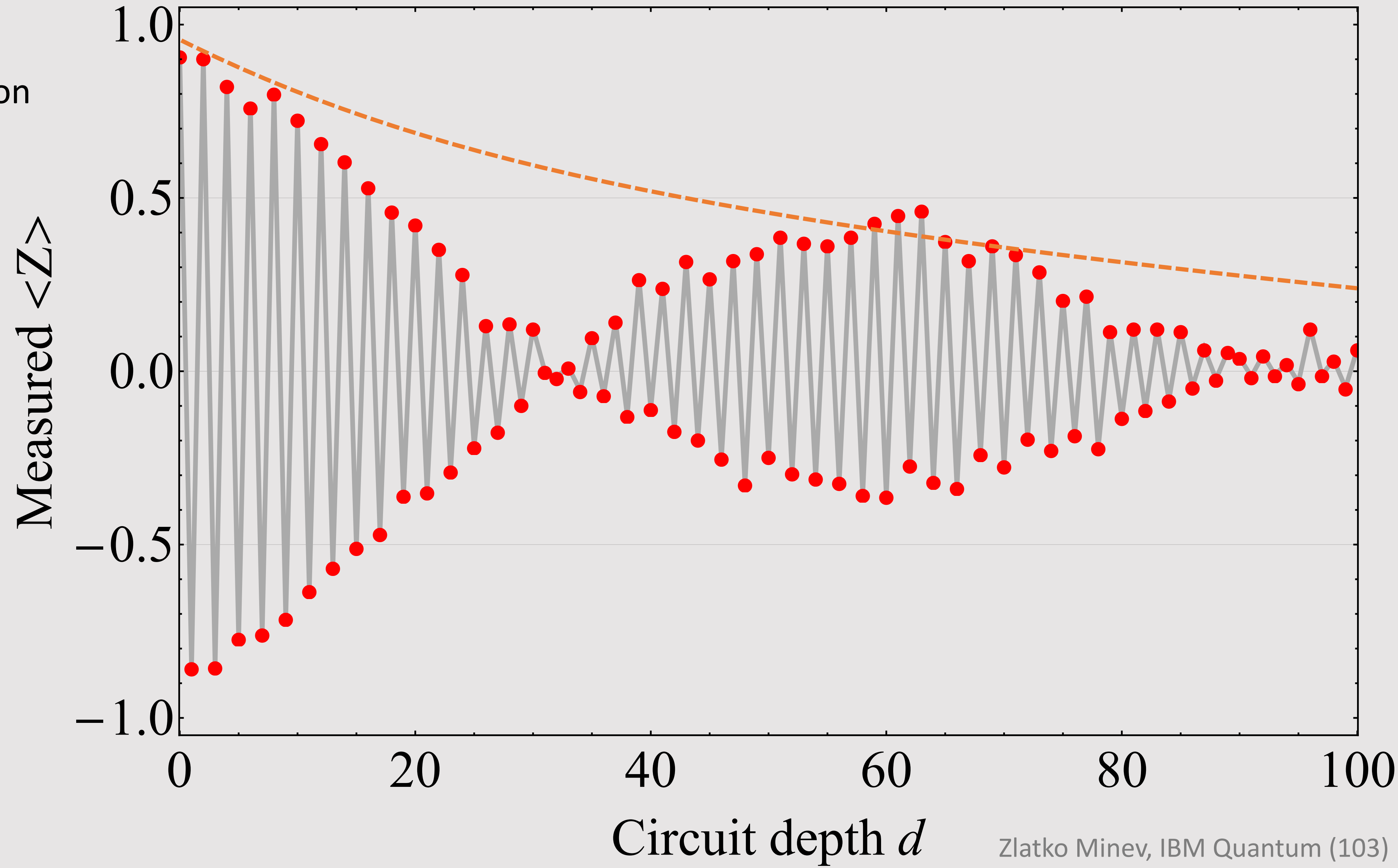
Coherent error
ε = 3 degrees over rotation

Projection noise
800 shots

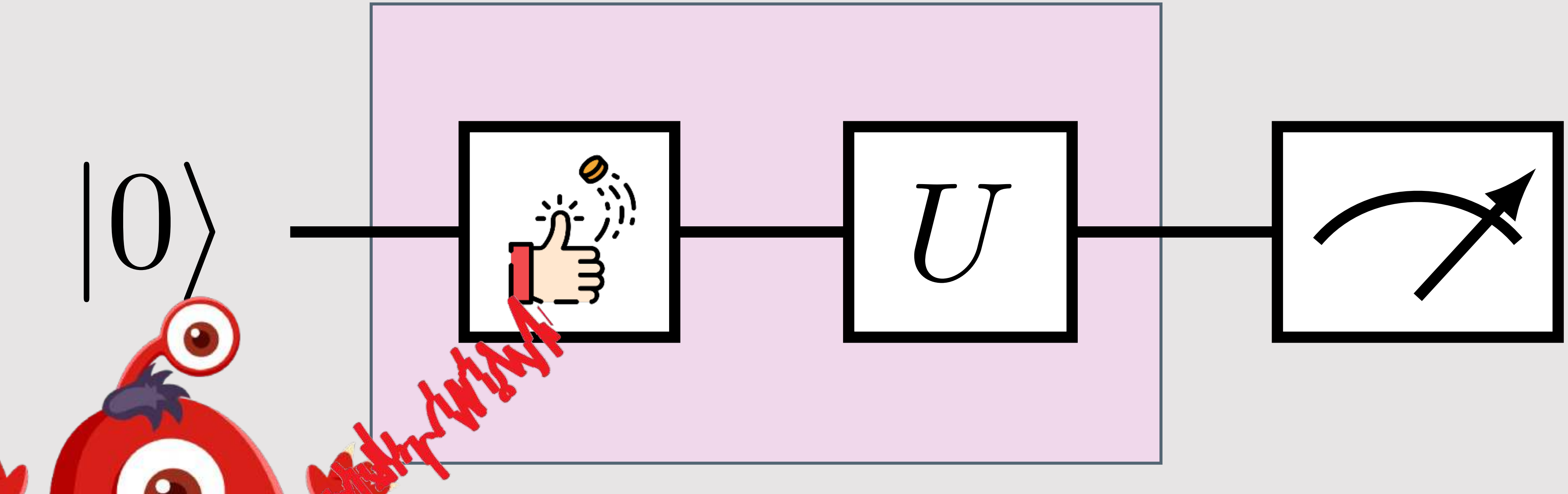
Readout error
ε = $p(1|0)$ = 0.02
ν = $p(0|1)$ = 0.05

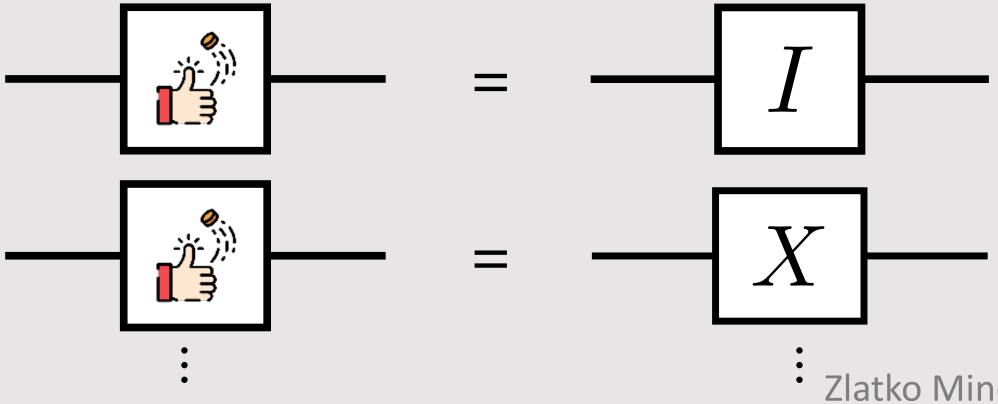
State prep error
p = 0.025

Incoherent bit flip noise
p = 0.008
$\Lambda(\rho) = (1-p)\,I\rho I + pX\rho X$



Other quantum noise models?





X bit-flip noise

$-\boxed{\mathcal{E}}-$

probability	circuit instance
$1-p$	I
p	X

Phase noise

$-\boxed{\mathcal{E}}-$

probability	circuit instance
$1-p$	I
p	Z

Depolarizing noise

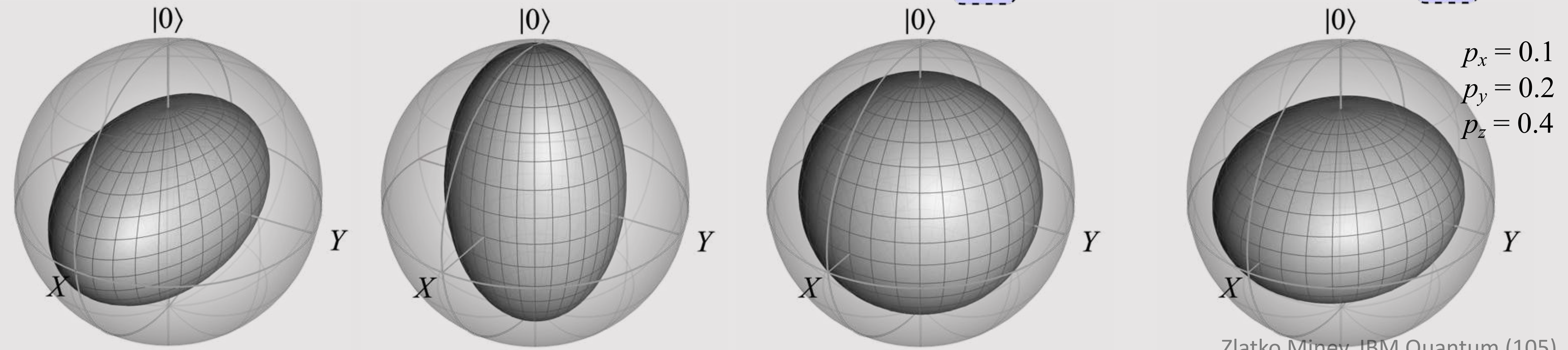
$-\boxed{\mathcal{E}}-$

probability	circuit instance
$1-3p/4$	I
$p/4$	X
$p/4$	Y
$p/4$	Z

Stochastic Pauli noise

$-\boxed{\mathcal{E}}-$

probability	circuit instance
$1-p_X-p_Y-p_Z$	I
p_X	X
p_Y	Y
p_Z	Z



(hard) Exercise on channels

Extra credit, harder problem to challenge yourself

1. Calculate the expectation values of <*I*>, <*X*>, <*Y*>, <*Z*> for the state after the following noise channels acted on a general qubit state (hint: see bit flip channel example and last slide)

   a) Phase noise

   b) Depolarizing noise

   c) Stochastic Pauli noise

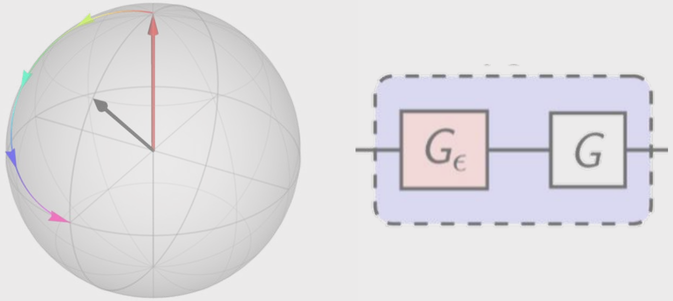
   d) For (a) – (c) summarize the Pauli fidelities.

2. Repeat the same calculation as (1) for the following channel known as *amplitude damping* and plot the effect on the Bloch sphere

$$\Lambda(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \,, K_0 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} , K_1 := \begin{pmatrix} 1 & \sqrt{p} \\ 0 & 0 \end{pmatrix} .$$

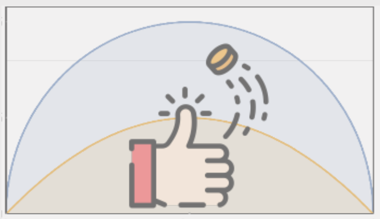
We now understand noise, how do we overcome it?

Coherent noise

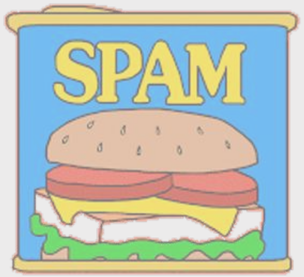


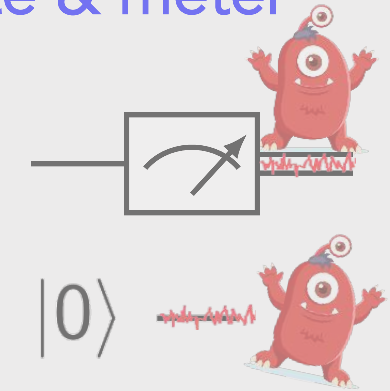
Measurements in quantum

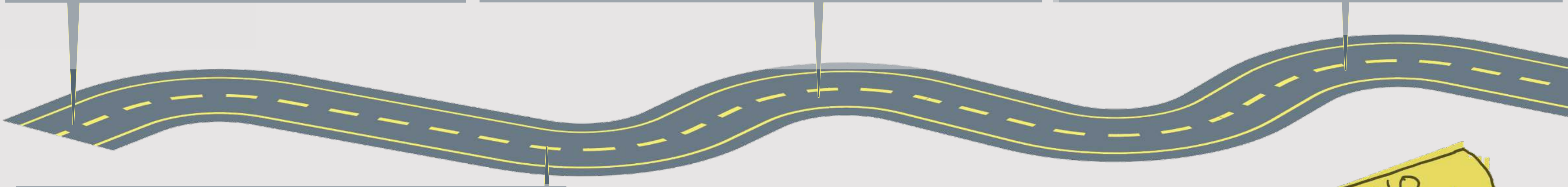
Projection noise



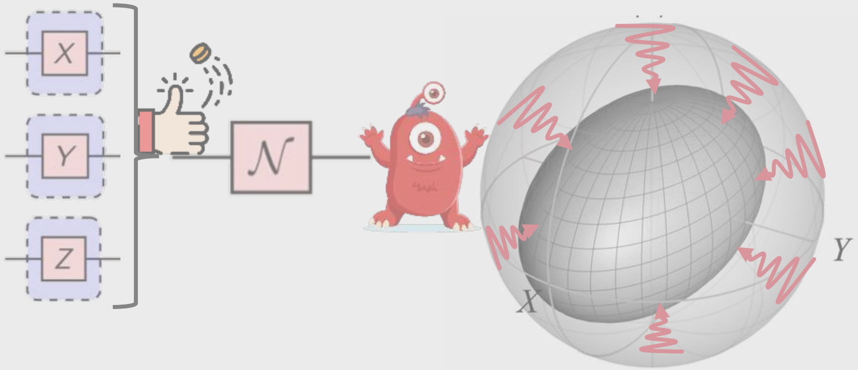
SPAM: Sate & meter





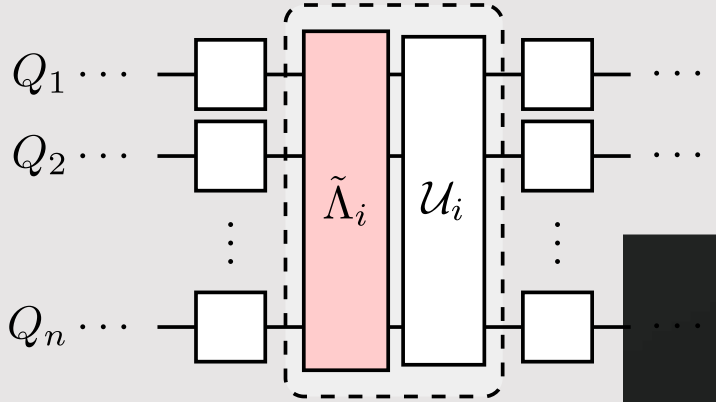


Incoherent noise

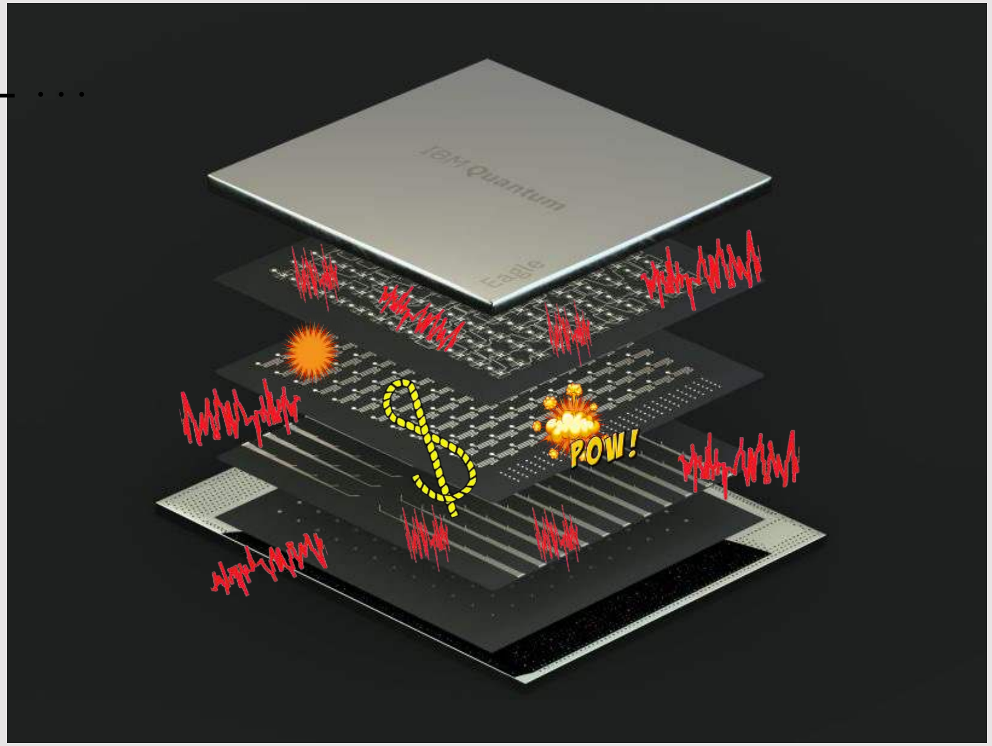


coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

Is it possible to understand & overcome quantum noise with accuracy, efficiency, and scalability?



Energy relaxation T_1
Dephasing T_2
Coherent errors ZZ
Classical crosstalk
Quantum crosstalk
State preparation error
Measurement correlated errors
…



Control errors
Photon shot noise
1/f charge noise
1/f flux noise
Nonequilibrium quasiparticles
Leakage
Cosmic rays
…

How to deal with errors due to noise?

Monitor
Error occurs
Error detect

Quantum error correction

Shor, PRA (1995), ...

Monitor
Error anticipated
Tell signal detected

Catch and reverse

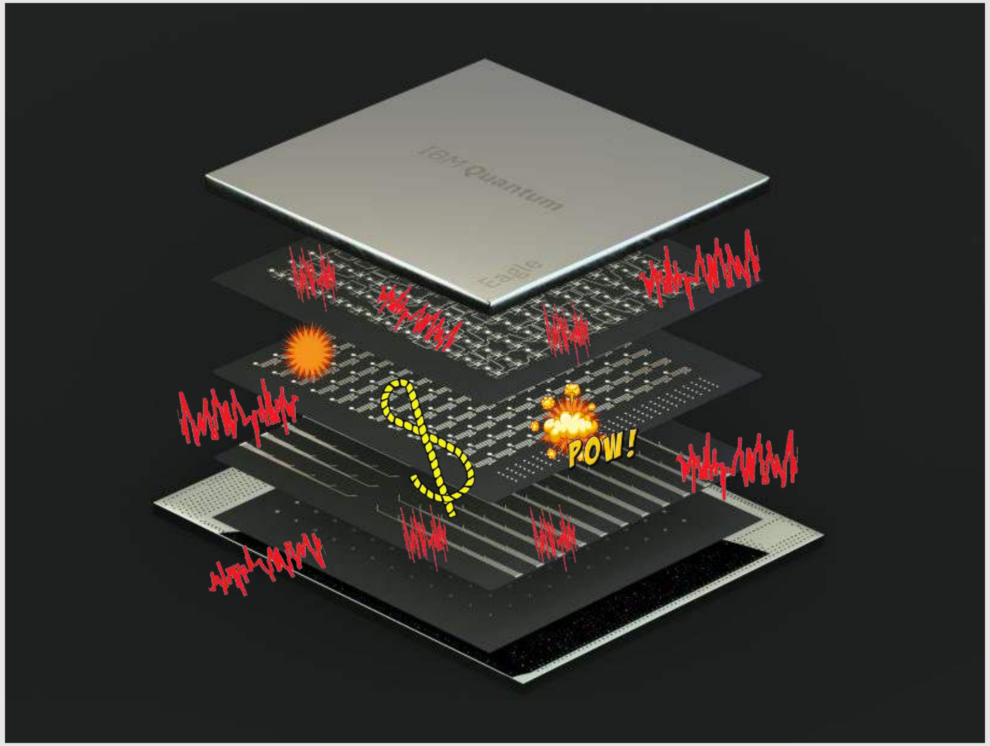
Minev, Nature (2019), ...

No monitor
Error occurs
Error undetected

Error mitigation

... subject of today

Error mitigation and error correction



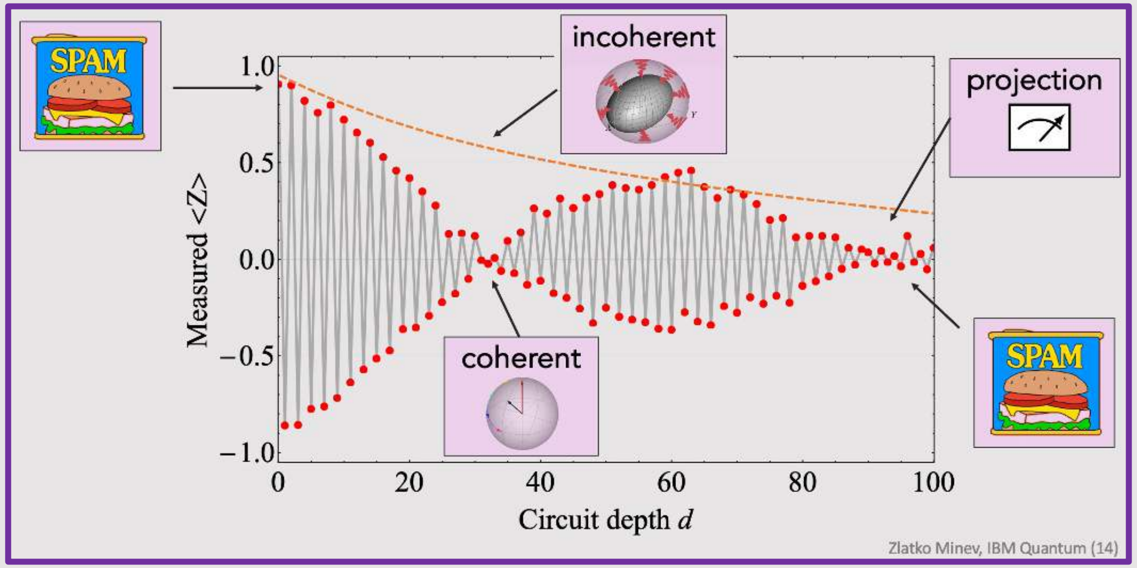
Error mitigation: working with what you have

- **benefit** suppress errors on classical results (expectation values)
- **q-cost** no extra qubits or hardware resources needed
- **c-cost** trades classical resources (post-processing) for lower error
- **limitation** bad asymptotic scaling: high number of samples & circuits

Error correction: protecting quantum information

- **benefit** suppress & correct errors to arbitrarily small level
- **q-cost** very large qubit and hardware overhead
- **c-cost** decoding and encoding can be classically costly
- **challenge** requires fault-tolerant operations and readout

Error mitigation landscape



more speed

more information, accuracy

Latest: Qiskit Quantum Seminar YouTube Series

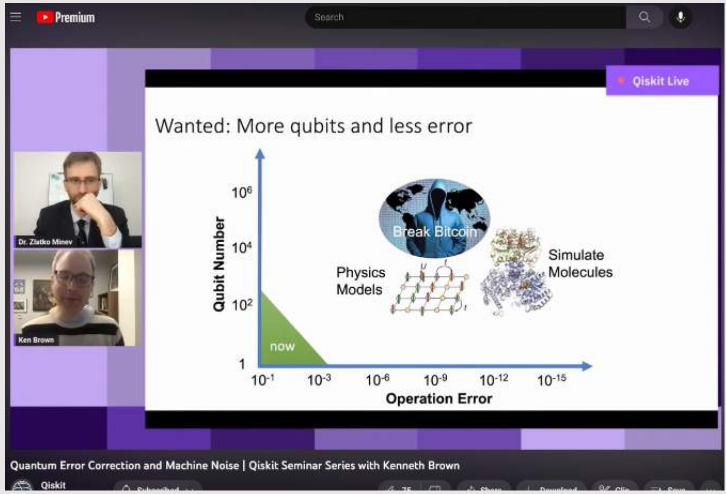
140+ years in 120+ seminars: Quantum research community connects on YouTube

Live every Friday

120+ seminar

244,461+ hours of total viewer watch time

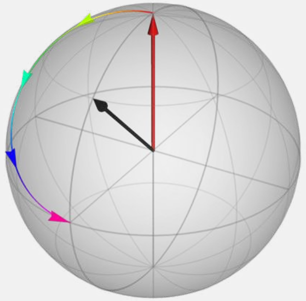
~ 1 million viewers (people who watched the video)

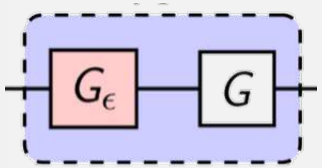


qiskit.org/events/seminar-series

We come the end of our journey today

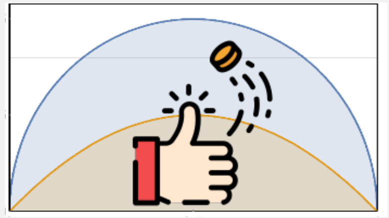
Coherent noise



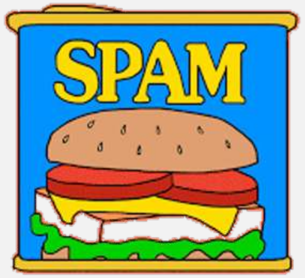


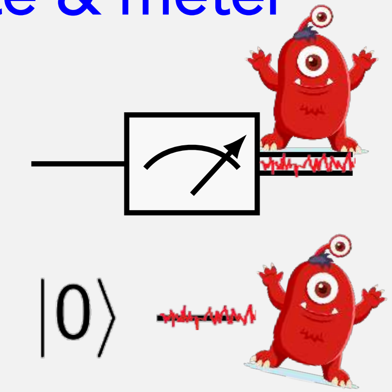
Measurements in quantum

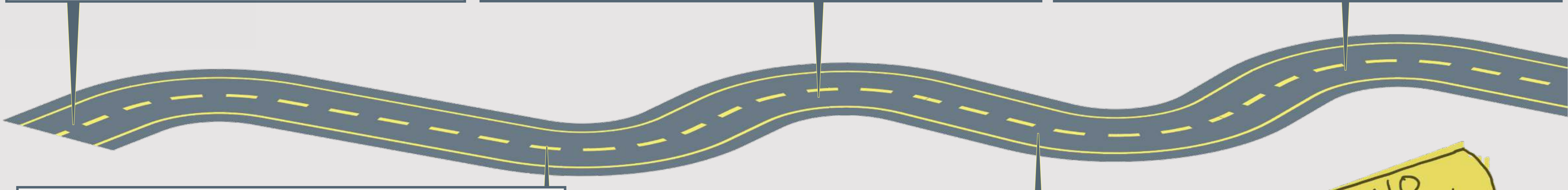
Projection noise



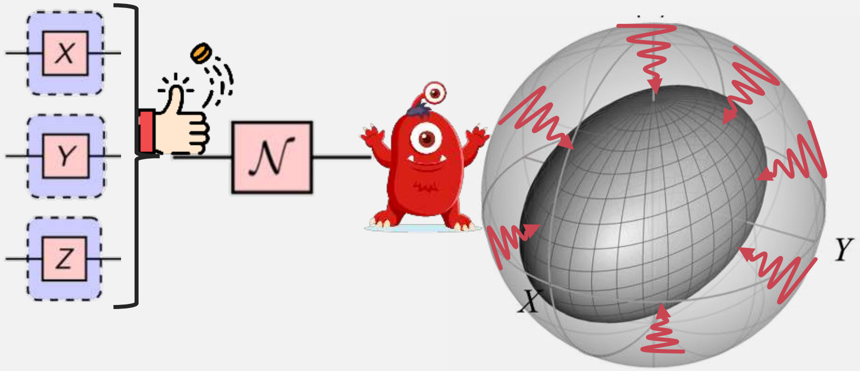
SPAM: Sate & meter







Incoherent noise



Bonus content

Coherent ZZ noise

...

coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

The important thing is not to stop questioning. Curiosity has its own reason for existence.

One cannot help but be in awe when they contemplate the mysteries of eternity, of life, of the marvelous structure of reality.

It is enough if one tries merely to comprehend a little of this mystery each day.

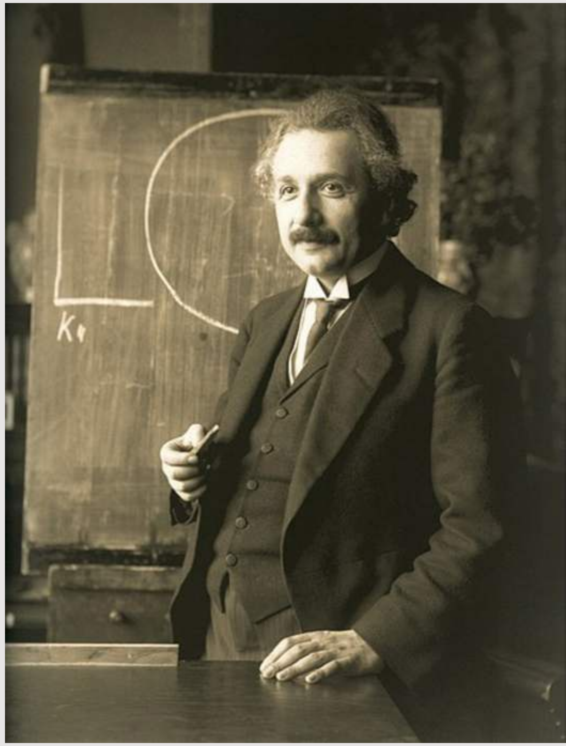
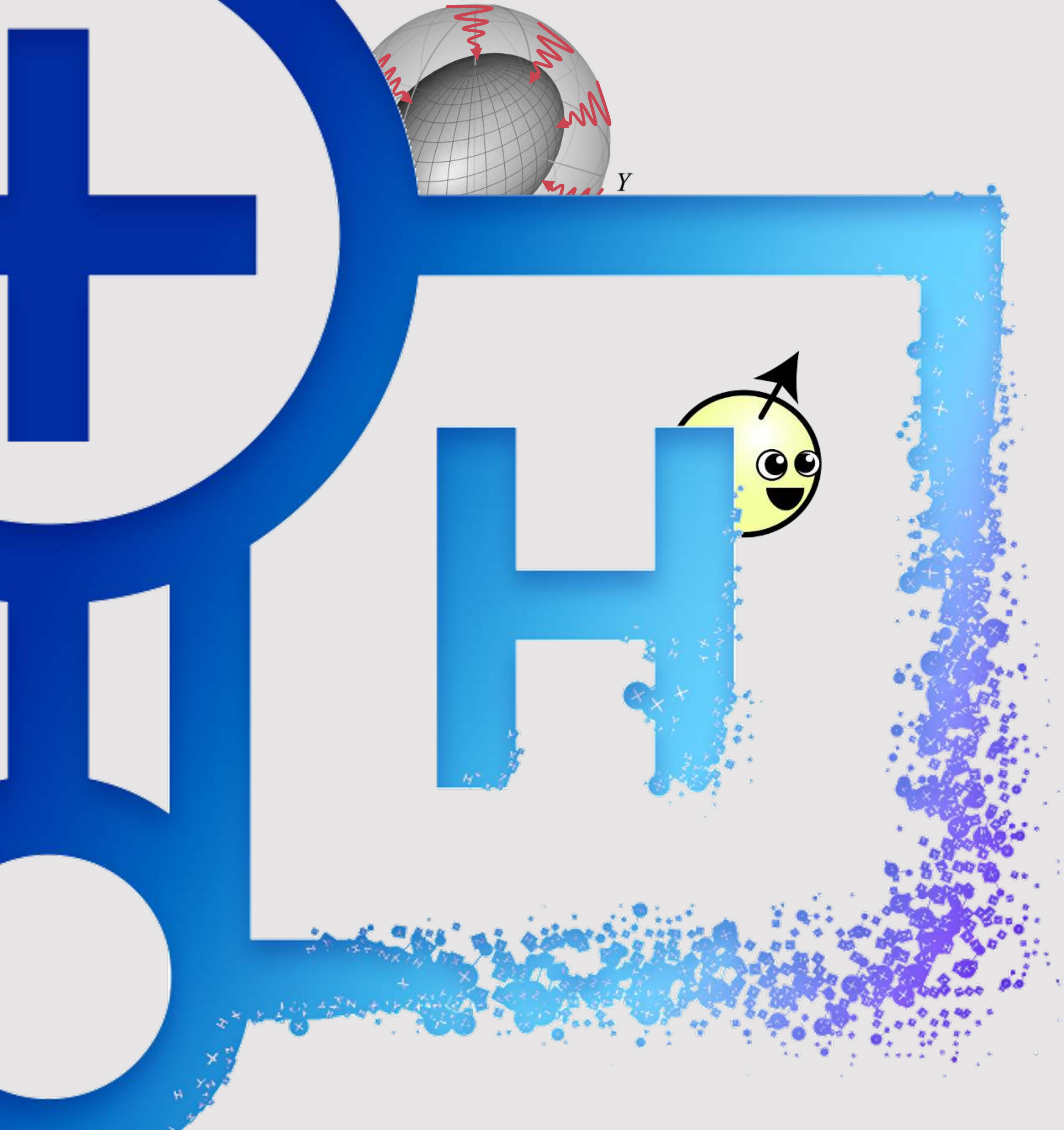


Photo: F. Schmutzer

Albert Einstein



Introduction to Quantum Noise

by Zlatko K. Minev

Thank you!

Next:
Lab work with Qiskit
Run experiments on real devices

Check out references, problems given in the lecture, dangerous bends

Stay in touch and have fun!

Qiskit | Global Summer School 2023

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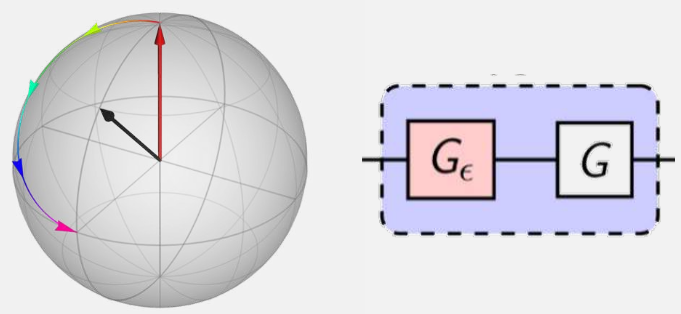
Bonus content

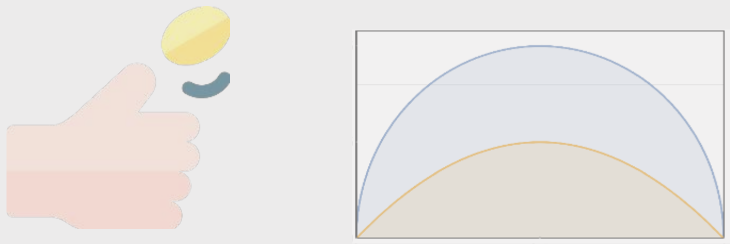
Bonus content

Coherent noise

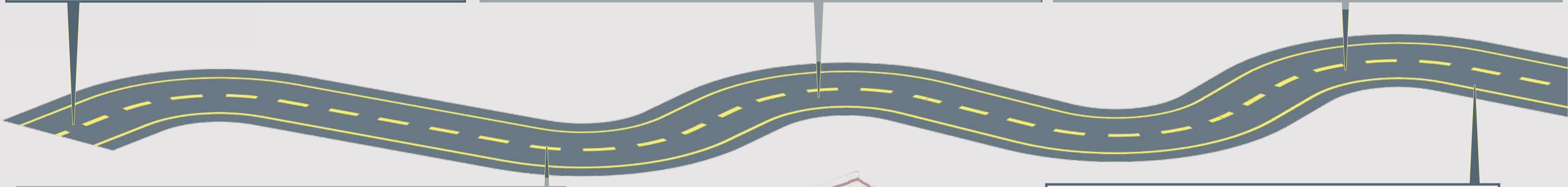


Measurement in a nutshell

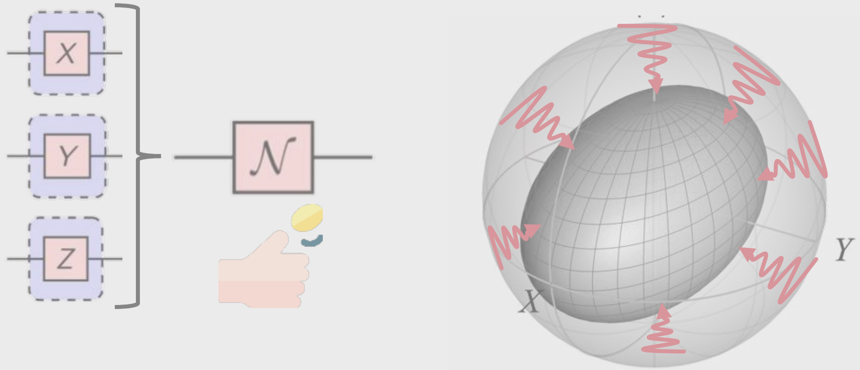
Projection noise



SPAM: Noisy meters



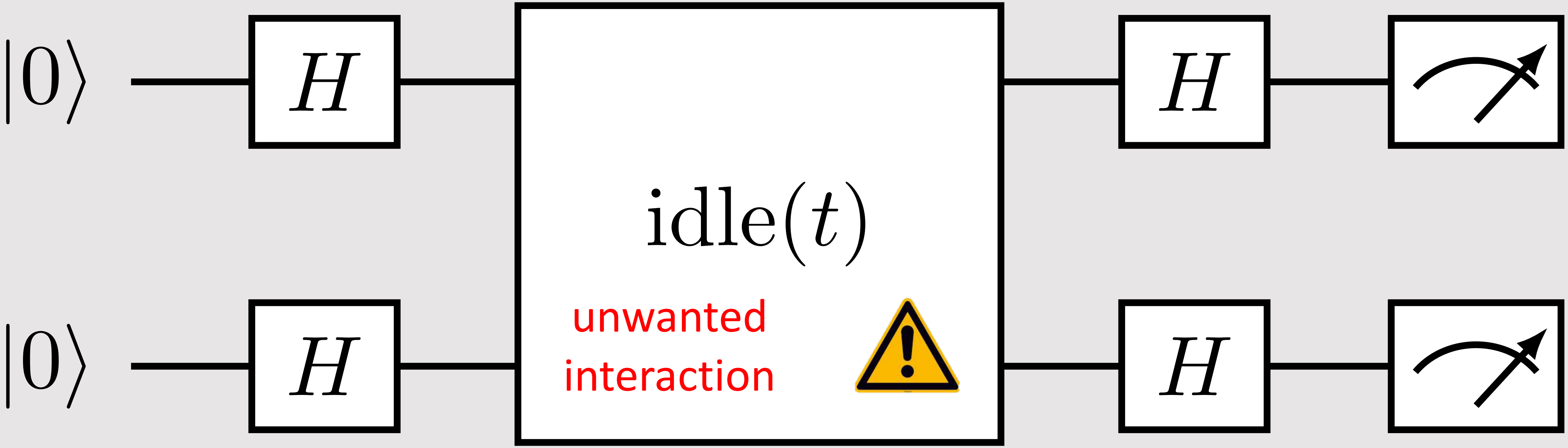
Incoherent noise



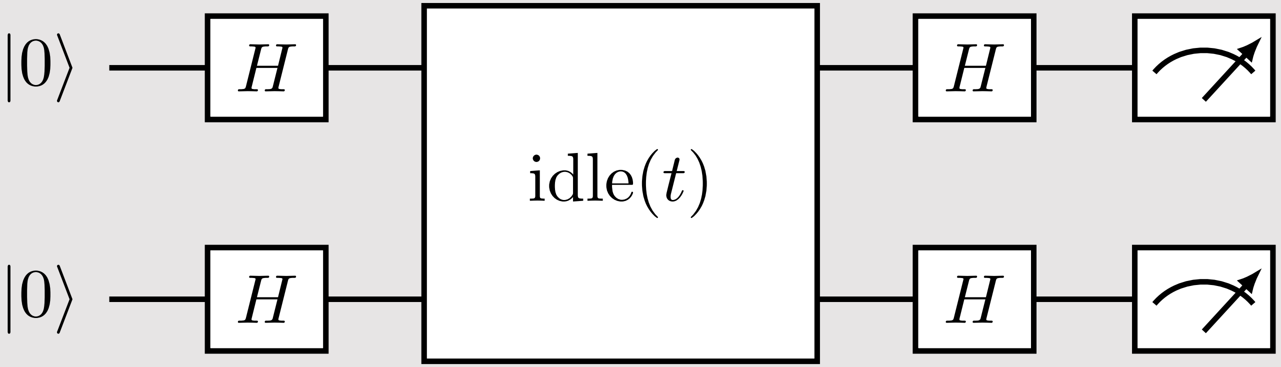
Bonus content

coin toss: flaticon; spam: make it move;
road based on: freepik

Bonus content: two-qubit coherent ZZ error



Bonus content: two-qubit ZZ error



Hadamard gate

$\boxed{H}$

$$H = \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \end{array}$$

$$H\,|0\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$H\,|1\rangle = |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

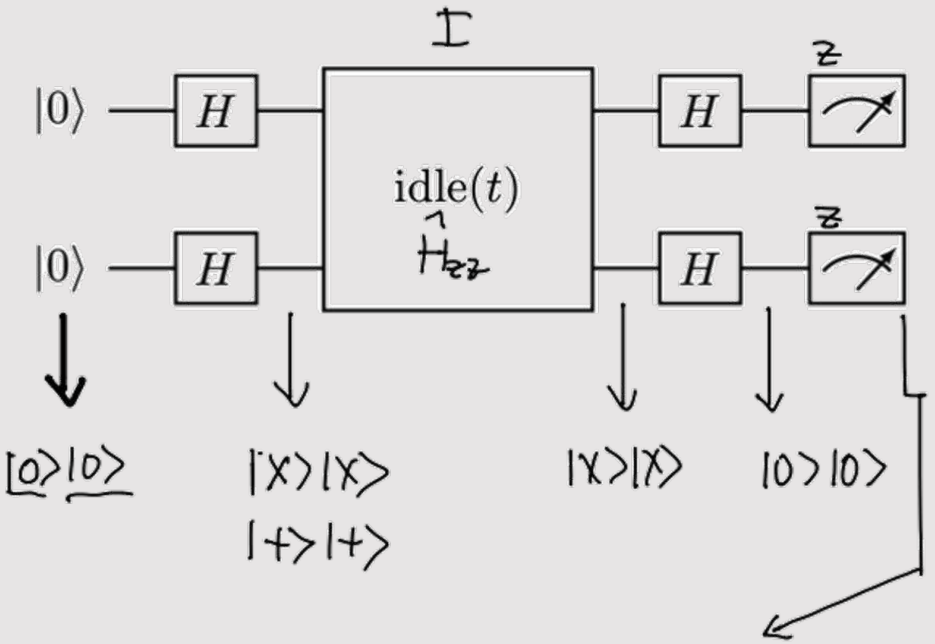
Breakout to notebook
(see next slides)

Introduction to quantum noise

Coherent errors

Qiskit Global Summer School on Quantum Machine Learning

Zlatko K. Minev



Hadamard gate

$$\boxed{H}$$

$$H = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ |0\rangle & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ |1\rangle & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}$$

$$\begin{cases} H|0\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ H|1\rangle = |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \end{cases}$$

$X|+x\rangle = +1|+x\rangle$

$X|-x\rangle = -1|-x\rangle$

$|+x\rangle :=: |+\rangle$

$|-x\rangle :=: |-\rangle$

$$\begin{cases} Z|0\rangle = +1|0\rangle \\ Z|1\rangle = -1|1\rangle \end{cases}$$

$\langle ZI\rangle = +1$

$\langle IZ\rangle = +1$

$\langle ZZ\rangle = \langle 0|\langle 0| Z Z |0\rangle|0\rangle$

$= \langle 0|Z|0\rangle \langle 0|Z|0\rangle$

$= (+1)(+1)$

$= +1$

NOISY

ZZ Interaction

$$\hat{H} = \frac{1}{2}\hbar\omega \hat{Z}\hat{Z}$$

$$\hat{U}(t) = \exp\left(-i\hbar^{-1}\hat{H}t\right)$$

$$= \exp\left(-i\frac{\omega t}{2}\hat{Z}\hat{Z}\right)$$

$$= \cos\left(\frac{\omega t}{2}\right)\hat{I} - i\sin\left(\frac{\omega t}{2}\right)\hat{Z}\hat{Z}$$

$$= \hat{R}_{ZZ}(\theta = \omega t)$$

$$\hat{R}_X(\theta) = \exp\left(-i\frac{\theta}{2}\hat{X}\right) \quad X^2 = I$$

$$= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)X$$

$$(\hat{Z}\hat{Z})^2 = Z^2Z^2 = I_2 \otimes I_2 = I_4$$

$$|0\rangle|0\rangle \xrightarrow{HH} |+\rangle|+\rangle \xrightarrow{\text{idle}} R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}|-\rangle|-\rangle \xrightarrow{HH} \cos\theta/2\,|0\rangle|0\rangle - i\sin\theta/2\,|1\rangle|1\rangle$$

$$R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}(Z|+\rangle)_1 \otimes (Z|+\rangle)_2$$

$$Z|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle$$

$$Z|+\rangle = |-\rangle$$

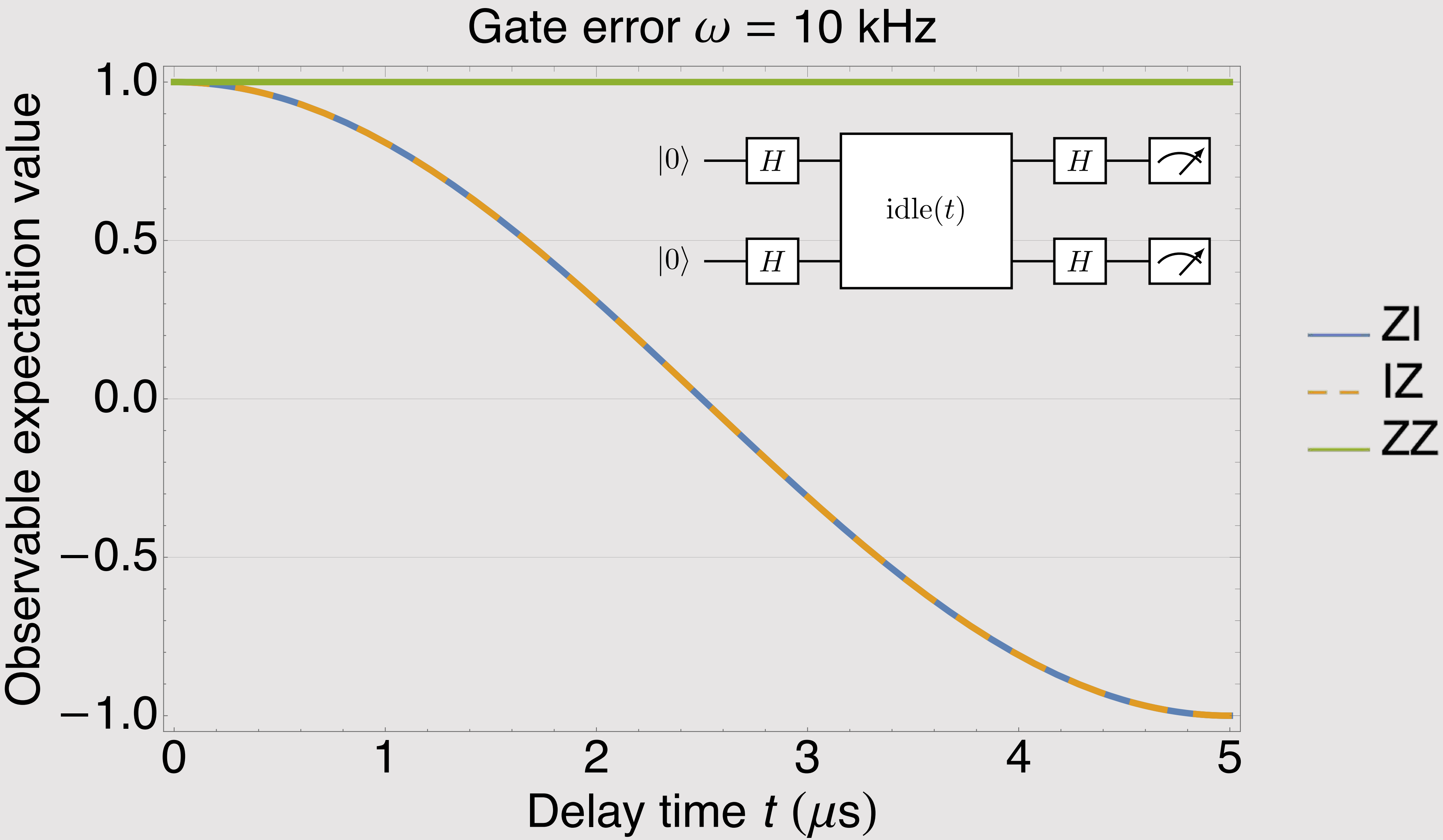
$$Z|-\rangle = |+\rangle$$

$$\langle ZI\rangle = \cos\omega t$$

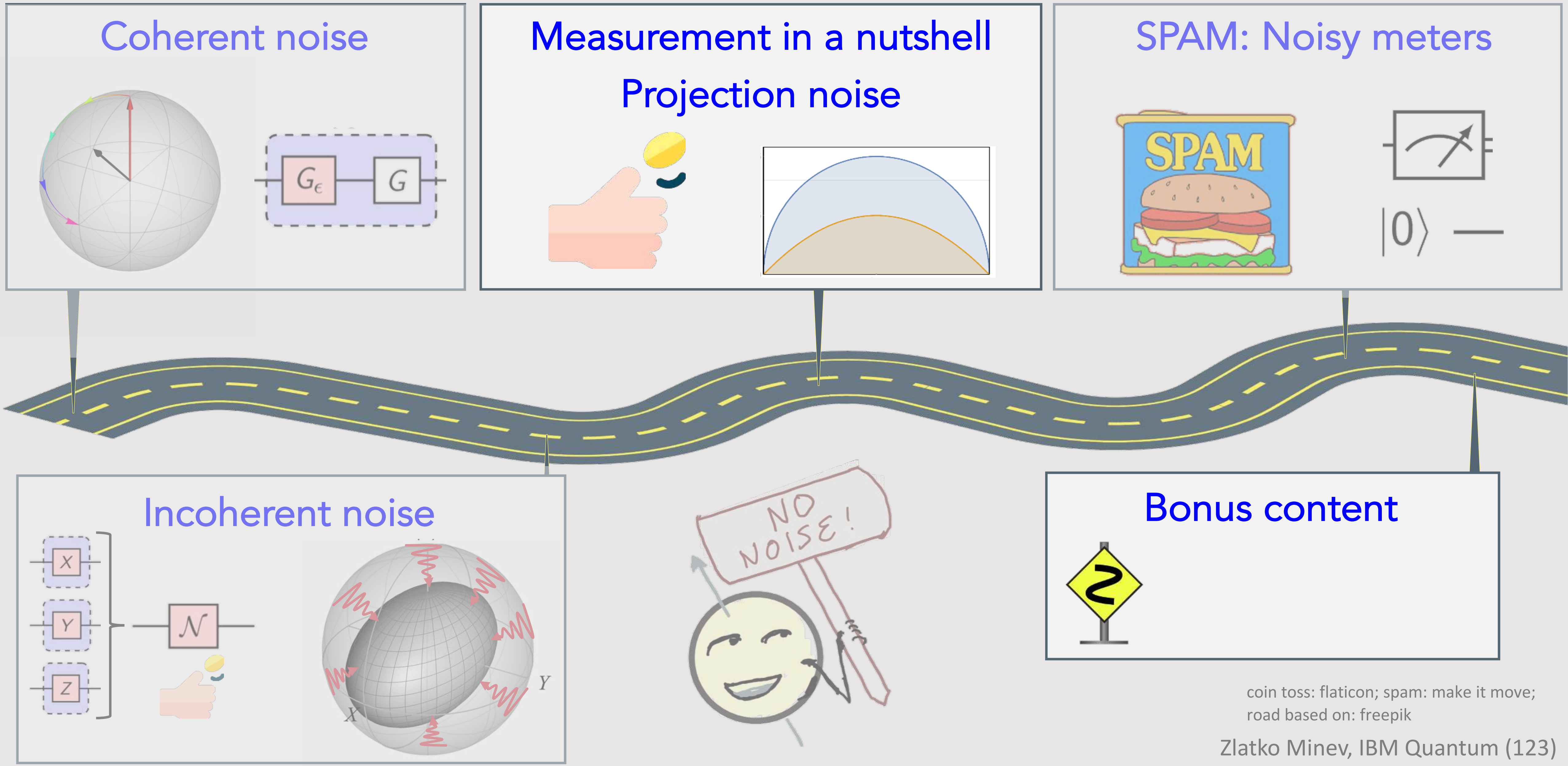
$$\langle IZ\rangle = \cos\omega t$$

$$\langle ZZ\rangle = 1$$

Two-qubit ZZ error plotted

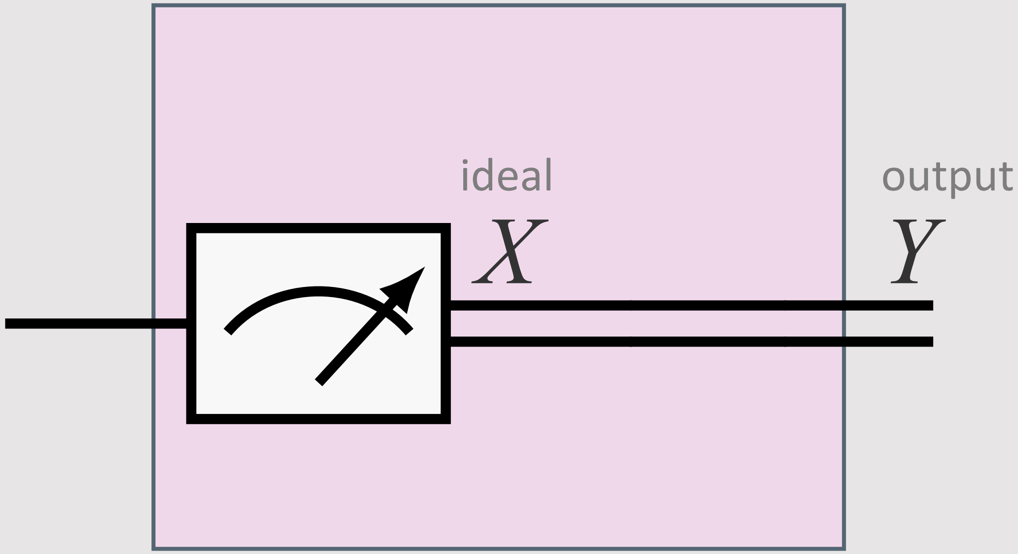


Bonus content



coin toss: flaticon; spam: make it move;
road based on: freepik

Recall the A matrix



$$\mathbf{A}_{\mathrm{ideal}} = \begin{array}{c|cc} & X=0 & X=1 \\ \hline Y=0 & 1 & 0 \\ Y=1 & 0 & 1 \end{array}$$

$$\mathbf{p}_{\mathrm{noisy}} = \mathbf{A}\mathbf{p}_{\mathrm{ideal}}$$

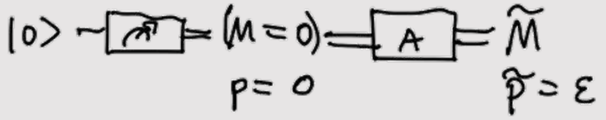
$$\mathbf{A} = \begin{array}{c|cc} & X=0 & X=1 \\ \hline Y=0 & 1-\epsilon & \nu \\ Y=1 & \epsilon & 1-\nu \end{array}$$

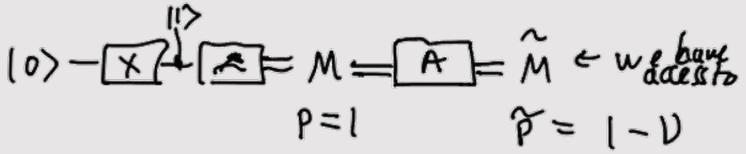
* Monster image by jcomp on Freepik

Deeper dive on the readout *A* matrix & Shannon entropy

Bonus section content:

Reconstruct A matrix





Noise mitigation

we know A

measure $\tilde{p}$, $\tilde{P}_M$ noisy

find p , P_M ideal

$\dim A = 2^n \times 2^n$ $n = \#\text{qubits}$

$$\tilde{P}_M = A P_M$$

$$P_M = A^{-1} \tilde{P}_M$$

$$p =$$

Assignment Fidelity

$$\mathcal{F}_0 = 1 - \tfrac{1}{2}\left[p(\tilde{M}=1 \mid M=0) + p(\tilde{M}=0 \mid M=1)\right]$$

$$= \tfrac{1}{d}\, \mathrm{Tr}(A) \qquad d = 2^n,\ n = \#\text{qubits}$$

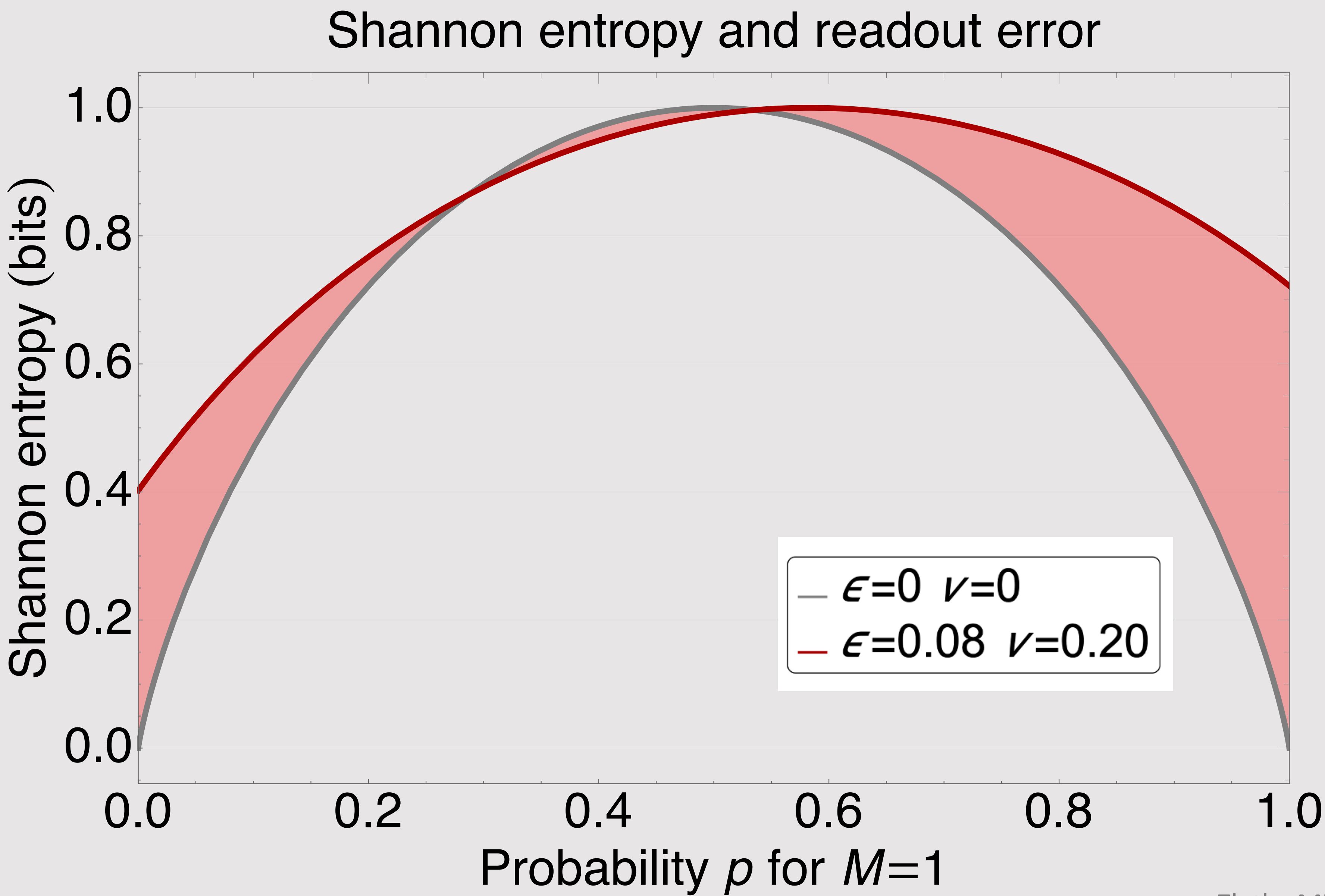
$$= 1 - \tfrac{1}{2}(\mu + \nu)$$

Shannon Entropy

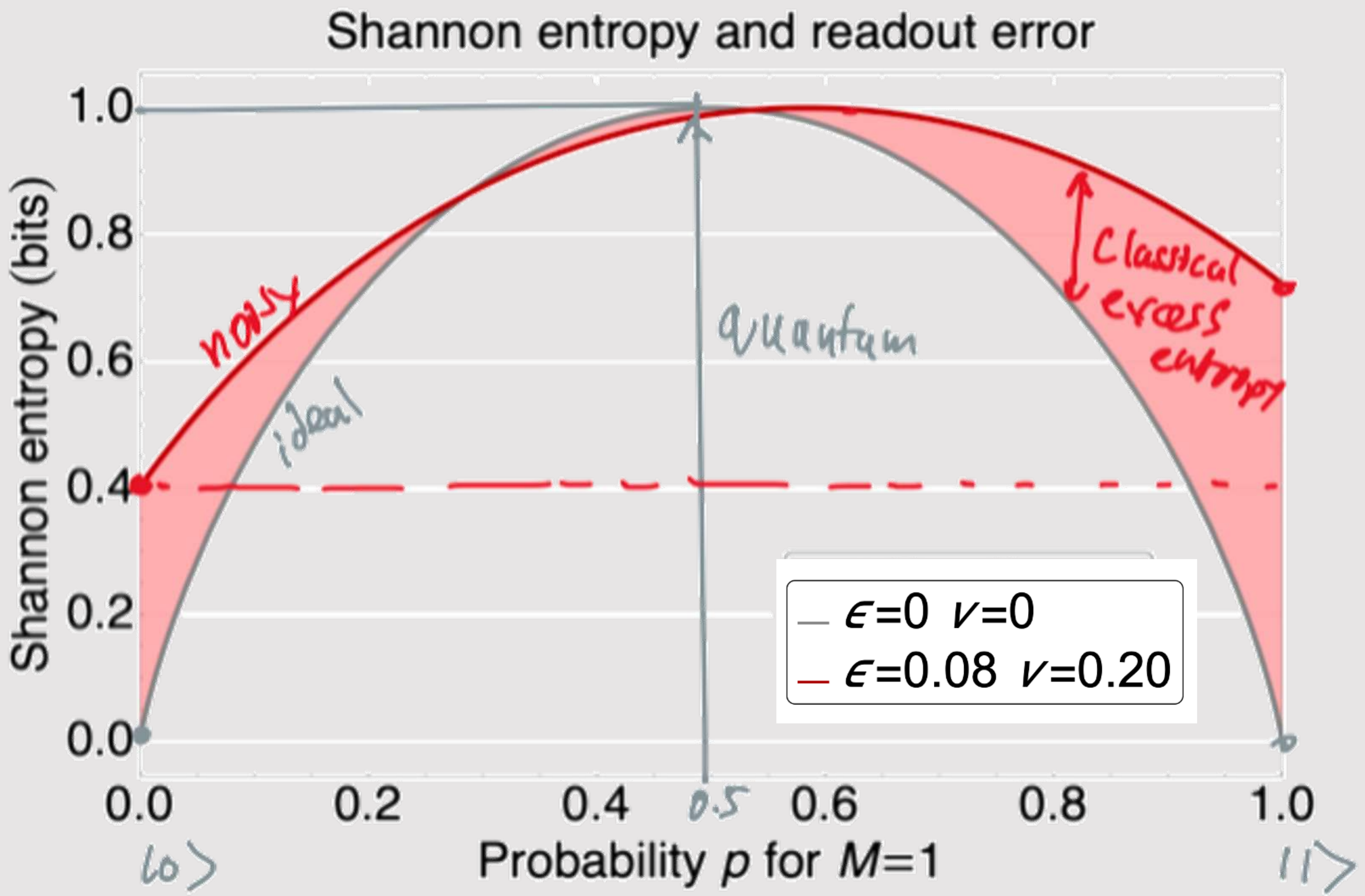
$$\mathcal{H}(M) = \mathcal{H}(P_M) = -\sum_m P_m \log_2 P_m = -(1-p)\log_2(1-p) - p\log_2 p$$

Binary entropy

Entropy


Shannon entropy and readout error
Shannon entropy (bits)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Probability p for M=1
ϵ=0 ν=0
ϵ=0.08 ν=0.20

Entropy



Shannon entropy and readout error
Shannon entropy (bits)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.5
0.6
0.8
1.0
Probability p for M=1
|0⟩
|1⟩
noisy
ideal
quantum
classical excess entropy
ϵ=0 ν=0
ϵ=0.08 ν=0.20

The End!