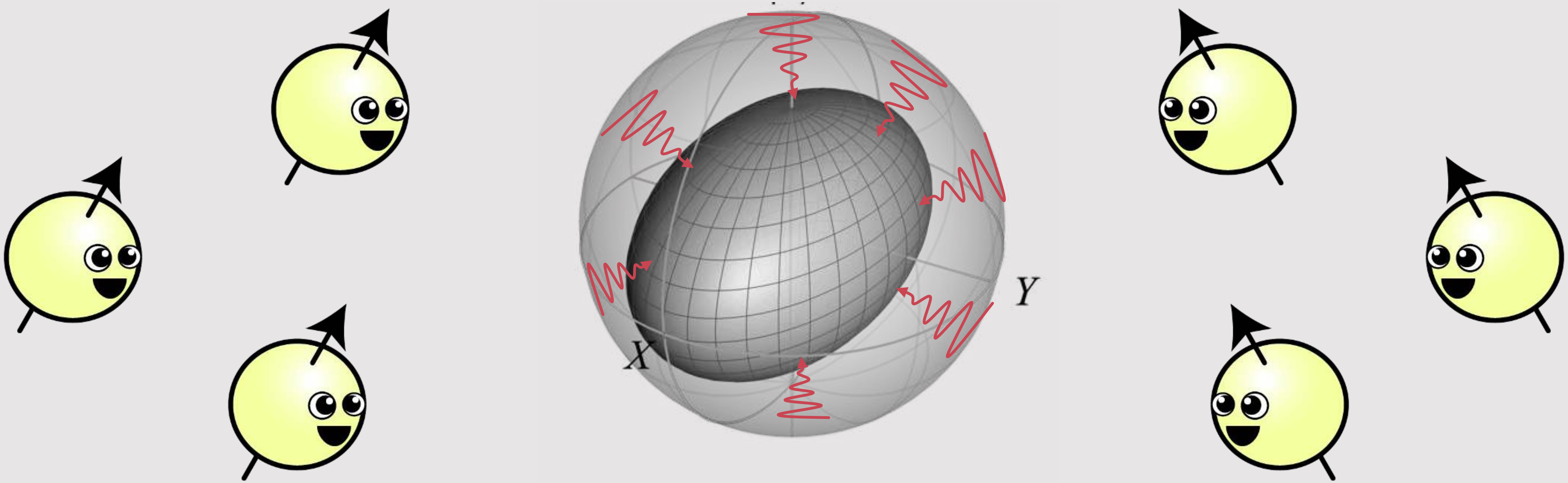
Introduction to Quantum Noise

IBM **Quantum**

Qiskit Global Summer School: Quantum Simulations

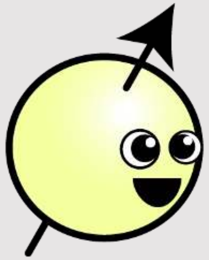


Zlatko K. Minev

IBM Quantum

@zlatko_minev   zlatko-minev.com

What do I need to know before I run quantum simulation on a real, noisy quantum processor?



Chapter 1: Hello World!

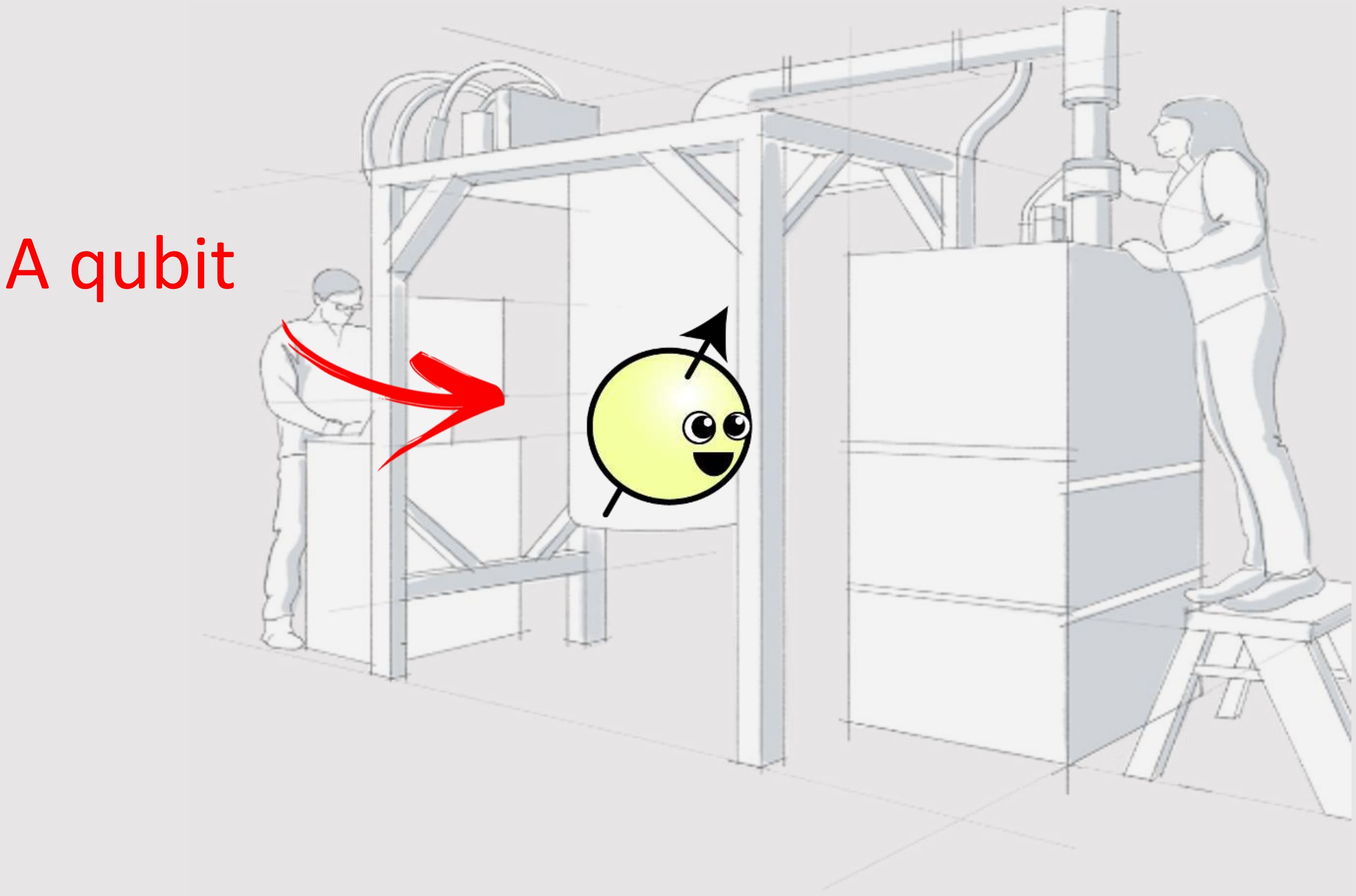
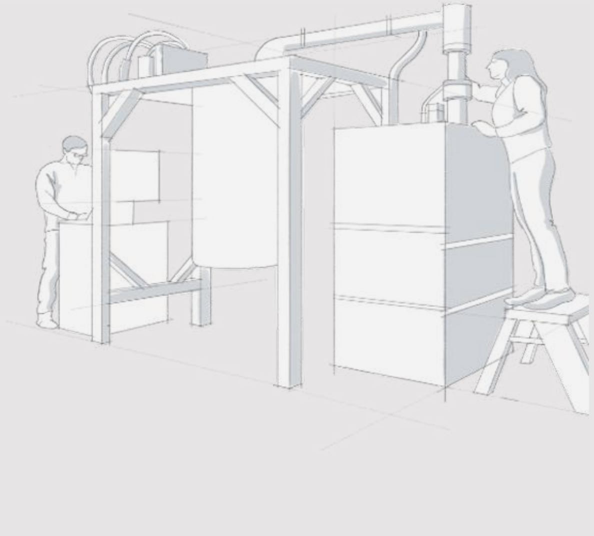
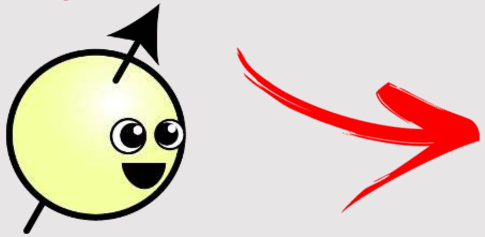


image: qiskit

Hello World! building blocks



A qubit



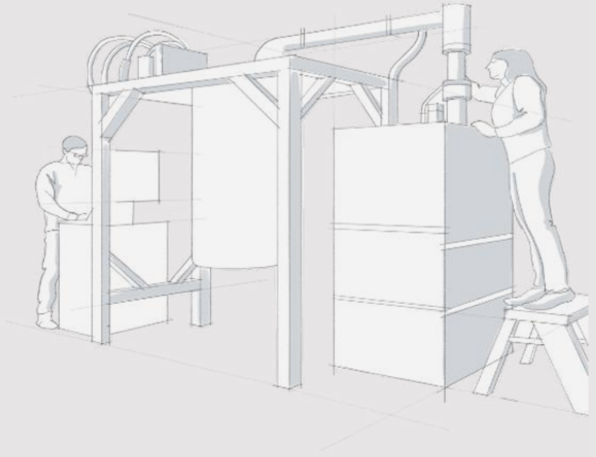
$|1\rangle$

$|0\rangle$

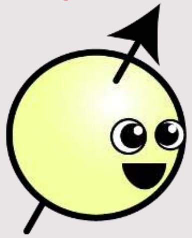
Computational basis states

image: qiskit

Hello World! building blocks



A qubit



$|1\rangle$

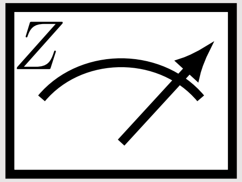
$|0\rangle$

Computational basis states

Operations: qubit gate

$$\boxed{X}$$

Measurements: qubit observable

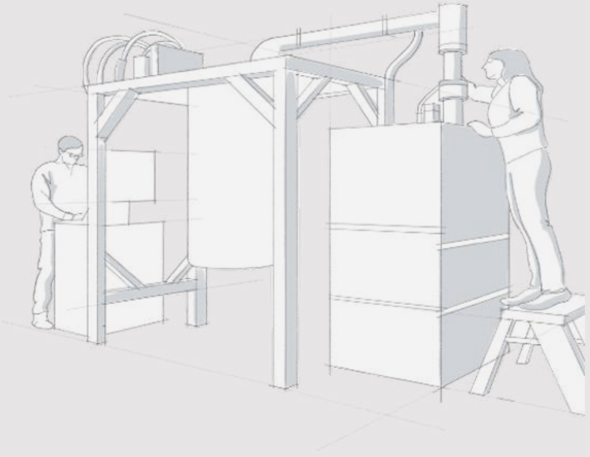


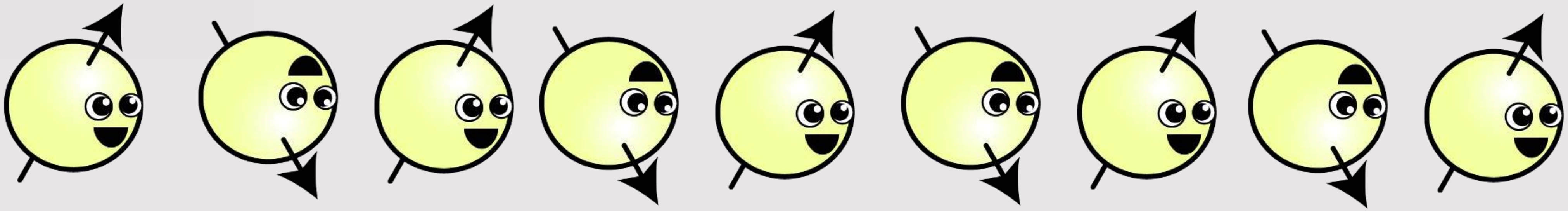
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! Even-odd algo: qubit flipper





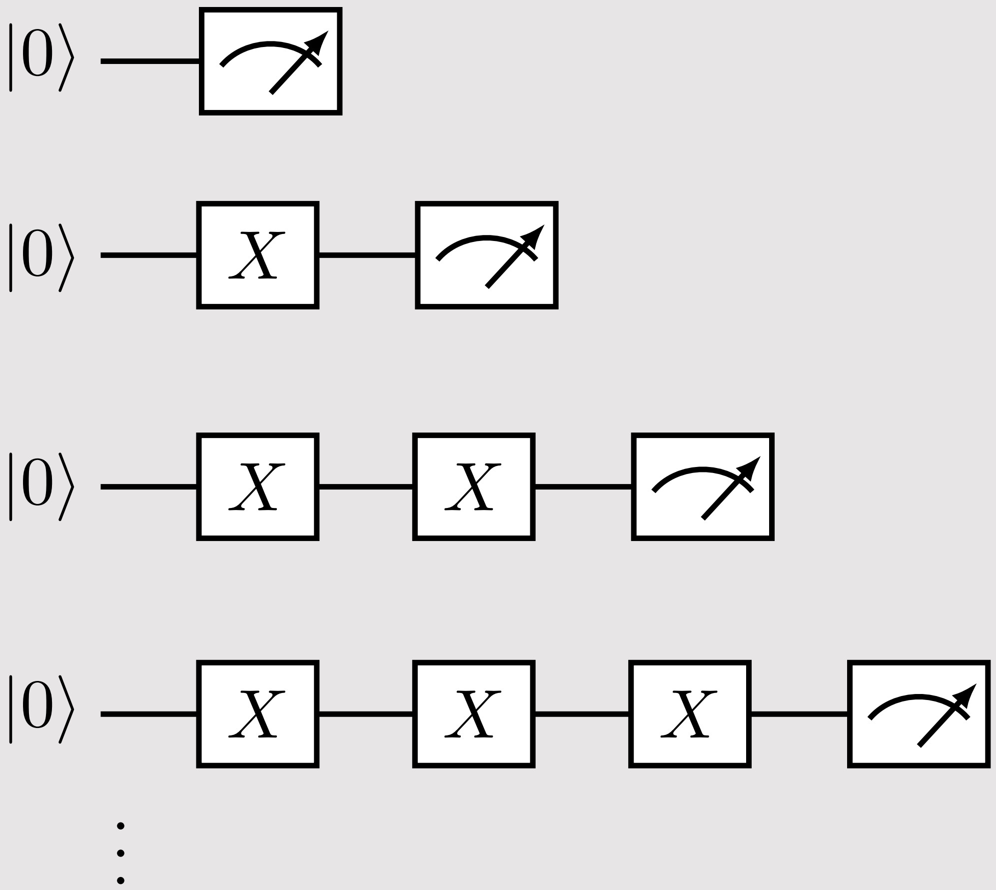
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! qubit flipper quantum circuits

depth



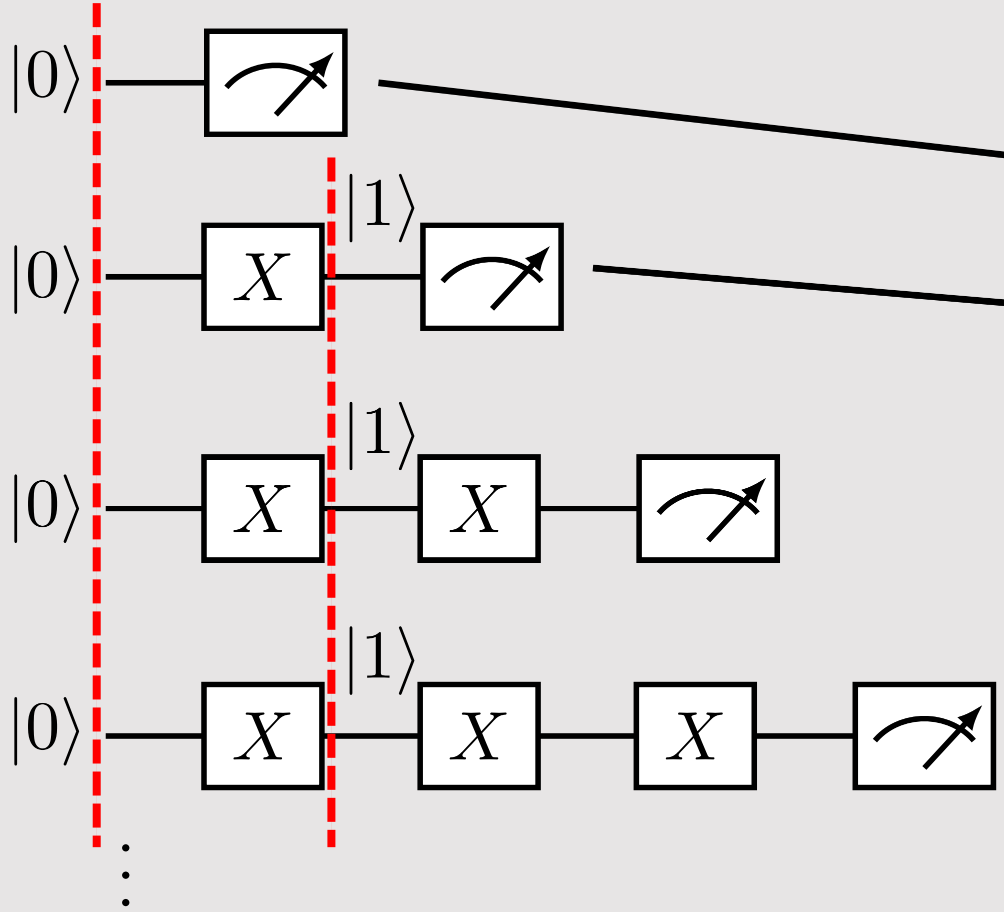
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

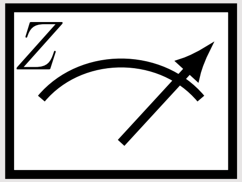
$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! "debugger" step through

depth



Measurement



$$\langle 0|Z|0\rangle = +1$$

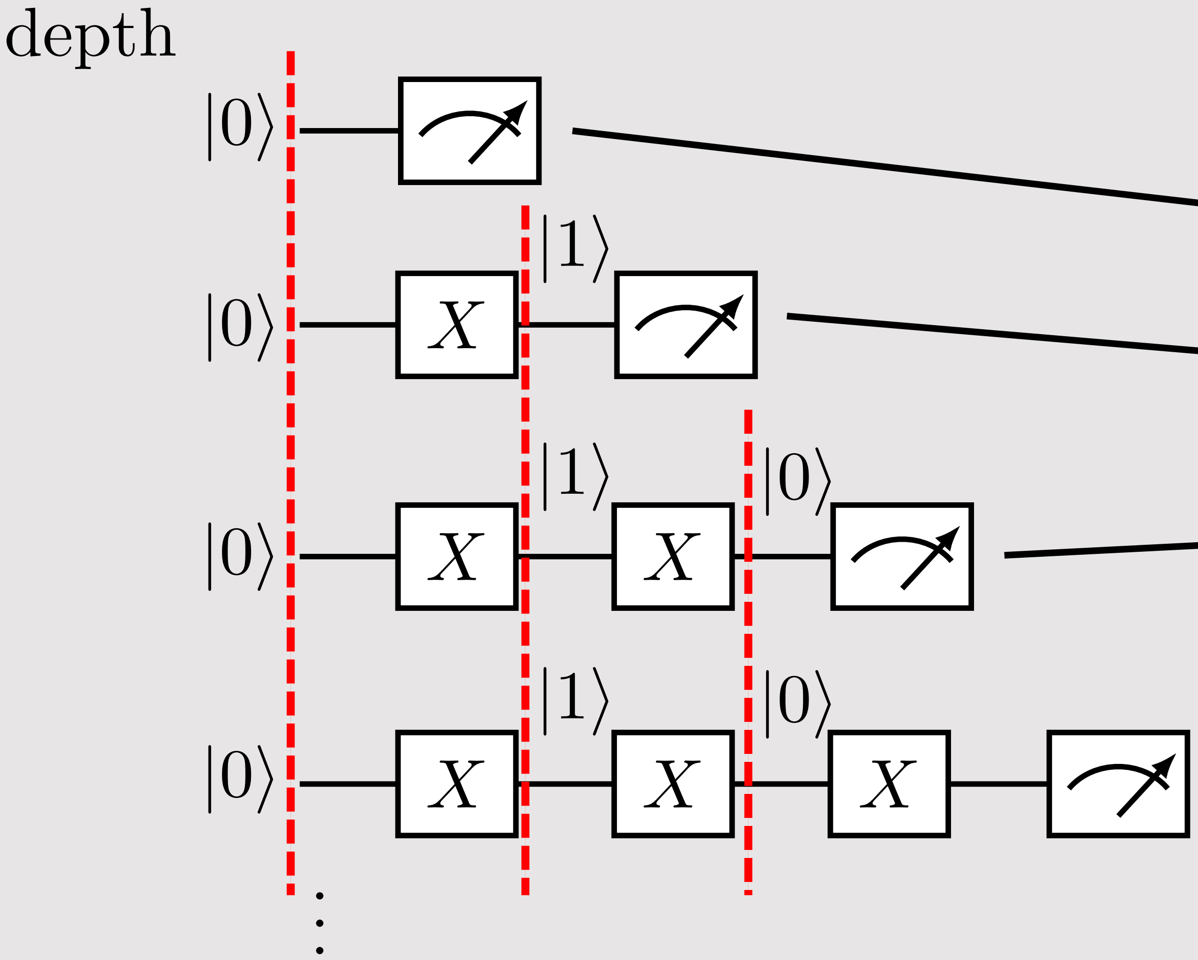
$$\langle 1|Z|1\rangle = -1$$

refresher:

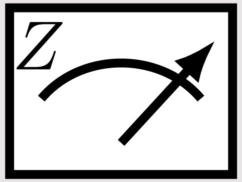
$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! "debugger" step through



Measurement



$$\langle 0|Z|0\rangle = +1$$

$$\langle 1|Z|1\rangle = -1$$

$$\langle 0|Z|0\rangle = +1$$

refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

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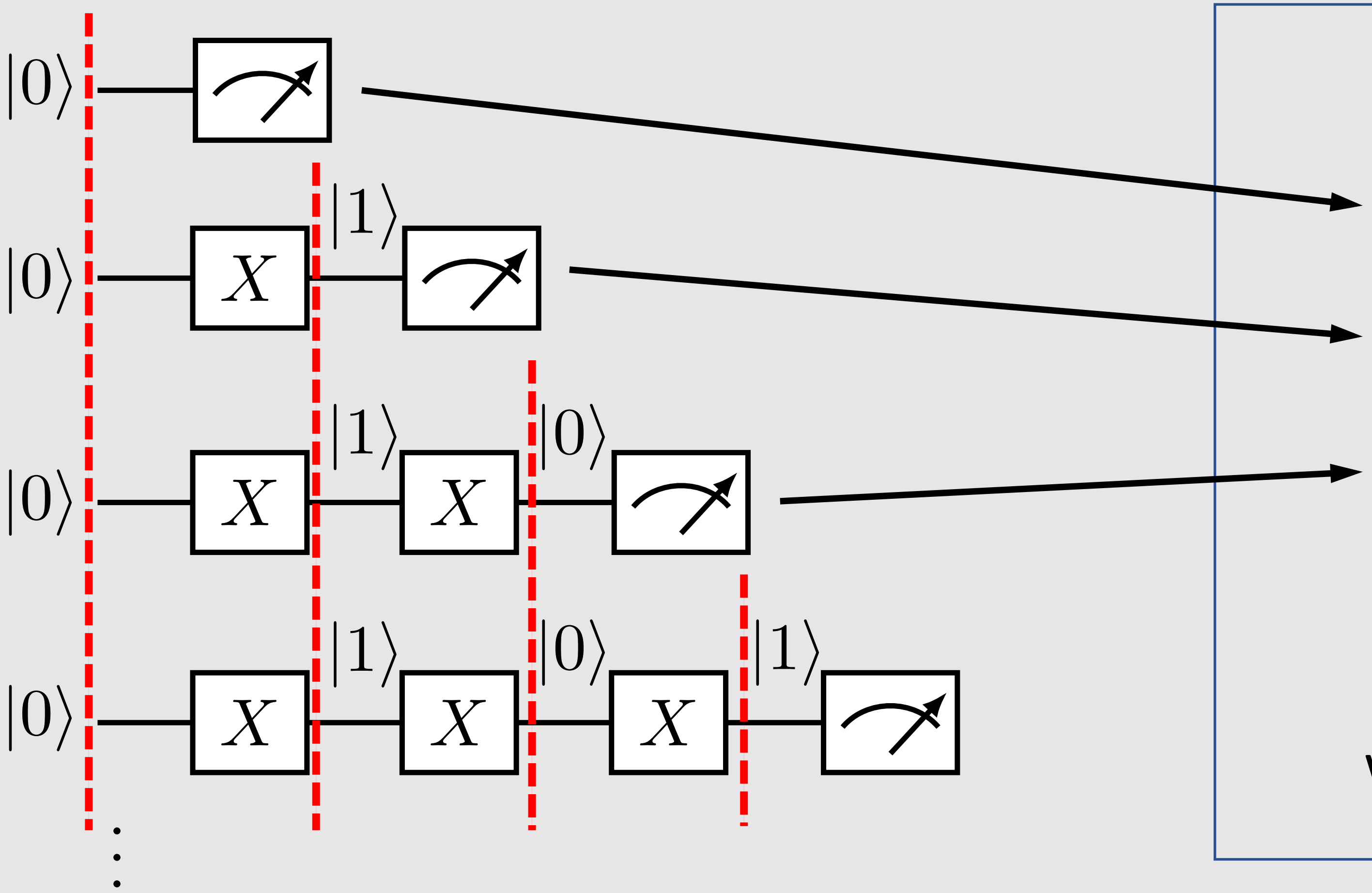
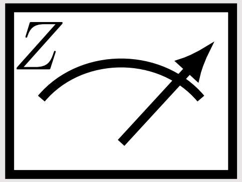
Hello World! "debugger" step through

depth



Measurement



$$\langle 0|Z|0\rangle = +1$$

$$\langle 1|Z|1\rangle = -1$$

$$\langle 0|Z|0\rangle = +1$$

$$\langle Z\rangle = (-1)^d$$

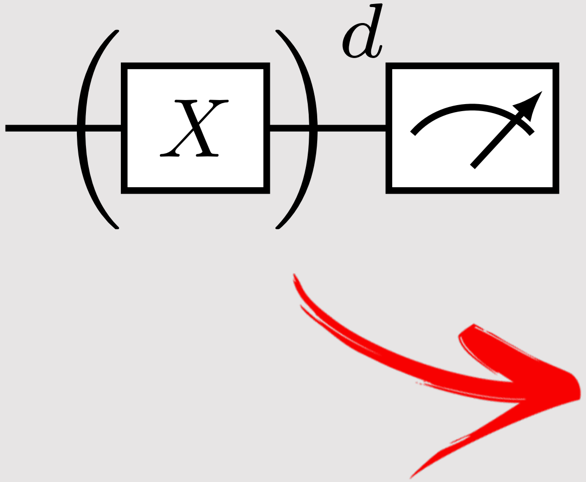
where d is the circuit depth

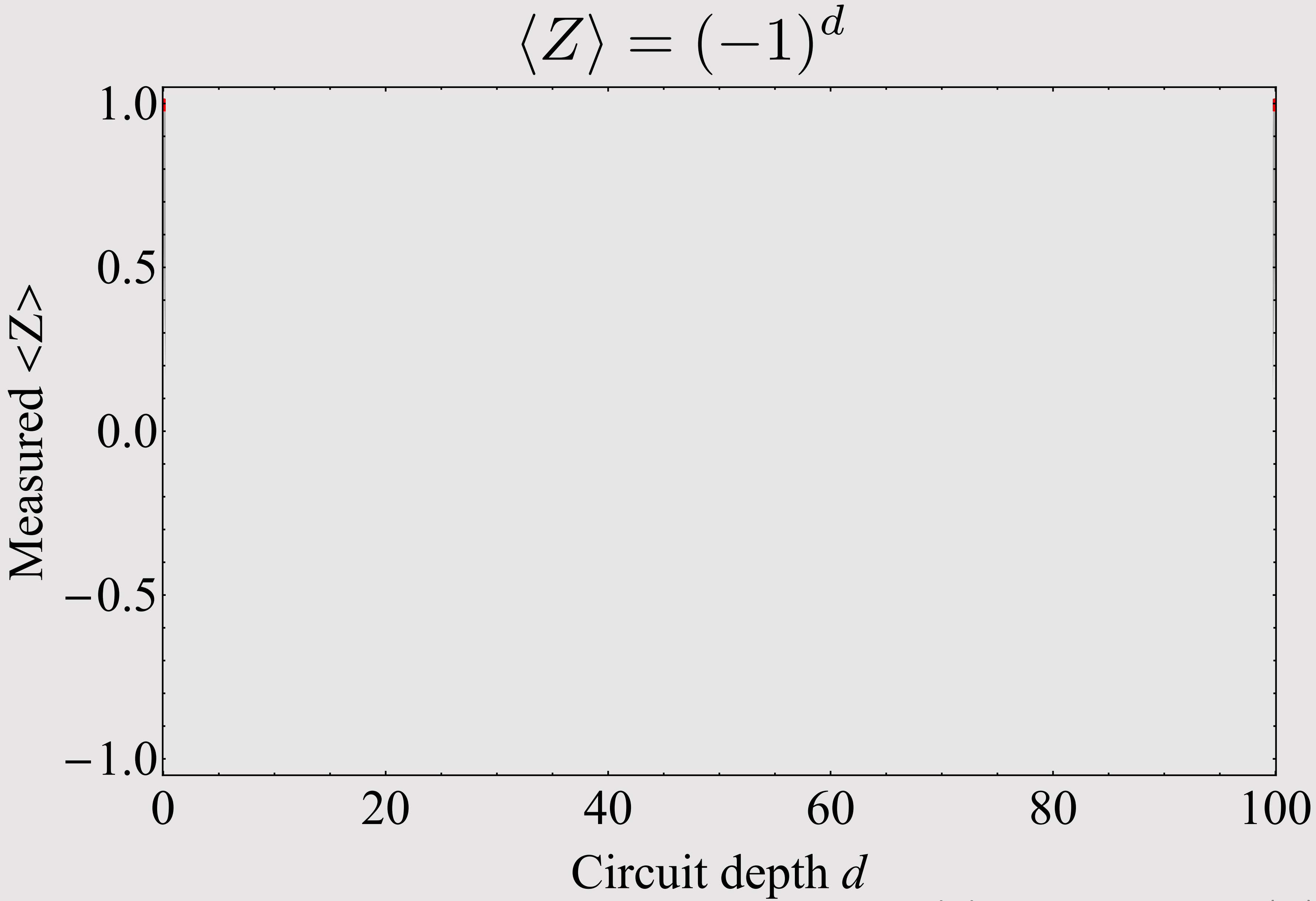
refresher:

$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

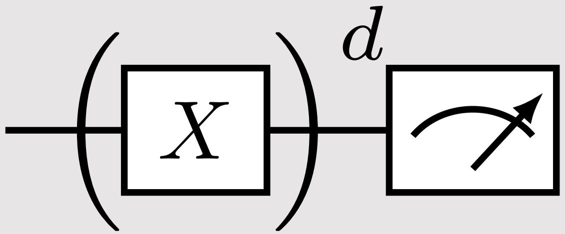
$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! Ideal expectation results

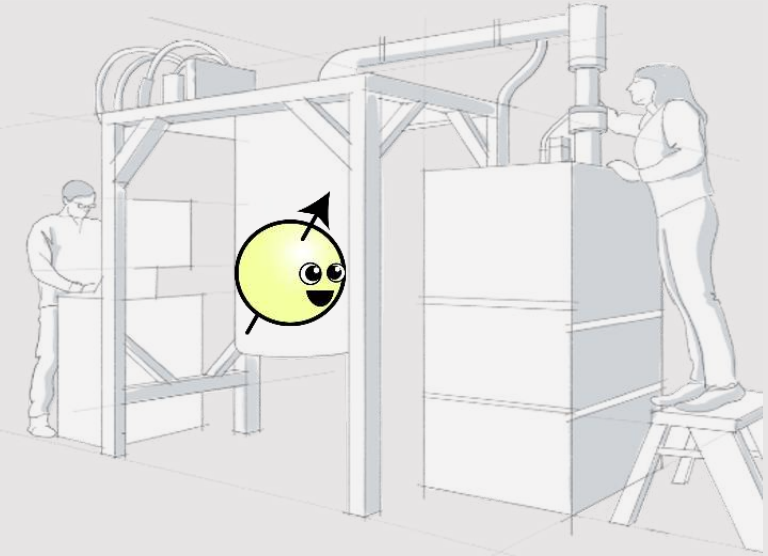




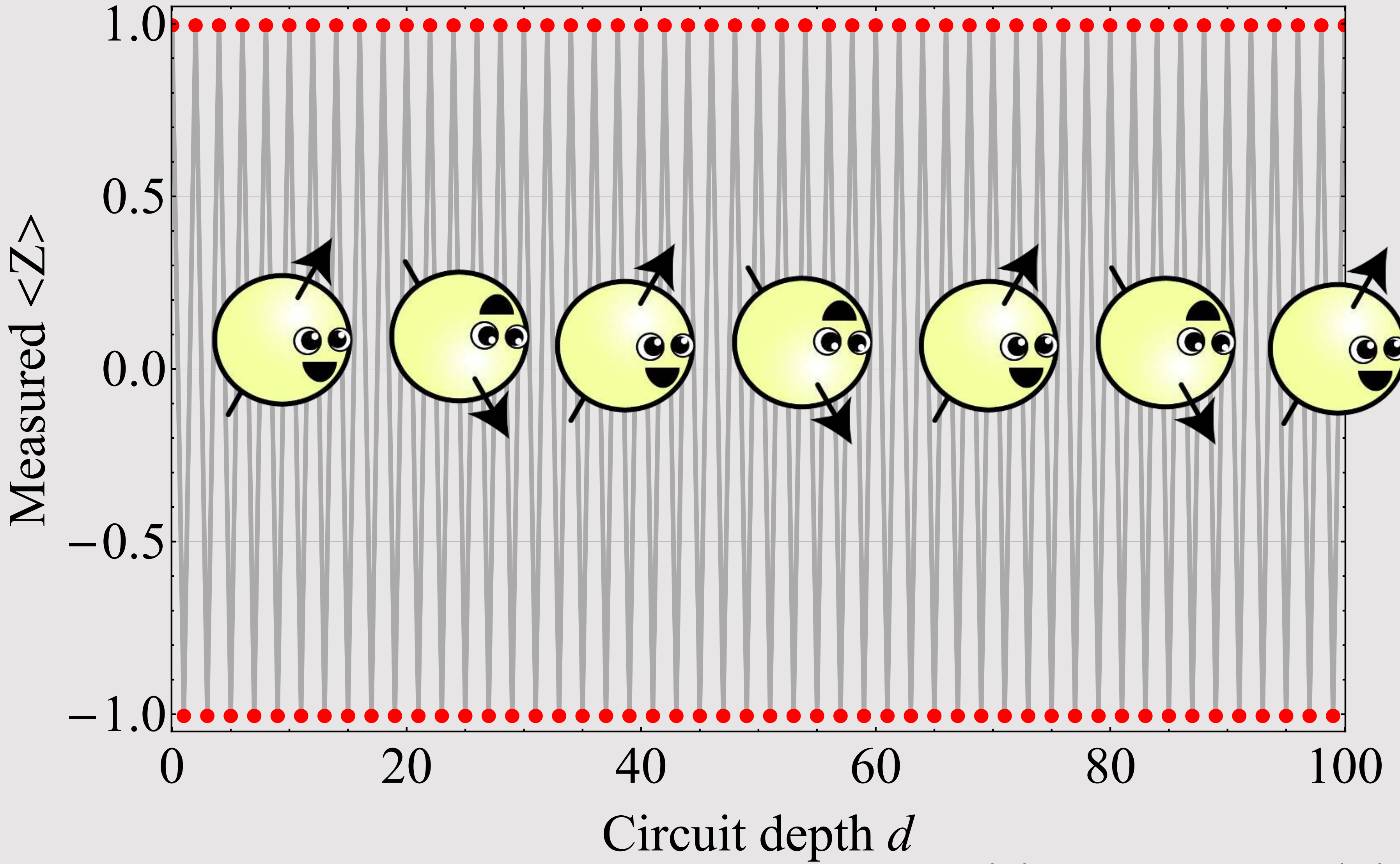
Hello World! Ideal expectation results



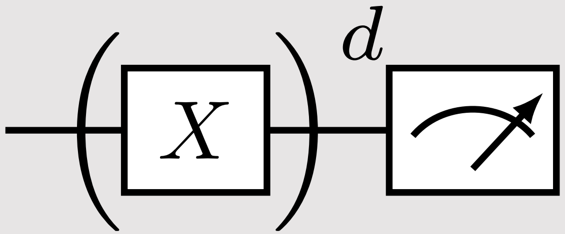
Let's run on a real device!



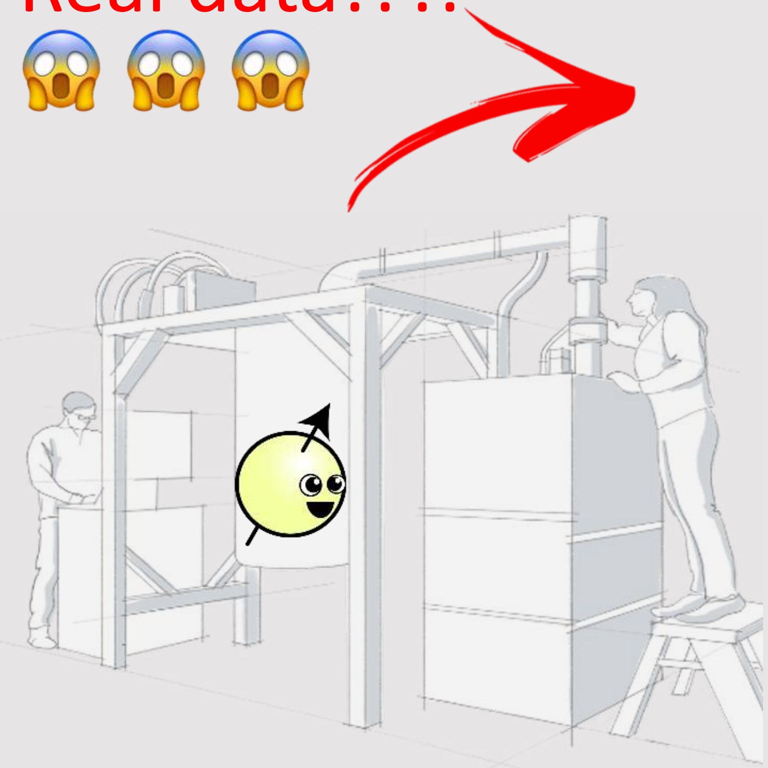
$$\langle Z \rangle = (-1)^d$$

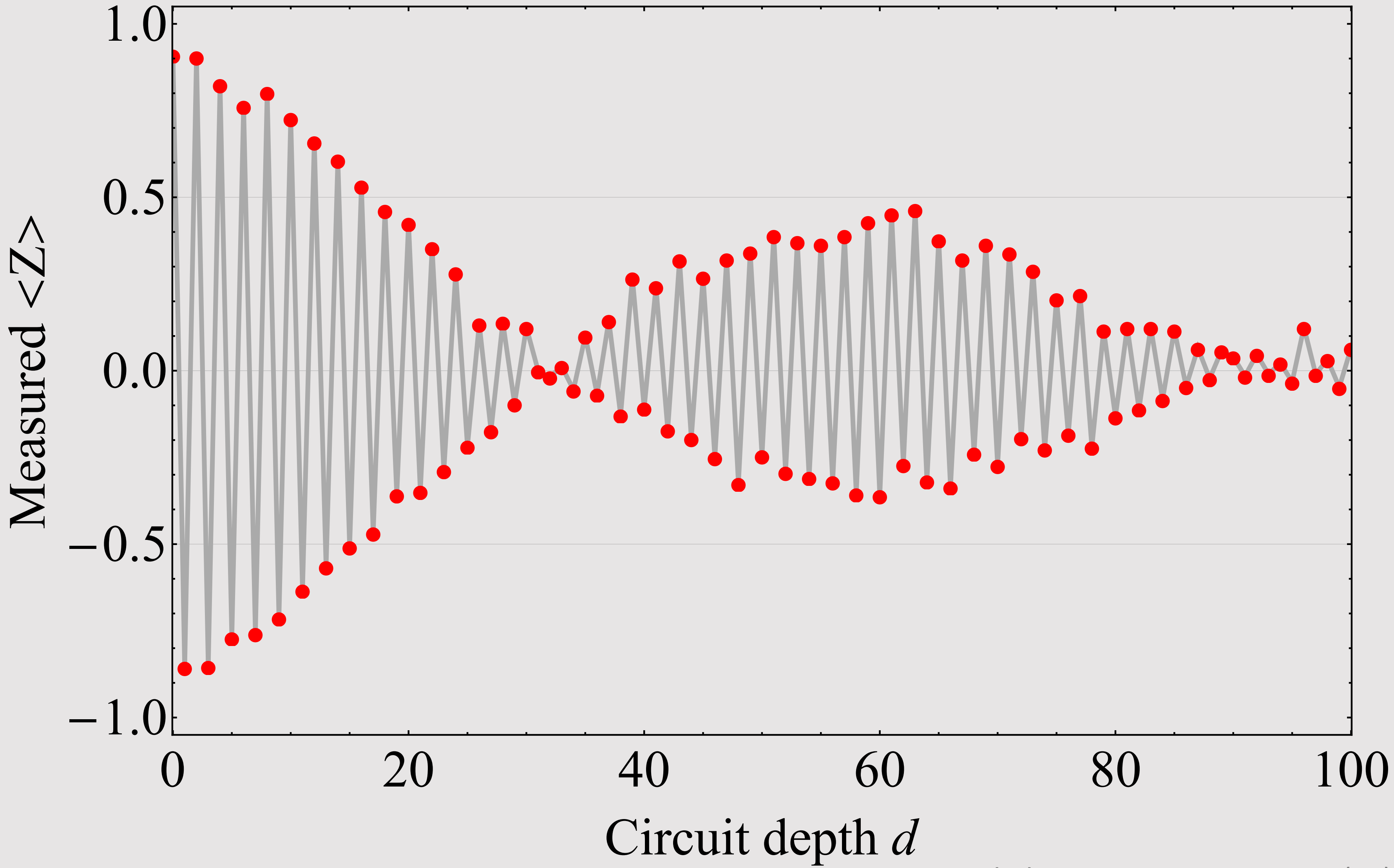


Hello World! Real expectation results



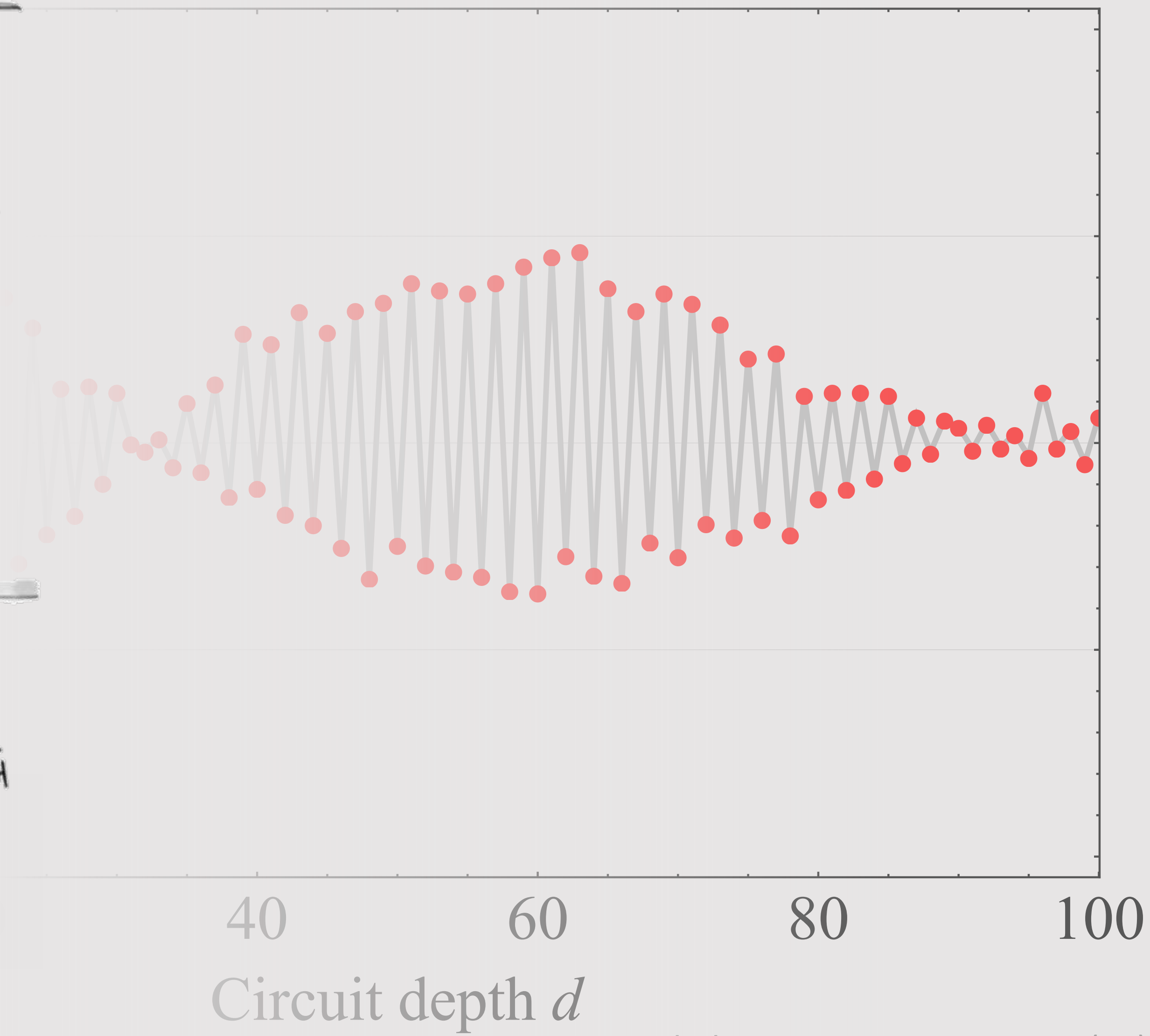
Real data!?!!



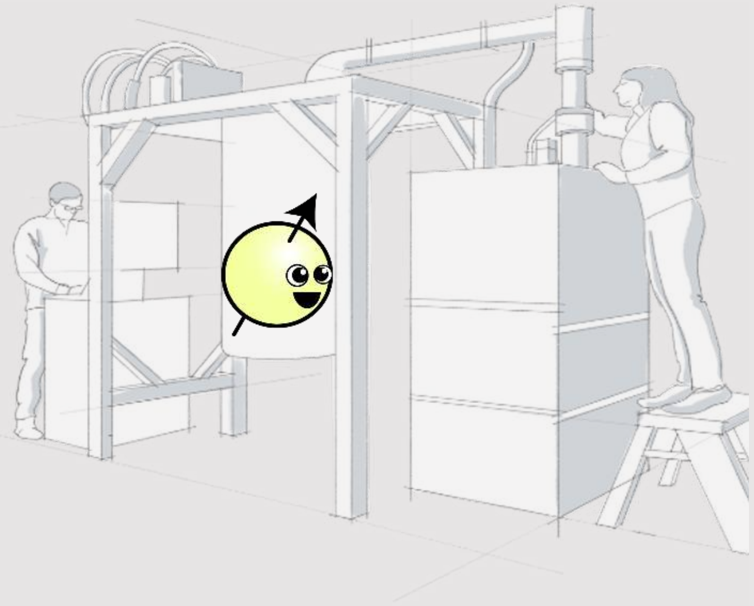


Real & noisy quantum processors: Why study noise?

"Well, your quantum computer is broken in every way possible simultaneously."



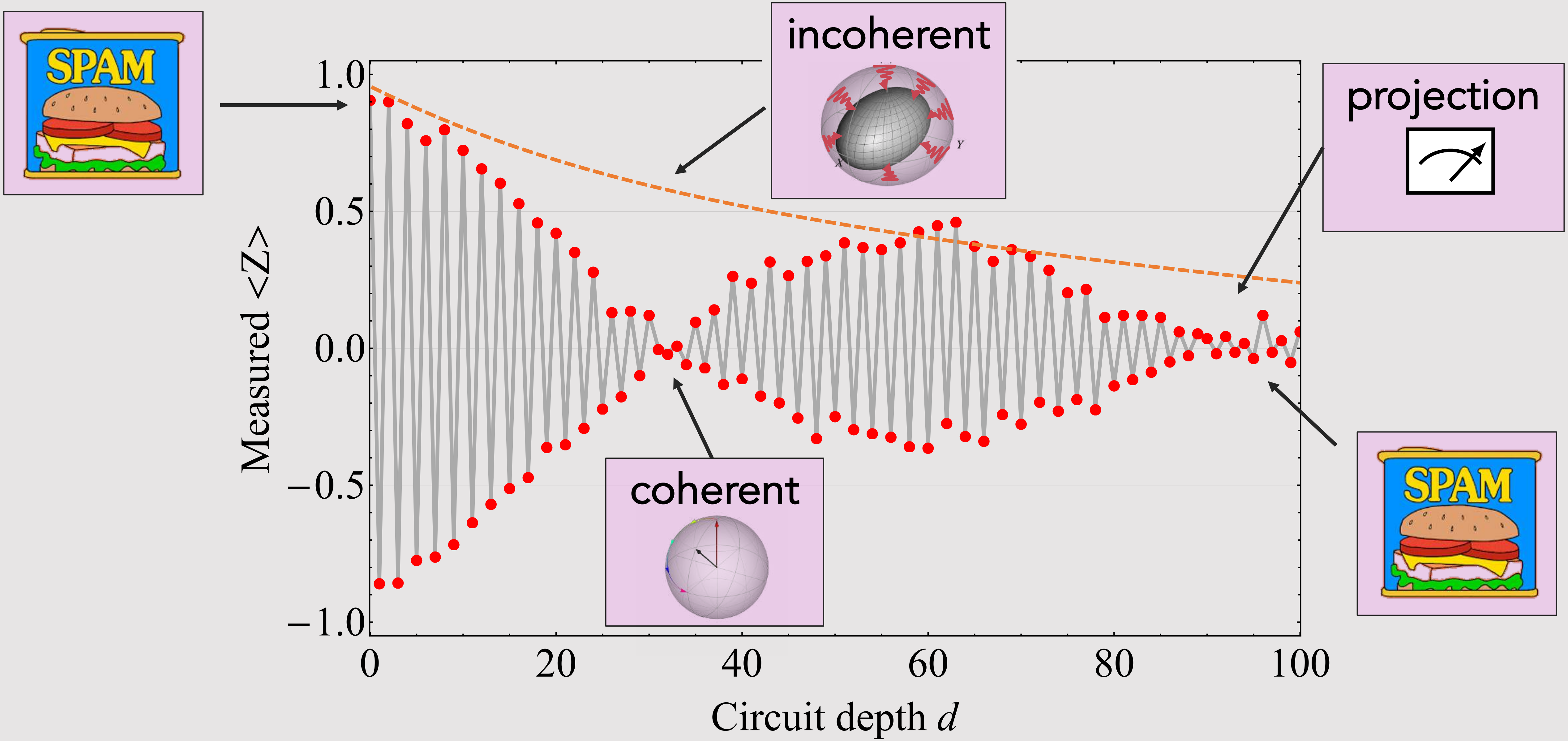
Cartoon: Paul Noth



"Quantum phenomena
do *not* occur in a Hilbert space,
they occur in a laboratory."

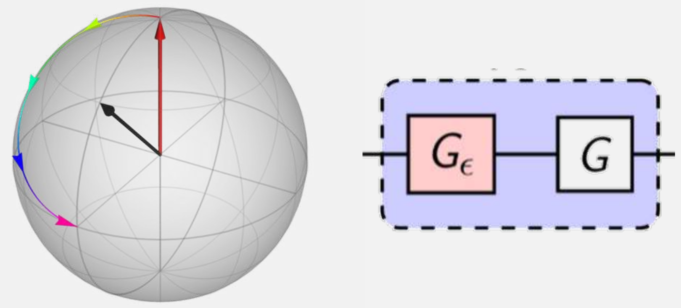
Asher Peres

Elements of 😱 noise



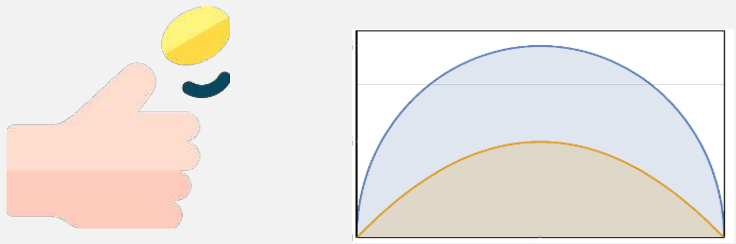
The road ahead

Coherent noise

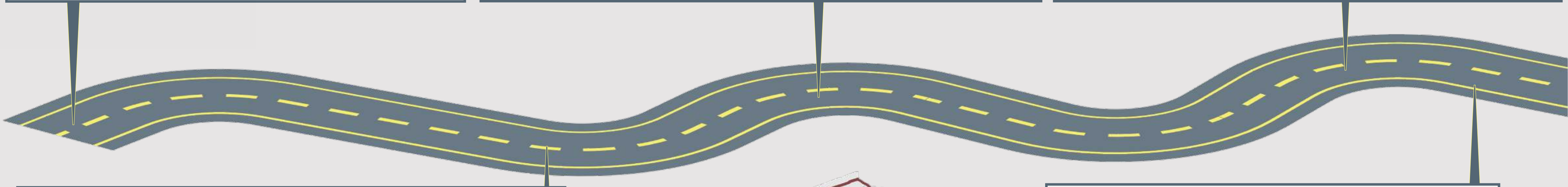


Measurement in a nutshell

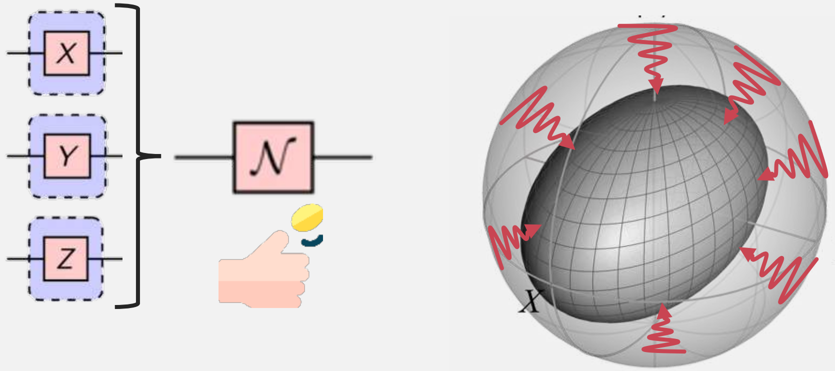
Projection noise



SPAM: Noisy meters



Incoherent noise



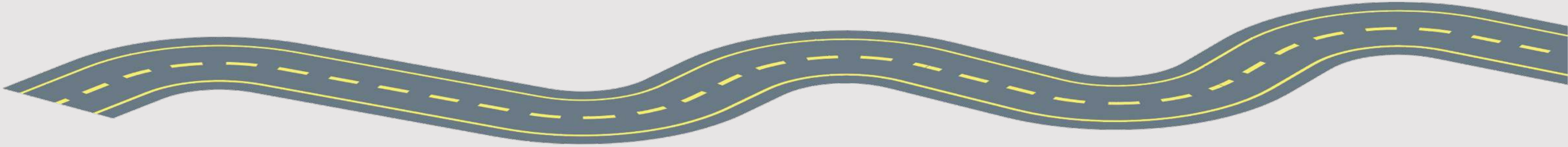
Bonus content

Coherent ZZ

State preparation

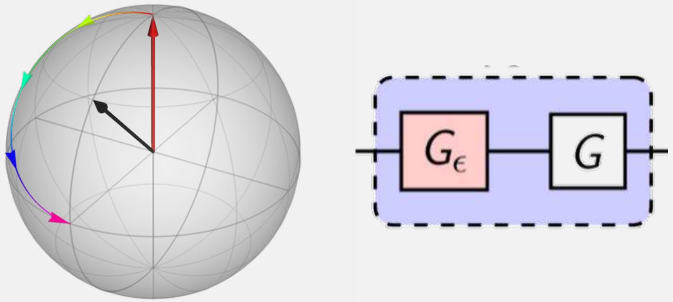
coin toss: flaticon; spam: make it move;
road based on: freepik

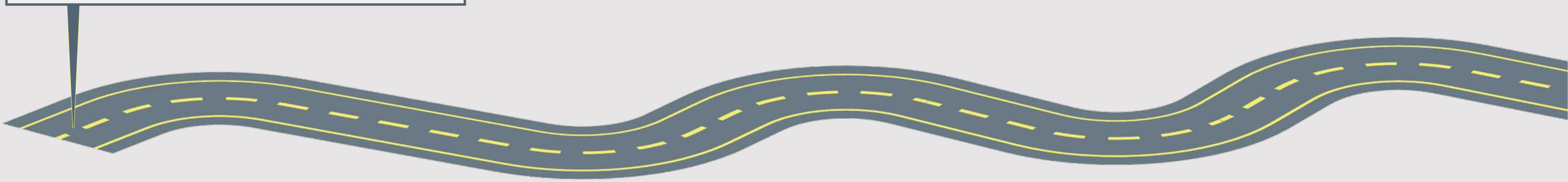
The destination is the journey



Chapter 2: Coherent noise

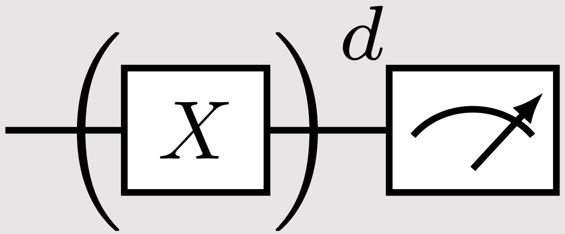
Coherent

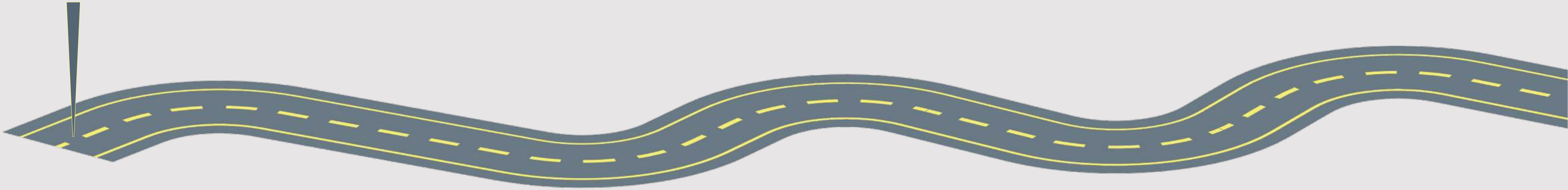




road based on: freepik

Return to the Hello World example



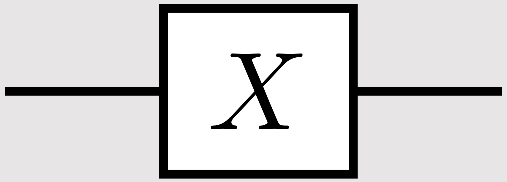


road based on: freepik

Origin of our X gate: time evolution

$$X = R_X(\pi)$$

(up to global phase)



Refresher:

$$\hat{H} = \frac{\hbar\omega}{2} X$$

$$= \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U(t) = \exp\left(-it\hat{H}/\hbar\right)$$

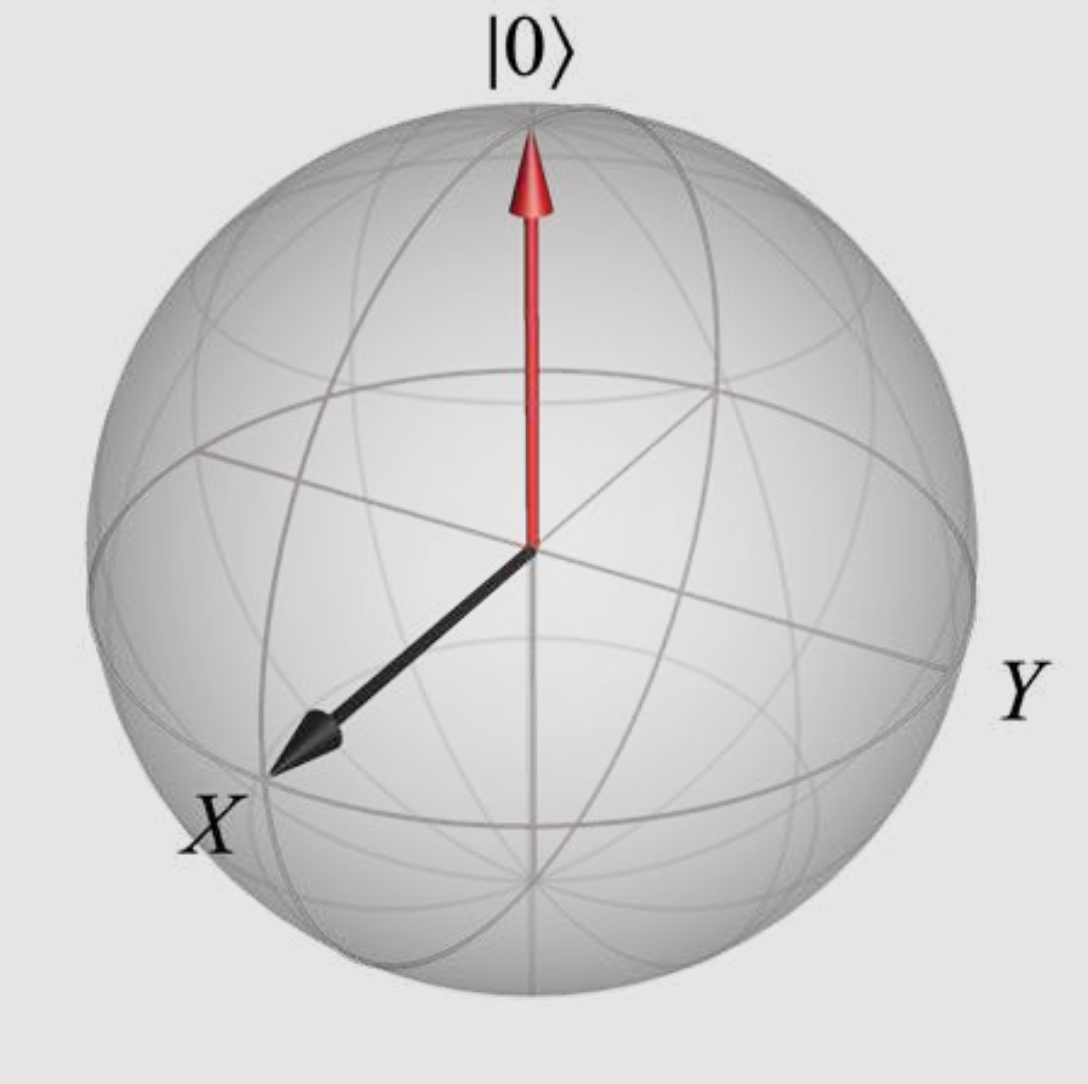
$$\theta := \omega t$$

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2} X\right)$$

$$= \cos\left(\theta/2\right) I - i \sin\left(\theta/2\right) X$$

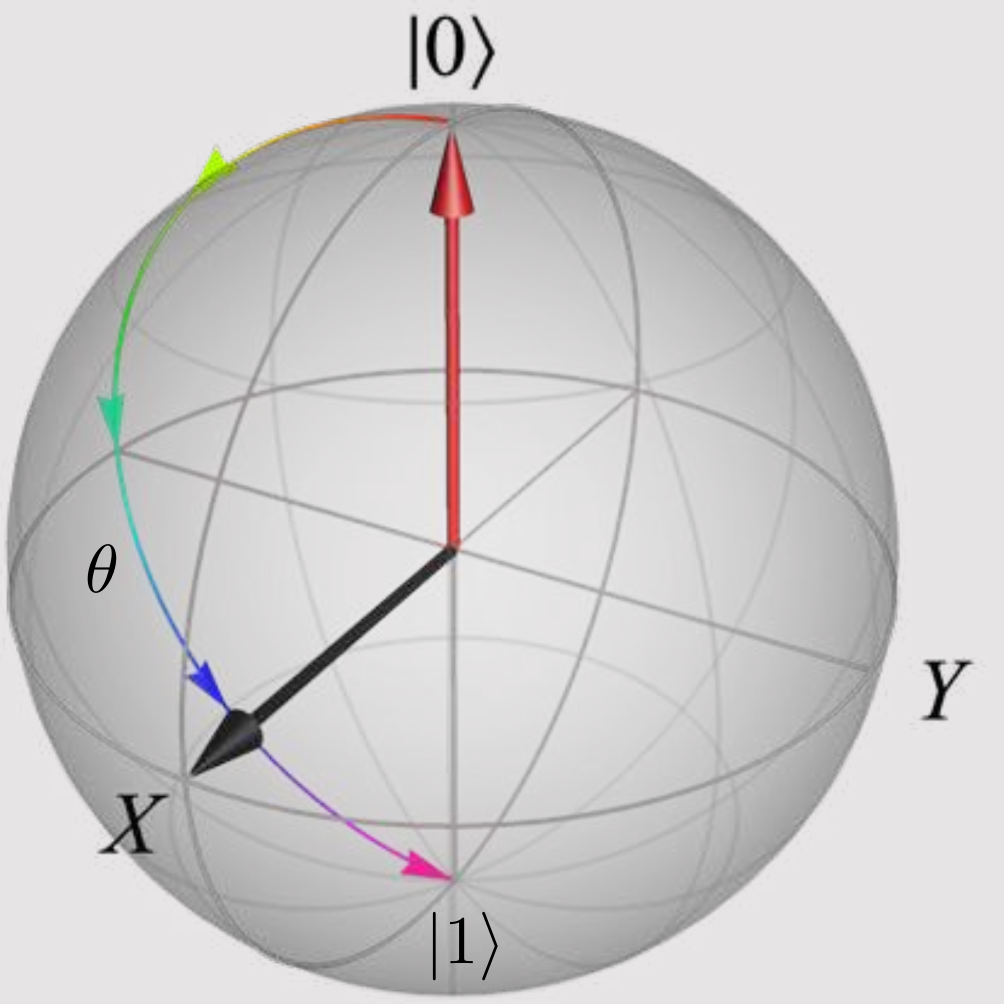
* We will often drop hats on Paulis *I*, *X*, *Y*, *Z*

Visualize: Bloch sphere



Visualize: Evolution on the Bloch sphere

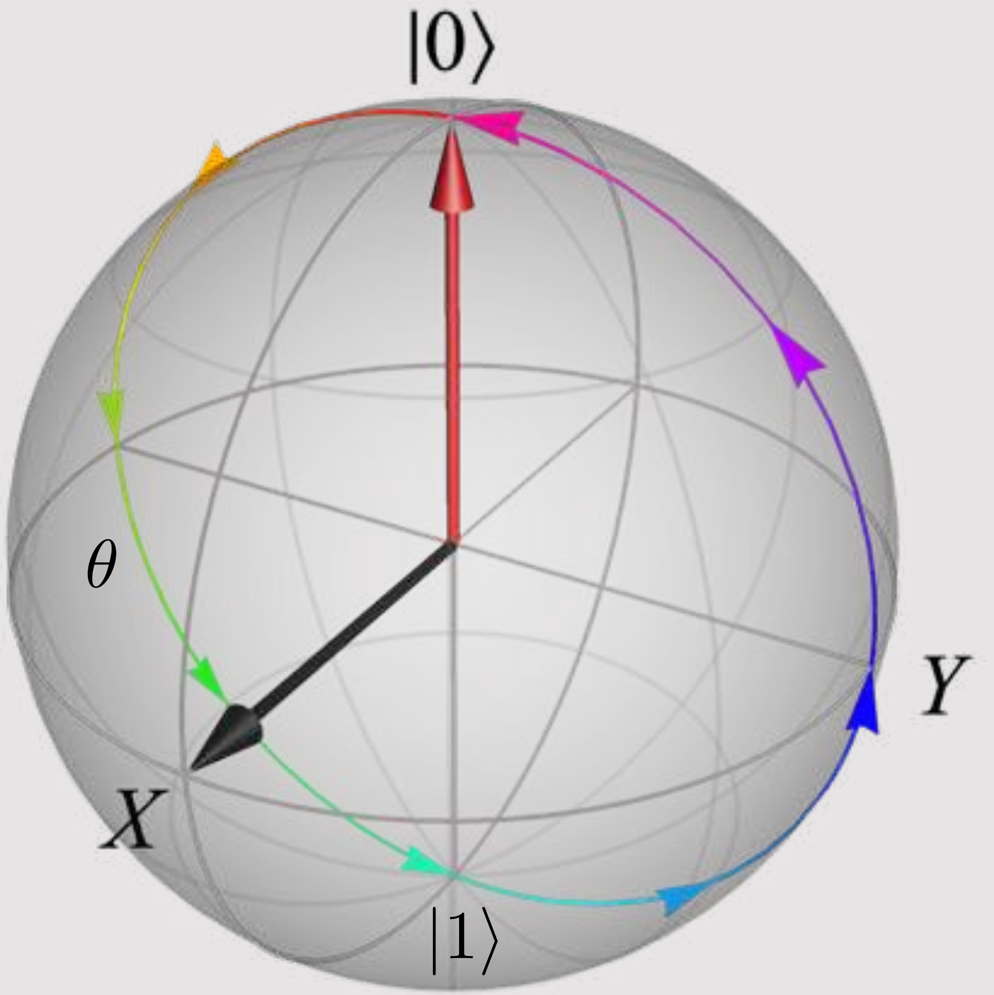
$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$



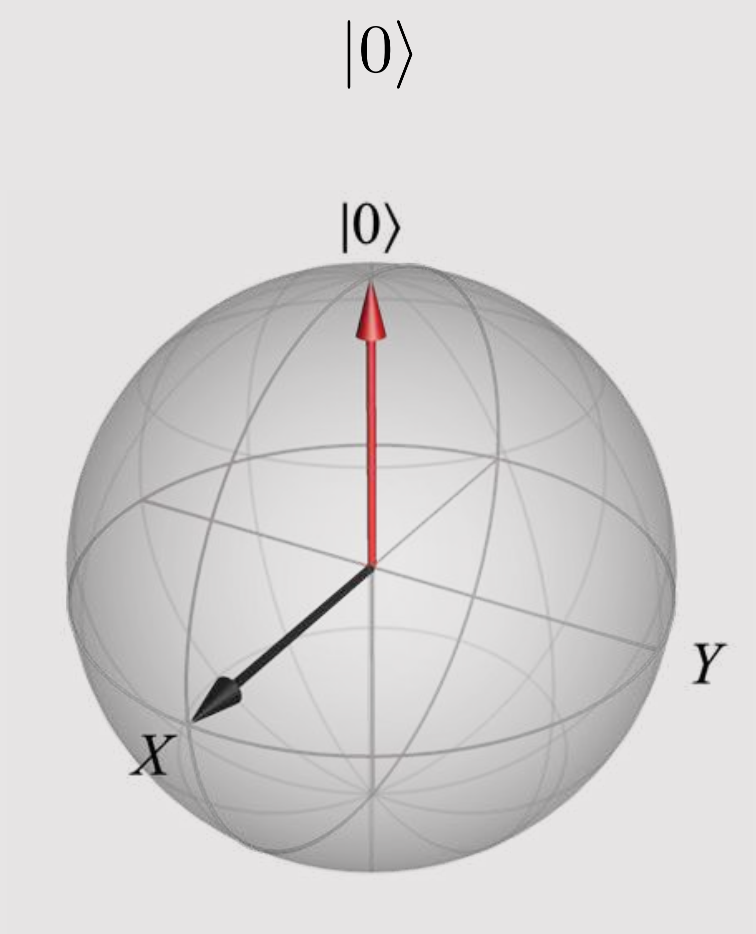
Visualize: Evolution on the Bloch sphere

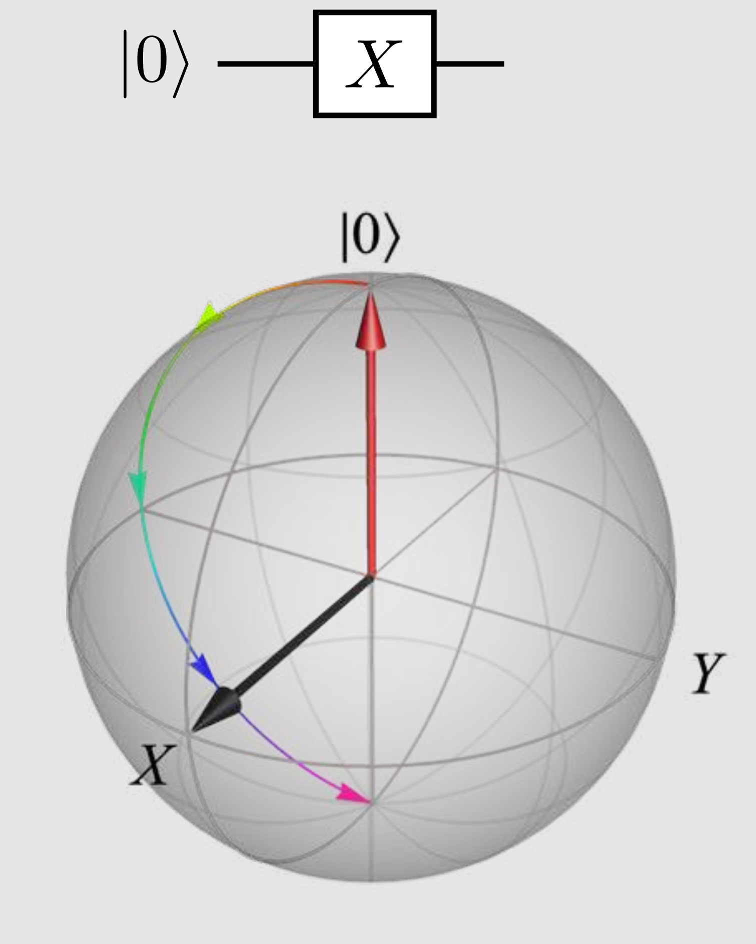
$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$

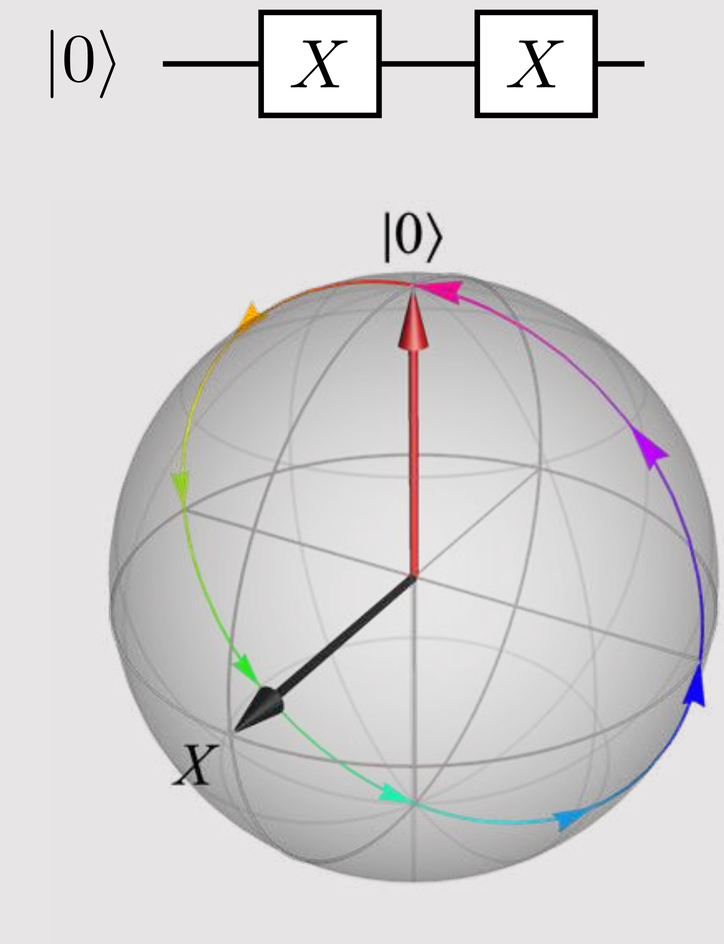
$$X\left|0\right\rangle = \left|1\right\rangle$$
$$X\left|1\right\rangle = \left|0\right\rangle$$



Evolution on the Bloch sphere

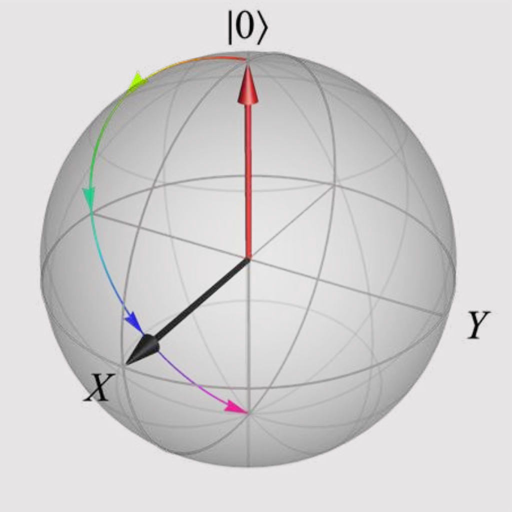


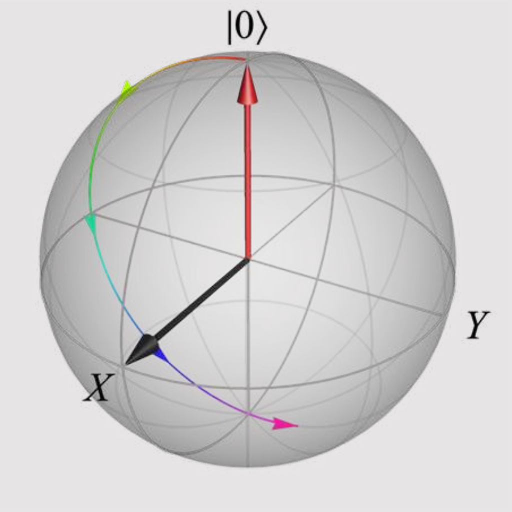




Miscalibrated gate

Ideal gate	Noisy gate
X	$\tilde{X}$



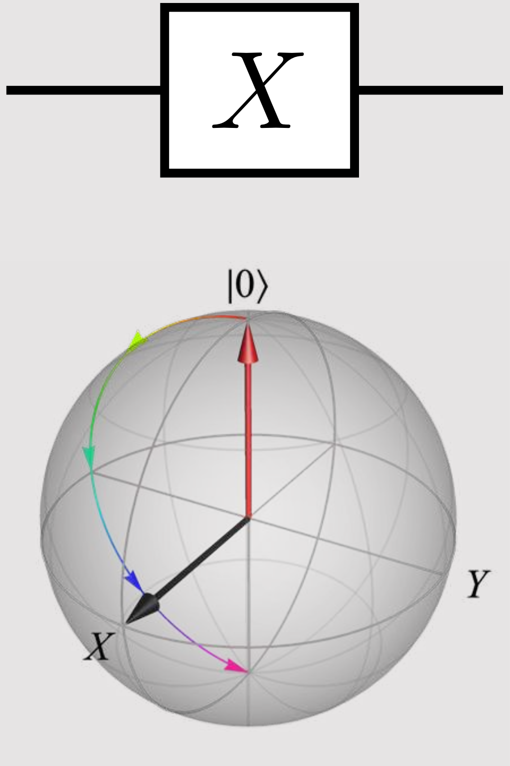


$$X = R_X(\pi)$$

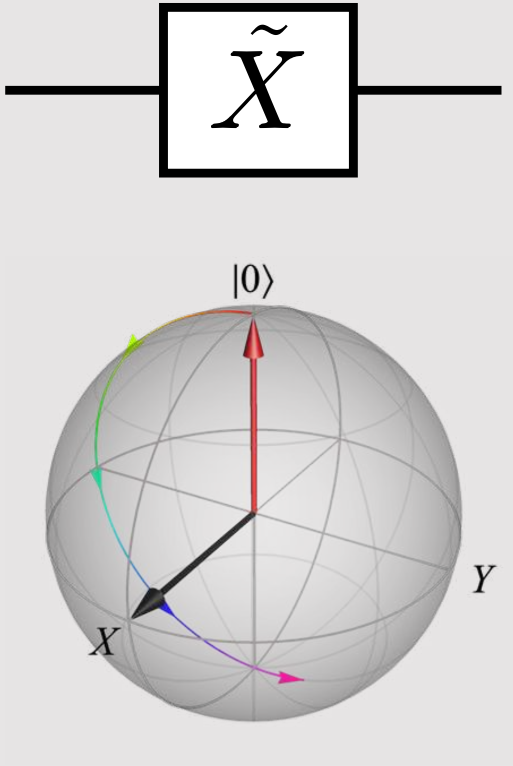
$$\tilde{X} := R_X\left(\pi + \epsilon\right)$$

Miscalibrated gate

Ideal gate



Noisy gate



Shown

$$R_X(\theta+\phi) = R_X(\theta)\,R_X(\phi)$$

$$X = R_X(\pi)$$

$$\tilde{X} := R_X(\pi+\epsilon)$$

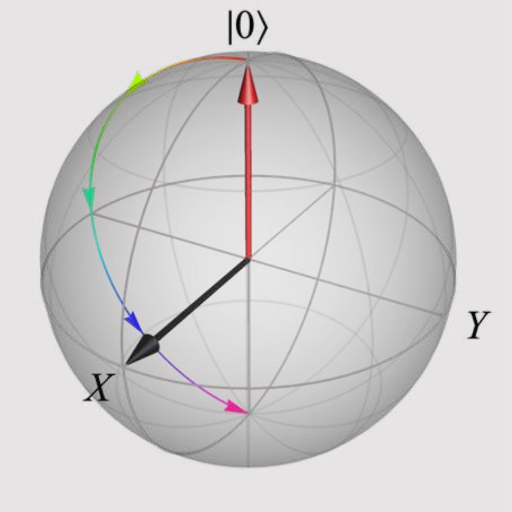
$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$
$$= \cos(\theta/2)I - i\sin(\theta/2)X$$

$$= \exp\left(-i\frac{\pi+\epsilon}{2}X\right)$$
$$= \exp\left(-i\frac{\pi}{2}X - i\frac{\epsilon}{2}X\right)$$
$$= \exp\left(-i\frac{\pi}{2}X\right)\exp\left(-i\frac{\pi}{2}X\right)$$
$$= R_X(\epsilon)\,R_X(\pi)$$
$$= X_\epsilon X$$

Noisy gate decomposition

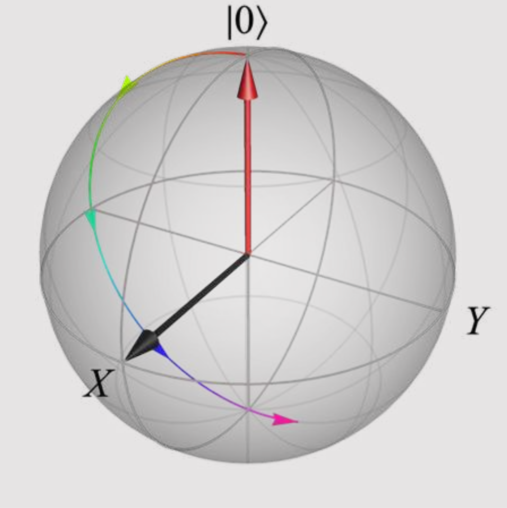
Ideal gate

X



Noisy gate

$\tilde{X}$



$$\tilde{X} = X_{\epsilon} \; X$$

Careful

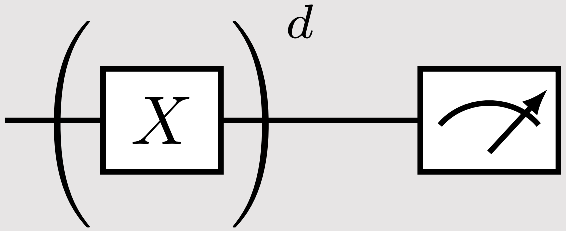
Common pitfall

The order in which gates appear in a schematic is the reverse of how they appear in the algebra.

$$-\boxed{X}-\boxed{Y}- \;=\; -\boxed{YX}-$$

Using a noisy gate in a quantum circuit

Ideal



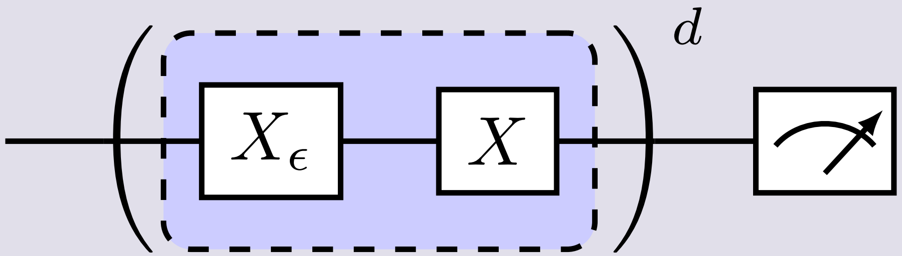
$$X = R_X(\pi)$$

$$U_{\text{total}} = X^d$$

$$= [R_X(\pi)]^d$$

$$= R_X(d\pi)$$

Noisy



$$\tilde{X} := R_X(\pi + \epsilon) = X_\epsilon X$$

$$\tilde{U}_{\text{total}} = \tilde{X}^d$$

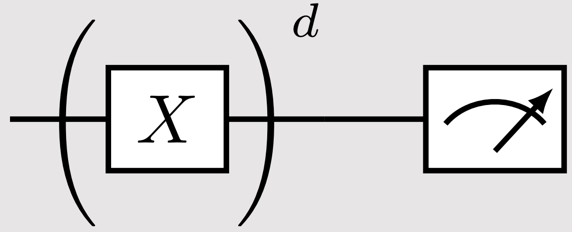
$$= [R_X(\epsilon)\, R_X(\pi)]^d$$

$$= R_X(d\epsilon)\, R_X(d\pi)$$

$$\tilde{U}_{\text{total}} = R_X(d\epsilon)\, U_{\text{total}}$$

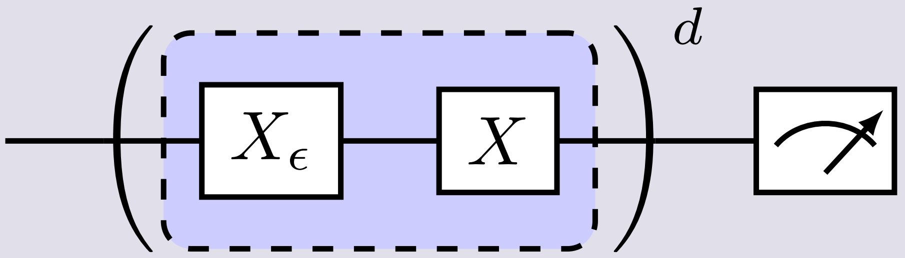
Using a noisy gate in a quantum circuit: final state

Ideal



$$U_{\text{total}} = X^d \; = R_X\left(d\pi\right)$$

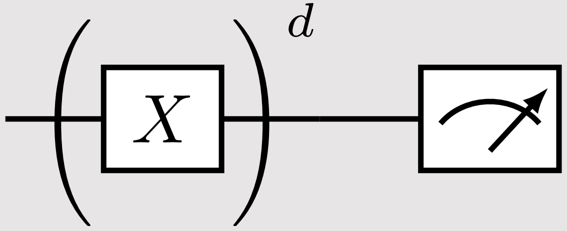
Noisy



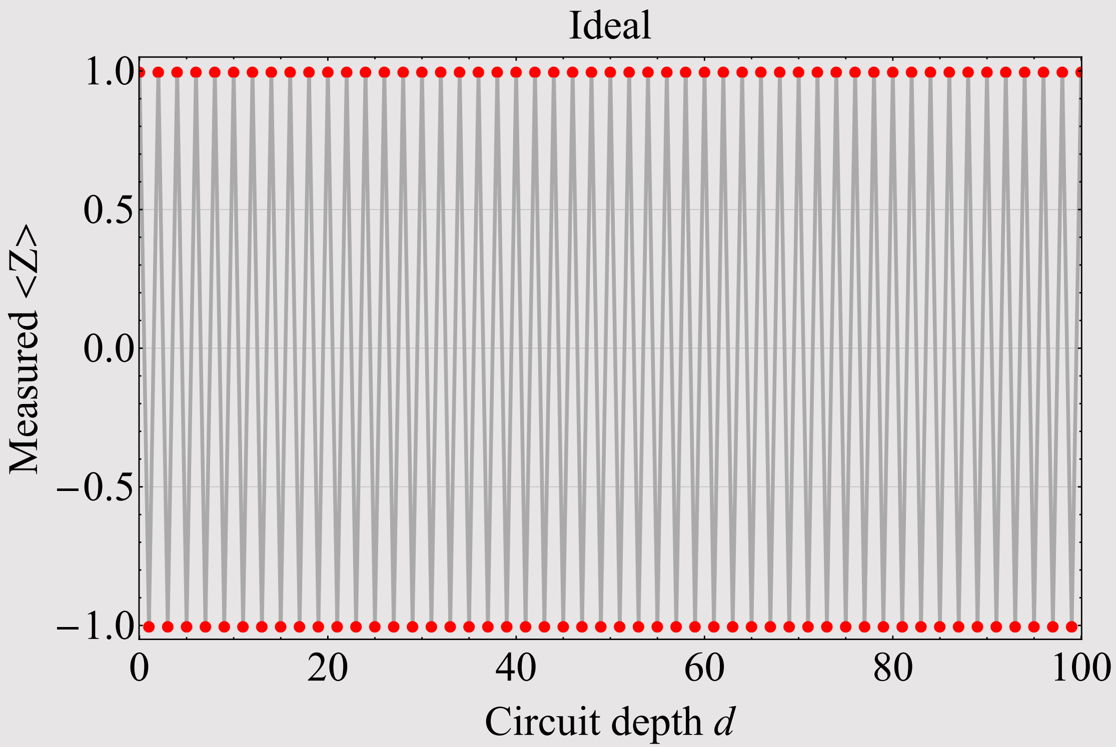
$$\tilde{U}_{\text{total}} = R_X\left(d\epsilon\right) U_{\text{total}}$$

Ideal vs. noisy observable

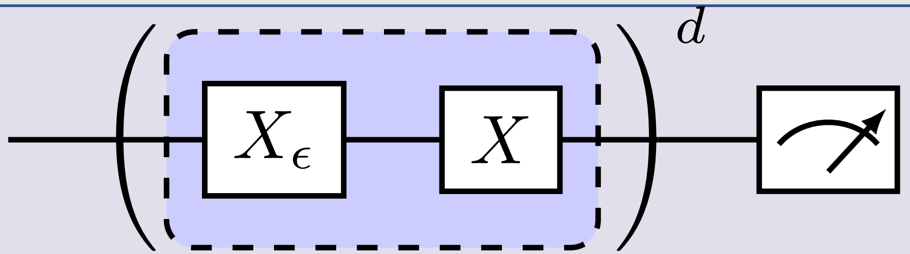
Ideal



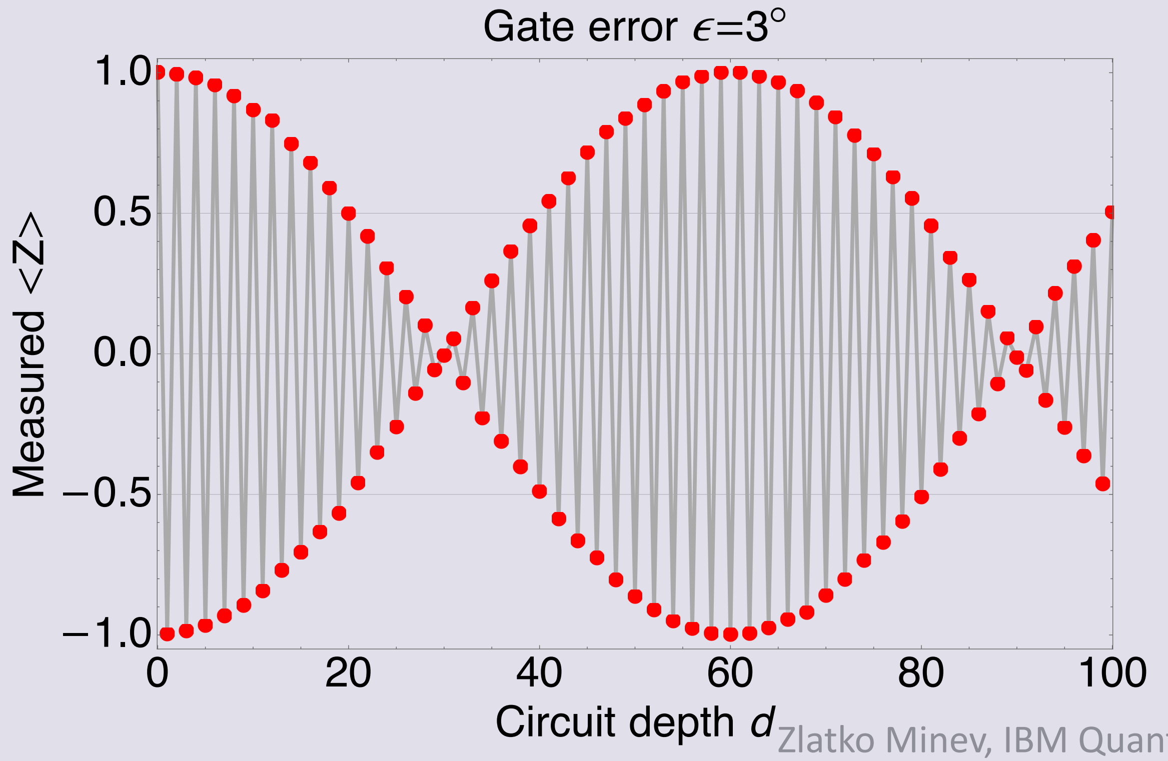
$$\langle \psi_f | Z | \psi_f \rangle = \cos(d\pi) = (-1)^d$$



Noisy

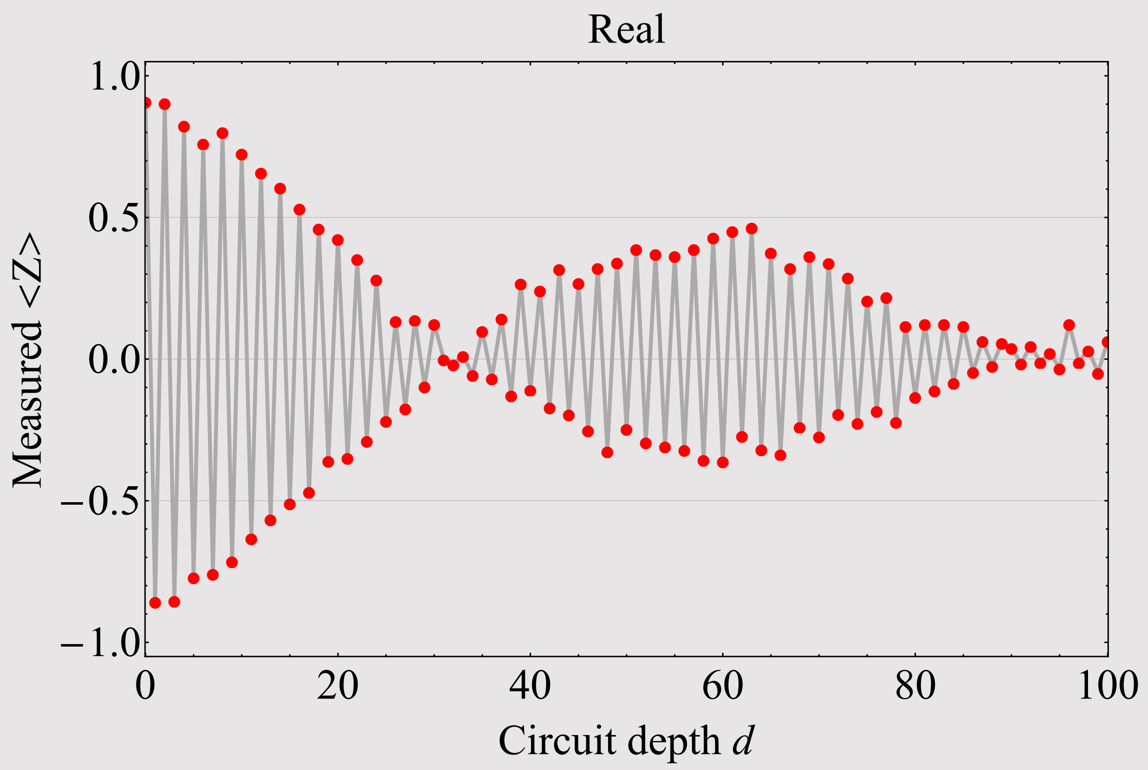


$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos(d\pi + d\epsilon)$$

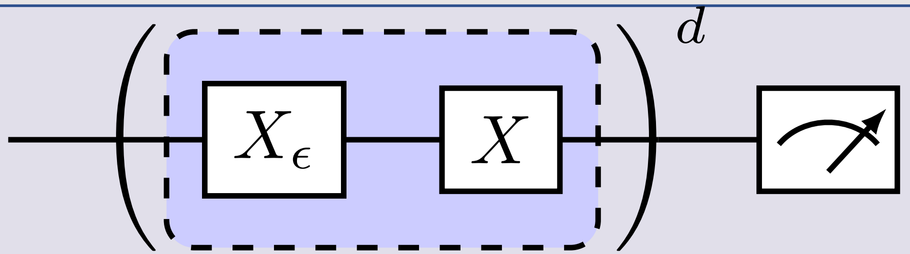


Compare to full experiment

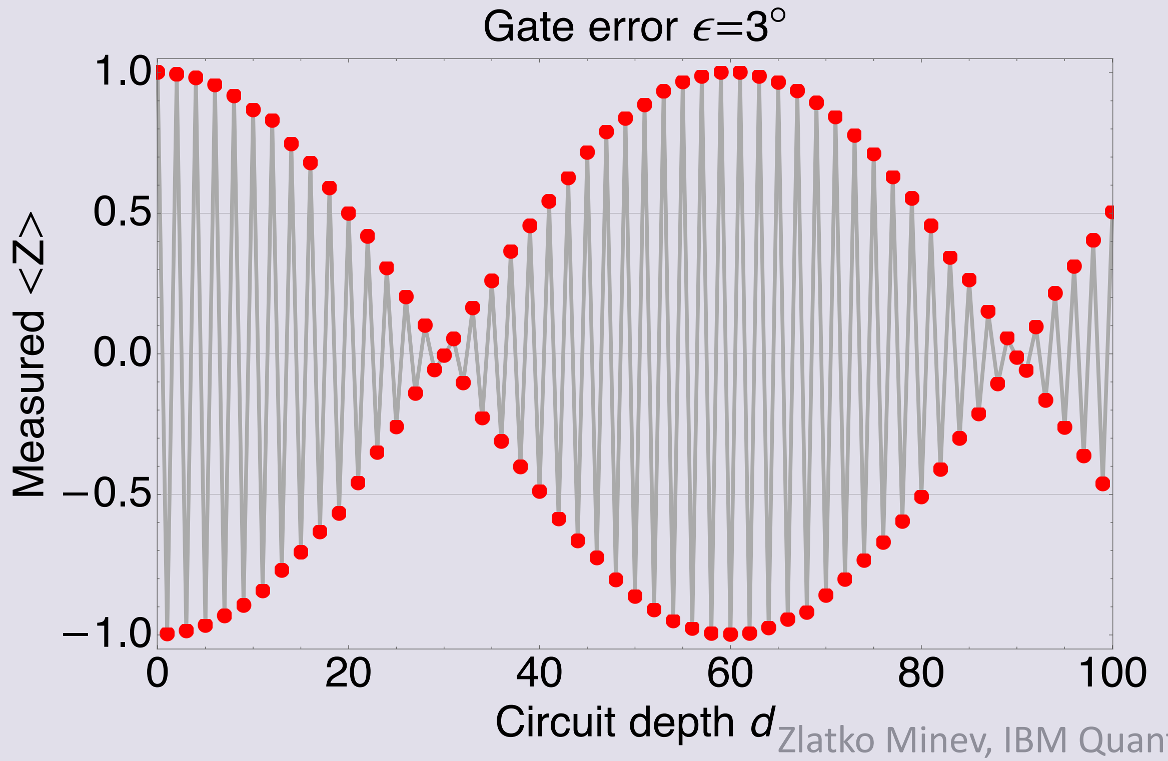
Full experiment



Noisy



$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



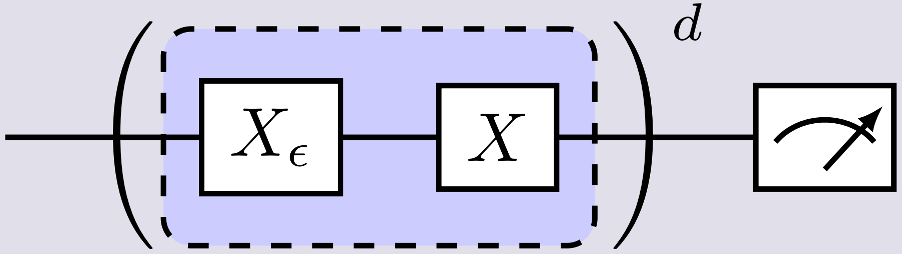
Coherent error is bad, quadratically so

Coherent errors have a *quadratic* impact on algorithmic accuracy (worst-case error)

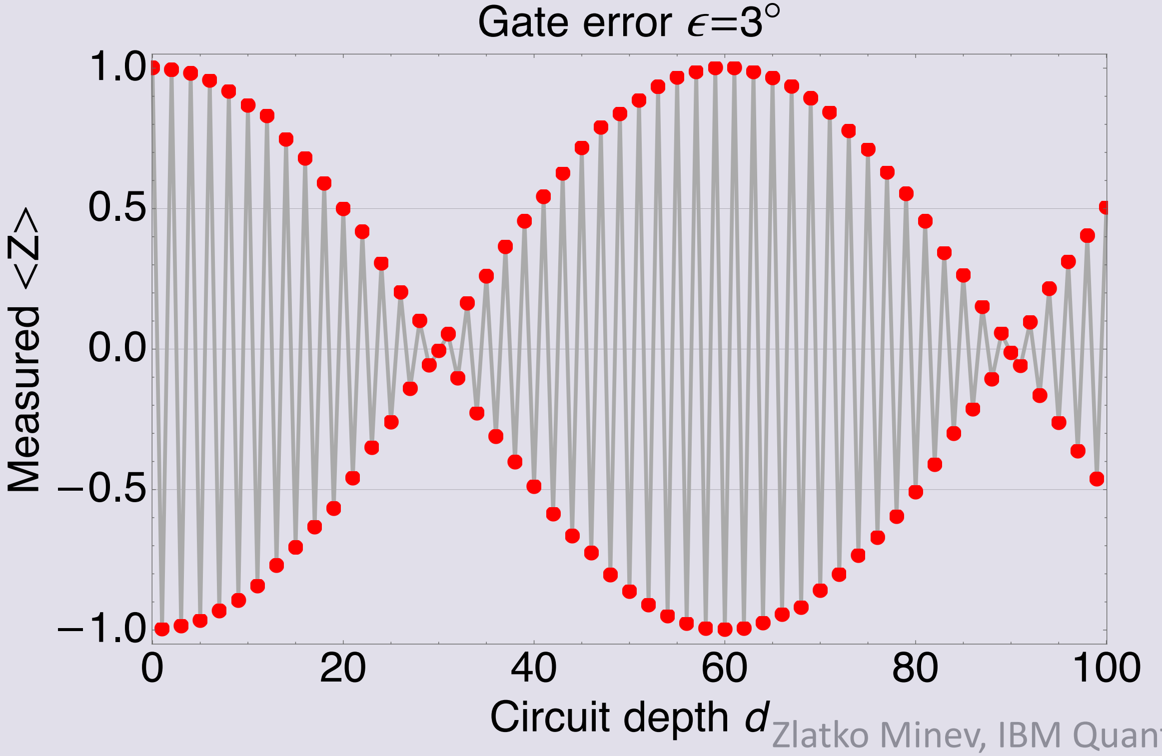
$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle - \langle \psi_f | Z | \psi_f \rangle$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^3)$$

Noisy

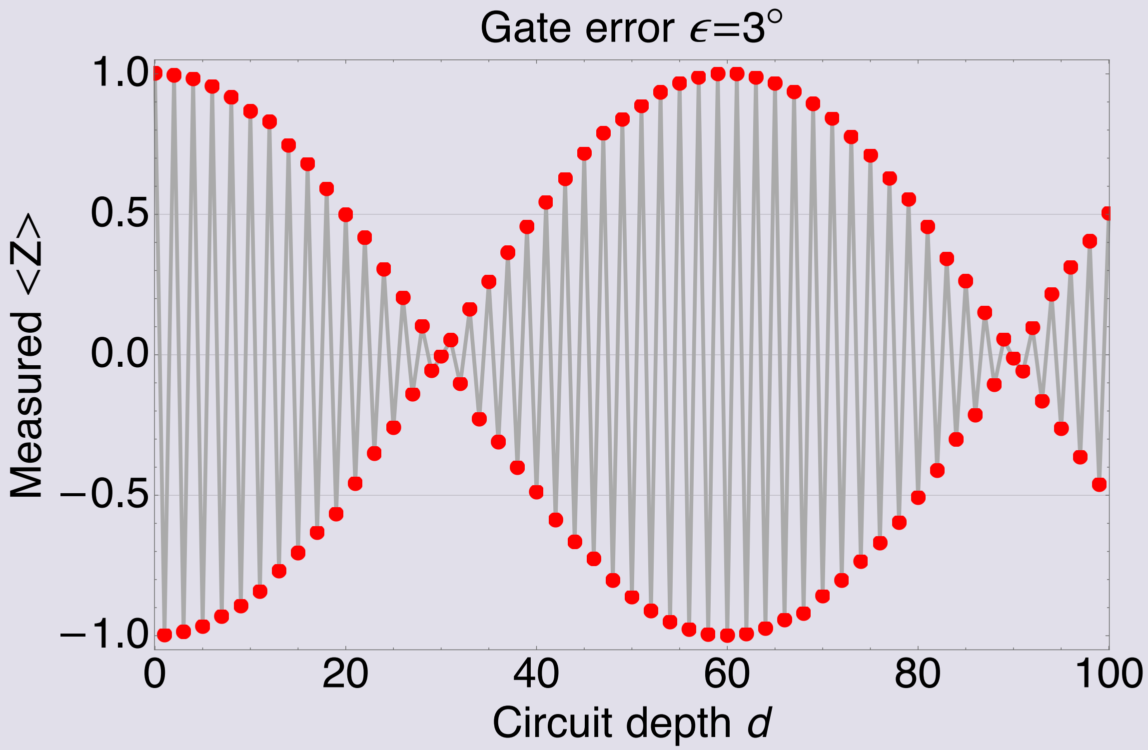


$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos(d\pi + d\epsilon)$$

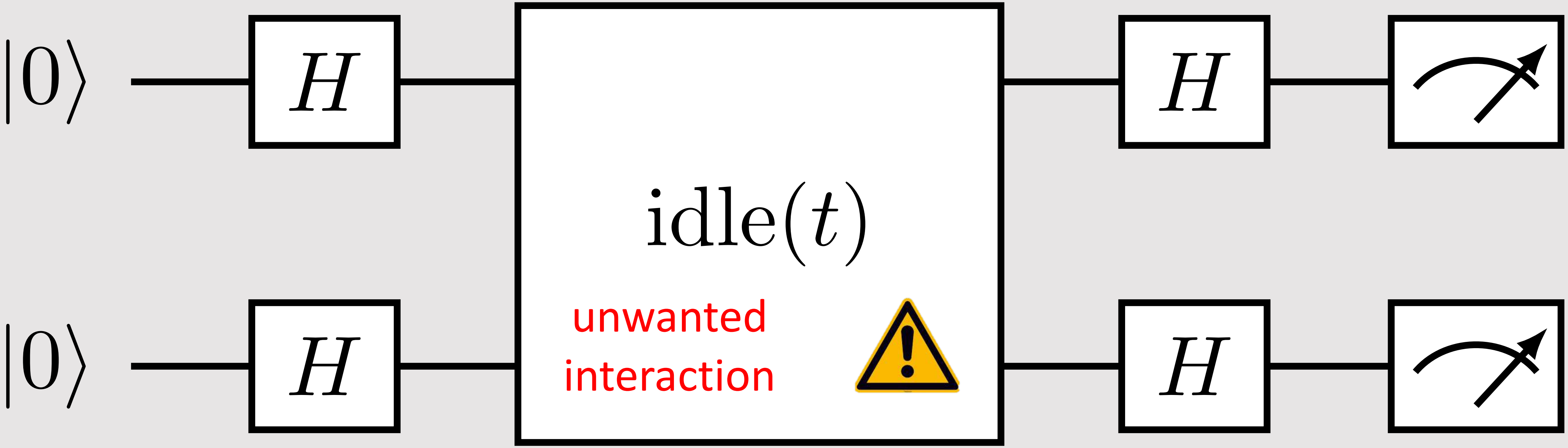


Coherent errors: brief summary

- are ubiquitous
- can be described by unitary operations
- do not loose quantum information
- data: can create oscillations in the data
- data: do not yield exponential decays
- have a *quadratic* impact on algorithmic accuracy (worst-case error)



Bonus content: two-qubit coherent ZZ error



Questions

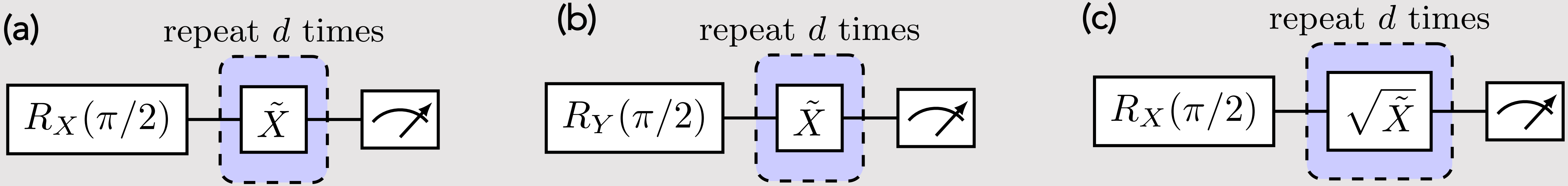
Answer these multiple-choice questions
in the chat; for example, type “1a 2b.”

1. Coherent noise can be caused by
   a) loss of energy of the qubit
   b) miscalibration, such as over-rotation
   c) wanted coupling to neighboring qubit

2. Coherent noise can be really bad because
   a) it results in loss of information
   b) you cannot undo it
   c) the worst-case error often grows quadratically

Advanced questions to dive deeper

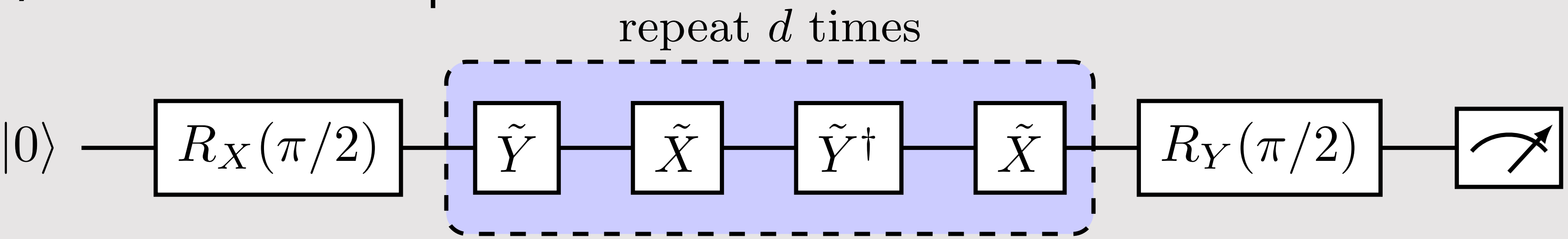
1. Amplitude-error amplification sequences.
   (A) Calculate the expectation value <Z (d)> as a function of depth d of the sequence for the following sequences. Assume the noise model $\tilde{X} := R_X\left(\pi + \epsilon\right) = X_\epsilon X$
   (B) How could you use this result to fine-tune your gates?
   (C) Can you come up with an alternative or more clever error-amplification sequence?



2. Phase-error amplification sequences. Use the noise model: Repeat (A), (B), and (C) for Exercise (1) for the following sequences and assuming a phase error between the X and Y gates, rather than an amplitude error.

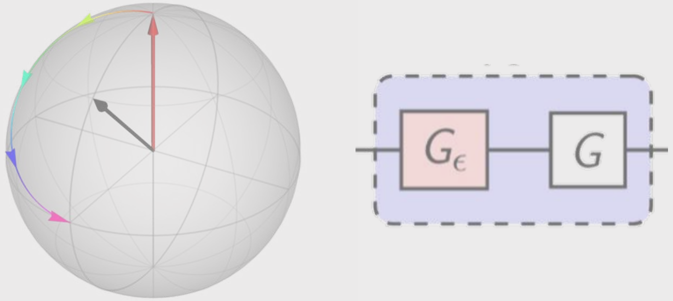
$$\tilde{X} = X \ ,$$

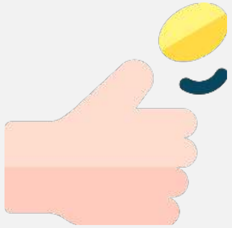
$$\tilde{Y} = \exp\left[-i\frac{\pi}{2}\left(\cos\epsilon\, Y + \sin\epsilon\, X\right)\right] \ .$$

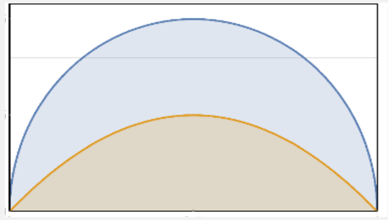


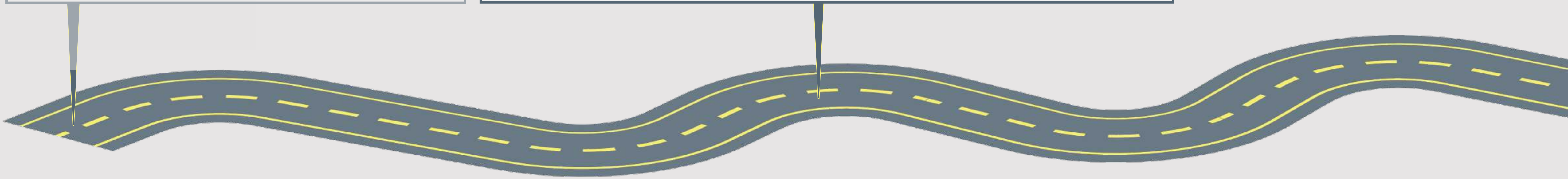
Chapter 3: Measurement theory in a nutshell + projection noise

Coherent



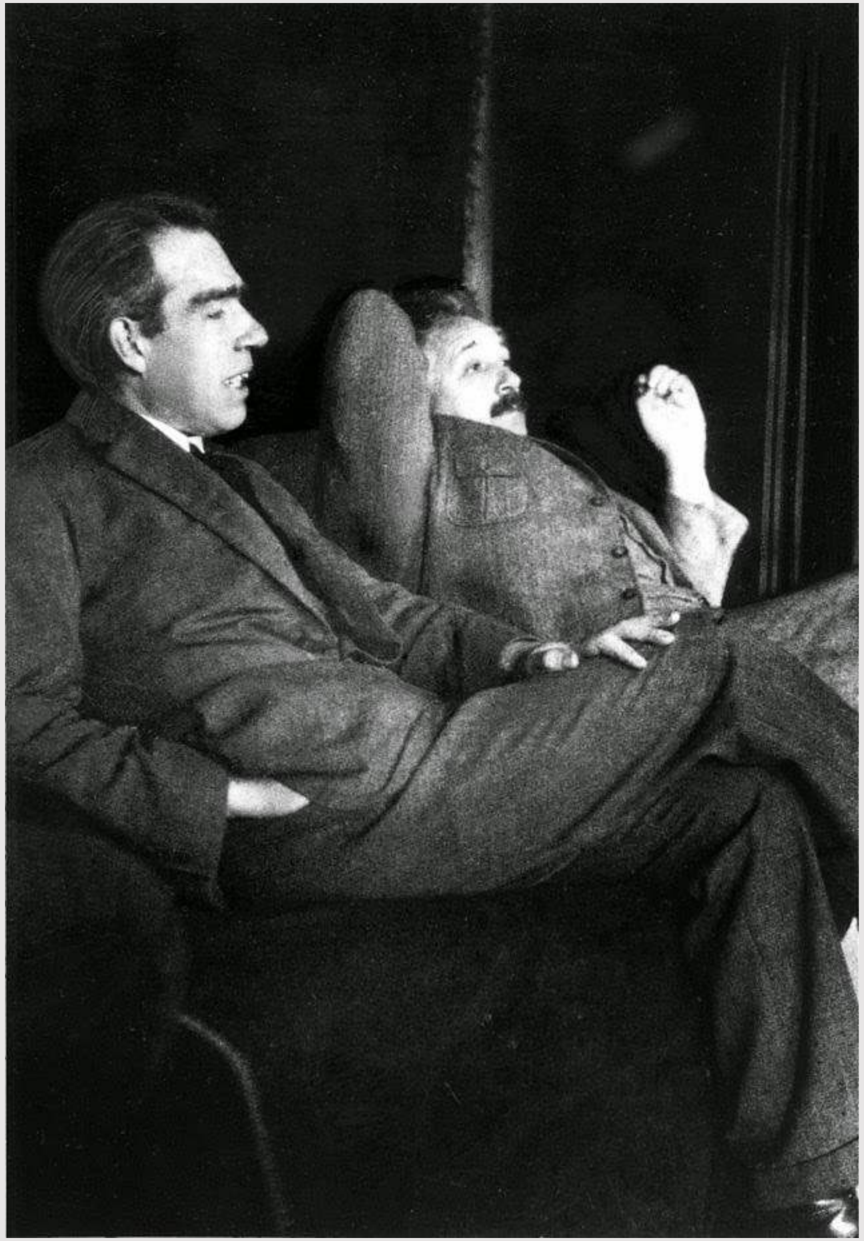






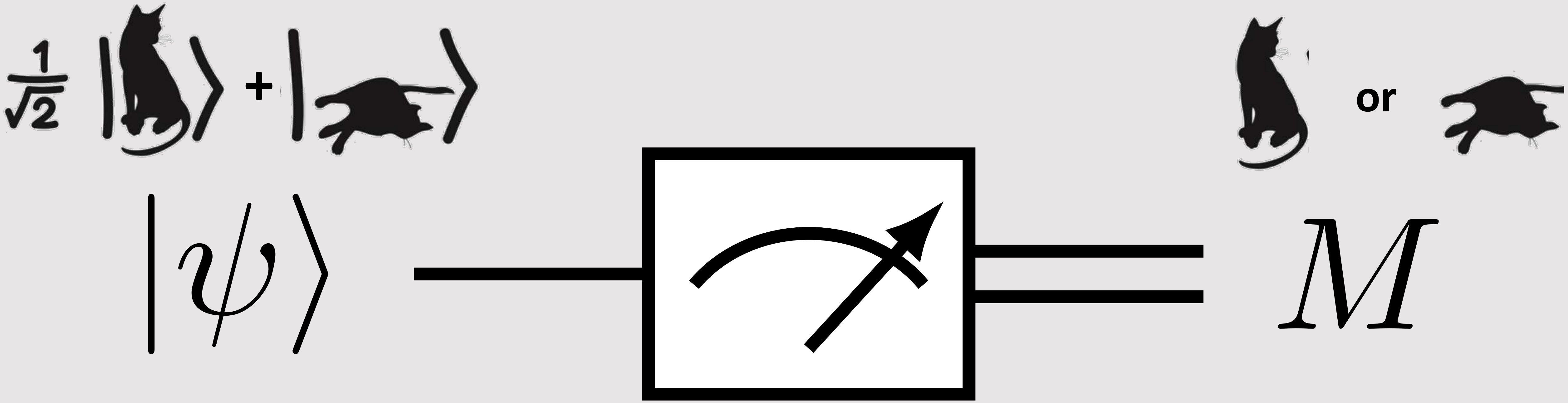
coin toss: flaticon; spam: make it move;
road based on: freepik

Quantum Measurement



Bohr and Einstein (1925)
Source: Wikimedia

Measurement apparatus: general



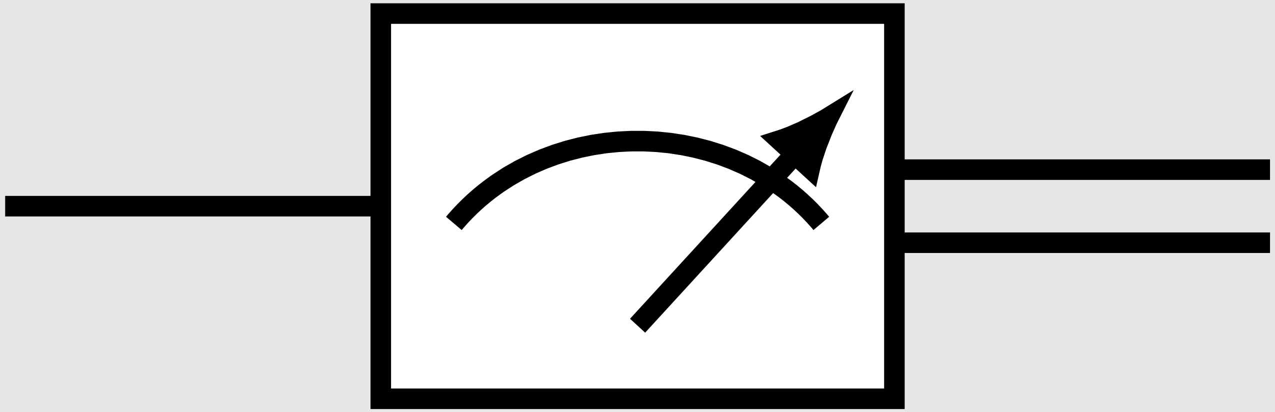
Quantum

Classical

Measurement basis

Computational basis $|0\rangle\,,|1\rangle$

$|\psi\rangle$



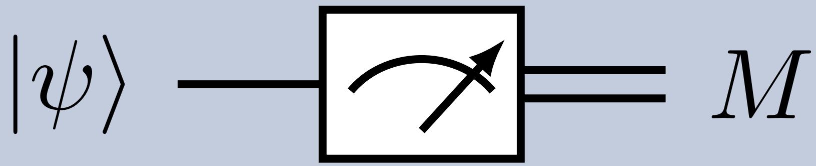
M

outcome | outcome probability

$$\begin{cases} M=0 & : \quad p(M=0)=|\langle 0|\psi\rangle|^2 \\ M=1 & : \quad p(M=1)=|\langle 1|\psi\rangle|^2 \end{cases}$$

indimart

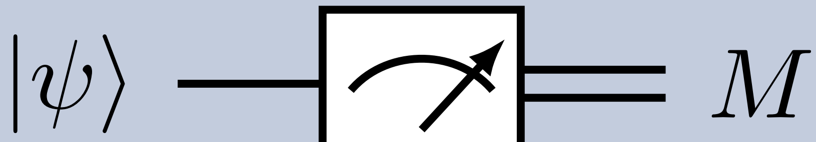
Resolution of the identity



outcome — outcome probability

$$\begin{cases} M = 0 & : \; p(M=0) = |\langle 0|\psi\rangle|^2 \\ M = 1 & : \; p(M=1) = |\langle 1|\psi\rangle|^2 \end{cases}$$

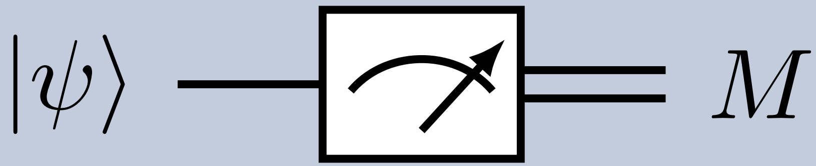
Resolution of the identity



outcome	outcome probability
$M = 0$	$p(M=0) = \lvert\langle 0 \vert \psi \rangle\rvert^2$
$M = 1$	$p(M=1) = \lvert\langle 1 \vert \psi \rangle\rvert^2$

$$
\begin{aligned}
&p\,(M = 0) + p\,(M = 1) \\
&= |\langle 0|\psi\rangle|^2 + |\langle 1|\psi\rangle|^2 \\
&= \langle\psi|0\rangle\,\langle 0|\psi\rangle + \langle\psi|1\rangle\,\langle 1|\psi\rangle \\
&= \langle\psi|\,(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|)\,|\psi\rangle \\
&= \langle\psi|\,\hat{I}\,|\psi\rangle \\
&= ||\psi\rangle|^2 \\
&= 1
\end{aligned}
$$

Projects and overlaps



outcome | outcome probability

$$\begin{cases} M=0 & : \; p(M=0) = |\langle 0|\psi\rangle|^2 \\ M=1 & : \; p(M=1) = |\langle 1|\psi\rangle|^2 \end{cases}$$

$$p\left(M=0\right)+p\left(M=1\right)$$

$$= \langle\psi|\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)|\psi\rangle$$

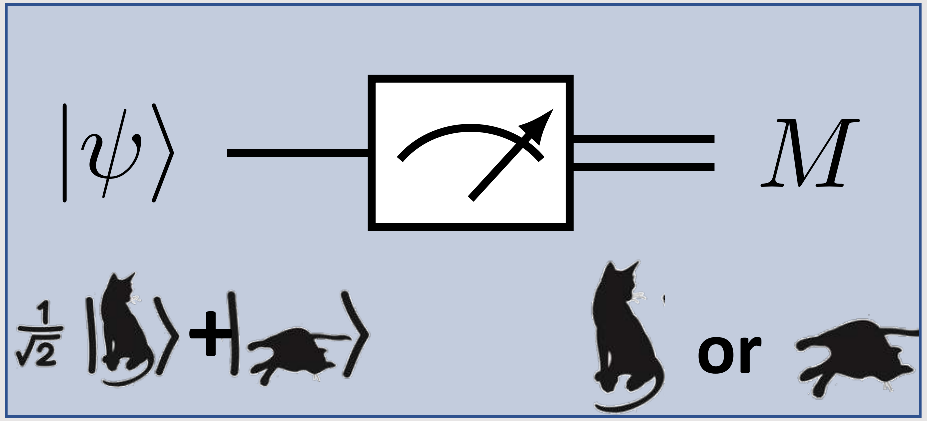
projectors

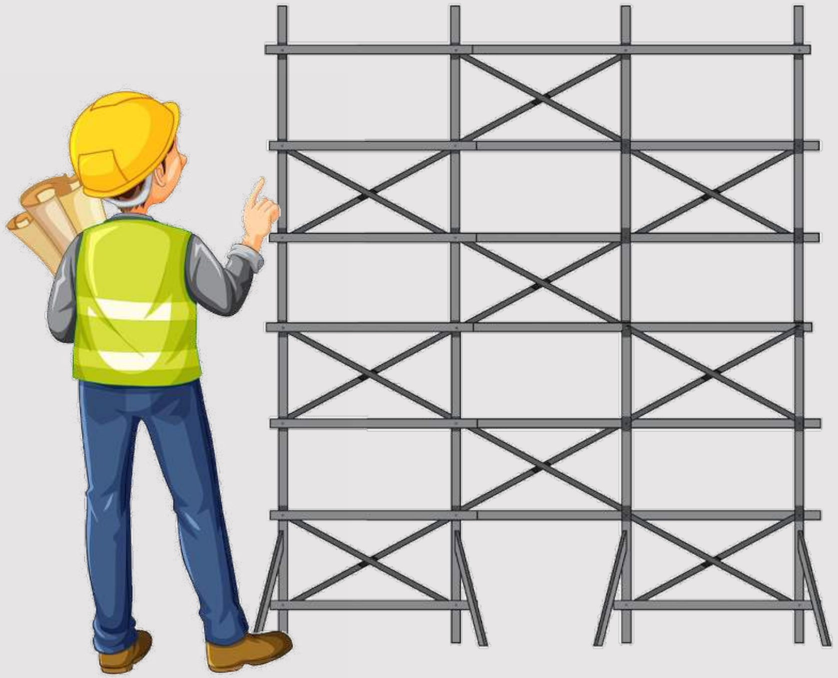
$$\hat{\Pi}_0 = |0\rangle\langle 0| \qquad \hat{\Pi}_1 = |1\rangle\langle 1|$$

inner-product in operator space (overlap between states)

$$p\left(m\right) = \left\langle \hat{\Pi}_m, |\psi\rangle \right\rangle = \mathrm{Tr}\left(\hat{\Pi}_m |\psi\rangle\langle\psi|\right)$$

Measurement theory 101 summary:
Scaffolding in terms of measurement operators

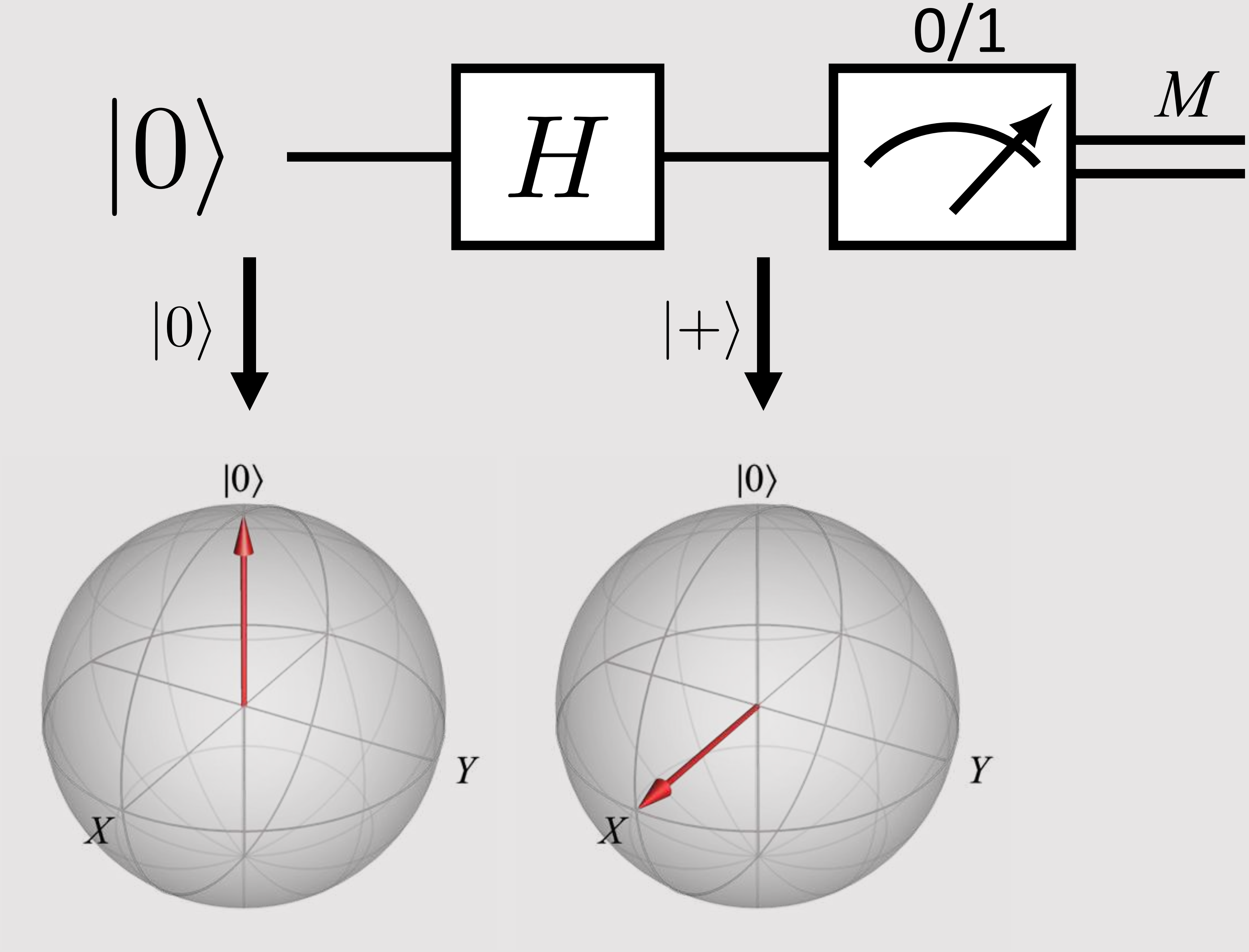




Measurement aspect	Example
Measurement outcome M	$M = 0$
Set of measurement outcomes Σ	$\Sigma = \{0,1\}$
Measurement operator Π_M	$\Pi_M = \|0\rangle\langle 0\|$
Probability of measuring $M = m$	$p(M{=}m) = \langle\Pi_m, \psi\rangle = \|\langle\Pi_m\|\psi\rangle\|^2$

cartoon: brgfx

Putting it to use: example ideal circuit



$M = 0$

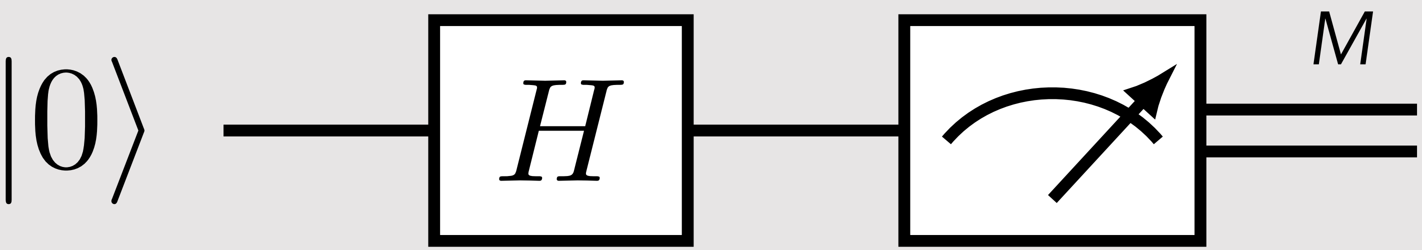
or

$M = 1$

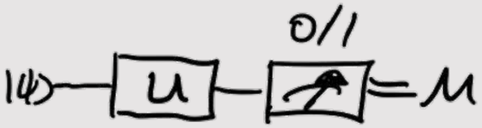
Quantum state

Classical random variable

Example ideal circuit



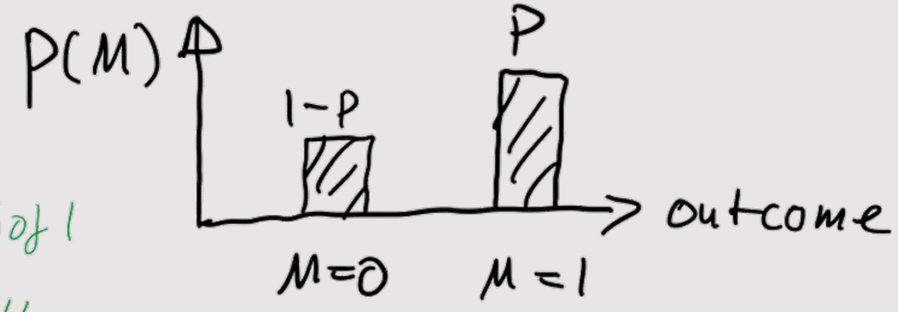
$M = 0$
or
$M = 1$



more general case

Prob to find $M=1$: $p(M=1) = p$ $\quad p \in [0,1]$

every possible qubit case



Bernoulli Distribution

$P(M) \sim B(p)$ prob of 1

classical rand variable

probability

Mean

Statistics

$$\mathbb{E}[M] = \sum_m m\,\mathbb{P}(m) = 0\,\mathbb{P}(m=0) + 1\,\mathbb{P}(m=1)$$

classical classical

$$= \mathbb{P}(m=1)$$

$$= p$$

$$= \sum_m m \langle \hat{\Pi}_m \rangle$$

$$= \sum_m m \langle |m\rangle\langle m| \rangle$$

$$= \langle \sum_m m |m\rangle\langle m| \rangle$$

$$= \langle \hat{M} \rangle$$

Quantum

$$\mathbb{V}[M] = \mathbb{E}[M^2] - \underbrace{\mathbb{E}[M]^2}_{} \approx \langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2$$

p^2

$$\mathbb{E}[M^2] = \sum_m m^2 \mathbb{P}(m)$$

$$= \sum_m m^2 \langle \, |m\rangle\langle m| \rangle$$

$$= \langle \sum_m m^2 |m\rangle\langle m| \rangle$$

$$= \langle \hat{M}^2 \rangle$$

$$= \cancel{0^2 (1-p)} \to 0 + 1^2 p$$

$$= p$$

$$\mathbb{V}[M] = p - p^2$$

$$= p(1-p)$$

$$= \sigma_M^2$$

$= 0$ if $p=0$

$= 1$ if $p=1$

Different observables

Different observables

For example

$\hat{M} = 0|0\rangle\langle 0| + 1\,|1\rangle\langle 1| = \frac{1}{2}(\hat{I}+\hat{Z}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\hat{M} = (+1)|0\rangle\langle 0| - 1\,|1\rangle\langle 1| = \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

For $\hat{M} = \hat{Z}$ $\qquad \Omega = \{-1, +1\}$ $\qquad$ Pauli Observables

$\hat{M} = \hat{X}$ $\qquad \Omega = \{+, -\}$

$\hat{\Pi}_+ = |+\rangle\langle +|$

$\hat{\Pi}_- = |-\rangle\langle -|$

$\hat{X} = \sum_{m \in \Omega} m\,|m\rangle\langle m|$

$= +1\,|+\rangle\langle +| - |-\rangle\langle -|$

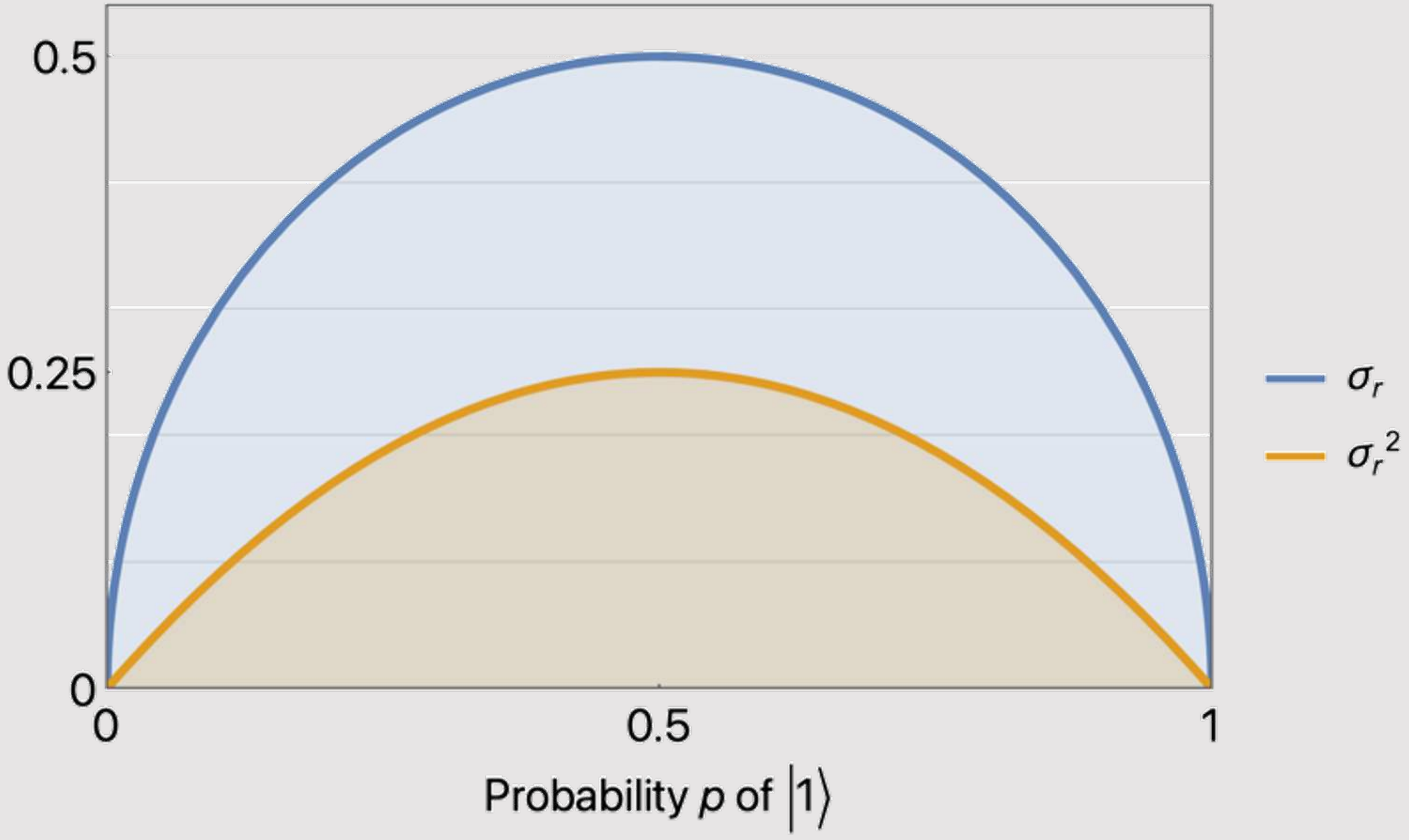
$\hat{X}|+\rangle = +1\,|x\rangle$

$\hat{X}|-\rangle = -1\,|x\rangle$

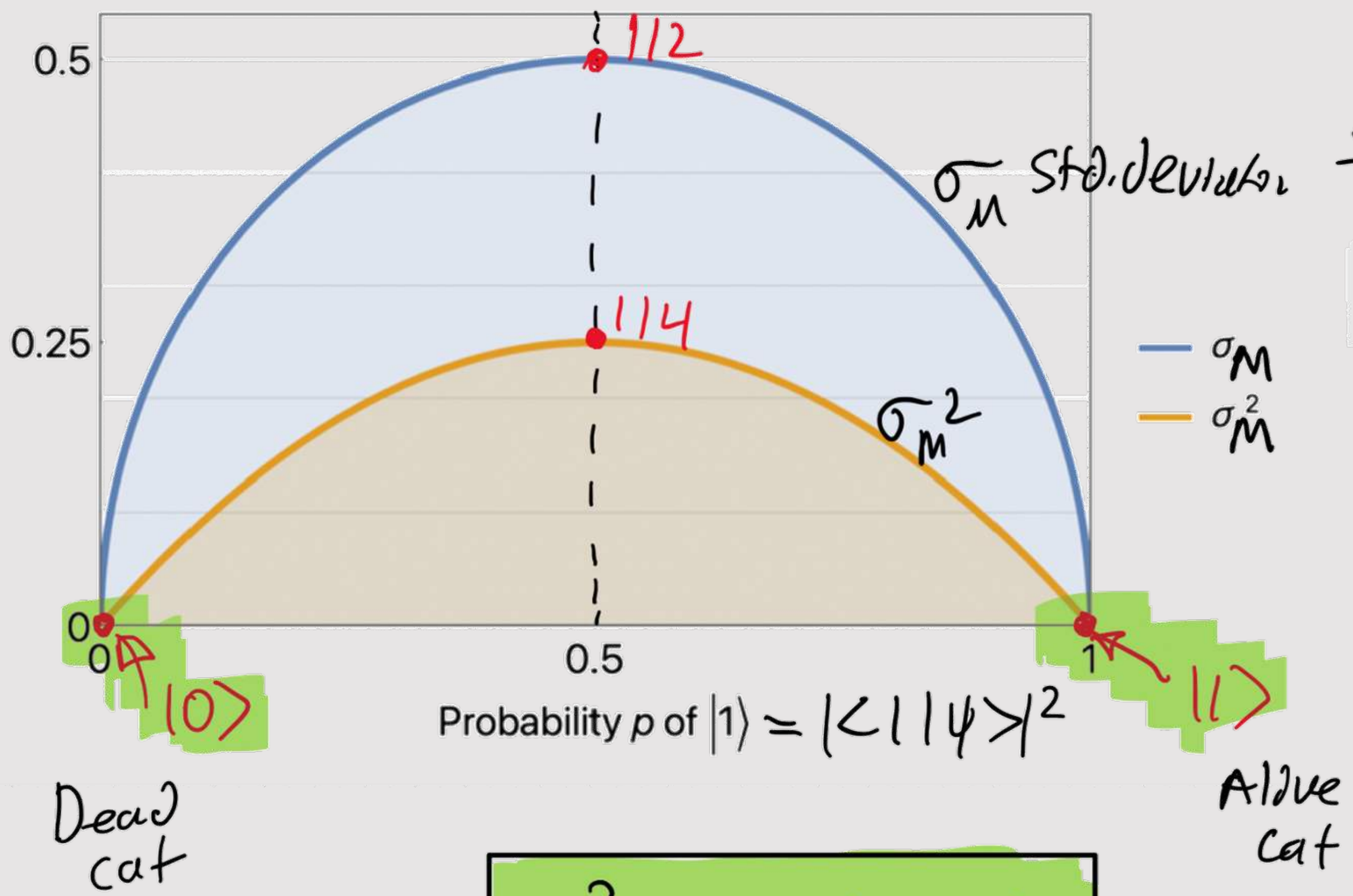
$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

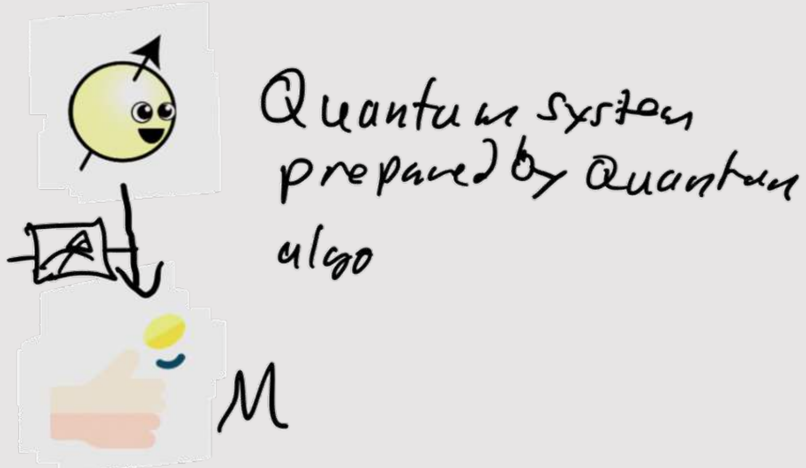
$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Variance of the random classical variable vs. probability to obtain 1



Variance of the random classical variable vs. probability to obtain 1





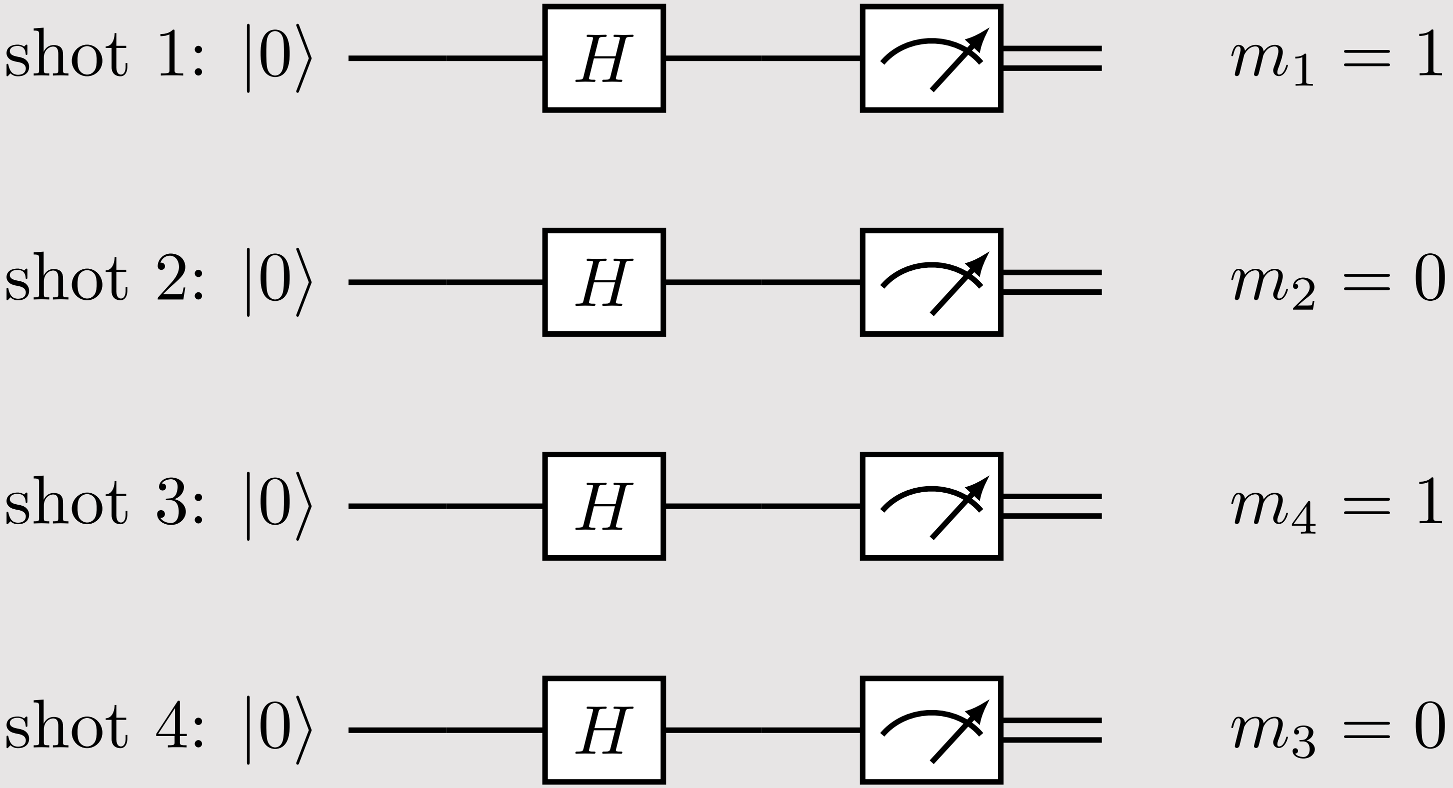
Random coin toss

Governed by Bernoulli distribution

Parameters determined by quantum algorithm

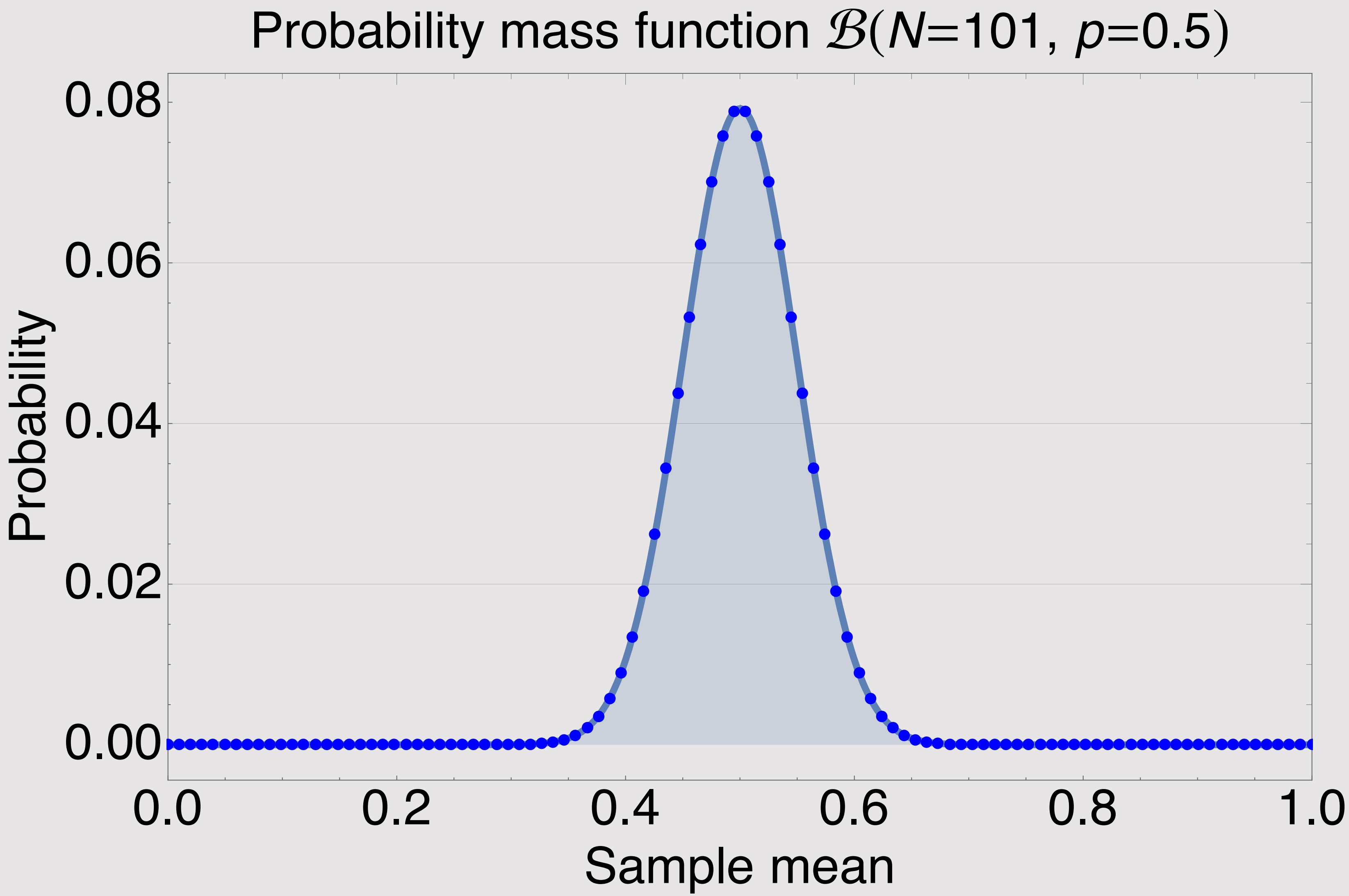
$$\boxed{\sigma_M^2 = p(1-p)}$$

Shots, shots, shots

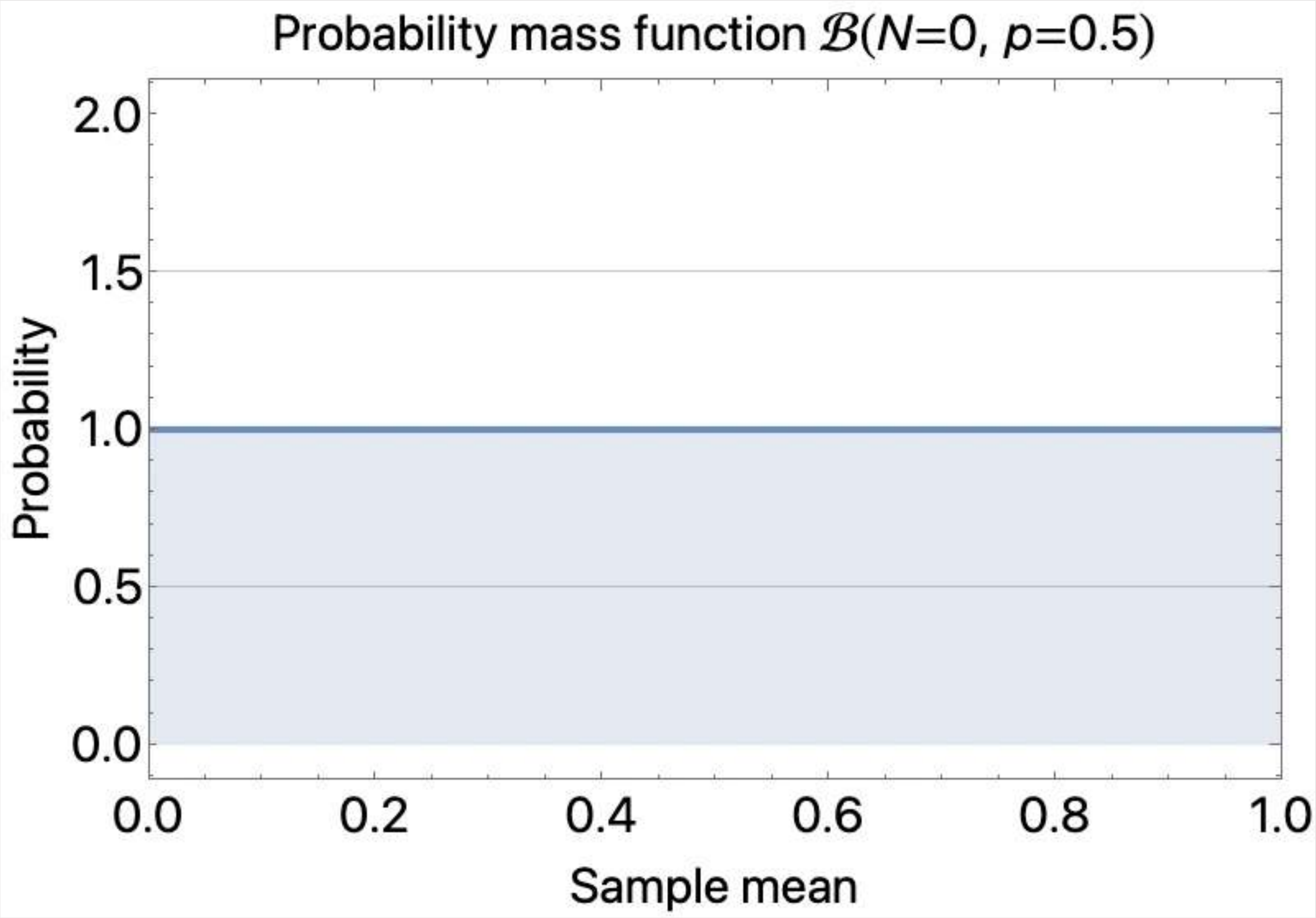


$\vdots$

Sampled output distribution



Animation of convergence of shots expectation value and mean



Concentration inequalities and tail bounds

Making a list,
checking it twice,
going to see
which inequality
is nice!

Markov? Hoeffding?
Jensen? Chebyshev?
Chernoff?

1. Probability (Technical note 11.9 v0.6)

1A. Concentration inequalities and tail bounds

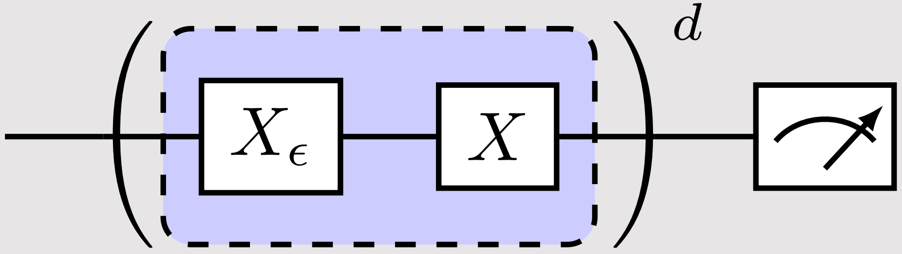
Unless otherwise specified, all variables are real $\mathbb{R}$. Inequalities come as one-sided $\Pr(\cdots \leq \cdots)$ and two-sided $\Pr(\lvert\cdots\rvert \leq \cdots)$. Notation: X is a random variable, $\mu := \mathbb{E}[X]$, $\sigma^2 := \mathrm{Var}[X]$, $S_n := X_1 + \cdots + X_n$.

Inequality	Conditions		Common form	Notes / Alternate form	
			Single random variable		
Markov[1]	Non-negative	$X \geq 0$	$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$ $\forall a > 0$	$\Pr[X \geq kE[X]] \leq \frac{1}{k}$ $k > 1$	[3, Sec. 5.1][6, Thm 1.13]
extension	+ non-negative, strictly increasing func Φ	$X \geq 0$ $\Phi(X) \geq 0$ increasing	$\Pr[X \geq a] = \Pr[\Phi(X) \geq \Phi(a)] \leq \frac{\mathrm{E}(\Phi(X))}{\Phi(a)}$	$\forall a > 0$	Wiki
Reverse Markov	upper-bounded by U (can be positive)	$\max X = U$	$\Pr[X \leq a] \leq \frac{U - \mathbb{E}[X]}{U-a}$	$\forall a > 0$	[1, Sec. 3.1]
Chebyshev[2]	Finite mean and variance	$\mathbb{E}[X], \mathrm{Var}[X]$ finite	$\Pr[\lvert X - \mathbb{E}[X]\rvert \geq a] \leq \frac{\sigma^2}{a^2}$	$\Pr[\lvert X - \mathbb{E}[X]\rvert \geq a \cdot \sigma] \leq \frac{1}{a^2}$ $\forall a > 0,\ \sigma^2 = \mathrm{Var}[X]$	[1, Sec. 3.2] [3, Sec. 5.1][2, Thm 18.11]
Cantelli	Improved Chebyshev	(same; but one-sided)	$\Pr[X - \mathbb{E}[X] \geq a]) \leq \frac{\sigma^2}{\sigma^2 + a^2}$	$\forall a > 0,\ \sigma^2 = \mathrm{Var}[X]$	Wiki
Chernoff[3]	Generic		$\Pr[X \geq a] = \Pr\left[e^{tX} \geq e^{ta}\right]$	$\forall t > 0,\quad a \in \mathbb{R}$	[1, Sec. 3.3]
Jensen		$f: \mathbb{R} \to \mathbb{R};\ f$ convex	$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$		[3, Prob. 5.3][6, Thm 1.14]
Hoeffding's lemma		$\mathbb{E}[X] = \mu$ $a \leq X \leq b$	$\mathbb{E}\left[e^{\lambda X}\right] \leq e^{\lambda\mu} e^{\frac{\lambda^2 (b-a)^2}{8}}$	$\lambda \in \mathbb{R}$	[1, Sec. 3.4]
			Sum of random variables		
Chernoff-Hoeffding (one-sided)	n independent random vars	$X_1, \ldots, X_n$ indep $S_n = X_1 + \cdots + X_n$ $X_i \in [a_i, b_i]$ $\forall i$	$\Pr[S_n - \mathbb{E}[S_n] \geq t] \leq \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$		[1, Sec. 3.5]
(two-sided)[4]	(same as above)		$\Pr[\lvert S_n - \mathbb{E}[S_n]\rvert > t] \leq 2\exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$	$\forall t \in \left(0, \frac{1}{2}\right)$	[5, Thm.1.1]
(two-sided iid)	same plus iid, range, mean μ for each	$X_1, \ldots, X_n \in [0,1]$ $\mathbb{E}[X_i] = \mu$ iid	$\Pr\left[\left\lvert \frac{S_n}{n} - \mu \right\rvert \geq \epsilon\right] \leq 2\exp\left(-2n\epsilon^2\right)$	$\forall \epsilon > 0$	[6, Thm 1.16]
Thm 1.3	n independent random vars	$X_1, \ldots, X_n$ indep $S_n = X_1 + \cdots + X_n$	$\Pr[S_n - \mathbb{E}[S_n] > \epsilon] \leq 2\exp\left(\frac{-\epsilon^2}{4\sum_{i=1}^n \mathrm{Var}[X_i]}\right)$	$\epsilon \in (0, 2\,\mathrm{Var}[S_n] / (\max_i \lvert X_i - \mathbb{E}[X_i]\rvert))$	[5, Thm. 1.3]
Azuma					
Weak law of large numbers	n independent iid random vars	$X_1, \ldots, X_n$ indep $\mathbb{E}[X_i] = \mu$ iid	$\lim_{n\to\infty} \Pr\left[\left\lvert \frac{1}{n} S_n - \mu \right\rvert \geq \epsilon\right] = 0$	$\forall \epsilon > 0$	[3, Sec. 5.2][6, Thm 1.15]
Strong law of large numbers	(same)	(same)	$\Pr\left[\lim_{n\to\infty} \frac{1}{n} S_n = \mu\right] = 1$		[3, Sec. 5.5]
	Advanced				
Bennett	n independent zero-mean	$X_1, \ldots, X_n$ indep $\mathbb{E}[X_i] = 0$ iid	$\Pr[S_n > \epsilon] \leq \exp\left(-n\sigma^2 h\left(\frac{\epsilon}{n\sigma^2}\right)\right)$	$\sigma^2 := \frac{1}{n}\sum_{i=1}^n \mathrm{Var}[X_i]\,, \forall \epsilon > 0\,,$ $h(a) := (1+a)\log(1+a) - a$ for $a \geq 0$	[1, 4.1]
Bernstein	(same)	(same)	$\Pr[S_n > \epsilon] \leq \exp\left(\frac{-n\epsilon^2}{2(\sigma^2 + \epsilon/3)}\right)$	(same)	[1, 4.2]
Efron-Stein	scalar func of vars $f: \chi^n \to \mathbb{R}$	$X_1, \ldots, X_n$ indep w/ values in set χ	$\mathrm{Var}[Z] \leq \sum_{i=1}^n \mathbb{E}\left[(Z - \mathbb{E}_i[Z])^2\right]$	$Z := g(X_1, \ldots, X_n)$ $\mathbb{E}_i[Z] := \mathbb{E}[Z \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_n]$	[1, 4.3]
McDiarmid's	scalar func of vars $f: \chi^n \to \mathbb{R}$	$X_1, \ldots, X_n$ indep w/ values in set χ	$\Pr[f(X_1, \ldots, X_n) - \mathbb{E}[f(X_1, \ldots, X_n)] \geq \epsilon] \leq \exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^n c_i^2}\right)$	condition: c-bounded difference property $\forall \epsilon > 0$ $\lvert f(X_1, \ldots, X_i, \ldots, X_n) - f(X_1, \ldots, X_i', \ldots, X_n)\rvert \leq c_i$	[1, 4.4]

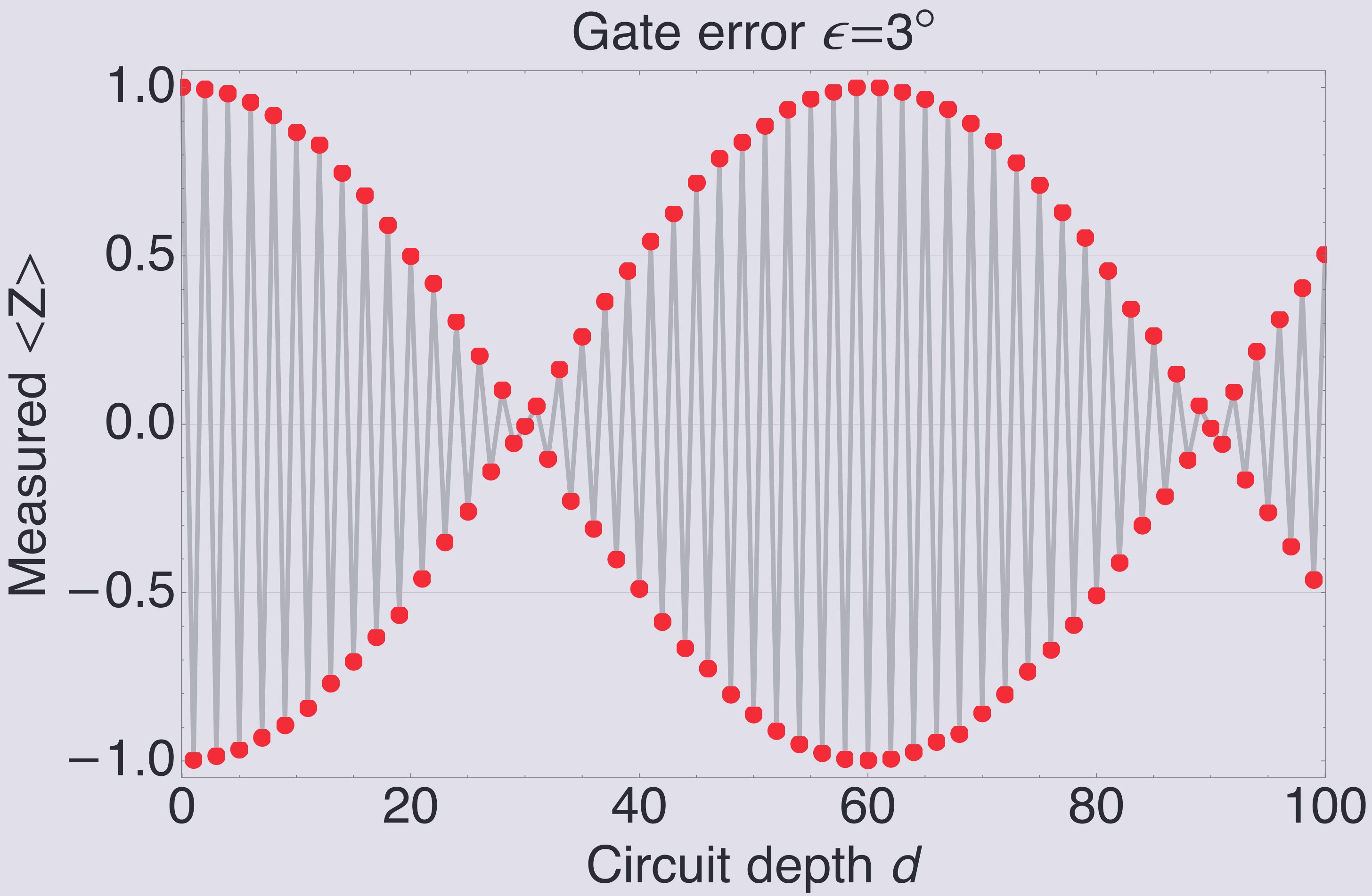
[1] Markov's inequality bounds the first moment of random variable. Use it when a constant probability bound is sufficient [1, Sec. 3.3].
[2] Chebyshev is derived from Markov. It bounds the second moment. It is the appropriate one when the variance σ is known. If σ is unknown, we can use the bounds of $X \in [a, b]$.
[3] Chernoff bound is used to bound the tails of the distribution for a sum of independent random variables. By far the most useful tool in randomized algorithms [1, Sec. 3.3].
[4] This probability can be interpreted as the level of significance ϵ (probability of making an error) for a confidence interval around the mean of size 2ϵ. Therefore, we require at least $\log(2\alpha)/2t^2$ samples to acquire $1 - \alpha$ confidence interval $\mathbb{E}[\bar{X}] \pm t$.

https://www.zlatko-minev.com/blog/inequalities

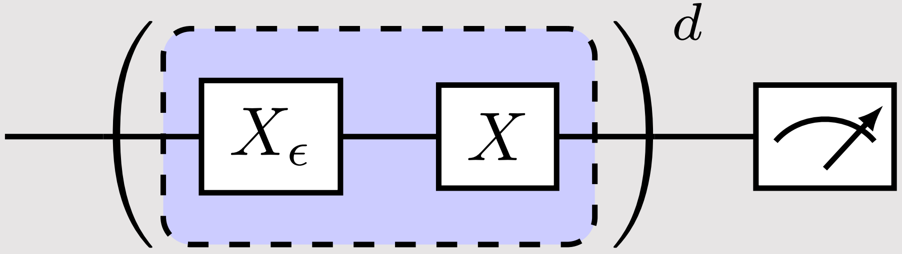
Recall gate error result



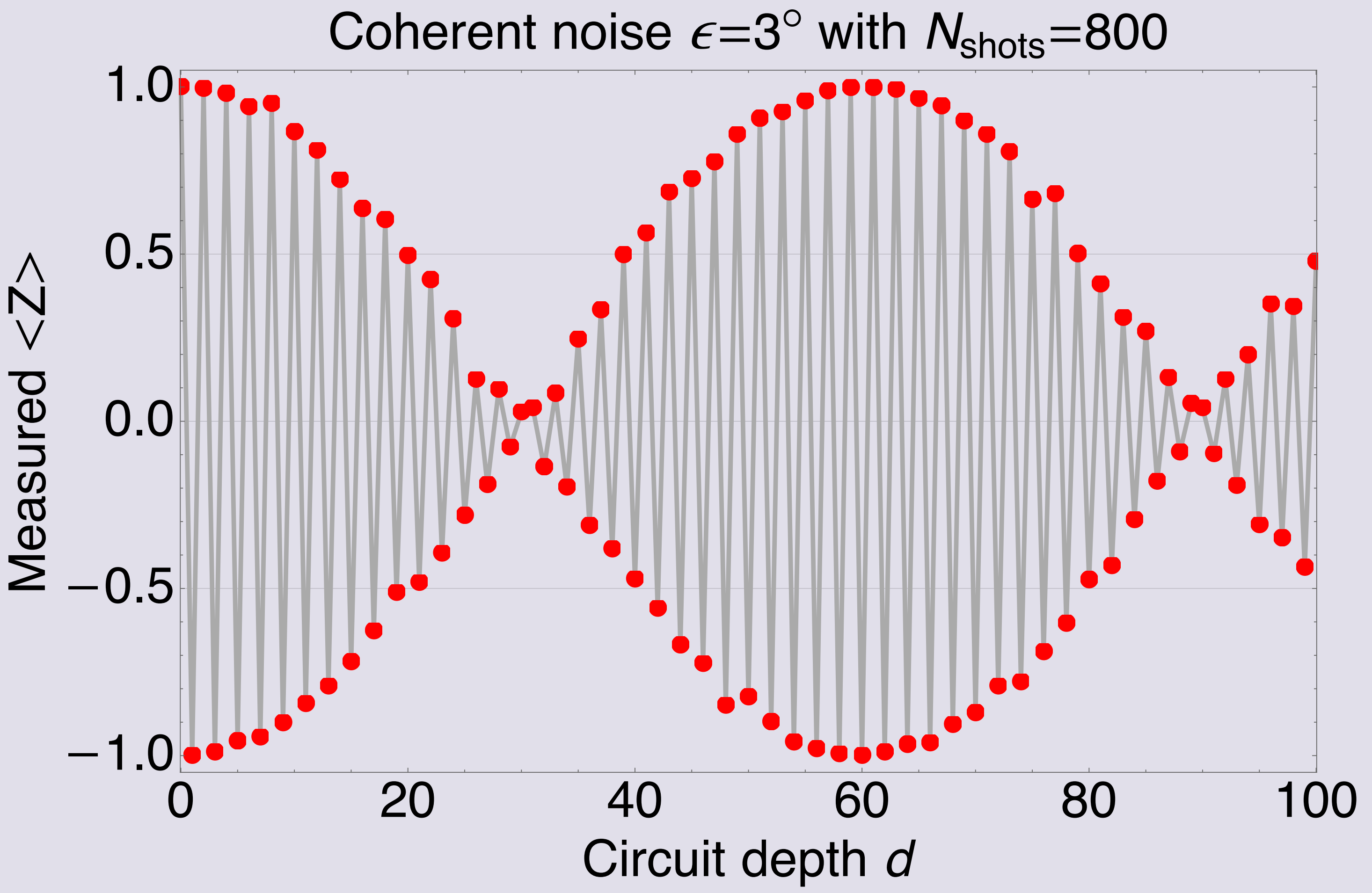
$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



Projection & sampling noise



$$\left\langle \tilde{\psi}_f \middle| Z \middle| \tilde{\psi}_f \right\rangle = \cos\left(d\pi + d\epsilon\right)$$



Questions

Answer these multiple-choice questions
in the chat; for example, type “1a 2b.”

1. Projection noise is due to
   a) measurement apparatus that could be made more efficient
   b) classical limitations
   c) core nature of quantum physics

2. To reduce projection noise
   a) increase the number of sample
   b) you cannot undo it
   c) apply readout error mitigation

Dive deeper? Try the following

1. Calculate the following for a qubit
   1. The expectation value of the sample variance for N shots of the observable |1><1|.

      where the sample mean is defined as $$S = \frac{1}{N}\sum_{n=1}^{N} M_n$$

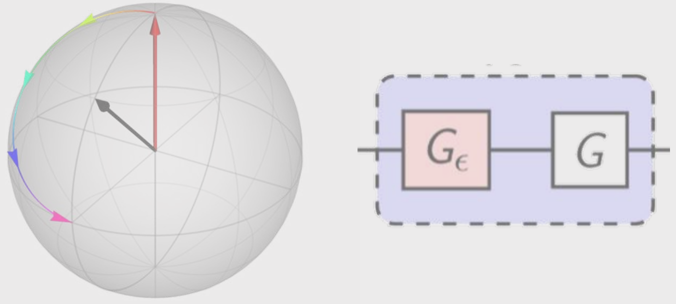
      and the sample variance is defined as $$V = \frac{1}{N}\sum_{n=1}^{N} \left(M_n - S\right)^2$$
   2. Is the estimate biased?
   3. The variance of V.
   4. Can you find an expression for an unbiased estimate of the sample variance?

2. What about two qubits?
   1. Can you find the projection operators for the observable ZZ?
   2. Find the probability distribution for the observables ZI, IZ, and ZZ for a general state.

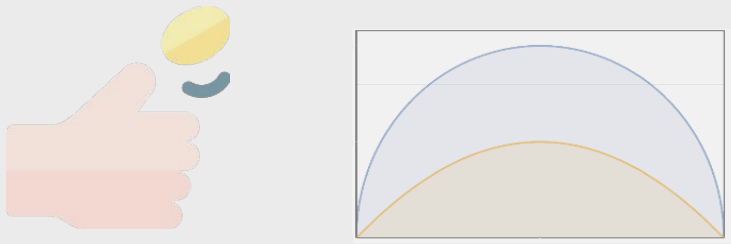
3. If you take 10 shots and find all 10 outcomes to be 1, what is the probability the qubit is in the |0> state? (hint: it’s not zero!)

State preparation & measurement

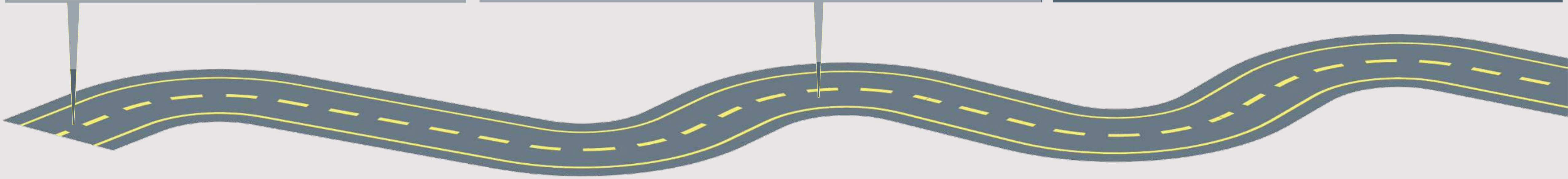
Coherent



Projection & measurement theory

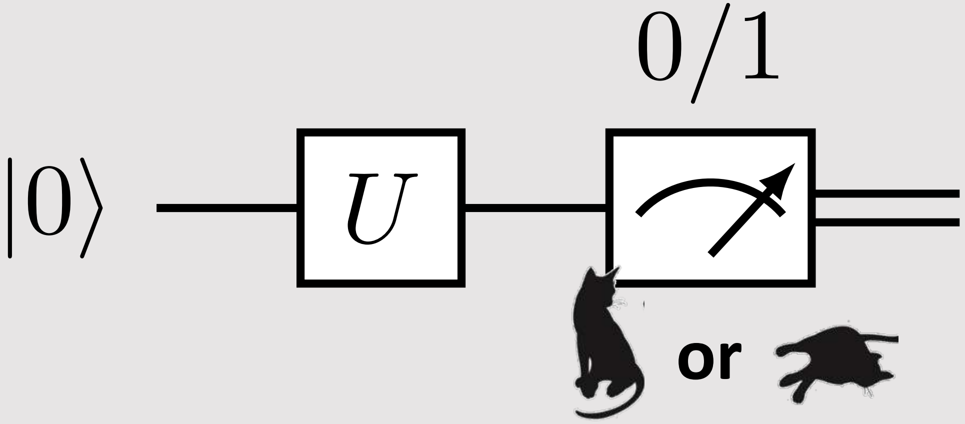


State preparation & measurement



coin toss: flaticon; spam: make it move;
road based on: freepik

Qubit example



$$\left\{\begin{array}{lll} \text{outcome} & \text{probability} & \text{quantum} \\ \mathbb{P}(M=0) = & 1-p & = |\langle 0|\psi\rangle|^2 \\ \mathbb{P}(M=1) = & p & = |\langle 1|\psi\rangle|^2 \end{array}\right.$$

Measurement error

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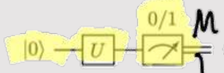
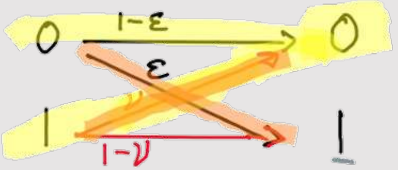


$M \begin{cases} \text{outcome} & \text{probability} & \text{quantum} \\ \mathbb{P}(M=0) = & 1-p & = |\langle 0|\psi\rangle|^2 \\ \mathbb{P}(M=1) = & p & = |\langle 1|\psi\rangle|^2 \end{cases}$

$\hat{M} = |1\rangle\langle 1| = \frac{1}{2}(\hat{I} - \hat{Z})$

M ideal classical outcome



$\tilde{M}$ noisy classical outcome

$$P_M = \begin{pmatrix} \mathbb{P}(M=0) \\ \mathbb{P}(M=1) \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix} \qquad P_{\tilde{M}} = \begin{pmatrix} \mathbb{P}(\tilde{M}=0) \\ \mathbb{P}(\tilde{M}=1) \end{pmatrix} = \begin{pmatrix} 1-\tilde{p} \\ \tilde{p} \end{pmatrix}$$

$$\begin{cases} \mathbb{P}(\tilde{M}=0) = \mathbb{P}(\tilde{M}=0|M=0)\,\mathbb{P}(M=0) + \overbrace{\mathbb{P}(\tilde{M}=0|M=1)}^{\text{error}}\,\mathbb{P}(M=1) \\ \mathbb{P}(\tilde{M}=1) = \mathbb{P}(\tilde{M}=1|M=0)\,\mathbb{P}(M=0) + \mathbb{P}(\tilde{M}=1|M=1)\,\mathbb{P}(M=1) = \varepsilon(1-p) + (1-\nu)p = \tilde{p} \end{cases}$$

$$\underset{\text{noisy}}{P_{\tilde{M}}} = \underset{\text{readout error matrix}}{A}\;\underset{\text{ideal}}{P_M}$$

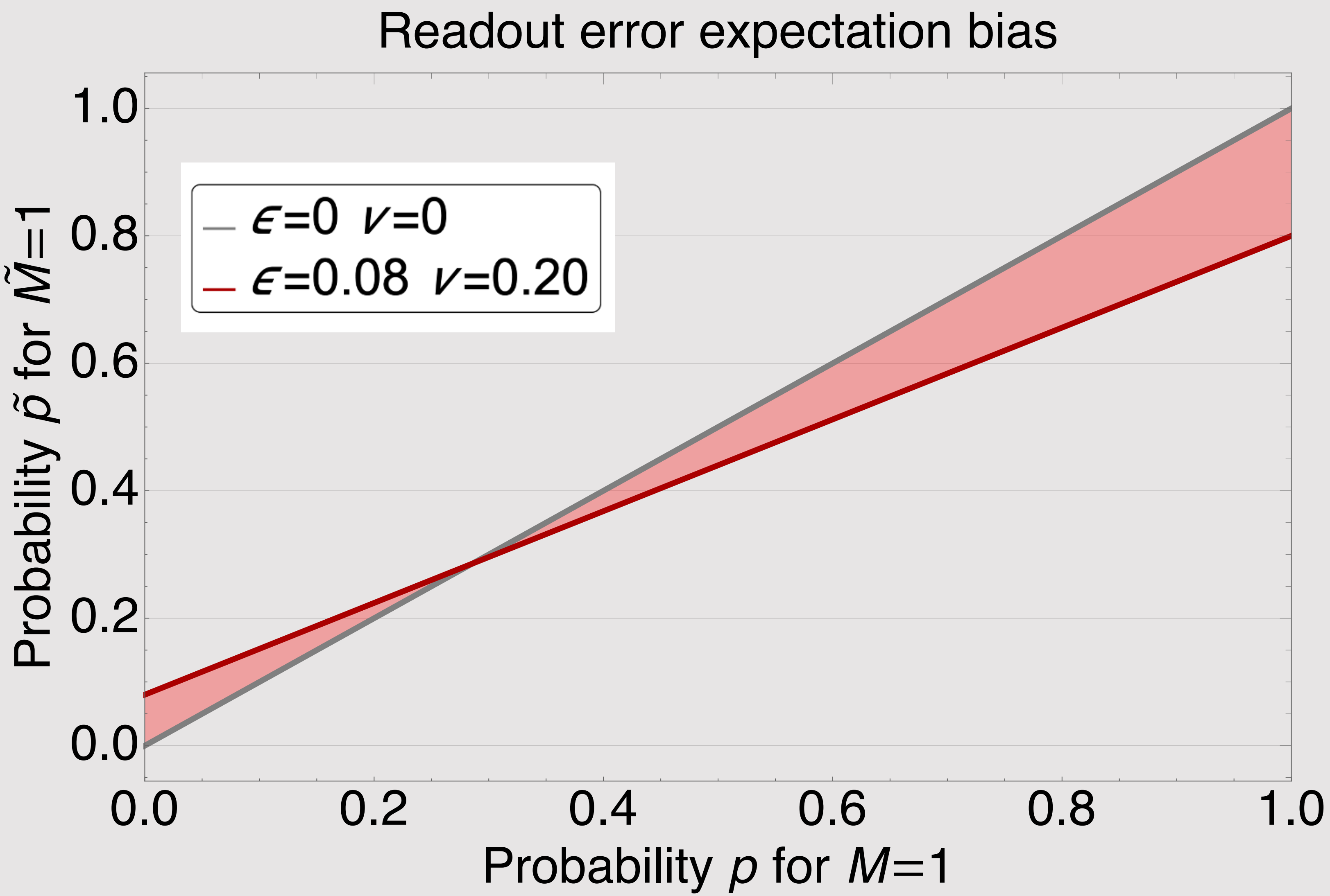
$$A = \begin{array}{c c} & \begin{array}{cc} M=0 & M=1 \end{array} \\ \begin{array}{c} \tilde{M}=0 \\ \tilde{M}=1 \end{array} & \begin{pmatrix} \mathbb{P}(\tilde{M}=0|M=0) & \mathbb{P}(\tilde{M}=0|M=1) \\ \mathbb{P}(\tilde{M}=1|M=0) & \mathbb{P}(\tilde{M}=1|M=1) \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 1-\varepsilon & \nu \\ \varepsilon & 1-\nu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for an ideal measurement}$$

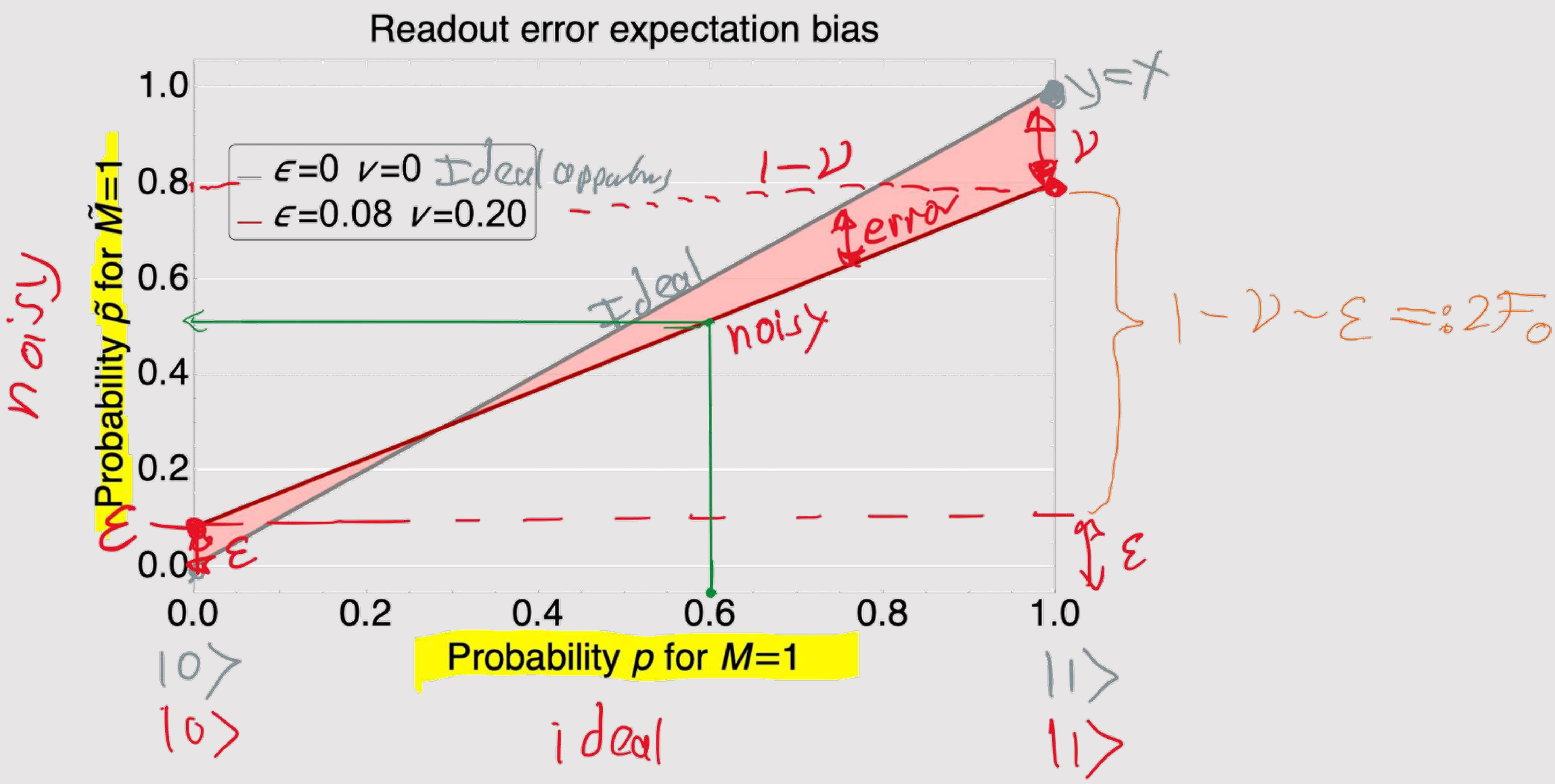
$\sum_m A_{mn} = 1$ for any n Stochastic matrix

$$\begin{aligned} \tilde{p} &= \varepsilon(1-p) + (1-\nu)p \\ &= \varepsilon - p\varepsilon + p - \nu p \\ &= p + \varepsilon - (\nu+\varepsilon)p \end{aligned}$$

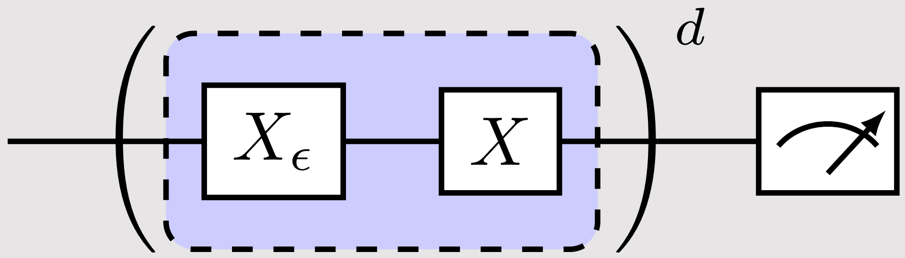
Readout error



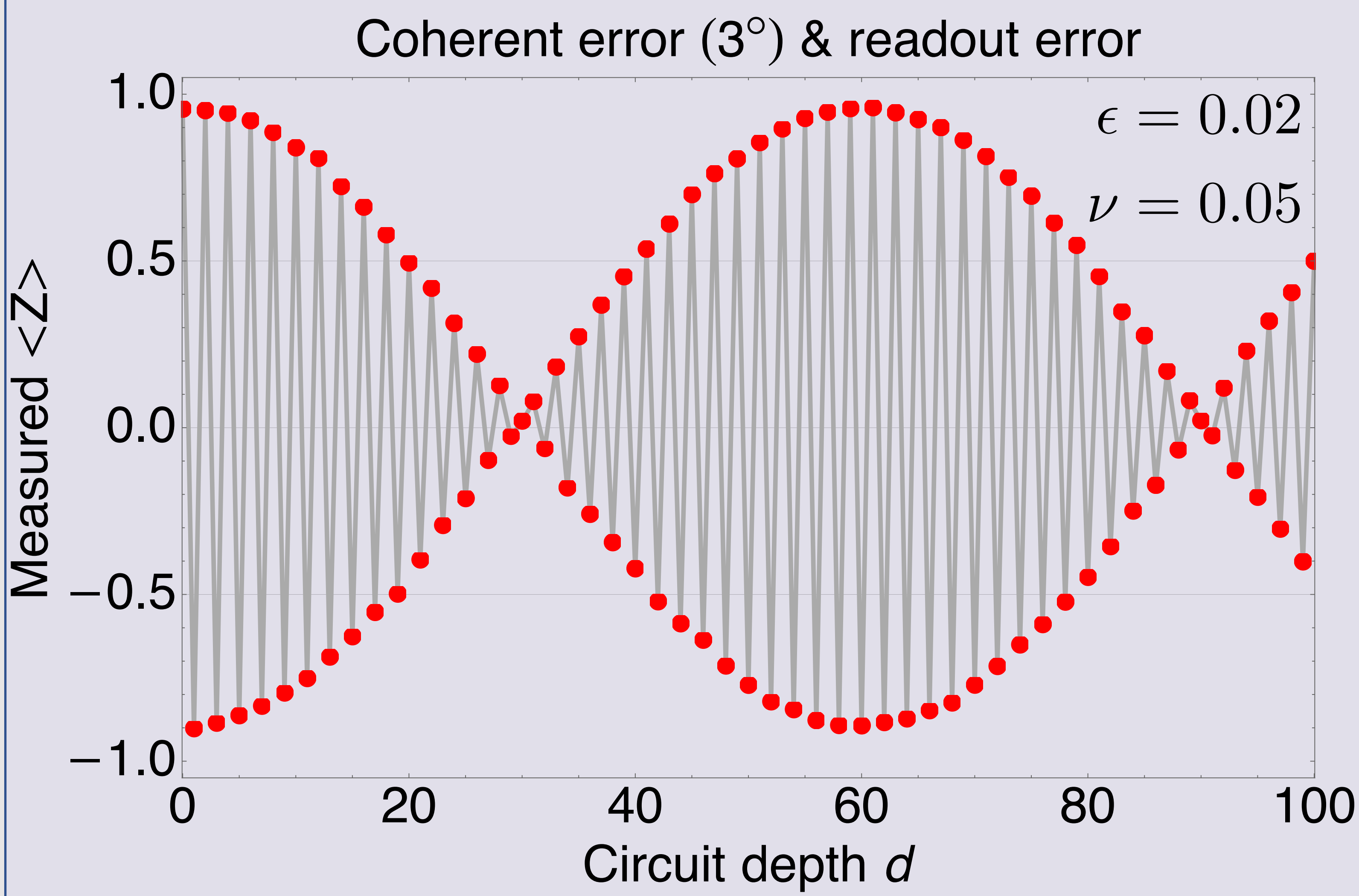
Assignment fidelity



Projection & sampling noise



$$A = \begin{array}{c c} & \begin{array}{cc} M=0 & M=1 \end{array} \\ \begin{array}{c} \tilde{M}=0 \\ \tilde{M}=1 \end{array} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{array}$$



Questions

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in the chat; for example, type “1a 2b.”

1. Readout error is due to
   a) measurement apparatus that could be made more efficient
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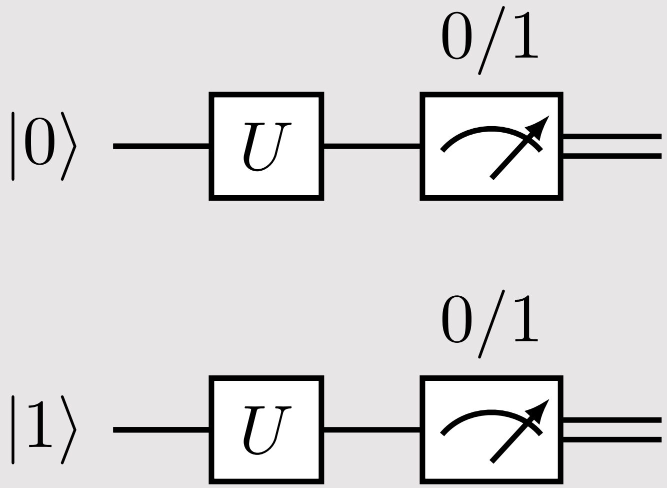
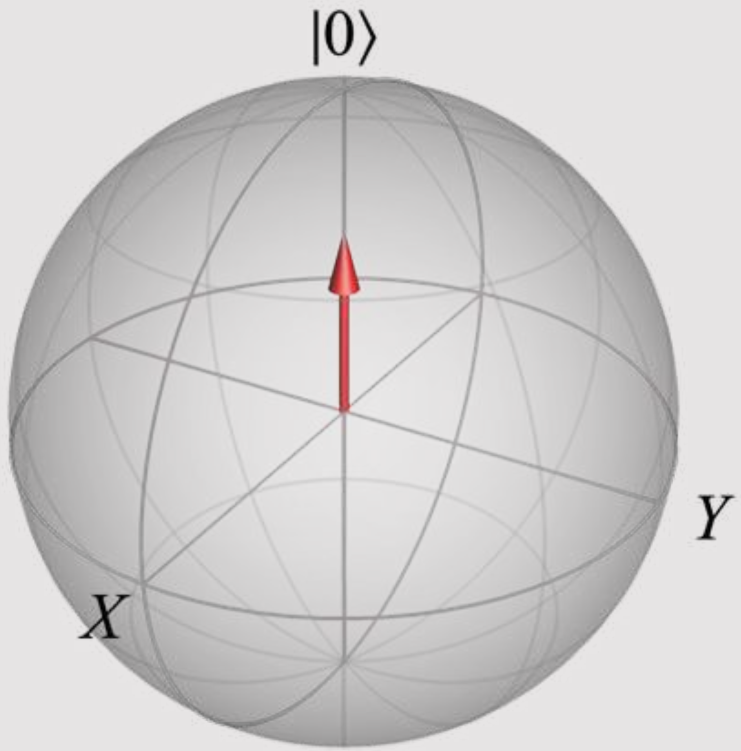
   c) core nature of quantum physics

2. To reduce readout error bias
   a) increase the number of sample
   b) you cannot undo it
   c) apply readout error mitigation

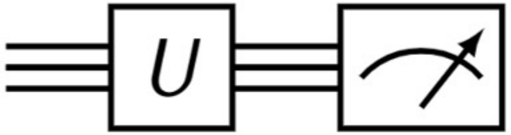
State prep



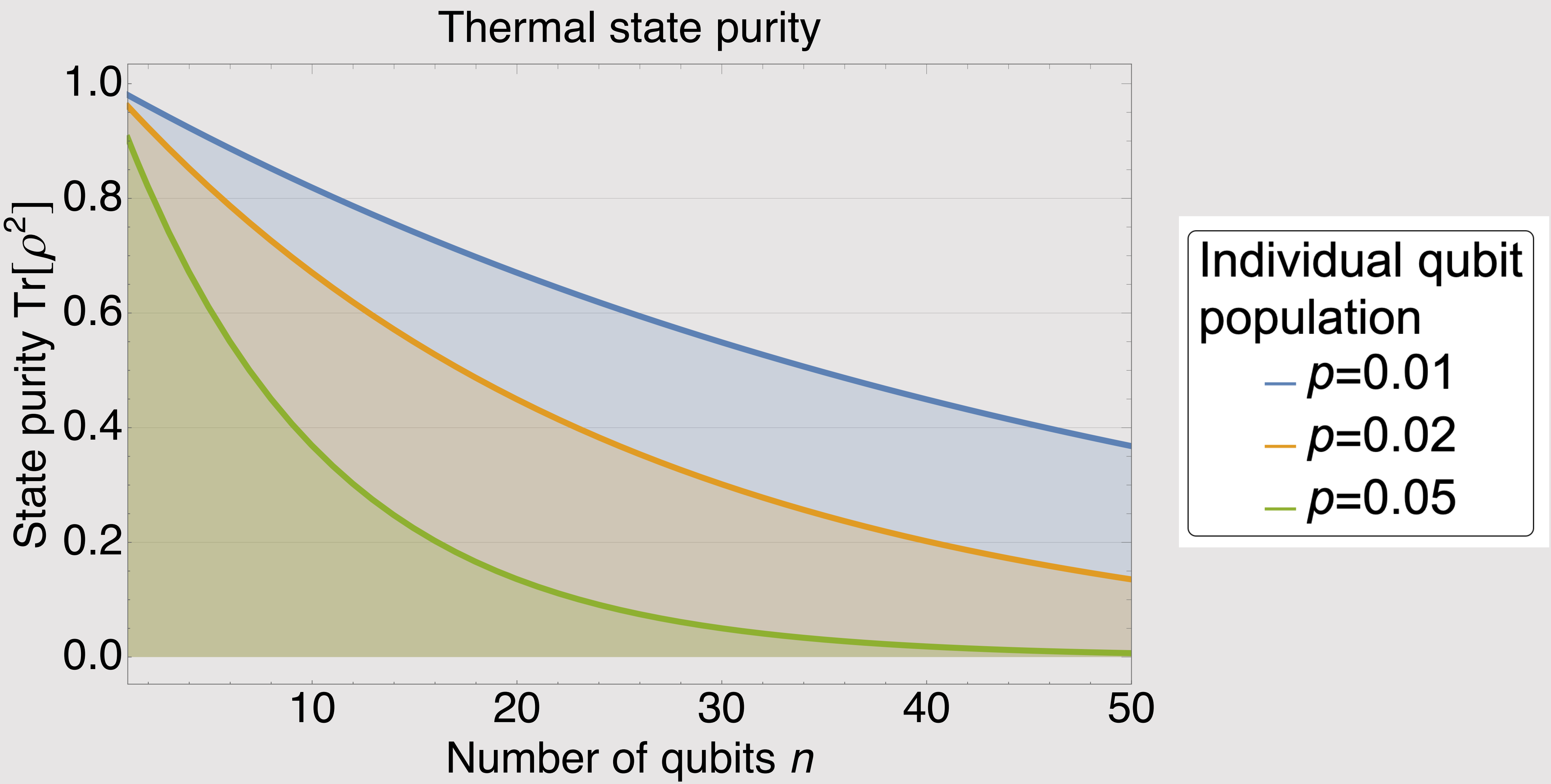


Multiple qubits

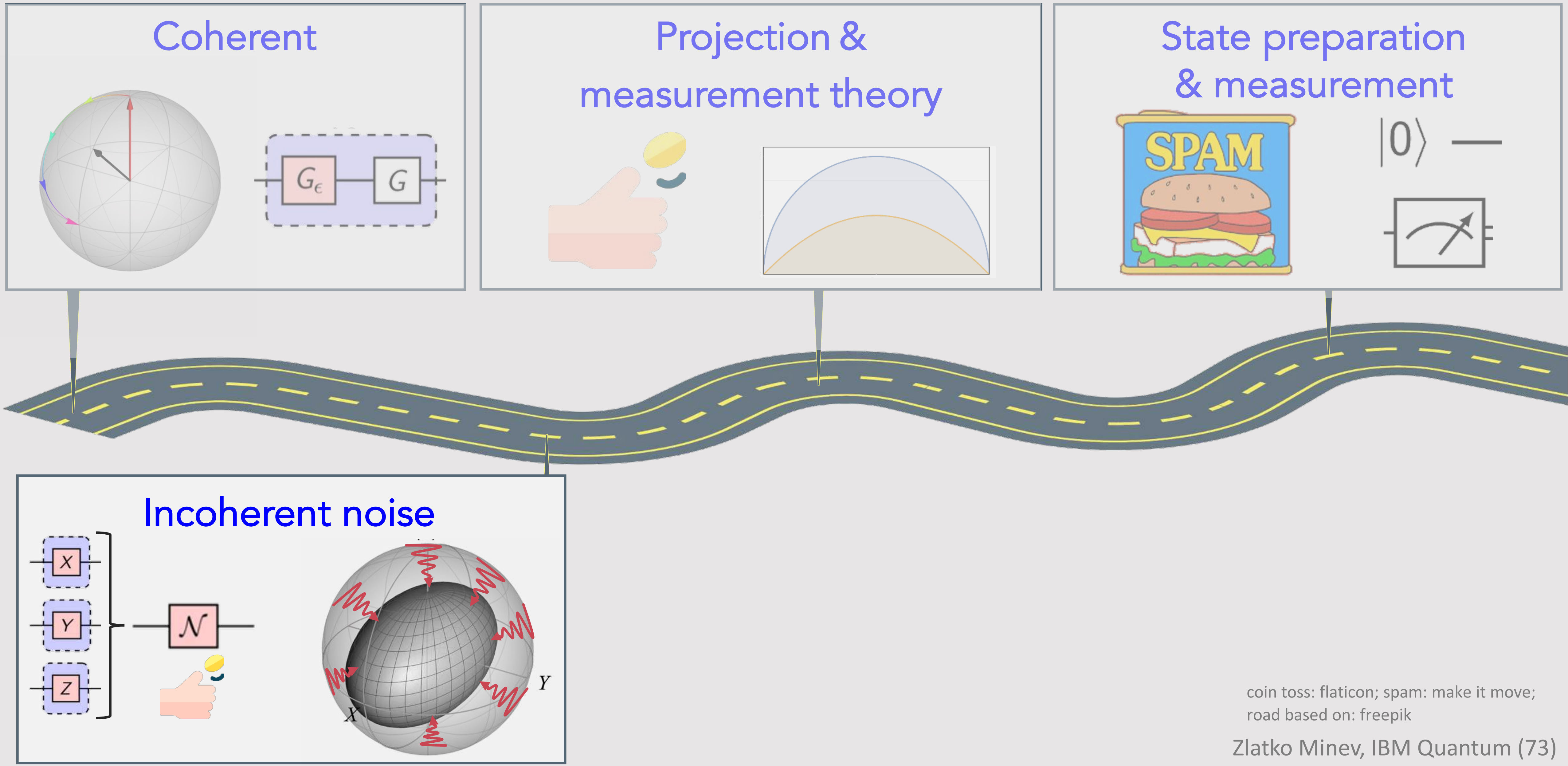
$$\left[(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|\right]^{\otimes n}$$



Thermal state

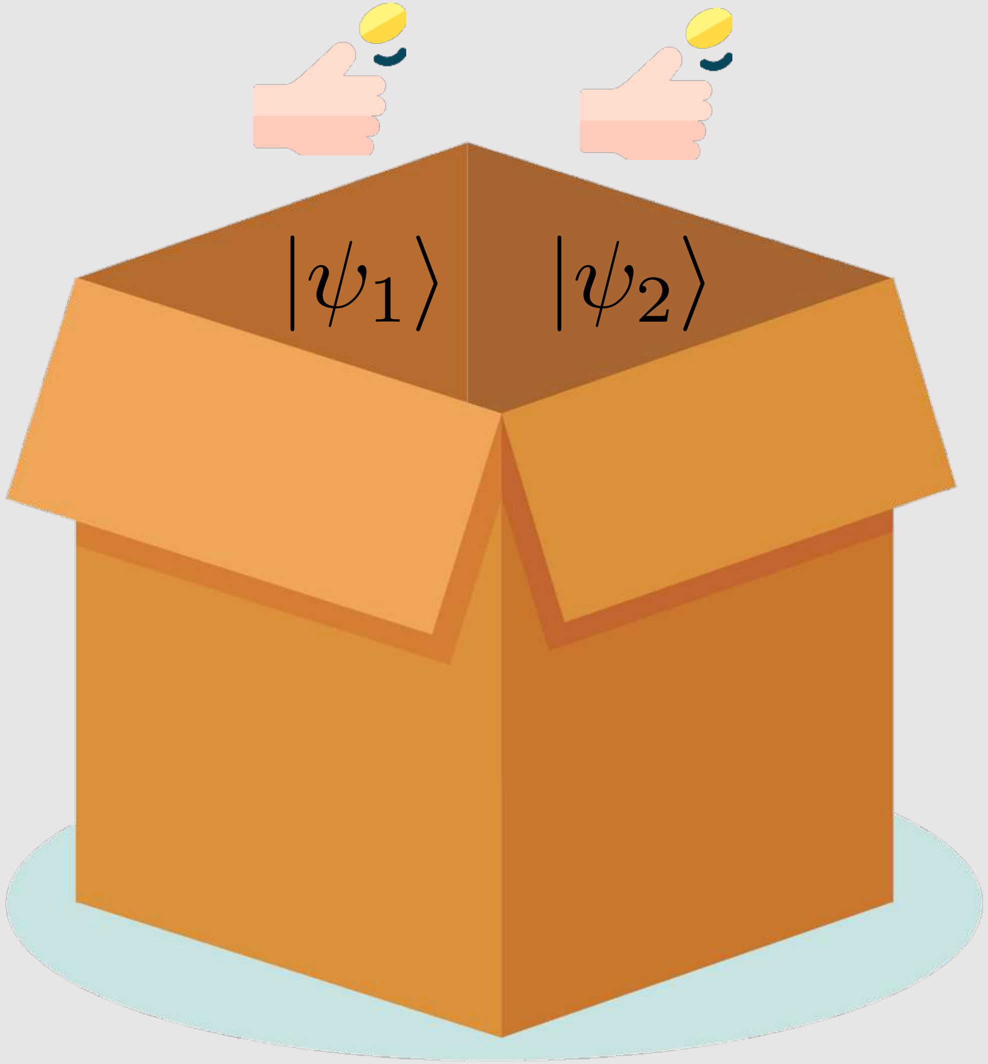


Incoherent noise

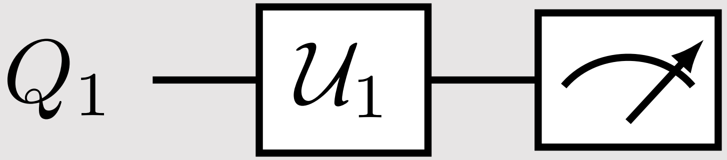


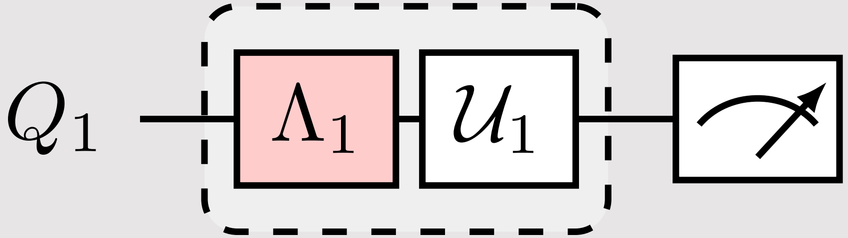
coin toss: flaticon; spam: make it move;
road based on: freepik

Review: mixed state (density matrix)

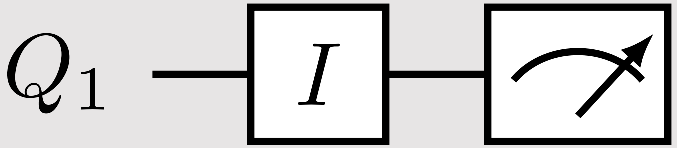


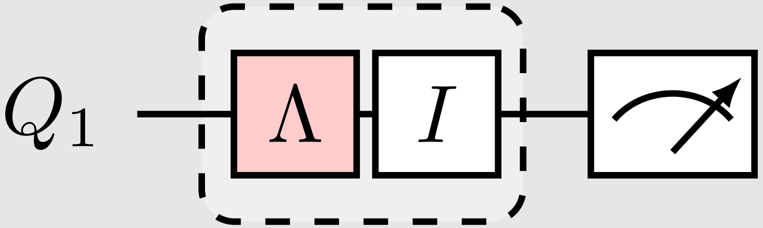
Toy model





Toy model: noise unraveling into quantum trajectories



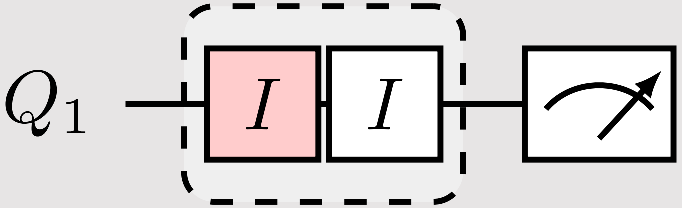


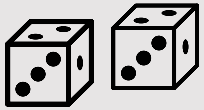
bit-flip noise

=

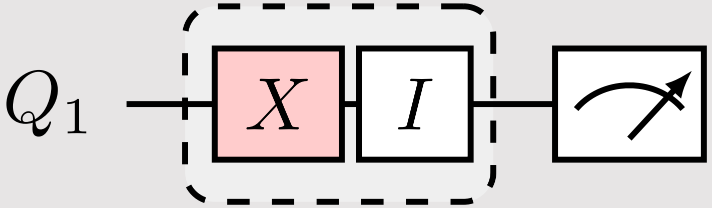
unraveling
(quantum trajectories)

probability 1-p



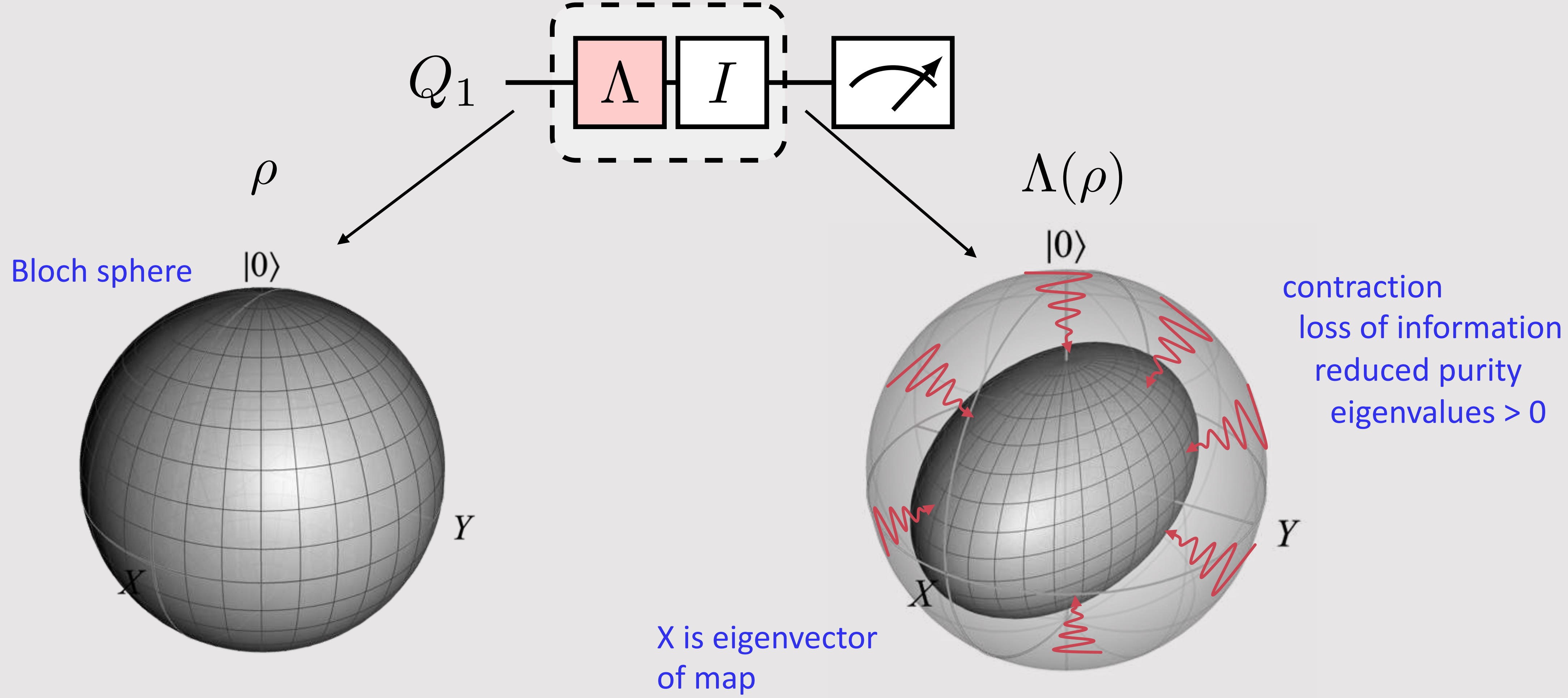


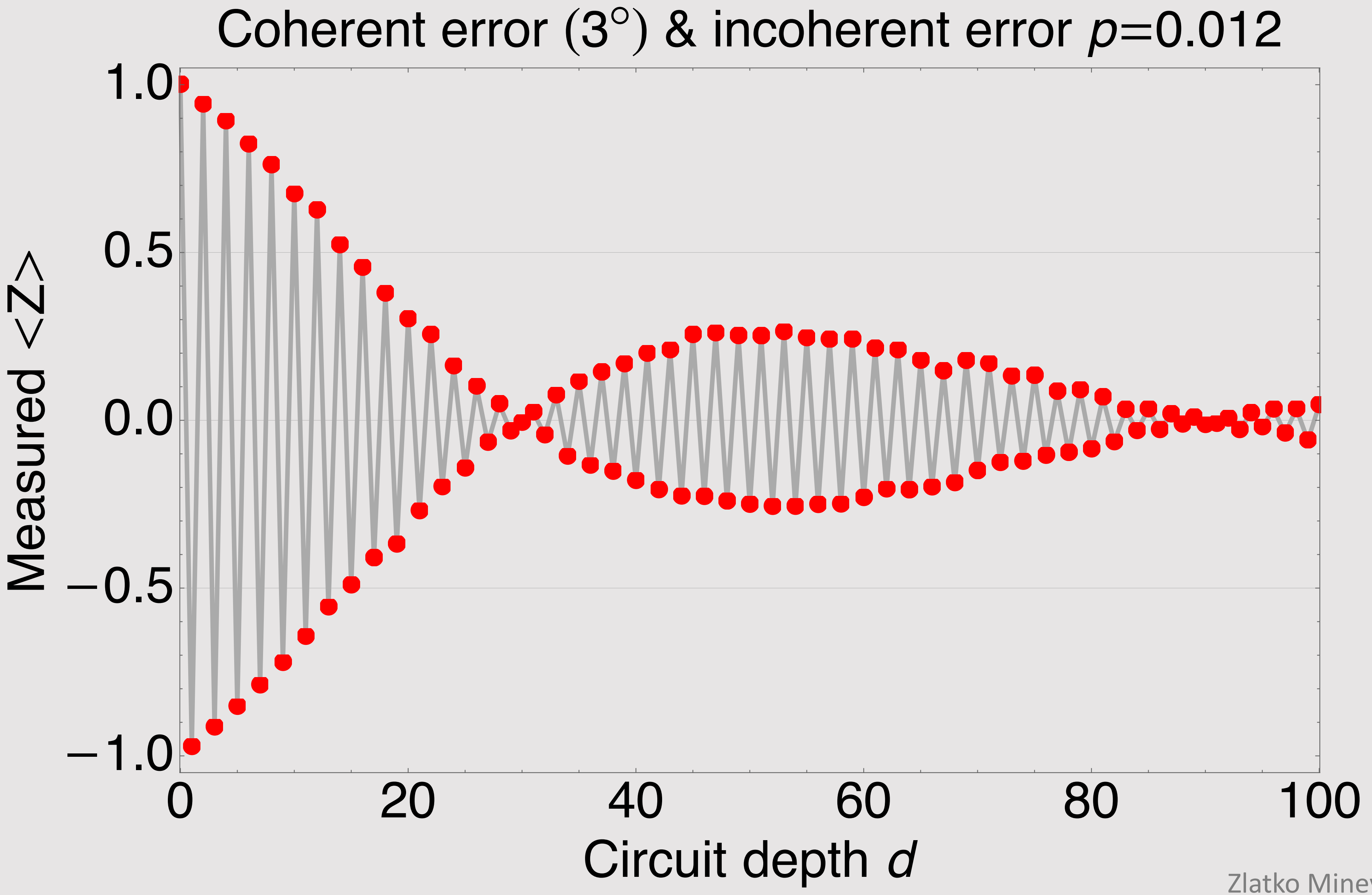
probability p

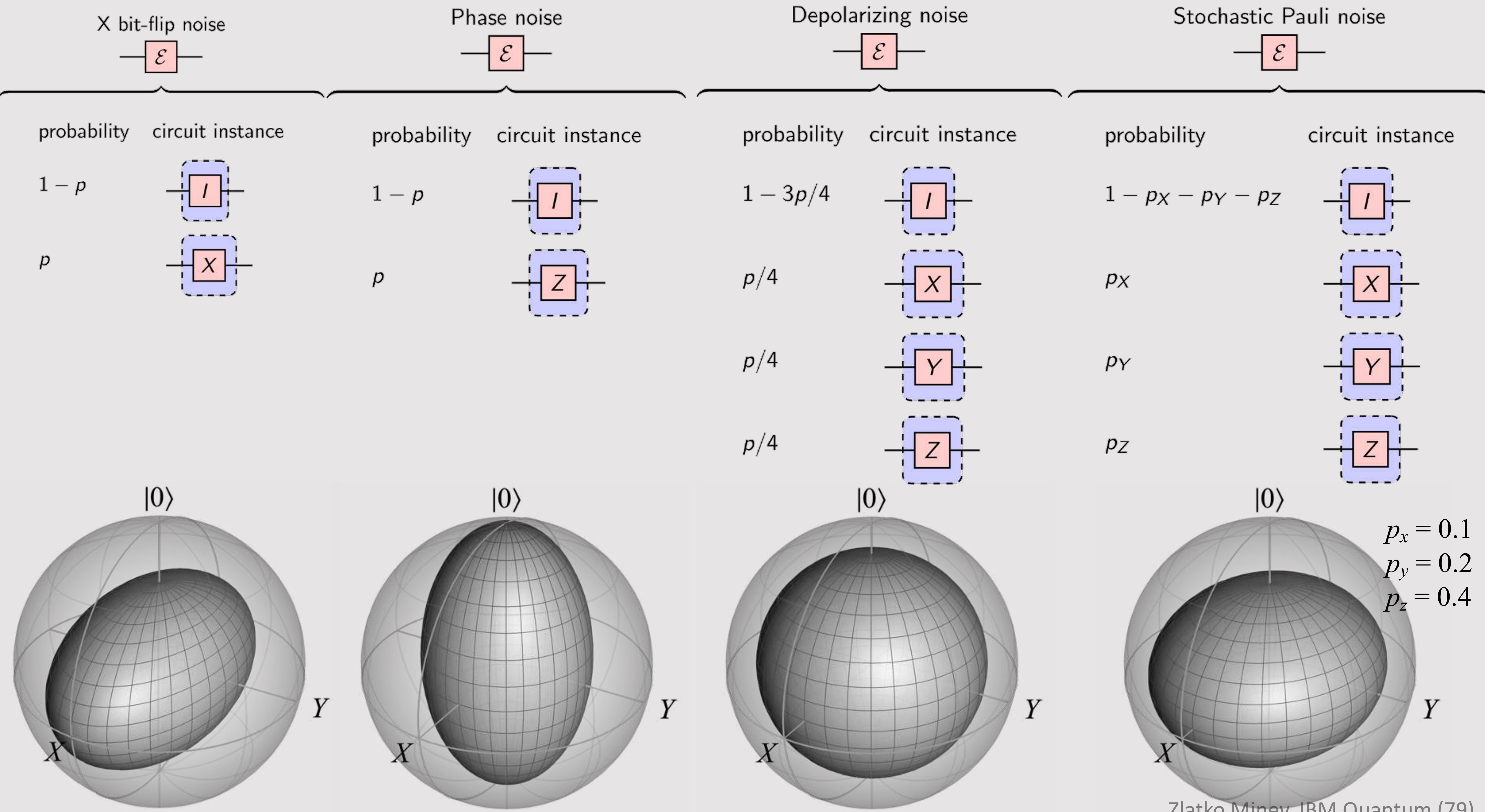


$$\Lambda\left(\rho\right) = (1-p)I\rho I + pX\rho X$$

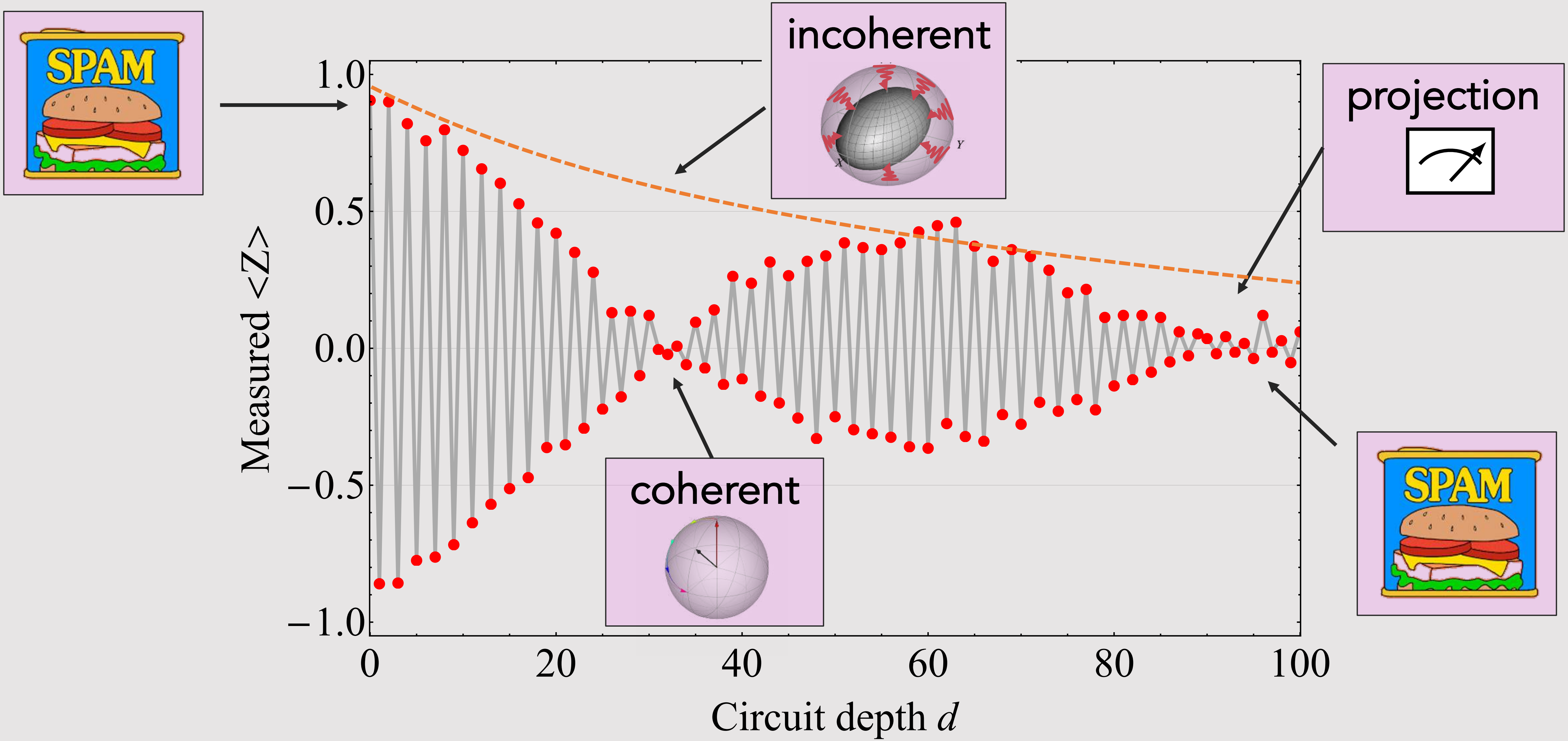
Toy model: noise unraveling into quantum trajectories




Coherent error (3°) & incoherent error p=0.012
1.0
0.5
0.0
−0.5
−1.0
Measured <Z>
0
20
40
60
80
100
Circuit depth d

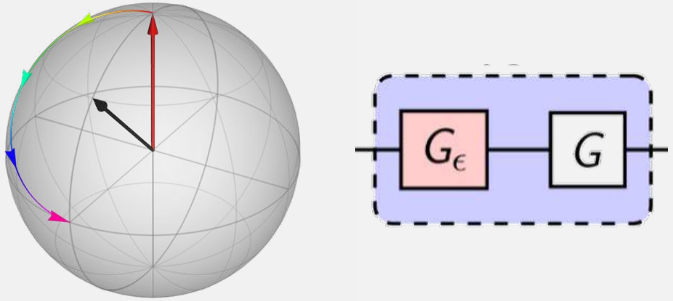

X bit-flip noise
$\mathcal{E}$
probability
circuit instance
$1-p$
I
p
X
Phase noise
$\mathcal{E}$
probability
circuit instance
$1-p$
I
p
Z
Depolarizing noise
$\mathcal{E}$
probability
circuit instance
$1-3p/4$
I
$p/4$
X
$p/4$
Y
$p/4$
Z
Stochastic Pauli noise
$\mathcal{E}$
probability
circuit instance
$1-p_X-p_Y-p_Z$
I
p_X
X
p_Y
Y
p_Z
Z
$|0\rangle$
X
Y
$|0\rangle$
X
Y
$|0\rangle$
X
Y
$|0\rangle$
X
Y
$p_x = 0.1$
$p_y = 0.2$
$p_z = 0.4$

Elements of noise



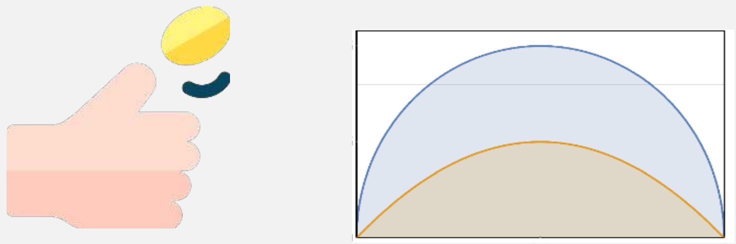
Our journey so far

Coherent noise

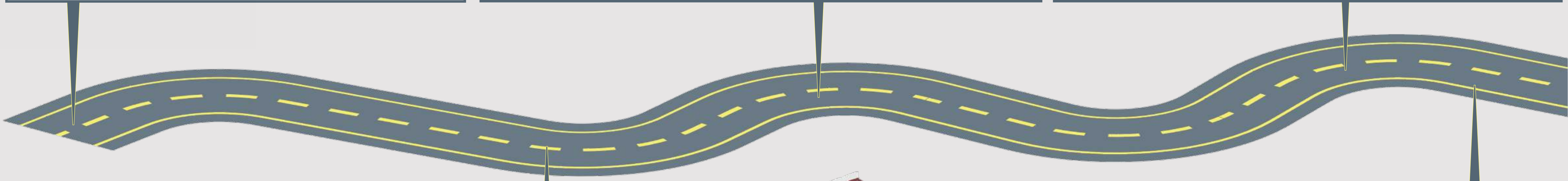


Measurement in a nutshell

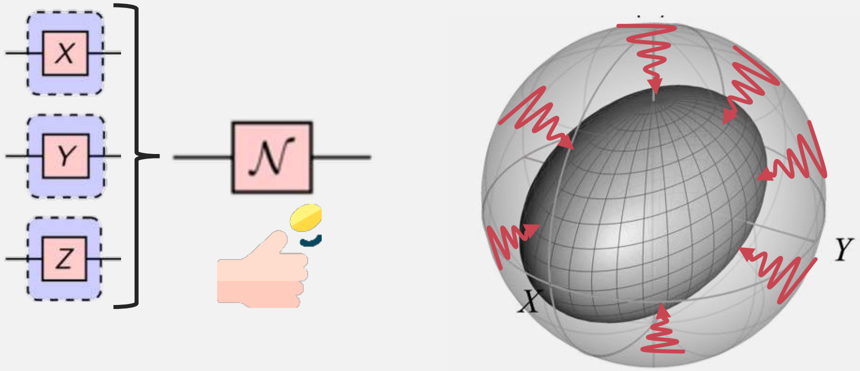
Projection noise



SPAM: Noisy meters



Incoherent noise

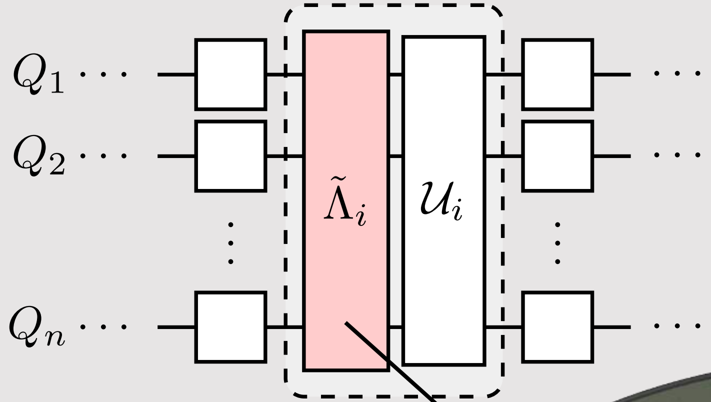


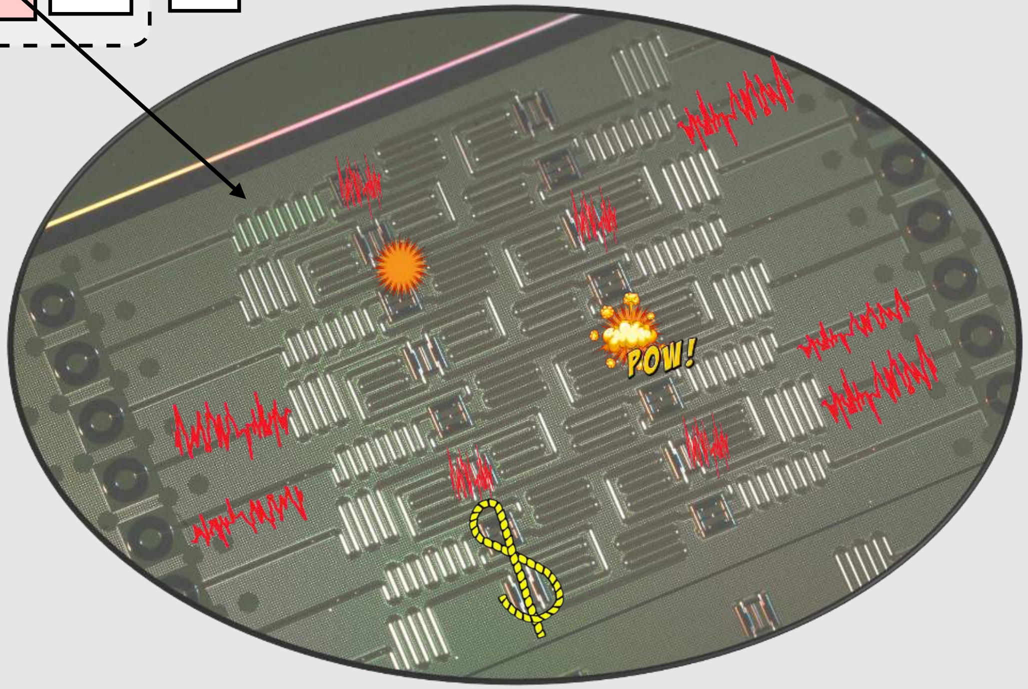
Bonus content

Coherent ZZ

State preparation

Outlook: Is it possible to learn & correct the noise with accuracy, efficiency, and scalability?

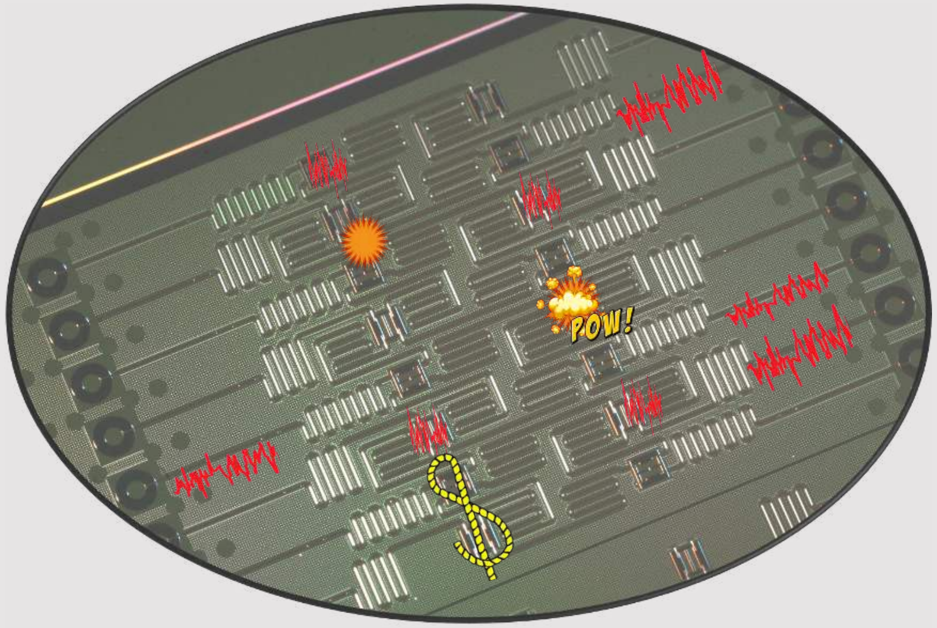




Energy relaxation T_1
Dephasing T_2
Coherent errors ZZ
Classical crosstalk
Quantum crosstalk
State preparation error
Measurement correlated errors
…

Control errors
Photon shot noise
1/f charge noise
1/f flux noise
Nonequilibrium quasiparticles
Leakage
Cosmic rays
…

Error mitigation and error correction



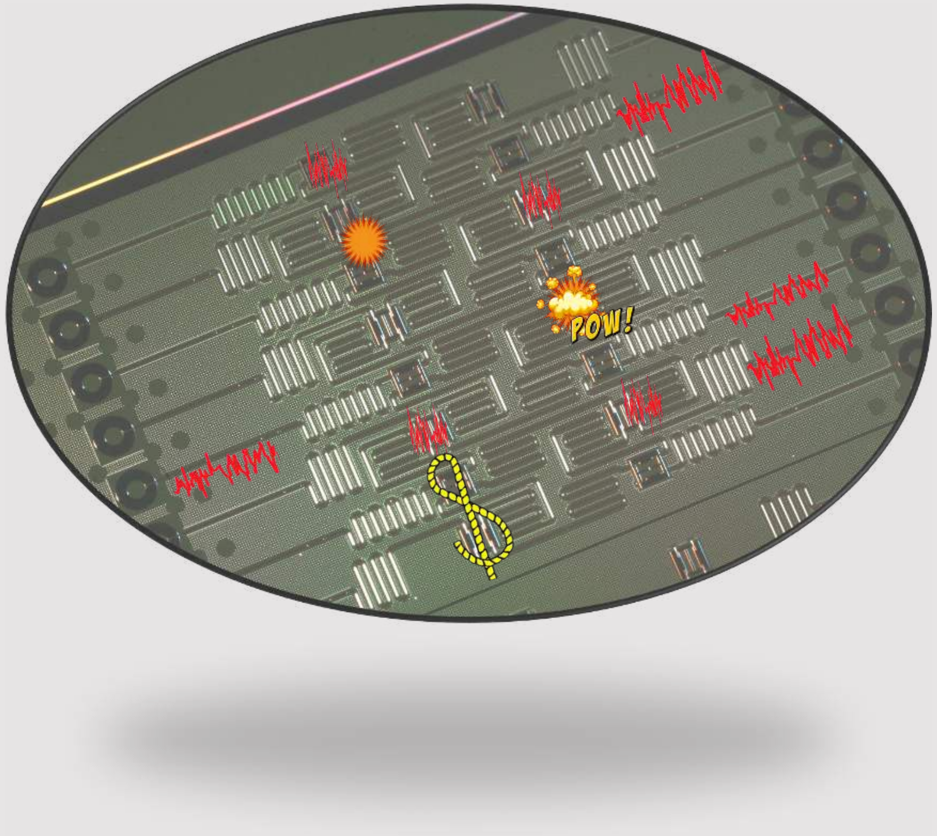
Error mitigation: working with what you have

- **benefit** suppress errors on classical results (expectation values)
- **q-cost** no extra qubits or hardware resources needed
- **c-cost** trades classical resources (post-processing) for lower error
- **limitation** bad asymptotic scaling: high number of samples & circuits

Error correction: protecting quantum information

- **benefit** suppress & correct errors to arbitrarily small level
- **q-cost** very large qubit and hardware overhead
- **c-cost** decoding and encoding can be classically costly
- **challenge** requires fault-tolerant operations and readout

Latest: Qiskit Quanutm Seminar YouTube Series



Ways to learn more
TODO

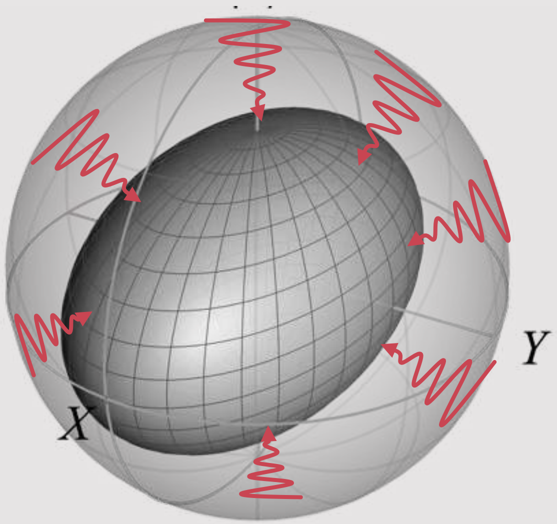
Introduction to Quantum Noise

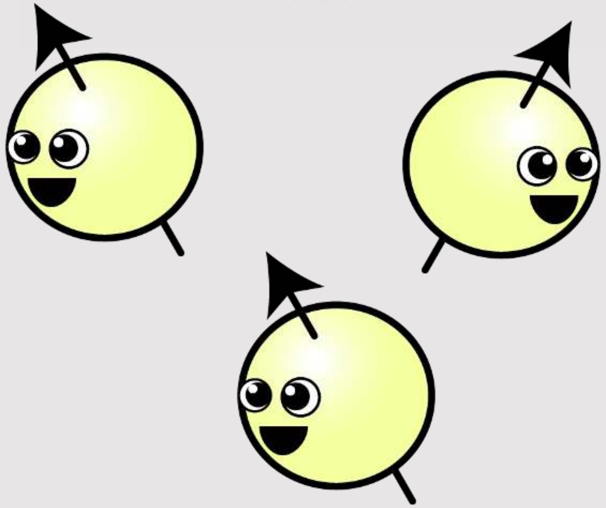
Lab work with Qiskit

Run experiments on real devices

Check out references, problems given in the lecture, dangerous bends

Stay in touch





Thank you!

Zlatko K. Minev

The important thing is not to stop questioning. Curiosity has its own reason for existence.

One cannot help but be in awe when he contemplates the mysteries of eternity, of life, of the marvelous structure of reality.

It is enough if one tries merely to comprehend a little of this mystery each day.

Albert Einstein

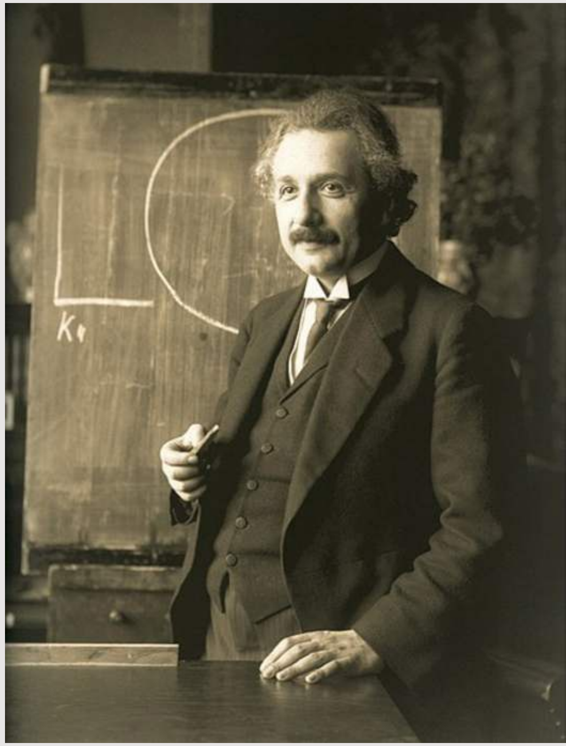
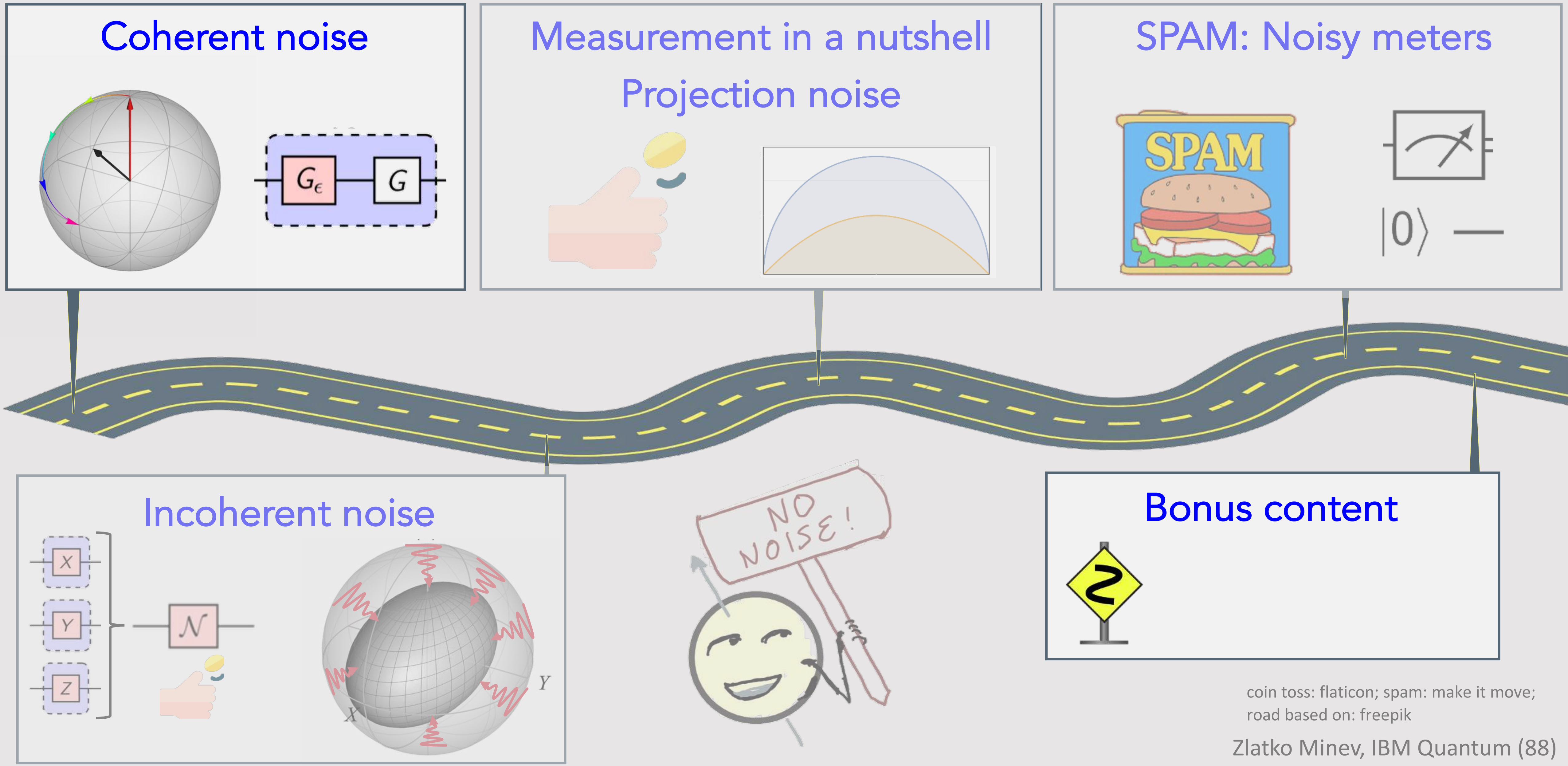


Photo: F. Schmutzer

Bonus content

Bonus content



coin toss: flaticon; spam: make it move;
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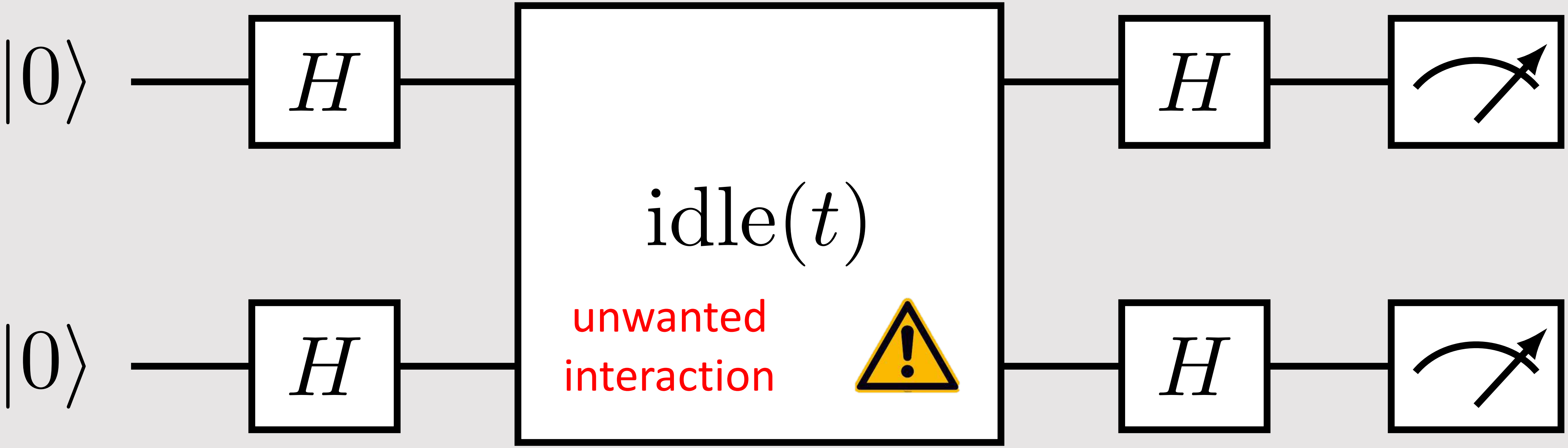
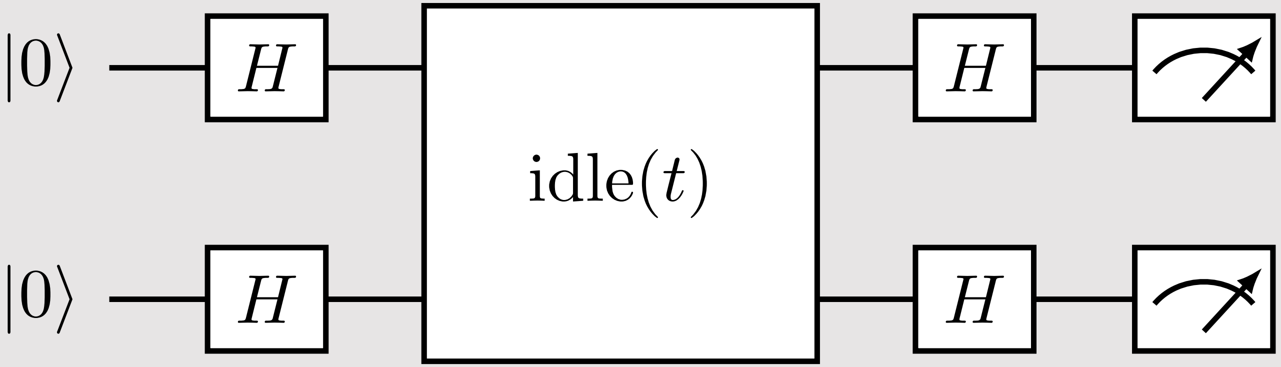
Bonus content: two-qubit coherent ZZ error



Bonus content: two-qubit ZZ error



Hadamard gate

$\boxed{H}$

$$H = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ |1\rangle & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}$$

$$H\,|0\rangle = |{+x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$H\,|1\rangle = |{-x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Breakout to notebook

Introduction to quantum noise

Coherent errors

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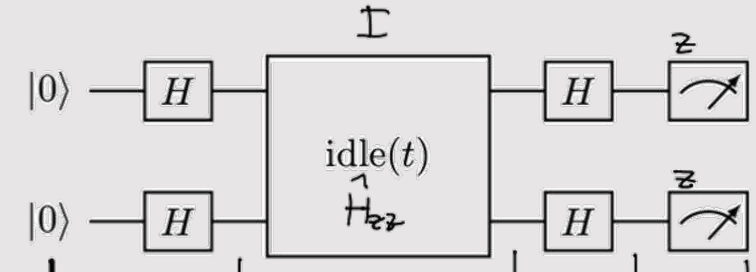


$|0\rangle|0\rangle$ $\qquad$ $|X\rangle|X\rangle$ $\qquad$ $|X\rangle|X\rangle$ $\qquad$ $|0\rangle|0\rangle$

$|+\rangle|+\rangle$

$$\langle ZI\rangle = +1$$
$$\langle IZ\rangle = +1$$
$$\begin{aligned}\langle ZZ\rangle &= \langle 0|\langle 0|\, Z\, Z|0\rangle|0\rangle \\ &= \langle 0|Z|0\rangle\langle 0|Z|0\rangle \\ &= (+1)(+1) \\ &= +1\end{aligned}$$

Hadamard gate

$\boxed{H}$

$$H = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ |1\rangle & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}$$

$$\begin{cases} H|0\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ H|1\rangle = |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \end{cases}$$

$$X|+x\rangle = +1\,|+x\rangle$$
$$X|-x\rangle = -1\,|-x\rangle$$
$$|+x\rangle =: |+\rangle$$
$$|-x\rangle =: |-\rangle$$

$$\begin{cases} Z|0\rangle = +1\,|0\rangle \\ Z|1\rangle = -1\,|1\rangle \end{cases}$$

NOISY

ZZ Interaction

$$\hat{H} = \frac{1}{2}\hbar\omega \hat{Z}\hat{Z}$$

$$\hat{U}(t) = \exp\left(-i\hbar^{-1}\hat{H}t\right)$$

$$= \exp\left(-i\frac{\omega t}{2}\hat{Z}\hat{Z}\right)$$

$$= \cos\left(\frac{\omega t}{2}\right)\hat{I} - i\sin\left(\frac{\omega t}{2}\right)\hat{Z}\hat{Z}$$

$$= \hat{R}_{ZZ}(\theta = \omega t)$$

$$\hat{R}_X(\theta) = \exp\left(-i\frac{\theta}{2}\hat{X}\right) \quad X^2 = I$$

$$= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)X$$

$$\left(\hat{Z}\hat{Z}\right)^2 = Z^2Z^2 = I_2 \otimes I_2 = \underline{I_4}$$

$$|0\rangle|0\rangle \xrightarrow{HH} |+\rangle|+\rangle \xrightarrow{\text{idle}} R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}|-\rangle|-\rangle \xrightarrow{HH} \underline{\cos\theta/2\,|0\rangle|0\rangle - i\sin\theta/2\,|1\rangle|1\rangle}$$

$$R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}\left(Z|+\rangle\right)_1 \otimes \left(Z|+\rangle\right)_2$$

$$Z|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle$$

$$Z|+\rangle = |-\rangle$$

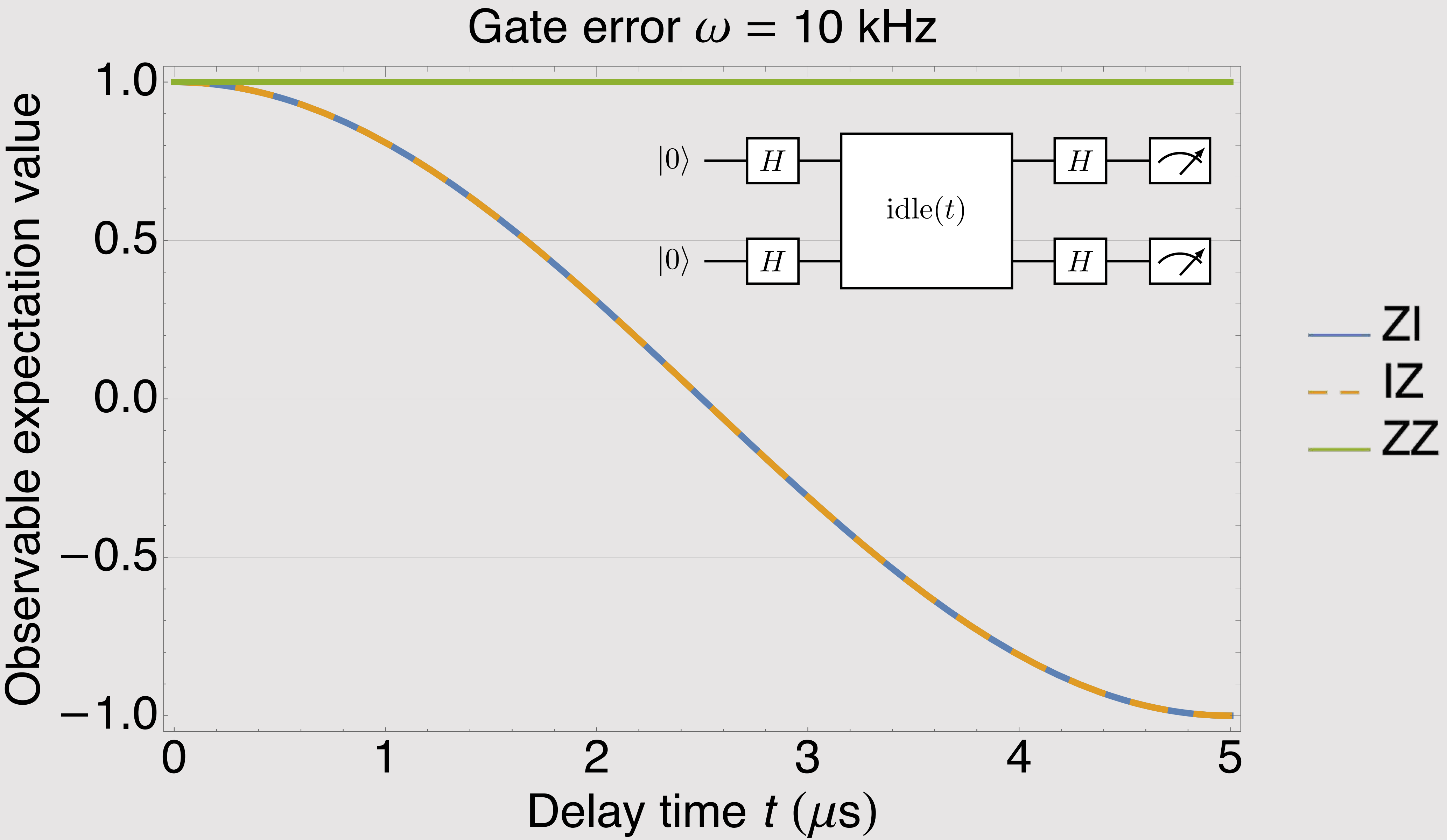
$$Z|-\rangle = |+\rangle$$

$$\langle ZI\rangle = \cos\omega t$$

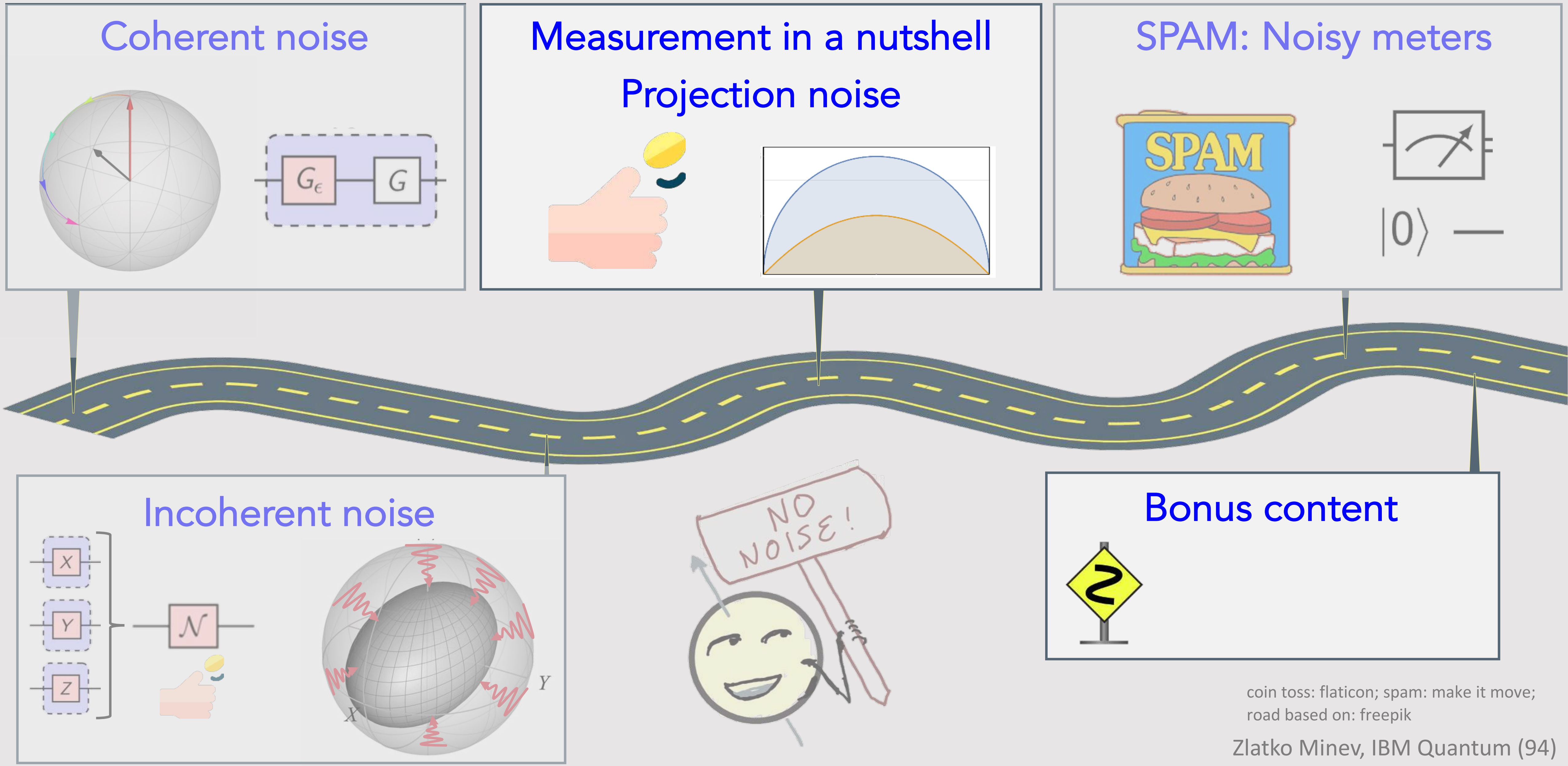
$$\langle IZ\rangle = \cos\omega t$$

$$\langle ZZ\rangle = 1$$

Bonus content: two-qubit ZZ error



Bonus content



coin toss: flaticon; spam: make it move;
road based on: freepik

A matrix

Bonus section content:

Reconstruct A matrix

$|0\rangle - \boxed{\text{meas}} = (M=0) = \boxed{A} = \tilde{M}$

$p = 0 \qquad \tilde{p} = \varepsilon$

$|0\rangle - \boxed{X} - |1\rangle - \boxed{\text{meas}} = M = \boxed{A} = \tilde{M}$ ← we have access to

$p = 1 \qquad \tilde{p} = 1 - \nu$

Noise mitigation

we know A

measure $\tilde{p}$, $\tilde{P}_M$ noisy

find p, P_M ideal

$\dim A = 2^n \times 2^n$ $\quad n = \#\text{qubits}$

$\tilde{P}_M = A P_M$

$P_M = A^{-1} \tilde{P}_M$

$p =$

Assignment Fidelity

$$F_0 = 1 - 1/2\,[p(\tilde{M}=1 \mid M=0) + p(\tilde{M}=0 \mid M=1)]$$

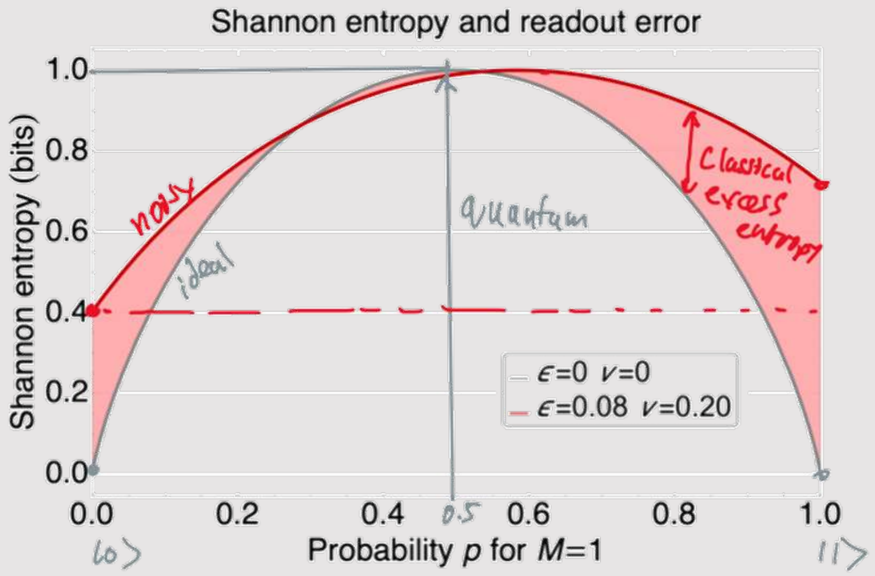
$$= 1/d \; \mathrm{Tr}(A) \qquad d = 2^n,\ n = \#\text{qubits}$$

$$= 1 - 1/2\,(\mu + \nu)$$

Shannon Entropy

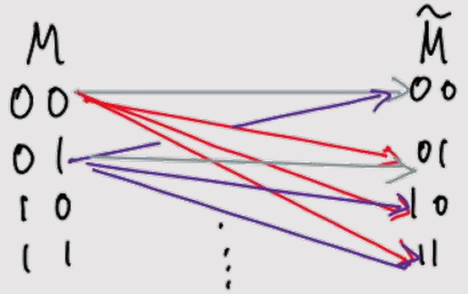
$$H(M) = H(P_M) = -\sum_m p_m \log_2 p_m = -(1-p)\log_2(1-p) - p \log_2 p$$

Binary entropy

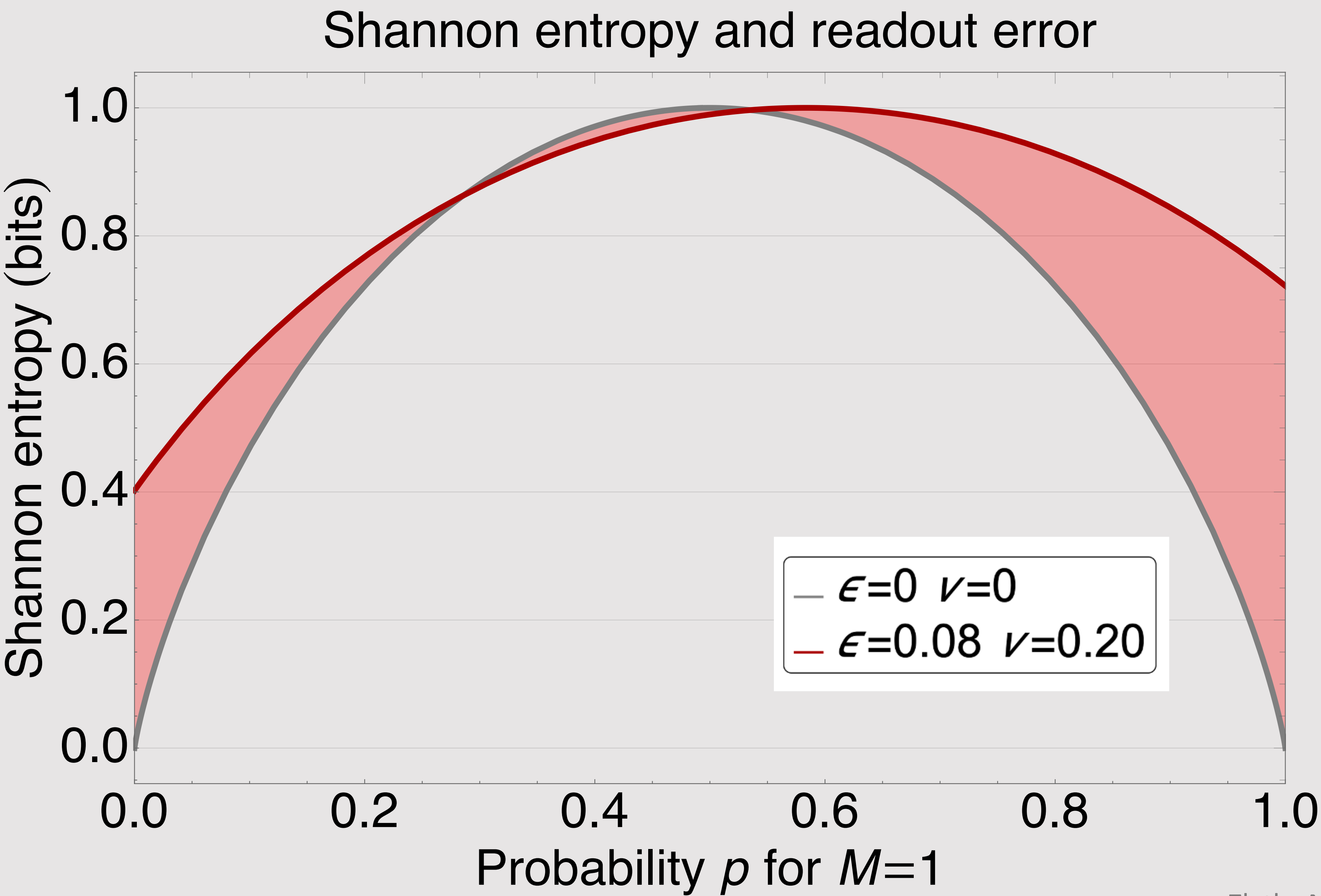


Larger System

n qubits $\qquad \dim A = 2^n \times 2^n$



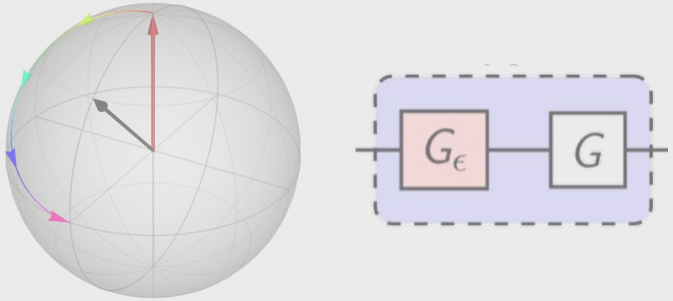
Entropy



Shannon entropy and readout error
Shannon entropy (bits)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Probability p for M=1
ϵ=0 ν=0
ϵ=0.08 ν=0.20

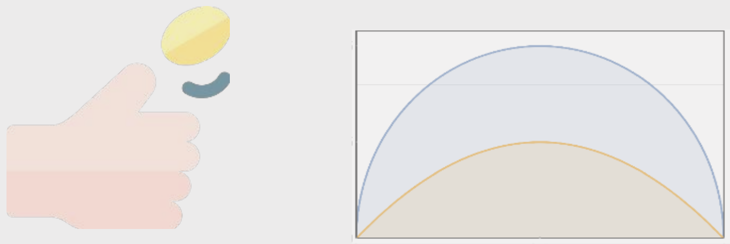
Bonus content

Coherent noise

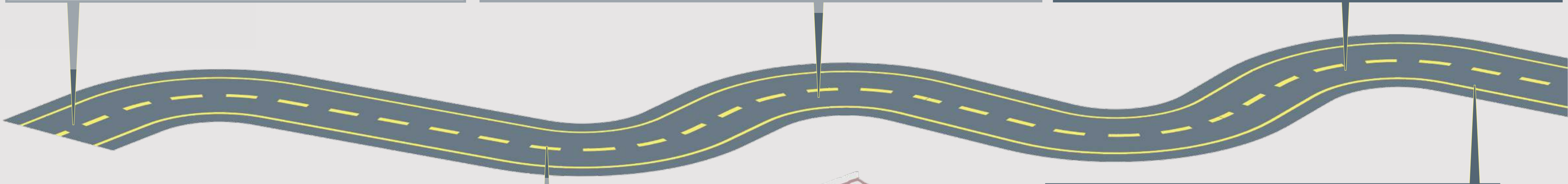


Measurement in a nutshell

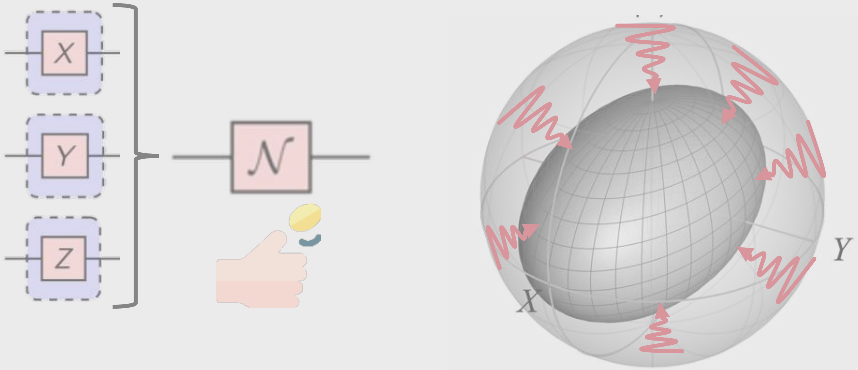
Projection noise



SPAM: Noisy meters



Incoherent noise



Bonus content

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road based on: freepik

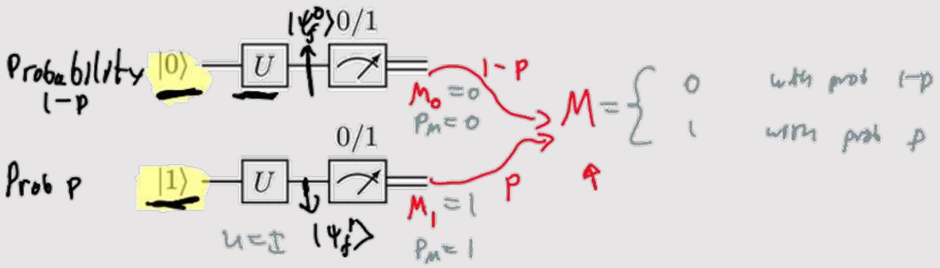
State prep noise

Introduction to quantum noise

State preparation noise

Qiskit Global Summer School on Quantum Machine Learning

Zlatko K. Minev



Density operator

$$\rho = (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$$
$$= P_{noerror}\,\rho_{ideal} + P_{error}\,\rho_{error}$$

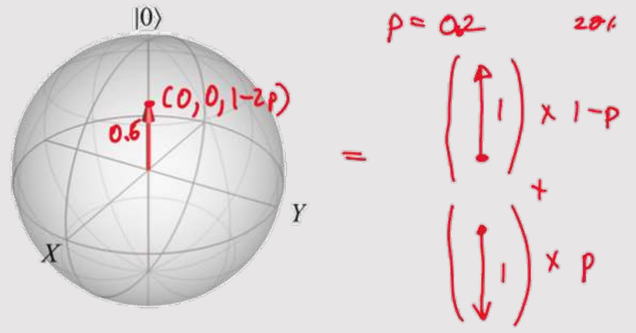
$$\rho_f = P_{noerror}\,\rho^{final}_{ideal} + P_{error}\,\rho^{final}_{error}$$
$$= (1-p)\,|\psi_f^0\rangle\langle\psi_f^0| + p\,|\psi_f^1\rangle\langle\psi_f^1|$$
$$= (1-p)\,U|0\rangle\langle 0|U^\dagger + p\,U|1\rangle\langle 1|U^\dagger$$
$$= U\left[(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|\right]U^\dagger$$
$$= U\rho U^\dagger$$
$$\rho_U = U\rho U^\dagger$$

$$(U|\psi\rangle)^\dagger = (|\psi_f\rangle)^\dagger$$
$$\langle\psi_f| = \langle\psi|U^\dagger$$

$$\rho_U = U\rho U^\dagger$$
$$|0\rangle\langle 0| = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$
$$|1\rangle\langle 1| = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$
$$\rho = \begin{pmatrix}1-p&0\\0&p\end{pmatrix} = \tfrac{1}{2}\left(\hat{I} + (1-2p)\hat{Z}\right)$$
$$\langle X\rangle = \mathrm{Tr}[X\rho] = 0$$
$$\langle Y\rangle = \mathrm{Tr}[Y\rho] = 0$$
$$\langle Z\rangle = \mathrm{Tr}[Z\rho] = 1-2p$$

$$\mathrm{Tr}[P_\alpha P_\beta] = 2\delta_{\alpha\beta} \quad \text{for } \alpha,\beta \in \{X,Y,Z\}$$

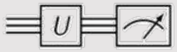
p = 0.2   20% take |1⟩



$$\mathrm{Tr}(\rho^2) = \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 0^2 + 0^2 + (1-2p)^2$$

State prep noise

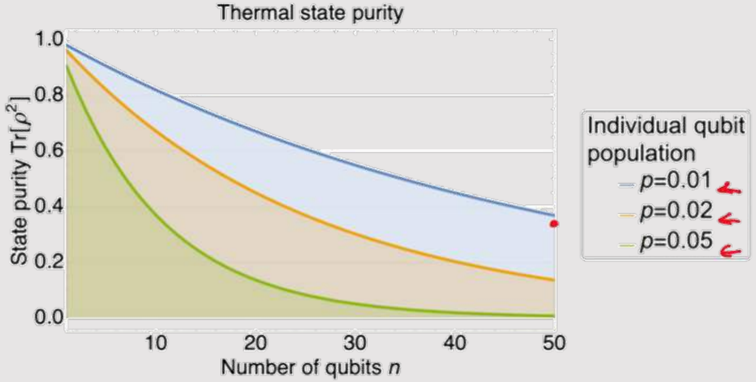
Scaling to larger number of qubits

$[(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|]^{\otimes n}$ 

$$\rho_0 = [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|]^{\otimes n}$$
$$= (1-p)^n |000\cdots 0\rangle\langle 0\cdots 0| + \cdots |0100\cdots\rangle\langle .1. \cdots$$

$$P(M = 0000\cdots 0) = \begin{cases} 1 & \text{if } p=0 \quad (\text{ideal case}) \\ (1-p)^n \simeq 1 - np + O(p^2) & \text{for } p>0 \end{cases}$$

$$\langle Z_k \rangle = \langle \mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1} \overset{\text{at } k}{Z} \mathbb{1}\mathbb{1}\mathbb{1}\cdots \rangle = 1-2p \xrightarrow{\text{much faster}}$$
$$\langle ZZZ\cdots Z \rangle = \mathrm{Tr}[Z^{\otimes n}\rho] = (1-2p)^n$$
$$\rho_0 = \prod_{k=1}^{n} \otimes \frac{1}{2}\left(\hat{I}_k + (1-2p)\hat{Z}_k\right)$$
$$= \frac{1}{2^n}(1+\cdots Z)\otimes(1+\cdots Z)\otimes \cdots$$
$$= \frac{1}{2^n}\left[(1-2p)^n Z^{\otimes n} + \cdots \right]$$
$$\mathrm{Tr}[\rho_0^2] = \prod_{k=1}^{n} \mathrm{Tr}(\rho_{0k}^2) = (1-2p)^{2n}$$



The End!