Solution manual - Scattering theory - Taylor

zlatomir.zhelev04

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Problem 4.1

We can write:

$$d\Omega = 2\pi d\left(\cos\theta\right)$$

From this we have to calculate:

$$\frac{d\Omega_{cm}}{d\Omega_{lab}} = \frac{d\left(\cos\theta_{cm}\right)}{d\left(\cos\theta_{lab}\right)} = \frac{1}{\frac{d(\cos\theta_{lab})}{d(\cos\theta_{cm})}}$$

Thus we have to find $\cos\theta_{lab}$ as a function of $\cos\theta_{cm}$. We can do this by using a formula such as one that can be seen in Landau & Lifschitz, Vol 1. Mechanics, which gives us:

$$\tan\theta_{lab} = \frac{\sin\theta_{cm}}{\lambda + \theta_{cm}}$$

Here $\lambda = \frac{m_1}{m_2}$ as in Taylor's text. We square the equation and obtain:

$$\tan^2\theta_{lab} = \frac{1}{\cos^2\theta_{lab}} - 1 = \frac{1-\cos^2\theta_{cm}}{\left(\lambda+\cos\theta_{cm}\right)^2}$$

$$\Longrightarrow \quad \frac{1}{\cos^2\theta_{lab}} = \frac{1-\cos^2\theta_{cm}+\cos^2\theta_{cm}+2\lambda\cos\theta_{cm}+\lambda^2}{\left(\lambda+\cos\theta_{cm}\right)^2} = \frac{1+2\lambda\cos\theta_{cm}+\lambda^2}{\left(\lambda+\cos\theta_{cm}\right)}$$

$$\Longrightarrow \quad \cos\theta_{lab} = \frac{\lambda+\cos\theta_{cm}}{\sqrt{1+2\lambda\cos\theta_{cm}+\lambda^2}}$$

$$\Longrightarrow \quad \frac{d\left(\cos\theta_{lab}\right)}{d\left(\cos\theta_{cm}\right)} = \frac{\sqrt{1+2\lambda\cos\theta_{cm}+\lambda^2} - \frac{\lambda(\lambda+\cos\theta_{cm})}{\sqrt{1+2\lambda\cos\theta_{cm}+\lambda^2}}}{\left(1+2\lambda\cos\theta_{cm}+\lambda^2\right)^{\frac{3}{2}}} = \frac{1+2\lambda\cos\theta_{cm}+\lambda^2-\lambda^2-\lambda\cos\theta_{cm}}{\left(1+2\lambda\cos\theta_{cm}+\lambda^2\right)^{\frac{3}{2}}}$$

$$\Longrightarrow \quad \frac{d\left(\cos\theta_{lab}\right)}{d\left(\cos\theta_{cm}\right)} = \frac{1+\lambda\cos\theta_{cm}}{\left(1+2\lambda\cos\theta_{cm}+\lambda^2\right)^{\frac{3}{2}}}$$

From which we finally obtain:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm}\frac{d\Omega_{cm}}{d\Omega_{lab}} = \left(\frac{d\sigma}{d\Omega}\right)_{cm}\frac{1+\lambda\cos\theta_{cm}}{\left(1+2\lambda\cos\theta_{cm}+\lambda^2\right)^{\frac{3}{2}}}$$

Problem 4.2

Using the notation from Taylor's book we have:

$$\sigma\left(d\Omega\leftarrow\phi\right) = \frac{d\Omega}{(2\pi m)^2}\int d^2\rho\, d^3\overline{p}\, d^3p'\, d^3p'' p^2 dp\, \delta\left(E_p - E_{p'}\right)\delta\left(E_p - E_{p''}\right) f^*\left(\mathbf{p}\leftarrow\mathbf{p}'\right) f^*\left(\mathbf{p}\leftarrow\mathbf{p}''\right)\Psi^*\left(\overline{\mathbf{p}},\mathbf{p}'\right)\Psi^*\left(\overline{\mathbf{p}},\mathbf{p}''\right)$$

Let us now look at the integral:

$$\begin{aligned}\int d^2\rho\Psi^*\left(\overline{\mathbf{p}},\mathbf{p}'\right)\Psi^*\left(\overline{\mathbf{p}},\mathbf{p}''\right) &= \int d^2\rho\; e^{i\rho\cdot\left(\mathbf{p}_1'-\mathbf{p}_1''\right)}\phi_1^*\left(\mathbf{p}_1'\right)\phi_2^*\left(\mathbf{p}_2'\right)\phi_1\left(\mathbf{p}_1''\right)\phi_2\left(\mathbf{p}_2''\right)\\ &= (2\pi)^2\,\delta_2\left(\mathbf{p}_{1\perp}'-\mathbf{p}_{1\perp}''\right)\phi_1^*\left(\mathbf{p}_1'\right)\phi_2^*\left(\mathbf{p}_2'\right)\phi_1\left(\mathbf{p}_1''\right)\phi_2\left(\mathbf{p}_2''\right)\end{aligned}$$

Due to the properties of the δ-function we can write:

$$\delta\left(E_p - E_{p'}\right)\delta\left(E_p - E_{p''}\right) = \delta\left(E_p - E_{p'}\right)\delta\left(E_{p'} - E_{p''}\right)$$

First in the CM frame the total momentum of both particles is strongly peaked about zero measurement (see Taylor), thus $\mathbf{p}_1 = -\mathbf{p}_2$, which means that there is a specific momentum where ϕ_2 is a small appreciably different from zero, which makes us write $\phi_2^*\left(\mathbf{p}_2'\right)\phi_2\left(\mathbf{p}_2''\right) \approx \left|\phi_2\left(\mathbf{p}_2'\right)\right|$ (this step is to be performed after calculating the integral over d^3p''). Second in this case measurement of $\mathbf{p}_1 = \mathbf{p}'$, thus we can write everywhere $\mathbf{p}'$ instead of $\mathbf{p}_1$. Third the energy $E_p = \frac{p^2}{2m}$, from which using the properties of the δ-functions:

$$\delta\left(\mathbf{p}_{1\perp}'-\mathbf{p}_{1\perp}''\right)\delta\left(\frac{\mathbf{p}'^2}{2m}-\frac{\mathbf{p}''^2}{2m}\right) = 2m\delta\left(\mathbf{p}_\perp'-\mathbf{p}_\perp''\right)\delta\left(\left(p_\parallel'+p_\parallel''\right)\left(p_\parallel'-p_\parallel''\right)\right)$$

We assume that the wavefunctions are sufficiently narrow so that the points corresponding to $p'_{\|} = -p''_{\|}$ do not contribute to the integral, so now when we integrate over d^3p'' we obtain:

$$\sigma\left(d\Omega \leftarrow \phi\right) = d\Omega \int d^3\overline{p}\ d^3p'\ p^2dp\frac{1}{p'_{\|}}\delta\left(\left(\mathbf{p}+\mathbf{p}'\right)\cdot\left(\mathbf{p}-\mathbf{p}'\right)\right)\left|f\left(\mathbf{p}\leftarrow\mathbf{p}'\right)\phi_1\left(\mathbf{p}'\right)\phi\left(\mathbf{p}'_2\right)\right|^2$$

We use similar arguments to above and evaluate:

$$\sigma\left(d\Omega \leftarrow \phi\right) = d\Omega \int d^3\overline{p}\ d^3p'\frac{p'}{p'_{\|}}\left|f\left(\mathbf{p}\leftarrow\mathbf{p}'\right)\phi_1\left(\mathbf{p}'\right)\phi\left(\mathbf{p}'_2\right)\right|^2$$

Given that the function change f changes slowly enough with respect to the peak of the wavefunction we can take it out of the integral ad obtain:

$$\sigma\left(d\Omega \leftarrow \phi\right) = d\Omega\left|f\left(\mathbf{p}\leftarrow\mathbf{p}'\right)\right|^2\int d^3\overline{p}\ d^3p'\left|\Psi\left(\overline{\mathbf{p}},\mathbf{p}'\right)\right|^2 = d\Omega\left|f\left(\mathbf{p}\leftarrow\mathbf{p}'\right)\right|^2$$

Where for small body angles we have:

$$\frac{d\sigma}{d\Omega}\left(d\Omega \leftarrow \phi\right) = \left|f\left(\mathbf{p}\leftarrow\mathbf{p}'\right)\right|^2$$

As for the second part it done by interchanging the indeces 1 and 2.

Chapter 5

Problem 5.1

When we transform $\langle \mathbf{p}_1', \mathbf{p}_2', \xi' | \, \mathbf{S} \, | \mathbf{p}_1, \mathbf{p}_2, \xi \rangle$ into CM frame we get a $\delta\left(\overline{\mathbf{p}'} - \overline{\mathbf{p}}\right)$, which we can factor out. If we neglect the no-scattering term then we can write:

$$\left\langle \mathbf{p}', \xi' \right| S \left| \mathbf{p}, \xi \right\rangle = \frac{i}{2\pi m} \delta\left(E_{p'} - E_p\right) f\left(\mathbf{p}', \xi' \leftarrow \mathbf{p}, \xi\right)$$

Now we use that:

$$|\chi\rangle = \sum_{\xi} \chi_\xi \left|\xi\right\rangle, \qquad \left\langle \chi' \right| = \sum_{\xi'} \chi_{\xi'}'^{*} \left\langle \xi \right|$$

Using this we can expand:

$$\left\langle \mathbf{p}', \chi' \right| S \left| \mathbf{p}, \chi \right\rangle = \sum_{\xi\ \xi'} \chi_{\xi'}'^{*} \chi_\xi \left\langle \mathbf{p}', \xi' \right| S \left| \mathbf{p}, \xi \right\rangle$$

From everything up to now it follows using similar arguments and calculations to those in **Problem 4.2** that we obtain:

$$\frac{d\sigma}{d\Omega}\left(\mathbf{p}', \chi' \leftarrow \mathbf{p}, \chi\right) = \left| \sum_{\xi\ \xi'} \chi_{\xi'}'^{*} f\left(\mathbf{p}', \xi' \leftarrow \mathbf{p}, \xi\right) \chi_\xi \right|$$

Problem 5.2

Let the spin operator be $\mathbf{s}$. The fact that the Hamiltonian H is spin-independent means that we have:

$$[\mathbf{s}, H] = 0$$

From this it easily follows that:

$$\left[\mathbf{s}, e^{-\frac{i}{\hbar}\int H dt}\right] = 0$$

For any limits of integration. Thus it should be true considering the integration from $-\infty$ to ∞, so we obtain:

$$[\mathbf{s}, S] = 0$$

Now taking the sandwich between two states we obtain (for brevity in the CM frame) and assuming that $|\xi\rangle$ are the eigenvectors of $\mathbf{s}$ with eigenvalues ξ:

$$\left\langle \mathbf{p}', \xi' \right| [\mathbf{s}, S] \left| \mathbf{p}, \xi \right\rangle = \left(\xi' - \xi\right) \left\langle \mathbf{p}', \xi' \right| S \left| \mathbf{p}, \xi \right\rangle = 0$$

This means that when $\xi \neq \xi'$, we necessarily have:

$$\left\langle \mathbf{p}', \xi' \right| S \left| \mathbf{p}, \xi \right\rangle = 0$$

From everything said up to now we can see that:

$$\left\langle \mathbf{p}', \xi' \right| S \left| \mathbf{p}, \xi \right\rangle = \delta_{\xi\xi'} \left\langle \mathbf{p}' \right| S \left| \mathbf{p} \right\rangle$$

As we can see the second term in independent of the spin, which is a direct consequence of the commutation between the spin operator and the Hamiltonian.
Which shows us that the amplitude matrix has the form $F\left(\mathbf{p}' \leftarrow \mathbf{p}\right) = f\left(\mathbf{p}' \leftarrow \mathbf{p}\right) I$. From this we can look at the differential cross section from an incoming unpolarized beam and look at how the particles are scattered for different spin states:

$$\frac{d\sigma}{d\Omega}\left(\mathbf{p}', \xi' \leftarrow \mathbf{p}\right) = \frac{1}{\left(2s_1 + 1\right)\left(2s_2 + 1\right)} \sum_{\xi} \frac{d\sigma}{d\Omega}\left(\mathbf{p}', \xi' \leftarrow \mathbf{p}, \xi\right) = \frac{1}{\left(2s_1 + 1\right)\left(2s_2 + 1\right)} \left| f\left(\mathbf{p}' \leftarrow \mathbf{p}\right)\right|^2$$

This shows the cross section for different final spins is the same and we can not differentiate the particles.

Problem 5.3

The differential cross section can be written out as:

$$\frac{d\sigma}{d\Omega}\left(\mathbf{p}',\chi'\leftarrow\mathbf{p},\chi\right)=\left|\chi'^{*}F\left(\mathbf{p}'\leftarrow\mathbf{p}\right)\chi\right|^{2}=\left|\sum_{\xi,\xi'}\chi'^{*}_{\xi'}f_{\xi',\xi}\chi_{\xi}\right|^{2}=$$

$$=\left(\left|\chi'^{*}_{\uparrow}f_{\uparrow\uparrow}\chi_{\uparrow}\right|^{2}+\left|\chi'^{*}_{\uparrow}f_{\uparrow\downarrow}\chi_{\downarrow}\right|^{2}+\left|\chi'^{*}_{\downarrow}f_{\downarrow\uparrow}\chi_{\uparrow}\right|^{2}+\left|\chi'^{*}_{\downarrow}f_{\downarrow\downarrow}\chi_{\downarrow}\right|^{2}\right)+$$
$$+2Re\left(\left(\chi'_{\uparrow}\chi^{*}_{\uparrow}\chi'^{*}_{\uparrow}\chi_{\downarrow}f^{*}_{\uparrow\uparrow}f_{\uparrow\downarrow}\right)+\left(\chi'_{\uparrow}\chi^{*}_{\uparrow}\chi'^{*}_{\downarrow}\chi_{\uparrow}f^{*}_{\uparrow\uparrow}f_{\downarrow\uparrow}\right)+\left(\chi'_{\uparrow}\chi^{*}_{\uparrow}\chi'^{*}_{\downarrow}\chi_{\downarrow}f^{*}_{\uparrow\uparrow}f_{\downarrow\downarrow}\right)\right)+$$
$$+2Re\left(\left(\chi'_{\uparrow}\chi^{*}_{\downarrow}\chi'^{*}_{\downarrow}\chi_{\uparrow}f^{*}_{\uparrow\downarrow}f_{\downarrow\uparrow}\right)+\left(\chi'_{\uparrow}\chi^{*}_{\downarrow}\chi'^{*}_{\downarrow}\chi_{\downarrow}f^{*}_{\uparrow\downarrow}f_{\downarrow\downarrow}\right)+\left(\chi'_{\downarrow}\chi^{*}_{\uparrow}\chi'^{*}_{\downarrow}\chi_{\downarrow}f^{*}_{\downarrow\uparrow}f_{\downarrow\downarrow}\right)\right)$$

We can see that in this equation we have ten different terms, accounting for the 10 different measurements that have to be done.

Problem 5.4

Similarly to before due to the unitarity of S we have $S^{\dagger}S=1$ and from $S=1+R$, we obtain:

$$R+R^{\dagger}=-R^{\dagger}R$$

Now we take the sandwich with $\langle\mathbf{p}',\chi'|$ and $|\mathbf{p},\chi\rangle$ and expansion of the unit operator in terms of the momentum eigenvectors and spins states we have:

$$\langle\mathbf{p}',\chi'|\,R\,|\mathbf{p},\chi\rangle+\left(\langle\mathbf{p},\chi|\,R\,|\mathbf{p}',\chi'\rangle\right)^{*}=\sum_{\xi''}\int d^{3}p''\left(\langle\mathbf{p}'',\xi''|\,R\,|\mathbf{p}',\chi'\rangle\right)^{*}\left(\langle\mathbf{p}'',\xi''|\,R\,|\mathbf{p},\chi\rangle\right)$$

Now using the expansion for R, and factoring out a common constant factor and a common δ-function we finally obtain:

$$f\left(\mathbf{p}',\chi'\leftarrow\mathbf{p},\chi\right)-f^{*}\left(\mathbf{p},\chi\leftarrow\mathbf{p}',\chi'\right)=\frac{i}{2\pi m}\sum_{\xi''}\int d^{3}p''\delta\left(E_{p}-E_{p''}\right)f^{*}\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p}',\chi'\right)f\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)$$

Now we consider $\mathbf{p}=\mathbf{p}'$ and $\chi=\chi'$ from which we have:

$$f\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)-f^{*}\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)=\frac{i}{2\pi m}\sum_{\xi''}\int d^{3}p''\delta\left(E_{p}-E_{p''}\right)f^{*}\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)f\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)$$

$$2iIm\left(f\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)\right)=\frac{i}{2\pi m}\sum_{\xi''}\int d^{3}p''\delta\left(E_{p}-E_{p''}\right)\left|f\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)\right|^{2}$$

$$Im\left(f\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)\right)=\frac{1}{4\pi m}\sum_{\xi''}\int d^{3}p''\delta\left(\frac{1}{2m}\left(p+p''\right)\left(p-p''\right)\right)\frac{d\sigma}{d\Omega}\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)$$

$$Im\left(f\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)\right)=\frac{p}{4\pi}\sum_{\xi''}\int d\Omega_{p''}\frac{d\sigma}{d\Omega}\left(\mathbf{p}'',\xi''\leftarrow\mathbf{p},\chi\right)=\frac{p}{4\pi}\sigma\left(\mathbf{p},\chi\right)$$

Finally we obtain the total cross section as:

$$\sigma\left(\mathbf{p},\chi\right)=\frac{4\pi}{p}Im\left(f\left(\mathbf{p},\chi\leftarrow\mathbf{p},\chi\right)\right)$$

Chapter 6

Problem 6.1

An antiunitary operator W is defined by:

$$U\left(a\left|\psi_1\right\rangle\right)=a^*\left(U\left|\psi_1\right\rangle\right)$$

$$|\langle W\psi|W\phi\rangle|=|\langle\psi|\phi\rangle|$$

We are going to do this problem in a different order, first we will do (c).
(c) Let the conjugation operator be denoted by K. If we are to use an orthonormal basis $\{|0\rangle,|1\rangle,...,|n\rangle,...\}$, we can then write:

$$K\left(a\left|\psi\right\rangle\right)=a^*\left(K\left|\psi\right\rangle\right)$$

$$|(K\psi,K\phi)|=\left|\sum_{nm}a_nb_m^*\left(n,m\right)\right|=\left|\sum_{n}a_nb_n^*\right|=\left|\sum_{n}a_n^*b_n\right|=|(\psi,\phi)|$$

From this we can see that K is antiunitary operator.
a) Any antiunitary operator W we can write as:

$$W=UK$$

Where U is some unitary operator. From this we obtain:

$$W\left|\psi\right\rangle=UK\sum_{i=0}^{\infty}a_i\left|i\right\rangle=\sum_{i=0}^{\infty}a_i^*U\left|i\right\rangle$$

From which we have:

$$(W\psi,W\phi)=\sum_{nm}\left(UKa_nn,UKb_mm\right)=\sum_{nm}a_nb_m^*\left(Un,Um\right)=\sum_{nm}a_nb_m^*\left(n,m\right)=\sum_{n}a_nb_n^*=(\phi,\psi)=(\psi,\phi)^*$$

b) From $\langle\psi|W^\dagger W|\psi\rangle=\langle\psi|\psi\rangle^*=\langle\psi|\psi\rangle$, we can conclude that in this case $W^\dagger W=1$. Now from the trick used in the text first letting $|\psi\rangle=|\phi\rangle+|\chi\rangle$ and then $|\psi\rangle=|\phi\rangle+i\,|\chi\rangle$, We obtain two equations:

$$\begin{cases}\langle\phi|W^\dagger W|\chi\rangle+\langle\chi|W^\dagger W|\phi\rangle=\langle\phi|\chi\rangle+\langle\chi|\phi\rangle\\ \langle\phi|W^\dagger W|\chi\rangle-\langle\chi|W^\dagger W|\phi\rangle=\langle\phi|\chi\rangle-\langle\chi|\phi\rangle\end{cases}$$

Adding them together we have:

$$\langle\phi|W^\dagger W|\chi\rangle=\langle\phi|\chi\rangle$$

Thus we can conclude that:

$$W^\dagger W=1$$

Similarly to what is done in the book one obtains:

$$WW^\dagger=1$$

Problem 6.2

Problem 6.3

Problem 6.4

Problem 6.5

Problem 6.6

Problem 6.7

Problem 6.8

Problem 6.9

Chapter 7

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