7.1

$$\partial_t P(n,t) = -(\lambda+\mu)P(n,t) + \mu P(n+1,t) + \lambda P(n-1,t)$$

(i) $P_k(t) = \frac{1}{\sqrt{N}}\sum_n P(n,t)\, e^{-ink} \Rightarrow$

$$\Rightarrow \partial_t P_k = -(\lambda+\mu)P_k + \mu e^{ik} P_k + \lambda e^{-ik} P_k \Rightarrow$$

$$\rightarrow P_k = P_k(0)\cdot \exp[-(\lambda+\mu)t + \mu e^{ik} t + \lambda e^{-ik} t]$$

$$P(n,0) = \delta(n) \Rightarrow P_k(0) = \frac{1}{\sqrt{N}} \Rightarrow$$

$$\Rightarrow P(n,t) = \frac{1}{\sqrt{N}}\sum P_k e^{ink} = \frac{N}{2\pi N}\int_{-\pi}^{\pi} dk\, e^{-(\lambda+\mu)t + (\mu+\lambda)t\cos k + i(\mu-\lambda)t\sin k + ink} \approx$$

$$\approx \frac{e^{-(\lambda+\mu)t}}{2\pi}\int_{-\pi}^{\pi} dk\, e^{(\lambda+\mu)t - \underbrace{\tfrac{1}{2}(\lambda+\mu)}_{D}tk^2 + i\underbrace{(\mu-\lambda)}_{\alpha}tk + ink} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-Dt\left(k^2 - ik\frac{n+\alpha t}{Dt}\right)} dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-Dt\left(k - i\frac{n+\alpha t}{2Dt}\right)^2 - \frac{(n+\alpha t)^2}{4Dt}} dk =$$

$$= \frac{e^{-\frac{(n+\alpha t)^2}{4Dt}}}{2\pi}\int e^{-Dt\xi^2} d\xi = \frac{e^{-\frac{(n+\alpha t)^2}{4Dt}}}{2\pi}\sqrt{\frac{\pi}{Dt}} = \boxed{\frac{e^{-\frac{(n+\alpha t)^2}{4Dt}}}{\sqrt{4\pi Dt}}} \quad (*)$$

(ii) $$\partial_t P = -(\lambda+\mu)P + \mu\left(P + 1\cdot\partial_x P + 1\cdot\tfrac{1}{2}\partial_x^2 P\right) + \lambda\left(P + (-1)\partial_x P + (-1)^2\partial_x^2 P\right) =$$

$$= \underbrace{\frac{\mu+\lambda}{2}}_{D}\partial_x^2 P + \underbrace{(\mu-\lambda)}_{\alpha}\partial_x P = D\partial_x^2 P + \alpha\partial_x P \Rightarrow$$

$$\Rightarrow \partial_t P = D\partial_x^2 P + \alpha\partial_x P \Rightarrow$$

$$\Rightarrow \partial_t P_k = -Dk^2 P_k + ik\alpha P_k \Rightarrow P_k = \overset{1}{P_{0k}}\, e^{-Dk^2 t + ik\alpha t} \Rightarrow$$

$$\Rightarrow P(x,t) = \frac{1}{2\pi}\int dk\, e^{-Dk^2 t + ik\alpha t - ikx} = \frac{1}{2\pi}\int dk\, e^{-Dt\left(k^2 + ik\frac{\alpha t + x}{Dt}\right)} =$$

$$= \frac{1}{2\pi}\int dk\, e^{-Dt\left(k + i\frac{x+\alpha t}{2Dt}\right)^2 - \frac{(x+\alpha t)^2}{4Dt}} = \boxed{\frac{e^{-\frac{(x+\alpha t)^2}{4Dt}}}{\sqrt{4\pi Dt}}} = (*)$$

совпали с дискретным при $\lambda t \gg 1$, $\mu t \gg 1$

(iii)

$$\dot{p}_n = -\lambda p_n + \lambda p_{n-1} \Rightarrow$$

$\Rightarrow$ пусть $\tilde{p}_n = \int_0^\infty e^{-st} p_n \, dt \Rightarrow$

н.у. $p(0,0)=1$

$$\Rightarrow \begin{cases} -1 + s\tilde{p}_0 = -\lambda \tilde{p}_0 + \lambda \tilde{p}_{N-1} \\ s\tilde{p}_1 = -\lambda \tilde{p}_1 + \lambda \tilde{p}_0 \\ \vdots \end{cases} \Rightarrow \begin{pmatrix} s+\lambda & \dots & & -\lambda \\ -\lambda & s+\lambda & \dots & 0 \\ & -\lambda & \ddots & \\ & & -\lambda & s+\lambda \end{pmatrix} \tilde{p} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \tilde{p}_k = \frac{\lambda}{s+\lambda} \tilde{p}_{k-1} = \left(\frac{\lambda}{s+\lambda}\right)^k \tilde{p}_0 ,$$

при этом $\sum p(n) = 1 \Rightarrow \sum \tilde{p} = \int_0^\infty 1 \cdot e^{-st} dt = \frac{1}{s} \Rightarrow$

$$\Rightarrow \sum_{k=0}^{N-1} \tilde{p}_0 \left(\frac{\lambda}{s+\lambda}\right)^k = \tilde{p}_0 \frac{\left(1 - \left(\frac{\lambda}{s+\lambda}\right)^{N-1}\right)}{1 - \frac{\lambda}{s+\lambda}} = \tilde{p}_0 \frac{(s+\lambda)\left(1 - \left(\frac{\lambda}{s+\lambda}\right)^{N-1}\right)}{s} = \frac{1}{s} \Rightarrow$$

$\Rightarrow$ при $N \to \infty$ (возвращаемся к исходной задаче про цепочку, а не кольцо):

$$\tilde{p}_0 \cdot \frac{s+\lambda}{s} = \frac{1}{s} \Rightarrow \tilde{p}_0 = \frac{1}{\lambda + s} \Rightarrow$$

$$\Rightarrow \tilde{p}_k(s) = \frac{\lambda^k}{(s+\lambda)^{k+1}}$$

возвращаясь обратно к $p_k(t)$, получим:

$$p_k(t) = \int \frac{ds}{2\pi i} e^{st} \frac{\lambda^k}{(s+\lambda)^{k+1}} = \lambda^k \operatorname*{res}_{s=-\lambda} \frac{e^{st}}{(s+\lambda)^{k+1}} = \lambda^k \cdot \frac{e^{-\lambda t}}{k!} \cdot t^k \Rightarrow$$

$$\Rightarrow \boxed{p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}}$$

П 11.1

$$\begin{cases} \dot{x}(t) = k\alpha(t)\,x(t) \\ x(0) = x_0 > 0 \end{cases}$$

$x(t) \geq 0$, $k > 0$

$0 \leq \alpha(t) \leq 1$

$$P = \int_0^T (1-\alpha)\,x\,dt$$

(i) хотим максимизировать P =>

=> введём $C(x,T,\alpha) = -\int_0^T (1-\alpha)\,x\,dt$ =>

$$\Rightarrow \begin{cases} \varphi(x) = 0 \\ R(x,t,\alpha) = (\alpha-1)\cdot x \end{cases}$$

ур-ие Г.-Я.-Б.:

$$-\partial_t J = \min_{\alpha}\left((\alpha-1)\cdot x + k\alpha x\,\partial_x J\right) =$$

$$= \begin{cases} -x & \text{при } \partial_x J > -\frac{1}{k} \\ kx\,\partial_x J & \text{при } \partial_x J < -\frac{1}{k} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \text{при } \partial_x J > -\frac{1}{k}: \ \partial_t J = x \Rightarrow J = xt + J_0, \quad J(T) = xT + J_0 = 0 \Rightarrow J(t) = x(t-T) \\ \text{при } \partial_x J < -\frac{1}{k}: \ \partial_t J = -kx\,\partial_x J \end{cases}$$

пусть $J = C_1(t) + C_2(t)\,x$ =>

$$\Rightarrow \dot{C}_1 + \dot{C}_2 x = -kx\,\partial_x(C_1 + C_2 x) = -kxC_2 \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{C}_2 = -kC_2 \\ \dot{C}_1 = 0 \end{cases} \Rightarrow \begin{cases} C_2 = C_{20}e^{-kt} \\ C_1 = C_{10} \end{cases}$$

$$\Rightarrow J = C_{10} + C_{20}e^{-kt}\cdot x$$

$$J(T) = 0 = C_{10} + C_{20}\,x\,e^{-kT} \Rightarrow$$

$$\Rightarrow J(t) = cx\left(e^{-kt} - e^{-kT}\right)$$

при этом:

$$\partial_x J_1 = t - T = -\frac{1}{k} \Rightarrow t = T - \frac{1}{k} \Rightarrow$$

$$\Rightarrow \partial_x J_2 = c\left(e^{-kt} - e^{-kT}\right) = c\left(e^{-kT+1} - e^{-kT}\right) = -\frac{1}{k} \Rightarrow$$

$$\Rightarrow c = -\frac{e^{kT}}{k(e-1)} \Rightarrow J_2 = \frac{x}{k(e-1)}\left(1 - e^{-k(t-T)}\right) \Rightarrow$$

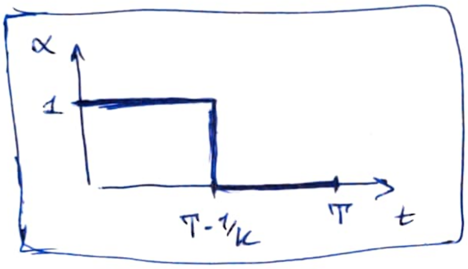
$$\Rightarrow \begin{cases} \text{при } t \geq T - \frac{1}{k}: & J = x(t-T) \\ \text{при } t < T - \frac{1}{k}: & J = \frac{x}{k(e-1)}\left(1 - e^{-k(t-T)}\right) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha = \arg\min\left((\alpha-1)x + (t-T)k\alpha x\right) = \boxed{0 \quad \text{при } t \geq T - \frac{1}{k}} \\ \alpha = \arg\min\left((\alpha-1)x + \frac{1}{k(e-1)}\left(1-e^{-k(t-T)}\right)k\alpha x\right) = \end{cases}$$

$$= \arg\min\left((\alpha-1)x + \frac{\alpha x}{(e-1)}\left(1 - e^{-k(T - \frac{1}{k} - \Delta t - T)}\right)\right) =$$

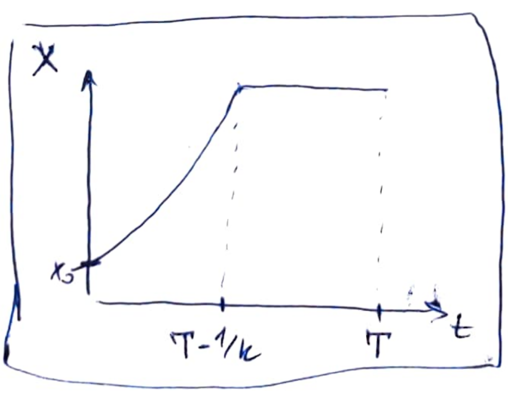
$$= \arg\min\left((\alpha-1)x + \frac{\alpha x}{e-1}\left(1 - e \cdot e^{k\Delta t}\right)\right) = \boxed{1 \quad \text{при } t < T - \frac{1}{k}}$$

$\Rightarrow$



при $k < \frac{1}{T} \Leftrightarrow T < \frac{1}{k} \Rightarrow T - \frac{1}{k} < 0 \Rightarrow$ контроль не нужен вовсе,
$\dot{x} = 0 \Rightarrow x = x_0$, $P = x_0 T$

(ii) $\dot{x} = k\alpha x \Rightarrow$ при $t < T - \frac{1}{k}$: $\dot{x} = kx \Rightarrow \boxed{x = x_0 e^{kt}}$
при $t \geq T - \frac{1}{k}$: $\dot{x} = 0 \Rightarrow \boxed{x = x_0 e^{kT-1}}$



(iii) $P = \int_0^T (1-\alpha)x\,dx = \int_{T-1/k}^{T} x_0 e^{kT-1}\,dT = \boxed{\frac{x_0}{k} e^{kT-1}}$

(iv)

$$\begin{cases} \dot{x} = k\,\alpha(t)\,x(t) + \xi(t) \\ \langle \xi \rangle = 0 \\ \langle \xi(t_1)\,\xi(t_2) \rangle = 2D\,\delta(t_1 - t_2) \end{cases} \Rightarrow$$

выведем уравнение Г.-Я.-Б. с шумом:

$$\langle J(t,x) \rangle = \min_\alpha \left(R + \langle J(t+dt, x + dx) \rangle \right) \approx$$

$$\approx \min_\alpha \left(R + \langle J(t,x) \rangle + \partial_t J \cdot \langle dt \rangle + \partial_x J \langle dx \rangle + \right.$$

$$\left. + \tfrac{1}{2} \partial_x^2 J \langle dx^2 \rangle \right) \Rightarrow$$

$$\Rightarrow -\partial_t J\, dt = \min \left(R + \langle dx \rangle \partial_x J + \tfrac{1}{2} \langle dx^2 \rangle \partial_x^2 J \right)$$

$$\begin{cases} \langle dx \rangle = f\, dt \\ \langle dx^2 \rangle = 2D\, dt \quad (\text{т.к. } \langle \xi(t_1)\,\xi(t_2) \rangle = 2D\,\delta(t_1 - t_2)) \end{cases} \Rightarrow$$

$$\Rightarrow -\partial_t J = \min \left(R + f \cdot \partial_x J + D \cdot \partial_x^2 J \right) \Rightarrow$$

$$\Rightarrow -\partial_t J = \min \left((\alpha - 1) x + k\alpha x\, \partial_x J + D \cdot \partial_x^2 J \right) = \min_\alpha \left((\alpha-1) x + k\alpha x\, \partial_x J \right) +$$

$$+ D\, \partial_x^2 J \Rightarrow$$

$\Rightarrow$ т.к. без шума $J \propto x$, то $\partial_x^2 J = 0 \Rightarrow$ мы получаем исходное ур-ние Г.-Я.-Б. $\Rightarrow$

$\Rightarrow$ ничего не изменяется

и $\alpha = \begin{cases} 1 & \text{при } t < T - 1/k \\ 0 & \text{при } t > T - 1/k \end{cases}$

(V) $\begin{cases} \dot{x} = k\alpha_0 x + \xi \\ x(0) > 0 \end{cases} \Rightarrow \partial_t n = D\,\partial_x^2 n - \partial_x (n\, k\alpha x) = D\,\partial_x^2 n + \beta\,\partial_x n x$,

где $\beta = -k\alpha \Rightarrow$

$$\Rightarrow \tilde{n} = \int dx\, e^{-iqx} n \Rightarrow \int dx\, e^{-iqx} \partial_x (nx) = -\int dx\, n x\, \partial_x e^{-iqx} =$$

$$= iq \int dx\, n x e^{-iqx} = -iq \frac{1}{i} \int dx\, n\, \partial_q e^{-iqx} =$$

$$= -q\,\partial_q \tilde{n} \Rightarrow$$

$$\Rightarrow \partial_t \tilde{n} = -Dq^2 \tilde{n} - \beta q\,\partial_q \tilde{n} \Rightarrow$$

$$\Rightarrow \partial_t \tilde{n} + \beta q\,\partial_q \tilde{n} = -Dq^2 \tilde{n}$$

метод хар-к: $\begin{cases} \frac{d\tilde{n}}{dt} = -Dq^2 \tilde{n} \\ \frac{dq}{d\tau} = \beta q \end{cases} \Rightarrow \begin{cases} \dot{\tilde{n}} = -Dq_0^2 e^{2\beta t} \tilde{n} \\ q = q_0 e^{\beta t} \end{cases} \Rightarrow$

$$\Rightarrow \tilde{n} = \tilde{n}(0) \cdot \cancel{e^{-\frac{D}{2\beta}q_0^2(e^{2\beta t}-1)}} \cdot \exp[-\tfrac{D}{2\beta} q_0^2 (e^{2\beta t} - 1)]$$

поглощающая "стенка" в $x=0$

$$\tilde{n}(q,0) = \int dx\, e^{-iqx} n(x,0) = \int dx\, e^{-iqx} (\delta(x-x_0) - \delta(x+x_0)) =$$

$$= e^{-iqx_0} - e^{iqx_0} \Rightarrow$$

$$\Rightarrow \tilde{n} = (e^{-iqx_0} - e^{iqx_0}) e^{-\frac{D}{2\beta} q_0^2 (e^{2\beta t} - 1)} = (e^{-iqx_0} - e^{iqx_0}) e^{-\frac{D}{2\beta} q^2 (1 - e^{-2\beta t})} \Rightarrow$$

$$\Rightarrow n(x,t) = \frac{1}{2\pi} \int dq\, e^{iqx} (e^{-iqx_0} - e^{iqx_0}) e^{-\frac{q^2}{2} \frac{D}{\beta} (1 - e^{-2\beta t})} =$$

$$= \frac{1}{\sqrt{2\pi \frac{D}{\beta}(1 - e^{-2\beta t})}} \left[\exp\left(-\frac{(x-x_0)^2}{2\frac{D}{\beta}(1-e^{-2\beta t})}\right) - \exp\left(-\frac{(x+x_0)^2}{2\frac{D}{\beta}(1-e^{-2\beta t})}\right) \right] \Rightarrow$$

$$\Rightarrow P_{surv}(t) = \int_0^\infty n(x,t)\,dx = \frac{1}{\sqrt{2\pi \frac{D}{\beta}(1-e^{-2\beta t})}} \left[\int_0^\infty e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx - \int_0^\infty e^{-\frac{(x+x_0)^2}{2\sigma^2}} dx \right] =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[\int_{-x_0}^\infty e^{-\frac{z^2}{2\sigma^2}} dz - \int_{x_0}^\infty e^{-\frac{z^2}{2\sigma^2}} dz \right] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{x_0} e^{-\frac{z^2}{2\sigma^2}} dz = \mathrm{Erf}\left[\frac{x_0}{\sqrt{2\sigma^2}}\right] \Rightarrow$$

$$\Rightarrow \quad P_{surv}(t) = \mathrm{Erf}\left[\frac{x_0}{\sqrt{2\sigma^2}}\right] = \mathrm{Erf}\left[\frac{x_0}{\sqrt{2\frac{D}{\beta}\cdot\left(1-e^{-2\beta t}\right)}}\right] = [\beta = -k\alpha] =$$

$$= \cancel{\mathrm{Erf}\left[\frac{x_0}{2\frac{D}{k\alpha}\left(e^{2k\alpha t}-1\right)}\right]}$$

$$= \boxed{\mathrm{Erf}\left[\frac{x_0}{\sqrt{2\frac{D}{k\alpha}\left(e^{2k\alpha t}-1\right)}}\right]} \Rightarrow \boxed{\text{при } t\to\infty \quad p_{surv} \to 0}$$

(при $k\alpha < 0$: $\mathrm{Erf}\left[x_0\sqrt{\frac{k\alpha}{2D}}\right]$)

(немного не понимаю: кажется, что ответ должен быть $p = \mathrm{Erf}\left[x_0\sqrt{\frac{k\alpha}{2D}}\right]$,

но мы знаем, что $p_{surv} = \mathrm{Erf}\left[\frac{x_0}{\sqrt{2\sigma^2}}\right]$,

из задачи о первом прохождении

а из ур-ния Ланжевена: $\sigma^2 = \int_0^t\int_0^t dt'dt'' e^{-\beta(t-t')} e^{-\beta(t-t'')} \langle \xi(t')\xi(t'')\rangle =$

$$= e^{-2\beta t}\, 2D\int_0^t dt'\, e^{2\beta t'} = \frac{D}{\beta} e^{-2\beta t}\left(e^{2\beta t}-1\right) =$$

$$= [\beta = -k\alpha] = -\frac{D}{k\alpha} e^{2k\alpha t}\left(e^{-2k\alpha t}-1\right) =$$

$$= \frac{D}{k\alpha}\left(e^{2k\alpha t}-1\right) \Rightarrow$$

$$\Rightarrow p = \mathrm{Erf}\left[\frac{x_0}{\sqrt{\frac{2D}{k\alpha}\left(e^{2k\alpha t}-1\right)}}\right] \to 0 \,)$$