2.4 Give context-free grammars that generate the following languages. In all parts, the alphabet $\sum$ is $\{0, 1\}$.

(b) {w | w starts and ends with the same symbol}

Solution.

A → 0B0 | 1B1 | 0 | 1
B → 0B | 1B | ϵ

■

(c) {w | the length of w is odd}

Solution.

A → 0 | 1 | 0A0 | 0A1 | 1A0 | 1A1

■

(e) {w | w = w^R, that is, w is a palindrome}

Solution.

A → 0 | 1 | 0A0 | 1S1 | ϵ

■

(f) The empty set

Solution.

A → A

■

2.9 Give a context-free grammar that generates the language

$$A = \{a^i b^j c^k \mid i = j \text{ or } j = k \text{ where } i, j, k \geq 0\}.$$

Is your grammar ambiguous? Why or why not?

Solution.

Language A is defined by two languages

$$A_1 = \{a^i b^j\ c^k \mid i, j, k \geq 0, i = j\}$$
$$A_2 = \{a^i b^j\ c^k \mid i, j, k \geq 0, j = k\}$$

The grammar for language A is the union of A_1 and A_2 or $A \rightarrow A_1 \mid A_2$ In the language A_1, the values of i and j are equal so there must be an equal number of a's and b's in the language A_1. Similarly, in the language A_2, the values of j and k are equal so there must be an equal number of b's and c's in the language A_2. Therefore, the language for A_1 and A_2 are:

$$A_1 \rightarrow A_1 c \mid X \mid \epsilon$$
$$X \rightarrow aXb \mid \epsilon$$
$$A_2 \rightarrow aA_2 \mid Y \mid \epsilon$$
$$Y \rightarrow bYc \mid \epsilon$$

For generating any string s = $a^i b^j\ c^k$ using the language A, either A_1 or A_2 can be used. Therefore, the grammar is ambiguous.

■

2.14 Convert the following CFG into an equivalent CFG in Chomsky normal form, using the procedure given in Theorem 2.9.

$$A \rightarrow BAB \mid B \mid \epsilon$$
$$B \rightarrow 00 \mid \epsilon$$

Solution.

(1) $S_0 \rightarrow A$
$A \rightarrow BAB \mid B \mid \epsilon$
$B \rightarrow 00 \mid \epsilon$

(2) $S_0 \rightarrow A$
$A \rightarrow BAB \mid BA \mid AB \mid A \mid B \mid \epsilon$
$B \rightarrow 00$

(3) $S_0 \rightarrow A \mid \epsilon$
$A \rightarrow BAB \mid BA \mid AB \mid BB \mid B$
$B \rightarrow 00$

(4) $S_0 \rightarrow A \mid \epsilon$
$A \rightarrow BAB \mid BA \mid AB \mid BB \mid 00$
$B \rightarrow 00$

(5) $S_0 \rightarrow BAB \mid BA \mid AB \mid BB \mid 00 \mid \epsilon$
$A \rightarrow BAB \mid BA \mid AB \mid BB \mid 00$
$B \rightarrow 00$

(6) $S_0 \rightarrow BAB \mid BA \mid AB \mid BB \mid N_0N_0 \mid \epsilon$
$A \rightarrow BAB \mid BA \mid AB \mid BB \mid N_0N_0$
$A_1 \rightarrow AB$
$B \rightarrow N_0N_0$
$N_0 \rightarrow 0$

■

2.33 Show that F = $\{a^i b^j \mid i = kj$ for some positive integer k$\}$ is not context-free.

Solution.

Let P be the pumping length for F that is guaranteed to exit by the pumping lemma. Select string s = $a^p b^{2p} \in$ F where k = 2 and divide the string into 5 parts uvxyz. According to the pumping lemma, v and y in s cannot be empty sets. Consider the cases where sub string v and y contain one or more than one type of alphabet symbol.
1) Both v and y contain only one type of alphabet symbol. In this case, both v and y does not contained mixed a's and b's. Thus the string does not violate any of the conditions of the lemma and does not contradict.
2) Either v or y contains more than one type of alphabet symbol. In this case, both v and y contain mixed a's and b's. Thus the string contains strings that are not a member of F. Thus violating the assumption and a contradiction is obtained, therefore language F is not context free.

■

2.43 For strings w and t, write w $\doteq$ t if the symbols of w are a permutation of the symbols of t. In other words, w $\doteq$ t if t and w have the same symbols in the same quantities, but possibly in a different order.

For any string w, define SCRAMBLE(w) = {t | t $\doteq$ w}. For any language A, let SCRAMBLE(A) = {t | t $\in$ SCRAMBLE(w) for some W in A}.

(a) Show that if $\sum = \{0, 1\}$, then the SCRAMBLE of a regular language is context free.

Solution.

The push down automata can be used to compare occurrences of two symbols, 1s and 0s, and can work to guess the transitions by reading the input. For language A, if w is a permutation for the string then there must be a path that leads to the accept with empty state. Therefore a PDA can be drawn and the SCRAMBLE of a regular language is context free.

■

(b) What happens in part (a) if $\sum$ contains three or more symbols? Prove your answer.

Solution.

The push down automata of a single stack can be used to compare the occurrences of two symbols, but for 3 or more symbols a PDA cannot be drawn. Therefore, the SCRAMBLE of a regular language with 3 or more symbols is not context free.

■