1. Let K be a subset of a topological space X. Show that the following two statements are logically equivalent:

   (a) For every family $\mathcal{F}$ of open sets in X:

   $$\text{if } \mathrm{K} \subseteq \bigcup_{\mathrm{O} \in \mathcal{F}} \mathrm{O}, \text{ then there exists a finite subfamily } \mathcal{F}_0 \subseteq \mathcal{F} \text{ such that } \mathrm{K} \subseteq \bigcup_{\mathrm{O} \in \mathcal{F}_0} \mathrm{O}.$$

   (b) Every family of closed subsets of K enjoying the finite intersection property has a nonempty intersection.

2. Prove that the image of a compact set under a continuous function is compact.

3. Consider a real-valued function f defined on a convex subset C of $\mathbb{R}^n$ such that for every $x_1, x_2 \in \mathrm{C}$, there exists $\alpha \in (0,1)$ for which $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1 + (1-\alpha)f(x_2)$.

   (a) Show that the function f is convex if it is continuous.

   (b) Does the converse of statement (a) also obtain in general? Justify your answer.

4. Let Σ be an alphabet. Prove that the family $\mathcal{P}\left(\Sigma^*\right)$ of all languages over alphabet Σ is uncountable.

Solution. That the set Σ^* of strings over Σ is countable has been shown during lecture. Accordingly consider a surjection $\phi : \mathbb{N} \longrightarrow \Sigma^*$. Define a mapping $\Phi : \mathcal{P}\left(\mathbb{N}\right) \longrightarrow \mathcal{P}\left(\Sigma^*\right)$ by setting for each $\mathrm{R} \in \mathcal{P}\left(\mathbb{N}\right)$:

$$\Phi\left(\mathrm{R}\right) \quad := \quad \left\{ w \in \Sigma^* : w = \phi(n) \text{ for some } n \in \mathrm{R} \right\}.$$

That is, the value of Φ when applied to R is the image of R under ϕ. It is readily verified that Φ is a bijection.

For *reductio ad absurdum*, assume that $\mathcal{P}\left(\Sigma^*\right)$ is countable. Then there is a surjection $\Psi : \mathbb{N} \longrightarrow \mathcal{P}\left(\Sigma^*\right)$, whereby it follows that the composition $\Phi^{-1} \circ \Psi$ is a surjective mapping from $\mathbb{N}$ onto $\mathcal{P}\left(\mathbb{N}\right)$, the existence of which is impossible since $\mathcal{P}\left(\mathbb{N}\right)$ is uncountable, as demonstrated during lecture. ■