0.11 Let $S(n) = 1+2+\ldots+n$ be the sum of the first n natural numbers and let $C(n) = 1^3+2^3+\ldots+n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = S^2(n)$ for every n.

(a) $S(n) = \frac{1}{2}n(n+1)$.

Solution.

Base case: $S(1) = 1$ and $\frac{1}{2}(1)(1+1) = 1$, thus $S(1) = \frac{1}{2}(1)(1+1)$

Inductive hypothesis: Assume that $S(n) = \frac{1}{2}(n)(n+1)$ is true

Inductive step: $S(n+1) = 1 + 2 + \ldots + n + (n+1)$

$S(n) = \frac{1}{2}(n)(n+1)$

$S(n+1) = S(n) + (n+1) = \frac{1}{2}(n)(n+1) + (n+1)$

$S(n+1) = \frac{1}{2}(n)(n+1) + \frac{2n+2}{2}$

$S(n+1) = \frac{1}{2}(n(n+1) + 2n + 2)$

$S(n+1) = \frac{1}{2}(n^2 + 3n + 2)$

$S(n+1) = \frac{1}{2}(n+1)(n+2)$

Thus, by principal of induction, S(n) is true for all n greater than or equal to 1. ■

(b) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Solution.

Base case: $C(1) = 1^3 = 1$ and $\frac{1}{4}1^2(1+1)^2 = 1$, thus $C(1) = \frac{1}{4}1^2(1+1)^2$

Inductive hypothesis: Assume that $C(n) = \frac{1}{4}n^2(n+1)^2$ is true.

Inductive step: $C(n+1) = 1^3 + 2^3 + \ldots + n^3 + (n+1)^3$

$C(n+1) = C(n) + (n+1)^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$

$C(n+1) = \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$

$C(n+1) = \frac{1}{4}(n+1)^2(n^2 + 4n + 4)$

$C(n+1) = \frac{1}{4}(n+1)^2(n+2)^2$

Thus, by principal of induction, C(n) is true for all n greater than or equal to 1. ■

0.12 Find the error in the following proof that all horses are the same color.

CLAIM: In any set of h horses, all horses are the same color.

PROOF: By induction on h.

Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

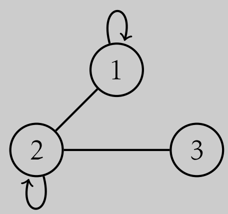
Induction step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k+1$. Take any set H of $k+1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore, all the horses in H must be the same color, and the proof is complete.

(a) The error lies in the induction step, where it assumes that the claim is true for $h = k$ and attempts to prove it for $h = k + 1$.

0.13 Show that every graph with two or more nodes contains two nodes that have equal degrees.

(a) Give a counterexample to the statement in the textbook.

Solution.



This is an undirected graph is an example of how the statement, "Every graph with two or more nodes contains two nodes that have equal degrees", is false. Node 1 has a degree of 2, node 2 has a degree of 3, and node 3 has a degree of 1. ■

(b) Formulate a true statement that is a charitable revision of the statement in the textbook.

Solution.

Every simple graph with two or more nodes contains two nodes that have equal degrees.

■

(c) Prove the revised statement is true.

Solution.

Assume that we have a simple graph with n nodes, where $n \geq 2$.
The degree of a node in a simple graph is the number of edges incident to that node.
The degrees of these n nodes can range from 0 to n - 1.
Since there are n nodes and n possible degrees, by the Pigeonhole Principle, if we have more than n nodes, then at least one degree must occur more than once.
Therefore, in a simple graph with two or more nodes, there exist two nodes that have equal degrees. ■