Logarithmic Regret Algorithms for Online Convex Optimization

Elad Hazan, Adam Kalai, Satyen Kale, and Amit Agarwal

Presenter: Zhe Li

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Background

Online Convex Optimization (OCO)

For $t = 1, 2, \cdots T$

- Learner (decision-maker) picks $\mathbf{x}_t \in K \subset \mathbb{R}^n$, where K is fixed convex set
- Environment responds with convex loss $f_t : K \to \mathbb{R}$
- Learner suffers loss $f_t(\mathbf{x}_t)$

The goal is to minimize

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x} \in K} \sum_{t=1}^{T} f_t(\mathbf{x}) = o(T) \tag{1}$$

Outline

- Online Gradient Descent: $GD\sqrt{T}$
- Online Gradient Descent for strongly convex $f_t(\cdot)$: $\frac{G^2}{2\alpha}(1+\log T)$
- Online Newton Step: $3(\frac{1}{\alpha}+4GD)n\log T$
- Exponentially Weighted Online Opt: $\frac{n}{\alpha}(1+\log(1+T))$

Where $||\nabla f_t(\mathbf{x})|| \leq G$ for all $\mathbf{x} \in K$, α is strongly convex constant for all $f_t(\mathbf{x})$ and D is the diameter of convex set $K, D = \sup_{\mathbf{x},\mathbf{y}\in K}||x-y||$

Online Gradient Descent

Gradient Descent(GD)

Input: convex set K, $T, \mathbf{x}_1 \in K$, and step sizes $\{\eta_t\}$

For $t = 1, 2, \cdots T$

- update and project

$$\mathbf{y}_{t+1} = \mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t) \quad \text{update step}$$
$$\mathbf{x}_{t+1} = \Pi_K[\mathbf{y}_{t+1}] \quad \text{project step}$$

End for

Online Gradient Descent

Online Gradient Descent (OGD)

Input: convex set K, $T, \mathbf{x}_1 \in K$, and step sizes $\{\eta_t\}$
For $t = 1, 2, \cdots T$

- play $\mathbf{x}_t$ and suffer loss $f_t(\mathbf{x}_t)$
- update and project

$$\mathbf{y}_{t+1} = \mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t) \quad \text{update step}$$
$$\mathbf{x}_{t+1} = \Pi_K[\mathbf{y}_{t+1}] \quad \text{project step}$$

End for

How to prove OGD can achieve $GD\sqrt{T}$?

Key points for analysis:

- $f_t(\mathbf{x}_t) - f_t(\mathbf{x}^*) \leq \bigtriangledown_t^T(\mathbf{x}_t - \mathbf{x}^*)$ from convexity of function $f_t(\cdot)$
- $||\mathbf{x}_{t+1} - \mathbf{x}^*||^2 = ||\Pi_K[\mathbf{x}_t - \eta_t \bigtriangledown_t] - \mathbf{x}^*||^2 \leq ||\mathbf{x}_t - \eta_t \bigtriangledown_t - \mathbf{x}^*||^2$
- Set step size as $\frac{D}{G\sqrt{t}}$

Why?

If restrict $f_t(\cdot)$ to be α-strongly convex?

Under this restriction, the regret bound can be improved to $\frac{G^2}{2\alpha}(1+\log T)$.
$O(\sqrt{T}) \rightarrow O(\log T)$

- $f_t(\mathbf{x}_t) - f_T(\mathbf{x}^*) \leq \bigtriangledown_t^T(\mathbf{x}_t - \mathbf{x}^*) - \frac{\alpha}{2}||\mathbf{x}_t - \mathbf{x}^*||^2$ from α-strongly convexity of function $f_t(\cdot)$
- Set step size $\frac{1}{\alpha t}$

Q: can we improve the regret bound if we make further assumption that $f_t(\cdot)$ is β-smooth?

Online Gradient Descent

Some extensions:

- Online mirror descent

$$\Phi(\mathbf{y}_{t+1}) = \Phi(\mathbf{x}_t) - \eta_t \nabla f_t(\mathbf{x}_t) \qquad \text{update step}$$
$$\mathbf{x}_{t+1} = \underset{\mathbf{x}\in K}{argmin} D_\Phi(\mathbf{x}||\mathbf{y}_{t+1}) \qquad \text{project step}$$

  For example: $\Phi(\mathbf{x}) = \frac{1}{2}||\mathbf{x}||^2$, Online Mirror Descent $\Longleftrightarrow$ Online Gradient Descent

- Stochastic Gradient Descent

Online Newton Step

Consider Portfolio management: $f_t(\mathbf{x}) = -\log(\mathbf{r}_t^T \mathbf{x})$,

- convex function but not strongly convex, Why?
- How to achieve regret bound $O(\log T)$?

Definition: α-exp-concave function

A convex function $f : \mathbb{R}^n \to \mathcal{R}$ is α-exp-concave over $K \subset \mathbb{R}^n$ if the function g is concave, where $g : K \to \mathbb{R}$ is defined as

$$g(\mathbf{x}) = e^{-\alpha f(\mathbf{x})}$$

Online Newton Step

Lemma

A twice differentiable function $f : \mathbb{R}^n \to \mathcal{R}$ is α-exp-concave over $K \subset \mathbb{R}^n$ if and only if

$$\nabla^2 f(\mathbf{x}) \succcurlyeq \alpha \nabla f(\mathbf{x}) \nabla f(\mathbf{x})^T$$

Lemma

Let $f : K \to \mathbb{R}$ be an α-concave function, and D, G denote the diameter of K and a bound on the (sub)-gradient of f respectively. The following holds for all $\gamma \leq \frac{1}{2}\min\{\frac{1}{4GD}, \alpha\}$

$$\forall \mathbf{x}, \mathbf{y} \in K : f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2}(\mathbf{x} - \mathbf{y})^T \nabla f(\mathbf{y}) \nabla f(\mathbf{y})^T (\mathbf{x} - \mathbf{y})$$

Looks similar? to what? How to use it?

Online Newton Step

Online Newton Step (ONS)

Input: convex set K, T, $\mathbf{x}_1 \in K$, and parameter $\gamma, \epsilon > 0$, $A_0 = \epsilon I_n$
For $t = 1, 2, \cdots T$

- play $\mathbf{x}_t$ and suffer loss $f_t(\mathbf{x}_t)$
- update $A_t = A_{t-1} + \nabla_t \nabla_t^T$ and project

$$\mathbf{y}_{t+1} = \mathbf{x}_t - \frac{1}{\gamma} A_t^{-1} \nabla_t \quad \text{update step}$$

$$\mathbf{x}_{t+1} = \Pi_K^{A_t}[\mathbf{y}_{t+1}] \quad \text{project step}$$

End for

Theorem

Online Newton Step algorithm with parameters $\gamma = \frac{1}{2}\min\{\frac{1}{GD}, \alpha\}, \epsilon = \frac{1}{\gamma^2 D^2}$ and $T > 4$ guarantees

$$\text{Regret}_T(\text{ONS}) \leq 5(\frac{1}{\alpha} + GD)n \log T \tag{2}$$

In order to prove the above regret bound,

Lemma

Online Newton Step algorithm with parameters $\gamma = \frac{1}{2}\min\{\frac{1}{GD}, \alpha\}, \epsilon = \frac{1}{\gamma^2 D^2}$ and $T > 4$ guarantees

$$\text{Regret}_T(\text{ONS}) \leq 4(\frac{1}{\alpha} + GD)(\sum_{t=1}^{T} \nabla_t^T A_t^{-1} \nabla_t + 1) \tag{3}$$

Online Newton Step

Key point for analysis:

- $f_t(\mathbf{x}_t) - f_t(x^*) \leq R_t := \nabla_t^T(\mathbf{x}_t - \mathbf{x}^*) - \frac{\gamma}{2}(\mathbf{x}^* - \mathbf{x}_t)^T \nabla_t \nabla_t^T (\mathbf{x}^* - \mathbf{x}_t)$ from α-exp-concave function $f_t(\cdot)$
- $(\mathbf{y}_{t+1} - \mathbf{x}^*)^T A_t (\mathbf{y}_{t+1} - \mathbf{x}^*) =$ $(\mathbf{x}_t - \mathbf{x}^*)^T A_t (\mathbf{y}_t - \mathbf{x}^*) - \frac{2}{\gamma} \nabla_t^T (\mathbf{x}_t - \mathbf{x}^*) + \frac{1}{\gamma^2} \nabla_t^T A_t^{-1} \nabla_t$
- $(\mathbf{y}_{t+1} - \mathbf{x}^*)^T A_t (\mathbf{y}_{t+1} - \mathbf{x}^*) \geq (x_{t+1} - \mathbf{x}^*)^T A_t (\mathbf{x}_{t+1} - \mathbf{x}^*)$
- $\nabla_t^T (\mathbf{x}_t - \mathbf{x}^*) \leq$ $\frac{1}{2\gamma} \nabla_t^T A_t^{-1} \nabla_t + \frac{\gamma}{2} (\mathbf{x}_t - \mathbf{x}^*)^T A_t (\mathbf{x}_t - \mathbf{x}^*) - \frac{\gamma}{2} (\mathbf{x}_{t+1} - \mathbf{x}^*)^T A_t (\mathbf{x}_{t+1} - \mathbf{x}^*)$

Online Newton Step

Remaining part:

- $\sum_{t=1}^{T} \nabla_t^T A_t^{-1} \nabla_t \leq n \log T$

Technical Lemma

Let $A \succcurlyeq B \succ 0$ be positive definite matrices. Then

$$A^{-1} \cdot (A - B) \leq \log \frac{|A|}{|B|} \tag{4}$$

Online Newton Step

Implementation and running time:

- $(A + \mathbf{x}\mathbf{x}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{x}\mathbf{x}^T A^{-1}}{1+\mathbf{x}^T A^{-1}\mathbf{x}}$
- Given A_{t-1}^{-1} and ∇_t, computing A_t^{-1} is $O(n^2)$.

Regret Bound summary

Table: Regret Bound Summary

	general	α-strongly convex	β-smooth	δ-exp-concave
upper bound	$\sqrt{T}$	$\frac{1}{\alpha}\log T$	$\sqrt{T}$	$\frac{n}{\delta}$
average bound	$\frac{1}{\sqrt{T}}$	$\frac{\log T}{\alpha T}$	$\frac{1}{\sqrt{T}}$	$\frac{n \log T}{\delta T}$
lower bound				

Question

- If $f_t(\cdot)$ is not convex?
- if we received $f_t(\mathbf{x}_t)$ instead of $f_t(\cdot)$?