Implementation of FDFD with Cylindrical Co-ordinates

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Relations between unit vectors

$$\hat{\mathbf{r}} = \cos(\theta)\hat{\mathbf{x}} + \sin(\theta)\hat{\mathbf{y}} \tag{1}$$

$$\hat{\theta} = -\sin(\theta)\hat{\mathbf{x}} + \cos(\theta)\hat{\mathbf{y}} \tag{2}$$

leading to

$$\hat{\mathbf{x}} = \cos(\theta)\hat{\mathbf{r}} - \sin(\theta)\hat{\theta} \tag{3}$$

$$\hat{\mathbf{y}} = \sin(\theta)\hat{\mathbf{r}} + \cos(\theta)\hat{\theta} \tag{4}$$

Curl operator in cylindrical coordinates

$$\begin{aligned}\nabla \times \mathbf{A} = &\left(-\frac{\partial A_\theta}{\partial z} + \frac{1}{r}\frac{\partial A_z}{\partial \theta}\right)\hat{\mathbf{r}} \\ &+ \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} \\ &+ \left(-\frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{1}{r}\frac{\partial (rA_\theta)}{\partial r}\right)\hat{\mathbf{z}}\end{aligned} \tag{5}$$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{6}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{7}$$

Let

$$
\begin{aligned}
\mathbf{E} = & \left(\sum_m E_r(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m E_r^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{r}} \\
& + \left(\sum_m E_\theta(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m E_\theta^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\theta} \\
& + \left(\sum_m E_z(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m E_z^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{z}} \qquad (8) \\
\mathbf{H} = & \left(\sum_m H_r(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m H_r^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{r}} \\
& + \left(\sum_m H_\theta(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m H_\theta^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\theta} \\
& + \left(\sum_m H_z(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m H_z^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{z}} \qquad (9) \\
\mathbf{J} = & \left(\sum_m J_r(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m J_r^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{r}} \\
& + \left(\sum_m J_\theta(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m J_\theta^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\theta} \\
& + \left(\sum_m J_z(r,z,m)e^{im\theta}e^{-i\omega t} + \sum_m J_z^*(r,z,m)e^{-im\theta}e^{i\omega t}\right)\hat{\mathbf{z}}, \qquad (10)
\end{aligned}
$$

where

$$m = 0, \pm 1, \pm 2, ...$$

$-\frac{\partial \tilde{E}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \tilde{E}_z}{\partial \theta} = -\mu \frac{\partial \tilde{H}_r}{\partial t}$ leads to:

$$
\begin{aligned}
& -\left(\sum_m \frac{\partial E_\theta}{\partial z}e^{im\theta}e^{-i\omega t} + \sum_m \frac{\partial E_\theta^*}{\partial z}e^{-im\theta}e^{i\omega t}\right) \\
& + \left(\sum_m \frac{im}{r}E_z e^{im\theta}e^{-i\omega t} - \sum_m \frac{im}{r}E_z^* e^{-im\theta}e^{i\omega t}\right) \\
& = \left(\sum_m i\omega\mu H_r e^{im\theta}e^{-i\omega t} - \sum_m i\omega\mu H_r^* e^{-im\theta}e^{i\omega t}\right). \qquad (11)
\end{aligned}
$$

Therefore,

$$-\frac{\partial E_\theta}{\partial z} + \frac{im}{r}E_z = i\omega\mu H_r, \quad -\frac{\partial E_\theta^*}{\partial z} - \frac{im}{r}E_z^* = -i\omega\mu H_r^* \qquad (12)$$

We can just consider the un-conjugated equation. Keep in mind that $m = 0, \pm 1, \pm 2, ...$

Altogether, the six Maxwell's equations are:

$$-\frac{\partial E_\theta}{\partial z} + \frac{im}{r} E_z = i\omega\mu H_r \tag{13}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = i\omega\mu H_\theta \tag{14}$$

$$-\frac{im}{r} E_r + \frac{1}{r}\frac{\partial (rE_\theta)}{\partial r} = i\omega\mu H_z \tag{15}$$

$$-\frac{\partial H_\theta}{\partial z} + \frac{im}{r} H_z = -i\omega\varepsilon E_r + J_r \tag{16}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = -i\omega\varepsilon E_\theta + J_\theta \tag{17}$$

$$-\frac{im}{r} H_r + \frac{1}{r}\frac{\partial (rH_\theta)}{\partial r} = -i\omega\varepsilon E_z + J_z. \tag{18}$$

If we let

$$\mathscr{D} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{im}{r} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial r} \\ -\frac{im}{r} & \frac{1}{r}\frac{\partial}{\partial r} r & 0 \end{pmatrix},$$

the Maxwell's equations are

$$\mathscr{D}_e \mathbf{E} = i\omega\mu\mathbf{H} \tag{19}$$

$$\mathscr{D}_h \mathbf{H} = -i\omega\varepsilon\mathbf{E} + \mathbf{J}, \tag{20}$$

so that

$$\mathscr{D}_h \mathscr{D}_e \mathbf{E} - \omega^2 \mu\varepsilon\mathbf{E} = i\omega\mu\mathbf{J}. \tag{21}$$

Keep in mind that the above equation is exclusively in the cylindrical co-ordinates (here, assuming constant μ and typically $\mu = 1$.)

Stretched Coordinate PMLs

Assume that PMLs are located at $r < r_s$, $r > r_b$, $z < z_s$, and $z > z_b$. The coordinates r and z shall be transformed according to [Taflove 1st book, Chapter 12, Page 538]

$$r \to r' = r_s - \int_r^{r_s} s_r(\tilde{r})\, d\tilde{r},\ r < r_s \tag{22}$$

$$r \to r' = r_b + \int_{r_b}^{r} s_r(\tilde{r})\, d\tilde{r},\ r > r_b \tag{23}$$

$$z \to z' = z_s - \int_z^{z_s} s_z(\tilde{z})\, d\tilde{z},\ z < z_s \tag{24}$$

$$z \to z' = z_b + \int_{z_b}^{z} s_z(\tilde{z})\, d\tilde{z},\ z > z_b \tag{25}$$

The differential operators will be transformed according to

$$\frac{\partial}{\partial r} \to \frac{1}{s_r}\frac{\partial}{\partial r} \tag{26}$$

$$\frac{\partial}{\partial z} \to \frac{1}{s_z}\frac{\partial}{\partial z} \tag{27}$$

The scale factors s_r and s_z should be chosen as [1]

$$s = 1 + i\sigma \left(\frac{l}{d}\right)^{\alpha}, \ \sigma = \frac{(\alpha+1)\ln R^{-1}}{2n_0\omega d} > 0. \tag{28}$$

Here, R is the desired reflection from the PML; n_0 is the refractive index of the background homogeneous medium; d is the thickness of the PML; Some typical choices are $m = 4$ and $R = e^{-16}$.

$l \leq d$ is the distance from the beginning of the PML and must be chosen as

$$\begin{aligned} l &= r_s - \tilde{r}, \ \tilde{r} < r_s \\ l &= \tilde{r} - r_b, \ \tilde{r} > r_b \\ l &= z_s - \tilde{z}, \ \tilde{z} < z_s \\ l &= \tilde{z} - z_b, \ \tilde{z} > z_b \end{aligned}$$

The integrals are then given by

$$\int_{r}^{r_s} s_r(\tilde{r})\, d\tilde{r} = \int_{r}^{r_s} 1 + i\sigma \left(\frac{r_s - \tilde{r}}{d}\right)^{\alpha} d\tilde{r} = r_s - r + i\frac{\sigma d}{\alpha+1}\left(\frac{r_s - r}{d}\right)^{\alpha+1} \tag{29}$$

$$\int_{r_b}^{r} s_r(\tilde{r})\, d\tilde{r} = \int_{r_b}^{r} 1 + i\sigma \left(\frac{\tilde{r} - r_b}{d}\right)^{\alpha} d\tilde{r} = r - r_b + i\frac{\sigma d}{\alpha+1}\left(\frac{r - r_b}{d}\right)^{\alpha+1} \tag{30}$$

$$\int_{z}^{z_s} s_z(\tilde{z})\, d\tilde{z} = \int_{z}^{z_s} 1 + i\sigma \left(\frac{z_s - \tilde{z}}{d}\right)^{\alpha} d\tilde{z} = z_s - z + i\frac{\sigma d}{\alpha+1}\left(\frac{z_s - z}{d}\right)^{\alpha+1} \tag{31}$$

$$\int_{z_b}^{z} s_z(\tilde{z})\, d\tilde{z} = \int_{z_b}^{z} 1 + i\sigma \left(\frac{\tilde{z} - z_b}{d}\right)^{\alpha} d\tilde{z} = z - z_b + i\frac{\sigma d}{\alpha+1}\left(\frac{z - z_b}{d}\right)^{\alpha+1} \tag{32}$$

The transformed coordinates then become

$$r' = r - i\frac{\sigma d}{\alpha+1}\left(\frac{r_s - r}{d}\right)^{\alpha+1}, \ r < r_s \tag{33}$$

$$r' = r + i\frac{\sigma d}{\alpha+1}\left(\frac{r - r_b}{d}\right)^{\alpha+1}, \ r > r_b \tag{34}$$

$$z' = z - i\frac{\sigma d}{\alpha+1}\left(\frac{z_s - z}{d}\right)^{\alpha+1}, \ z < z_s \tag{35}$$

$$z' = z + i\frac{\sigma d}{\alpha+1}\left(\frac{z - z_b}{d}\right)^{\alpha+1}, \ z > z_b. \tag{36}$$

With SC-PML, the curl operator becomes

$$\mathscr{D} = \begin{pmatrix} 0 & -s_z^{-1}\frac{\partial}{\partial z} & \frac{im}{r'} \\ s_z^{-1}\frac{\partial}{\partial z} & 0 & -s_r^{-1}\frac{\partial}{\partial r} \\ -\frac{im}{r'} & \frac{s_r^{-1}}{r'}\frac{\partial}{\partial r}r' & 0 \end{pmatrix}. \tag{37}$$

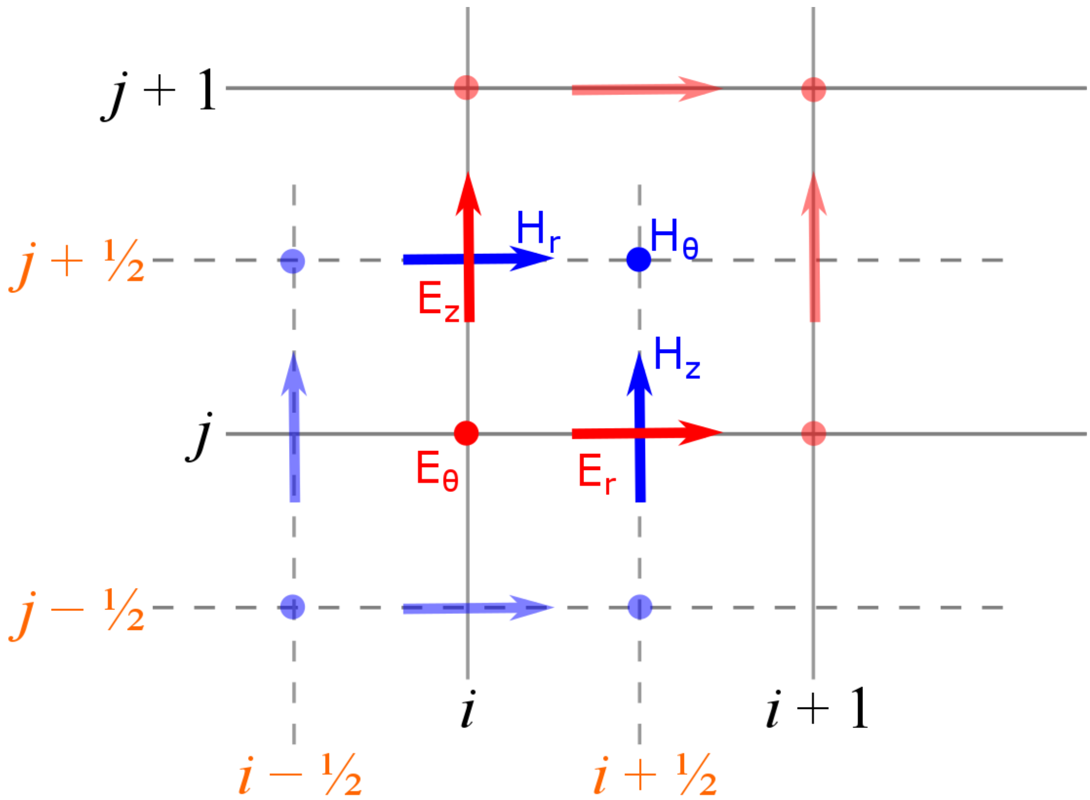

Fig. 1. Note the staggered Curl-conformng arrangement between **E** and **H** fields. We can focus our attention on the components at (i,j), $(i+\frac{1}{2},j)$, $(i,j+\frac{1}{2})$, $(i+\frac{1}{2},j+\frac{1}{2})$.

Discretization

To be clear, we re-write the Maxwell's equations with SC-PML:

$$-s_z^{-1}\frac{\partial E_\theta}{\partial z}+\frac{im}{r'}E_z=i\omega\mu H_r \tag{38}$$

$$s_z^{-1}\frac{\partial E_r}{\partial z}-s_r^{-1}\frac{\partial E_z}{\partial r}=i\omega\mu H_\theta \tag{39}$$

$$-\frac{im}{r'}E_r+\frac{s_r^{-1}}{r'}\frac{\partial(r'E_\theta)}{\partial r}=i\omega\mu H_z \tag{40}$$

$$-s_z^{-1}\frac{\partial H_\theta}{\partial z}+\frac{im}{r'}H_z=-i\omega\varepsilon E_r+J_r \tag{41}$$

$$s_z^{-1}\frac{\partial H_r}{\partial z}-s_r^{-1}\frac{\partial H_z}{\partial r}=-i\omega\varepsilon E_\theta+J_\theta \tag{42}$$

$$-\frac{im}{r'}H_r+\frac{s_r^{-1}}{r'}\frac{\partial(r'H_\theta)}{\partial r}=-i\omega\varepsilon E_z+J_z. \tag{43}$$

Note: you should compare these with the equations on Taflove 1st book, Chapter 12, Page 538.

Let's look at the Yee grid (Fig. 1).

Note that $r_i=R+i\Delta r,\quad z_j=j\Delta z$ where $R\geq 0$.

Let (i,j) be the implementation index (ie the actual index of C arrays). One can relate it to the real-space grid points as follows:

$$
\begin{aligned}
E_r,\ i,\ j \quad &\longleftrightarrow \quad i+\tfrac{1}{2},\ j \\
E_\theta,\ i,\ j \quad &\longleftrightarrow \quad i,\ j \\
E_z,\ i,\ j \quad &\longleftrightarrow \quad i,\ j+\tfrac{1}{2} \\
H_r,\ i,\ j \quad &\longleftrightarrow \quad i,\ j+\tfrac{1}{2} \\
H_\theta,\ i,\ j \quad &\longleftrightarrow \quad i+\tfrac{1}{2},\ j+\tfrac{1}{2} \\
H_z,\ i,\ j \quad &\longleftrightarrow \quad i+\tfrac{1}{2},\ j
\end{aligned}
$$

Now we can simply refer to the Yee grid (Fig. 1), the SC-PML Maxwell's equations and the index relations to write down the discretization scheme and its implementation in C. Note that discretization should be handled with care at the origin as opposed to elsewhere.

Grid points at $i > 0$

$$
\begin{aligned}
i\omega\mu\, H_r &= -s_z^{-1}\frac{\partial E_\theta}{\partial z} + \frac{im}{r'}E_z \\
i\omega\mu\, H_r[i, j+\tfrac{1}{2}] &= -\frac{s_z^{-1}[j+\tfrac{1}{2}]}{\Delta z}\, E_\theta[i, j+1] \\
&\quad + \frac{s_z^{-1}[j+\tfrac{1}{2}]}{\Delta z}\, E_\theta[i, j] \\
&\quad + \frac{im}{r'_i}\, E_z[i, j+\tfrac{1}{2}].
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
i\omega\mu\, H_\theta &= s_z^{-1}\frac{\partial E_r}{\partial z} - s_r^{-1}\frac{\partial E_z}{\partial r} \\
i\omega\mu\, H_\theta[i+\tfrac{1}{2}, j+\tfrac{1}{2}] &= +\frac{s_z^{-1}[j+\tfrac{1}{2}]}{\Delta z}\, E_r[i+\tfrac{1}{2}, j+1] \\
&\quad - \frac{s_z^{-1}[j+\tfrac{1}{2}]}{\Delta z}\, E_r[i+\tfrac{1}{2}, j] \\
&\quad - \frac{s_r^{-1}[i+\tfrac{1}{2}]}{\Delta r}\, E_z[i+1, j+\tfrac{1}{2}] \\
&\quad + \frac{s_r^{-1}[i+\tfrac{1}{2}]}{\Delta r}\, E_z[i, j+\tfrac{1}{2}]
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
i\omega\mu\, H_z &= -\frac{im}{r'}E_r + \frac{s_r^{-1}}{r'}\frac{\partial (r' E_\theta)}{\partial r} \\
i\omega\mu\, H_z[i+\tfrac{1}{2}, j] &= -\frac{im}{r'_{i+1/2}}\, E_r[i+\tfrac{1}{2}, j] \\
&\quad + \frac{s_r^{-1}[i+\tfrac{1}{2}]}{r'_{i+1/2}}\frac{r'_{i+1}}{\Delta r}\, E_\theta[i+1, j] \\
&\quad - \frac{s_r^{-1}[i+\tfrac{1}{2}]}{r'_{i+1/2}}\frac{r'_i}{\Delta r}\, E_\theta[i, j]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
-i\omega\varepsilon\, E_r &= -s_z^{-1}\frac{\partial H_\theta}{\partial z} + \frac{im}{r'}H_z - J_r \\
-i\omega\varepsilon\, E_r[i+\tfrac{1}{2},j] &= -\frac{s_z^{-1}[j]}{\Delta z}\, H_\theta[i+\tfrac{1}{2},j+\tfrac{1}{2}] \\
&\quad + \frac{s_z^{-1}[j]}{\Delta z}\, H_\theta[i+\tfrac{1}{2},j-\tfrac{1}{2}] \\
&\quad + \frac{im}{r'_{i+1/2}}\, H_z[i+\tfrac{1}{2},j] \\
&\quad - J_r[i+\tfrac{1}{2},j]
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
-i\omega\varepsilon\, E_\theta &= s_z^{-1}\frac{\partial H_r}{\partial z} - s_r^{-1}\frac{\partial H_z}{\partial r} - J_\theta \\
-i\omega\varepsilon\, E_\theta[i,j] &= +\frac{s_z^{-1}[j]}{\Delta z}\, H_r[i,j+\tfrac{1}{2}] \\
&\quad - \frac{s_z^{-1}[j]}{\Delta z}\, H_r[i,j-\tfrac{1}{2}] \\
&\quad - \frac{s_r^{-1}[i]}{\Delta r}\, H_z[i+\tfrac{1}{2},j] \\
&\quad + \frac{s_r^{-1}[i]}{\Delta r}\, H_z[i-\tfrac{1}{2},j] \\
&\quad - J_\theta[i,j]
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
-i\omega\varepsilon\, E_z &= -\frac{im}{r'}H_r + \frac{s_r^{-1}}{r'}\frac{\partial(r'H_\theta)}{\partial r} - J_z \\
-i\omega\varepsilon\, E_z[i,j+\tfrac{1}{2}] &= -\frac{im}{r'_i}\, H_r[i,j+\tfrac{1}{2}] \\
&\quad + \frac{s_r^{-1}[i]}{r'_i}\frac{r'_{i+1/2}}{\Delta r}\, H_\theta[i+\tfrac{1}{2},j+\tfrac{1}{2}] \\
&\quad - \frac{s_r^{-1}[i]}{r'_i}\frac{r'_{i-1/2}}{\Delta r}\, H_\theta[i-\tfrac{1}{2},j+\tfrac{1}{2}]
\end{aligned} \tag{49}
$$

Grid points at $i = 0$ and $R > 0$

Everything is fine! You can just set $H_z[-\frac{1}{2},j] = 0$ and $H_\theta[-\frac{1}{2},j+\frac{1}{2}] = 0$ since there will be PML.

Grid points at $i = 0$ and $R = 0$ and $|m| \geq 2$

Simply set

$$
H_r[0,j+\tfrac{1}{2}] = E_\theta[0,j] = E_z[0,j+\tfrac{1}{2}] = 0 \tag{50}
$$

so that you won't have to worry about $1/r_0$, $H_z[-\frac{1}{2},j]$ or $H_\theta[-\frac{1}{2},j+\frac{1}{2}]$. The difference equations for the remaining three components H_θ, H_z, E_r do not pose any issue.

Grid points at $i = 0$ and $R = 0$ and $m = 0$

See Taflove 1st book, Chapter 12, Page 529. Set

$$H_r[0, j+\tfrac{1}{2}] = E_\theta[0, j] = 0. \tag{51}$$

The rate of change of electric flux through a circular loop of radius $\frac{\Delta r}{2}$ is equal to the path integral of the magnetic field along the loop.

$$\begin{aligned} -i\omega\varepsilon\, E_z[0, j+\tfrac{1}{2}]\pi\frac{\Delta r^2}{4} &= 2\pi\frac{\Delta r}{2}\, H_\theta[\tfrac{1}{2}, j+\tfrac{1}{2}] \\ -i\omega\varepsilon\, E_z[0, j+\tfrac{1}{2}] &= \frac{4}{\Delta r}\, H_\theta[\tfrac{1}{2}, j+\tfrac{1}{2}] \end{aligned} \tag{52}$$

Grid points at $i = 0$ and $R = 0$ and $|m| = 1$

Set

$$E_z[0, j+\tfrac{1}{2}] = 0. \tag{53}$$

Recall the vector and integration directions in the Stoke's theorem: the right-hand rule!

Consider a rectangular loop around E_θ in Fig. 1. By the right-hand rule, if you go around the loop clock-wise (cw), the surface normal points into the page, same as $+\hat{\theta}$. Therefore,

$$\int_S (\nabla\times\mathbf{H})\cdot\, d\mathbf{S} = \int_S \left(\varepsilon\frac{\partial E_\theta}{\partial t}\hat{\theta}\right)\cdot\hat{\theta}\, dS = -i\omega\varepsilon E_\theta[0, j]\Delta r\Delta z$$

$$\int_{\partial S,\mathrm{cw}} \mathbf{H}\cdot d\mathbf{r} = H_r[0, j+\tfrac{1}{2}]\Delta r - H_z[\tfrac{1}{2}, j]\Delta z - H_r[0, j-\tfrac{1}{2}]\Delta r + H_z[-\tfrac{1}{2}, j]\Delta z$$

The trick is assuming $H_z[-\frac{1}{2}, j] = -H_z[\frac{1}{2}, j]$. Therefore,

$$\int_{\partial S,\mathrm{cw}} \mathbf{H}\cdot d\mathbf{r} = -2H_z[\tfrac{1}{2}, j]\Delta z + H_r[0, j+\tfrac{1}{2}]\Delta r - H_r[0, j-\tfrac{1}{2}]\Delta r$$

leading to

$$\begin{aligned} -i\omega\varepsilon E_\theta[0, j] = &+\frac{1}{\Delta z}\, H_r[0, j+\tfrac{1}{2}] \\ &-\frac{1}{\Delta z}\, H_r[0, j-\tfrac{1}{2}] \\ &-\frac{2}{\Delta r}\, H_z[\tfrac{1}{2}, j] \end{aligned} \tag{54}$$

On the other hand, the trick to set H_r is simply to approximate the term E_z/r at the next nearest neighbor (see Taflove 1st book, page 535).

$$\begin{aligned} i\omega\mu\, H_r[0, j+\tfrac{1}{2}] = &-\frac{s_z^{-1}[j+\frac{1}{2}]}{\Delta z}\, E_\theta[0, j+1] \\ &+\frac{s_z^{-1}[j+\frac{1}{2}]}{\Delta z}\, E_\theta[0, j] \\ &+\frac{im}{\Delta r}\, E_z[1, j+\tfrac{1}{2}]. \quad (i \text{ is imaginary number}) \end{aligned} \tag{55}$$

Planar current sheets in cylindrical coordinates

$$\begin{aligned}
\mathbf{J} = & \left[J_x(x,y)\hat{\mathbf{x}} + J_y(x,y)\hat{\mathbf{y}} \right] \delta(z-z_{\rm src}) e^{-i\omega t} \\
= & \left[J_x(r\cos\theta, r\sin\theta) \left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\theta} \right) \right. \\
& \left. + J_y(r\cos\theta, r\sin\theta) \left(\sin\theta \hat{\mathbf{r}} + \cos\theta \hat{\theta} \right) \right] \delta(z-z_{\rm src}) e^{-i\omega t} \\
= & \left[J_x(r\cos\theta, r\sin\theta)\cos\theta + J_y(r\cos\theta, r\sin\theta)\sin\theta \right] \delta(z-z_{\rm src}) e^{-i\omega t} \hat{\mathbf{r}} \\
& + \left[J_y(r\cos\theta, r\sin\theta)\cos\theta - J_x(r\cos\theta, r\sin\theta)\sin\theta \right] \delta(z-z_{\rm src}) e^{-i\omega t} \hat{\theta} \qquad (56) \\
= & \sum_m J_r(r,z,m) e^{im\theta} e^{-i\omega t} \hat{\mathbf{r}} + \sum_m J_\theta(r,z,m) e^{im\theta} e^{-i\omega t} \hat{\theta} \qquad (57)
\end{aligned}$$

leading to

$$\sum_m J_r(r,z,m) e^{im\theta} = \delta(z-z_{\rm src}) \left[J_x(r\cos\theta, r\sin\theta)\cos\theta + J_y(r\cos\theta, r\sin\theta)\sin\theta \right] \qquad (58)$$

$$\sum_m J_\theta(r,z,m) e^{im\theta} = \delta(z-z_{\rm src}) \left[J_y(r\cos\theta, r\sin\theta)\cos\theta - J_x(r\cos\theta, r\sin\theta)\sin\theta \right] \qquad (59)$$

Therefore,

$$J_r(r,z,m) = \tfrac{\delta(z-z_{\rm src})}{2\pi} \int_0^{2\pi} \left[J_x(r\cos\theta, r\sin\theta)\cos\theta + J_y(r\cos\theta, r\sin\theta)\sin\theta \right] e^{-im\theta}\, d\theta \quad (60)$$

$$J_\theta(r,z,m) = \tfrac{\delta(z-z_{\rm src})}{2\pi} \int_0^{2\pi} \left[J_y(r\cos\theta, r\sin\theta)\cos\theta - J_x(r\cos\theta, r\sin\theta)\sin\theta \right] e^{-im\theta}\, d\theta \quad (61)$$

Special case: normal incident plane wave source with x polarization

Let $J_x = A,\ J_y = 0$. Then $J_r = J_\theta = 0$ for $m \neq \pm 1$.

$$J_r(r,z,+1) = \frac{A}{2}\,\delta(z-z_{\rm src}), \quad J_\theta(r,z,+1) = \frac{iA}{2}\,\delta(z-z_{\rm src}) \qquad (62)$$

$$J_r(r,z,-1) = \frac{A}{2}\,\delta(z-z_{\rm src}), \quad J_\theta(r,z,-1) = -\frac{iA}{2}\,\delta(z-z_{\rm src}) \qquad (63)$$

General oblique incident plane wave source

Let $J_x = A_x e^{ik_x x + ik_y y}$, $J_y = A_y e^{ik_x x + ik_y y}$.

We need to evaluate the following two integrals:

$$P_m(r) = \int_0^{2\pi} \exp\left[ik_x r\cos\theta + ik_y r\sin\theta - im\theta \right] \cos\theta\, d\theta \qquad (64)$$

$$Q_m(r) = \int_0^{2\pi} \exp\left[ik_x r\cos\theta + ik_y r\sin\theta - im\theta \right] \sin\theta\, d\theta. \qquad (65)$$

Therefore,

$$J_r^m = \tfrac{\delta(z-z_{\rm src})}{2\pi} \left(A_x P_m + A_y Q_m \right) \qquad (66)$$

$$J_\theta^m = \tfrac{\delta(z-z_{\rm src})}{2\pi} \left(A_y P_m - A_x Q_m \right) \qquad (67)$$

Note that

$$\mathbf{k} = (\ k\sin\alpha\cos\phi,\ k\sin\alpha\sin\phi,\ k\cos\alpha\ ) \tag{68}$$

$$\tag{69}$$

Therefore,

$$\begin{aligned} k_x\cos\theta + k_y\sin\theta &= k\sin\alpha\,(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &= k\sin\alpha\cos(\theta-\phi). \end{aligned} \tag{70}$$

leading to

$$\begin{aligned} P_m(r) &= \tfrac{1}{2}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m+1)\theta\Big]\,d\theta \\ &\quad + \tfrac{1}{2}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m-1)\theta\Big]\,d\theta \end{aligned} \tag{71}$$

$$\begin{aligned} Q_m(r) &= \tfrac{i}{2}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m+1)\theta\Big]\,d\theta \\ &\quad - \tfrac{i}{2}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m-1)\theta\Big]\,d\theta. \end{aligned} \tag{72}$$

We will use the identity of the Bessel's first integral:

$$\int_0^{2\pi} \exp\Big[ia\cos\theta - in\theta\Big]\,d\theta = 2\pi i^n \mathfrak{J}_n(a). \tag{73}$$

Here $\mathfrak{J}_n(a)$ is the nth order Bessel's function of the first kind.

Note that

$$\begin{aligned} &\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m+1)\theta\Big]\,d\theta \\ &= e^{-i(m+1)\phi}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos(\theta-\phi) - i(m+1)(\theta-\phi)\Big]\,d\theta \\ &= e^{-i(m+1)\phi}\int_{-\phi}^{2\pi-\phi} \exp\Big[ikr\sin\alpha\cos\theta' - i(m+1)\theta'\Big]\,d\theta' \\ &= e^{-i(m+1)\phi}\int_0^{2\pi-\phi} + \int_{-\phi}^{0} \quad \exp\Big[ikr\sin\alpha\cos\theta' - i(m+1)\theta'\Big]\,d\theta' \\ &= e^{-i(m+1)\phi}\int_0^{2\pi-\phi} + \int_{2\pi-\phi}^{2\pi} \quad \exp\Big[ikr\sin\alpha\cos\theta' - i(m+1)\theta'\Big]\,d\theta' \\ &= e^{-i(m+1)\phi}\int_0^{2\pi} \exp\Big[ikr\sin\alpha\cos\theta' - i(m+1)\theta'\Big]\,d\theta' \\ &= 2\pi i^{m+1} e^{-i(m+1)\phi}\ \mathfrak{J}_{m+1}(kr\sin\alpha) \end{aligned} \tag{74}$$

Therefore,

$$P_m(r) = \pi\Big[\ i^{m+1}e^{-i(m+1)\phi}\ \mathfrak{J}_{m+1}(kr\sin\alpha) +\ i^{m-1}e^{-i(m-1)\phi}\ \mathfrak{J}_{m-1}(kr\sin\alpha)\Big] \tag{75}$$

$$Q_m(r) = i\pi\Big[\ i^{m+1}e^{-i(m+1)\phi}\ \mathfrak{J}_{m+1}(kr\sin\alpha) -\ i^{m-1}e^{-i(m-1)\phi}\ \mathfrak{J}_{m-1}(kr\sin\alpha)\Big] \tag{76}$$

Let's prove that P_m is even in m and Q_m is odd in m. Note that

$$J_x^m = \begin{bmatrix} P_m \\ -Q_m \\ 0 \end{bmatrix} \tag{77}$$

$$J_x^{-m} = \begin{bmatrix} P_{-m} \\ -Q_{-m} \\ 0 \end{bmatrix} = \begin{bmatrix} P_m \\ Q_m \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} J_x^m = \hat{\sigma} J_x^m \tag{78}$$

Therefore, we have that

$$\mathscr{D}_h^m \mathscr{D}_e^m E^m - \omega^2 \varepsilon E^m = i\omega J_x^m \tag{79}$$

$$\mathscr{D}_h^{-m} \mathscr{D}_e^{-m} E^{-m} - \omega^2 \varepsilon E^{-m} = i\omega J_x^{-m} \tag{80}$$

$$\hat{\sigma} \mathscr{D}_h^{-m} \mathscr{D}_e^{-m} \hat{\sigma}\hat{\sigma} E^{-m} - \omega^2 \hat{\sigma}\varepsilon\hat{\sigma}\hat{\sigma} E^{-m} = i\omega \hat{\sigma} J_x^{-m} \tag{81}$$

$$\mathscr{D}_h^m \mathscr{D}_e^m (\hat{\sigma} E^{-m}) - \omega^2 \varepsilon (\hat{\sigma} E^{-m}) = i\omega J_x^m \tag{82}$$

$$\implies \quad E^{-m} = \hat{\sigma} E^m \tag{83}$$

Far Field Transformation

By surface equivalence principle,

$$\mathbf{K} = -\hat{\mathbf{z}} \times \mathbf{E} = (\ E_y,\ -E_x,\ 0\ ) \tag{84}$$

$$\mathbf{J} = \hat{\mathbf{z}} \times \mathbf{H} = (\ -H_y,\ H_x,\ 0\ ) \tag{85}$$

Fix the coordinate notations:

$$\mathbf{r}_n = (\ r\cos\theta,\ r\sin\theta,\ z_0\ ) \qquad \text{(Source point)}$$

$$\mathbf{r}_f = (\ x,\ y,\ z\ ) \qquad \text{(Far field point)}$$

$$\mathbf{R} = (\ x - r\cos\theta,\ y - r\sin\theta,\ z - z_0\ ) \qquad \text{(Displacement from source to far)}$$

$$R = \sqrt{(x - r\cos\theta)^2 + (y - r\sin\theta)^2 + (z - z0)^2} \qquad \text{(Distance between source and far)}$$

The scalar Green's function is

$$g(\mathbf{r}_f, \mathbf{r}_n) = \frac{e^{ikR}}{4\pi R}, \quad k = \omega\sqrt{\mu\varepsilon} \tag{86}$$

The vector potentials are

$$\mathbf{A} = \mu \int\ g(R)\ \mathbf{J}(r,\theta)\ r d\theta dr \tag{87}$$

$$\mathbf{F} = \varepsilon \int\ g(R)\ \mathbf{K}(r,\theta)\ r d\theta dr \tag{88}$$

The far fields are given by

$$\mathbf{E}^f = \mu \int \left[i\omega\ g\mathbf{J} + \frac{i\omega}{k^2} \nabla(\nabla g \cdot \mathbf{J}) - \frac{1}{\varepsilon}(\nabla g) \times \mathbf{K} \right]\ r d\theta dr \tag{89}$$

$$\mathbf{H}^f = \varepsilon \int \left[i\omega\ g\mathbf{K} + \frac{i\omega}{k^2} \nabla(\nabla g \cdot \mathbf{K}) + \frac{1}{\mu}(\nabla g) \times \mathbf{J} \right]\ r d\theta dr \tag{90}$$

Let $g_\nu = \frac{\partial g}{\partial \nu}$ for $\nu = x,y,z$. Then

$$\begin{aligned}
\nabla(\nabla g \cdot \mathbf{J}) &= (\ g_{xx}J_x + g_{xy}J_y,\ g_{yx}J_x + g_{yy}J_y,\ g_{zx}J_x + g_{zy}J_y\ ) && (91)\\
(\nabla g) \times \mathbf{K} &= (\ -g_zK_y,\ g_zK_x,\ g_xK_y - g_yK_x\ ) && (92)\\
\nabla(\nabla g \cdot \mathbf{K}) &= (\ g_{xx}K_x + g_{xy}K_y,\ g_{yx}K_x + g_{yy}K_y,\ g_{zx}K_x + g_{zy}K_y\ ) && (93)\\
(\nabla g) \times \mathbf{J} &= (\ -g_zJ_y,\ g_zJ_x,\ g_xJ_y - g_yJ_x\ ) && (94)
\end{aligned}$$

Now let's write down the far field components separately.

$$\begin{aligned}
E_x^f &= \mu \int \left[i\omega g J_x + \frac{i\omega}{k^2}(g_{xx}J_x + g_{xy}J_y) + \frac{1}{\varepsilon} g_z K_y \right] r d\theta dr && (95)\\
E_y^f &= \mu \int \left[i\omega g J_y + \frac{i\omega}{k^2}(g_{yx}J_x + g_{yy}J_y) - \frac{1}{\varepsilon} g_z K_x \right] r d\theta dr && (96)\\
E_z^f &= \mu \int \left[\frac{i\omega}{k^2}(g_{zx}J_x + g_{zy}J_y) - \frac{1}{\varepsilon}(g_xK_y - g_yK_x) \right] r d\theta dr && (97)\\
H_x^f &= \varepsilon \int \left[i\omega g K_x + \frac{i\omega}{k^2}(g_{xx}K_x + g_{xy}K_y) - \frac{1}{\mu} g_z J_y \right] r d\theta dr && (98)\\
H_y^f &= \varepsilon \int \left[i\omega g K_y + \frac{i\omega}{k^2}(g_{yx}K_x + g_{yy}K_y) + \frac{1}{\mu} g_z J_x \right] r d\theta dr && (99)\\
H_z^f &= \varepsilon \int \left[\frac{i\omega}{k^2}(g_{zx}K_x + g_{zy}K_y) + \frac{1}{\mu}(g_xJ_y - g_yJ_x) \right] r d\theta dr && (100)
\end{aligned}$$

In terms of the near fields,

$$\begin{aligned}
E_x^f &= \mu \int \left[-i\omega g H_y - \frac{i\omega}{k^2}(g_{xx}H_y - g_{xy}H_x) - \frac{1}{\varepsilon} g_z E_x \right] r d\theta dr && (101)\\
E_y^f &= \mu \int \left[+i\omega g H_x - \frac{i\omega}{k^2}(g_{yx}H_y - g_{yy}H_x) - \frac{1}{\varepsilon} g_z E_y \right] r d\theta dr && (102)\\
E_z^f &= \mu \int \left[-\frac{i\omega}{k^2}(g_{zx}H_y - g_{zy}H_x) + \frac{1}{\varepsilon}(g_xE_x + g_yE_y) \right] r d\theta dr && (103)\\
H_x^f &= \varepsilon \int \left[+i\omega g E_y + \frac{i\omega}{k^2}(g_{xx}E_y - g_{xy}E_x) - \frac{1}{\mu} g_z H_x \right] r d\theta dr && (104)\\
H_y^f &= \varepsilon \int \left[-i\omega g E_x + \frac{i\omega}{k^2}(g_{yx}E_y - g_{yy}E_x) - \frac{1}{\mu} g_z H_y \right] r d\theta dr && (105)\\
H_z^f &= \varepsilon \int \left[+\frac{i\omega}{k^2}(g_{zx}E_y - g_{zy}E_x) + \frac{1}{\mu}(g_xH_x + g_yH_y) \right] r d\theta dr && (106)
\end{aligned}$$

Now E_x, E_y, H_x, H_y must be expressed in terms of $E_r, E_\theta, H_r, H_\theta$ and (r,θ).

$$\begin{aligned}
\mathbf{E} &= E_r \hat{\mathbf{r}} + E_\theta \hat{\theta}\\
&= E_r(\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}}) + E_\theta(-\sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{y}})\\
&= \Big[E_r\cos\theta - E_\theta \sin\theta \Big]\hat{\mathbf{x}} + \Big[E_r \sin\theta + E_\theta \cos\theta \Big]\hat{\mathbf{y}} && (107)\\
&= \Big[\sum_m E_r(r,z_0,m)e^{im\theta}\cos\theta - \sum_m E_\theta(r,z_0,m)e^{im\theta}\sin\theta \Big]\hat{\mathbf{x}}\\
&+ \Big[\sum_m E_r(r,z_0,m)e^{im\theta}\sin\theta + \sum_m E_\theta(r,z_0,m)e^{im\theta}\cos\theta \Big]\hat{\mathbf{y}} && (108)
\end{aligned}$$

Therefore

$$E_x = \sum_m E_r(r,z_0,m)e^{im\theta}\cos\theta - \sum_m E_\theta(r,z_0,m)e^{im\theta}\sin\theta \tag{109}$$
$$E_y = \sum_m E_r(r,z_0,m)e^{im\theta}\sin\theta + \sum_m E_\theta(r,z_0,m)e^{im\theta}\cos\theta \tag{110}$$
$$H_x = \sum_m H_r(r,z_0,m)e^{im\theta}\cos\theta - \sum_m H_\theta(r,z_0,m)e^{im\theta}\sin\theta \tag{111}$$
$$H_y = \sum_m H_r(r,z_0,m)e^{im\theta}\sin\theta + \sum_m H_\theta(r,z_0,m)e^{im\theta}\cos\theta \tag{112}$$

For ready evaluation,

$$E_x^f = \mu\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[-i\omega g H_y^m - \frac{i\omega}{k^2}(g_{xx}H_y^m - g_{xy}H_x^m) - \frac{1}{\varepsilon}g_z E_x^m\right] re^{im\theta}\ d\theta dr \tag{113}$$
$$E_y^f = \mu\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[+i\omega g H_x^m - \frac{i\omega}{k^2}(g_{yx}H_y^m - g_{yy}H_x^m) - \frac{1}{\varepsilon}g_z E_y^m\right] re^{im\theta}\ d\theta dr \tag{114}$$
$$E_z^f = \mu\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[-\frac{i\omega}{k^2}(g_{zx}H_y^m - g_{zy}H_x^m) + \frac{1}{\varepsilon}(g_x E_x^m + g_y E_y^m)\right] re^{im\theta}\ d\theta dr \tag{115}$$
$$H_x^f = \varepsilon\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[+i\omega g E_y^m + \frac{i\omega}{k^2}(g_{xx}E_y^m - g_{xy}E_x^m) - \frac{1}{\mu}g_z H_x^m\right] re^{im\theta}\ d\theta dr \tag{116}$$
$$H_y^f = \varepsilon\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[-i\omega g E_x^m + \frac{i\omega}{k^2}(g_{yx}E_y^m - g_{yy}E_x^m) - \frac{1}{\mu}g_z H_y^m\right] re^{im\theta}\ d\theta dr \tag{117}$$
$$H_z^f = \varepsilon\int_{r_0}^{r_1}\sum_m\int_0^{2\pi}\left[+\frac{i\omega}{k^2}(g_{zx}E_y^m - g_{zy}E_x^m) + \frac{1}{\mu}(g_x H_x^m + g_y H_y^m)\right] re^{im\theta}\ d\theta dr \tag{118}$$

where

$$E_x^m = E_r^m\cos\theta - E_\theta^m\sin\theta \tag{119}$$
$$E_y^m = E_r^m\sin\theta + E_\theta^m\cos\theta \tag{120}$$
$$H_x^m = H_r^m\cos\theta - H_\theta^m\sin\theta \tag{121}$$
$$H_y^m = H_r^m\sin\theta + H_\theta^m\cos\theta \tag{122}$$
$$g = \frac{e^{ikR}}{4\pi R}, \quad k = \omega\sqrt{\mu\varepsilon} \tag{123}$$
$$g_x = R_x(-1+ikR)\frac{e^{ikR}}{4\pi R^3} \tag{124}$$
$$g_y = R_y(-1+ikR)\frac{e^{ikR}}{4\pi R^3} \tag{125}$$
$$g_z = R_z(-1+ikR)\frac{e^{ikR}}{4\pi R^3} \tag{126}$$

Here, R_x, etc is the xth component of $\mathbf{R}$, etc; NOT $\partial_x R$, etc.

Also,

$$g_{xx} = \frac{e^{ikR}}{4\pi R^2}\left[\frac{3R_x^2}{R^3} - \frac{3ikR_x^2}{R^2} - \frac{(1+k^2R_x^2)}{R} + ik\right] \tag{127}$$

$$g_{yy} = \frac{e^{ikR}}{4\pi R^2}\left[\frac{3R_y^2}{R^3} - \frac{3ikR_y^2}{R^2} - \frac{(1+k^2R_y^2)}{R} + ik\right] \tag{128}$$

$$g_{xy} = g_{yx} = \frac{e^{ikR}R_xR_y}{4\pi R^3}\left[\frac{3}{R^2} - \frac{3ik}{R} - k^2\right] \tag{129}$$

$$g_{zx} = \frac{e^{ikR}R_xR_z}{4\pi R^3}\left[\frac{3}{R^2} - \frac{3ik}{R} - k^2\right] \tag{130}$$

$$g_{zy} = \frac{e^{ikR}R_yR_z}{4\pi R^3}\left[\frac{3}{R^2} - \frac{3ik}{R} - k^2\right] \tag{131}$$

References

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